10. Differentiability

Exercise 10.1

1. Question

Show that f(x) = |x - 3| is continuous but not differentiable at x = 3.

Answer

f(x) = |x – 3|

Therefore we can write it as,

$$f(x) = \begin{cases} -(x-3), x < 3 \\ x-3, x \geq 3 \end{cases}$$

f(3) = 3 - 3 = 0

LHL = $\lim_{x \to 3} f(x)$

= $\lim_{h \to 0} f(3-h)$

= $\lim_{h \to 0} 3-(3-h)$

= $\lim_{h \to 0} 0$

RHL = $\lim_{x \to 3} f(x)$

= $\lim_{h \to 0} f(3+h)$

= $\lim_{h \to 0} 3+h-3$

= 0

LHL = RHL = f(3)

Since, f(x) is continuous at x = 3

(LHD at x = 3) = $\lim_{x-3^-} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h-0^-} \frac{f(3-h)-f(3)}{3-h-3}$

= $\lim_{h-0^-} \frac{3-(3-h)-0}{-h}$

= $\lim_{h-0^-} \frac{h}{-h}$

= - 1

(RHD at x = 3) = $\lim_{x-3^+} \frac{f(x)-f(3)}{x-3}$

= $\lim_{h-0^+} \frac{f(3+h)-f(3)}{3+h-3}$

= $\lim_{h-0^+} \frac{3+h-3-0}{h}$

= $\lim_{h-0^+} \frac{h}{h}$

= 1

(LHD at x = 3)$\neq$(RHD at x = 3)

Hence, f(x) is continuous but not differentiable at x = 3.

2. Question

Show that f(x) = $x^{\frac{1}{3}}$ is not differentiable at x = 0.

Answer

For differentiability,

LHD(at x = 0) = RHD (at x = 0)

(LHD at x = 0) = $\lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h \to 0^-} \frac{f(0-h)-f(0)}{0-h-0}$

= $\lim_{h \to 0^-} \frac{(-h)^{\frac{1}{3}}-0}{-h}$

= $\lim_{h \to 0^-} \frac{(-h)^{\frac{1}{3}}}{-h}$

= $\lim_{h \to 0^-} \frac{(-1)^{\frac{1}{3}}(h)^{\frac{1}{3}}}{(-1)h}$

= $\lim_{h \to 0} (-1)^{\frac{-2}{3}} \, h^{\frac{-2}{3}}$

= Not defined

(RHD at x = 3) = $\lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h \to 0^-} \frac{f(0+h)-f(0)}{0+h-0}$

= $\lim_{h \to 0^+} \frac{(h)^{\frac{1}{3}}-0}{+h}$

= $\lim_{h \to 0^+} \frac{(h)^{\frac{1}{3}}}{+h}$

= $\lim_{h \to 0} \, h^{\frac{-2}{3}}$

= Not defined

Since, LHD and RHD does not exist at x = 0

Hence, f(x) is not differentiable at x = 0

3. Question

Show that $f(x)=\begin{cases} 12x-13, & x<3 \\ 2x^2+5, & x \geq 3 \end{cases}$ is differentiable at x = 3. Also, find f'(3).

Answer

For differentiability,

LHD(at x = 3) = RHD (at x = 3)

(LHD at x = 3) = $\lim_{x \to 3^-} \frac{f(x)-f(3)}{x-3}$

$= \lim_{h \to 0^-} \frac{f(3-h)-f(3)}{3-h-3}$

$= \lim_{h \to 0^-} \frac{[12(3-h)-13]-[12(3)-13]}{-h}$

$= \lim_{h \to 0^-} \frac{36-12h-13-36+13}{-h}$

$= \lim_{h \to 0^-} \frac{-12h}{-h}$

= 12

(RHD at x = 3) = $\lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$

$= \lim_{h \to 0^+} \frac{f(3+h)-f(3)}{3+h-3}$

$= \lim_{h \to 0^+} \frac{[2(3+h^2)+5]-[12(3)-13]}{3+h-3}$

$= \lim_{h \to 0^+} \frac{18+12h+2h^2+5-36+13}{h}$

$= \lim_{h \to 0^+} \frac{2h^2+12h}{h}$

$= \lim_{h \to 0^+} \frac{h(2h+12)}{h}$

= 12

Since, (LHD at x = 3) = (RHD at x = 3)

Hence, f(x) is differentiable at x = 3.

4. Question

Show that the function f defined as follows,

$$f(x) = \begin{cases} 3x-2, & 0 < x \le 3 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x-4, & x > 2 \end{cases}$$

Is continuous at x = 2, but not differentiable there at x = 2.

Answer

For continuity,

LHl(at x = 2) = RHL (at x = 2)

f(2) = $2(2)^2$ - 2

= 8 - 2 = 6

LHL = $\lim_{x \to 2^-} f(x)$

= $\lim_{h \to 0^-} f(2-h)$

= $\lim_{h \to 0^-} [2(2-h)^2 - (2-h)]$

= 8 - 2

= 6

RHL = $\lim_{x \to 2^+} f(x)$

$= \lim_{h\to0^+} f(2 + h)$

$= \lim_{h\to0} 5(2 + h) - 4$

= 6

Since, LHL = RHL = f(2)

Hence, F(x) is continuous at x = 2

For differentiability,

LHD(at x = 2) = RHD (at x = 2)

(LHD at x = 2) = $\lim_{x-2^-} \frac{f(x)-f(2)}{x-2}$

$= \lim_{h\to0} \frac{f(2-h)-f(2)}{2-h-2}$

$= \lim_{h\to0} \frac{[2(2-h)^2-(2-h)]-[8-2]}{-h}$

$= \lim_{h\to0} \frac{8-8h+2h^2-h-6}{-h}$

$= \lim_{h\to0} \frac{2h^2-6h}{-h}$

$= \lim_{h\to0} \frac{h(2h-6)}{-h}$

$= \lim_{h\to0} (6-2h)$

= 6

(RHD at x = 2) = $\lim_{x-2^+} \frac{f(x)-f(2)}{x-2}$

$= \lim_{h\to0} \frac{f(2+h)-f(2)}{2+h-2}$

$= \lim_{h\to0} \frac{[5(2+h)-4]-[8-2]}{h}$

$= \lim_{h\to0} \frac{10+5h-4-6}{h}$

= 5

Since, (RHD at x = 2)≠(LHD at x = 2)

Hence, f(2) is not differentiable at x = 2.

5. Question

Discuss the continuity and differentiability of f(x) = |x| + |x - 1| in the interval (- 1,2).

Answer

f(x) = $\begin{cases} x + x + 1, -1 < x < 0 \\ 1, 0 \le x \le 1 \\ -x - x + 1, 1 < x < 2 \end{cases}$

f(x) = $\begin{cases} 2x + 1, -1 < x < 0 \\ 1, 0 \le x \le 1 \\ -2x + 1, 1 < x < 2 \end{cases}$

We know that a polynomial and a constant function is continuous and differentiable every where. So, f(x) is continuous and differentiable for x ∈(- 1,0) and x ∈(0,1) and (1,2).

We need to check continuity and differentiability at x = 0 and x = 1.

Continuity at x = 0

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-} 2x + 1$ = 1

$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} 1$ = 1

F(0) = 1

$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = f(0)$

Since, f(x) is continuous at x = 0

Continuity at x = 1

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 1$ = 1

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} 1$ = 1

F(1) = 1

$\log_{x\to0^+} f(x) = \log_{x\to0^-} f(x)$ = 1

Since, f(x) is continuous at x = 1

For differentiability,

LHD(at x = 0) = RHD (at x = 0)

Differentiability at x = 0

(LHD at x = 0) = $\lim\limits_{x-0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim\limits_{x-0^-} \frac{2x+1-1}{x-0}$

= $\lim\limits_{x-0^-} \frac{2x}{x}$

= 2

(RHD at x = 0) = $\lim\limits_{x-0^+} \frac{f(x)-f(0)}{x-0}$

= $\lim\limits_{x-0^+} \frac{1-1}{x}$

= $\lim\limits_{x-0^+} \frac{0}{x}$

= 0

Since,(LHD at x = 0)≠(RHD at x = 0)

So, f(x) is differentiable at x = 0.

For differentiability,

LHD(at x = 1) = RHD (at x = 1)

Differentiability at x = 1

(LHD at x = 1) = $\lim\limits_{x-1^-} \frac{f(x)-f(1)}{x-1}$

= $\lim\limits_{x-1^-} \frac{1-1}{x-1}$

= 0

(RHD at x = 1) = $\lim_{x-1^+} \frac{f(x)-f(1)}{x-1}$

= $\lim_{x-1^+} \frac{-2x+1-1}{x-1}$

= ∞

Since, f(x) is not differentiable at x = 1.

So, f(x) is continuous on (- 1,2) but not differentiable at x = 0, 1

6. Question

Find whether the following function is differentiable at x = 1and x= 2 or not:

f(x) = $\begin{cases} x, & x \leq 0 \\ 2-x, & 1 \leq x \leq 2 \\ -2+3x-x^2, & x > 2 \end{cases}$

Answer

Differentiability at x = 1

For differentiability,

LHD(at x = 1) = RHD (at x = 1)

(LHD at x = 1) = $\lim_{x-1^-} \frac{f(x)-f(1)}{x-1}$

= $\lim_{x-1^-} \frac{x-1}{x-1}$

= 1

(RHD at x = 1) = $\lim_{x-1^+} \frac{f(x)-f(1)}{x-1}$

= $\lim_{x-1^+} \frac{2-x-1}{x-1}$

= $\lim_{x-1^+} \frac{1-x}{x-1}$

= - 1

Since, (LHD at x = 1)≠(RHD at x = 1)

So, f(x) is not differentiable at x = 1

Differentiable at x = 2

(LHD at x = 2) = $\lim_{x-2^-} \frac{f(x)-f(2)}{x-2}$

= $\lim_{x-2^-} \frac{2-x-0}{x-2}$

= $\lim_{x-2^-} \frac{2-x}{x-2}$

= - 1

(RHD at x = 2) = $\lim_{x-2^+} \frac{f(x)-f(2)}{x-2}$

= $\lim_{x-2^+} \frac{-2+3x-x^2-0}{x-2}$

$= \lim_{x-2^+} \frac{(1-x)(x-2)}{x-2}$

= - 1

Since, (LHD at x = 2) = (RHD at x = 2)

So, f(x) is differentiable at x = 2.

7 A. Question

Show that the function F(x) = $\begin{cases} x^m \sin\left(\frac{1}{x}\right), x \neq 0 \\ 0 \qquad\qquad , x = 0 \end{cases}$ is

Differentiable at x = 0, if m > 1.

Answer

f(x) = $\begin{cases} x^m \sin\left(\frac{1}{x}\right), x \neq 0 \\ \quad 0, x = 0 \end{cases}$

(LHD at x = 0) = $\lim_{x-0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{h\to 0^-} \frac{f(0-h)-f(0)}{(0-h)-0}$

$= \lim_{h\to 0^-} \frac{(0-h)^m \sin\left(\frac{1}{-h}\right)-0}{-h}$

$= \lim_{h\to 0^-} \frac{(-h)^m \sin\left(-\frac{1}{h}\right)-0}{-h}$

$= \lim_{h\to 0^-} (-h)^{m-1} \sin(-\frac{1}{h})$

$= \lim_{h\to 0^-} -(-h)^{m-1} \sin(\frac{1}{h})$

= 0 * k [when - 1$\leq K \leq 1$]

(RHD at x = 0) = $\lim_{x-0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{(0+h)-0}$

$= \lim_{h\to 0^+} \frac{h^m \sin\left(\frac{1}{h}\right)-0}{h}$

$= \lim_{h\to 0^+} (h)^{m-1} \sin(\frac{1}{h})$ = 0

Since, (LHD at x = 0) = (RHD at x = 0)

Hence, f(x) is differentiable at x = 0

7 B. Question

Show that the function F(x) = $\begin{cases} x^m \sin\left(\frac{1}{x}\right), x \neq 0 \\ 0 \qquad\qquad , x = 0 \end{cases}$ is

Continuous but not differentiable at x = 0, if 0<m<1.

Answer

LHL = $\lim_{x \to 0^-} f(x)$

= $\lim_{h \to 0} f(0-h)$

= $\lim_{h \to 0} (-h)^m \sin(-\frac{1}{h})$

= $\lim_{h \to 0} -(-h)^m \sin(\frac{1}{h})$

= 0 x k [when $-1 \leq k \leq 1$]

= 0

RHL = $\lim_{x \to 0^+} f(x)$

= $\lim_{h \to 0} f(0 + h)$

= $\lim_{h \to 0} (0 + h)^m \sin(\frac{1}{0+h})$

= $\lim_{h \to 0} (h)^m \sin(\frac{1}{h})$

= 0 x k [when $-1 \leq k' \leq 1$]

= 0

LHL = RHL = f(0)

Since, f(x) is continuous at x = 0

For Differentiability at x = 0

(LHD at x = 0) = $\lim_{x - 0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h \to 0} \frac{f(0-h)-f(0)}{(0-h)-0}$

= $\lim_{h \to 0^-} \frac{(-h)^m \sin\left(-\frac{1}{h}\right)}{h}$

= $\lim_{h \to 0^-} -(-h)^{m-1} \sin(\frac{1}{h})$

= Not Defined [since 0<m<1]

(RHD at x = 0) = $\lim_{x - 0^+} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{(0+h)-0}$

= $\lim_{h \to 0^+} \frac{h^m \sin\left(\frac{1}{h}\right)}{0+h-0}$

= $\lim_{h \to 0^+} (h)^{m-1} \sin(\frac{1}{h})$

= 0

Since , (LHD at x = 0)$\neq$(RHD at x = 0)

Hence, f(x) is continuous at x = 0 but not differentiable.

7 C. Question

Show that the function F(x) = $\begin{cases} x^m \sin\left(\frac{1}{x}\right), x \neq 0 \\ 0 \qquad\qquad\quad, x = 0 \end{cases}$ is

Neither continuous and nor differentiable, if m≤0.

Answer

LHL = $\lim_{x \to 0^-} f(x)$

= $\lim_{h \to 0} f(0-h)$

= $\lim_{h \to 0} (-h)^m \sin(-\frac{1}{h})$

= Not defined, as m $\leq$0

RHL = $\lim_{x \to 0^+} f(x)$

= $\lim_{h \to 0} f(0 + h)$

= $\lim_{h \to 0} (h)^m \sin(\frac{1}{h})$

= Not Defined, as $m \leq 0$

Since, LHL and RHL are not Defined ,So f(x) is not continuous.

Let x = 0 for $m \leq 0$

Now,

(LHD at x = 0) = $\lim_{x - 0^-} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h \to 0^-} \frac{f(0-h)-f(0)}{(0-h)-0}$

= $\lim_{h \to 0^-} \frac{(0-h)^m \sin\left(\frac{1}{-h}\right)-0}{-h}$

= $\lim_{h \to 0^-} \frac{(-h)^m \sin\left(-\frac{1}{h}\right)-0}{-h}$

= $\lim_{h \to 0^-} (-h)^{m-1} \sin(-\frac{1}{h})$

= $\lim_{h \to 0^-} -(-h)^{m-1} \sin(\frac{1}{h})$

= Not Defined , as $m \leq 0$

(RHD at x = 0) = $\lim_{x - 0^+} \frac{f(x)-f(0)}{x-0}$

= $\lim_{h \to 0^+} \frac{f(0+h)-f(0)}{(0+h)-0}$

= $\lim_{h \to 0^+} \frac{h^m \sin\left(\frac{1}{h}\right)}{h}$

= $\lim_{h \to 0^+} (h)^{m-1} \sin(\frac{1}{h})$

= Not Defined , as $m \leq 0$

Hence, f(x) is not differentiable at x = 0 for $m \leq 0$

8. Question

Find the values of a and b so that the function

f(x) = $\begin{cases} x^2 + 3x + a, x \leq 1 \\ bx + 2, x > 1 \end{cases}$

is differentiable at each x € R.

Answer

For differentiability at x = 1,

LHD(at x = 1) = RHD(at x = 1)

(LHD at x = 1) = $\lim_{x-1^-} \frac{f(x)-f(1)}{x-1}$

= $\lim_{h\to 0} \frac{f(1-h)-f(1)}{1-h-1}$

= $\lim_{h\to 0} \frac{[(1-h)^2 + 3(1-h) + a]-[1+3+a]}{-h}$

= $\lim_{h\to 0} \frac{h^2-5h}{-h}$

= - 5

(RHD at x = 1) = $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

= $\lim_{h\to 0} \frac{f(1+h)-f(1)}{1+h-1}$

= $\lim_{h\to 0} \frac{[b(1+h)+2]-[b+2]}{h}$

= $\lim_{h\to 0} \frac{h^2-5h}{-h}$

= b

Since, f(x) is differentiable , So

(LHD at x = 1) = (RHD at x = 1)

-5 = b

F(1) = 1 + 3 + a

= 4 + a

LHL = $\lim_{x\to 1^{-1}} f(x)$

= $\lim_{h\to 0} f(1-h)$

= $\lim_{h\to 0}(1-h)^2 + 3(1-h) + a$

= $\lim_{h\to 0} 4 + a$

RHL = $\lim_{x\to 1^+} f(x)$

= $\lim_{h\to 0} f(1+h)$

= $\lim_{h\to 0} b(1+b) + 2$

= b + 2

9. Question

Show that the function

$$F(x) = \begin{cases} |2x-3|[x], \geq 0 \\ \sin\left(\frac{\Pi x}{2}\right) \quad , x < 0 \end{cases}$$

Is continuous but not differentiable at x = 1.

Answer

$$F(x) = \begin{cases} (2x-3)[x], \geq \frac{3}{2} \\ -(2x-3), 1 \leq x \leq \frac{3}{2} \\ \sin\left(\frac{\Pi x}{2}\right), x < 1 \end{cases}$$

For continuity at x = 1

F(1) = - (2(1) - 3) = 1

LHL = $\lim_{x \to 1^{-1}} f(x)$

$= \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} sin(\frac{\pi(1-h)}{2})$

$= \sin\frac{\pi}{2}$

= 1

RHL = $\lim_{x \to 1^{+}} f(x)$

$= \lim_{h \to 0} f(1 + h)$

$= \lim_{h \to 0} -(2(1 + h) - 3)$

= - 1(- 1)

= 1

LHL = RHL = f(1)

So, f(x) is continuous at x = 1

For differentiability at x = 1

(LHD at x = 1) = $\lim_{x-1^{-}} \frac{f(x)-f(1)}{x-1}$

$= \lim_{h \to 0} \frac{f(1-h)-f(1)}{1-h-1}$

$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi(1-h)}{2}\right)-1}{-h}$

$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}h\right)-1}{-h}$

$= \lim_{h \to 0} \frac{\cos(\frac{\pi}{2}h)-1}{-\frac{h}{2}}$

= 0

(RHD at x = 1) = $\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}$

$= \lim_{h\to0}\frac{f(1+h)-f(1)}{1+h-1}$

$= \lim_{h\to0}\frac{-[2(1+h)-3]-1}{h}$

$= \lim_{h\to0}\frac{-2-2h+3-1}{h}$

$= \lim_{h\to0}\frac{-2h}{h}$

= - 2

(LHD at x = 1)≠(RHD at x = 1)

Hence, f(x) is continuous but not differentiable at x = 1.

10. Question

If $f(x)=\begin{cases} ax^2-b & , \text{if}\ |x|<1 \\ \frac{1}{|x|} & , |x|\ge1 \end{cases}$ is differentiable at x = 1, find a and b.

Answer

$$f(x) = \begin{cases} -\frac{1}{x}, if\ |x| < -1 \\ ax^2 - b, -1 < x < 1 \\ \frac{1}{|x|}, |x| \ge 1 \end{cases}$$

LHL = $\lim_{x\to1^-} f(x)$

$= \lim_{h\to0} f(1-h)$

$= \lim_{h\to0} a(1-h)^2 - b$

= a - b

RHL = $\lim_{x\to1^+} f(x)$

$= \lim_{h\to0} f(1+h)$

$= \lim_{h\to0}\frac{1}{1+h}$

= 1

Since, f(x) is continuous ,so

LHL = RHL

a - b = 1 - - - - (i)

(LHD at x = 1) = $\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}$

$= \lim_{h\to0}\frac{f(1-h)-f(1)}{1-h-1}$

$= \lim_{h \to 0} \frac{a(1-h)^2 - b - 1}{-h}$

$= \lim_{h \to 0} \frac{a(1-h)^2 - (a-1) - 1}{-h}$

Using equation (i)

$= \lim_{h \to 0} \frac{a + ah^2 - 2ah - a + 1 - 1}{-h}$

$= \lim_{h \to 0} \frac{ah^2 - 2ah}{-h}$

$= \lim_{h \to 0} (2a - ah)$

2a

(RHD at x = 1) = $\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$

$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$

$= \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h}$

$= \lim_{h \to 0} \frac{1 - 1 - h}{(1+h)h}$

$= \lim_{h \to 0} \frac{-1}{1+h}$

= - 1

Since f(x) is differentiable at x = 1,

(LHD at x = 1) = (RHD at x = 1)

2a = - 1

a = $-\frac{1}{2}$

put a = $-\frac{1}{2}$ in equation (i),

a - b = 1

$\left(\frac{-1}{2}\right)$ - b = 1

b = $-\frac{1}{2} - 1$

b = $-\frac{3}{2}$

a = $-\frac{1}{2}$

11. Question

Find the values of a and b, if the function f(x) defined by

$$f(x) = \begin{cases} x^2 + 3x + a, & x \le 1 \\ bx + 2, & x > 1 \end{cases}$$

is differentiable at x = 1.

Answer

Let us find left hand and right-hand derivative of the function f(x).

L.H.D at (x = 1)

(LHD at x = 1) = $\lim_{x-1^-} \frac{f(x)-f(1)}{x-1}$

$= \lim_{h\to 0} \frac{f(1-h)-f(1)}{1-h-1}$

$= \lim_{h\to 0} \frac{(1-h)^2+3\ (1-h)+a-[1^2+3.1+a]}{1-h-1}$

$= \lim_{h\to 0} \frac{h^2-5\ h}{-\ h}$

= 5

Now let us find Right-hand Derivative of f(x).

(R.H.D at x = 1)

(RHD at x = 1) = $\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{h\to 0} \frac{f(1+h)-f(1)}{1+h-1}$

$= \lim_{h\to 0} \frac{b\ (1+h)+2-b-2}{1+h-1}$

$= \lim_{h\to 0} \frac{bh}{h}$

= b

Derivative at (x = 1) = $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1} \frac{x^2+3\,x+a-1-3-a}{x-1}$

$= \lim_{x\to 1} \frac{x^2+4\,x-x-4}{x-1}$

$= \lim_{x\to 1} x+4$

= 5

Now by definition of differentiability,

L.H.D = R.H.D = Derivative at that point

Therefore, 5 = b = 5

So, a can have any value and b = 5.

Exercise 10.2

1. Question

If f is defined by f (x) = x^2, find f ‘ (2).

Answer

We are given with a polynomial function f(x) = x^2, and we have to find whether it is derivable at x = 2 or not, so by using the formula, f ‘(c) = $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$, we get,

f ‘(2) = $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$

f ‘(2) = $\lim_{x\to2}\frac{x^2-2^2}{x-2}$

f ‘(2) = $\lim_{x\to2}\frac{(x+2)(x-2)}{x-2}$

[using $a^2 - b^2 = (a + b)(a - b)$]

f ‘(2) = $\lim_{x\to2} x + 2 = 4$

Hence, the function is differentiable at x = 2 and its derivative equals to 4.

2. Question

If f is defined by f (x) = x^2 – 4x + 7, show that $f'(5) = 2f'\left(\frac{7}{2}\right)$

Answer

We are given with a polynomial function f(x) = x^2 - 4x + 7, and we have to f ‘(x) it’s value at

x = 5 and x = $\frac{7}{2}$, so by using the formula, f ‘(c) = $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, we get,

f ‘(5) = $\lim_{x\to5}\frac{f(x)-f(5)}{x-5}$

f ‘(5) = $\lim_{x\to5}\frac{x^2-4x+7-(5^2-4\times5+7)}{x-5}$

f ‘(5) = $\lim_{x\to5}\frac{x^2-4x-5}{x-5}$

f ‘(5) = $\lim_{x\to5}\frac{x(x-5)+1(x-5)}{x-5}$

f ‘(5) = $\lim_{x\to5} (x + 1) = 6$

Hence to function is differentiable at x = 5 and has value 6 .

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{f(x)-f(\frac{7}{2})}{x-\frac{7}{2}}$

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+7-[(\frac{7}{2})^2-4\times\frac{7}{2}+7]}{x-\frac{7}{2}}$

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+7-[(\frac{7}{2})^2-4\times\frac{7}{2}+7]}{x-\frac{7}{2}}$

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+\frac{7}{4}}{x-\frac{7}{2}}$

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{x^2-4x+\frac{7}{4}}{x-\frac{7}{2}}$

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{(2x-1)(2x-7)}{2(2x-7)}$

f ‘($\frac{7}{2}$) = $\lim_{x\to\frac{7}{2}}\frac{(2x-1)}{2}$ = 3

Therefore f ‘(5) = 2f ‘($\frac{7}{2}$) = 6,

Hence, proved.

3. Question

Show that the derivative of the function f given by f (x) = $2x^3 - 9x^2 + 12x + 9$, at x = 1 and x = 2 are equal.

Answer

We are given with a polynomial function f(x) = $2x^3 - 9x^2 + 12x + 9$, and we have to find f ‘(x) at x = 1 and x = 2, so by using the formula, f ‘(c) = $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, we get,

f ‘(1) = $\lim_{x \to 1} \frac{f(x)-f(1)}{x-1}$

f ‘(1) = $\lim_{x \to 1} \frac{2x^3-9x^2+12x+9-[2(1)^3-9(1)^2+12(1)+9]}{x-1}$

f ‘(1) = $\lim_{x \to 1} \frac{2x^3-9x^2+12x-5}{x-1}$

f ‘(1) = $\lim_{x \to 1} \frac{(x-1)(2x^2-7x+5)}{x-1}$

f ‘(1) = $\lim_{x \to 1} 2x^2 - 7x + 5 = 0$

for x = 2, we get,

f ‘(2) = $\lim_{x \to 2} \frac{f(x)-f(2)}{x-2}$

f ‘(2) = $\lim_{x \to 2} \frac{2x^3-9x^2+12x+9-[2(2)^3-9(2)^2+12(2)+9]}{x-2}$

f ‘(2) = $\lim_{x \to 2} \frac{2x^3-9x^2+12x-4}{x-2}$

f ‘(2) = $\lim_{x \to 2} \frac{(x-2)(2x^2-5x+2)}{x-2}$

f ‘(2) = $\lim_{x \to 2} 2x^2 - 5x + 2 = 0$

Hence they are equal at x = 1 and x = 2 .

4. Question

If for the function Φ (x) = $\lambda x^2 + 7x - 4$, Φ’ (5) = 97, find λ.

Answer

We have to find the value of λ given in the real function and we are given with the differentiability of the function f(x) = $\lambda x^2 + 7x - 4$ at x = 5 which is f ‘(5) = 97, so we will adopt the same process but with a little variation.

So by using the formula, f ‘(c) = $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, we get,

f ‘(5) = $\lim_{x \to 5} \frac{f(x)-f(5)}{x-5}$

f ‘(5) = $\lim_{x \to 5} \frac{\lambda x^2+7x-4-[\lambda(5)^2+7(5)-4]}{x-5}$

f ‘(5) = $\lim_{x \to 5} \frac{\lambda x^2+7x-4-[\lambda(5)^2+7(5)-4]}{x-5}$

f ‘(5) = $\lim_{x \to 5} \frac{\lambda x^2+7x-35-25\lambda}{x-5}$

as the limit has some finite value, then there must be the formation of some indeterminate form like $\frac{0}{0}, \frac{\infty}{\infty}$, so if we put the limit value, then the numerator will also be zero as the denominator, but there must be a factor (x - 5) in the numerator, so that this form disappears.

$f'(5) = \lim_{x\to5}\frac{(x-5)(\lambda x + 5\lambda+7)}{x-5}$

$f'(5) = \lim_{x\to5} \lambda x + 5\lambda + 7 = 97$

$f'(5) = 10\lambda + 7 = 97$

$10\lambda = 90$

$\lambda = 9$

5. Question

If $f(x) = x^3 + 7x^2 + 8x - 9$, find $f'(4)$.

Answer

We are given with a polynomial function $f(x) = x^3 + 7x^2 + 8x - 9$, and we have to find whether it is derivable at x = 4 or not, so by using the formula, $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, we get,

$f'(4) = \lim_{x\to4}\frac{f(x)-f(4)}{x-4}$

$f'(4) = \lim_{x\to4}\frac{x^3 + 7x^2 + 8x-9-[4^3 + 7(4)^2 + 8(4)-9]}{x-4}$

$f'(4) = \lim_{x\to4}\frac{(x-4)(x^2 + 11x + 52)}{x-4}$

$f'(4) = \lim_{x\to4} x^2 + 11x + 52$

$f'(4) = 112$.

6. Question

Find the derivative of the function f defined by $f(x) = mx + c$ at $x = 0$.

Answer

We are given with a polynomial function $f(x) = mx + c$, and we have to find whether it is derivable at x = 0 or not, so by using the formula, $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, we get,

$f'(0) = \lim_{x\to0}\frac{f(x)-f(0)}{x-0}$

$f'(0) = \lim_{x\to0}\frac{mx + c-[m(0) + c]}{x-0}$

$f'(0) = \lim_{x\to0}\frac{mx + c-c}{x-0}$

$f'(0) = \lim_{x\to0} m = m$

This is the derivative of a function at x = 0, and also this is the derivative of this function at every value of x.

Which you can check also by simply derivating the f(x) function and putting the value of x afterwards.

7. Question

Examine the differentiability of the function f defined by

$$f(x) = \begin{cases} 2x + 3 & , \text{ if } -3 \le x < -2 \\ x + 1 & , \text{ if } -2 \le x < 0 \\ x + 2 & , \text{ if } 0 \le x \le 1 \end{cases}$$

Answer

To find whether the function f(x) is derivable at a point x = c we have to check that $f'(c^-) = f'(c^+) =$ finite quantity, this condition must be fulfilled in order the function to be derivable.

As discussed above the top of this document where the description of the topic is given.

$$f(x) = \begin{cases} 2x+3 & , \text{if } -3 \le x < -2 \\ x+1 & , \text{if } -2 \le x < 0 \\ x+2 & , \text{if } 0 \le x \le 1 \end{cases}$$

To prove the first two functions of f(x) to be derivable we must take the point x = - 2, around this point we will prove the derivability of this sub - functions because this point is common in them both.

So by using the formula, f '(c) = $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$, we get,

f '(- 2 $^-$) = $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

f '(- 2 $^-$) = $\lim_{x \to -2^-} \frac{2x+3-(-2+1)}{x-(-2)}$

value of f(c) = - 1 because when we put exact value of x = c then we have to take the second sub function because it is only defined when x = - 2.

f '(- 2 $^-$) = $\lim_{x \to -2^-} \frac{2x+4}{x+2}$ = 2

if the limit would not have been simplified to a number before putting the limit then we would put x = - 2 - h,

f '(- 2 $^+$) = $\lim_{x \to -2^+} \frac{x+1-(-2+1)}{x-(-2)}$

f '(- 2 $^+$) = $\lim_{x \to -2^+} \frac{x+2}{x+2}$ = 1

Therefore the function is not differentiable at x = - 2.

if the limit would not have been simplified to a number before putting the limit, then we would put x = - 2 + h.

Now we have to find the derivability at point x = 0, because at this point or near it, the next two sub-functions are defined and hence, we get,

f '(0 $^-$) = $\lim_{x \to -0^-} \frac{x+1-(0+2)}{x-(0)}$

f '(0 $^-$) = $\lim_{x \to -0^-} \frac{x-1}{x-(0)} = \lim_{x \to -0^-} \frac{x-1}{x}$

putting x = 0 - h,

= $\lim_{h \to 0} \frac{h+1}{h} = \infty$

f '(0 $^+$) = $\lim_{x \to 0^+} \frac{x+2-(0+2)}{x-(0)}$

f '(0 $^+$) = $\lim_{x \to 0^+} \frac{x}{x}$ = 1

as we can see that the value around a point is not the same, so we will say that the function is not derivable at x = 0.

At the end we conclude that at x = 0 and x = - 2 function is non differentiable.

8. Question

Write an example of a function which is everywhere continuous but fails to be differentiable exactly at five points.

Answer

As we know that polynomial functions are always continuous and differentiable, we also know that the point of derivability is a sharp corner or a stop on the on going curve.

Hence f(x) = |x| is a function which is non-derivable at a point which is the sharp corner, at that point, there are more than one slopes possible due to this it is non-derivable but the modulus function is always continuous.

So a function which is non derivable at exactly 5 points and continuous always is,

f(x) = |x - 1| + |x - 2| + |x - 3| + |x - 4| + |x - 5|

this can be done by drawing the graph of the function or by algebraic method also.

We can choose any value along with x in the modulus function as we only need the points.

To draw the graph, we will solve the function by taking some points or intervals to open the modulus one by one.

9. Question

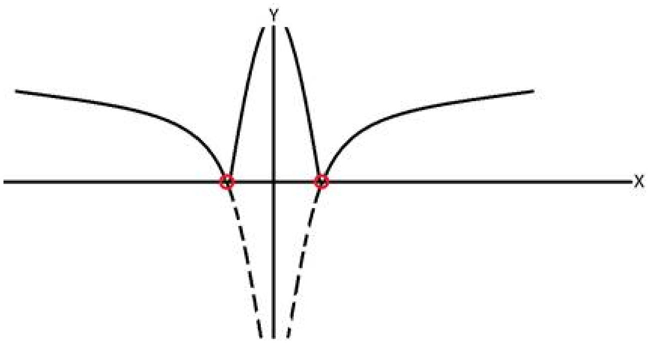
Discuss the continuity and differentiability of f (x) = |log |x| |.

Answer

We have to find the differentiability and the continuity of the function which we will do by considering both graphical and algebraic methods.

Now f(x) = |log|x||, as we know that log x function is continuous in its domain and hence it is also derivable, but after we introduce modulus function, there might be the formation of sharp corners as the graph's direction changes to its mirror reflection.

The graph of the function is,



Now we observe the two red dots which are sharp end corners due to mirror image formation because of outer modulus function due to which we observe that the function is non-derivable and there are two curves due to inner modulus function which does not connect and hence the function is not continuous.

We can also prove this question by the algebraic method by reducing the function to a much simpler form, which is,

f (x) = { $\begin{matrix} \log x, \text{for } x > 1 \\ -\log x, \text{for } 0 < x < 1 \end{matrix}$

{ $\begin{matrix} \log(-x), \text{for } x < -1 \\ -\log(-x), \text{for } -1 < x < 0 \end{matrix}$

Therefore this is the resolved function which is to be used to prove that weather this function is continuous and derivable or not.

10. Question

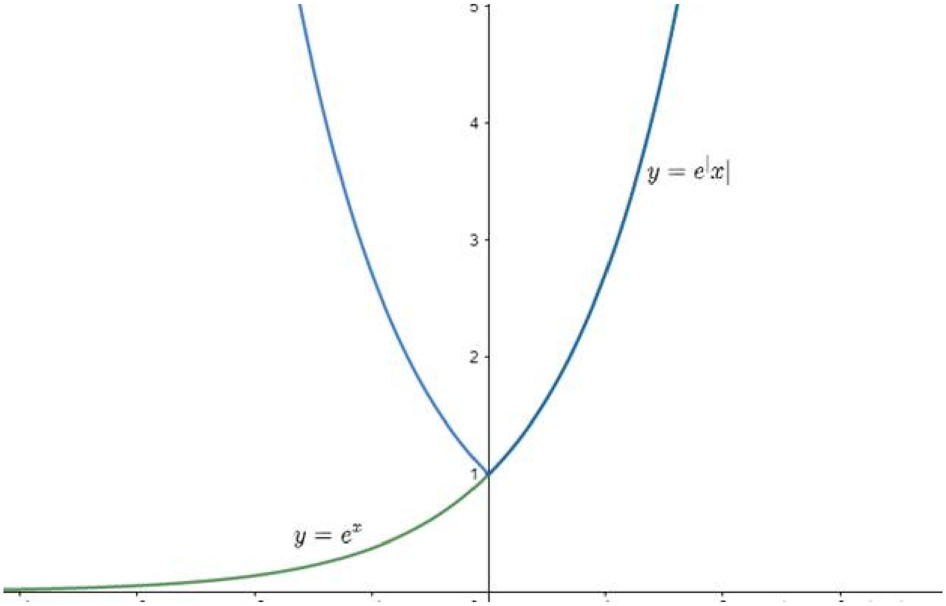
Discuss the continuity and differentiability of f (x) = e $^{|x|}$.

Answer

We have to find the differentiability and the continuity of the function which we will do by considering both graphical and algebraic methods.

Now f(x) = $e^{|x|}$, as we know that log x function is continuous in its domain and hence it is also derivable, but after we introduce modulus function, there might be the formation of sharp corners as the graph's direction changes to its mirror reflection.

The graph of the function is,



As we know that,

f(x) = $e^{|x|}$, so by opening the modulus function by limiting the domain we get,

f(x) = $\{\begin{matrix} e^{x}, \text{for } x \geq 0 \\ e^{-x}, \text{for } x < 0 \end{matrix}$

We can put equality to any one of the two sub-functions.

Now we need to find the continuity around the point x = 0 because it is the common point between the two sub-functions, now the condition for the continuity to be there-there must be the satisfaction of the one condition which is,

L.H.L = R.H.L = f(x = c)

Which can further be elaborated as, at x = c

$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ = f(c) = finite quantity,

L.H.L. = $\lim_{x \to 0^-} e^{-x}$

Put x = 0 - h

= $\lim_{h \to 0} e^{h}$ = 1

R.H.L. = $\lim_{x \to 0^+} e^{x}$

Put x = 0 + h

= $\lim_{h \to 0} e^{h}$ = 1

f(x = 0) = e^{x} = e^{-x} = 1

Hence the function is a continuous function because it satisfies the required condition of continuity.

Now we have to check the function for derivability.

For which the condition is,

L.H.D. = R.H.D. = finite quantity.

The rule for derivability.

So by using the formula,

f '(c) = $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$, we get,

f '(0 $^-$) = $\lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$

f '(0 $^-$) = $\lim\limits_{x \to 0^-} \frac{e^{-x}-1}{x-0}$

put x = 0 - h,

f '(0 $^-$) = $\lim\limits_{h \to 0} \frac{e^{h}-1}{-h}$ = - 1

This above is for the L.H.D.

Now we are doing for the R.H.D.,

f '(0 $^+$) = $\lim\limits_{x \to 0^+} \frac{e^{x}-1}{x-0}$

put x = 0 + h,

f '(0 $^+$) = $\lim\limits_{h \to 0} \frac{e^{h}-1}{h}$ = 1

As we can see that L.H.D. ≠ R.H.D.,

Hence we will say that this function is not derivable but it is continuous.

11. Question

Discuss the continuity and differentiability of

$$f(x) = \begin{cases} (x-c) \quad \cos\left(\frac{1}{x-c}\right), & x \neq c \\ 0, & x = c \end{cases}$$

Answer

To find weather the function f(x) is derivable at a point x = c we have to check that

f '(c $^-$) = f '(c $^+$) = finite quantity, this condition must be fulfilled in order the function to be derivable.

As discussed above the top of this document where the description of the topic is given.

The given function is,

$$f(x) = \begin{cases} (x-c) \quad \cos\left(\frac{1}{x-c}\right), & x \neq c \\ 0, & x = c \end{cases}$$

and we have to find weather it is derivable at x = c or not, so by using the formula,

f '(c) = $\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$, we get,

L.H.D. is given by,

f '(c $^-$) = $\lim\limits_{x \to c^-} \frac{f(x)-f(c)}{x-c}$

f '(c $^-$) = $\lim\limits_{x \to c^-} \frac{(x-c)\cos(\frac{1}{x-c})-0}{x-c}$

$f'(c^-) = \lim_{x \to c^-} \cos(\frac{1}{x-c})$

put x = c - h,

$f'(c^-) = \lim_{h \to 0} \cos(\frac{1}{c-h-c})$

$f'(c^-) = \lim_{h\to0} \cos(\frac{1}{h})$

as the angle of cos is going to infinity, so it’s limit will oscillate between - 1 and 1, i.e. within the range of cos .

R.H.D. is given by,

$f'(c^+) = \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$

$f'(c^+) = \lim_{x \to c^+} \frac{(x-c)\cos(\frac{1}{x-c})-0}{x-c}$

$f'(c^+) = \lim_{h \to 0} \cos(\frac{1}{c+h-c})$

$f'(c^+) = \lim_{h\to0} \cos(\frac{1}{h})$

as the angle of cos e is going to infinity, so it’s limit will oscillate between - 1 and 1 i.e. within the range of cos e.

As the results are also not equal to finite quantity, so hence the function is non-derivable.

Now we have to check the function for the continuity,

For a function to be continuous it should follow one condition fully, which is,

L.H.L = R.H.L. = f(x = c), at any point x = c.

Which can be further elaborated as,

$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(x = c)$

Now we have to find L.H.L.,

$f(c^-) = \lim_{x \to c^-} (x-c)\cos(\frac{1}{x-c})$

put x = c - h,

$f(c^-) = \lim_{h\to0} (c - h - c)\cos(\frac{1}{c-h-c})$

$f(c^-) = \lim_{h\to0} (-h)\cos(\frac{1}{h})$

As the cose function’s angle is approaching to infinity so it’s value will be any between - 1 to 1 and it is multiplied by h which is approaching to zero, so the whole limit will approach to zero.

$f(c^-) = 0$

Now we have to find R.H.L.,

$f(c^+) = \lim_{x \to c^+} (x-c)\cos(\frac{1}{x-c})$

put x = c + h,

$f(c^+) = \lim_{h \to 0} (c + h - c)\cos(\frac{1}{c+h-c})$

$f(c^+) = \lim_{h\to0} (h)\cos(\frac{1}{h})$

As the cose function’s angle is approaching to infinity so it’s value will be any between - 1 to 1 and it is multiplied by h which is approaching to zero, so the whole limit will approach to zero.

$f(c^{+}) = 0$

Hence as the L.H.L = R.H.L. = f(x = c) = 0

So our function is continuous but is non derivable.

12. Question

Is |sin x| differentiable? What about cos |x|?

Answer

To find weather the function f(x) is derivable at a point x = c we have to check that

$f'(c^{-}) = f'(c^{+})$ = finite quantity, this condition must be fulfilled in order the function to be derivable.

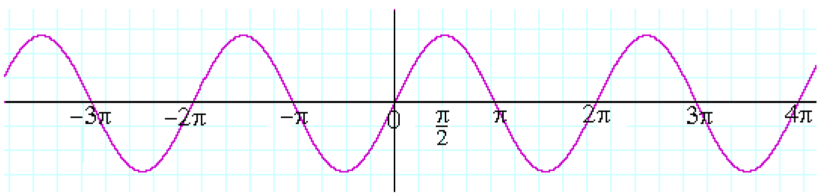
As discussed above the top of this document where description of the topic is given.

The given function is,

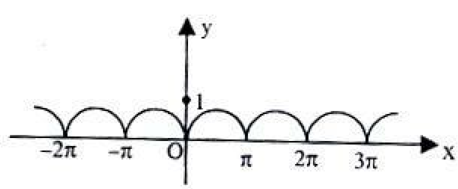
$f(x) = |\sin x|$,

To find weather the function is derivable or not we can either make graph of the function and look for pointed edges, not smooth edges, where there are multiple slopes present which makes the function non derivable.

The graph of the function sin x is,



The graph of $|\sin x|$ is,



As you can see at every integral multiple of π there is a pointed turn which shows multiple slopes at one point which makes the function non derivable.

We can also solve the problem by algebraic method, which is,

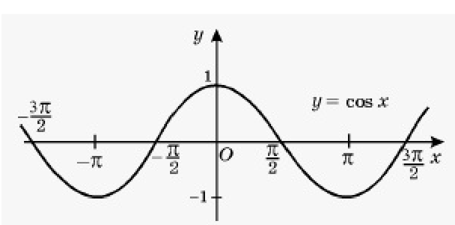
$$f(x) = \begin{cases} \sin x, \text{ for } 0 < x \le \pi \\ -\sin x, \text{ for } \pi < x \le 2\pi \end{cases}$$

this is the resolved function with which we can solve by using limits method.

$f(x) = \cos|x|$,

To find weather the function is derivable or not we can either make graph of the function and look for pointed edges, not smooth edges, where there are multiple slopes present which makes the function non derivable.

The graph of the cos x is,



As we know the property of cos x which is cos(- x) = cosx which means that weather there is modulus with the angle of cose or not there is no difference.

Hence the curve do not have any sharp points, so it is derivable hence cos x is differentiable everyhere so │cos x│ is also differentiable everywhere.

We can also prove by algebraic method by taking a general angle ‘x’ and prove for it .

Very short answer

1. Question

Define differentiability of a function at a point.

Answer

Let f(x) be a real valued function defined on an open interval (a, b) and let c € (a, b), then f(x) is said to be differentiable at x = c iff

$\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists finitely or $f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$.

2. Question

Is every differentiable function continuous?

Answer

Yes, if a function is differentiable at a point then it is necessarily continuous at that point.

Let f(x) be a function differentiable at x = c.

Then,

$\lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists finitely

Let $f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$

To prove that f(x) is continuous at x = c, it is sufficient to show that $\lim\limits_{x \to c} f(x) = f(c)$

$$\lim_{x \to c} f(x) = [\lim_{x \to c}\left\{\frac{f(x)-f(c)}{x-c}\right\}(x-c)] + f(c)$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left\{\frac{f(x)-f(c)}{x-c}\right\}\lim_{x \to c}(x-c) + f(c)$$

$$\lim_{x \to c} f(x) = f'(c) \times 0 + f(c)$$

$$\lim_{x \to c} f(x) = f(c)$$

Hence, f(x) is continuous at x = c.

3. Question

Is every continuous function differentiable?

Answer

No, a function may be continuous at a point but not differentiable at that point.

For example,

Function f(x)=|x| is continuous at x = 0 but not differentiable at x =0.

4. Question

Give an example of a function which is continuous but not differentiable at a point.

Answer

Consider the function f(x) = |x| where

$$f(x) = \begin{cases} -x, x \le 0 \\ x, x > 0 \end{cases}$$

This function is continuous at x = 0 but not differentiable at x = 0.

LHL at x =0,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} -(0-h) = 0$$

RHL at x =0,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h) = 0$$

And f(0)=0

Hence, f(x) is continuous at x =0.

LHD at x =0,

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0} \frac{(0-h)-(0)}{-h} = -1$$

RHD at x =0,

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0}$$

$$= \lim_{h\to 0} \frac{(0+h)-(0)}{h} = 1$$

∵ LHD ≠RHD

∴ f(x) is not differentiable at x =0.

5. Question

If f(x) is differentiable at x = c, then write the value of $\lim_{x\to c} f(x)$.

Answer

Given that f(x) is differentiable at x = c,

Then,

$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists finitely

or $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$

$$\lim_{x\to c} f(x) = [\lim_{x\to c} \left\{\frac{f(x)-f(c)}{x-c}\right\}(x-c)] + f(c)$$

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left\{\frac{f(x)-f(c)}{x-c}\right\} \lim_{x\to c}(x-c) + f(c)$$

$$\lim_{x\to c} f(x) = f'(c) \times 0 + f(c)$$

$$\lim_{x\to c} f(x) = f(c)$$

6. Question

If f(x) = |x – 2| write whether f’ (2) exists or not.

Answer

Given that f(x) = |x – 2| where

$$f(x) = \begin{cases} 2 - x, x \le 2 \\ x - 2, x > 2 \end{cases}$$

LHD at x =2,

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{h \to 0} \frac{f(2 - h) - f(2)}{2 - h - 2}$$

$$= \lim_{h \to 0} \frac{(2 - h) - (2)}{-h} = -1$$

RHD at x =2,

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{2 + h - 2}$$

$$= \lim_{h \to 0} \frac{(2 + h) - (2)}{h} = 1$$

∵ LHD ≠RHD

∴ f' (2) does not exists.

7. Question

Write the points where f(x) = $|\log_e x|$ is not differentiable.

Answer

Given that f(x) = $|\log_e x|$ where

$$f(x) = \begin{cases} -\log_e x, \ 0 < x < 1 \\ \log_e x, \ x \ge 1 \end{cases}$$

Clearly f(x) is differentiable for x>1 and x<1. So, we will check the differentiability at x = 1.

LHD at x =1,

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{f(1 - h) - f(1)}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{\log(1 - h) - \log 1}{-h} = -1$$

RHD at x =1,

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{\log(1 + h) - \log 1}{h} = 1$$

∵ LHD ≠RHD

∴ f(x) = $|\log_e x|$ is not differentiable.

8. Question

Write the points of non-differentiability of f(x) = |log|x||.

Answer

Given that the f(x) = |log|x|| where

$$|x| = \begin{cases} -x, -\infty < x < -1 \\ -x, -1 < x < 0 \\ x, 0 < x < 1 \\ x, 1 < x < \infty \end{cases}$$

$$\log|x| = \begin{cases} \log(-x), -\infty < x < -1 \\ \log(-x), -1 < x < 0 \\ \log x, 0 < x < 1 \\ \log x, 1 < x < \infty \end{cases}$$

$$|\log|x|| = \begin{cases} \log(-x), -\infty < x < -1 \\ -\log(-x), -1 < x < 0 \\ -\log x, 0 < x < 1 \\ \log x, 1 < x < \infty \end{cases}$$

LHD at x =1,

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{\log(1-h) - \log 1}{-h} = -1$$

RHD at x =1,

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{\log(1+h) - \log 1}{h} = 1$$

∵ LHD ≠RHD

So, function is not differentiable at x =1.

LHD at x =-1,

$$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-1 - h - (-1)}$$

$$= \lim_{h \to 0} \frac{\log(-1-h) - \log(-1)}{-h} = -1$$

RHD at x =-1,

$$\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x - 1} = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{(-1) + h - (-1)}$$

$$= \lim_{h \to 0} \frac{\log(-1+h) - \log(-1)}{h} = 1$$

∵ LHD ≠RHD

So, function is not differentiable at x =-1.

At x =0 function is not defined.

∴ Function is not differential at x = 0 and ±1.

9. Question

Write the derivative of f(x) = $|x|^3$ at x = 0.

Answer

Given that f(x) = $|x|^3$ where

$f(x)=\begin{cases}-x^3, x\le 0\\ x^3, x>0\end{cases}$

LHD at x =0,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$=\lim_{h\to 0}\frac{(h^3)}{-h}=0$$

RHD at x =0,

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$=\lim_{h\to 0}\frac{h^3}{h}=0$$

Also, f(0) = 0

⇒ LHD at (x = 0) = RHD (at x = 0) = f(0)

⇒ f'(0)= 0

10. Question

Write the number of points where f(x) = |x| + |x – 1| is continuous but not differentiable.

Answer

Given f(x) = |x|+|x – 1|

$$f(x)=\begin{cases}-x-(x-1), x<0\\ x-(x-1), 0\le x\le 1\\ x+x-1, x>1\end{cases}$$

$$f(x)=\begin{cases}-2x+1, x<0\\ 1, 0\le x\le 1\\ 2x-1, x>1\end{cases}$$

For x<0,

f(x) = -2x+1 which being a polynomial function is continuous and differentiable.

For x € (0, 1),

f(x) = 1 which being a constant function is continuous and differentiable.

For 1<x,

f(x) = 2x - 1 which being a polynomial function is continuous and differentiable.

So, the possible points where function is continuous but not differentiable are 0 and 1.

LHD (at x = 0):

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$=\lim_{h\to 0}\frac{-2(-h)+1-(-2\times 0+1)}{-h}=-2$$

RHD (at x =0):

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$=\lim_{h\to 0}\frac{1-1}{h}=0$$

∵ LHD ≠RHD

So, function is not differentiable at x =0.

Similarly,

LHD at x =1,

$$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{h\to0}\frac{f(1-h)-f(1)}{1-h-1}$$

$$=\lim_{h\to0}\frac{1-1}{-h}=0$$

RHD at x =1,

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{h\to0}\frac{f(1+h)-f(1)}{1+h-1}$$

$$=\lim_{h\to0}\frac{2(1+h)-1-(2-1)}{h}=2$$

∵ LHD ≠RHD

So, function is not differentiable at x =1.

The number of points where f(x) = |x| + |x – 1| is continuous but not differentiable are two i.e. x = 1 and x = 0.

11. Question

If $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists finitely, write the value of $\lim_{x\to c}f(x)$.

Answer

$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists finitely

Let $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

$$\lim_{x\to c}f(x)=[\lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}\right\}(x-c)]+f(c)$$

$$\lim_{x\to c}f(x)=\lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}\right\}\lim_{x\to c}(x-c)+f(c)$$

$$\lim_{x\to c}f(x)=f'(c)\times0+f(c)$$

$$\lim_{x\to c}f(x)=f(c)$$

12. Question

Write the value of the derivative of f(x) = |x – 1| + |x – 3| at x = 2.

Answer

Given that f(x) = |x – 1| + |x – 3|.

$$f(x)=\begin{Bmatrix}-(x-1)-(x-3), x<1\\ x-1-(x-3), 1\le x\le3\\ x-1+x-3, x>3\end{Bmatrix}$$

$$f(x)=\begin{Bmatrix}-2x+4, x<1\\ 2, 1\le x\le3\\ 2x-4, x>3\end{Bmatrix}$$

We check differentiability at x = 2,

LHD at x =2,

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2}$$

$$=\lim_{h\to 0}\frac{2-2}{-h}=0$$

Hence, f'(2) = 0

13. Question

If $f(x)=\sqrt{x^2+9}$, write the value of $\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}$.

Answer

Given that $f(x)\ =\sqrt{x^2+9}$

Now, $f(4)=\sqrt{16+9}$

⇒ f(4) = 5

$$\frac{f(x)-f(4)}{x-4}=\frac{\sqrt{x^2+9}\ -5}{x-4}$$

$$\Rightarrow\frac{f(x)-f(4)}{x-4}=\frac{\sqrt{x^2+9}\ -5}{x-4}\times\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}$$

$$\Rightarrow\frac{f(x)-f(4)}{x-4}=\frac{x^2+9-25}{x-4(\sqrt{x^2+9}+5)}$$

$$\Rightarrow\frac{f(x)-f(4)}{x-4}=\frac{x^2-16}{x-4(\sqrt{x^2+9}+5)}$$

$$\Rightarrow\frac{f(x)-f(4)}{x-4}=\frac{(x+4)(x-4)}{x-4(\sqrt{x^2+9}+5)}$$

$$\Rightarrow\frac{f(x)-f(4)}{x-4}=\frac{(x+4)}{(\sqrt{x^2+9}+5)}$$

Taking limit x→ 4,

$$\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\lim_{x\to 4}\frac{(x+4)}{(\sqrt{x^2+9}+5)}$$

$$\Rightarrow\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\frac{(4+4)}{(\sqrt{16+9}+5)}$$

$$\Rightarrow\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=\frac{8}{10}$$

$$\Rightarrow\lim_{x\to 4}\frac{f(x)-f(4)}{x-4}=0.8$$

MCQ

1. Question

Choose the correct answer.

Let f(x) = |x| and g(x) = $|x^3|$, then

A. f(x) and g(x) both are continuous at x = 0

B. f(x) and g(x) both are differentiable at x = 0

C. f(x) is differentiable but g(x) is not differentiable at x = 0

D. f(x) and g(x) both are not differentiable at x = 0

Answer

Given f(x) = |x| and g(x) = $|x^3|$,

$$f(x) = \begin{Bmatrix} -x, x \leq 0 \\ x, x > 0 \end{Bmatrix}$$

Checking differentiability and continuity,

LHL at x =0,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} -(0-h) = 0$$

RHL at x =0,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h) = 0$$

And f(0)=0

Hence, f(x) is continuous at x =0.

LHD at x =0,

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0} \frac{(0-h)-(0)}{-h} = -1$$

RHD at x =0,

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0}$$

$$= \lim_{h\to 0} \frac{(0+h)-(0)}{h} = 1$$

∵ LHD ≠RHD

∴ f(x) is not differentiable at x =0.

$$g(x) = \begin{Bmatrix} -x^3, x \leq 0 \\ x^3, x > 0 \end{Bmatrix}$$

Checking differentiability and continuity,

LHL at x =0,

$$\lim_{x\to 0^-} g(x) = \lim_{h\to 0} g(0-h) = \lim_{h\to 0} -(0-h)^3 = 0$$

RHL at x =0,

$$\lim_{x\to 0^+} g(x) = \lim_{h\to 0} g(0+h) = \lim_{h\to 0} (0+h)^3 = 0$$

And g(0)=0

Hence, g(x) is continuous at x =0.

LHD at x =0,

$$\lim_{x\to 0^-} \frac{g(x)-g(0)}{x-0} = \lim_{h\to 0} \frac{g(0-h)-g(0)}{0-h-0}$$

$$= \lim_{h \to 0} \frac{(0-h)^3 - (0)}{-h} = -1$$

RHD at x =0,

$$\lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = \lim_{h \to 0} \frac{g(0+h) - g(0)}{0 + h - 0}$$

$$= \lim_{h \to 0} \frac{(0+h)^3 - (0)}{h} = 0$$

∵ LHD = RHD

∴ g(x) is differentiable at x =0.

Hence, option A is correct.

2. Question

Choose the correct answer.

The function f(x) = $\sin^{-1}$(cos x) is

A. discontinuous at x = 0

B. continuous at x = 0

C. differentiable at x = 0

D. none of these

Answer

Given f(x) = $\sin^{-1}$(cos x),

Checking differentiability and continuity,

LHL at x =0,

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \sin^{-1}(\cos(0-h)) = \lim_{h \to 0} \sin^{-1}(\cos(-h)) = \sin^{-1} 1 = \frac{\pi}{2}$$

RHL at x =0,

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \sin^{-1}(\cos(0+h)) = \lim_{h \to 0} \sin^{-1}(\cos(h)) = \sin^{-1} 1 = \frac{\pi}{2}$$

And $f(0) = \frac{\pi}{2}$

Hence, f(x) is continuous at x =0.

LHD at x =0,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(\cos(0-h)) - \left(\frac{\pi}{2}\right)}{-h} = 1$$

RHD at x =0,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0+h) - f(0)}{0 + h - 0}$$

$$= \lim_{h \to 0} \frac{\sin^{-1}(\cos(0+h)) - \left(\frac{\pi}{2}\right)}{h} = -1$$

∵ LHD ≠RHD

∴ f(x) is not differentiable at x =0.

Hence, option B is correct.

3. Question

Choose the correct answer.

The set of points where the function f(x) = x|x| is differentiable is

A. (–∞, ∞)

B. (–∞, 0) ∪ (0, ∞)

C. (0, ∞)

D. [0, ∞]

Answer

We have f(x) = x|x|

Where $f(x) = \begin{Bmatrix} -x^2, x < 0 \\ 0, x = 0 \\ x^2, x > 0 \end{Bmatrix}$

We have $-x^2$ and x^2 which being polynomial functions are continuous and differentiable.

The only possible point of non-differentiability can be x =0.

LHD at x =0,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{(0 - h)^2 - (0)}{-h} = 0$$

RHD at x =0,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{0 + h - 0}$$

$$= \lim_{h \to 0} \frac{(0 + h)^2 - (0)}{h} = 0$$

∵ LHD = RHD = f(0)

∴ f(x) is differentiable at x =0.

Hence f(x) is differentiable at (–∞, ∞).

4. Question

Choose the correct answer.

If $f(x) = \begin{cases} \dfrac{|x+2|}{\tan^{-1}(x+2)} & , \ x \neq -2 \\ 2 & , \ x = -2 \end{cases}$, then f(x) is

A. continuous at x = –2

B. not continuous at x = –2

C. differentiable at x = –2

D. continuous but bot derivable at x = –2

Answer

Given that $f(x) = \begin{Bmatrix} \frac{|x+2|}{\tan^{-1}(x+2)}, x \neq -2 \\ 2, x = -2 \end{Bmatrix}$

$$\Rightarrow f(x) = \begin{Bmatrix} \frac{-(x+2)}{\tan^{-1}(x+2)}, x < -2 \\ \frac{(x+2)}{\tan^{-1}(x+2)}, x > -2 \\ 2, x = -2 \end{Bmatrix}$$

Checking the continuity at x = -2,

LHL:

$$\lim_{x \to -2^-} f(x) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} \frac{-(-2-h+2)}{\tan^{-1}(-2-h+2)} = \frac{h}{\tan^{-1}(-h)} = -1$$

RHL:

$$\lim_{x \to -2^+} f(x) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} \frac{(-2+h+2)}{\tan^{-1}(-2+h+2)} = \frac{h}{\tan^{-1}(h)} = 1$$

And f(-2)= 2

∵ LHL ≠ RHL ≠f(-2)

Hence, f(x) is not continuous at x =-2.

So, option B is correct.

5. Question

Choose the correct answer.

Let f(x) = (x + |x|) |x|. Then, for all x

A. f is continuous

B. f is differentiable for some x

C. f’ is continuous

D. f’’ is continuous

Answer

Given that f(x) = (x + |x|) |x|

$$\Rightarrow f(x) = \begin{Bmatrix} (x-x)(-x), x < 0 \\ 0, x = 0 \\ (x+x)(x), x > 0 \end{Bmatrix}$$

$$\Rightarrow f(x) = \begin{Bmatrix} 0, x < 0 \\ 0, x = 0 \\ 2x^2, x > 0 \end{Bmatrix}$$

So, we can say that f(x) is continuous for all x.

Now, checking the differentiability at x =0

LHD at x =0,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$

$$= \lim_{h \to 0} \frac{0 - (0)}{-h} = 0$$

RHD at x =0,

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$=\lim_{h\to 0}\frac{2h^2-(0)}{h}=\lim_{h\to 0}\frac{2h}{1}=0$$

∵ LHD = RHD

So, f(x) is differentiable for all x.

Hence, option A is correct.

6. Question

Choose the correct answer.

The function $f(x) = e^{-|x|}$ is

A. continuous everywhere but not differentiable at x = 0

B. continuous and differentiable everywhere

C. not continuous at x = 0

D. none of these

Answer

Given that $f(x) = e^{-|x|}$

$$\Rightarrow f(x)=\begin{cases} e^{x}, x<0 \\ 1, x=0 \\ e^{-x}, x>0 \end{cases}$$

Checking continuity and differentiability at x = 0,

LHL:

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} e^{-h}=1$$

RHL:

$$\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} e^{-h}=1$$

And f(0)= 1

∵ LHL = RHL = f(0)

f(x) is continuous at x = 0.

LHD at x =0,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$=\lim_{h\to 0}\frac{e^{-h}-(0)}{-h}=\infty$$

∵ LHD does not exist, so f(x) is not differentiable at x =0.

Hence, option A is correct.

7. Question

Choose the correct answer.

The function f(x) = |cos x| is

A. everywhere continuous and differentiable

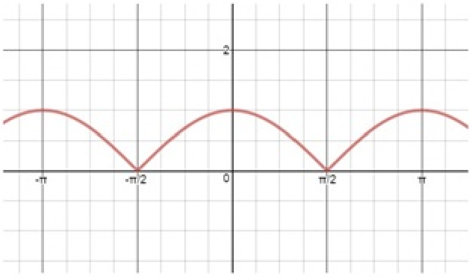
B. everywhere continuous but not differentiable at (2n + 1) π/2, n∊Z

C. neither continuous nor differentiable at (2n + 1) π/2, nϵZ

D. none of these

Answer

Given that f(x) = |cos x|



From the graph it is evident that it is everywhere continuous but not differentiable at (2n + 1) π/2, nϵZ.

8. Question

Choose the correct answer.

If $f(x)=\sqrt{1-\sqrt{1-x^2}}$, then f(x) is

A. continuous on [–1, 1] and differentiable on (–1, 1)

B. continuous on [–1, 1] and differentiable on (–1, 0) ∪ϕ (0, 1)

C. continuous and differentiable on [–1, 1]

D. none of these

Answer

Given that $f(x)=\sqrt{1-\sqrt{1-x^2}}$

So, the function will be defined for those values of x for which

$1-x^2 \geq 0$

$\Rightarrow x^2 \leq 1$

⇒ |x| ≤ 1

⇒ -1 ≤ x≤ 1

∴ Function is continuous in [-1, 1].

Now, we will check the differentiability at x =0

LHD at x =0,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$=\lim_{h\to 0}\frac{\sqrt{1-\sqrt{1-(0-h)^2}-(0)}}{-h}=-\infty$$

∵ LHD does not exist, so f(x) is not differentiable at x =0.

∴ f(x) is not differentiable at x =0.

Hence, option B is correct.

9. Question

Choose the correct answer.

If f(x) = a |sinx| + b $e^{|x|}$ + c|x^3| and if f(x) is differentiable at x = 0, then

A. a = b = c = 0

B. a = 0, b = 0, c ϵ R

C. b = c = 0, a ϵ R

D. c = 0, a = 0, b ϵ R

Answer

Given that f(x) = a |sinx| + b $e^{|x|}$ + c|x^3| and f(x) is differentiable at x = 0.

⇒ LHD = RHD at x = 0.

$$\Rightarrow \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$\Rightarrow \lim_{h\to 0}\frac{-a\sin(-h)+be^h-ch^3-b}{0-h}=\lim_{h\to 0}\frac{a\sin(h)+be^h+ch^3-b}{0+h}$$

$$\Rightarrow \lim_{h\to 0}\frac{a\cos h+be^h+3ch^2}{-1}=\lim_{h\to 0}\frac{a\cos h+be^h+3ch^2}{1}$$

By L'Hospital Rule,

⇒ -(a+b) = a+b

⇒ -2(a+b) = 0

⇒ a + b =0

This is the value for all c € R.

Hence, option B is correct.

10. Question

Choose the correct answer.

If $f(x)=x^2+\frac{x^2}{1+x^2}+\frac{x^2}{\left(1+x^2\right)^2}+\ldots+\frac{x^2}{\left(1+x^2\right)^n}+\ldots$, then at x = 0, f(x)

A. has no limit

B. is discontinuous

C. in continuous but not differentiable

D. is differentiable

Answer

Given $f(x)=x^2+\frac{x^2}{1+x^2}+\frac{x^2}{(1+x^2)^2}+\cdots\ldots+\frac{x^2}{(1+x^2)^n}+\cdots$

For x =0, x^2 = 0

⇒ f(x) = 0

For x≠ 0,

$x^2 + 1 > x^2$

$\Rightarrow 0 < \frac{x^2}{1+x^2} < 1$

$\therefore \lim_{x\to 0} f(x) = \lim x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots \ldots + \frac{x^2}{(1+x^2)^n} + \cdots$

$\therefore \lim_{x\to 0} f(x) = \lim x^2 \left(1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots \ldots + \frac{1}{(1+x^2)^n} + \cdots\right)$

$\therefore \lim_{x\to 0} f(x) = \lim\{x^2 (\frac{1}{1-\frac{1}{1+x^2}})\}$

$\because$ Sum of infinite series where $r = \frac{1}{1+x^2}$

$\Rightarrow \lim_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 (\frac{1+x^2}{x^2})$

$\Rightarrow \lim_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 + 1$

$\Rightarrow \lim_{x\to 0} f(x) = 1 \neq f(0)$

So, f(x) is discontinuous at x =0.

11. Question

Choose the correct answer.

If f(x) = $|\log_e x|$, then

A. f'(1^+) = 1

B. f'(1^-) = –1

C. f'(1) = 1

D. f'(1) = –1

Answer

Given that $f(x) = \begin{Bmatrix} -\log_e x, 0 < x < 1 \\ \log_e x, x \geq 1 \end{Bmatrix}$

Differentiability at x =1,

LHD at x =1,

$\lim\limits_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim\limits_{h\to 0} \frac{f(1-h) - f(1)}{1 - h - 1}$

$= \lim\limits_{h\to 0} \frac{\log 1 - h}{-h} = -1$

RHD at x =1,

$\lim\limits_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim\limits_{h\to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$

$= \lim\limits_{h\to 0} \frac{\log(1+h)}{h} = 1$

So, f'(1^+) = 1 and f'(1^-) = –1

12. Question

Choose the correct answer.

f(x) = |$\log_e$ |x||, then

A. f(x) is continuous and differentiable for all x in its domain

B. f(x) is continuous for all for all x in its domain but bot differentiable at x = ±1

C. f(x) is neither continuous nor differentiable at x = ±1

D. none of these

Answer

Given that the f(x) = |log|x|| where

$$|x| = \begin{cases} -x, -\infty < x < -1 \\ -x, -1 < x < 0 \\ x, 0 < x < 1 \\ x, 1 < x < \infty \end{cases}$$

$$\log|x| = \begin{cases} \log(-x), -\infty < x < -1 \\ \log(-x), -1 < x < 0 \\ \log x, 0 < x < 1 \\ \log x, 1 < x < \infty \end{cases}$$

$$|\log|x|| = \begin{cases} \log(-x), -\infty < x < -1 \\ -\log(-x), -1 < x < 0 \\ -\log x, 0 < x < 1 \\ \log x, 1 < x < \infty \end{cases}$$

We can see that function is continuous for all x. Now, checking the points of non differentiability.

LHD at x =1,

$$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1} = \lim_{h\to0}\frac{f(1-h)-f(1)}{1-h-1}$$

$$= \lim_{h\to0}\frac{\log(1-h)-\log 1}{-h} = -1$$

RHD at x =1,

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{h\to0}\frac{f(1+h)-f(1)}{1+h-1}$$

$$= \lim_{h\to0}\frac{\log(1+h)-\log 1}{h} = 1$$

∵ LHD ≠RHD

So, function is not differentiable at x =1.

LHD at x =-1,

$$\lim_{x\to-1^-}\frac{f(x)-f(-1)}{x-(-1)} = \lim_{h\to0}\frac{f(-1-h)-f(-1)}{-1-h-(-1)}$$

$$= \lim_{h\to0}\frac{\log(-1-h)-\log(-1)}{-h} = -1$$

RHD at x =-1,

$$\lim_{x\to-1^+}\frac{f(x)-f(-1)}{x-1} = \lim_{h\to0}\frac{f(-1+h)-f(-1)}{(-1)+h-(-1)}$$

$$= \lim_{h\to0}\frac{\log(-1+h)-\log(-1)}{h} = 1$$

∵ LHD ≠RHD

So, function is not differentiable at x =-1.

At x =0 function is not defined.

∴ Function is not differential at x = 0 and ±1.

Hence, option B is correct.

13. Question

Choose the correct answer.

Let $f(x)=\begin{cases}\frac{1}{|x|} & \text{for } |x|\ge 1\\ ax^2+b & \text{for } |x|<1\end{cases}$. If f(x) is continuous and differentiable at any point, then

A. $a=\frac{1}{2}, b=-\frac{3}{2}$

B. $a=\frac{1}{2}, b=\frac{3}{2}$

C. a = 1, b = –1

D. none of these

Answer

Given that $f(x)=\begin{Bmatrix}\frac{-1}{x}, x\le -1\\ ax^2+b, -1<x<1\\ \frac{1}{x}, x\ge 1\end{Bmatrix}$

∵ f(x) is continuous and differentiable at any point, consider x =1.

$\lim_{x\to 1}\frac{1}{x}=\lim_{x\to 1}ax^2+b$

$\Rightarrow a+b=1$

Also,

$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$

$\Rightarrow \lim_{x\to 1}\frac{ax^2-a}{x-1}=\lim_{x\to 1}\frac{1-x}{x(x-1)}$

$\Rightarrow \lim_{x\to 1}a(x+1)=\lim_{x\to 1}(-x)$

$\Rightarrow a=\frac{-1}{2}$

Putting above value in a+b =0,

$b=\frac{3}{2}$

Hence, none of the options is correct.

14. Question

Choose the correct answer.

The function f(x) = x – [x], where [.] denotes the greatest integer function is

A. continuous everywhere

B. continuous at integer points only

C. continuous at non-integer points only

D. differentiable everywhere

Answer

Given function f(x) = x – [x]

For any integer n,

$$f(x) = \begin{cases} x - (n-1), n-1 \le x < n \\ 0, x = n \\ x - n, n \le x < n+1 \end{cases}$$

LHL:

$$\lim_{x \to n^-} x - n + 1 = n - n + 1 = 1$$

RHL:

$$\lim_{x \to 1^+} x - n = n - n = 0$$

Hence, f(x) is not continuous at integer points.

∴ Given function is continuous on non integer points only.

15. Question

Choose the correct answer.

Let $f(x) = \begin{cases} ax^2 + 1 & , x > 1 \\ x + 1/2 & , x \le 1 \end{cases}$. Then, f(x) is derivable at x = 1, is

A. a = 2

B. a = 1

C. a = 0

D. a = 1/2

Answer

Given function $f(x) = \begin{cases} ax^2 + 1, x > 1 \\ x + \frac{1}{2}, x \le 1 \end{cases}$ and f(x) is derivable at x = 1.

⇒ LHD (at x = 1) = RHD (at x=1)

$$\Rightarrow \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(1-h) - f(1)}{1-h-1} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{1+h-1}$$

$$\Rightarrow \lim_{h \to 0} \frac{\left(1 - h + \frac{1}{2}\right) - \frac{3}{2}}{-h} = \lim_{h \to 0} \frac{a(1+h)^2 + 1 - \frac{3}{2}}{h}$$

$$\Rightarrow a - \frac{1}{2} = 0$$

$$\Rightarrow a = \frac{1}{2}$$

Hence, option D is correct.

16. Question

Choose the correct answer.

Let f(x) = |sin x|. Then,

A. f(x) is everywhere differentiable.

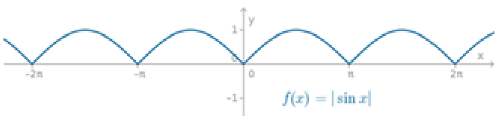
B. f(x) is everywhere continuous but not differentiable at x = nπ, nϵZ

C. f(x) is everywhere continuous but not differentiable at x = (2n +1) π/2, nϵZ.

D. none of these

Answer

Given that f(x) = |sin x|



From the graph it is evident that it is continuous everywhere but not differentiable at x = nπ, nϵZ

17. Question

Choose the correct answer.

Let f(x) = |cos x|. Then,

A. f(x) is everywhere differentiable

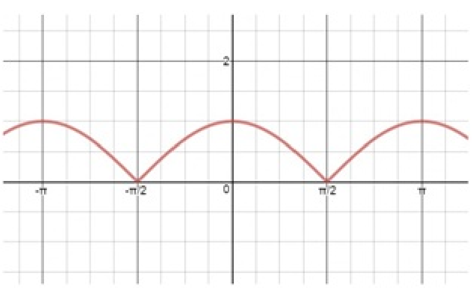
B. f(x) is everywhere continuous but not differentiable at x = nπ, nϵZ

C. f(x) is everywhere continuous but not differentiable at x = (2n + 1) π/2, nϵZ

D. None of these

Answer

Given that f(x) = |cos x|



From the graph it is evident that f(x) is everywhere continuous but not differentiable at x = (2n + 1) π/2, nϵZ.

18. Question

Choose the correct answer.

The function f(x) = 1 + |cos x| is

A. continuous no where

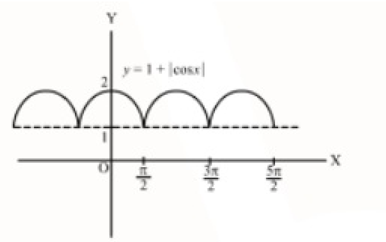
B. continuous everywhere

C. not differentiable at x = 0

D. not differentiable at x = n π, nϵZ

Answer

Given that f(x) = 1 + |cos x|.



From the graph it is evident that it is continuous everywhere.

19. Question

Choose the correct answer.

The function f(x) = |cos x| is

A. differentiable at x (2n + 1) π/2, nϵZ

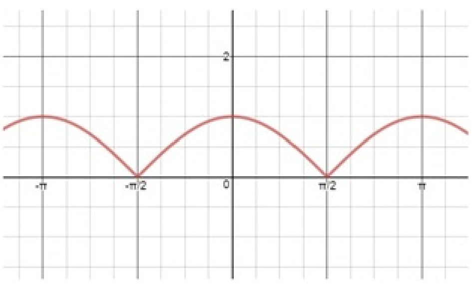
B. continuous but bot differentiable at x = (2n + 1) π/2, nϵZ

C. neither differentiable nor continuous at x = nπ,nϵZ

D. none of these

Answer

Given that f(x) = |cos x|



From the graph it is evident that it is continuous but bot differentiable at x = (2n + 1) π/2, nϵZ.

20. Question

Choose the correct answer.

The function $f(x)=\frac{\sin(\pi[x-\pi])}{4+[x]^2}$, where [.] denotes the greatest integer function, is

A. continuous as well as differentiable for all xϵR

B. continuous for all x but not differentiable at some x

C. differentiable for all x but not continuous at some x.

D. none of these

Answer

Given that $f(x)=\frac{\sin(\pi[x-\pi])}{4+[x]^2}$

∵ We know that π (x - π) = nπ and sin nπ = 0

So, $4+[x]^2 \neq 0$

⇒ f(x) = 0 for all x

Thus f(x) is a constant function and continuous as well as differentiable for all x∈R

21. Question

Choose the correct answer.

Let f(x) = a + b |x| + c $|x|^4$, where a, b, and c are real constants. Then, f(x) is differentiable at x = 0, if

A. a = 0

B. b = 0

C. c = 0

D. none of these

Answer

Given that f(x) = a + b |x| + c $|x|^4$, where a, b, and c are real constants and f(x) is differentiable at x = 0.

$$f(x) = \begin{cases} a + bx + cx^4, x \geq 0 \\ a - bx + cx^4, x < 0 \end{cases}$$

∵ f(x) is differentiable at x =0

∴ LHD = RHD

$$\Rightarrow \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \lim_{x \to 0^-} \frac{a - bx + cx^4 - a}{x} = \lim_{x \to 0^+} \frac{a + bx + cx^4 - a}{x}$$

$$\Rightarrow \lim_{h \to 0} \frac{a - b(-h) + c(-h)^4 - a}{-h} = \lim_{h \to 0} \frac{a + bh + ch^4 - a}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{a - b(-h) + c(-h)^4 - a}{-h} = \lim_{h \to 0} \frac{a + bh + ch^4 - a}{h}$$

$$\Rightarrow \lim_{h \to 0} -b - ch^3 = \lim_{h \to 0} b + ch^3$$

⇒ - b = b

⇒ 2b = 0

⇒ b = 0

22. Question

Choose the correct answer.

If f(x) = |3 – x| + (3 + x), where (x) denotes the least integer greater than or equal to x, then f(x) is

A. continuous and differentiable at x = 3

B. continuous but not differentiable at x = 3

C. differentiable but not continuous at x = 3

D. neither differentiable nor continuous at x = 3

Answer

Given that f(x) = |3 – x| + (3 + x), where (x) denotes the least integer greater than or equal to x.

$$f(x) = \begin{cases} 3 - x + 3 + 3, 2 < x < 3 \\ x - 3 + 3 + 4, 3 < x < 4 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 9 - x, 2 < x < 3 \\ x + 4, 3 < x < 4 \end{cases}$$

Checking continuity at x =3,

Here, LHL at x = 3

$\lim_{x \to 3^-} 9 - x = 6$

RHL at x = 3

$\lim_{x \to 3^+} x + 4 = 7$

∵ LHL ≠ RHL

∴ f(x) is neither continuous nor differentiable at x = 3.

23. Question

Choose the correct answer.

If $f(x) = \begin{cases} \frac{1}{1+e^{1/x}} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$, then f(x) is

A. continuous as well as differentiable at x = 0

B. continuous but not differentiable at x = 0

C. differentiable but not continuous at x = 0

D. none of these

Answer

Given that $f(x) = \begin{Bmatrix} \frac{1}{1+e^{\frac{1}{x}}}, x \neq 0 \\ 0, x = 0 \end{Bmatrix}$

Checking continuity at x = 0,

LHL:

$\lim_{x \to 0^-} \frac{1}{1 + e^{\frac{1}{x}}} = 1$

But f(x =0) =0

Hence, function is neither continuous nor differentiable at x =0.

So, option D is correct.

24. Question

Choose the correct answer.

If $f(x) = \begin{cases} \frac{1-\cos x}{x \sin x} & , x \neq 0 \\ \frac{1}{2} & , x = 0 \end{cases}$, then at x = 0, f(x) is

A. continuous and differentiable

B. differentiable but not continuous

C. continuous but not differentiable

D. neither continuous nor differentiable

Answer

Given that $f(x) = \left\{\begin{matrix} \frac{1-\cos x}{x\sin x}, x \neq 0 \\ \frac{1}{2}, x = 0 \end{matrix}\right\}$

Checking continuity and differentiability at x = 0,

LHL:

$$\lim_{x\to 0^-}\frac{1-\cos x}{x\sin x} = \lim_{x\to 0^-}\frac{1-\cos x}{x\sin x}\times\frac{x}{x} = \lim_{x\to 0^-}\frac{1-\cos x}{x^2}\times\frac{x}{\sin x} = \frac{1}{2}$$

LHL = f(x=0)

Hence, f is continuous at x = 0.

LHD at x =0,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{h\to 0}\frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0}\frac{\frac{1-\cos(-h)}{(-h)\sin(-h)} - \left(\frac{1}{2}\right)}{-h} = 0$$

RHD at x =0,

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{0+h-0}$$

$$= \lim_{h\to 0}\frac{\frac{1-\cos(h)}{(h)\sin(h)} - \left(\frac{1}{2}\right)}{h} = 0$$

∵ LHD = RHD = f(0)

∴ f(x) is differentiable at x =0.

So, option A is correct.

25. Question

Choose the correct answer.

The set of points where the function f(x) given by f(x) = |x – 3| cos x is differential then,

A. R

B. R – {3}

C. (0, ∞)

D. none of these

Answer

Given that the function f(x) given by f(x) = |x – 3| cos x is differential then

LHD at x =3,

$$\lim_{x\to 3^-}\frac{f(x)-f(3)}{x-3} = \lim_{h\to 0}\frac{f(3-h)-f(3)}{3-h-3}$$

$$= \lim_{h\to 0}\frac{h\cos(3-h)-0}{-h} = -\cos 3$$

RHD at x =3,

$$\lim_{x\to 3^+}\frac{f(x)-f(3)}{x-3} = \lim_{h\to 0}\frac{f(3+h)-f(3)}{3+h-3}$$

$= \lim_{h \to 0} \frac{h\cos(3+h) - 0}{h} = \cos 3$

∵ LHD ≠ RHD

∴ f(x) is not differentiable at x = 3 but since it is a product of modular and cosine function, it is differentiable at all other points.

Hence, option B is correct.

26. Question

Choose the correct answer.

Let $f(x) = \begin{cases} 1, & x \leq -1 \\ |x|, & -1 < x < 1 \\ 0, & x \geq 1 \end{cases}$. Then, f is

A. continuous at x = –1

B. differentiable at x = –1

C. everywhere continuous

D. everywhere differentiable

Answer

Given that $f(x) = \begin{Bmatrix} 1, x \leq -1 \\ |x|, -1 < x < 1 \\ 0, x \geq 1 \end{Bmatrix}$

Checking the continuity at x = -1:

LHL at x =-1,

$\lim_{x \to -1^-} f(x) = \lim_{h \to 0} f(-1-h) = \lim_{h \to 0} 1 = 1$

RHL at x =-1,

$\lim_{x \to -1^+} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} (1+h) = 1$

Hence, f(x) is continuous at x =-1.

Checking the differentiability at x =-1:

LHD at x =-1,

$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-1-h-(-1)}$

$= \lim_{h \to 0} \frac{1-1}{-h} = 0$

RHD at x =-1,

$\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x - 1} = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{(-1)+h-(-1)}$

$= \lim_{h \to 0} \frac{1-1}{h} = 0$

∵ LHD = RHD

Hence, options A and B are correct.