Probability – matching subjects to objects with one object unused

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A textbook question asks

Five diners in a restaurant choose randomly from a menu featuring five main courses. Find the probability that exactly one of the main courses is not chosen by any of the diners.

Repeat the question if there are n *diners and a choice of* n *main courses.*

(Cambridge Mathematics Extension 1 Year 12, Pender *et al*)

This is a specific example of the more general problem of allocating *n* subjects to *n* objects with one object unused. This note describes two methods of solution.

Method *A*

1) Let the subjects be denoted by {1, 2, 3, 4, 5} and the objects as {*A, B, C, D, E*}. There are 5! ways of arranging the subjects. For illustrative purposes, take one of the permutations as 34125.
2) We need to combine two of the subjects as only four of the objects are utilised. For example, we could allocate subjects {3, 4, 1, 2, 5} to objects {*A*, *A*, *B*, *C*, *D*} respectively. Table 1 shows the 4 possible ways this can be done *for the permutation* 34125.

Object *A*	Object *B*	Object *C*	Object *D*
3 4	1	2	5
3	4 1	2	5
3	4	1 2	5
3	4	1	2 5

3) We need to halve the number of ways because, for example, the allocation (34)125 is the same as (43)125 (subjects 3 and 4 are allocated to Object *A* in both cases).
4) There are five ways of choosing the object not utilised.
5) The total number of ways of allocating 5 subjects to 4 objects is then $(5!)(4)(\frac{1}{2})(5) = (10)(5!)$

Method *B*

1) There are $\binom{5}{2} = 10$ ways of choosing the two subjects to share the same object
2) There are 4! ways of allocating the four units (3 single subjects and a double) to the four objects
3) There are 5 ways of selecting the object not utilised
4) The total number of ways of allocating 5 subjects to 4 objects is then (10)(4!)(5)=(10)(5!)

There are 5^5 ways of allocating five subjects to five objects without restriction and hence the probability that one object will not be utilised is $\frac{10\times5!}{5^5} = \frac{48}{125}$

Generalising this to *n* subjects and *n* objects we have the probability that one object is not utilised is

$$\frac{\binom{n}{2}(n-1)!\,n}{n^n} = \frac{n^2(n-1)^2(n-2)!}{2n^n}$$