A note on Span

Suppose you want to find whether the vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$ span R^3. That is, we want to see whether an arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in R^3 can be expressed as a linear combination of these three vectors.

Set up a system of equations:

$$x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + z\begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In matrix form we have

$$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Row reduce

$$\begin{pmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

So the equations become

$$x + y + 4z = a \qquad 2y + 5z = b - 2a$$

Since we don't have a third equation we cannot solve to find values for (x, y, z). This is because the original vectors are linearly depepndent.

Now consider another example with vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$. The system becomes

$$x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} + z\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In matrix form we have

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Row reduce

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 3 & 7 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b - 2a \\ c - 3a \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b - 2a \\ c - a - b \end{pmatrix}$$

(I've left out the $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ vector for clarity).

So the equations become

$$x + 2y + z = a \qquad y + z = b - 2a \qquad z = c - a - b$$

Back substitution gives

$$x = 4a - 3b + c \qquad y = 2b - a - c \qquad z = c - a - b$$

Test this with arbitrary vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$

This gives $x = 23$, $y = -12$ and $z = 4$

ie $$23 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 12 \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

Thus $\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ can be expressed as a linear combination of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

Since this is an arbitrary choice then the three given vectors span R^3.

The test for spanning is whether the given vectors are linearly independent ie that the reduced form of the matrix does not have a row of zeroes.