Derivation of Simpson’s Rule

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The formula

Simpson's Rule is a method for numerically evaluating an integral:

$$\int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

where x_1 is the average of x_0 and x_2 ie

$$x_1 = \frac{x_0 + x_2}{2}$$

and h is the difference between successive x values ie

$$h = x_1 - x_0$$

The formula

The expression

$$\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

calculates the exact area under a parabola that passes through three points on the curve $y = f(x)$

The three points are $(x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))$

Interpreting the formula

Now $\frac{h}{3} = \frac{2h}{6}$ and, putting the 6 inside the brackets,

$$\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] = 2h[\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}]$$

or

$$2h[\frac{1}{6}f(x_0) + \frac{4}{6}f(x_1) + \frac{1}{6}f(x_2)]$$

which can be thought of as finding the area of a rectangle with width $2h$ and height

$$\frac{1}{6}f(x_0) + \frac{4}{6}f(x_1) + \frac{1}{6}f(x_2)$$

Interpreting the formula

Thus the area under the parabola is exactly equal to the area of the rectangle.

The expression

$$\frac{1}{6}f(x_0) + \frac{4}{6}f(x_1) + \frac{1}{6}f(x_2)$$

is a *weighted average* of the function values $f(x_0), f(x_1), f(x_2)$ with weights $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$

The following slides show how these weights are determined.

Determining the weights

A parabola fitted to a function will give the *exact* area when the function is a straight line or a parabola. Therefore there are 3 simple functions we can use to determine the weights (a, b, c) in the weighted average of function values given by

$$a[f(x_0)] + b[f(x_1)] + c[f(x_2)]$$

These functions are

(i) $f(x) = 1$ (ii) $f(x) = x$ (iii) $f(x) = x^2$

We find the exact areas bounded by the function and the x-axis between $x = 0$ and $x = 1$

Determining the weights

Note that in the following examples $h = \frac{1}{2}$ so $2h = 1$.

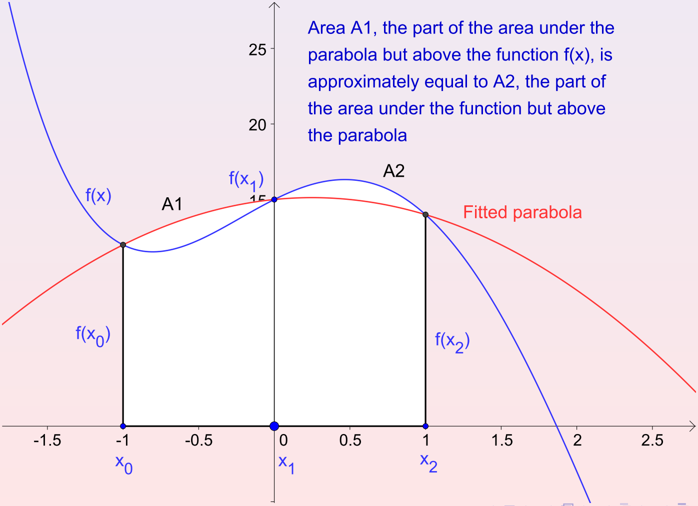
Hence the weighted value of the function values,

$$a[f(x_0)] + b[f(x_1)] + c[f(x_2)]$$

will give the area as determined by Simpson's Rule

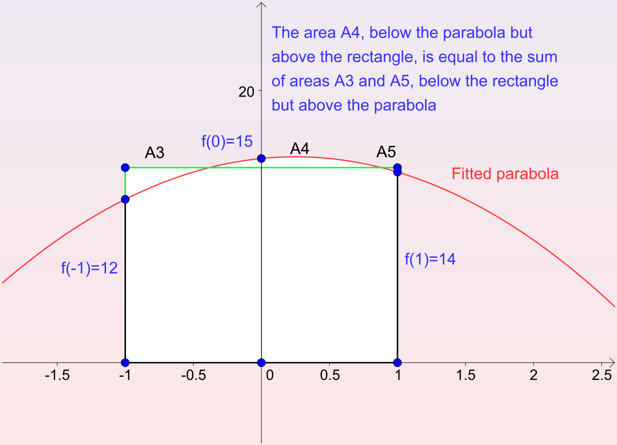
Parabola approximates the function

Consider an arbitrary function $f(x)$ and a parabola fitted to three points:



Areas under parabola and rectangle equal

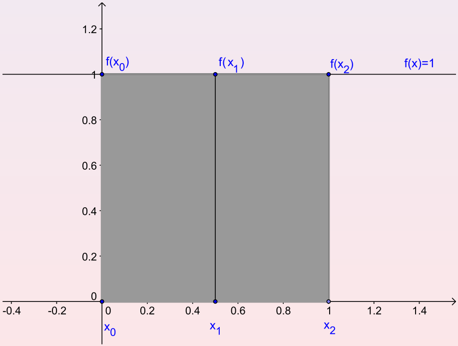
$$\text{Height of rectangle}=\frac{1}{6}(12)+\frac{4}{6}(15)+\frac{1}{6}(14)=14.33$$



Determining the weights

(i) $f(x) = 1$

We use Simpson's Rule to calculate the area of the square:



Determining the weights

(i) $f(x) = 1$

The region bounded by the function and the x-axis between $x = 0$ and $x = 1$ is a square with area 1 sq unit.

Using $(x_0, x_1, x_2) = (0, \frac{1}{2}, 1)$ in $a[f(x_0)] + b[f(x_1)] + c[f(x_2)]$ we have

$$a[f(0)] + b[f(\frac{1}{2})] + c[f(1)] = a[1] + b[1] + c[1] = a + b + c$$

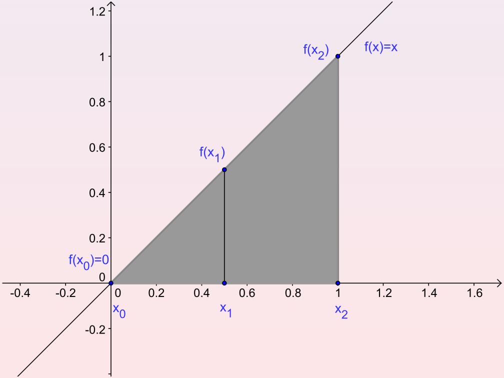
and since this equals the area we have

$$a + b + c = 1$$

Determining the weights

(ii) $f(x) = x$

We use Simpson's Rule to calculate the area of the triangle:



Determining the weights

(ii) $f(x) = x$

The region bounded by the function and the x-axis between $x = 0$ and $x = 1$ is a triangle with area $\frac{1}{2}$ sq unit.

Using $(x_0, x_1, x_2) = (0, \frac{1}{2}, 1)$ in $a[f(x_0)] + b[f(x_1)] + c[f(x_2)]$ we have

$$a[f(0)] + b[f(\frac{1}{2})] + c[f(1)] = a[0] + b[\frac{1}{2}] + c[1] = \frac{1}{2}b + c$$

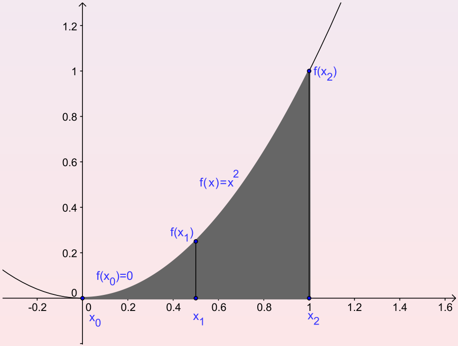
and since this equals the area we have

$$\frac{1}{2}b + c = \frac{1}{2}$$

Determining the weights

(iii) $f(x) = x^2$

We use Simpson's Rule to calculate the area under the parabola:



Determining the weights

(iii) $f(x) = x^2$

The area of region bounded by the function and the x-axis between $x = 0$ and $x = 1$ is a found from $\int_0^1 x^2 dx = \frac{1}{3}$ sq unit.

Using $(x_0, x_1, x_2) = (0, \frac{1}{2}, 1)$ in $a[f(x_0)] + b[f(x_1)] + c[f(x_2)]$ we have

$$a[f(0)] + b[f(\frac{1}{2})] + c[f(1)] = a[0] + b[\frac{1}{4}] + c[1] = \frac{1}{4}b + c$$

and since this equals the area we have

$$\frac{1}{4}b + c = \frac{1}{3}$$

Determining the weights

Now we have three equations:

$$a + b + c = 1 \tag{1}$$

$$\frac{1}{2}b + c = \frac{1}{2} \tag{2}$$

$$\frac{1}{4}b + c = \frac{1}{3} \tag{3}$$

Solving (2) and (3) simultaneously for b gives $b = \frac{4}{6}$, then $c = \frac{1}{6}$ and then $a = \frac{1}{6}$ from(1)

Thus the weights are $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$