Rotations, reflections and the Klein four-group

An introduction to Abstract Algebra for secondary school students

Dr Richard Kenderdine

Kenderdine Maths Tutoring

Senior mathematics students learn about transforming functions by a 180° rotation or reflection in either the *x*- or *y*-axis. These transformations and their combinations provide an application of the Klein four-group studied in Abstract Algebra and hence provide a suitable introduction for secondary school students.

Consider the function $f(x) = 4(x-1)^2 + 2$, plotted as Function A in Figure 1.

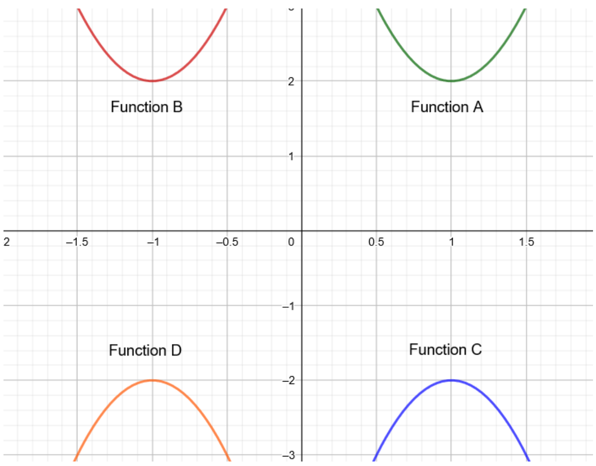

Figure 1: Original function shown as Function A together with reflections and rotation

Function B is the result of reflection in the *y*-axis (horizontal reflection), Function C is the result of reflection in the *x*-axis (vertical reflection) and Function D arises from rotation by 180°.

To make it simpler a code is assigned to each operation, as shown in Table 1.

Code	Operation
e	Leave unchanged
a	Reflect horizontally
b	Reflect vertically
c	Rotate 180°

Table 1: Coded operations

Now we can look at the outcomes when two successive operations occur. Reflecting horizontally twice or reflecting vertically twice or rotating 180° twice is equivalent to leaving unchanged.

More interesting results occur when we combine the reflections and rotation. Reflecting horizontally and then vertically (or vice versa) is equivalent to rotating 180°, reflecting horizontally and rotating

180° is equivalent to reflecting vertically. Similarly, reflecting vertically and rotating 180° is equivalent to reflecting horizontally.

These combinations are be summarised in Table 2.

	e	***a***	***b***	***c***
e	*e*	*a*	*b*	*c*
a	*a*	*e*	*c*	*b*
b	*b*	*c*	*e*	*a*
c	*c*	*b*	*a*	*e*

Table 2: Combinations of reflections and rotation

To help interpret the table, take the second row. Here '*a*' refers to reflecting horizontally. We then apply operations '*e*' that leaves it unchanged, '*a*' that undoes the horizontal reflection, '*b*' that applies a vertical reflection to result in effectively a 180° rotation and finally a 180° rotation ('*c*') that effectively results in a vertical reflection.

In symbolic terms we have

$$ae = ea = a,\ be = eb = b,\ ce = ec = c$$

$$e^2 = a^2 = b^2 = c^2 = e$$

$$ab = ba = c,\ ac = ca = b,\ bc = cb = a$$

The operation '*e'* leaves the input unchanged. This operation is called the identity. In arithmetic we are used to 1 being the multiplicative identity (multiplying by 1 leaves the input unchanged) and 0 being the additive identity (adding 0 leaves the input unchanged).

An operation that undoes another operation is called the inverse operation. The combination of an operation and its inverse results in the identity. For example, the inverse of multiplying by 2 is multiplying by $\frac{1}{2}$ so multiplying by both 2 and $\frac{1}{2}$ results in the input being unchanged. Similarly, adding 5 then subtracting 5 effectively adds 0, leaving the input unchanged; subtracting 5 is the inverse of adding 5.

In our example we have $a^2 = b^2 = c^2 = e$, this means that '*a*', '*b*', and '*c'* are their own inverses.

Also, *ab = ba = c, ac = ca = b, bc = cb = a.* This means that the operations are commutative ie the order in which they are performed doesn't matter. An algebraic object is said to be Abelian when the commutative law holds.

Groups

A group is a fundamental concept in Abstract Algebra that combines a set with an operation. The operation is termed a binary operation because it operates on two elements of the set and the result is also a member of the set.

The binary operation, given the symbol *, must satisfy three properties:

(1) it is associative ie *a* * (*b* * *c*) = (*a* * *b*) * *c* for elements *a, b* and *c* in the set

(2) there is an identity element in the set
(3) every element of the set has an inverse in the set

As an example, the integers form a group with addition as the binary operation. The identity is 0 and the inverse of any integer is the negative of the integer.

However, integers do not form a group with multiplication. Although multiplication is associative and the identity is 1, the multiplicative inverse of an integer is a fraction.

The Klein four-group

The Klein four-group is a set with four elements combined with a binary operation that has two properties:

(1) every element is its own inverse
(2) combining any two non-identity elements yields the third non-identity element

If we let the set be {e, *a, b, c*} where 'e' is the identity then property (1) tells us that

$$e^2 = a^2 = b^2 = c^2 = e$$

while property (2) gives

$$ab = ba = c,\ ac = ca = b,\ bc = cb = a$$

and these are precisely the outcomes given in Table 2 that in Abstract Algebra is called a Cayley table.

This is where the abstraction comes in: we are not saying anything specific about the elements '*a*', '*b*' or '*c*', all we know is that they satisfy the two properties given above. We have seen one application with horizontal and vertical reflections combined with 180° rotation. Other applications are

(1) the set {(0,0), (1,0), (0,1), (1,1)} with addition modulo 2 (numbers in mod 2 are the remainders when an integer is divided by 2)

(2) the set of matrices $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\}$ with matrix multiplication (2 by 2 matrices are multiplied as: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$)

(3) the set {1, 3, 5, 7} with multiplication modulo 8 (mod 8 numbers are remainders when an integer is divided by 8)

(4) the set {Ø, {α}, {β}, {α, β}} which is the power set of the set {α, β} with symmetric difference as the binary operation. (the power set is the set of all the possible subsets of a given set. The symmetric difference of two sets is the set of elements belonging to one but not both sets)

Cosets

A subgroup consists of some elements of the group such that the properties of the group are maintained (associativity, existence of an identity and inverses). If we apply the binary operation to every element of the group with every element in the subgroup we create a series of sets. These are called cosets of the subgroup.

For example, suppose the subgroup is {*e, a*}. Table 3 shows the results of applying the binary operation to all the elements of the group and the subgroup.

	{*e, a*}
e	{*e, a*}
a	{*a, e*}
b	{*b, c*}
c	{*c, b*}

Table 3: Cosets of the subgroup {*e, a*}

We see that there are two cosets, {*e, a*} (the subgroup itself) and {*b, c*} with no common elements. These cosets are said to partition the group – this means that all the elements in the group are included in a coset and no element is in more than one coset.

The cosets for the subgroup {*e, b*} are {*e, b*} and {*a, c*} while those for the subgroup {*e, c*} are {*e, c*} and {*a, b*}. Table 4 is the Cayley table for {{*e, a*}, {*b, c*}}. For example, to evaluate {*e, a*}{*b, c*} we find $eb = b$, $ec = c$, $ab = c$ and $ac = b$. Hence there are two distinct repeated results. Note that the identity is {*e, a*} and each coset is its own inverse.

	{*e, a*}	**{*b, c*}**
{*e, a*}	{*e, a*}	{*b, c*}
{*b, c*}	{*b, c*}	{*e, a*}

Table 4: Cayley table for the cosets of {*e, a*}

The above cosets were obtained using *gH* where *g* is an element of the group and *H* is the subgroup. These are called left cosets. Right cosets are obtained from *Hg* and may differ from left cosets. However left and right cosets are the same when the group is Abelian and that is the case with the Klein four-group.