

Real Numbers

- **Euclid's Division Lemma**

For any given positive integers a and b , there exists unique integers q and r such that

$$a = bq + r \text{ where } 0 \leq r < b$$

Note: If b divides a , then $r = 0$

Example 1:

For $a = 15$, $b = 3$, it can be observed that

$$15 = 3 \times 5 + 0$$

Here, $q = 5$ and $r = 0$

If b divides a , then $0 < r < b$

Example 2:

For $a = 20$, $b = 6$, it can be observed that $20 = 6 \times 3 + 2$

Here, $q = 3$, $r = 2$, $0 < 2 < 6$

- **Euclid's division algorithm**

Euclid's division algorithm is a series of well-defined steps based on "Euclid's division lemma", to give a procedure for calculating problems.

Steps for finding HCF of two positive integers a and b ($a > b$) by using Euclid's division algorithm:

Step 1: Applying Euclid's division lemma to a and b to find whole numbers q and r , such that $a = bq + r$, $0 \leq r < b$

Step 2: If $r = 0$, then $\text{HCF}(a, b) = b$

If $r \neq 0$, then again apply division lemma to b and r

Step 3: Continue the same procedure till the remainder is 0. The divisor at this stage will be the HCF of a and b .

Note: $\text{HCF}(a, b) = \text{HCF}(b, r)$

Example:

Find the HCF of 48 and 88.

Solution:

Take $a = 88$, $b = 48$

Applying Euclid's division lemma, we get

$$88 = 48 \times 1 + 40 \quad (\text{Here, } 0 \leq 40 < 48)$$

$$48 = 40 \times 1 + 8 \quad (\text{Here, } 0 \leq 8 < 40)$$

$$40 = 8 \times 5 + 0 \quad (\text{Here, } r = 0)$$

$$\text{HCF}(48, 88) = 8$$

- For any positive integer a, b , $\text{HCF}(a, b) \times \text{LCM}(a, b) = a \times b$

Example 1:

Find the LCM of 315 and 360 by the prime factorisation method. Hence, find their HCF.

Solution:

$$315 = 3 \times 3 \times 5 \times 7 = 3^2 \times 5 \times 7$$

$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$$

$$\text{LCM} = 3^2 \times 5 \times 7 \times 2^3 = 2520$$

$$\therefore \text{HCF}(315, 360) = \frac{315 \times 360}{\text{LCM}(315, 360)} = \frac{315 \times 360}{2520} = 45$$

Example 2:

Find the HCF of 300, 360 and 240 by the prime factorisation method.

Solution:

$$300 = 2^2 \times 3 \times 5^2$$

$$360 = 2^3 \times 3^2 \times 5$$

$$240 = 2^4 \times 3 \times 5$$

$$\text{HCF}(300, 360, 240) = 2^2 \times 3 \times 5 = 60$$

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• **Using Euclid's division lemma to prove mathematical relationships**

Result 1:

Every positive even integer is of the form $2q$, while every positive odd integer is of the form $2q + 1$, where q is some integer.

Proof:

Let a be any given positive integer.

Take $b = 2$

By applying Euclid's division lemma, we have

$$a = 2q + r \text{ where } 0 \leq r < 2$$

As $0 \leq r < 2$, either $r = 0$ or $r = 1$

If $r = 0$, then $a = 2q$, which tells us that a is an even integer.

If $r = 1$, then $a = 2q + 1$

It is known that every positive integer is either even or odd.

Therefore, a positive odd integer is of the form $2q + 1$.

Result 2:

Any positive integer is of the form $3q$, $3q + 1$ or $3q + 2$, where q is an integer.

Proof:

Let a be any positive integer.

Take $b = 3$

Applying Euclid's division lemma, we have

$a = 3q + r$, where $0 \leq r < 3$ and q is an integer

Now, $0 \leq r < 3 \Rightarrow r = 0, 1, \text{ or } 2$

$\therefore a = 3q + r$

$\Rightarrow a = 3q + 0, a = 3q + 1, a = 3q + 2$

Thus, $a = 3q$ or $a = 3q + 1$ or $a = 3q + 2$, where q is an integer.

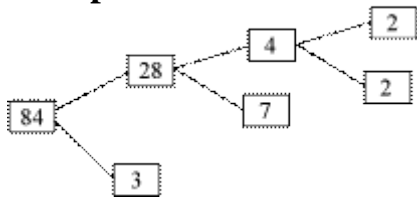
- Fundamental theorem of arithmetic states that every composite number can be uniquely expressed (factorised) as a product of primes apart from the order in which the prime factors occur.

Example: 1260 can be uniquely factorised as

$$\begin{array}{r|l} 2 & 1260 \\ \hline 2 & 630 \\ \hline 3 & 315 \\ \hline 3 & 105 \\ \hline 5 & 35 \\ \hline & 7 \end{array}$$

$$1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7$$

Example: Factor tree of 84



$$84 = 2 \times 2 \times 3 \times 7$$

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- According to fundamental theorem of arithmetic, a number can be represented as the product of primes having a unique factorisation.

Example:

Check whether 15^n is divisible by 10 or not for any natural number n . Justify your answer.

Solution:

A number is divisible by 10 if it is divisible by both 2 and 5.

$$15^n = (3 \cdot 5)^n$$

3 and 5 are the only primes that occur in the factorisation of 15^n

By uniqueness of fundamental theorem of arithmetic, there is no other prime except 3 and 5 in the factorisation of 15^n .

2 does not occur in the factorisation of 15^n .

Hence, 15^n is not divisible by 10.

- Every number of the form \sqrt{p} , where p is a prime number is called an irrational number. For example, $\sqrt{3}$, $\sqrt{11}$, $\sqrt{12}$ etc.

Theorem: If a prime number p divides a^2 , then p divides a , where a is a positive integer.

Example:

Prove that $\sqrt{7}$ is an irrational number.

Solution:

If possible, suppose $\sqrt{7}$ is a rational number.

Then, $\sqrt{7} = \frac{p}{q}$, where p, q are integers, $q \neq 0$.

If $\text{HCF}(p, q) \neq 1$, then by dividing p and q by $\text{HCF}(p, q)$, $\sqrt{7}$ can be reduced as

$$\sqrt{7} = \frac{a}{b} \text{ where } \text{HCF}(a, b) = 1 \quad \dots (1)$$

$$\Rightarrow \sqrt{7}b = a$$

$$\Rightarrow 7b^2 = a^2$$

$$\Rightarrow a^2 \text{ is divisible by } 7$$

$$\Rightarrow a \text{ is divisible by } 7 \quad \dots (2)$$

$$\Rightarrow a = 7c, \text{ where } c \text{ is an integer}$$

$$\therefore \sqrt{7}c = b$$

$$\Rightarrow 7b^2 = 49c^2$$

$$\Rightarrow b^2 = 7c^2$$

$\Rightarrow b^2$ is divisible by 7

$\Rightarrow b$ is divisible by 7 ... (3)

From (2) and (3), 7 is a common factor of a and b . which contradicts (1)

$\therefore \sqrt{7}$ is an irrational number.

Example:

Show that $\sqrt{12} - 6$ is an irrational number.

Solution:

If possible, suppose $\sqrt{12} - 6$ is a rational number.

Then $\sqrt{12} - 6 = \frac{p}{q}$ for some integers p, q ($q \neq 0$)

Now,

$$\sqrt{12} - 6 = \frac{p}{q}$$

$$\Rightarrow 2\sqrt{3} = \frac{p}{q} + 6$$

$$\Rightarrow \sqrt{3} = \frac{1}{2} \left(\frac{p}{q} + 6 \right)$$

As $p, q, 6$ and 2 are integers, $\frac{1}{2} \left(\frac{p}{q} + 6 \right)$ is rational number, so is $\sqrt{3}$.

This conclusion contradicts the fact that $\sqrt{3}$ is irrational.

Thus, $\sqrt{12} - 6$ is an irrational number.

- Decimal expansion of a rational number can be of two types:

(i) Terminating

(ii) Non-terminating and repetitive

In order to find decimal expansion of rational numbers we use long division method.

For example, to find the decimal expansion of $\frac{1237}{25}$.

We perform the long division of 1237 by 25.

$$\begin{array}{r}
 49.48 \\
 25 \overline{) 1237.00} \\
 \underline{100} \\
 237 \\
 \underline{225} \\
 120 \\
 \underline{100} \\
 200 \\
 \underline{200} \\
 0
 \end{array}$$

$$\frac{1237}{25}$$

Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal number is terminating.

- If x is a rational number with terminating decimal expansion then it can be expressed in

$\frac{p}{q}$ the form, where p and q are co-prime (the HCF of p and q is 1) and the prime factorisation of q is of the form $2^n 5^m$, where n and m are non-negative integers.

$$\frac{p}{q}$$

- Let $x = \frac{p}{q}$ be any rational number.
 - If the prime factorization of q is of the form $2^m 5^n$, where m and n are non-negative integers, then x has a terminating decimal expansion.
 - If the prime factorisation of q is not of the form $2^m 5^n$, where m and n are non-negative integers, then x has a non-terminating and repetitive decimal expansion.

$$\frac{17}{1600} = \frac{17}{2^6 \times 5^2}$$

For example, $\frac{17}{1600} = \frac{17}{2^6 \times 5^2}$ has the denominator in the form $2^n 5^m$, where $n = 6$ and $m = 2$ are non-negative integers. So, it has a terminating decimal expansion.

$\frac{723}{392} = \frac{3 \times 241}{2^3 \times 7^2}$ has the denominator not in the form $2^n 5^m$, where n and m are non-negative integers. So, it has a non-terminating decimal expansion.