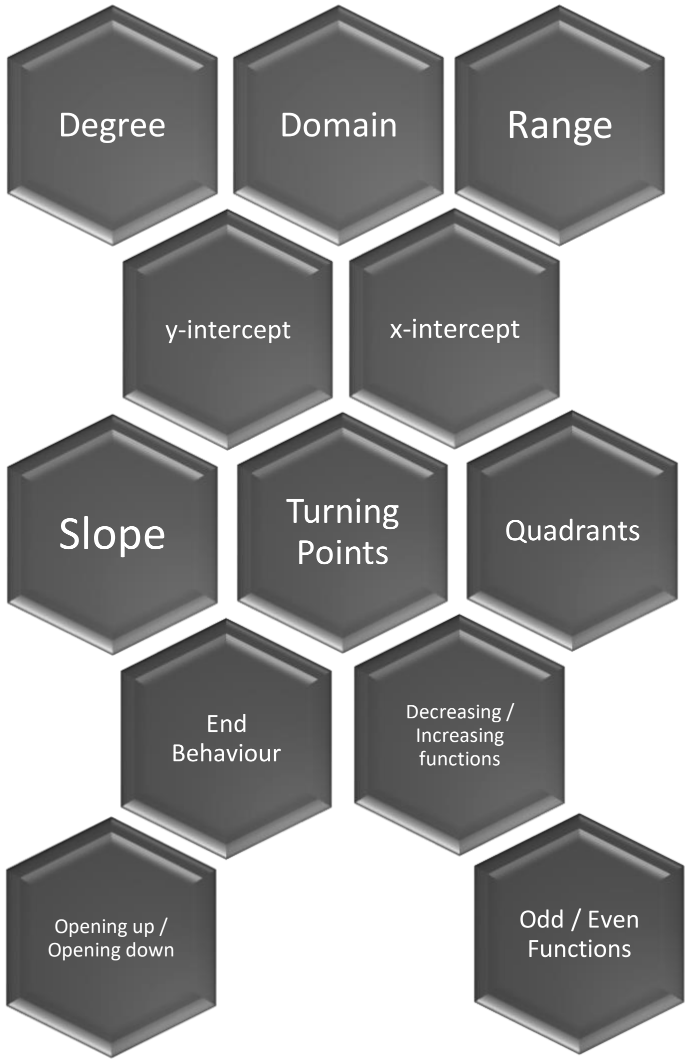
POLYNOMIAL FUNCTIONS NOTES

At the end of this unit you will be able to describe polynomial functions using all of these characteristics:



Personally, these terms make the most sense when I see them in examples and practice questions. Check out the videos on the *Learn with Pi* YouTube channel for some quick and dirty visual examples!

First, what is a polynomial function?

It is a function where the exponents on our variables are whole numbers.

For example,

$$f(x) = x^3 + 5x^2 + x - 2$$

The exponents are **3**, **2**, and **1** on our variable *x*.

Something like:

$$f(x) = x^{1/2}$$

has an exponent of ½ and is not considered a polynomial function.

WHAT ARE WE DOING?

There is a lot of information in this unit – and a lot of it seems like random things to memorize. BUT, all of it can be linked to how we **DRAW** our functions.

If you can remember the general shape of what polynomials of different degrees look like, then remembering the rest of the information in this unit becomes **A LOT** easier. This unit is all about **describing** polynomials, and if we can *picture* a polynomial it becomes a lot easier to describe.

Check out the video ***Graphing Polynomial Functions – Quick and Dirty*** on the *Learn with Pi* YouTube channel for the most important things to remember!

DEGREE: The degree of a polynomial function is the largest exponent that appears in the function.

Example 1:

What degree are each of the following polynomials?

a) $f(x) = x^4 + 5x^2 - 2$
   Degree of 4.

b) $y = x^6 + 5x^2 - 2x^7$

Degree of 7. Seven is the highest exponent, it doesn't matter that there is a negative sign in front of the coefficient, or that it wasn't written at the front of the expression.

c) $z = 3w^3 + w^2 + 6w$

Degree of 3.

DOMAIN AND RANGE:

The **DOMAIN** of a function is the set of *x* values (independent variable) that the function is defined for. It includes all *x* values that we are allowed to plug into our function. Very often, the domain of a polynomial function is:

$$\{x \in R\}$$

Which means that "*x* may be any real number".

That just means that if you were to graph a function, like $y = x$:

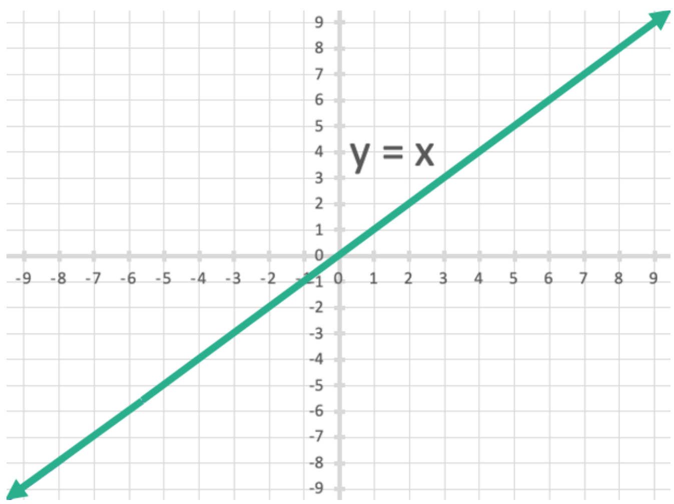


Figure 1: y = x

Your function will continue to be defined for all negative and positive values of *x*, going toward infinity in both directions. That's what the arrows on each end of the function mean. Thus, the domain for this function is:

$$\{x \in R\}$$

The **RANGE** of a function is the set of all possible *y* values (dependent variable) for the function. For polynomial functions the range is often **either**:

$$\{y \in R\}$$

(Eg. in *y = x* in Figure 1, *y* continues to infinity in both the up and down directions)

Or the range may be defined by some minimum or maximum value of the function:

$$\{y \leq maximum\} \text{ or } \{y \geq minimum\}$$

Other domains and ranges are possible, but these are what you will see most often in basic problems.

Example 2:

Determine the domain and range for the following function:

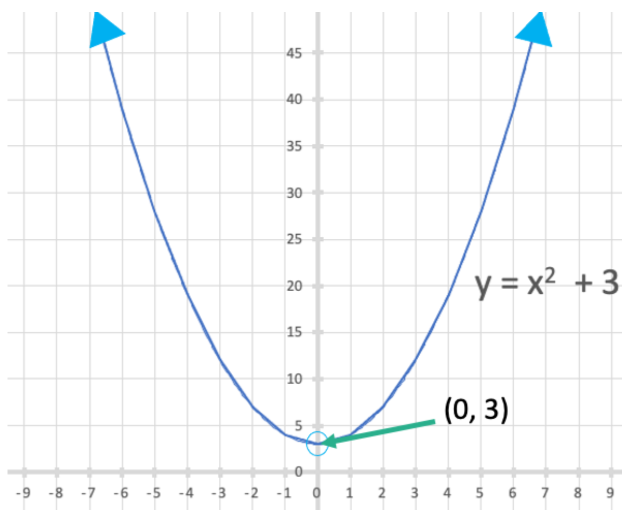


Figure 2: $x^2 + 3$

Domain: $\{x \in R\}$

The *x* values for the function continue to infinity in both directions.

Range: $\{y \geq 3\}$

The *y* values continue to infinity only in the upward direction. No *y* values are possible below the function's **minimum** of *y = 3*.

***X*-Intercepts and *Y*-Intercept:**

An ***x*-intercept** is any point where a function crosses the x-axis. Along the x-axis, all y-values are equal to **zero**. Thus, at all x-intercepts, y = 0.

The function in Figure 2 has a minimum at y = 3, which is larger than zero. It is a **quadratic** that "opens up" (is U-shaped). That means the function will never go below y = 3, so it has no x-intercepts.

The function y = x (Figure 1) crosses the x-axis once, at (0,0). This means it has one x-intercept, at x = 0.

A function may have multiple x-intercepts. **The maximum number of x-intercepts possible is equal to the degree of the function,** however, it may have less. Sometimes it has none.

A ***y-intercept*** is any point where a function crosses the y-axis. A polynomial function will never have more than one y-intercept. Each x value in the domain of a function can only have one y-value associated with it. Thus, at the y-axis where x = 0, we can only have one point.

SLOPE:

The slope of a polynomial is only constant for a **LINEAR** function (functions with a degree of one). The slope is often represented by the letter 'm' in a linear equation.

$$y = mx + b$$

$$m = \frac{rise}{run}$$

Check out some of the linear equations videos on the *Learn with Pi* YouTube channel for more information.

Example 3:

What is the slope of the function in Figure 3? How would you calculate it using just the graph?

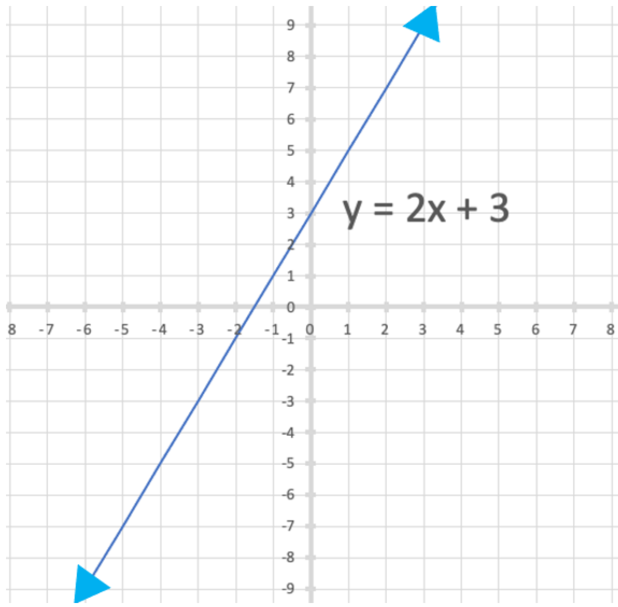


Figure 3: y = 2x + 3

The ‘***m***’ value in front of ***x***, using *y = mx + b*, is 2.

The slope is equal to 2.

Using just the graph, we could calculate the slope using two points on along the line.

We’ll use the coordinates (1, 5) and (2, 7). The ‘rise’ is the difference in the y-coordinates, or the vertical change between them. The ‘run’ is the difference in the x-coordinates, or the horizontal difference between them. Remember to always keep the order of your coordinates the same for your subtractions on the top and the bottom. That will correctly give you either a positive or a negative slope.

$$m = \frac{rise}{run} = \frac{7-5}{2-1} = \frac{2}{1} = 2$$

TURNING POINTS:

A turning point is where the graph of a function changes direction. It goes from increasing to decreasing in the y-direction, or from decreasing to increasing. The maximum number of turning points a function may have is equal to the degree of the function minus one.

$$max\ number\ of\ turning\ points\ possible = degree - 1$$

Example 4:

How many turning points are in the quadratic function $y = x^2 + 3$, shown in Figure 2?

Degree of function: 2

$$\#\ of\ turning\ points = 2 - 1 = 1$$

There is one turning point. It is located at (0,3). Going from **LEFT** to **RIGHT**, the y-values of this function are decreasing. When it gets to (0,3) it *turns* and begin increasing.

QUADRANTS & END BEHAVIOUR:

A cartesian plane (ie the graph we plot our functions on) is divided into four spaces by the crossing of the *x* and *y* axes. Each of these four spaces is numbered. We do this so that we can describe the behaviour of functions and understand what other people are talking about…

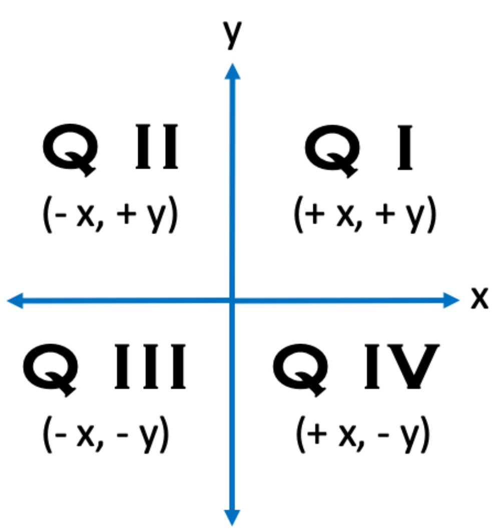


Figure 4: Cartesian Quadrants

If I say that the **END BEHAVIOUR** of a function is that it extends from QII to QIV you should be able to picture a rough shape for the function. The left end of the function will be somewhere in QII and the right end will be somewhere in QIV. That could give you something like what's shown in Figure 5.

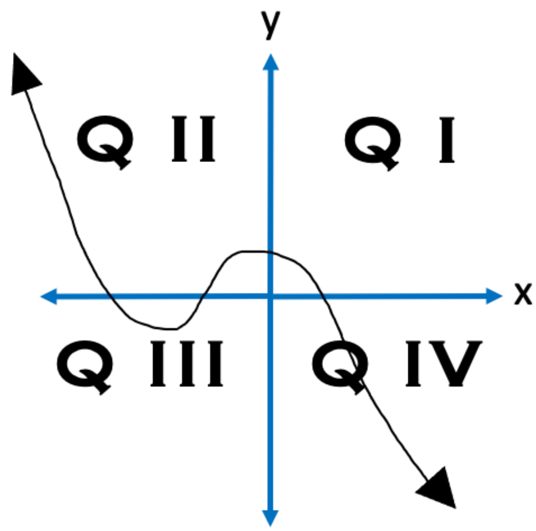


Figure 5: Example of QII to QIV End Behaviour

The **LEADING COEFFICIENT** of a polynomial is the constant term in front of the variable with the highest exponent. You can tell what the end behaviour of a function will be by looking at the **largest exponent** and the **leading coefficient**. If the:

Largest exponent is an **EVEN** number and

- Leading coefficient is positive = End behaviour: QII to QI
- Leading coefficient is negative = End behaviour: QIII to QIV

Or largest exponent is an **ODD** number and

- Leading coefficient is positive = End behaviour: QIII to QI
- Leading coefficient is negative = End behaviour: QII to QIV

ODD/EVEN FUNCTIONS:

A function is '**ODD**' when:

$$-\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(-\boldsymbol{x})$$

This means we have *origin symmetry*. For example, if we have an odd function like:

$$y = x^3$$

Then each symmetric point will be the same distance from the origin, (0, 0).

A function is '**EVEN**' when:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(-\boldsymbol{x})$$

This produces a symmetry about the y-axis. An example of an even function is:

$$y = x^2$$

Using these two examples, we can observe these two types of symmetry by plugging in values for our 'x' and seeing what the corresponding 'y' is. Table 1 shows a set of coordinates for each function. Their graphs, in Figure 6, help illustrate the symmetries of odd and even functions.

Table 1: Illustrative Coordinates of Odd and Even Functions

ODD FUNCTION		EVEN FUNCTION	
$-f(x) = f(-x)$		$f(x) = f(-x)$	
EG.	$y = x^2$	EG.	$y = x^3$
X	**Y**	**X**	**Y**
-5	25	-5	-125
-4	16	-4	-64
-3	9	-3	-27
-2	4	-2	-8
-1	1	-1	-1
0	0	0	0
1	1	1	1
2	4	2	8
3	9	3	27
4	16	4	64
5	25	5	125

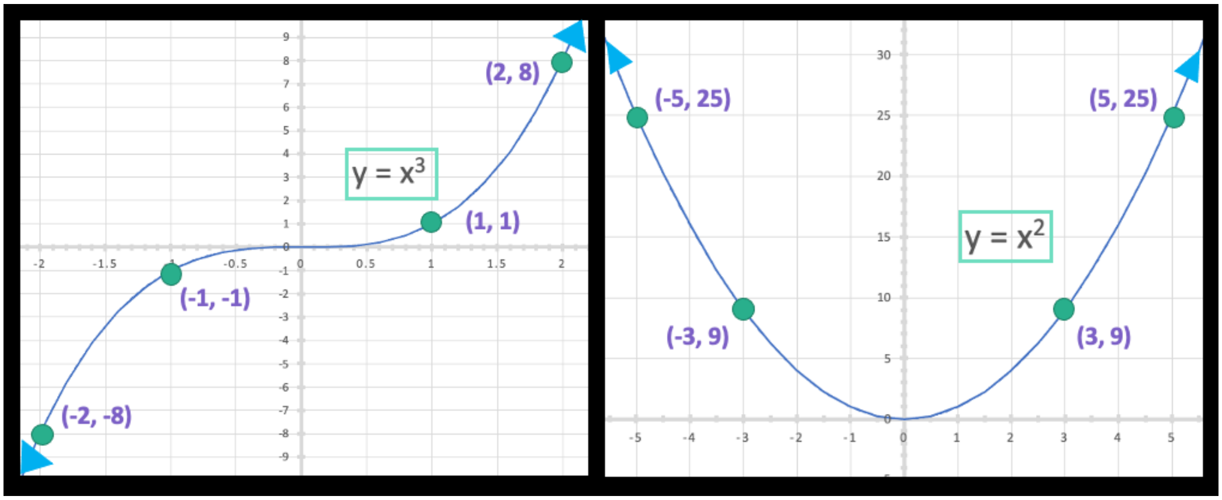


Figure 6: Graphs of Illustrative Odd (left) and Even (right) Functions

For typical polynomials, an **ODD** function will only have exponents that are **ODD** numbers, while an **EVEN** function will have **EVEN** exponents. However, there are other functions that may be even or odd. For example, $\cos(x)$ is an even function, while $\sin(x)$ is an odd function, as defined by the symmetry criteria.

Note, a function is **not always** either even or odd. In fact, **most functions are NEITHER**. Still, due to their cool symmetry properties, odd and even functions are important to learn about because they end up being useful in real life applications.

Just like with odd and even *numbers*, we can predict some things when we add/subtract or multiply/divide functions that we know are even or odd.

- $(EVEN\ function) + (EVEN\ function) = EVEN\ function$
- $(ODD\ function) + (ODD\ function) = ODD\ function$
- $(EVEN\ function) + (ODD\ function) =$
  $\boldsymbol{NEITHER}\ even\ nor\ odd\ (unless\ 0\ is\ one\ of\ your\ functions)$
- $(EVEN\ function)\ \times (EVEN\ function) = EVEN\ function$
- $(ODD\ function) \times (ODD\ function) = EVEN function$
- $(EVEN\ function) \times (ODD\ function) = ODD\ function$

To help myself remember these rules, I think of a simple example of what happens to the exponents in each of the following cases:

EG.	EVEN + EVEN	$x^2 + x^2 = 2x^2$	Result: EVEN
EG.	ODD + ODD	$x^3 + x^3 = 2x^3$	Result: ODD
EG.	EVEN + ODD	$x^2 + x = x^2 + x$	Result: NEITHER (no symmetry)
EG.	EVEN x EVEN	$x^2 + x^2 = x^{\mathbf{2+2}} = x^4$	Result: EVEN
EG.	ODD x ODD	$x^3 + x^3 = x^{\mathbf{3+3}} = x^6$	Result: EVEN
EG.	EVEN x ODD	$x^2 + x^3 = x^{\mathbf{2+3}} = x^5$	Result: ODD

DECREASING/INCREASING FUNCTIONS:

Increasing function: As the x-values of the function **INCREASE**, the y-values of the function also **INCREASE**.

Decreasing function: As the x-values of the function **INCREASE**, the y-values of the function **DECREASE**.

If a function is **STRICTLY** increasing or decreasing, then there can be no interval where it is flat or where part of the function has a reversed increasing/decreasing trend.

A STRICTLY increasing or STRICTLY decreasing function is also considered an INJECTIVE function. This is because each y-value is **unique**. Since the y-values are either continuously getting larger of smaller, no y-value will ever repeat.

OPENING UP/OPENING DOWN:

A quadratic function is said to **OPEN UP** if its graph roughly looks like a “U” shape. It **OPENS DOWN** if the “U” looks upside down.

If a quadratic function is in *standard* or *vertex* form, you can tell if its graph will open up or down by the number in the ***‘a’*** spot of the function. (Check out the YouTube videos on quadratic functions for more info).

Standard form: $\boldsymbol{y = ax^2 + bx + c}$
Vertex form: $\boldsymbol{y = a(x-h)^2 + k}$

If $a \neq 0$, then the function will OPEN UP if $a > 0$ and it will OPEN DOWN if $a < 0$.

If a function opens up, then it has a MINIMUM. If it opens down, it has a MAXIMUM. Figure 7 and Figure 8 show examples of functions that open up and down.

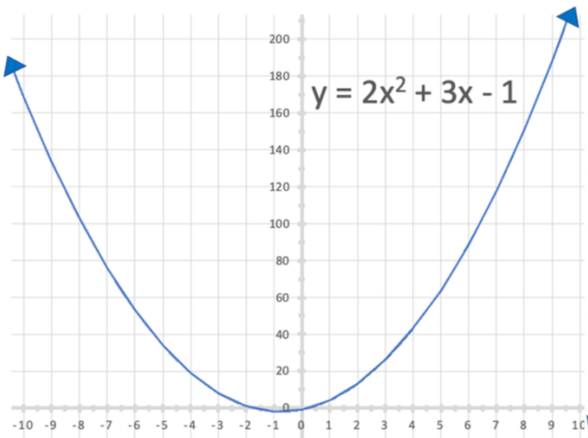


Figure 7: Example Opens Up (a = 2)

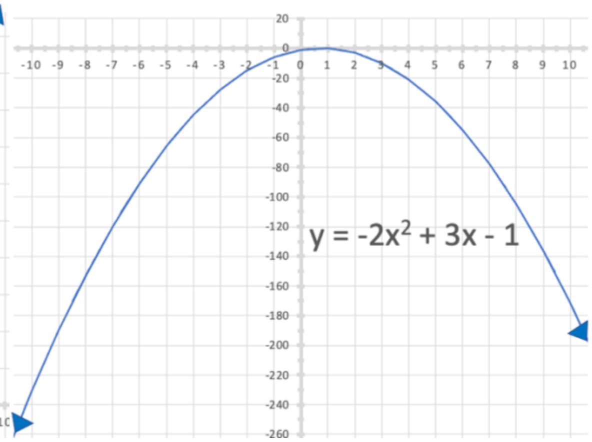


Figure 8: Example Opens Down (a = -2)

EXAMPLE 5:

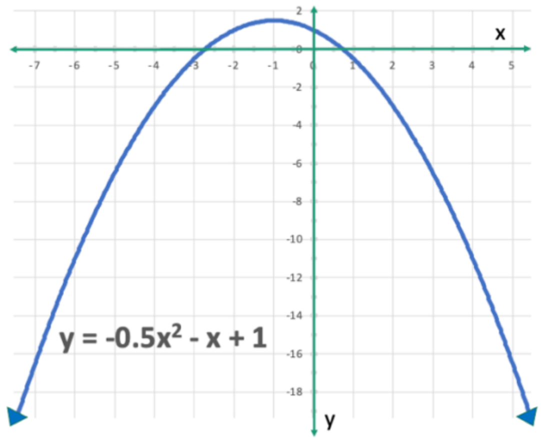


Figure 9: $-0.5x^2 - x + 1$

a) Is the function in Figure 9 an even function?
b) Does the function in Figure 9 open up or down?

ANSWER:

a) No. An even function is symmetric about the y-axis with $f(x) = f(-x)$. You can see that at x = 2, y = -3, but at x = -2, y = 1 (NOT +3). Additionally, from the graph you can see that the function is mirrored at x = -1, and NOT at x = 0. Both of these things indicate that this is not an even function.
b) The function opens down. It makes an upside down ‘U’ shape.

EXAMPLE 6:

Use these functions to answer the following questions:

$$f(x) = -x^3 + x$$

$$g(x) = 3x^3 - 2x$$

$$h(x) = 4x^2 + 8$$

Which of the following would give us an ODD function:

A. f(x) + g(x)
B. g(x) + h(x)
C. f(x) + h(x)
D. f(x) + g(x) + h(x)

ANSWER:

A. f(x) + g(x) -> Correct! ODD + ODD = ODD
B. g(x) + h(x) -> Incorrect: ODD + EVEN gives us neither an odd or even function
C. f(x) + h(x) -> Incorrect: ODD + EVEN gives us neither an odd or even function
D. f(x) + g(x) + h(x) -> ODD + ODD = ODD, then we + EVEN means we have ODD + EVEN which is NEITHER odd or even!

Which of the following would give us an EVEN function:

A. f(x)g(x) + h(x)
B. g(x)h(x) + f(x)
C. f(x)h(x) + g(x)
D. f(x) – g(x) + h(x)

ANSWER:

A. f(x)g(x) + h(x) -> Correct! ODD x ODD = EVEN, then EVEN + EVEN = EVEN
B. g(x)h(x) + f(x) -> Incorrect: ODD x EVEN = ODD, then ODD + ODD = ODD
C. f(x)h(x) + g(x) -> ODD x EVEN = ODD, then ODD + ODD = ODD
D. f(x) – g(x) + h(x) -> Incorrect: ODD – ODD = ODD, then ODD + EVEN = neither