Solving systems of differential equations by matrix exponentials

Dr Richard Kenderdine
Kenderdine Maths Tutoring

Suppose we have a nonhomogeneous linear system of differential equations with constant coefficients defined by

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

where A is a constant $n \times n$ coefficient matrix, $\vec{x}$ is a vector $(x_1(t), \; \ldots.. x_n(t))$ and $\vec{f}(t) = (f_1(t), \; \ldots.. f_n(t))$.

The general solution is given by

$$\vec{x}(t) = e^{tA}\begin{pmatrix} C_1 \\ . \\ C_n \end{pmatrix} + \int^t e^{(t-s)A}\,\vec{f}(s)\,ds$$

If we have the initial conditions $\vec{x}(t_0) = \vec{x}_0$ then the solution is

$$\vec{x}(t) = e^{(t-t_o)A}\,\vec{x}_0 + \int_{t_0}^t e^{(t-s)A}\,\vec{f}(s)\,ds$$

Example

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\vec{x} + \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix} \quad \text{with } \vec{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

The matrix $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$ has eigenvalues (5,3) and corresponding eigenvectors ((-1,1), (-3,1)).

Using *Mathematica*, the matrix exponential is given by

$$e^{tA} = \begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{5t} & 0 \\ 0 & e^{3t} \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{3e^{3t}}{2} - \frac{e^{5t}}{2} & \frac{3e^{3t}}{2} - \frac{3e^{5t}}{2} \\ -\frac{e^{3t}}{2} + \frac{e^{5t}}{2} & -\frac{e^{3t}}{2} + \frac{3e^{5t}}{2} \end{pmatrix}$$

Then, using $t_0 = 0$, $\vec{x}_0 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\vec{f}(s) = \begin{pmatrix} e^{-2s} \\ e^{-2s} \end{pmatrix}$, the solution is

$$\vec{x}(t) = \begin{pmatrix} \frac{3e^{3t}}{2} - \frac{e^{5t}}{2} & \frac{3e^{3t}}{2} - \frac{3e^{5t}}{2} \\ -\frac{e^{3t}}{2} + \frac{e^{5t}}{2} & -\frac{e^{3t}}{2} + \frac{3e^{5t}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{3e^{3(t-s)}}{2} - \frac{e^{5(t-s)}}{2} & \frac{3e^{3(t-s)}}{2} - \frac{3e^{5(t-s)}}{2} \\ -\frac{e^{3(t-s)}}{2} + \frac{e^{5(t-s)}}{2} & -\frac{e^{3(t-s)}}{2} + \frac{3e^{5(t-s)}}{2} \end{pmatrix}\begin{pmatrix} e^{-2s} \\ e^{-2s} \end{pmatrix}, ds$$

$$= \begin{pmatrix} -\frac{1}{35} e^{-2t} \left(11 - 126 e^{5t} + 10 e^{7t}\right) \\ \frac{1}{35} e^{-2t} \left(-3 - 42 e^{5t} + 10 e^{7t}\right) \end{pmatrix}$$

We can check this by comparing both sides of the original system:

LHS $\partial_t \vec{x}(t)$ // FullSimplify

Out[30]//MatrixForm=

$$\begin{pmatrix} \frac{1}{35} e^{-2t} \left(22 + 378 e^{5t} - 50 e^{7t}\right) \\ \frac{2}{35} e^{-2t} \left(3 - 63 e^{5t} + 25 e^{7t}\right) \end{pmatrix}$$

RHS $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$

Out[33]//MatrixForm=

$$\begin{pmatrix} \frac{1}{35} e^{-2t} \left(22 + 378 e^{5t} - 50 e^{7t}\right) \\ \frac{2}{35} e^{-2t} \left(3 - 63 e^{5t} + 25 e^{7t}\right) \end{pmatrix}$$