Solutions for competition problems

Dr Richard Kenderdine

These are my solutions:

(1) First, note that we require an expression that gives the n^{th} term in terms of n and not a recurrence relation where a term is expressed as a function of other terms. For example, since 6 – (3 + 9) = -6 we could conclude that any term beyond the third could be found from the previous three terms in a similar way.

However, that is not what is required here. Obviously the first three terms are the first three multiples of 3 whereas the fourth term is not. We need an expression that is added to multiples of 3 but is 0 for the first three multiples of 3. Such an expression is

$$k(n-1)(n-2)(n-3) \quad \text{for } n > 0$$

Hence we can express the n^{th} term as

$$T_n = 3n + k(n-1)(n-2)(n-3) \quad \text{for some constant } k$$

Then we find k by using the fourth term

$$T_4 = 3(4) + k(4-1)(4-2)(4-3) = 12 + 6k = -6 \Rightarrow k = -3$$

Finally, we can express the n^{th} term as

$$T_n = 3n - 3(n-1)(n-2)(n-3)$$

(2) First, note that the proper divisors of a number are all the integers that divide into the number except the number itself. For example, the proper divisors of 12 are 1, 2, 3, 4 and 6.

Let the two prime numbers be m and n. Then $1 + m + n = 2014$ therefore $m + n = 2013$.

Now 2013 is an odd number and if two numbers add to give an odd number then the two numbers have different parity (one is odd and the other even). Two even numbers or two odd numbers always add to an even number.

Now 2 is the only one even prime number. Hence the two numbers are 2 and 2011 and the answer to the question is the product, ie 4022.

(3) Let f be the number of females and m the number of males in the queue. Suppose you insert yourself into the queue so that there are j females and k males behind. Therefore there are $m-k$ males in front of you. You want to position yourself so that the number of females behind equals the number of males in front. That is, $j = m - k$, or $j + k = m$.

So you need to insert yourself into the queue such that the total number of females and males behind you equals the total number of males in the queue. This can always be done.

(4) First note that $x, x+1, x+2, x+3$ are consecutive integers. Also, every second integer is even (thus divisible by 2), every third integer is divisible by 3 and every fourth integer is divisible by 4 (2^2).

We want to find the Highest Common Factor of m which means we want to minimise the product of the maximum powers of 2 and 3 contained in m.

For example, consider $(1)(2)^2(3)^3(4)^4 = (1)(2)^2(3)^3(2)^8 = 2^{10}3^3$

Now use $(4)(5)^2(6)^3(7)^4 = (2^2)(5)^2(2)^3(3)^3(7)^4 = 2^5 3^3 5^2 7^4$

Some thought reveals that the minimum powers of 2 and 3 occur when the first term is a multiple of 4 (but not a power of 2), resulting in the third term being a multiple of 2 (but not of 4), and the second term is a multiple of 3 (but not a power of 3).

For example, $20(21)^2(22)^3(23)^4 = (2^2)(5)(3^2)(7^2)(2^3)(11^3)(23)^4 = 2^5 3^2 (5) 7^2 11^3 23^4$

Thus the highest integer that is a factor of m for all values of x is $2^5 3^2$ = 288