

Transformations

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There are three types of transformation:

- (1) translation a units right or left in the horizontal direction or a units up or down in the vertical direction
- (2) reflection in the y - or x -axis
- (3) dilation by a scale factor of a either in the horizontal or vertical direction.

The table shows the replacements needed in each case:

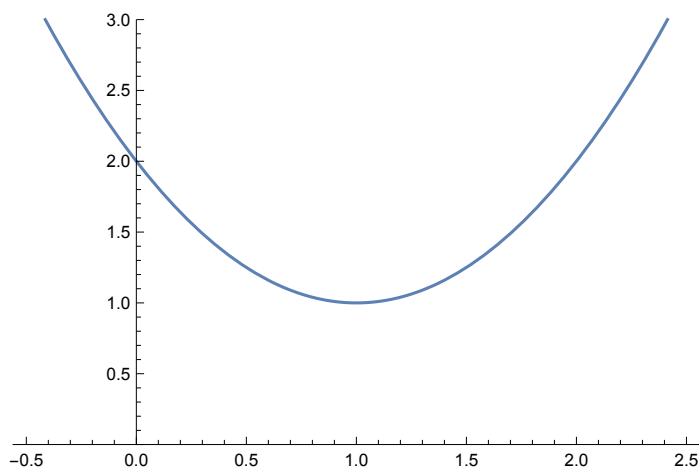
Transformation	Horizontal	Vertical
Translation	Replace x with $x - a$ (right) or $x + a$ (left)	Replace y with $y - a$ (up) or $y + a$ (down)
Reflection	Replace x with $-x$	Replace y with $-y$
Dilation	Replace x with $\frac{x}{a}$	Replace y with $\frac{y}{a}$

Although there are shortcuts for some of these transformation it is better to always use these replacements as they are consistent for all functions. Students sometimes are confused when faced with a function not in the form $y = f(x)$.

Example

Suppose we have the function $f(x) = (x - 1)^2 + 1$:

`Plot[(x - 1)^2 + 1, {x, -0.5, 2.5}, PlotRange -> {0, 3}]`



Now consider the following transformations:

- (1) Horizontal reflection (HR), replace x with $-x$
- (2) Vertical reflection (VR), replace y with $-y$ or alternatively multiply the function by -1

(3) Horizontal translation 3 units to the left (HT), replace x with $x + 3$

(4) Vertical translation 1 unit up (VT), replace y with $y - 1$ or alternatively add 1 to the function

(5) Horizontal dilation with scale factor 2 (HD), replace x with $\frac{x}{2}$

(6) Vertical dilation with scale factor 2 (VD), replace y with $\frac{y}{2}$ or alternatively multiply the function by 2

First we'll obtain the new equations with these six transformations:

(1) HR: $((-x) - 1)^2 + 1 = (x + 1)^2 + 1$

(2) VR: $-[(x - 1)^2 + 1] = -(x - 1)^2 - 1$

(3) HT: $((x + 3) - 1)^2 + 1 = (x + 2)^2 + 1$

(4) VT: $[(x - 1)^2 + 1] + 1 = (x - 1)^2 + 2$

(5) HD: $(\frac{x}{2} - 1)^2 + 1 = \frac{1}{4}(x - 2)^2 + 1$

(6) VD: $2[(x - 1)^2 + 1] = 2(x - 1)^2 + 2$

Now to each of these transformations we'll apply the six transformations. Consequently we will then be able to see which transformations commute.

Transformations	HR $(x + 1)^2 + 1$	VR $-(x - 1)^2 - 1$	HT $(x + 2)^2 + 1$	VT $(x - 1)^2 + 2$	HD $\frac{1}{4}(x - 2)^2 + 1$	VD $2(x - 1)^2 + 2$
HR	$(x - 1)^2 + 1$ (1)	$-(x + 1)^2 - 1$ (2)	$(x - 2)^2 + 1$	$(x + 1)^2 + 2$ (3)	$\frac{1}{4}(x + 2)^2 + 1$ (4)	$2(x + 1)^2 + 2$ (5)
VR	$-(x + 1)^2 - 1$ (2)	$(x - 1)^2 + 1$ (1)	$-(x + 2)^2 - 1$ (6)	$-(x - 1)^2 - 2$	$-\frac{1}{4}(x - 2)^2 - 1$ (7)	$-2(x - 1)^2 - 2$ (8)
HT	$(x + 4)^2 + 1$	$-(x + 2)^2 - 1$ (6)	$(x + 5)^2 + 1$	$(x + 2)^2 + 2$ (9)	$\frac{1}{4}(x + 1)^2 + 1$	$2(x + 2)^2 + 2$ (10)
VT	$(x + 1)^2 + 2$ (3)	$-(x - 1)^2$	$(x + 2)^2 + 2$ (9)	$(x - 1)^2 + 3$	$\frac{1}{4}(x - 2)^2 + 2$ (11)	$2(x - 1)^2 + 3$
HD	$\frac{1}{4}(x + 2)^2 + 1$ (4)	$-\frac{1}{4}(x - 2)^2 - 1$ (7)	$\frac{1}{4}(x + 4)^2 + 1$	$\frac{1}{4}(x - 2)^2 + 2$ (11)	$\frac{1}{16}(x - 4)^2 + 1$	$\frac{1}{2}(x - 2)^2 + 2$ (12)
VD	$2(x + 1)^2 + 2$ (5)	$-2(x - 1)^2 - 2$ (8)	$2(x + 2)^2 + 2$ (10)	$2(x - 1)^2 + 4$	$\frac{1}{2}(x - 2)^2 + 2$ (12)	$4(x - 1)^2 + 4$

The numbers in the cells indicate identical transformations and therefore show transformations that commute:

(1) and (2): Horizontal and vertical reflections

(3): Horizontal reflection and vertical translation

(4): Horizontal reflection and horizontal dilation

(5) Horizontal reflection and vertical translation

(6) Vertical reflection and horizontal translation

(7) Vertical reflection and horizontal dilation

(8): Vertical reflection and vertical dilation

(9): Horizontal and vertical translations

(10): Horizontal translation and vertical dilation

(11): Horizontal dilation and vertical translation

(12): Horizontal and vertical dilations

The transformations that don't commute are the following ***in the same direction***:

(i) a translation and a reflection

(ii) a translation and a dilation