

Complex locus – difference of arguments

The algebraic method

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Previously the locus obtained from the difference of arguments was obtained using a result from circle geometry (angle at the centre of a circle is twice the angle at the circumference). This note uses a purely algebraic method.

Background

Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ with arguments θ_1 and θ_2 respectively, then $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$ and if we have to solve the equation $\arg\left(\frac{z_1}{z_2}\right) = \alpha$ then we solve $\tan(\theta_1 - \theta_2) = \tan(\alpha)$.

This becomes $\frac{\tan(\theta_1) - \tan(\theta_2)}{1 + \tan(\theta_1)\tan(\theta_2)} = \tan(\alpha)$ and since $\tan(\theta_1) = \frac{y_1}{x_1}$ and $\tan(\theta_2) = \frac{y_2}{x_2}$ we can find the locus in Cartesian coordinates.

Example

Find the locus defined by $\arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{3}$.

Let $\theta_1 = \arg(z+1)$ and $\theta_2 = \arg(z-3)$ then $\tan(\theta_1) = \frac{y}{x+1}$ and $\tan(\theta_2) = \frac{y}{x-3}$

Then $\tan\left(\arg\left(\frac{z+1}{z-3}\right)\right) = \tan\left(\frac{\pi}{3}\right)$ becomes $\tan(\theta_1 - \theta_2) = \tan\left(\frac{\pi}{3}\right)$, leading to

$$\frac{\frac{y}{x+1} - \frac{y}{x-3}}{1 + \left(\frac{y}{x+1}\right)\left(\frac{y}{x-3}\right)} = \sqrt{3}$$

The LHS simplifies to

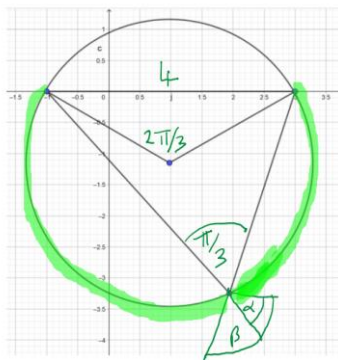
$$\frac{-4y}{x^2 - 2x - 3 + y^2}$$

Eliminating the fraction and completing the square for both x and y eventually results in

$$(x-1)^2 + \left(y + \frac{2}{\sqrt{3}}\right)^2 = \frac{16}{3}$$

This is a circle with centre $\left(1, -\frac{2}{\sqrt{3}}\right)$ and radius $\frac{4}{\sqrt{3}}$. However only an arc of this circle with endpoints $z = -1$ and 3 (not included in the arc) is the locus. In this case it is the lower arc:

$$(d) \arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{3}$$



A quick way to determine which arc is the solution is to choose two points on the circle vertically above and below the centre and calculate the arguments of the interval between these points and the endpoints of the arc. In the above

example the argument of the interval joining any point on the lower arc to $z = -1$ is less negative than the argument of the interval joining the same point to $z = 3$, resulting in a positive difference of $\frac{\pi}{3}$ as required.

Connection to co-ordinate geometry

In co-ordinate geometry the angle, θ , between two lines with gradients m_1 and m_2 is given by

$$\theta = \tan^{-1} \left[\frac{m_1 - m_2}{1 + m_1 m_2} \right]$$

This is the same expression as used in the complex case except m_1 and m_2 are the gradients of the intervals between z and the two fixed points.