Using vector methods to find the area of a triangle

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We know that the angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is found from

$$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}$$

while the area of a triangle is given by $A = \frac{1}{2}\,a\,b\sin\theta$

We can find $\sin\theta$ from $\sqrt{1-\cos^2\theta}$ using the above:

$$\sin\theta = \sqrt{1-\left(\frac{\boldsymbol{u}.\boldsymbol{v}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}\right)^2}$$

$$= \frac{\sqrt{(|\boldsymbol{u}|\,|\boldsymbol{v}|)^2-(u.v)^2}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}$$

$$= \frac{\sqrt{(u.u)\,(v.v)-(u.v)^2}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}$$

So $$A = \frac{1}{2}\,|\boldsymbol{u}|\,|\boldsymbol{v}|\,\frac{\sqrt{(u.u)\,(v.v)-(u.v)^2}}{|\boldsymbol{u}|\,|\boldsymbol{v}|}$$

$$A = \frac{1}{2}\sqrt{(\boldsymbol{u}.\boldsymbol{u})\,(\boldsymbol{v}.\boldsymbol{v}) - (\boldsymbol{u}.\boldsymbol{v})^2} \qquad (1)$$

For two dimensional vectors we can take this further.

Using $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ we have

$$(\boldsymbol{u}.\boldsymbol{u})(\boldsymbol{v}.\boldsymbol{v}) = (x_1^2+x_2^2)(y_1^2+y_2^2) = x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2$$

$$(\boldsymbol{u}.\boldsymbol{v})^2 = (x_1 y_1 + x_2 y_2)^2 = x_1^2 y_1^2 + 2\,x_1 y_1 x_2 y_2 + x_2^2 y_2^2$$

Thus $$(\boldsymbol{u}.\boldsymbol{u})(\boldsymbol{v}.\boldsymbol{v}) - (\boldsymbol{u}.\boldsymbol{v})^2 = x_1^2 y_2^2 + x_2^2 y_1^2 - 2\,x_1 y_1 x_2 y_2$$

$$= (x_1 y_2 - x_2 y_1)^2$$

Hence $$A = \frac{1}{2}\sqrt{(x_1 y_2 - x_2 y_1)^2} = \frac{1}{2}\,|x_1 y_2 - x_2 y_1| \qquad (2)$$

This is the equivalent to half of the vector cross product in 2 dimensions (the cross product of two vectors is a vector, usually defined in 3 dimensional space, and is denoted as $\boldsymbol{u} \times \boldsymbol{v}$. The modulus of the cross product equals the product of the moduli of the two vectors and the sine of the angle between them ie $|\,\boldsymbol{u}\times\boldsymbol{v}\,| = |\boldsymbol{u}|\,|\boldsymbol{v}|\sin\theta$)

Lagrange's identity states that for vectors in 3-space

$$|\,\boldsymbol{u}\times\boldsymbol{v}\,|^2 = |\boldsymbol{u}|^2\,|\boldsymbol{v}|^2 - (\boldsymbol{u}\,.\,\boldsymbol{v})^2$$

I have shown that $(x_1 y_2 - x_2 y_1)^2 = |(\boldsymbol{u.u})(\boldsymbol{v.v}) - (\boldsymbol{u}\,.\,\boldsymbol{v})^2|$ which is Lagrange's identity in 2-space.

In 3-space the area is half the magnitude of the vector with components

$$(x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1)$$

Example in 2-space:

Let $\boldsymbol{u} = \begin{pmatrix}3\\2\end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix}2\\-1\end{pmatrix}$

Then $\boldsymbol{u.u} = 13$ $\quad \boldsymbol{v.v} = 5$ $\quad \boldsymbol{u.v} = 4$ $\qquad |(\boldsymbol{u.u})(\boldsymbol{v.v}) - (\boldsymbol{u}\,.\,\boldsymbol{v})^2| = (13)(5) - 4^2 = 49$

So $\mathit{Area} = \frac{1}{2}\sqrt{49} = \frac{7}{2}\,u^2$

Alternatively, $\quad \frac{1}{2}|x_1 y_2 - x_2 y_1| = \frac{1}{2}|3\times -1 - 2\times 2| = \frac{7}{2}$

Example in 3-space:

Now extend to 3 dimensions: Let $\boldsymbol{u} = \begin{pmatrix}3\\2\\-4\end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix}2\\-1\\5\end{pmatrix}$

Then $\boldsymbol{u.u} = 29$ $\quad \boldsymbol{v.v} = 30$ $\quad \boldsymbol{u.v} = -16$ $\qquad |(\boldsymbol{u.u})(\boldsymbol{v.v}) - (\boldsymbol{u}\,.\,\boldsymbol{v})^2| = (29)(30) - (-16)^2 = 614$

So $\mathit{Area} = \frac{1}{2}\sqrt{614} = 12.390\ u^2$

Alternatively, $\quad \cos\theta = \frac{-16}{\sqrt{29}\,\sqrt{30}} = -\,0.54245 \Rightarrow \sin\theta = \sqrt{1 - 0.54245^2} = 0.84009$

Then $\mathit{Area} = \frac{1}{2}\sqrt{29}\,\sqrt{30}\,(0.84009) = 12.390\,u^2$

(Note: Joseph Louis Lagrange (1736 - 1813) was a French-Italian mathematician and astonomer. By the age of 25 he was regarded by many of his contemporaries as the greatest living mathematician. One of his most famous works is a memoir on mechanics. The *Lagrangian* is the difference between the potential and kinetic energies of a set of particles. He made important contributions to number theory and the solution of differential equations (and introduced the notation f ' (x) fo denote the derivative) as well as the introduction of the metric system. Perhaps his name is best known through the use of Lagrange multipliers to solve optimisation problems with constraints, often studied in economics)