PROBABILITY PART C: Determining Probabilities

In this section we bring together everything we've learned all they way from Set Theory to Part B of this unit.

When you hear someone say "The odds are good that my team will win", what does that mean? Likely, you think that person is saying their team will probably win. **ODDS** and **PROBABILITY** are related. They are two different ways of expressing something. In fact, you can write an odds expression as a probability statement and vice versa.

- **ODDS** express the ratio of the likelihood of one event occurring relative to another. The event that you care about goes first. The statement

  A:B

  expresses the odds "**in favour**" of event A occurring relative to event B.

- This is also expressed as:
  - **Odds in favour of A = # outcomes for A : # of outcomes against A**

For example, if the odds in favour of horse A winning a race against horse B are 2:6 (A:B) then the odds in favour of horse B winning the race are 6:2 (B:A). The order you write the numbers changes depending on which horse you care about. However, the probability that each horse would win remains constant (25% chance horse A wins, 75% chance horse B wins).

- Odds expressions are equivalent to the ratios of the probabilities of events happening.

The odds in favour of horse A winning were 2:6. We can simplify this by dividing both sides of the ratio by 2, which gives 1:3.

The probability of horse A winning to the probability of horse B winning is:

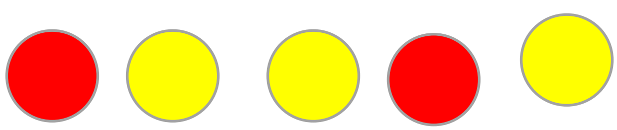
0.25 : 0.75. We can simplify this by dividing both sides of the ratio by 0.25 which gives 1:3. This matches the above.

Do you remember our bag with 5 balls in it from Example 1? Let's bring that out again to explain this further.

EXAMPLE 18

There is a bag with 5 balls in it. Three are yellow and two are red. You reach in and draw 1 ball out.

a) What is the probability of drawing a red ball?
b) What are the odds in favour or drawing a red ball?
c) What are the odds in favour of drawing a yellow ball?



a) $P(desired\ outcome) = \frac{\#\ of\ desired\ outcomes}{total\ \#\ of\ possible\ outcomes} = \frac{2}{5} = 40\%$
b) Our "in favour" event would be drawing a red ball. There are 2 red balls. The not "in favour" event would be drawing a yellow ball. There are 3 yellow balls. So the odds of drawing a red ball are 2:3 ("two to three").
c) We care about yellow for this question, so we switch the order to put the yellow numbers first: 3:2

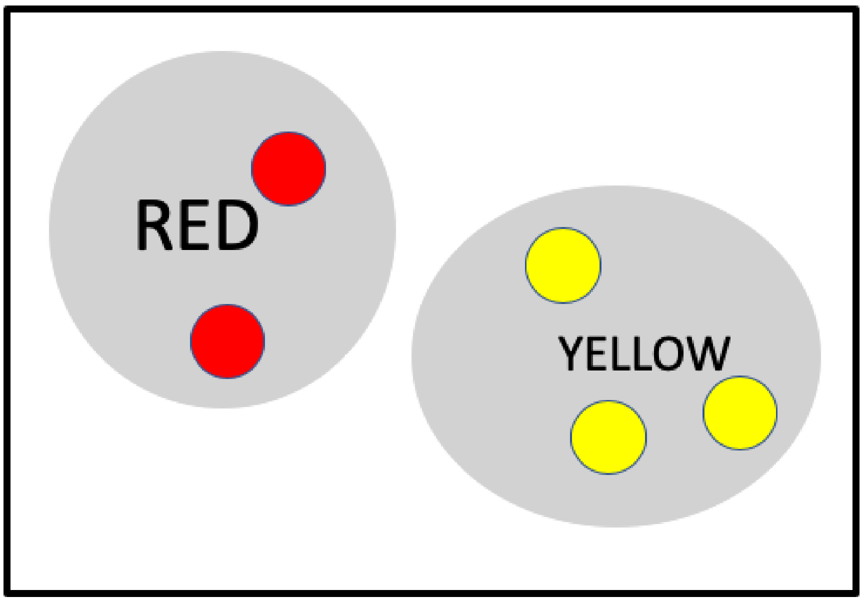
Notice that the numbers in our odds expressions add together to equal the number on the bottom of our probability fraction: 3+2 = 5.

PROBABILITY PART C NOTES

In a general form, the odds in favour of a red ball (R) being drawn are:

R:Y and the probability is: $P(R) = \frac{R}{R+Y}$

In a Venn Diagram, all the balls in the bag could be grouped like this:



The following points may seem obvious and strange to point out, but they lead to general rules that are very helpful for solving problems:

- The **COMPLEMENT** of the red balls would be all the yellow balls (every ball that is not red in the bag is yellow): R' = Y
- All the red balls and all the balls that are not red add up to all the balls. If you are drawing a ball, so the probability of drawing a red ball plus the probability of not drawing a red ball is 100%:
  P(R) +P(R') = 1

ODDS summary:

- Odds in favour of event A happening to event B are A:B
- A:B = P(A) :P(B)
- If 'a' are the number of times event A could occur and 'b' are the number of times event B could occur:

$$P(A) = \frac{a}{a+b} = \frac{P(A)}{P(A)+P(B)}$$

- P(A) + P(A') = 1

EXAMPLE 19

The odds in favour of event A occurring are shown. Convert them to the probability that event A will happen.

a) 1:4
b) 4:1
c) 62:50

Solution:

a) a = 1, a' = 4

$$P(A) = \frac{a}{a+a'} = \frac{1}{1+4} = \frac{1}{5} = 0.20 \; or \; 20\%$$

b) a = 4, a' = 1

$$P(A) = \frac{a}{a+a'} = \frac{4}{4+1} = \frac{4}{5} = 0.80\ or\ 80\%$$

c) a = 62, a' = 50

$$P(A) = \frac{a}{a+a'} = \frac{62}{62+50} = \frac{62}{112} = 0.554\ or\ 55.4\%$$

EXAMPLE 20

The probability of event A occurring are shown.

a) P(A) = 15% What are the odds **in favour** of event A occurring?
b) P(A) = 90% What are the odds **in favour** of event A occurring?
c) P(A) = 20% What are the odds **against** event A occurring?

Solution:

a) P(A) + P(A') = 1, so P(A') = 1 – P(A) = 1 – 0.15 = 0.85
a : a' = P(A) : P(A') = 0.15 : 0.85
Simplify the ratios. Multiply each side by 100
15 : 85
Divid each side by their largest common denominator, 5:
3 : 17

b) P(A) + P(A') = 1, so P(A') = 1 – P(A) = 1 – 0.90 = 0.10
a : a' = P(A) : P(A') = 0.90 : 0.10
Simplify the ratios. Multiply each side by 100
90 : 10
Divid each side by their largest common denominator, 10:
9 : 1

c) P(A) + P(A’) = 1, so P(A’) = 1 – P(A) = 1 – 0.20 = 0.80
We want the odds against, so switch the order:
a' : a = P(A’) : P(A) = 0.80 : 0.20
Simplify the ratios. Multiply each side by 100
80 : 20
Divide each side by their largest common denominator, 20:
4 : 1
(Note, the odds **in favour** of event A would be 1:4)

EXAMPLE 21

An animal shelter has cats and dogs. The **odds against** a cat being the next animal to be adopted are 36:51. What is the probability that the next animal adopted will be a cat?

Solution:

- “odds against” means that cats are represented by the 2nd number in the expression 36:51
- Total number of animals = 36 + 51 = 87

$$P(cat) = \frac{a}{a+a'} = \frac{36}{36+51} = \frac{36}{87} = 0.414$$

There is a 41.4% chance a cat will be the next animal adopted.

INCLUSION-EXCLUSION PRINCIPLE

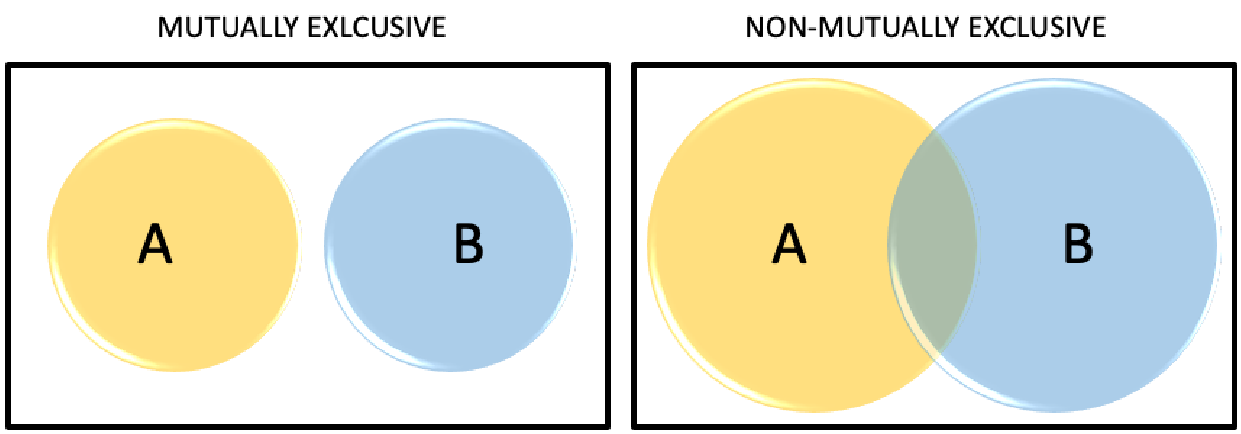
We saw this principle at the end of the unit on Set Theory. Remember:

$|A \cup B| = |A| + |B| - |A \cap B|$

"The number of elements in the union of A and B is equal to the sum of the elements in set A and Set B, less the number of elements in both sets A and B."

The same principle applies when A and B represent events.

MUTUALLY EXCLUSIVE events are events that cannot occur at the same time (e.g. drawing a black card from a deck or drawing a red card; a card is either red or black, it is never both colours). **NON-MUTUALLY EXCLUSIVE EVENTS** have outcomes that can overlap (e.g. drawing a card that is red and also a diamond).



When events are **MUTUALLY EXLCUSIVE** there are no overlapping outcomes:

- $P(A \cap B) = 0$
- $P(A \cup B) = P(A) + P(B)$

For events with overlapping outcomes, the **INCLUSION-EXCLUSION PRINCIPLE** states:

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

This is useful for determining probabilities of non-mutually exclusive

CONDITIONAL PROBABILITY

Pretend you're watching a TV show. The character gets up, goes to their closet and picks out what they're going to wear. Do you think they will wear a jacket? In most episodes they don't wear a jacket, so you would guess that today they wouldn't either. However, what if we know more information? Is it raining? If we know it's raining then the probability they wear a jacket will go up.

So if we had to guess without knowing the weather we would assume a different probability for them wearing a jacket than if we had to guess and we did know the weather.

What this shows us is that sometimes the probability of an event occurring is related to one or more other events occurring. How likely is it that it is raining? This impacts how likely it is that the person wears a jacket. These types of events are called **DEPENDENT**.

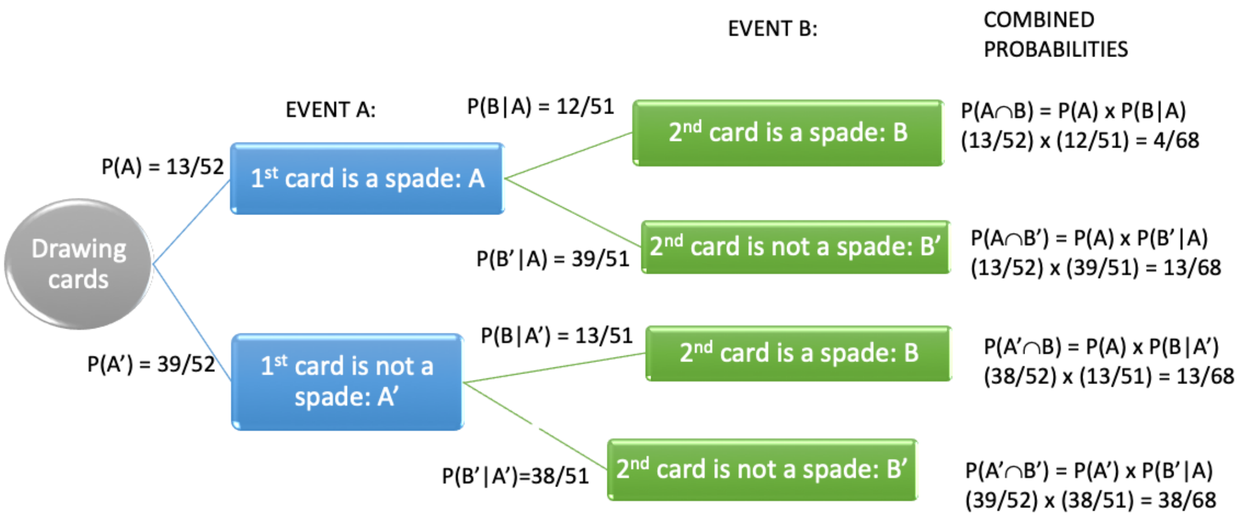
Dependent events also occur when you are drawing multiple objects blindly from a group **without replacement**.

EXAMPLE 22

You draw one card from a deck. Without putting it back you draw a second card. What is the probability that the second card is a spade?

Let's use a decision tree to show this.

- $P(B|A)$ means "the probability that B will occur, **given that** A has already occurred.



- To get the total probability of drawing a spade as the second card, we must consider that the **first** card could have been a spade or not.
- We sum the probabilities, considering both Event A outcomes

$P(B) = P(A \cap B) + P(A' \cap B) = (4/68) + (13/68) = 17/68 = 0.25$

There is a 25% chance that the second card you draw is a spade.

CONDITIONAL PROBABILITY Summary

- If you use a decision tree, your end row will still always sum to one
- Drawing items **without replacement** results in dependent events
- $P(A \cap B) = P(A) \times P(B|A)$
- The above rearranges to a useful form of:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

INDEPENDENT EVENTS

With conditional probability, the probability of one event impacts the probability of another event. When that **ISN'T** the case, the events are called **INDEPENDENT**. For example, whether or not Pi wakes me up in the morning is independent from whether or not it's raining.

For independent events:

- With a decision tree, your end row will still sum to one
- Drawing items **with replacement** results in independent events
- The probability of BOTH events occurring is equal to the probabilities of each independent event multiplied together
- $P(A \cap B) = P(A) \times P(B)$

EXAMPLE PROBLEMS:

Tips:

- Determine if events are mutually exclusive or not
- Determine if the events are independent or dependent
- Draw a table, tree diagram or adjacent lines to “fill in the blanks”. Do whatever you need to visualize the problem.

EXAMPLE 23

There are 42 cats and dogs. 18 of the animals are cats. One animal will be selected randomly from the group.

a) What is the probability the animal will be a dog?
b) What are the odds against the animal being a cat?

Solution:

a) 42 – 18 = 24 of the animals are dogs.

$$P(dog) = \frac{\#\, dogs}{\#\, all\ animals} = \frac{24}{42} = \frac{4}{7} = 0.571$$

There is a 57.1% chance it will be a dog.

b) Odds against it being a cat = # dogs : # cats

18:24 or simplified (divide both sides by largest common denominator) = 3:8

EXAMPLE 24

A casino is testing new games of chance: Rodeo Roll, Cannon Fire and Teeter Toss. The odds in favour of winning each of the three games are:
Rodeo Roll 2:7
Cannon Fire 3:11
Teeter Toss 1:3
If someone were to play all three games, which game would they be most likely to win?

Solution:
Since the odds expressions are all expressed with different numbers on both sides of the expression, we first need to convert them all to probabilities to be able to compare them.

- Rodeo Roll 2:7 odds have a $\frac{2}{2+7} = \frac{2}{9} = 0.222 = 22.2\%$ chance of winning
- Cannon Fire 3:11 odds have a $\frac{3}{3+11} = \frac{3}{14} = 0.214 = 21.4\%$ chance of winning
- Teeter Toss 1:3 odds have a $\frac{1}{1+3} = \frac{1}{4} = 0.25 = 25\%$ chance of winning

The highest percent chance of winning comes from the Teeter Toss game. They would be most likely to win the Teeter Toss.

EXAMPLE 25

You draw a card from a standard deck of cards. What is the probability the card will be a face card or a heart?

Solution:

- It is possible for a card to be both a face card and a heart. The events are **non-mutually exclusive**
- We can apply the inclusion-exclusion principle
- We will call drawing a face card 'Event A' and drawing a heart 'Event B'

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = \frac{12}{52} + \frac{13}{52} - \frac{4}{52} = \frac{21}{52} = 0.404$$

There is a probability of 40.4% that the card you draw will be a face card or a heart.

EXAMPLE 26

Riley is playing in a badminton tournament. He will play 12 games. The probability that he will win any given game is 72%. To the nearest whole number, how many games is he likely to lose?

Solution:

- We are given the probability of winning, but we want to know how many losses. We need to convert it to a probability of losing
- P(losing) = 1 – 0.72 = 0.28
- # likely loses = P(losing) x # games played = 0.28 x 12 = 3.36 games

He will likely lose 3 games.

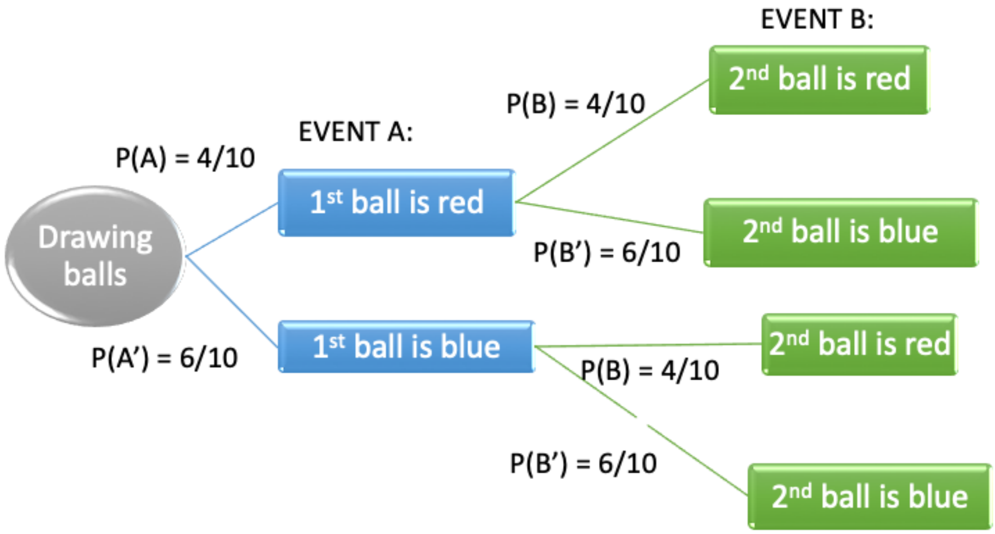
EXAMPLE 27

Candice conducts two experiments. She has a bag with 10 balls. Four of the balls are red and six of the balls are blue. In Experiment One she draws one ball and puts it back, then she draws a second ball. In Experiment Two she draws one ball, does not put it back, then draws a second ball. For both experiments we will call the first ball drawn ‘Event A’ and the second ball as ‘Event B’.

a) For Experiment One, what is the probability that the first and second balls are both red?
b) For Experiment Two, what is the probability that the first and second balls are both red?

Solution:

a) First ask yourself, are these events independent or dependent? If Candice puts the first ball back in the bag, then the conditions for drawing the second ball are the same as the first time she drew a ball. The events are **independent**.



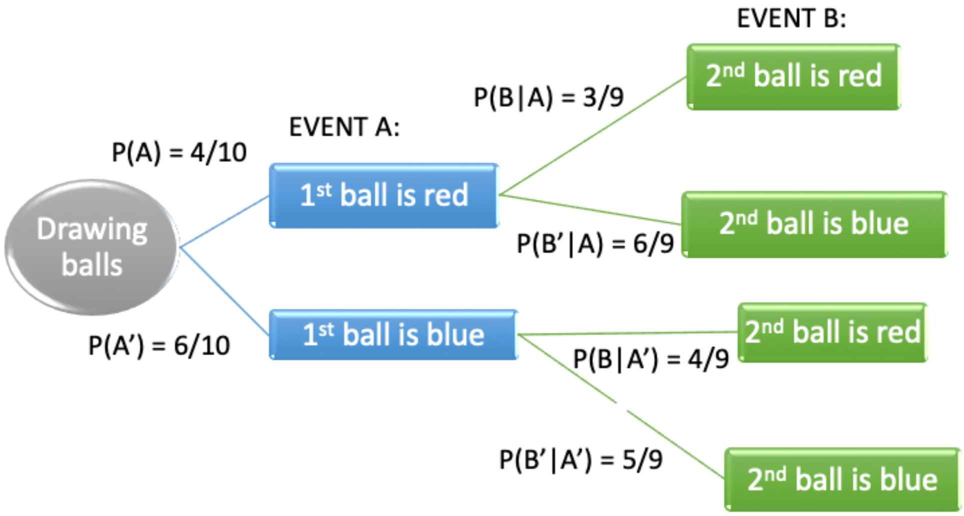
Using a decision tree we can follow the line of two red balls in a row and multiply the probabilities together:

$$P(A \cap B) = P(A) \times P(B) = \left(\frac{4}{10}\right) \times \left(\frac{4}{10}\right) = \frac{16}{100} = 0.16$$

There is a 16% probability that both balls will be red.

b) Are these events independent or dependent? The conditions for the second draw are different than the first draw, so the events are **dependent**. Conditional probability will apply.

PROBABILITY PART C **NOTES**



$$P(A \cap B) = P(A) \times P(B|A) = \left(\frac{4}{10}\right) \times \left(\frac{3}{9}\right) = \frac{12}{90} = \frac{2}{15} = 0.133$$

There is a 13.3% probability that both balls will be red.