Past Problems from PoTW

AperioMAT

Week 13

Level 1

Let a, b, c, d, and e be positive integers such that

$$abcde = a + b + c + d + e.$$

Find the maximum possible value of $\max\{a, b, c, d, e\}$.

Answer: 5
Solution:
Suppose that $a \le b \le c \le d \le e$. We need to find the maximum value of e. Since

$$e < a + b + c + d + e \le 5e,$$

then $e < abcde \le 5e$, i.e. $1 < abcd \le 5$. Hence $(a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 1, 4), (1, 1, 2, 2)$, or $(1, 1, 1, 5)$, which leads to $\max\{e\} = 5$.

Level 2

We start with the three numbers $0, 1, \sqrt{2}$, to which the following operation is repeatedly applied: one of the numbers is chosen and an arbitrary rational multiple of the difference of the two others is added. Is it possible to obtain the triple $0, \sqrt{2}-1, \sqrt{2}+1$ after a number of applications of this operation?

Answer: No
Solution:

Proof. Since $\sqrt{2}$ is irrational, all numbers that are obtained during this process must have the form $a + b\sqrt{2}$ for some rational a and b. Such a number can be represented by the point (a, b) in the plane. We consider the triangle that is formed by the three numbers. In the beginning, its vertices are $(0, 0), (1, 0)$ and $(0, 1)$. The described operation amounts to a translation of one of the points along a line parallel to the opposite side of the triangle. This operation does not change the area of the triangle, so the area remains constant. In the beginning, the area is $\frac{1}{2}$. The triangle that is formed by the three points $(0, 0), (-1, 1)$ and $(1, 1)$, however, has area 1 , so it is impossible to reach the triple $0, \sqrt{2} - 1, \sqrt{2} + 1$. $\square$

Week 8

Level 1

The tetrahedron $ABCD$ is divided into five convex polyhedra so that each face of $ABCD$ is a face of one of the polyhedra (no faces are divided), and the intersection of any two of the five polyhedra is either a common vertex, a common edge, or a common face. What is the smallest possible sum of the number of faces of the five polyhedra?

Answer: 22

Solution: No polyhedron shares two faces with $ABCD$; otherwise, its convexity would imply that it is $ABCD$. Then exactly one polyhedron P must not share a face with $ABCD$, and has its faces in $ABCD$'s interior. Each of P 's faces must then be shared with another polyhedron, implying that P shares at least 3 vertices with each of the other polyhedra. Also, any polyhedron face not shared with $ABCD$ must be shared with another polyhedron. This implies that the sum of the number of faces is even. Each polyhedron must have at least four faces for a sum of at least 20. Assume this is the sum. Then each polyhedron is a four-vertex tetrahedron, and P shares at most 2 vertices with $ABCD$. Even if it did share 2 vertices with $ABCD$, say A and B, it would then share at most 2 vertices with the tetrahedron containing ACD, a contradiction. Therefore, the sum of the faces must be at least 22. This sum can indeed be obtained. Let P and Q be very close to A and B, respectively; then the five polyhedra $APCD, PQCD, BQCD, ABDPQ$, and $ABCPQ$ satisfy the requirements.

Level 2

Express the sum

$$\sum_{k=0}^{n} \frac{(-1)^k}{k^3+9k^2+26k+24} \binom{n}{k}$$

in the form $p(n)/q(n)$, where p, q are polynomials with integer coefficients.

Answer: $\frac{1}{2(n+3)(n+4)}$

Solution: We have

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(-1)^k}{k^3+9k^2+26k+24} \binom{n}{k} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{(k+2)(k+3)(k+4)} \binom{n}{k} \\
&= \sum_{k=0}^{n} (-1)^k \frac{k+1}{(n+1)(n+2)(n+3)(n+4)} \binom{n+4}{k+4} \\
&= \frac{1}{(n+1)(n+2)(n+3)(n+4)} \sum_{k=4}^{n+4} (-1)^k (k-3) \binom{n+4}{k}
\end{aligned}$$

and

$$\sum_{k=0}^{n+4}(-1)^k(k-3)\binom{n+4}{k}$$
$$=\sum_{k=0}^{n+4}(-1)^k k\binom{n+4}{k}-3\sum_{k=0}^{n+4}(-1)^k\binom{n+4}{k}$$
$$=\sum_{k=1}^{n+4}(-1)^k k\binom{n+4}{k}-3(1-1)^{n+4}$$
$$=\frac{1}{n+4}\sum_{k=1}^{n+4}(-1)^k\binom{n+3}{k-1}$$
$$=\frac{1}{n+4}(1-1)^{n+3}=0.$$

Therefore

$$\sum_{k=4}^{n+4}(-1)^k(k-3)\binom{n+4}{k}$$
$$=-\sum_{k=0}^{3}(-1)^k(k-3)\binom{n+4}{k}$$
$$=3\binom{n+4}{0}-2\binom{n+4}{1}+\binom{n+4}{2}=\frac{(n+1)(n+2)}{2}$$

and the given sum equals $\frac{1}{2(n+3)(n+4)}$.

Week 7

Level 1

How many pairs (x, y) of positive integers with $x \leq y$ satisfy $\gcd(x, y) = 5!$ and $\operatorname{lcm}(x, y) = 50!$?

Solution: First, note that there are 15 primes from 1 to 50 :

$$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47).$$

To make this easier, let's define $f(a, b)$ to be the greatest power of b dividing a. (Note $g(50!, b) > g(5!, b)$ for all $b < 50$.)

Therefore, for each prime p, we have either $f(x, p) = f(5!, p)$ and $f(y, p) = f(50!, p)$ OR $f(y, p) = f(5!, p)$ and $f(x, p) = f(50!, p)$. Since we have 15 primes, this gives 2^{15} pairs, and clearly $x \neq y$ in any such pair (since the gcd and lcm are different), so there are 2^{14} pairs with $x \leq y$.

Level 2

For any real number b, let $f(b)$ denote the maximum of the function

$$\left|\sin x + \frac{2}{3+\sin x} + b\right|$$

over all $x \in \mathbb{R}$. Find the minimum of $f(b)$ over all $b \in \mathbb{R}$.

Solution: The minimum value is $3/4$. Let $y = 3 + \sin x$; note $y \in [2,4]$ and assumes all values therein. Also let $g(y) = y + 2/y$; this function is increasing on $[2,4]$, so $g(2) \le g(y) \le g(4)$. Thus $3 \le g(y) \le 9/2$, and both extreme values are attained. It now follows that the minimum of $f(b) = \max(|g(y) + b - 3|)$ is $3/4$, which is attained by $b = -3/4$; for if $b > -3/4$ then choose $x = \pi/2$ so $y = 4$ and then $g(y) + b - 3 > 3/4$, while if $b < -3/4$ then choose $x = -\pi/2$ so $y = 2$ and $g(y) + b - 3 = -3/4$; on the other hand, our range for $g(y)$ guarantees $-3/4 \le g(y) + b - 3 \le 3/4$ for $b = -3/4$.

Week 6

Level 1

Let n be a positive integer. Call a nonempty subset S of $\{1, 2, \ldots, n\}$ 'good' if the arithmetic mean of the elements of S is also an integer. Further let T_n denote the number of 'good' subsets of $\{1, 2, \ldots, n\}$. Prove that T_n and n are either both odd or both even.

Source: INMO 2013; Problem 4.

Solution: The official solution can be found at this link. Here's my solution:

Let us look for a recursion for general term T_{n+1}. Here, the sets $\{1, 2, \ldots, n\}$ and $\{2, 3, \ldots, n+1\}$ are denoted by X_1 and X_2, respectively.

By definition, there are T_n 'good' subsets $S \in X_1$. Observe that there are also T_n 'good' subsets of X_2, as every such subset can be formed by adding 1 to all elements in the corresponding subset of $\{1, 2, \ldots, n\}$.

If $n+1$ is even, all 'good' subsets of T_{n+1} are either in X_1, X_2, or both. The overlap contains 'good' subsets of $\{2, 3, \ldots, n\} = \{1+1, 2+1, \ldots, (n-1)+1\}$. Hence, we subtract the overcount of T_{n-1} elements. This gives $T_{n+1} = 2T_n - T_{n-1}$. Thus, T_{n+1} and T_{n-1} have the same parity (i.e. for all even n, $T_n \equiv T_2 (= 2) \equiv 0 \pmod 2$).

If $n+1$ is odd, the full set $\{1, 2, \ldots, n+1\}$ must also be counted in T_{n+1} (so we add 1). This means that T_n for odd n has opposite parity to T_n for even n, and we are done.

Level 2

If $n \geq 2$ is an integer and $0 < a_1 < a_2 \cdots < a_{2n+1}$ are real numbers, prove the inequality:

$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}.$$

Source: 1998 Balkan MO; Problem 2.

Solution: Again, we use induction. This may seem unusual, but we will treat the exponents and subscripts as two separate variables, and only perform induction on the latter. Let the exponent remain denoted by n, but call the subscript variable m.

We do this because $m = 1$ is now valid, which is much simpler to evaluate than $m = 2$. In this case, the inequality becomes $\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} < \sqrt[n]{a_1 - a_2 + a_3}$. By definition, both LHS and RHS are positive. Thus, we can take the n^{th} power of both sides and rewrite as an equation:

$$(\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3})^n < a_1 - a_2 + a_3$$

$$(a_1 - a_2 + a_3) - (\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3})^n > 0$$

This may also seem strange, but taking the derivative of both sides with respect to a_1 gives $1 - ((\sqrt[n]{a_1})/(\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3}))^{n-1} > 0$. It is given that $a_3 > a_2$, so the denominator of the expression is greater than the numerator (i.e., this becomes $1 - (\text{some } x < 1)^{n-1} > 0$). For $n \geq 2$, this is evidently true.

If instead $m = 2$, we add the terms a_4 and a_5. Given $a_5 - a_4 > 0$, following the same procedure and taking the derivative gives the same result. Similarly, for every subsequent m, we add two terms with difference $a_{2m+1} - a_{2m} > 0$. Thus, the statement holds. (Note that this is only a rough sketch of the induction step and not enough for a formal proof.)

Week 5

Level 1

For each natural number $n \geq 2$, determine the largest possible value of the expression

$$V_n = \sin x_1 \cos x_2 + \sin x_2 \cos x_3 + \cdots + \sin x_n \cos x_1,$$

where $x_1, x_2, \ldots, x_n$ are arbitrary real numbers.

Solution:
By the inequality $2ab \leq a^2 + b^2$, we get

$$V_n \leq \frac{\sin^2 x_1 + \cos^2 x_2}{2} + \cdots + \frac{\sin^2 x_n + \cos^2 x_1}{2} = \frac{n}{2},$$

with equality for $x_1 = \cdots = x_n = \pi/4$.

Level 2

For any natural number $n \geq 3$, let $m(n)$ denote the maximum number of points lying within or on the boundary of a regular n-gon of side length 1 such that the distance between any two of the points is greater than 1. Find all n such that $m(n) = n - 1$.

Answer: $n = 4, 5, 6$

Solution:

We can easily show that $m(3) = 1$, e.g. dissect an equilateral triangle ABC into 4 congruent triangles and then for two points P, Q there is some corner triangle inside which neither lies; if we assume this corner is at A then the circle with diameter BC contains the other three small triangles and so contains P and Q; $BC = 1$ so $PQ \leq 1$. This method will be useful later; call it a lemma.

On the other hand, $m(n) \geq n - 1$ for $n \geq 4$ as the following process indicates. Let the vertices of our n-gon be $A_1, A_2, \ldots, A_n$. Take $P_1 = A_1$. Take P_2 on the segment A_2A_3 at an extremely small distance d_2 from A_2; then $P_2P_1 > 1$, as can be shown rigorously, e.g. using the Law of Cosines in triangle $P_1A_2P_2$ and the fact that the cosine of the angle at A_2 is nonnegative (since $n \geq 4$). Moreover P_2 is on a side of the n-gon other than A_3A_4, and it is easy to see that as long as $n \geq 4$, the circle of radius 1 centered at A_4 intersects no side of the n-gon not terminating at A_4, so $P_2A_4 > 1$ while clearly $P_2A_3 < 1$. So by continuity there is a point P_3 on the side A_3A_4 with $P_2P_3 = 1$. Now slide P_3 by a small distance d_3 on A_3A_4 towards A_4; another trigonometric argument can easily show that then $P_2P_3 > 1$. Continuing in this manner, obtain P_4 on A_4A_5 with $P_3P_4 = 1$ and slide P_4 by distance d_4 so that now $P_3P_4 > 1$, etc. Continue doing this until all points P_i have been defined; distances P_iP_{i+1} are now greater than by construction, $P_{n-1}P_1 > 1$ because $P_1 = A_1$ while P_{n-1} is in the interior of the side $A_{n-1}A_n$; and all other P_iP_j are greater than 1 because it is easy to see that the distance between any two points of nonadjacent sides of the n-gon is at least 1 with equality possible only when (among other conditions) P_i, P_j are endpoints of their respective sides, and in our construction this never occurs for distinct i, j. So our construction succeeds. Moreover, as all the distances d_i tend to 0 each P_i tends toward A_i, so it follows that the maximum of the distances A_iP_i can be made as small as desired by choosing d_i sufficiently small. On the other hand, when $n > 6$ the center O of the n-gon is at a distance greater than 1 from each vertex, so if the P_i are sufficiently close to the A_i then we will also have $OP_i > 1$ for each i. Thus we can add the point O to our set, showing that $m(n) \geq n$ for $n > 6$.

It now remains to show that we cannot have more than $n - 1$ points at mutual distances greater than 1 for $n = 4, 5, 6$. As before let the vertices of the polygon be A_1, etc. and the center O; suppose we have n points $P_1, \ldots, P_n$ with $P_iP_j > 1$ for i not equal to j. Since $n \leq 6$ it follows that the circumradius of the polygon is not greater than 1 , so certainly no P_i can be equal to O. Let the ray from O through P_i intersect the polygon at Q_i and assume WLOG our numbering is such that $Q_1, Q_2, \ldots, Q_n$ occur in that order around the polygon, in the same orientation as the vertices were numbered. Let Q_1 be on the side A_kA_{k+1}.

A rotation by angle $2\pi/n$ brings A_k into A_{k+1}; let it also bring Q_1 into Q_1', so triangles $Q_1Q_1'O$ and $A_kA_{k+1}O$ are similar. We claim P_2 cannot lie inside or on the boundary of quadrilateral $OQ_1A_{k+1}Q_1'$. To see this, note that $P_1Q_1A_{k+1}$ and $P_1A_{k+1}Q_1'$ are triangles with an acute angle at P_1, so the maximum distance from P_1 to any point on or inside either of these triangles is attained when that point is some vertex; however $P_1Q_1 \leq OQ_1 \leq 1$, and $P_1A_{k+1} \leq O_1A_{k+1} \leq 1$ (e.g. by a trigonometric argument similar to that mentioned earlier), and as for P_1Q_1', it is subsumed in the following case: we can show that $P_1P \leq 1$ for any P on or inside OQ_1Q_1', because $n \leq 6$ implies that $\angle Q_1OQ_1' = 2\pi/n \geq \pi/3$, and so we can erect an equilateral triangle on Q_1Q_1' which contains O, and the side of this triangle is less than $A_kA_{k+1} = 1$ (by similar triangles OA_kA_{k+1} and OQ_1Q_1') so we can apply the lemma now to show that two points inside this triangle are at a distance at most 1 . The result of all this is that P_2 is not inside the quadrilateral $OQ_1A_{k+1}Q_1'$, so that $\angle P_1OP_2 = \angle Q_1OP_2 > 2\pi/n$. On the other hand, the label P_1 is not germane to this argument; we can show in the same way that $\angle P_iOP_{i+1} > 2\pi/n$ for any i (where $P_{n+1} = P_1$). But then adding these n inequalities gives $2\pi > 2\pi$, a contradiction, so our points P_i cannot all exist. Thus $m(n) \leq n-1$ for $n = 4, 5, 6$, completing the proof.

Week 4

Level 1

Solve the equation

$$e^{x^{1/e}} \cdot x^{(1/e)^x} = 1$$

in positive x.

Answer: $\frac{1}{e}$

Solution:
Notice that $x = \frac{1}{e}$ satisfies the equation:

$$\begin{aligned} e^{\left(\frac{1}{e}\right)^{\frac{1}{e}}} \cdot \left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)^{\frac{1}{e}}} &= e^{\left(\frac{1}{e}\right)^{\frac{1}{e}}} \cdot e^{-\left(\frac{1}{e}\right)^{\frac{1}{e}}} \\ &= e^{\left(\frac{1}{e}\right)^{\frac{1}{e}} - \left(\frac{1}{e}\right)^{\frac{1}{e}}} \\ &= e^0 \\ &= 1 \end{aligned}$$

Let us show that it is the only solution.

Let us take the natural logarithm of both sides of the equation and rewrite it as

$$\ln\left(e^{\left(x^{\frac{1}{e}}\right)}\cdot x^{\left(\frac{1}{e}\right)^x}\right)=\ln(1)$$
$$\ln\left(e^{\left(x^{\frac{1}{e}}\right)}\right)+\ln\left(x^{\left(\frac{1}{e}\right)^x}\right)=0$$
$$x^{\frac{1}{e}}+\left(\frac{1}{e}\right)^x\ln x=0$$
$$x^{\frac{1}{e}}e^x+\ln x=0$$

Let us now consider the function

$$f(x)=x^{\frac{1}{e}}e^x+\ln x$$

Its first derivative is

$$f'(x)=\left(x^{\frac{1}{e}}e^x+\ln x\right)'=\frac{e^x}{x^{\frac{e-1}{e}}}+x^{\frac{1}{e}}e^x+\frac{1}{x}$$

Notice that $f'(x)>0$ for all positive x. This implies that the left-hand side of the last equation represents a strictly increasing function, which takes each of its values exactly once. Therefore, it takes the value 0 only at $x=\frac{1}{e}$, which implies that $x=\frac{1}{e}$ is the only solution of the initial equation.

Level 2

Let n be the number of all functions $f:\{1,2,...,12\}\to\{1,2,...,12\}$,such that for any positive integers i and j, where $1\le i<j\le 12$ it holds true $\min\{f(i),f(i+1),...,f(j)\}=\min(f(1),f(j))$ and $f(i)\ne f(j)$. Find $\frac{n}{4}$.

Answer: 512

Solution:
Let $f(j)=12$, then let us prove that

$$f(1)<f(2)<\ldots<f(j-1)$$

and

$$f(j+1)>f(j+2)>\ldots>f(12).$$

Let us prove that if $j>1$ and $f(j)=12$, then

$$f(j-1)>\ldots>f(1)$$

We proceed the proof by contradiction argument and let i (where $i<j$) be the greatest number, such that

$$f(i+1)<f(i)$$

Thus, it follows that

$$f(i)>f(i+1)<f(i+2)<\ldots<f(j)$$

Hence, we obtain that

$$\min\{f(i), f(i+1), \ldots, f(j)\} = f(i+1)$$

This leads to a contradiction. In a similar way, one can prove that

$$f(j+1) > f(j+2) > \ldots > f(12)$$

Note that the number of functions simultaneously satisfying the assumptions of the problem and the condition $f(j) = 12$ is equal to $\left(\begin{array}{c} 11 \\ 12-j \end{array} \right)$. Therefore, we obtain that the number of functions satisfying the assumptions of the problem is:

$$n = \sum_{j=1}^{12} \left(\begin{array}{c} 11 \\ 12-j \end{array} \right) = 2^{11}$$

Thus, it follows that

$$\frac{n}{4} = \frac{2^{11}}{4} = 512$$

Week 3

Level 1

Find all positive integers n such that $3_{n-1} + 5_{n-1}$ divides $3_n + 5_n$.

Answer: 1

Source: St. Petersburg 1996

Solution:

This only occurs for $n = 1$. Let $s_n = 3^n + 5^n$ and note that

$$s_n = (3+5)s_{n-1} - 3 \cdot 5 \cdot s_{n-2}$$

So s_{n-1} must also divide $3 \cdot 5 \cdot s_{n-2}$.

If $n > 1$, then s_{n-1} is coprime to 3 and 5, then s_{n-1} must divide s_{n-2}, which is impossible since $s_{n-1} > s_{n-2}$.

Level 2

Let m, n and p are real numbers such that

$$(m+n+p)\left(\tfrac{1}{m} + \tfrac{1}{n} + \tfrac{1}{p}\right) = 1$$

Find all possible values of

$$\frac{1}{(m+n+p)^{2023}} - \frac{1}{m^{2023}} - \frac{1}{n^{2023}} - \frac{1}{p^{2023}}.$$

Answer: 0
Source: BdMO 2023 Secondary National

Solution:
Claim: $(m+n+p)\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{p}\right)=1 \Rightarrow (m+n)(n+p)(p+m)=0.$

Proof: $(m+n+p)\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{p}\right) = 3+\frac{m}{n}+\frac{m}{p}+\frac{n}{m}+\frac{n}{p}+\frac{p}{m}+\frac{p}{n} = 1$, note that this implies, $(m+n+p)(mn+mp+np)-mnp=(m+n)(m+p)(n+p)=0.$

Now, without loss of generality, assume $m+p=0 \Rightarrow m=-p$
Hence,
$\frac{1}{(m+n+p)^{2023}}-\frac{1}{m^{2023}}-\frac{1}{n^{2023}}-\frac{1}{p^{2023}}$
$=\frac{1}{(m+n+p)^{2023}}-\frac{1}{p^{2023}}=\frac{1}{p^{2023}}-\frac{1}{p^{2023}}=0.$ **(Answer)**

1 Week 2

Level 1

For each positive integer n, let $f_1(n)$ be 2 times the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n)=f_1(f_{j-1}(n))$. If $n \leq 49$ and $f_{49}(n)=12$, find the difference between the largest and smallest values of n.

Answer: $\frac{1}{2}$
Source: AMC

Solution:

Level 2

Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI=6$, $CI=5$, $DI=3$, determine the value of $(\frac{DP}{DQ})^2$.

Answer: $\frac{1}{2}$
Source: HMMT

Solution:

Week 1

Level 1

Amy tosses a fair coin 2024 times, and Bill tosses a fair coin 2023 times. What is the probability that Amy gets more heads than Bill does?

Answer: $\frac{1}{2}$
Source: Mandelbrot

Solution:
For simplicity, assume a scenario in which Amy tosses her coin once and Bill does not flip his at all. Amy will have more heads only if her one coin lands on heads ($P = \frac{1}{2}$).

Now, let's say that Amy and Bill toss their coins n more times each. By symmetry, the probability that Amy flips more heads than Bill in these n tosses is $\frac{1}{2}$.

Therefore, the probability ($P = \frac{1}{2}$) of Amy flipping more heads is not dependent on the value of n (only the difference in the number of tosses).

Level 2

Solve the following system of three equations for the unknown x, y, and z (a, b, c are given):

$$x^2y^2 + x^2z^2 = axyz$$
$$y^2z^2 + y^2x^2 = bxyz$$
$$z^2x^2 + z^2y^2 = cxyz$$

Source: Stanford Mathematics Competition (1946-1965)

Solution:
If $x = 0$, the second, or third, equation yields $y^2 * z^2 = 0$, so one more unknown must be zero. If two variables are equal to zero, the equations are satisfied. So, let's consider the case where no one of the three unknowns is zero. By diving xyz, we obtain the following system -

$$\frac{zx}{y} + \frac{xy}{z} = a$$
$$\frac{yz}{x} + \frac{xy}{z} = b$$
$$\frac{yz}{x} + \frac{xz}{y} = c$$

Adding these three equations and diving by 2, we get

$$\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} = \frac{a+b+c}{2}$$

From this equation, we subtract each of the three equations of the foregoing system and obtain

$$\frac{yz}{x} = \frac{-a+b+c}{2}$$
$$\frac{zx}{y} = \frac{a-b+c}{2}$$
$$\frac{xy}{z} = \frac{a+b-c}{2}$$

The product of these

$$xyz = \frac{(-a+b+c)(a-b+c)(a+b-c}{8}$$

Diving this by each equation of the foregoing system to get (after extracting a square root) -

$$x = \frac{(a-b+c)(a+b-c)^{\frac{1}{2}}}{2}$$
$$y = \frac{(-a+b+c)(a+b-c)^{\frac{1}{2}}}{2}$$
$$z = \frac{(-a+b+c)(a-b+c)^{\frac{1}{2}}}{2}$$

[Answer]