# Summing a series of sine functions with arguments in arithmetic progression

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The aim of this note is to prove the identity, for $n \gtrsim 0$,

$$\sin(x) + \sin(x + \alpha) + \sin(x + 2\alpha) + \ldots\ldots + \sin(x + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\sin\left(x + \frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

**(1) Proof by induction**

Step 1: Let $n = 0$. LHS = $\sin(x)$ RHS = $\sin(x)$ therefore true for $n = 0$

Step 2: assume true for $n = k$

$$\sin(x) + \sin(x + \alpha) + \sin(x + 2\alpha) + \ldots\ldots + \sin(x + k\alpha) = \frac{\sin\left(\frac{(k+1)\alpha}{2}\right)\sin\left(x + \frac{k\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

Step 3: prove true for $n = k + 1$

$$\sin(x) + \sin(x + \alpha) + \ldots\ldots + \sin(x + k\alpha) + \sin(x + (k+1)\alpha) = \frac{\sin\left(\frac{(k+1)\alpha}{2}\right)\sin\left(x + \frac{k\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} + \sin(x + (k+1)\alpha)$$

The RHS becomes

$$\frac{1}{\sin\left(\frac{\alpha}{2}\right)}\left[\sin\left(\frac{(k+1)\alpha}{2}\right)\sin\left(x + \frac{k\alpha}{2}\right) + \sin\left(\frac{\alpha}{2}\right)\sin(x + (k+1)\alpha)\right]$$

Now we use the products-to-sums relationship $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

Thus we have, from the two products,

$$\frac{1}{2}\left[\cos\left(\frac{(k+1)\alpha}{2} - \left(x + \frac{k\alpha}{2}\right)\right) - \cos\left(\frac{(k+1)\alpha}{2} + \left(x + \frac{k\alpha}{2}\right)\right) + \cos\left(\frac{\alpha}{2} - x - (k+1)\alpha\right) - \cos\left(\frac{\alpha}{2} + x + (k+1)\alpha\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2} - x\right) - \cos\left(x + \left(k + \frac{1}{2}\right)\alpha\right) + \cos\left(-x - \left(k + \frac{1}{2}\right)\alpha\right) - \cos\left(x + \left(k + \frac{3}{2}\right)\alpha\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\alpha}{2} - x\right) - \cos\left(x + \left(k + \frac{3}{2}\right)\alpha\right)\right]$$

(using $\cos(-x) = \cos(x)$ to eliminate the middle two terms)

Now reversing, using sums-to-products, we have

$$\sin\left(\frac{\frac{\alpha}{2} - x + x + \left(k + \frac{3}{2}\right)\alpha}{2}\right)\sin\left(\frac{x + (k + 3/2)\alpha - \alpha/2 + x}{2}\right) = \sin\left(\frac{(k+2)\alpha}{2}\right)\sin\left(x + \frac{(k+1)\alpha}{2}\right)$$

Finally, the RHS becomes $\frac{1}{\sin\left(\frac{\alpha}{2}\right)}\left[\sin\left(\frac{(k+2)\alpha}{2}\right)\sin\left(x + \frac{(k+1)\alpha}{2}\right)\right]$, the required result for $n = k + 1$

**(2) Using complex numbers**

We have $\cos(x + n\alpha) + i\sin(x + n\alpha) = e^{i(x+n\alpha)}$ and can use the exponential form to deliver the result.

Now $\quad e^{i(x)} + e^{i(x+\alpha)} + e^{i(x+2\alpha)} +,,,,,+ e^{i(x+n\alpha)} = e^{i(x)}\left[1 + e^{i(\alpha)} + e^{i(2\alpha)} + \ldots\ldots e^{i(n\alpha)}\right].$

This is a geometric series with first term $e^{ix}$ and common ratio $e^{i\alpha}$ . The sum of the first $n$ + 1 terms is given by $\frac{e^{ix}\left[e^{i(n+1)\alpha}-1\right]}{e^{i\alpha}-1}$

We then realize the denominator: $\quad \frac{e^{ix}\left[e^{i(n+1)\alpha}-1\right]}{e^{i\alpha}-1}\times\frac{e^{-i\alpha}-1}{e^{-i\alpha}-1}$

The denominator becomes $2 - 2\cos\alpha = 4\sin^2\frac{\alpha}{2}$ while the numerator is

$$e^{i(x+n\alpha)} - e^{i(x+(n+1)\alpha} - e^{i(x-\alpha)} + e^{ix}$$

We need the imaginary part of the numerator: $\sin(x + n\alpha) - \sin(x + (n + 1)\alpha) - \sin(x - \alpha) + \sin x$

Now we use the sums-to-products result $\sin(A + B) - \sin(A - B) = 2\cos A\sin B$ to give

$$\sin(x + n\alpha) - \sin(x + (n + 1)\alpha) = 2\cos\left(x + \left(n + \tfrac{1}{2}\right)\alpha\right)\sin\left(-\tfrac{1}{2}\alpha\right)$$

and

$$\sin x - \sin(x - \alpha) = 2\cos\left(x - \tfrac{1}{2}\alpha\right)\sin\left(\tfrac{1}{2}\alpha\right)$$

Therefore we have, using $\sin(-x) = -\sin(x)$,

$$2\sin\left(\tfrac{1}{2}\alpha\right)\left[\cos\left(x - \tfrac{1}{2}\alpha\right) - \cos\left(x + \left(n + \tfrac{1}{2}\right)\alpha\right)\right]$$

Now using $\cos(A - B) - \cos(A + B) = 2\sin A\sin B$ we have

$$4\sin\left(\tfrac{1}{2}\alpha\right)\sin\left(x + \tfrac{n\alpha}{2}\right)\sin\left(\left(\tfrac{n+1}{2}\right)\alpha\right)$$

Now including the denominator we have the result

$$\frac{4\sin\left(\frac{\alpha}{2}\right)\sin\left(x+\frac{n\alpha}{2}\right)\sin\left(\left(\frac{n+1}{2}\right)\alpha\right)}{4\sin^2\frac{\alpha}{2}} = \frac{\sin\left(\left(\frac{n+1}{2}\right)\alpha\right)\sin\left(x+\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$

The corresponding result for cos is

$$\cos(x) + \cos(x + \alpha) + \cos(x + 2\alpha) +\ldots\ldots+ \cos(x + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right)\cos\left(x+\frac{n\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}$$