Linear transformation of Poisson Distribution

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The Poisson distribution for a random variable X with parameter λ has the probability function

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad (1)$$

Now suppose we have another random variable Y defined by $Y = aX + b$ ie a linear transformation of X. The probability function for Y is given by

$$P(Y = y) = \frac{\lambda^{\frac{y-b}{a}} e^{-\lambda}}{\left(\frac{y-b}{a}\right)!} \qquad (2)$$

It is important to recognise that while X can take on the values 0, 1, 2, ... ie non-negative integers, the values that Y can take are determined by the transformation $Y = aX + b$. Hence to calculate the expected value of Y, $E(Y)$, we cannot use the usual expression for a Poisson distributed random variable:

$$E(Y) = \sum_{y=0}^{\infty} y\, P(Y = y) \qquad (3)$$

Instead we have to use only those values that Y can take under the transformation and we use

$$E(Y) = \sum_{i=1}^{\infty} y_i\, P(Y = y_i) \qquad (4)$$

Calculation of Expected Value and Variance

The calculation of the Expected Value and Variance of the transformed variable Y follow the usual rules:

$$E(Y) = E(aX + b) = aE(X) + b = a\lambda + b \qquad (5)$$

$$Var(Y) = Var(aX + b) = a^2\, Var(X) = a^2 \lambda \qquad (6)$$

We can also calculate

$$E(Y^2) = E\left[(aX + b)^2\right] = Var(Y) + (E(Y))^2 = a^2\lambda + (a\lambda + b)^2 \qquad (7)$$

A function to calculate Expected Value and Variance

Here is a function in *Mathematica* to calculate the Expected Value and Variance for a general linear transformation of the Poisson distribution:

```
lintrpoisson[λ_, a_, b_] := (

  y = Table[a x + b, {x, 0, 99}];

  expect = Sum[y[[k]] (e^-λ λ^((y[[k]]-b)/a)) / (((y[[k]]-b)/a)!), {k, 1, 100}];

  expsq = Sum[(y[[k]])^2 (e^-λ λ^((y[[k]]-b)/a)) / (((y[[k]]-b)/a)!), {k, 1, 100}];

  Print["E(Y)= ", expect, "    E(Y^2)=   ",
   expsq, "   Var(Y)=   ", expsq - expect^2]
)
```

$$\text{expect} = \sum_{k=1}^{100} \left(y[[k]] \frac{e^{-\lambda} \lambda^{\frac{y[[k]]-b}{a}}}{\left(\frac{y[[k]]-b}{a}\right)!} \right);$$

$$\text{expsq} = \sum_{k=1}^{100} \left((y[[k]])^2 \frac{e^{-\lambda} \lambda^{\frac{y[[k]]-b}{a}}}{\left(\frac{y[[k]]-b}{a}\right)!} \right);$$

Example

Suppose we have the transformation $Y = 0.25\, X + 7$.

The values of Y corresponding to $X = 0, 1, 2, 3,.....$ are $Y = 7, 7.25, 7.5, 7.75,......$

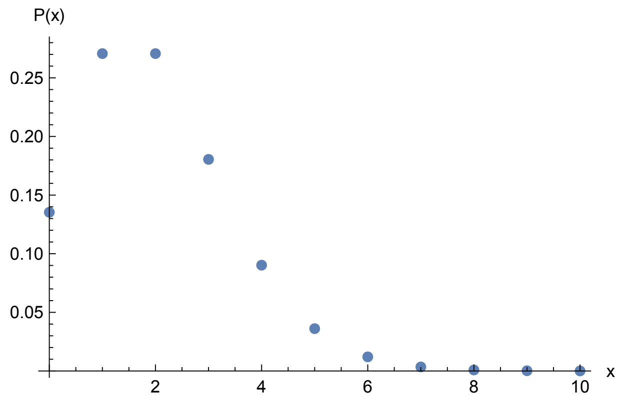
We run the function *lintrpoisson* with the input parameters $\lambda = 2$, a = 0.25 and b = 7:

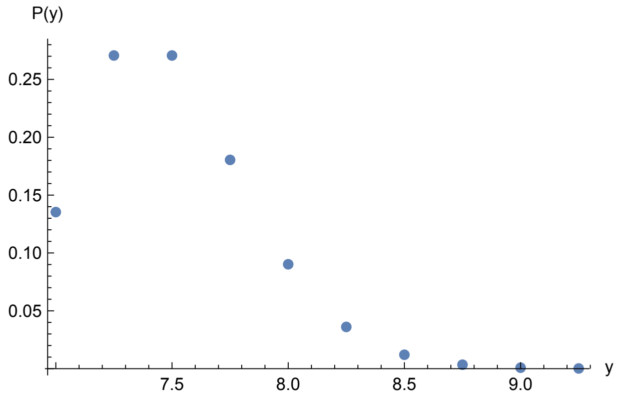
```
lintrpoisson[2, 0.25, 7]
```

```
E(Y)= 7.5    E(Y^2)=  56.375   Var(Y)=  0.125
```

The calculated values agree with (5) and (6): $E(Y) = 0.25 \times 2 + 7 = 7.5$ and $Var(Y) = 0.25^2 \times 2 = 0.125$

Probability density plots for X and Y are shown below. The only difference is the scale on the horizon- tal axis as defined by the transformation:





A conditional probability concerning two Poisson processes

Suppose we have two independent random variables X and Z that have Poisson distributions with parameters λ_1 and λ_2 respectively and we calculate $Y = X + Z$.

We want the probability that $X = k$ given that $Y = n$ ie $P(X = k \mid Y = n)$ with $0 \leq k \leq n$.

The standard result for conditional probability is $P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$ so here we have

$$P(X = k \mid Y = n) = \frac{P(X = k \text{ and } Y = n)}{P(Y = n)} \tag{8}$$

Now the sum of two Poisson distributions with parameters λ_1 and λ_2 is also a Poisson distribution with parameter $\lambda_1 + \lambda_2$, hence

$$P(Y = n) = \frac{(\lambda_1 + \lambda_2)^n e^{-(\lambda_1 + \lambda_2)}}{n!} \tag{9}$$

If $X = k$ and $Y = n$ then $Z = n - k$ and, since both X and Z are independent,

$$P(X = k \text{ and } Y = n) = P(X = k \text{ and } Z = n - k) = \frac{\lambda_1^k e^{-\lambda_1}}{k!} \times \frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!} \tag{10}$$

Then

$$P(X = k \mid Y = n) = \frac{\frac{\lambda_1^k e^{-\lambda_1}}{k!} \times \frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!}}{\frac{(\lambda_1+\lambda_2)^n e^{-(\lambda_1+\lambda_2)}}{n!}}$$

$$= \binom{n}{k} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \qquad (11)$$

$$= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$$

which is a binomial probability function with $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$