The complications of a constant in a differential equation

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The inclusion of a constant in a differential equation can change the solution from simple to complex where a closed form solution may not even exist. Thsi note looks at one example.

1. An easily solved differential equation

Consider the equation $\frac{dy}{dx} = -\frac{x}{y}$ with $y(1) = 2$

This is solved by separation of variables:

$$\int y \, dy = -\int x \, dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

From the boundary condition we have $C = \frac{5}{2}$ and hence the solution is

$$y = \sqrt{5 - x^2}$$

The slope field in Figure 1 shows the solution as a collection of semi-circles (dependent upon the constant of integration) and is obtained from:

```
Show[StreamPlot[{1, -x/y}, {x, -3, 3}, {y, -3, 3}],
 Plot[-x, {x, -3, 3}], Plot[0, {x, -3, 3}]]
```

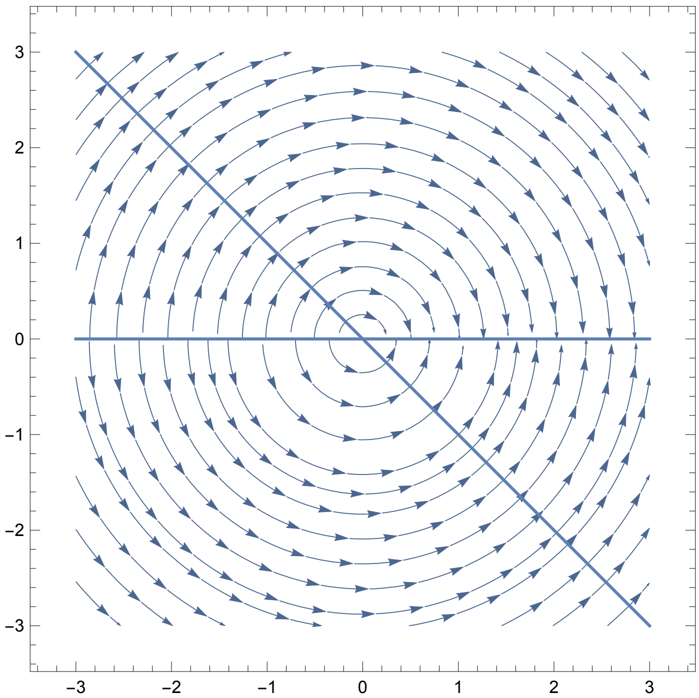

Figure 1: Slope field for $y' = -\frac{x}{y}$ with the line $y = -x$ (isocline with $y' = 1$)

2. A non-so-easily solved differential equation

If we alter our DE by including a constant then we find that it is not so easy to solve For simplicity we let the constant be 1. Thus we have $\frac{dy}{dx} = 1 - \frac{x}{y}$ with $y(1) = 2$

The slope field now looks a lot different, as shown in Figure 2..

```
Show[StreamPlot[{1, 1 - x/y}, {x, -3, 3}, {y, -3, 3}],
 Plot[x, {x, -3, 3}], Plot[0, {x, -3, 3}]]
```

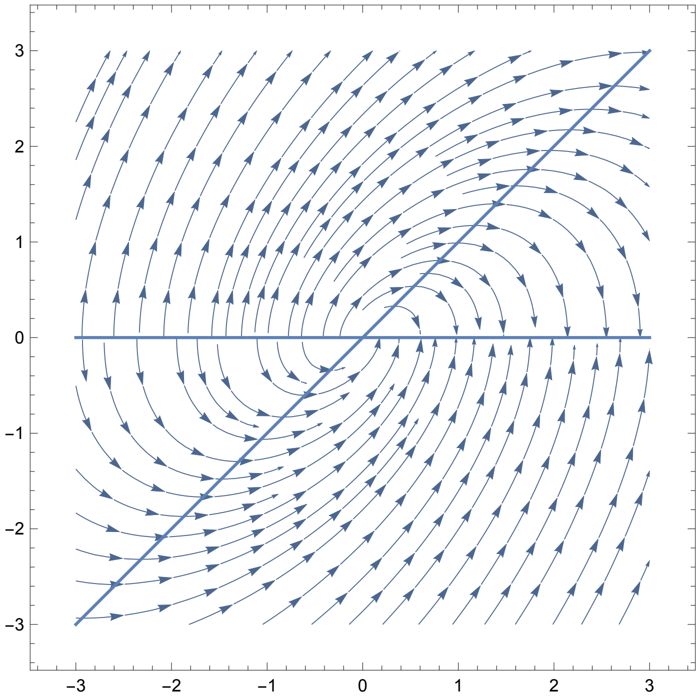

Figure 2: Slope field for $y' = 1 - \frac{x}{y}$ with the line $y = x$ (isocline with $y' = 0$)

The standard way of solving a DE when $\frac{dy}{dx}$ is a function of $\frac{x}{y}$ or $\frac{y}{x}$ is to introduce a new variable $v = \frac{y}{x}$ so that $y = v\,x$.

Then $\frac{dy}{dx} = v + \frac{dv}{dx}x$ and substituting into the DE we have $v + \frac{dv}{dx}x = 1 - \frac{1}{v}$

This becomes

$$\frac{dv}{dx}x = \frac{v-1-v^2}{v} \quad \Longrightarrow \quad \frac{v}{v^2-v+1}dv = -\frac{1}{x}dx$$

We can manipulate the numerator on LHS in order to integrate:

$$\frac{1}{2}\int\frac{2v-1+1}{v^2-v+1}\,dv = -\int\frac{1}{x}\,dx$$

$$\frac{1}{2}\int\frac{2v-1}{v^2-v+1}\,dv + \frac{1}{2}\int\frac{1}{\left(v-\frac{1}{2}\right)^2+\frac{3}{4}}\,dv = -\,\mathrm{Ln}(x) + C$$

$$\frac{1}{2}\,\mathrm{Ln}\left(v^2 - v + 1\right) + \frac{1}{\sqrt{3}}\,\mathrm{ArcTan}\left(\frac{2v-1}{\sqrt{3}}\right) = -\,\mathrm{Ln}(x) + C$$

Replacing v with $\frac{y}{x}$ $\quad \frac{1}{2}\text{Ln}\left(\left(\frac{y}{x}\right)^2 - \frac{y}{x} + 1\right) + \frac{1}{\sqrt{3}}\text{ArcTan}\left(\frac{2\frac{y}{x}-1}{\sqrt{3}}\right) = -\text{Ln}(x) + C$

Using the condition $y(1) = 2$ yields $C = \frac{1}{2}\text{Ln}(3) + \frac{\pi}{3\sqrt{3}}$

Therefore the final answer is

$$\frac{1}{2}\text{Ln}\left(\left(\frac{y}{x}\right)^2 - \frac{y}{x} + 1\right) + \frac{1}{\sqrt{3}}\text{ArcTan}\left(\frac{2\frac{y}{x}-1}{\sqrt{3}}\right) = -\text{Ln}(x) + \frac{1}{2}\text{Ln}(3) + \frac{\pi}{3\sqrt{3}}$$

Obviously there is no closed form solution for *y*. If we try to solve it using *Mathematica* we just obtain the same equation:

```
DSolve[{y'[x] == 1 - x/y[x], y[1] == 2}, y[x], x]
```

```
Solve[
 ArcTan[(-1 + 2 y[x]/x)/Sqrt[3]]/Sqrt[3] + 1/2 Log[1 - y[x]/x + y[x]^2/x^2] == 1/18 (2 Sqrt[3] π + 9 Log[3]) - Log[x], y[x]]
```

However we can at least solve it numerically and plot the solution:

```
s = NDSolve[{y'[x] == 1 - x/y[x], y[1] == 2}, y[x], {x, 0.0001, 4.5}]
```

```
{{y[x] → InterpolatingFunction[ Domain: {{0.0001, 4.29}} Output: scalar ][x]}}
```

This solution can be plotted in Figure 3.The *x*-intercept is 4.28965 (5dp). The function value when $x = 0.0001$ is 1.28029.

```
Plot[Evaluate[y[x] /. s], {x, 0.0001, 4.289}, PlotRange → All]
```

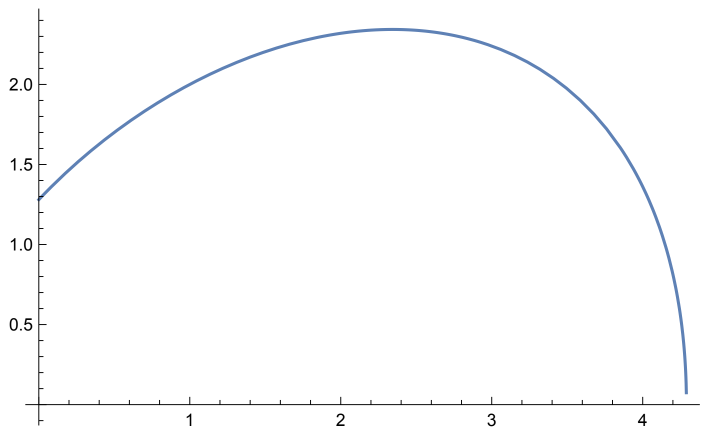

Figure 3: Solution curve for $y' = 1 - \frac{x}{y}$ with $y(1) = 2$

Figure 2 shows that there are possible solutions for negative values of *y*. We set up an equation from the solution off the DE, substitute values for *y*, equate to 0 and solve for *x*. The results are shown in Table 1 and plotted in Figure 4.

```
eqny[y_] := ArcTan[(-1 + 2 y/x)/Sqrt[3]]/Sqrt[3] + 1/2 Log[1 - y/x + y^2/x^2] + Log[x] - 1/18 (2 Sqrt[3] π + 9 Log[3])
```

$$\text{eqny[y\_]} := \frac{\text{ArcTan}\left[\frac{-1+\frac{2y}{x}}{\sqrt{3}}\right]}{\sqrt{3}} + \frac{1}{2}\text{Log}\left[1 - \frac{y}{x} + \frac{y^2}{x^2}\right] + \text{Log}[x] - \frac{1}{18}\left(2\sqrt{3}\,\pi + 9\,\text{Log}[3]\right)$$

```
t = Table[FindRoot[eqny[j], {x, 1}], {j, -8, 0, 0.5}];

TableForm[Table[{t[[j]][[1]][[2]], -8 + 0.5 (j - 1)}, {j, 1, 17}],
 TableHeadings → {None, {"x", "y"}}]
```

x	y
1.60242×10^{-14}	-8.
0.344503	-7.5
0.807543	-7.
1.2415	-6.5
1.64749	-6.
2.02628	-5.5
2.37837	-5.
2.70396	-4.5
3.00295	-4.
3.27496	-3.5
3.51925	-3.
3.73468	-2.5
3.91958	-2.
4.07162	-1.5
4.18751	-1.
4.26252	-0.5
4.28965	0.

Table 1: Solutions of $y' = 1 - \frac{x}{y}$ with y(1) = 2 for fixed negative values of *y*

```
ListPlot[%, Joined → True]
```

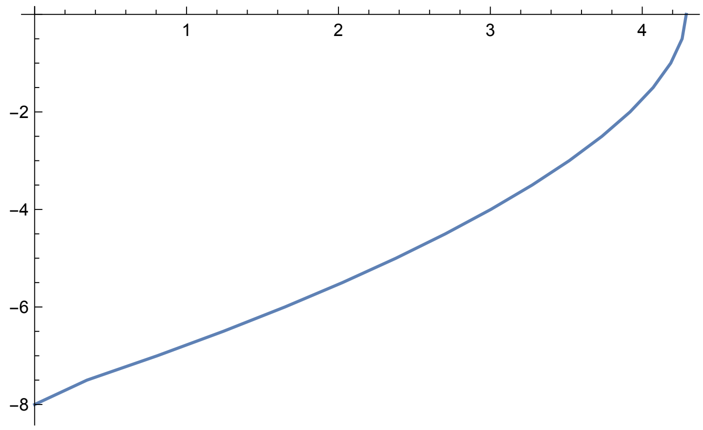

Figure 4: Solution curve for $y' = 1 - \frac{x}{y}$ with y(1) = 2 for fixed negative values of *y*

Figure 5 shows the full solution for positive *x*-values.

```
Show[Plot[Evaluate[y[x] /. s],
  {x, 0.0001, 4.289}, PlotRange → All, AspectRatio → 2],
 ListPlot[Table[{t[[j]][[1]][[2]], -8 + 0.5 (j - 1)}, {j, 1, 17}],
  Joined → True, AspectRatio → 2]]
```

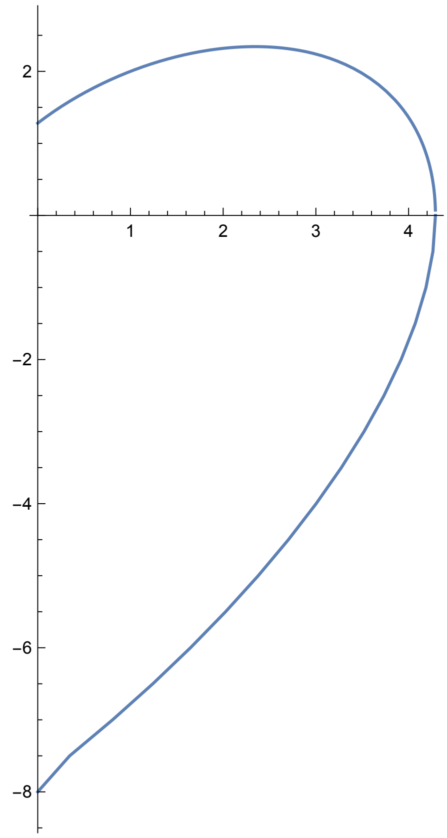

Figure 5: Full solution for $y' = 1 - \frac{x}{y}$ **with** $y(1) = 2$