PROBABILITY PART B: Learning to Count

I bet you thought you already knew how to count, right? Hopefully you do, but a big part of probability problems is knowing how to count LOTS of things. Like how many ways could a deck of 52 cards be shuffled? If you had to count that one way at a time it would take you forever. In this section we're going to learn how to use math formulas and shortcuts to help us count things we don't want to do the long way.

If you didn't like the Fundamental Counting Principle, then I've got bad news – it is a helpful tool in understanding the next thing. On the bright side, the more you learn the cooler the problems become.

A **PERMUTATION** is one way in which a number of things could be ordered. We often want to know how many ways something can be arranged, IE how many permutations of something there are.

<u>EXAMPLE 5</u>

Say your teacher is drawing names for the order of people who have to give a presentation in class. Abby, Ben and Carrol have to present. How many permutations (orders) can they present in?

1. Abby, Ben, Carrol
2. Abby, Carrol, Ben
3. Ben, Abby, Carrol
4. Ben, Carrol, Abby
5. Carrol, Abby, Ben
6. Carrol, Ben, Abby

There are 6 possible orders. Instead of thinking through all the orders, you can "fill in the blanks". Three names will be drawn, so we need to fill in three blanks:

_____ x _____ x _____

When she draws the first name, there are three possibilities:

3 x ___ x ___

For the second name, she hasn't put the first name back (**NO REPLACEMENT),** so there are two possibilities:

3 x 2 x _____

For the third name, only one person remains to be drawn, so there is only one possibility.

3 x 2 x 1 = 6

That means there are 6 possible orders that the students may present in.

With permutations, the number of possible orders increases very quickly with each new decision. Sometimes it's mind boggling. For example, what if there were 12 people presenting? How many orders could they present in?

12 x 11 x 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 479 001 600 possibilities

THAT'S A BIG NUMBER. Also, writing out that whole thing may seem like overkill. If you don't want to write out all the numbers from 12 to 1 then you're in luck! Math loves to use symbols to represent other things so that we don't have to write so much. Really, math had the original emojis ….

Instead of writing out all of the numbers from 12 to 1, we could use **FACTORIAL** notation to say:

12! = 479 001 600

A factorial means that:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \ldots.(n-i) \text{ where } (n-i) = 1$$

where ‘n’ is a whole number and the last term is equal to 1.

<u>EXAMPLE 6</u>

$n = 4$

$4! = 4 \times (4-1) \times (4-2) \times (4-3)$

$4-3 = 1$ so we know it is our last term. Simplifying:

$4! = 4 \times 3 \times 2 \times 1 = 24$

Notice that in the final form it looks like we are just counting down to one from the highest number and multiplying all the numbers together. Also, if you have a graphing calculator there is likely a “!” button that you can press after a number to calculate the factorial of it.

FACTORIALS: Let’s get mathy

Factorials can be used to simplify and expand lots of different math statements – so here’s some examples to illustrate some of the common problem types:

<u>EXAMPLE 7</u>

Evaluate

a) $5 \times 6 \times 4!$
   - We can rearrange it to make it look sequential

   $6 \times 5 \times 4! =$ **$6!$**

b) $\frac{6!4!}{9!}$

- We want to expand one or more terms on the top or bottom until we have terms on the top and bottom that match
- When they match it means we have a number being divided by itself, which equals 1 – so we can cancel them
- Here 9 is the biggest factorial number so we choose it. It is on the bottom, so we look on the top for its biggest factorial number, which is 6.
- We expand the 9 until we get to 6! and then we can cancel it and calculate our answer.

$$\frac{6!\,4!}{9!} = \frac{6!\,4!}{9 \times 8 \times 7 \times 6!} = \frac{4!}{9 \times 8 \times 7} = \frac{24}{504} = \mathbf{\frac{1}{21}}$$

c) $\frac{7!}{5!} = \frac{7 \times 6 \times 5!}{5!} = 7 \times 6 = \mathbf{42}$

EXAMPLE 8

Write the expressions using factorial notation.

a) $4 \times 3 \times 2 \times 1 = \mathbf{4}!$

b) 12×11

- We can use the same idea from EXAMPLE 6; if two matching factorials are on the top and bottom of a fraction they will cancel. Now we do that backwards.
- We are multiplying our fraction by 1 when we do this, so the numerical value is not changing.
- This expression looks like everything from 10! has been cancelled (it would be the next term after 11)

$$12 \times 11 = \frac{12 \times 11 \times 10!}{10!} = \frac{\mathbf{12}!}{\mathbf{10}!}$$

c) $\dfrac{12\times11\times10}{3\times2\times1} = \dfrac{12\times11\times10\times9!}{9!\times3!} = \dfrac{\mathbf{12!}}{\mathbf{9!3!}}$

<u>EXAMPLE 9</u>

Simplify

a) $\frac{(n-3)!}{n!}$

- Use the same strategy as before, taking the biggest number and expanding it to equal something on the opposite side of the fraction
- n > (n – 3) so we expand the bottom

$$\frac{(n-3)!}{n!} = \frac{(n-3)!}{n\times(n-1)\times(n-2)\times(n-3)!} = \frac{\mathbf{1}}{\boldsymbol{n\times(n-1)\times(n-2)}}$$

b) $\frac{(n+3)!}{n!} = \frac{(n+3)\times(n+2)\times(n+1)\times n!}{n!} = (n+3)\times(n+2)\times(n+1)$

c) $\frac{(n+4)\times(n+2)!}{(n+3)!}$

- The biggest term with a factorial on it is (n+3)! since (n+3) > (n+2) and the term (n+4) does not have a factorial.

$$\frac{(n+4)\times(n+2)!}{(n+3)!} = \frac{(n+4)\times(n+2)!}{(n+3)\times(n+2)!} = \frac{(n+4)}{(n+3)}$$

PERMUTATIONS

Now that we've crushed factorials, we're ready to level up on our permutation game. Here's the formula:

$$_nP_r = \frac{n!}{(n-r)!} \ where\ 0 \leq r \leq n$$

Think of "n" as the total number of things you have and "r" as the number of those things that you are picking and arranging in an order. This is often called the "pick" formula. It may look strange, but it's actually just another way to write things you've already learned.

EXAMPLE 10

Your teacher is drawing names for the order of people who have to give a presentation in class. Abby, Ben and Carrol have to present. There is only time for 2 of them to present today, the other person will have to present tomorrow. How many permutations (orders) can the two selected people present in?

Start with the "fill in the blanks" strategy that we learned with the Fundamental Counting Principle. Draw one line for each name that will be drawn today.

____ x ____

Fill in the blanks with the number of possible names each time she draws. First, she has 3 to pick from. She draws one. There are two left. She draws another. Now she has her two presenters.

$\underline{3} \times \underline{2} = 6$ orders

Rewrite that using factorial notation:

$$3 \times 2 = \frac{3 \times 2 \times 1!}{1!} = \frac{3!}{1!}$$

Now let's use our permutations formula to solve the problem. There are 3 total names being drawn from, so n =3. 2 students will present today, r = 2.

$${}_nP_r = \frac{n!}{(n-r)!} = {}_3P_2 = \frac{3!}{(3-2)!} = \frac{3!}{1!}$$

Using our "pick" formula we end up with the same answer as when we use the "fill in the blanks" method and write it as a factorial!

If they're the same why would we bother with a complicated formula? Sometimes the numbers get so big in our problems that if we don't use the formula, we end up writing lines all day long to fill in the blanks.

If you have a graphing calculator, there is likely a "${}_nP_r$ " button in the probability section of your functions too. See if you can use your calculator buttons to check:

"3" "${}_nP_r$ " " 2" = 6

(If you're curious, the possible orders are: 1. Abby, Ben 2. Abby, Carrol 3. Ben, Abby 4. Ben, Carrol 5. Carrol, Abby 6. Carrol, Ben)

PERMUTATION EXAMPLES:

When doing permutation problems, don't rush to use the "pick" formula. Think about what the problem is asking.

- Are repeats allowed? Eg, if you're drawings names from a hat and after you draw a name you put it back in the hat before you draw the next name then the "pick" formula won't apply. Use the "fill in the blanks" strategy.

EXAMPLE 11

Jerome coaches a hockey team. His team is in a shoot-out to win the game. He must pick from 6 players who will take 3 shots against the other team's goalie. How many shoot-out orders are possible if

a) The same player can't take more than one shot?
b) Jerome can pick the same player to take multiple shots?

Solution:

a) Each player can only take one shot, so there are no repeats. Filling in the blanks for 3 shots:

___ x ___ x ___ = $\underline{6}$ x $\underline{5}$ x $\underline{4}$ = 120 shoot out orders

Or since there are no repeats, we can use the "pick" formula:

$${}_nP_r = \frac{n!}{(n-r)!} = {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120 \text{ shoot out orders}$$

b) One player can take more than one shot, so the number of players he has to pick from each time stays the same.

___ x ___ x ___ = $\underline{6}$ x $\underline{6}$ x $\underline{6}$ = 216 shoot out orders

EXAMPLE 12

Mina has to make a password for his e-mail account. It needs to have 8 characters. The first 5 characters will be letters and the last 3 characters will be digits from 0 to 9. How many passwords are possible if

a) He cannot use the same letter or number more than once?
b) He can repeat a letter or number?

Solution:

a) There are **26 letters** to pick from and **10 digits**.

___ x ___ x ___ x ___ x ___ x ___ x ___ x ___

26 x 25 x 24 x 23 x 22 x **10 x 9 x 8** = **5 683 392 000 possible passwords**

Or since there are no repeats, we can use the "pick" formula. Note that when we are picking letters it is different from our step of picking numbers; we have to use the formula in two places.

${}_{26}P_5 \times {}_{10}P_3$ = 5 683 392 000 possible passwords

b) He can repeat letters and numbers, which means that the number he picks from each time stays the same. We cannot use the "pick" formula here.

26 x 26 x 26 x 26 x 26 x **10 x 10 x 10** = **11 881 376 000 possible passwords**

PERMUTATIONS: SAMESIES

If you are arranging a group of objects but some of them are the same, then the total number of different orders you can have **decreases** because it doesn't matter what order the identical objects go in.

When you use the "pick" formula to determine the arrangements of objects, you are taking the number of arrangements possible for all objects (n) and reducing it to only the arrangements with the number of objects you want (r).

For example, if you had 6 letters (a, b, c, d, e, and f) and wanted to order all 6 you would have:

$${}_{6}P_{6} = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! = 720 \text{ possible arrangements}$$

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

(0! = 1 because there is only 1 way to arrange 0 items)

If you wanted to arrange only 4 of the 6 in an order you would have

$$_6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360 \text{ possible arrangements}$$

$$6 \times 5 \times 4 \times 3 \cancel{\times 2 \times 1} = 360$$

When you select fewer than the total number, you are **removing** those unused decisions from your calculation. You remove them by dividing **n!** by the number of arrangements possible from your unused decisions. In this example, when we select only 4 of the 6 letters we don't need to consider the last 2 decisions, so we have fewer total arrangements possible.

The same concept applies when you have multiple objects that are the **same**. If I am arranging the letters **a, b, c, d, e** and **e** then the order I place my two **e**'s in relative to each other doesn't matter because the final order would look the same. The decisions involved in arranging these two letters needs to be **removed** from our total number of arrangements since it does not impact the final appearance of the order. We do this in the same way as we did in the "pick" formula – by dividing.

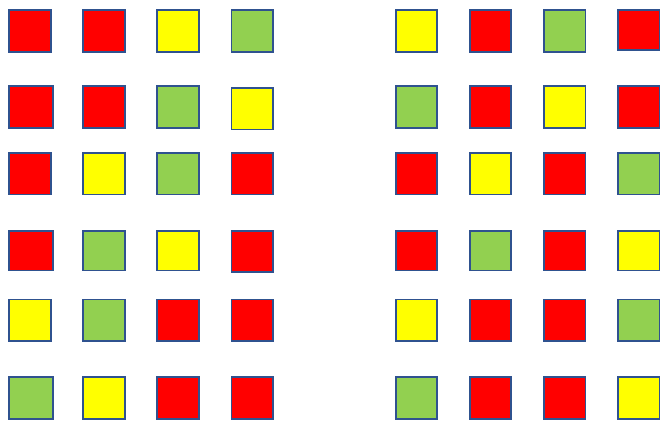
<u>EXAMPLE 13</u>

There are 4 coloured blocks. You want to arrange them in different orders based on their colours. How many arrangements are possible?

a) The blocks are all different colours: red (R), blue (B), yellow (Y) and green (G).
b) Two of the blocks have the same colour: two red blocks (R), one blue (B), one yellow (Y) and one green (G).

a) $number\ of\ objects\ to\ arrange! = 4! = 24$ arrangements

b) $\frac{Number\ of\ objects\ to\ arrange!}{Number\ of\ objects\ that\ are\ the\ same!} = \frac{4!}{2!} =$ 12 arrangements



EXAMPLE 14

a) How many ways can the letters of KITTEN be arranged?

$$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2!\,(2\ identical\ T's)} = \frac{6!}{2!} = 360\ ways$$

b) How many ways can the letters of KITTEN be arranged if the first letter must be K?

If the first letter must be K then we only have one possibility for the first decision. For the second letter, since K has been used, we have 5 possibilities left and so on.

$$\frac{1 \times 5 \times 4 \times 3 \times 2 \times 1}{2!\,(2\ identical\ T's)} = \frac{1 \times 5!}{2!} = 60\ ways$$

EXAMPLE 15

How many ways can 3 red, 1 blue, 2 yellow and 4 green blocks be arranged?

$$\frac{10!\,(10\ total\ blocks)}{3!\,2!\,4!\,(identical\ blocks)} = 12600$$

COMBINATIONS

Combinations are similar to permutations, except that they do not care about the order of things – only the chosen group. Say you were selecting your starting line-up for a sports team. The team has 20 players and you need 5 to start the game. Using permutations, you could determine how many orders that you could pick those 5 players in.

$$ {}_{n}P_{r} = {}_{20}P_{5} = \frac{n!}{(n-r)!} = \frac{20!}{(20-5)!} = 1\ 860\ 480\ orders $$

With permutations, if you pick Abby, Ben, Carrol, Dean and Frank you are counting that line-up multiple times. That group of players is counted in all the different orders they could be selected in. When you go to start the game though, do you care if Abby was picked before Ben or Ben before Abby? Not really. All that matters is that you selected those players to play together. That's where combinations come in. Using combinations eliminates the times you count the same line-up more once.

So how many combinations of 5 players can be selected from your team of 20? There is a very similar formula to the "pick" formula that we can use:

$$ {}_{n}C_{r} = \binom{n}{r} = \frac{n!}{r!(n-r)!} where\ 0 \leq r \leq n $$

"n" is still the total number of things you have, and "r" is the number of things you are choosing. This is often called the "choose" formula. The only difference is the "r!" on the bottom, which we learned in the last section is used to eliminate the outcomes from the decisions we don't care about. In this case, that's the order the 5 players are chosen in.

$$ {}_{20}C_{5} = \frac{20!}{5!(20-5)!} = 15504\ possible\ starting\ line\ ups $$

Notice that there are fewer possibilities than when we cared about order with the pick formula. For the same number selected, there will always be fewer outcomes we care about with the "choose" formula compared to the "pick" formula.

<u>EXAMPLE 16</u>

Mark is making a three-scoop ice cream cone. There are 12 flavours to choose from.

a) How many combinations of flavours can he make?
b) If he wants to layer the flavours in a certain order, how many arrangements of his three scoops can he make?

Solution:

a) ${}_{12}C_3 = \frac{12!}{3!(12-3)!} = 220\ flavour\ combinations$
b) Order matters, so we will use a permutation:

${}_{12}P_3 = \frac{12!}{(12-3)!} = 1320\ flavour\ arrangements$

<u>EXAMPLE 17</u>

At a conference there are 5 students from Alberta and 7 students from Ontario. How many ways can you make a group of 6 students that has **at least 4** students from Alberta?

Solution:

Think of what the different group of 6 could look like with **at least 4** students from Alberta:

a) 4 from Alberta, 2 from Ontario
b) 5 from Alberta, 1 from Ontario

For the total, we want to add all the combinations possible to form **both** of the above. Think of the "fill in the blanks" method where each decision to get to a grouping is multiplied together.

a) We need to choose 4 out of 5 possible students from Alberta and 2 out of 7 possible students from Ontario: ${}_5C_4 \times {}_7C_2$
b) We need to choose 5 out of 5 possible students from Alberta and 1 out of 7 possible students from Ontario: ${}_5C_5 \times {}_7C_1$

Then we add the groupings for each possible scenario together:

Using the "choose" function on your calculator:

$${}_5C_4 \times {}_7C_2 + {}_5C_5 \times {}_7C_1 = 112 \text{ possible groups}$$

Or using the formula:

$$\frac{5!}{4!\,(5-4)!} \times \frac{7!}{2!\,(7-2)!} + \frac{5!}{5!\,(5-5)!} \times \frac{7!}{1!\,(7-1)!} =$$

$$\frac{5!}{4!\,1!} \times \frac{7!}{2!\,5!} + \frac{5!}{5!\,0!} \times \frac{7!}{1!\,6!} = \frac{5 \times 4!}{4!\,1!} \times \frac{7 \times 6 \times 5!}{2!\,5!} + \frac{5!}{5!\,0!} \times \frac{7 \times 6!}{1!\,6!} =$$

$$5 \times \frac{42}{2} + 1 \times 7 = 112 \text{ possible groups}$$

PROBLEM SOLVING TIP:

- Understand what the question is asking you. Seems basic right? I don't think it is. The trick I use is to pretend that I have to ask that same question to someone else, and I need to explain to them what they need to do. If you take the time to do this **BEFORE** you start trying to calculate numbers you will save yourself lots of headaches by calculating numbers and then forgetting why you were calculating them…