Vectors in 3D - equations of planes

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The Cambridge text briefly describes how to obtain the equation of a plane given the coordinates of three points P_0 , P_1 and P_2. Let $\boldsymbol{u}$ be the vector $\overrightarrow{\boldsymbol{P_0 P_1}}$ and $\boldsymbol{v}$ be the vector $\overrightarrow{\boldsymbol{P_0 P_2}}$. Then the equation of the plane containing the three points is given by

$$r = P_0 + \lambda \boldsymbol{u} + \mu \boldsymbol{v}$$

There is another methd that uses a point in the plane and a vector normal to the plane.

In the following derivations we use the fixed point (2, 1, 4) and normal vector (1, - 2, 3).

(A) Equation using a point and two vectors in the plane normal to another vector

The equation will be of the form $P_0 + \lambda \boldsymbol{u} + \mu \boldsymbol{v}$

Let $\boldsymbol{u} = (a, b, c)$ and $\boldsymbol{v} = (d, e, f)$ where $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular to (1, -2, 3)

We can choose the values for two of the components of each of $\boldsymbol{u}$ and $\boldsymbol{v}$ and then calculate the third such that the dot product is 0.

We have $\boldsymbol{u}.(1, -2, 3) = a - 2b + 3c = 0$ and letting $b = 1$, $c = 1$ yields $a = -1$, Similarly, letting $f = 2$ and $e = 1$ yields $d = - 4$.

Thus (- 1, 1, 1) and (- 4, 1, 2) are vectors in the plane perpendicular to (1, - 2, 3) and therefore the equation of the plane through (2, 1, 4) is

$$\boldsymbol{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} \qquad \textbf{(1)}$$

(B) Using a point and a normal to the plane

Let $P\ (x, y, z)$ be an arbitrary point in the plane, $P_0(x_0,\ y_0,\ z_0)$ a fixed point in the plane and $\boldsymbol{n}$ be a vector normal to the plane. Then the vector $\overrightarrow{\boldsymbol{P_0 P}}$ is perpendicular to $\boldsymbol{n}$.

Thus $\boldsymbol{n}.\overrightarrow{\boldsymbol{P_0 P}} = 0$ and therfore the equation of the plane is, using $\boldsymbol{n} = (a, b, c)$

$$(a, b, c)\ ,\ (x - x_0, y - y_0, z - z_0) = 0$$

Thus reduces to $\quad a\,x + b\,y + c\,z = k$

In our example, using $\boldsymbol{n} = (1, - 2, 3)$, we have $x - 2y + 3z = k$ and using (2, 1, 4) in the plane we calculate 2 - 2(1) + 3(4) = 12 yielding the equation of the plane as

$$x - 2y + 3z = 12 \qquad \textbf{(2)}$$

Note that we did not need to use the dot product as the coefficients are just the components of the normal vector.

That is, if (a, b, c) is a normal vector to the plane then the equation of the plane takes the form

$$a\,x + b\,y + c\,z = k$$

where k is determined by substituting the coordinates of a point in the plane.

Aside: Suppose we have a line with gradient $\frac{a}{b}$ then the equation of the line is $y = \frac{a}{b}x + c$ or $a\,x - b\,y = k$. The gradient has horizontal component b and vertical component a. The normal therefore has horizontal component a and vertical component $-b$.

Thus the components of the normal are the coefficients for the equation of the line.

(C) Connecting the two methods

In **(1)**, let $\lambda = 1$ and $\mu = 1$, then $\boldsymbol{r} = (-3, 3, 7)$. Check that this satisfies **(2)**:

$$-3 - 2\,(3) + 3\,(7) = 12$$

Now use $\lambda = 2$ and $\mu = -1$, then $\boldsymbol{r} = (4, 2, 4)$. Check that this satisfies **(2)**:

$$4 - 2\,(2) + 3\,(4) = 12$$

Thus two arbitrarily chosen values for λ and μ in Eqn **(1)** yield points in the plane defined by **(2)** as expected.

Now do the reverse by elimating the parameters λ and μ in Eqn **(1)**. Start with the parametric equations

$$x = 2 - \lambda - 4\mu \qquad y = 1 + \lambda + \mu \qquad z = 4 + \lambda + 2\mu$$

These yield $\quad x + y = 3 - 3\mu \quad$ and $\quad x + z = 6 - 2\mu \implies \mu = 3 - \frac{1}{2}(x - z)$

Then $\quad x + y = 3 - 3\left(3 - \frac{1}{2}(x - z)\right) \implies x - 2y + 3z = 12$