

Complex numbers – plotting differences of arguments

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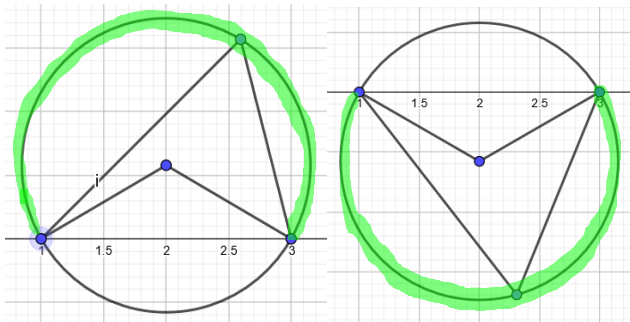
These questions are of the form $\text{Arg}\left(\frac{z-w}{z-v}\right) = \theta$

The points w and v will lie on an arc of a circle and usually you need to find the centre and radius except when $\theta = \pi/2$, in this case the centre is the midpoint of w and v .

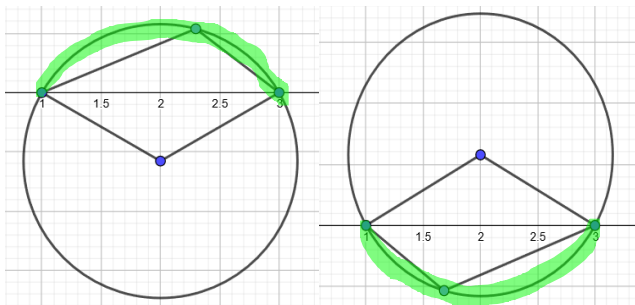
The arc of the circle depends upon whether the real or imaginary parts of w and v differ, whether w or v has the larger relevant coordinate and whether ϑ is acute or obtuse.

(1) Real parts differ.

These two plots show the arcs when the real parts differ and ϑ is acute. On the left $w = 3$ and $v = 1$ while on the right $w = 1$ and $v = 3$:

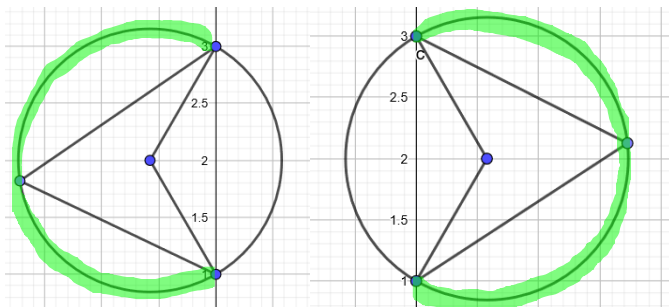


These two plots have ϑ obtuse and hence the solution is the minor arc. On the left $w = 3$ and $v = 1$ while on the right $w = 1$ and $v = 3$.

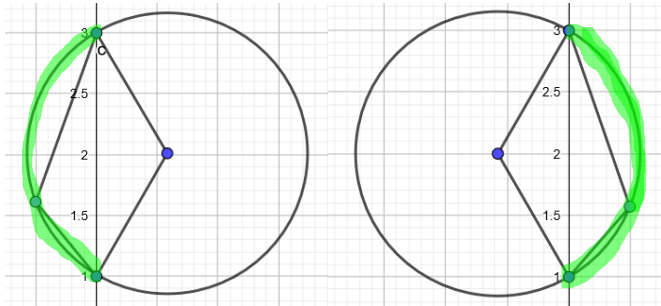


(2) Imaginary parts differ.

These two plots have ϑ acute. On the left $w = 3i$ and $v = i$ while on the right $w = i$ and $v = 3i$.



Here ϑ is obtuse. On the left $w = 3i$ and $v = 1$ while on the right $w = 1$ and $v = 3i$.



To summarise, acute angles give major arcs as solutions while obtuse angles yield minor arcs; for differences in real parts, if the larger real part is in the numerator the solution is the upper arc while for differences in imaginary parts if the larger part is in the numerator then the solution is the left part of the arc.

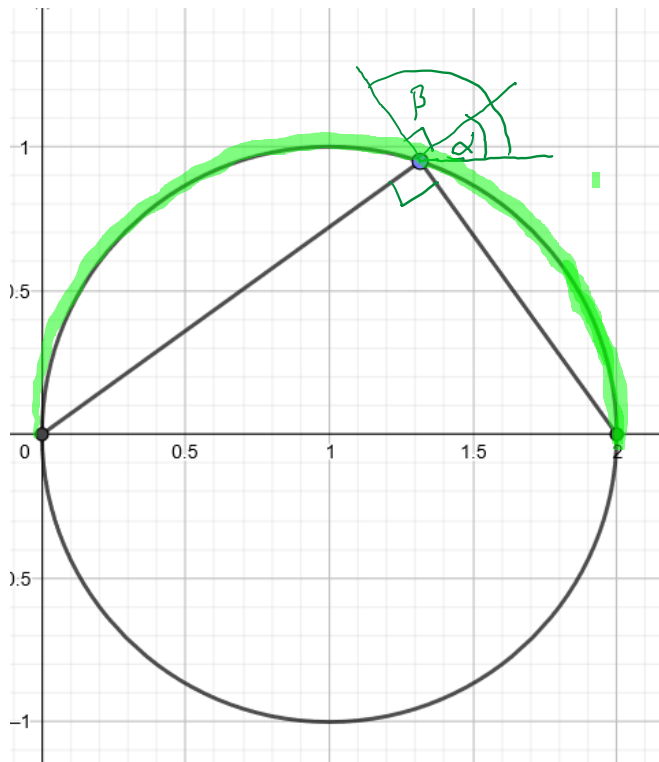
Note that the points themselves are not included in the solution (not shown in the above diagrams but in exams use open circles to denote the endpoints of the solution arcs).

Calculating the radius and location of the centre of the circle requires knowledge of the fact that for angles standing on the same arc the angle at the centre is twice the angle at the circumference. The special case arises when the angle at the centre is 180° , giving a right-angle at the circumference ie the angle in a semi-circle is a right-angle.

The triangle with vertices the centre of the circle and endpoints of the chord is isosceles with known angle at the centre and length of chord, therefore the radius and distance from the centre to the chord can be calculated. Since the equation of the chord is known the co-ordinates of the centre can be determined.

Examples (Q9 Exercise 1F Cambridge)

(a) $\arg\left(\frac{z-2}{z}\right) = \frac{\pi}{2}$

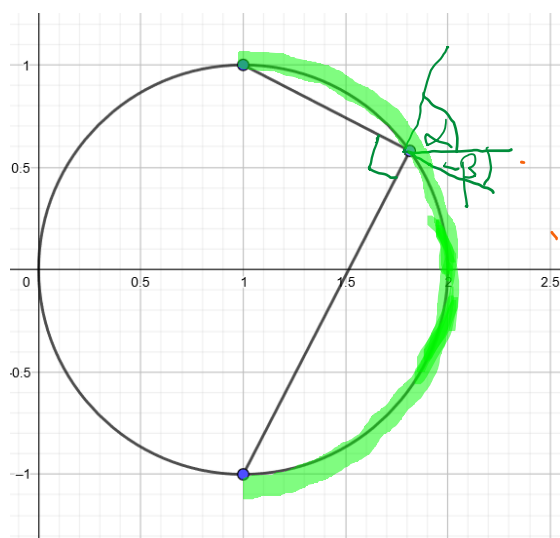


$\text{Arg}(z-2) = \beta$

$\text{Arg}(z) = \alpha$

$\text{Arg}(z-2) - \text{Arg}(z) = \beta - \alpha$
 $= \frac{\pi}{2}$

(b) $\arg\left(\frac{z-1+i}{z-1-i}\right) = \frac{\pi}{2}$

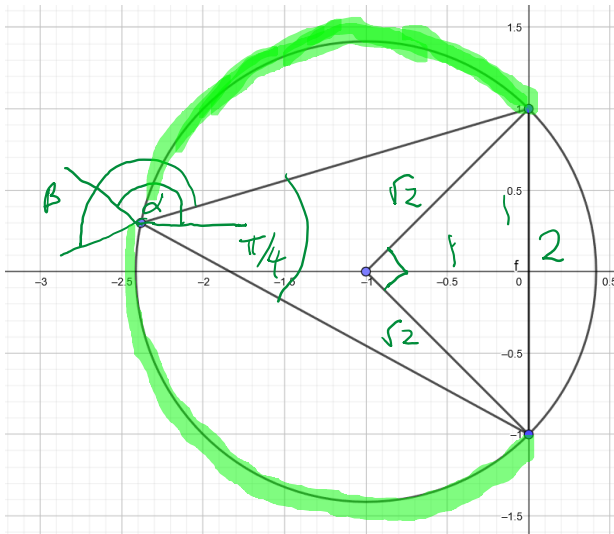


$\text{Arg}(z-1+i) = \alpha$

$\text{Arg}(z-1-i) = -\beta$

$\text{Arg}(z-1+i) - \text{Arg}(z-1-i) = \alpha - (-\beta)$
 $= \alpha + \beta$
 $= \frac{\pi}{2}$

$$(c) \arg\left(\frac{z-i}{z+i}\right) = \frac{\pi}{4}$$



$$\text{Arg}(z-i) = \beta$$

$$\text{Arg}(z+i) = \alpha$$

$$\text{Arg}(z-i) - \text{Arg}(z+i) = \beta - \alpha = \frac{\pi}{4}$$

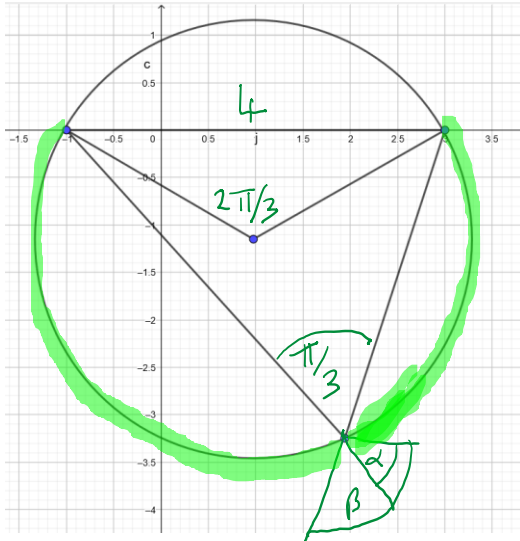
Since angle at centre

$$= \frac{\pi}{2} \Rightarrow \text{radius} = \sqrt{2}$$

and distance centre \rightarrow chord =

$$\therefore \text{Centre} = (-1, 0)$$

$$(d) \arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{3}$$



$$\text{Arg}(z+1) = -\alpha$$

$$\text{Arg}(z-3) = -\beta$$

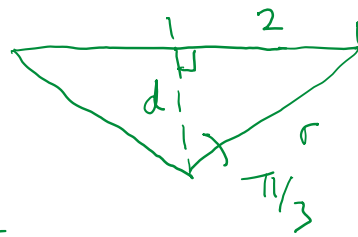
$$\text{Arg}(z+1) - \text{Arg}(z-3) = -\alpha + \beta = \frac{\pi}{3}$$

$$\therefore \text{Angle at centre} = 2\pi/3$$

$$d^2 + 2^2 = \left(\frac{4}{\sqrt{3}}\right)^2$$

$$d = \sqrt{\frac{16}{3} - 4} = \frac{2}{\sqrt{3}}$$

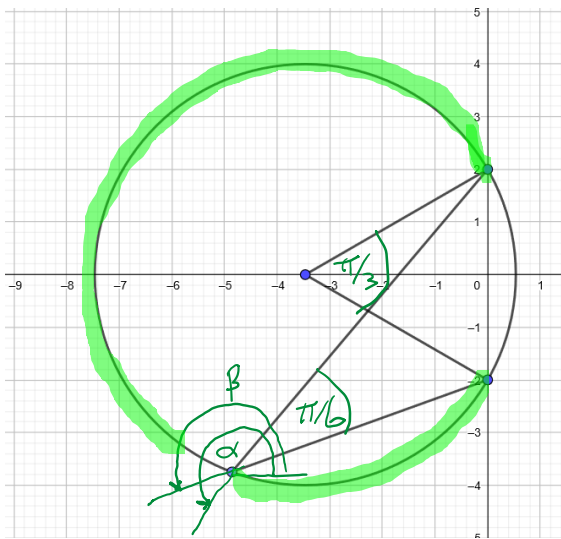
$$\therefore C = \left(1, -\frac{2}{\sqrt{3}}\right)$$



$$\sin \frac{\pi}{3} = \frac{2}{r}$$

$$r = \frac{2/\sqrt{3}}{2} = \frac{4}{\sqrt{3}}$$

$$(e) \arg\left(\frac{z-2i}{z+2i}\right) = \frac{\pi}{6}$$

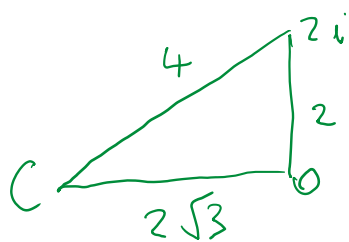


$$\text{Arg}(z-2i) = \alpha$$

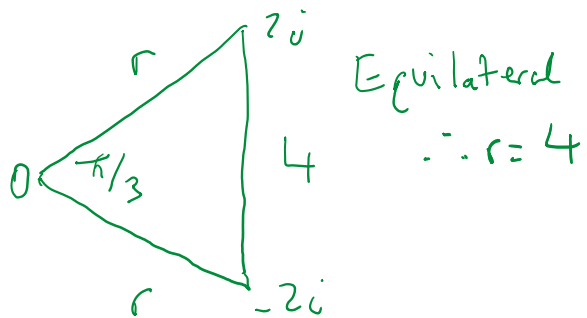
$$\text{Arg}(z+2i) = \beta$$

$$\text{Arg}(z-2i) - \text{Arg}(z+2i) = \alpha - \beta = \frac{\pi}{6}$$

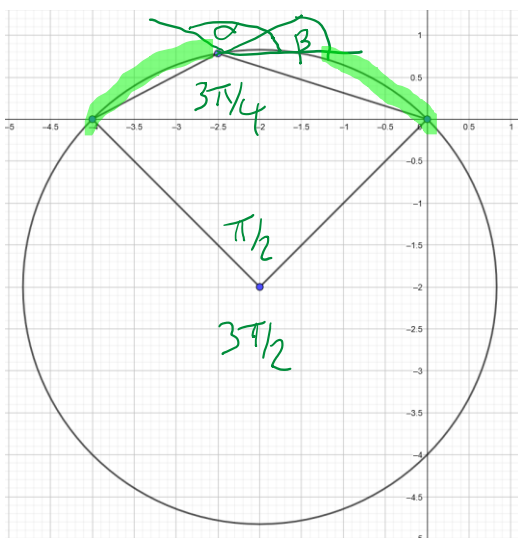
$$\therefore \text{Angle at centre} = \frac{\pi}{3}$$



$$\text{Centre} = (-2\sqrt{3}, 0)$$



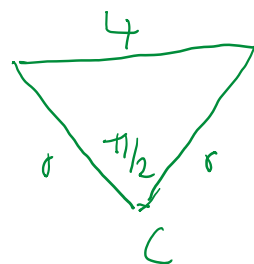
$$(f) \arg\left(\frac{z}{z+4}\right) = \frac{3\pi}{4}$$



$$\text{Arg}(z) = \alpha$$

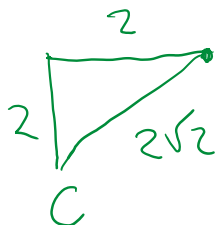
$$\text{Arg}(z+4) = \beta$$

$$\text{Arg}(z) - \text{Arg}(z+4) = \alpha - \beta = \frac{3\pi}{4}$$



$$2r^2 = 16$$

$$r = 2\sqrt{2}$$



$$C = (-2, -2)$$