

A **rational expression** is a fraction made up polynomials – meaning that it includes variables. When you solve or simplify them it is important to remember that they must remain exactly what their name says – **rational**. Rational numbers are always able to be written as a fraction of two integers.

An example of a rational expression would be:

$$\frac{3 + 2x}{8 - x}, x \neq 8$$

- If $x = 8$ then the bottom of this fraction (the denominator) would equal zero
- We can't divide by zero – it is undefined
- Thus 8 is a **non-permissible value** for 'x'

Non-permissible values are variable values that make the expression **undefined**.

We state what the non-permissible values are because they **restrict** what the variable for our expression can be.

To solve rational expressions we will do multiplication, division, addition and subtraction – just like with fractions.

- Tip: if you **multiply** or **divide** both the top and the bottom of a fraction by the same number, the **value** of the fraction stays the same

EXAMPLE 1: Multiplication

$$\frac{5x}{2x + 1} \times 2 = \frac{5x \times 2}{2x + 1} = \frac{10x}{2x + 1}$$

We have changed the value of this rational expression by multiplying it by two. If $x = 1$, then before multiplying our expression it equals:

$$\frac{5(1)}{2(1) + 1} = \frac{5}{3}$$

After multiplying by two it equals:

$$\frac{10(1)}{2(1) + 1} = \frac{10}{3}$$



EXAMPLE 2: Equivalent Expressions

$$\frac{5x}{2x+1} \times \frac{2}{2} = \frac{5x \times 2}{(2x+1) \times 2} = \frac{10x}{4x+2}$$

Here, since we multiplied **both** the top and the bottom by two, the value of our expression does not change. This is because $2 \div 2 = 1$ and anything multiplied by one equals itself. If $x = 1$:

$$\frac{5(1)}{2(1)+1} = \frac{5}{3} \text{ and } \frac{10(1)}{4(1)+2} = \frac{10}{6} = \frac{5}{3}$$

We say that $\frac{5x}{2x+1}$ and $\frac{10x}{4x+2}$ are **equivalent expressions**. They look different, but for any variable value they will give you the same answer.

EXAMPLE 3: Non-Permissible Values

What are the non-permissible values of the following expressions:

a) $\frac{2x}{2x+1}$

$$2x + 1 \neq 0$$

$$2x \neq -1$$

$$\frac{2x}{2} \neq -\frac{1}{2}$$

$$x \neq -\frac{1}{2}$$

b) $\frac{x+2}{x(x-1)}$

$$x(x-1) \neq 0$$

Since x and $(x-1)$ are being multiplied together and **zero** multiplied by anything is equal to zero, then if **either** x or $(x-1)$ are equal to zero the entire denominator will be zero. Thus, there are **two** non-permissible values.

$$x \neq 0 \text{ and } x \neq 1$$

c) $\frac{x^2-5}{x^2-x-6}$

$$x^2 - x - 6 \neq 0$$

Factor using any method of choice to find the roots.



$$(x - 3)(x + 2) \neq 0$$

$$x \neq 3 \text{ and } x \neq -2$$

SIMPLIFYING

We often want to write rational expressions in their simplest form. To do this, we try to expand the numerator and denominator of the expression and see if we can cancel matching terms. In other words, we want to divide the top and bottom of the expression by their **greatest common factor (GCF)**.

Steps to simplifying:

1. Factor the expression. This helps us find our non-permissible values and makes simplifying much easier.
2. State all non-permissible values.
3. Determine the common factors of the numerator and denominator.
4. Divide.

EXAMPLE 4: Simplifying

$$\text{Simplify } \frac{4x + 16}{8x^2 + 32x}$$

$$\text{Factor: } \frac{4x + 16}{2x(4x + 16)}$$

Non – permissible values: $2x \neq 0$ and $4x + 16 \neq 0$

$$\text{NPV: } x \neq 0 \text{ and } x \neq -4$$

$$\text{Divide: } \frac{4x + 16}{2x(4x + 16)}$$

$$\text{Final simplified expression: } \frac{1}{2x}, x \neq 0, -4$$

If we tried to find the non-permissible values **after** simplifying, we would only be able to identify $x \neq 0$; we would miss $x \neq -4$. This is why we always determine the NPV **before** simplifying.

Sometimes it can be difficult to determine the GCF. If you're stuck, you can break it down into more steps by identifying *any* common factor, dividing and repeating with another factor until



you cannot identify any other factors except one. At that point, you will have reached the same answer as using the GCF. For example, doing the same problem:

$$\text{Pull a 4 out of the top and bottom and divide: } \frac{4(x+4)}{4(2x^2+8x)} = \frac{(x+4)}{(2x^2+8x)}$$

$$\text{Factor the bottom to see the next common factor: } \frac{(x+4)}{(x+4)2x}$$

$$\text{Divide to get your final answer: } \frac{1}{2x}, x \neq 0, -4$$

EXAMPLE 5: Simplifying

$$\text{Simplify } \frac{15x^2 + 3x}{9x^2 + 6x}$$

$$\text{Factor: } \frac{3x(5x + 1)}{3x(3x + 2)}$$

Non – permissible values: $3x \neq 0$ and $3x + 2 \neq 0$

$$x \neq 0 \text{ and } x \neq -\frac{2}{3}$$

$$\text{Divide: } \frac{3x(5x + 1)}{3x(3x + 2)}$$

$$\text{Final simplified expression: } \frac{5x + 1}{3x + 2}, x \neq 0, -\frac{2}{3}$$

MULTIPLYING RATIONAL EXPRESSIONS

Multiplying rational expressions follows the same rules as multiplying fractions.

To multiply two fractions the numerators get multiplied together and the denominators get multiplied together. Next, the expression is simplified.

EXAMPLE 6: Multiplying Fractions

$$\frac{5}{7} \times \frac{2}{5} = \frac{5 \times 2}{7 \times 5} = \frac{10}{35}$$

Simplify by determining the GCF and dividing:

$$\text{GCF} = 5$$

$$\frac{10 \div 5}{35 \div 5} = \frac{2}{7}$$



When multiplying rational expressions there is an additional step: Identify the **non-permissible values**. To make this easier, we **factor** both expressions first if possible.

Multiplying Rational Expressions - Steps:

- 1) Factor
- 2) Determine NPV
- 3) Multiply
- 4) Simplify

EXAMPLE 7: Multiplying Rational Expressions

$$\frac{3x}{2x-1} \times \frac{x+4}{x}$$

Each term is already factored.

Non – permissible values: $x \neq \frac{1}{2}, 0$

$$\text{Multiply: } \frac{3x \times (x+4)}{(2x-1) \times x} = \frac{3x(x+4)}{x(2x-1)}$$

$$\text{Simplify: } \frac{3x(x+4)}{x(2x-1)}$$

$$= \frac{3x+12}{2x-1}, x \neq \frac{1}{2}, 0$$

EXAMPLE 8: Multiplying Rational Expressions

$$\frac{x^2-4}{3x} \times \frac{5}{x^2-x-2}$$

$$\text{Factor: } \frac{(x+2)(x-2)}{3x} \times \frac{5}{(x-2)(x+1)}$$

Non – permissible values: $x \neq 0, 2, -1$

$$\text{Multiply: } \frac{(x-2)(x+2)5}{3x(x-2)(x+1)}$$

Since $\frac{(x-2)}{(x-2)} = 1$ the expression simplifies to:



$$= \frac{(x+2)5}{3x(x+1)}$$

$$= \frac{5x+10}{3x^2+3x}, x \neq 0, 2, -1$$

DIVIDING RATIONAL EXPRESSIONS

Dividing rational expressions is like dividing regular fractions, but with more steps. To divide a fraction, you take the fraction you are dividing by and flip the top and bottom numbers. Then, you multiply this with your first fraction.

EXAMPLE 9: Dividing Two Fractions

$$\frac{5}{7} \div \frac{2}{5}$$

Flip the 2nd term and multiply:

$$\frac{5}{7} \div \frac{2}{5} = \frac{5}{7} \times \frac{5}{2} = \frac{25}{14}$$

Since 25 and 14 do not have any common factors other than 1, this is the simplified form of this expression.

Now let's try dividing a rational expression.

Dividing Rational Expressions – Steps:

- 1) Factor
- 2) Determine NPV
- 3) Flip the 2nd fraction you are dividing by
- 4) Determine if flipping introduces new NPV
- 5) Multiply
- 6) Simplify

EXAMPLE 10: Dividing Rational Expressions

$$\frac{3x-3}{13} \div \frac{3x+6}{x+1}$$

$$\text{Factor: } \frac{3(x-1)}{13} \div \frac{3(x+2)}{x+1}$$

Non – permissible values: $x \neq -1$



$$\text{Flip: } \frac{3(x-1)}{13} \div \frac{3(x+2)}{x+1} = \frac{3(x-1)}{13} \times \frac{x+1}{3(x+2)}$$

Now $3(x+2)$ is on the bottom of a fraction.

It cannot equal 0, which gives us another NPV:

$$(x+2) \neq 0, \text{ so } x \neq -2$$

$$\text{Multiply and cancel matching terms on top \& bottom: } \frac{3(x-1)(x+1)}{13 \times 3(x+2)}$$

$$\text{Simplify: } \frac{(x-1)(x+1)}{13(x+2)}$$

$$= \frac{x^2 - 1}{13x + 26}, x \neq -1, -2$$

EXAMPLE 11: Dividing Rational Expressions

Assuming a is your variable, and b is a constant, divide:

$$\frac{a^2 - ab + 2a - 2b}{a + 2} \div \frac{(a + b)}{(a^2 - b^2)}$$

$$\text{Factor: } \frac{(a+2)(a-b)}{(a+2)} \div \frac{(a+b)}{(a+b)(a-b)}$$

Determine non-permissible values of 'a':

$$(a + 2) \neq 0, \text{ therefore } a \neq -2$$

$$(a + b) \neq 0 \text{ and } (a - b) \neq 0, \text{ therefore } a \neq \pm b$$

$$\text{Flip: } \frac{(a+2)(a-b)}{(a+2)} \times \frac{(a+b)(a-b)}{(a+b)}$$

Did flipping introduce new NPV? No, because the term $(a + b)$ was previously already on the bottom of a fraction and used to determine an NPV of 'a'. We can move to the next step.

$$\text{Multiply \& cancel matching terms: } \frac{(a+2)(a-b)}{(a+2)} \times \frac{(a+b)(a-b)}{(a+b)}$$

$$= (a - b)^2$$

$$= a^2 - 2ab + b^2, a \neq -2, b, -b$$

ADDITION AND SUBTRACTION OF RATIONAL EXPRESSIONS

For adding and subtracting rational expressions, we go back to our trusty rules for fractions. Unlike multiplying and dividing, where we can use the expressions as we see them, to add or subtract fractions we first need to make sure they have a **common denominator**.



EXAMPLE 12: Adding Fractions

$$\frac{5}{7} + \frac{2}{5}$$

We need to make the bottom of both these fractions the same so we can add them. We want to use the lowest common multiple (LCM) of 7 and 5.

The multiples of 7 are: 7, 14, 21, 28, **35**, 42 ... etc.

The multiples of 5 are: 5, 10, 15, 20, 25, 30, **35**, 40 ... etc.

The lowest common multiple is **35**. We can now multiply each of the fractions by a fraction equal to 1 in order to change the bottom.

$$\frac{5}{7} \times \frac{5}{5} + \frac{2}{5} \times \frac{7}{7} = \frac{25}{35} + \frac{14}{35}$$

Now the denominators match and we can finish adding:

$$= \frac{25 + 14}{35} = \frac{39}{35}$$

The same idea applies for rational expressions! Remember that equivalent expressions have the **same value** but look different. We will use them to rewrite our expressions with the same denominators, without changing their value.

Adding or Subtracting Rational Expressions – Steps:

- 1) Factor all denominators
- 2) Determine the NPV
- 3) Use the lowest common multiple (LCM) of the denominators to rewrite the expressions as equivalent expressions with the same denominator (fraction bottoms)
- 4) Combine the expressions to add or subtract the numerators (fraction tops)
- 5) Simplify

EXAMPLE 13: Subtracting Rational Expressions

$$\frac{4x + 2}{x + 1} - \frac{2x - 1}{x + 1}$$

Factor: Both denominators are already factored; go to the next step.

NPV: $x + 1 \neq 0$, therefore $x \neq -1$



LCM of denominators: Both expressions already have the same denominator; go to the next step.

$$\text{Combine: } \frac{(4x+2)-(2x-1)}{x+1} = \frac{4x+2-2x-(-1)}{x+1} = \frac{2x+3}{x+1}$$

Expression cannot be further simplified. Our solution is:

$$\frac{2x+3}{x+1}, x \neq -1$$

EXAMPLE 14: Adding Rational Expressions

$$\frac{x+1}{x^2-4} + \frac{2}{x^2+2x}$$

$$\text{Factor: } \frac{x+1}{(x-2)(x+2)} + \frac{2}{x(x+2)}$$

NPV: $(x+2)$, x and $(x-2) \neq 0$, therefore $x \neq -2, 0$ and 2

LCM: $x(x-2)(x+2)$, applying to expressions to make bottoms match:

$$\frac{x+1}{(x-2)(x+2)} \times \frac{x}{x} + \frac{2}{x(x+2)} \times \frac{(x-2)}{(x-2)}$$

$$\frac{x^2+x}{x(x-2)(x+2)} + \frac{2x-4}{x(x-2)(x+2)}$$

$$\text{Combine: } \frac{(x^2+x)+(2x-4)}{x(x-2)(x+2)} = \frac{x^2+x+2x-4}{x(x-2)(x+2)} = \frac{x^2+3x-4}{x(x-2)(x+2)}$$

Simplify: By factoring the top of our expression we can see that no terms may be canceled from both the numerator and denominator.

$$\frac{(x+4)(x-1)}{x(x-2)(x+2)}$$

Therefore our solution may be written as:

$$\frac{x^2+3x-4}{x(x-2)(x+2)}, x \neq -2, 0, 2$$

SOLVING RATIONAL EXPRESSIONS

If you multiply BOTH sides of an equation by the same number then we haven't changed the fact that each side is equal to the other. We use that fact to get rid of ugly fractions in our equations and make it easier to solve for variables.

$$\text{For example: } \frac{2x}{3} = 4$$

Has the solution: $x = 6$



If you multiply **BOTH** sides of the equation by 3 to get rid of the fraction:

$$\frac{2x}{3} \times 3 = 4 \times 3$$

$$2x = 12$$

The solution is easier to find, $x = 6$.

Since we are messing around with the bottoms of our expressions we have to remember to find our NPV **before** we start making things look pretty.

Solving Rational Expressions – Steps:

- 1) Factor expressions if needed.
- 2) Determine the NPV.
- 3) Multiply all terms by the lowest common denominator (LCD). This gets rid of fractions and makes the problem easier to solve.
- 4) Simplify.
- 5) Solve for your variable.
- 6) Check your answer.

EXAMPLE 15: Solving Rational Expressions

$$\frac{3x - 2}{x - 1} = \frac{2}{x + 1}$$

Factor: Both expressions cannot be further factored; go to next step.

$$\text{NPV: } x \neq 1, -1$$

Multiply both sides by LCD. $LCD = (x - 1)(x + 1)$

$$\frac{3x - 2}{x - 1} \times (x - 1)(x + 1) = \frac{2}{x + 1} \times (x - 1)(x + 1)$$

All denominators will be able to cancel.

$$(3x - 2)(x + 1) = 2(x - 1)$$

Simplify and solve.

$$3x^2 + x - 2 = 2x - 2$$

$$3x^2 - x = 0$$

$$x(3x - 1) = 0$$



$$x = 0 \text{ and } x = \frac{1}{3}$$

Check:

- Solutions are not NPV values. They are valid solutions.
- Substituting the solutions into the original expression yields equal values.

$$\frac{3(0) - 2}{(0) - 1} = \frac{2}{(0) + 1}$$

$$2 = 2$$

AND

$$\frac{3\left(\frac{1}{3}\right) - 2}{\left(\frac{1}{3}\right) - 1} = \frac{2}{\left(\frac{1}{3}\right) + 1}$$

$$\frac{-1}{-\left(\frac{2}{3}\right)} = \frac{2}{\left(\frac{4}{3}\right)}$$

$$\frac{3}{2} = \frac{3}{2}$$

Both solutions, $x = 0$ and $x = \frac{1}{3}$ are valid and correct.

EXAMPLE 16: Solving Rational Expressions

$$\frac{4}{x^2 - 4} = \frac{1}{x + 2}$$

$$\text{Factor: } \frac{4}{(x + 2)(x - 2)} = \frac{1}{x - 2}$$

$$\text{NPV: } x \neq 2, -2$$

Multiply both sides by LCD. $LCD = (x - 2)(x + 2)$

$$\frac{4}{(x + 2)(x - 2)} \times (x + 2)(x - 2) = \frac{1}{x - 2} \times (x + 2)(x - 2)$$

All denominators will be able to cancel.

$$4 = x + 2$$

Simplify and solve.

$$x - 2 = 0$$

$$x = 2$$



Check:

- Solution is an NPV. This is NOT a valid solution, as the equation becomes undefined when $x = 2$.
- $x = 2$ is called an **EXTRANEIOUS ROOT**

Thus, there are no valid solutions, only one extraneous root.

EXAMPLE 17: Solving Rational Expressions

$$\frac{7}{x-4} = \frac{3}{4x-16}$$

$$\text{Factor: } \frac{7}{x-4} = \frac{3}{4(x-4)}$$

$$\text{NPV: } x \neq 4$$

Multiply both sides by LCD. $LCD = 4(x-4)$

$$\frac{7}{x-4} \times 4(x-4) = \frac{3}{4(x-4)} \times 4(x-4)$$

All denominators cancel.

$$28 = 3$$

$$\text{BUT.... } 28 \neq 3 !!!$$

Therefore, there is NO SOLUTION.

WORD PROBLEMS

The last step of working with rational expressions is using them in real world (or math world...) word problems !!

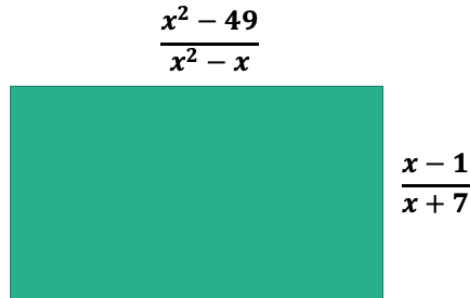
Here's some steps to think about when doing word problems:

- Summarize the information you are being told
- Think about how you would solve the problem in general *before* you start calculating.
- What steps will you need to take?
- After you solve it, think about if your answer makes sense in the real world.
- Check your answer.



EXAMPLE 18: Sample Word Problem

There is a rectangular spot on Karen's wall where a picture used to hang. She wants to paint over it. The dimensions of this rectangle are provided with rational expressions. [units: m]



- To order the right amount of paint, Karen needs to know the area of the rectangle. Determine an expression for the area.
- In addition to the NPV for the area expression, are there any other restrictions that should be put on the variable?
- If the area of the rectangle is $\frac{2}{9} \text{ m}^2$, what was the value of the variable?

Solution:

- The area of a rectangle is Length x Width. The length and width are each given to us as rational expressions; we can multiply these together to get an expression for the area.

$$\begin{aligned} \text{Area} &= \frac{x^2 - 49}{x^2 - x} \times \frac{x - 1}{x + 7} \\ \text{Area} &= \frac{(x - 7)(x + 7)}{x(x - 1)} \times \frac{x - 1}{x + 7}, \text{NPV: } x \neq 0, 1, -7 \\ \text{Area} &= \frac{(x - 7)(x + 7)(x - 1)}{x(x - 1)(x + 7)} \\ \text{Area} &= \frac{x - 7}{x}, x \neq 0, 1, -7 \end{aligned}$$

- If you look at the expression for the area, you can see that if $x = 7$, the area will be zero. That means the rectangle wouldn't exist. If $x < 7$ there would be a negative area. A negative area isn't possible in the real world. So we should only consider x :

$$x \geq 7$$



Also, since all the NPV we found are less than 7, stating $x \geq 7$ will ensure all solutions are valid.

- c) We have an expression for the area (from part A) and a solution for the area. We can set them equal to each other and solve for the value of x that occurs at that area.

$$\frac{x-7}{x} = \frac{2}{9} \text{ where } x \geq 7$$

$$\frac{x-7}{x} \times 9x = \frac{2}{9} \times 9x$$

$$9(x-7) = 2x$$

$$9x - 63 = 2x$$

$$7x = 63$$

$$x = 9$$

