Slope Fields for First-order Differential Equations

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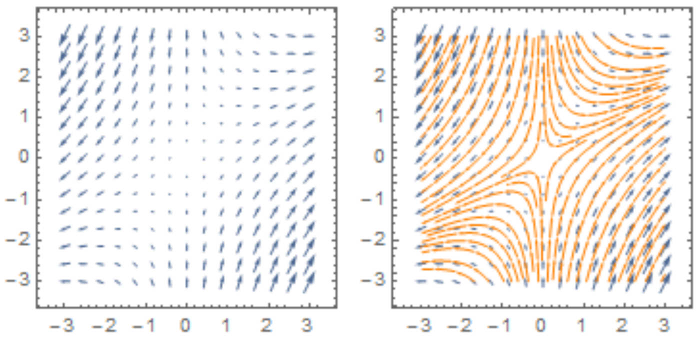
This note provides details for sketching and interpreting slope fields for first-order differential equations when expressed in the form $y' = f(x, y)$. Examples are taken from the Cambridge Year 12 Extension 1 textbook.

The text has diagrams of slope fields that consist of short intervals showing the gradient at various points. *Mathematica* plots a vector field that is similar, shown on the left below, and also provides a streamplot that shows a selection of solutions to the differential equation (DE). Integration constants exist in solutions to DEs and we need to know the coordinates of a point to find the particular solution. If all the possible solutions were shown on a plot then the plot would just be a filled region as the constant is continuous. The streamplot applicable to the vector plot is shown on the right:



Isoclines are curves that join points on the streamlines (solutions) for which the gradients of the tangents are constant.
In each of the examples the solution of the DE is provided and the streamplot is shown together with at least one solution and isocline.

Example 1: $y' + y = x$

Consider the differential equation $y' + y = x$ so that $y' = x - y$. To sketch the slope field note that:

(1) $y' = 0$ along the line $y = x$
(2) for fixed $x > y + 1$ we have $y' > 0$ and decreasing as y increases
(3) for fixed $x < y$ we have $y' < 0$ and $\longrightarrow -\infty$ as $y \longrightarrow \infty$

To solve $y' + y = x$ we need to find two solutions, one to $y' + y = 0$ and the other to $y' + y = x$. This is because the first solution effectively hides in the background but obviously effects the solution.

We solve $y' + y = 0$ by writing $y' = -y$ ie $\frac{1}{y}\frac{dy}{dx} = -1$

$$\text{so } \int \frac{1}{y}\, dy = \int -1\, dx \Longrightarrow \ln(y) = -x + c \Longrightarrow y = K e^{-x}$$

To solve $y' + y = x$ we let $y = A x + B \Longrightarrow y' = A$

$$\text{Then } y' + y = A + A x + B \Longrightarrow A = 1 \text{ and } B = -1$$

So the full solution is $\boldsymbol{y = K e^{-x} + x - 1}$

```
Show[StreamPlot[{1, x - y}, {x, -3, 3}, {y, -3, 3}],
 Plot[e^-x + x - 1, {x, -3, 3}, PlotStyle → {Blue, Thick}],
 Plot[-2 e^-x + x - 1, {x, -3, 3}, PlotStyle → {Green, Thick}],
 Plot[x - 1, {x, -3, 3}, PlotStyle → {Red, Thick}],
 Plot[x, {x, -3, 3}, PlotStyle → {Orange, Dashed, Thick}]]
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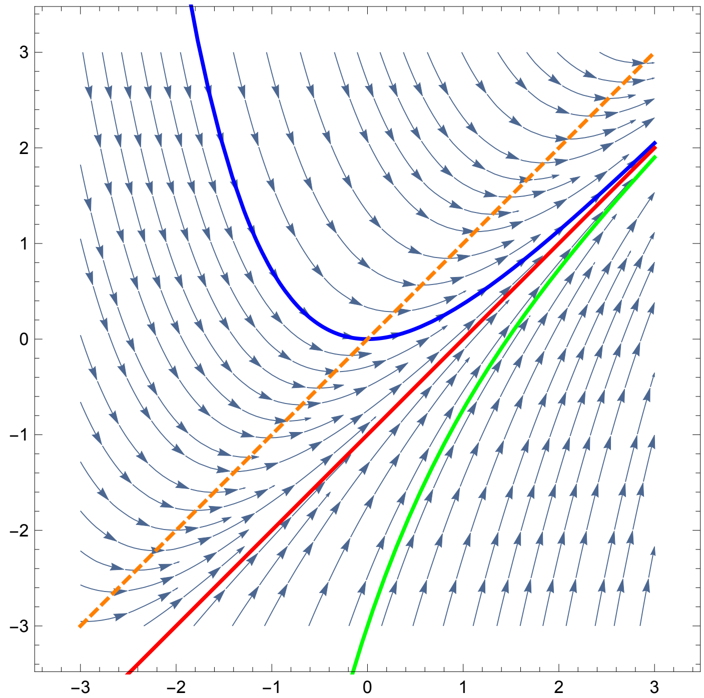

The plot shows the streamlines together with the line $y = x$ (where $y' = 0$), the line $y = x - 1$ (the particular solution) and two full solutions, one for $K = 1$ (blue) and the other for $K = -2$ (green). Note that $y = x - 1$ is an asymptote as the influence of the $K\,e^{-x}$ term in the solution decays quickly for increasing positive x.

Example 2: $y' = x\,y$

The equation $y' = x\,y$ has $y' = 0$ when either x or y are 0, is positive when both x and y are the same sign and negative when different signs.

To solve $y' = x\,y$ we write $\frac{1}{y}\frac{dy}{dx} = x$ so $\int \frac{1}{y}\,dy = \int x\,dx$

$$\Longrightarrow \ln(y) = \frac{1}{2}x^2 + C$$

$$\Longrightarrow y = K e^{\frac{1}{2}x^2}$$

This is an even function, positive when $K > 0$ and negative when $K < 0$, equal to K when $x = 0$ and rapidly approaching the appropriate $\pm\infty$

In[93]:=
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Show[StreamPlot[{1, x y}, {x, -3, 3}, {y, -3, 3}, PlotLabel → "Isocline for y'=1"],
 Plot[x + 2.1, {x, -3, 3}], Plot[x + 0.9, {x, -3, 3}],
 Plot[x + 0.6, {x, -3, 3}], Plot[x - 1, {x, -3, 3}], Plot[x - 2.1, {x, -3, 3}],
 Plot[ⅇ^(1/2 x^2), {x, -3, 3}, PlotStyle → {Red, Thick}],
 Plot[1/x, {x, -3, 3}, PlotStyle → Green]]
```
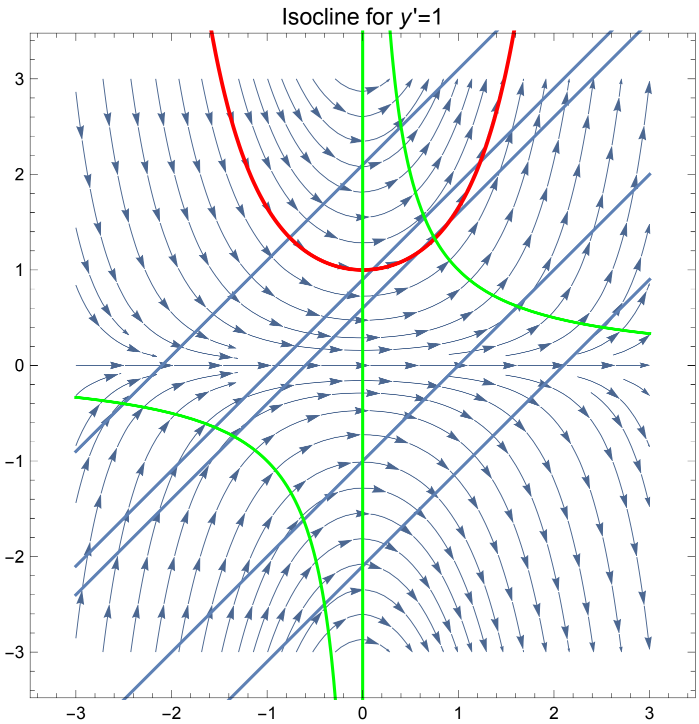
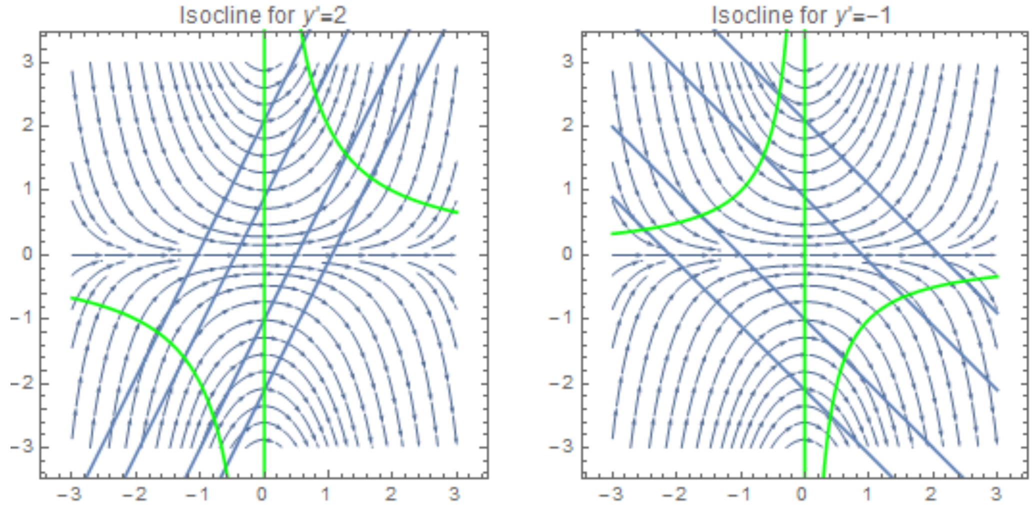



The plot shows the streamlines for the family of solutions $y = K e^{\frac{1}{2}x^2}$, the solution when $K = 1$ (red), together with the isocline joining points where $y' = 1$ (the hyperbola $y = \frac{1}{x}$ in green - the vertical line is just the asymptote of the hyperbola) and five lines with gradient 1, noting that these lines are tangential to the streamlines at the intersection with the hyperbola.

To show two further examples, here are plots for $y' = 2$ (the isocline is $y = \frac{2}{x}$ and the lines have gradient 2) and *y*’ = -1 (the isocline is $y = \frac{-1}{x}$ and the lines have gradient -1). Again the lines are tangential to the streamlines at the point of intersection with the isocline.



Example 3: $y' = 1 - \left(\frac{y}{3}\right)^2$

This non-linear differential equation has *y*' as a function of *y* only and is constant for fixed values of *y*. Therefore the isoclines are horizontal lines. There are two horizontal asymptotes, at $y = \pm 3$. The gradient is 1 when $y = 0$. The gradient is positive for $-3 < y < 3$ and negative for $y > |3|$

To solve $y' = 1 - \left(\frac{y}{3}\right)^2$ we write $y' = \frac{9-y^2}{9}$ then separate to give $\frac{9}{9-y^2}\, d\,y = d\,x$

$$\text{Then } \int \frac{9}{9-y^2}\, d y = \int 1\, d x$$

$$\frac{3}{2}\int\left(\frac{1}{3+y} + \frac{1}{3-y}\right) d y = x + C$$

$$\frac{3}{2}(\ln(3+y) - \ln(3-y)) = x + C \implies \frac{3}{2}\ln\left(\frac{3+y}{3-y}\right) = x + C \implies \frac{3+y}{3-y} = K\, e^{\frac{2}{3}x}$$

Re-arranging to give *y* the subject we have

$$y = \frac{3\left(K\, e^{\frac{2}{3}x} - 1\right)}{K\, e^{\frac{2}{3}x} + 1}$$

For $x > 0$ the dominant term is the exponential (ie ignore the 1 in numerator and denominator) and therefore the limiting value for *y* is 3.
For $x < 0$ the exponential terms decay rapidly and the limiting value for *y* is -3

In[20]:=
```
Show[StreamPlot[{1, 1 - (y/3)^2}, {x, -10, 10}, {y, -10, 10}],
 Plot[0, {x, -10, 10}, PlotStyle → Blue],
 Plot[3 (ⅇ^(2/3 x) - 1)/(ⅇ^(2/3 x) + 1), {x, -10, 10}, PlotStyle → {Red, Thick}],
 Plot[3 (-2 ⅇ^(2/3 x) - 1)/(-2 ⅇ^(2/3 x) + 1), {x, -10, 10}, PlotStyle → {Green, Thick}]]
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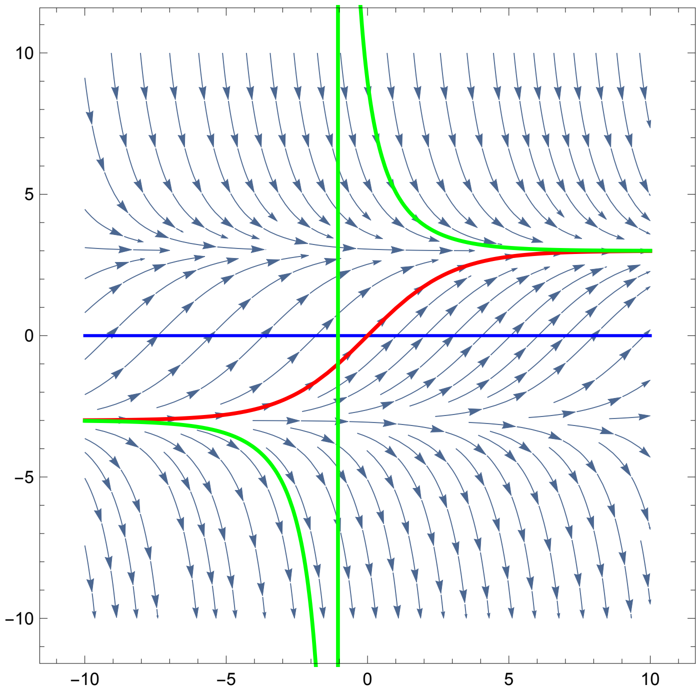

The plot shows the streamlines, the isocline for $y = 0$ (blue) and the solutions when $K = 2$ (red) and -2 (green). All the isoclines are horizontal.

Example 4: $y' = -1 - y$

This is a simple linear DE with gradient 0 when $y = -1$, negative gradient for $y > -1$ and positive gradient for $y < -1$.

The solution to the DE is $y = -1 + K e^{-x}$. The plot shows solutions for $K = 1$ (red) and $K = -2$ (green). The isoclines are all horizontal as y' is constant for fixed values of y.

In[25]:=
```
Show[StreamPlot[{1, -1 - y}, {x, -3, 3}, {y, -3, 3}],
 Plot[-1 + e^-x, {x, -3, 3}, PlotStyle → {Red, Thick}],
 Plot[-1 - 2 e^-x, {x, -3, 3}, PlotStyle → {Green, Thick}]]
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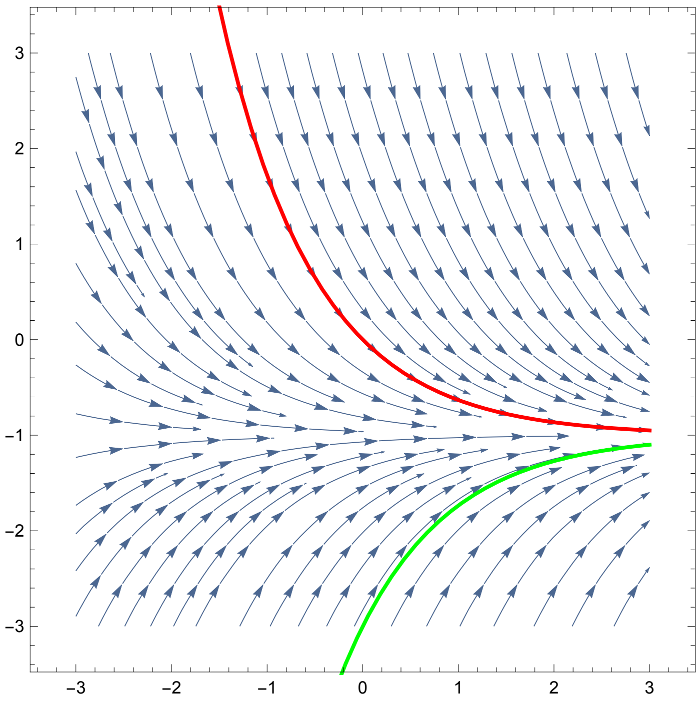
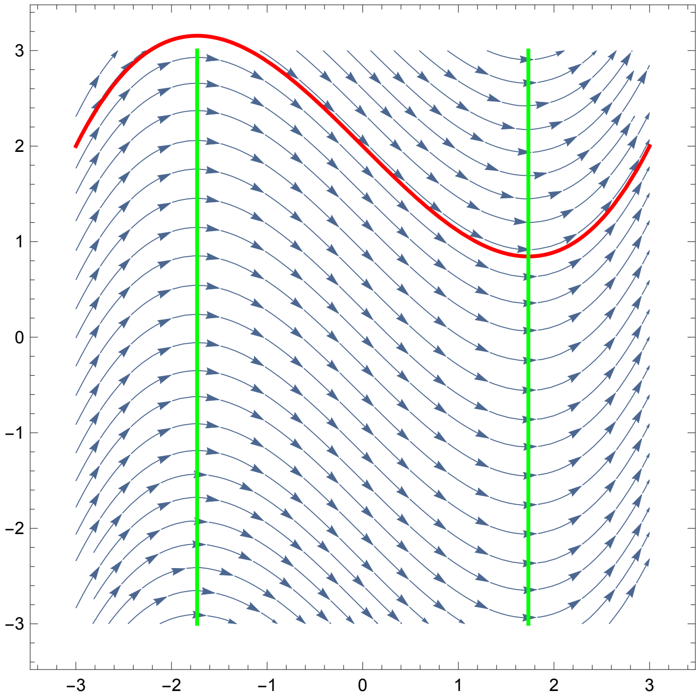


Example 5: $y' = \frac{1}{3}(x^2 - 3)$

This DE has zero gradient at $x = \pm\sqrt{3}$, positive gradient for $|x| > \sqrt{3}$ and negative gradient for $-\sqrt{3} < x < \sqrt{3}$.

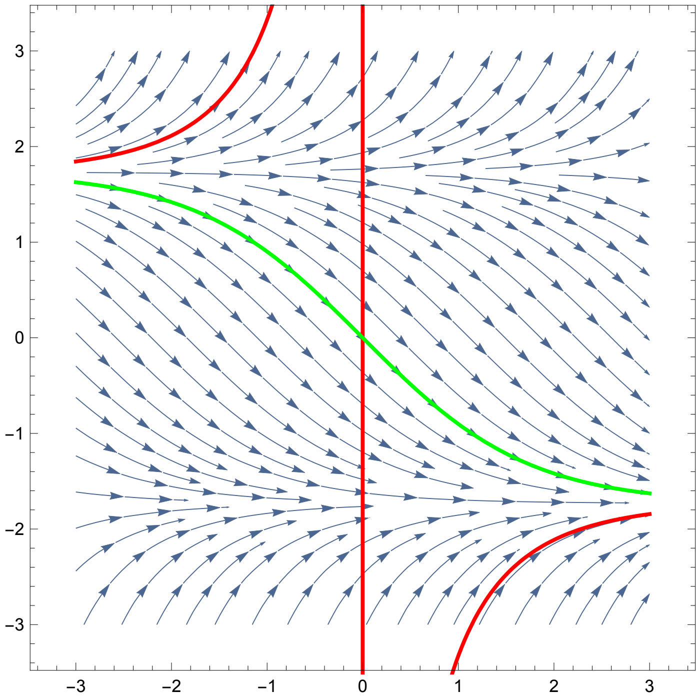
The solution is $y = \frac{x^3}{9} - x + C$. The plot shows the isoclines where the gradient is 0 and the solution when $C = 2$. All the isoclines are vertical.



Example 6: $y' = \frac{1}{3}\left(y^2 - 3\right)$

This DE has replaced the *x* in Example 5 with *y*. It is similar to Example 2.
The gradient is 0 when $y = \pm\sqrt{3}$, positive for $|y| > \sqrt{3}$ and negative for $-\sqrt{3} < y < \sqrt{3}$

The solution is $y = -\sqrt{3}\left(\frac{1+K e^{\frac{-2}{\sqrt{3}}x}}{1-K e^{\frac{-2}{\sqrt{3}}x}}\right)$ and the plot shows solutions for *K* = 1 (red) and *K* = -1 (green).
The isoclines are horizontal as *y*' is constant for fixed values of *y*.

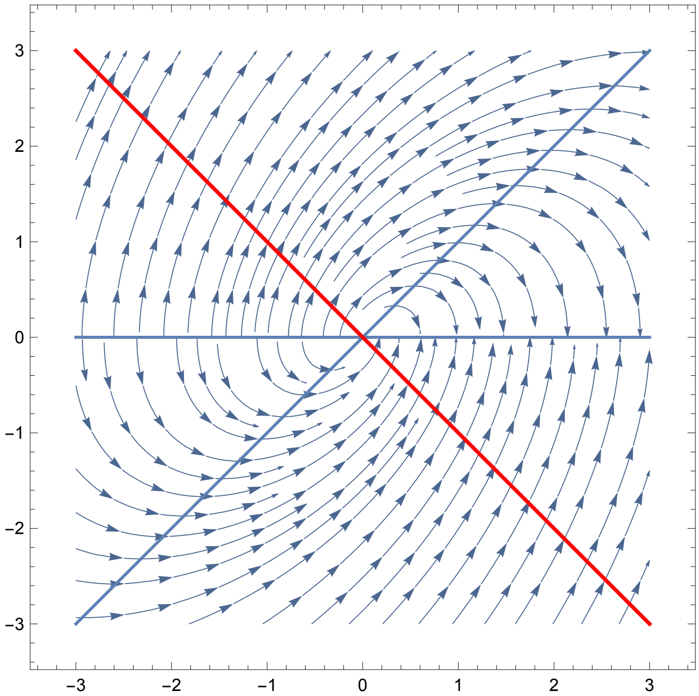


Example 7: $y' = 1 - \frac{x}{y}$

This DE has $y' = 0$ along the line $y = x$ and no solution for $y = 0$ (the gradient tends to infinite ie vertical along the x-axis),

The plot shows the asymptote ($y = 0$) and the isoclines where the gradient is 0 ($y = x$) and where the gradient is 2 ($y = -x$), in red.

The solution to the DE is complicated and not shown.



Example 8: $y' = x - \frac{1}{2}y$

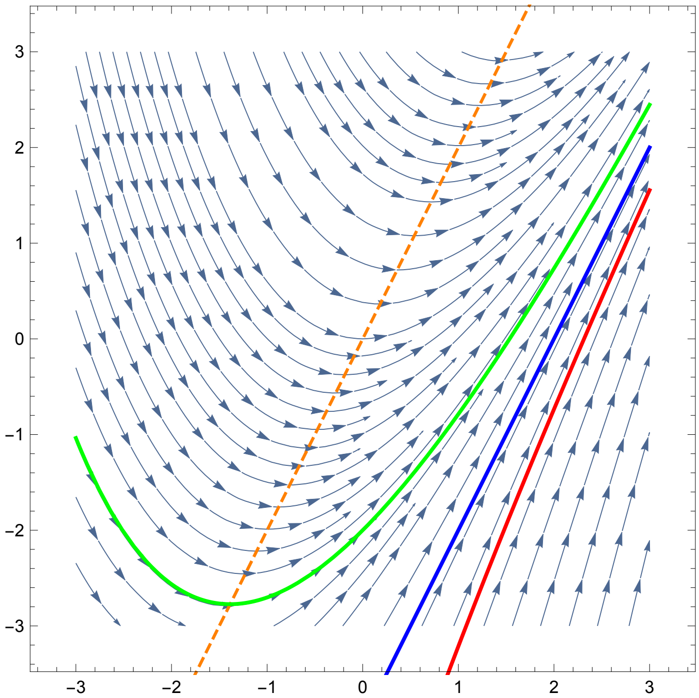
Here the gradient is 0 along the line $y = 2x$, it is positive for $x > \frac{1}{2}y$ and negative for $x < \frac{1}{2}y$.

To solve the DE we write as $y' + \frac{1}{2}y = x$ and first solve $y' + \frac{1}{2}y = 0$, as in Example 1.

The solution is $y = K e^{-\frac{1}{2}x}$ while the solution with x on the RHS is $y = 2x - 4$.

Hence the full solution is $y = K e^{-\frac{1}{2}x} + 2x - 4$

The plot shows the asymptote ($2x - 4$, blue), the isocline where the gradient is 0 (orange), the solution when $K = -2$ (red) and $K = 2$ (green)



Example 9: $y' = \frac{-2\,x\,y}{1+x^2}$

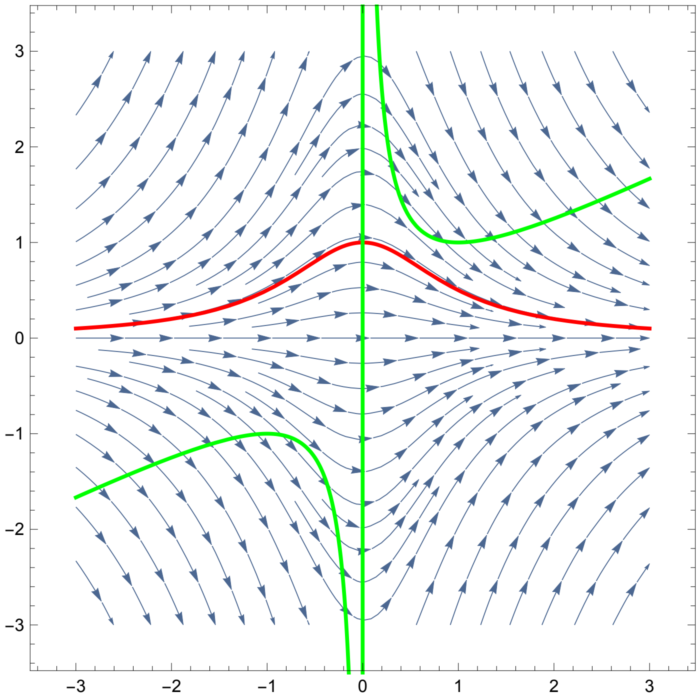
This DE has y ' equal to zero along both x and y axes, negative when x and y have the same sign and positive otherwise.

To solve the DE we write it as $\frac{y'}{y} = \frac{-2\,x}{1+x^2}$

$$\text{Then } \int \frac{1}{y}\,dy = \int \frac{-2\,x}{1+x^2}\,dx$$

$$\ln(y) = -\ln(1 + x^2) + C \quad \Longrightarrow \quad \ln(y) + \ln(1 + x^2) = C \quad \Longrightarrow \quad y = \frac{K}{1+x^2}$$

The plot shows the streamlines and the solution when K = 1 (red) and an isocline for y' = -1 (green).



Example 10: $y' = \frac{-2xy}{1+y^2}$

A similar equation to Example 9 but with *y* instead of *x* in the denominator. Again *y*' = 0 on the axes.

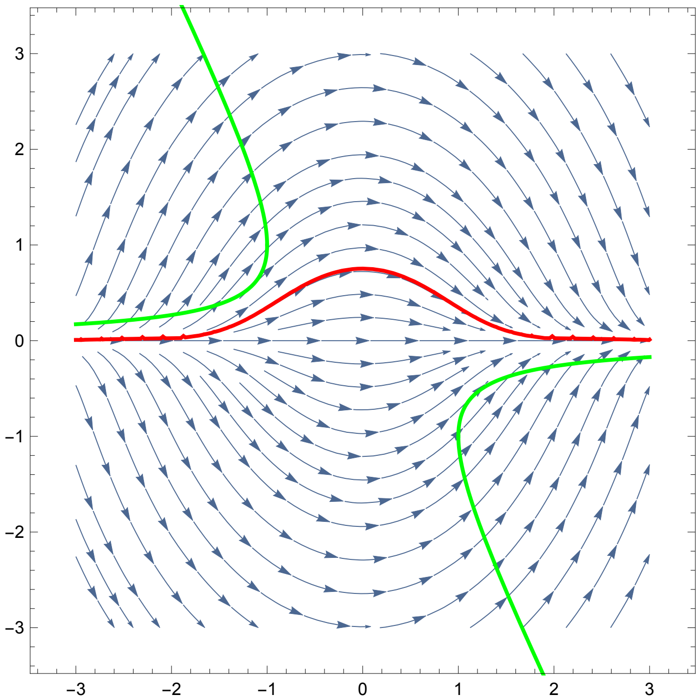
To solve the DE we write $\frac{1+y^2}{y}\,d\,y = -2\,x\,d\,x$

$$\int \frac{1+y^2}{y}\,dy = -\int 2\,x\,dx \quad \text{or} \quad \int\left(\frac{1}{y}+y\right)dy = -\int 2\,x\,dx$$

$$\ln(y) + \frac{1}{2}y^2 = -x^2 + C$$

This equation cannot be easily solved explicitly for *y* but can be plotted for a given value of *C*.

The plot shows the streamlines with a solution with *C* = 0 (red) and an isocline for *y*'=1, this is the solution to $1 = \frac{-2xy}{1+y^2}$. Solving this as a quadratic in *y* gives $y = -x \pm \sqrt{x^2-1}$ (green).



Example 11: $y' = \frac{-y}{x}$

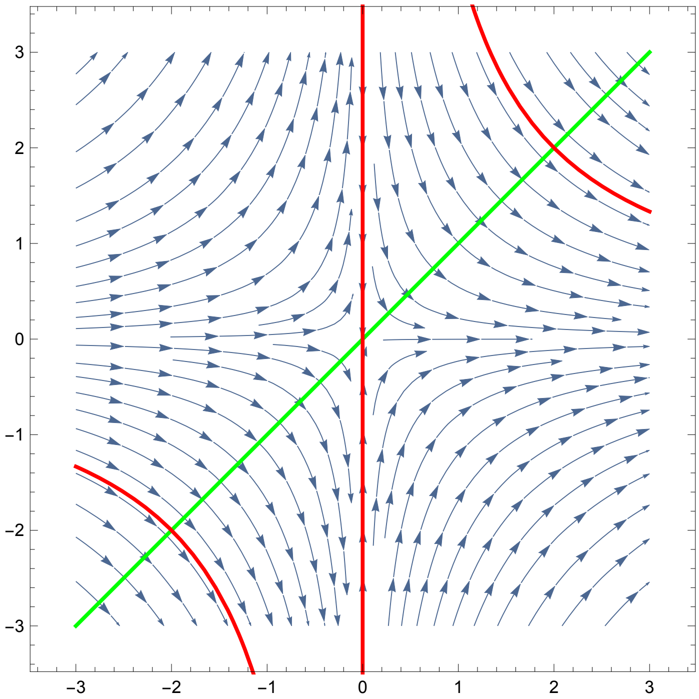
For this DE y' = 0 on the x-axis and undefined on the y-axis, is -1 on the line $y = x$, positive when x and y are different signs and negative when the same sign.

To solve the DE we write $\frac{1}{y}\, d\, y = \frac{-1}{x}\, d\, x \implies \int \frac{1}{y}\, dy = -\int \frac{1}{x}\, dx$

Then $\ln(y) = -\ln(x) + C$

$\ln(y) = \ln\left(\frac{1}{x}\right) + C \implies y = \frac{K}{x}$

The plot shows the streamlines with one solution $y = \frac{4}{x}$ (red) and the isocline for y' = -1 (green).

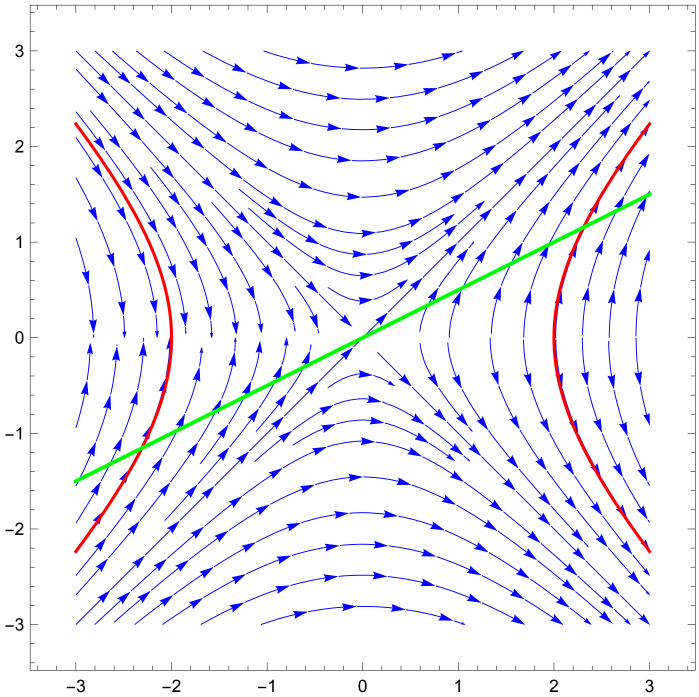


Example 12: $y' = \frac{x}{y}$

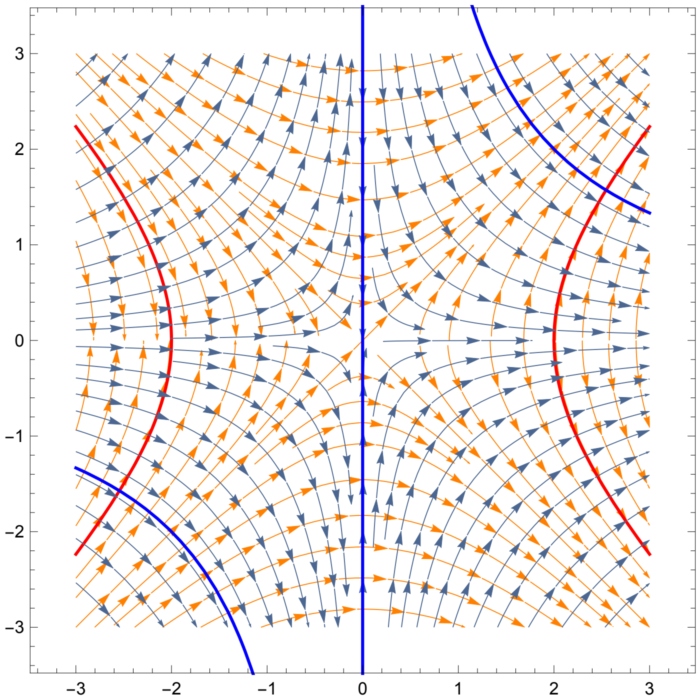
This DE has y' the negative reciprocal of Example 11. This time we have $y' = 0$ on the y-axis and undefined on the x-axis, equal to 1 on the line $y = x$, positive when x and y are have the same sign and negative when they have different signs.

To solve the DE we write $y\,dy = x\,dx \implies \int y\,dy = \int x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$
The solution can be written as $x^2 - y^2 = K$ which is the equation of a hyperbola.

The plot shows the streamlines with a solution when $K = 4$ (red) and an isocline for $y' = 2$ ($y = \frac{1}{2}x$) (green).



The following plot shows the solutions in Examples 11 and 12 that are perpendicular at the intersection points. This occurs because the definitions of y' in the DEs are negative reciprocals and therefore the solutions will be perpendicular.

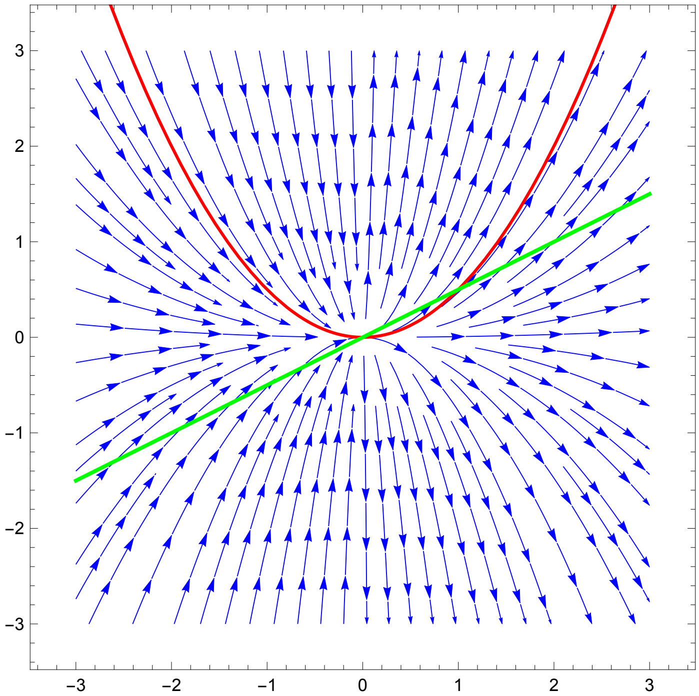

3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

Example 13: $y' = \frac{2y}{x}$

This DE is similar to Example 11. We have *y*' positive when *x* and *y* are the same sign and negative otherwise. Again *y*' is zero on the *x*-axis and undefined on the *y*-axis.

The solution to the DE is $y = K\,x^2$ and the isoclines are of the form $y = \frac{c}{2}\,x$.

The plot shows a solution when $K = \frac{1}{2}$ (red) and an isocline when $c = 1$ (green) ie $y' = 1$.



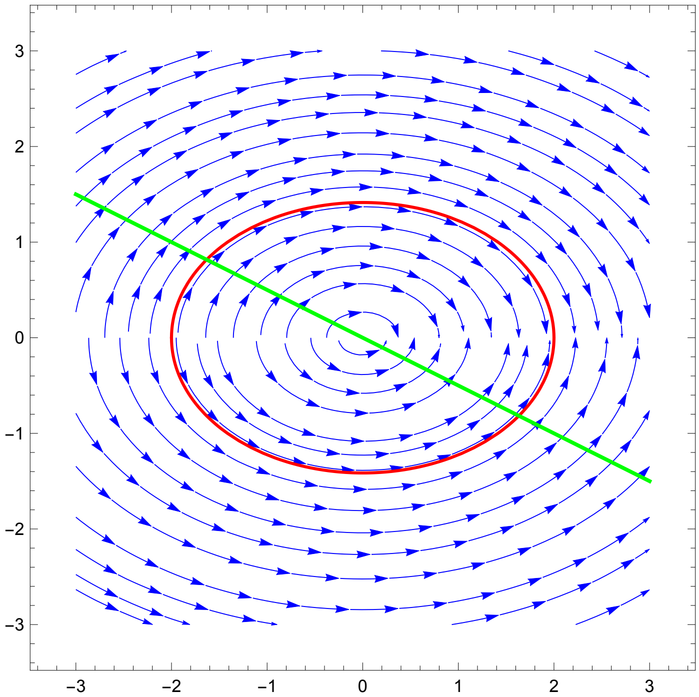
Example 14: $y' = \frac{-x}{2y}$

This DE is the negative reciprocal of Example 13. We have zero gradient on the y-axis and undefined on the x-axis. When x and y are the same sign y' is negative and when they are different signs y' is positive. For fixed x the gradient reduces for larger $|y|$.

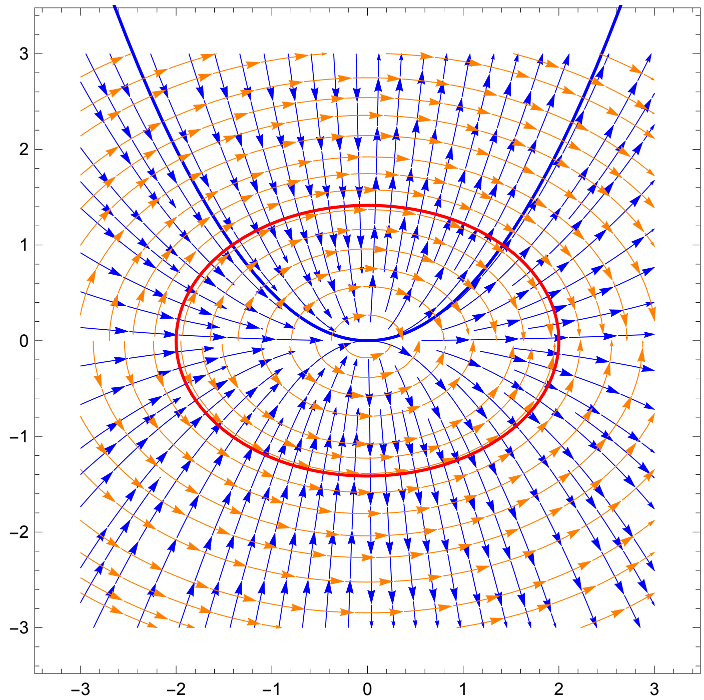
To solve the DE we write $2\,y\,dy = -x\,dx \implies \int 2\,y\,dy = \int -x\,dx \implies y^2 = -\frac{1}{2}x^2 + C$

The solution can be written as $x^2 + 2\,y^2 = K$ which is the equation of an ellipse.

The plot shows the streamlines, the solution when $K = 4$ (red) and the isocline for $y' = 1$ ($y = -\frac{1}{2}x$) (green)



The following plot combines the solutions to Examples 13 and 14. As expected, the solutions are perpendicular at their points of intersection.


3
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