Integrals of Sec θ and Cosec θ

Dr Richard Kenderdine

Integrating sec θ and cosec θ requires the use of the t-results ($t = \tan\frac{\theta}{2}$) and trigonometric identities.

We have $\cos\theta = \frac{1-t^2}{1+t^2}$, $\sin\theta = \frac{2t}{1+t^2}$ and $d\theta = \frac{2\,dt}{1+t^2}$

1) **Integrating Sec θ**

$$\int \sec\theta\, d\theta = \int \frac{1}{\cos\theta} d\theta = \int \frac{2}{1-t^2} dt$$

Using partial fractions we have

$$\int \frac{2}{1-t^2} dt = \int \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \ln\left|\frac{1+t}{1-t}\right| + C$$

We can replace t with $\tan\frac{\theta}{2}$ as one solution and also use the addition compound formula for $\tan(A+B)$ with $A = \frac{\pi}{4}$ and $B = \frac{\theta}{2}$:

$$\ln\left|\frac{1+t}{1-t}\right| = \ln\left|\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right| = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right|$$

Alternatively, we can multiply both numerator and denominator of $\frac{1+t}{1-t}$ with $1+t$

$$\frac{1+t}{1-t} \times \frac{1+t}{1+t} = \frac{1+t^2+2t}{1-t^2} = \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} = \sec\theta + \tan\theta$$

Hence we have another, probably more useful, solution:

$$\ln\left|\frac{1+t}{1-t}\right| = \ln|\sec\theta + \tan\theta|$$

To summarise,

$$\int \sec\theta\, d\theta = \ln\left|\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right| = \ln\left|\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right| = \ln|\sec\theta + \tan\theta| + C$$

2) **Integrating cosec θ**

$$\int \operatorname{cosec}\theta\, d\theta = \int \frac{1}{\sin\theta} d\theta = \int \frac{1}{t} dt = \ln|t| = \ln\left|\tan\left(\frac{\theta}{2}\right)\right| + C$$

To convert this back into functions with argument θ we use $\tan\theta = \frac{2t}{1-t^2}$ and solve for t:

$$(\tan\theta)t^2 + 2t - \tan\theta = 0$$

The solution is $t = \frac{-1+\sqrt{1+\tan^2\theta}}{\tan\theta} = -\cot\theta + \frac{\sec\theta}{\tan\theta} = \operatorname{cosec}\theta - \cot\theta$

Hence we have another solution: $\int \operatorname{cosec}\theta\, d\theta = \ln|\operatorname{cosec}\theta - \cot\theta| + C$