11. Differentiation

Exercise 11.1

1. Question

Differentiate the following functions from first principles :

e^{-x}

Answer

We have to find the derivative of e^{-x} with the first principle method, so,

$f(x) = e^{-x}$

by using the first principle formula, we get,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{-(x+h)}-e^{-x}}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{-x}(e^{-h}-1)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{-x}(e^{-h}-1)(-1)}{h(-1)}$

[By using $\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$]

$f'(x) = -e^{-x}$

2. Question

Differentiate the following functions from first principles :

e^{3x}

Answer

We have to find the derivative of e^{3x} with the first principle method, so,

$f(x) = e^{3x}$

by using the first principle formula, we get,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{3(x+h)}-e^{3x}}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{3x}(e^{3h}-1)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{3x}(e^{3h}-1)3}{3h}$

[By using $\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$]

$f'(x) = 3e^{3x}$

3. Question

Differentiate the following functions from first principles :

$e^{ax + b}$

Answer

We have to find the derivative of e^{ax+b} with the first principle method, so,

$f(x) = e^{ax+b}$

by using the first principle formula, we get,

$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to0} \frac{e^{a(x+h)+b}-e^{ax+b}}{h}$

$f'(x) = \lim_{h\to0} \frac{e^{ax+b}(e^{ah}-1)a}{ah}$

[By using $\lim_{x\to0} \frac{e^{x}-1}{x} = 1$]

$f'(x) = a\, e^{ax+b}$

4. Question

Differentiate the following functions from first principles :

$e^{\cos x}$

Answer

We have to find the derivative of $e^{\cos x}$ with the first principle method, so,

$f(x) = e^{\cos x}$

by using the first principle formula, we get,

$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to0} \frac{e^{\cos(x+h)}-e^{\cos x}}{h}$

$f'(x) = \lim_{h\to0} \frac{e^{\cos x}(e^{\cos(x+h)-\cos x}-1)}{h}$

$f'(x) = \lim_{h\to0} \frac{e^{\cos x}(e^{\cos(x+h)-\cos x}-1)}{\cos(x+h)-\cos x}\ \frac{\cos(x+h)-\cos x}{h}$

[By using $\lim_{x\to0} \frac{e^{x}-1}{x} = 1$]

$f'(x) = \lim_{h\to0} e^{\cos x}\ \frac{\cos(x+h)-\cos x}{h}$

$f'(x) = \lim_{h\to0} e^{\cos x}\ \frac{\cos x \cos h-\sin x \sin h-\cos x}{h}$

[By using $\cos(x+h) = \cos x \cosh - \sin x \sinh$]

$f'(x) = \lim_{h\to0} e^{\cos x}\ [\frac{\cos x(\cos h-1)}{h} - \frac{\sin x \sin h}{h}]$

[By using $\lim_{x\to0} \frac{\sin x}{x} = 1$ and

$\cos 2x = 1–2\sin^2 x$]

$f'(x) = \lim_{h\to0} e^{\cos x}\ [\frac{\cos x\left(-2\sin^2\frac{h}{2}\right)(\frac{h}{4})}{h(\frac{h}{4})} - \sin x]$

f '(x) = $\lim_{h\to 0} e^{\cos x}\ [\frac{\cos x\left(-2\sin^2\frac{h}{2}\right)(\frac{h}{4})}{\frac{h^2}{2^2}} - \sin x]$

f '(x) = $-e^{\cos x} \sin x$

5. Question

Differentiate the following functions from first principles :

$e^{\sqrt{2x}}$

Answer

We have to find the derivative of $e^{\sqrt{2x}}$ with the first principle method, so,

f(x) = $e^{\sqrt{2x}}$

by using the first principle formula, we get,

f '(x) = $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{2(x+h)}}-e^{\sqrt{2x}}}{h}$

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{2x}}(e^{\sqrt{2(x+h)}-\sqrt{2x}}-1)}{h}$

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{2x}}(e^{\sqrt{2(x+h)}-\sqrt{2x}}-1)}{h} \times \frac{\sqrt{2(x+h)}-\sqrt{2x}}{\sqrt{2(x+h)}-\sqrt{2x}}$

[By using $\lim_{x\to 0} \frac{e^x-1}{x}$ = 1]

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{2x}}}{h} \times (\sqrt{2(x+h)} - \sqrt{2x}\,) \times \frac{\sqrt{2(x+h)}+\sqrt{2x}}{\sqrt{2(x+h)}+\sqrt{2x}}$

[By rationalising]

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{2x}}}{h} \times \frac{(2(x+h)-2x)}{\sqrt{2(x+h)}+\sqrt{2x}}$

f '(x) = $\frac{e^{\sqrt{2x}}}{\sqrt{2x}}$

6. Question

Differentiate each of the following functions from the first principal :

log cos x

Answer

We have to find the derivative of log cosx with the first principle method, so,

f(x) = log cos x

by using the first principle formula, we get,

f '(x) = $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

f '(x) = $\lim_{h\to 0} \frac{\log\cos(x+h)-\log\cos x}{h}$

f '(x) = $\lim_{h\to 0} \frac{\log(\frac{\cos(x+h)}{\cos x})}{h}$

f '(x) = $\lim_{h\to 0} \frac{\log(1+\frac{\cos(x+h)}{\cos x}-1)}{h}$

[Adding and subtracting 1]

f '(x) = $\lim_{h\to 0} \frac{\log(1+\frac{\cos(x+h)-\cos x}{\cos x})}{h}$

[Rationalising]

f '(x) = $\lim_{h\to 0} \frac{\log(1+\frac{\cos(x+h)-\cos x}{\cos x})}{h} \times \frac{\frac{\cos(x+h)-\cos x}{\cos x}}{\frac{\cos(x+h)-\cos x}{\cos x}}$

[By using $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$]

f '(x) = $\lim_{h\to 0} \frac{\frac{\cos(x+h)-\cos x}{\cos x}}{h}$

[cosC – cosD = –2 $\sin\frac{C-D}{2} \sin\frac{C+D}{2}$]

f '(x) = $\lim_{h\to 0} \frac{\frac{-2\sin\frac{2x+h}{2}\sin\frac{h}{2}}{\cos x}}{\frac{2h}{2}}$ [By using $\lim_{x\to 0} \frac{\sin x}{x} = 1$]

f '(x) = $\lim_{h\to 0} \frac{-2\sin\frac{2x+h}{2}}{2\cos x}$

f '(x) = – tan x

7. Question

Differentiate each of the following functions from the first principal :

$e^{\sqrt{\cot x}}$

Answer

We have to find the derivative of $e^{\sqrt{\cot x}}$ with the first principle method, so,

f(x) = $e^{\sqrt{\cot x}}$

by using the first principle formula, we get,

f '(x) = $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{\cot(x+h)}}-e^{\sqrt{\cot x}}}{h}$

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{\cot x}}(e^{\sqrt{\cot(x+h)}-\sqrt{\cot x}}-1)}{h}$

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{\cot x}}(e^{\sqrt{\cot(x+h)}-\sqrt{\cot x}}-1)}{h} \times \frac{(\sqrt{\cot(x+h)}-\sqrt{\cot x})}{\sqrt{\cot(x+h)}-\sqrt{\cot x}}$

[By using $\lim_{x\to 0} \frac{e^x-1}{x} = 1$]

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{\cot x}}}{h} \times (\sqrt{\cot(x+h)} - \sqrt{\cot x}) \times \frac{(\sqrt{\cot(x+h)}+\sqrt{\cot x})}{\sqrt{\cot(x+h)}+\sqrt{\cot x}}$

[Rationalizing]

f '(x) = $\lim_{h\to 0} \frac{e^{\sqrt{\cot x}}}{h} \times (\cot(x+h) - \cot x) \times \frac{1}{\sqrt{\cot(x+h)}+\sqrt{\cot x}}$

$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{\cot x}}}{h} \times \frac{\cos(x+h)\sin x - \sin(x+h)\cos x}{\sin x \ \sin(x+h)} \times \frac{1}{\sqrt{\cot(x+h)}+\sqrt{\cot x}}$

[sinA cosB – cosA sinB = sin(A–B)]

$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{\cot x}}}{h} \times \frac{\sin(x-x-h)}{\sin x \ \sin(x+h)} \times \frac{1}{\sqrt{\cot(x+h)}+\sqrt{\cot x}}$

[By using $\lim_{x \to 0} \frac{\sin x}{x} = 1$]

$f'(x) = \lim_{h \to 0} \frac{e^{\sqrt{\cot x}}}{\sin x \ \sin(x+h)} \times \frac{-1}{\sqrt{\cot(x+h)}+\sqrt{\cot x}}$

$f'(x) = \frac{-\text{cosec}^2 x \, e^{\sqrt{\cot x}}}{2\sqrt{\cot x}}$

8. Question

Differentiate each of the following functions from the first principal :

$x^2 \ e^x$

Answer

We have to find the derivative of x^2e^x with the first principle method, so,

$f(x) = x^2e^x$

by using the first principle formula, we get,

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 e^{(x+h)} - x^2 e^x}{h}$

$f'(x) = \lim_{h \to 0} \frac{(x^2+h^2+2hx)e^{(x+h)} - x^2 e^x}{h}$

[By using $(a+b)^2 = a^2+b^2+2ab$]

$f'(x) = \lim_{h \to 0} \frac{x^2 e^x\{(h^2+2hx+1)e^{(h)}-1\}}{h}$

$f'(x) = \lim_{h \to 0} \frac{x^2 e^x(e^h-1)}{h} + \lim_{h \to 0} \frac{e^{(x+h)}[h^2+2hx]}{h}$

[By using $\lim_{x \to 0} \frac{e^x-1}{x} = 1$]

$f'(x) = x^2e^x + \lim_{h \to 0} e^{(x+h)} [h+2x]$

$f'(x) = x^2e^x + 2x \ e^x$

9. Question

Differentiate each of the following functions from the first principal :

log cosec x

Answer

We have to find the derivative of log cosec x with the first principle method, so,

$f(x) = \log \text{cosec} x$

by using the first principle formula, we get,

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

f ‘(x) = $\lim_{h\to 0}\frac{\log \operatorname{cosec}(x+h)-\log \operatorname{cosec} x}{h}$

f ‘(x) = $\lim_{h\to 0}\frac{\log \sin x-\log \sin(x+h)}{h}$

f ‘(x) = $\lim_{h\to 0}\frac{\log\frac{\sin x}{\sin(x+h)}}{h}$

[By using log a – log b = log $\frac{a}{b}$]

f ‘(x) = $\lim_{h\to 0}\frac{\log[1+\frac{\sin x}{\sin(x+h)}-1]}{h}$

[adding and subtracting 1]

f ‘(x) = $\lim_{h\to 0}\frac{\log[1+\frac{\sin x-\sin(x+h)}{\sin(x+h)}]}{h}$

f ‘(x) = $\lim_{h\to 0}\frac{\log[1+\frac{\sin x-\sin(x+h)}{\sin(x+h)}]}{h}\times\frac{\frac{\sin x-\sin(x+h)}{\sin(x+h)}}{\frac{\sin x-\sin(x+h)}{\sin(x+h)}}$

[Rationalising]

f ‘(x) = $\lim_{h\to 0}\frac{\sin x-\sin(x+h)}{h\sin(x+h)}$

f ‘(x) = $\lim_{h\to 0}\frac{2\cos\frac{2x+h}{2}\sin\frac{-h}{2}}{h\sin(x+h)}$

[sin C – sin D = 2 sin $\frac{C-D}{2}$ cos$\frac{C+D}{2}$]

f ‘(x) = $\lim_{h\to 0}\frac{-2\cos\frac{2x+h}{2}\sin\frac{-h}{2}}{(-1)h\sin(x+h)}$

[By using $\lim_{x\to 0}\frac{\sin x}{x}$ = 1]

f ‘(x) = – cot x

10. Question

Differentiate each of the following functions from the first principal :

$\sin^{-1}(2x + 3)$

Answer

We have to find the derivative of $\sin^{-1}(2x+3)$ with the first principle method, so,

f(x) = $\sin^{-1}(2x+3)$

by using the first principle formula, we get,

f ‘(x) = $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

f ‘(x) = $\lim_{h\to 0}\frac{\sin^{-1}(2[x+h]+3)-\sin^{-1}(2x+3)}{h}$

Let $\sin^{-1}[2(x+h)+3]$ = A and $\sin^{-1}(2x+3)$ = B, so,

sinA = [2(x+h)+3] and sinB = (2x+3),

2h = sinA – sinB, when h→0 then sinA→sinB we can also say that **A→B** and hence A–B→0,

f ‘(x) = $\lim_{A-B\to 0}\frac{2(A-B)}{\sin A-\sin B}$

f ‘(x) = $\lim_{A-B\to 0}\frac{2(A-B)}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}$

[sinC – sinD = 2 sin $\frac{C-D}{2}$ cos$\frac{C+D}{2}$]

f ‘(x) = $\lim_{A-B\to 0}\frac{2}{1\cos\frac{A+B}{2}}$

[By using $\lim_{x\to 0}\frac{\sin x}{x}$ = 1]

f ‘(x) = $\frac{2}{\cos B}$

f ‘(x) = $\frac{2}{\cos[\sin^{-1}(2x+3)]}$

[By using Pythagoras theorem, in which H = 1 and P = 2x+3, so, we have to find B, which comes out to be $\sqrt{1-(2x+3)^2}$ by the relation $H^2 = P^2 + B^2$]

f ‘(x) = $\frac{2}{\sqrt{1-(2x+3)^2}}$

Exercise 11.2

1. Question

Differentiate the following functions with respect to x:

sin(3x + 5)

Answer

Let y = sin(3x + 5)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(3x+5)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\frac{d}{dx}(3x+5)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\left[\frac{d}{dx}(3x) + \frac{d}{dx}(5)\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\left[3\frac{d}{dx}(x) + \frac{d}{dx}(5)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\,[3\times 1 + 0]$$

$$\therefore \frac{dy}{dx} = 3\cos(3x+5)$$

Thus, $\frac{d}{dx}[\sin(3x+5)] = 3\cos(3x+5)$

2. Question

Differentiate the following functions with respect to x:

$\tan^2 x$

Answer

Let y = $\tan^2 x$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^2 x)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2\tan^{2-1} x \frac{d}{dx}(\tan x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = 2\tan x \frac{d}{dx}(\tan x)$$

However, $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = 2\tan x\,(\sec^2 x)$$

$$\therefore \frac{dy}{dx} = 2\tan x \sec^2 x$$

Thus, $\frac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$

3. Question

Differentiate the following functions with respect to x:

tan(x° + 45°)

Answer

Let y = tan(x° + 45°)

First, we will convert the angle from degrees to radians.

We have $1^\circ = \left(\frac{\pi}{180}\right)^c \Rightarrow (x+45)^\circ = \left[\frac{(x+45)\pi}{180}\right]^c$

$$\Rightarrow y = \tan\left[\frac{(x+45)\pi}{180}\right]$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\tan\left[\frac{(x+45)\pi}{180}\right]\right\}$$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$

$\Rightarrow \frac{dy}{dx} = \sec^2\left[\frac{(x+45)\pi}{180}\right]\frac{d}{dx}\left[\frac{(x+45)\pi}{180}\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \sec^2(x^\circ + 45^\circ)\frac{\pi}{180}\frac{d}{dx}(x+45)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)\left[\frac{d}{dx}(x) + \frac{d}{dx}(45)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)\,[1+0]$$

$$\therefore \frac{dy}{dx} = \frac{\pi}{180}\sec^2(x^\circ + 45^\circ)$$

Thus, $\frac{d}{dx}[\tan(x° + 45°)] = \frac{\pi}{180}\sec^2(x° + 45°)$

4. Question

Differentiate the following functions with respect to x:

sin(log x)

Answer

Let y = sin(log x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x)$ [using chain rule]

However, $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = \cos(\log x) \times \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x}\cos(\log x)$$

Thus, $\frac{d}{dx}[\sin(\log x)] = \frac{1}{x}\cos(\log x)$

5. Question

Differentiate the following functions with respect to x:

$e^{\sin\sqrt{x}}$

Answer

Let $y = e^{\sin\sqrt{x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin\sqrt{x}}\right)$$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\frac{d}{dx}\left(\sin\sqrt{x}\right)$ [using chain rule]

We have $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\cos\sqrt{x}\frac{d}{dx}\left(\sqrt{x}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\cos\sqrt{x}\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = e^{\sin\sqrt{x}}\cos\sqrt{x}\left[\frac{1}{2}x^{\left(\frac{1}{2}-1\right)}\right]$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} e^{\sin\sqrt{x}} \cos\sqrt{x}\, x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} e^{\sin\sqrt{x}} \cos\sqrt{x}$

Thus, $\frac{d}{dx}\left(e^{\sin\sqrt{x}}\right) = \frac{1}{2\sqrt{x}} e^{\sin\sqrt{x}} \cos\sqrt{x}$

6. Question

Differentiate the following functions with respect to x:

$e^{\tan x}$

Answer

Let y = $e^{\tan x}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}(e^{\tan x})$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\tan x} \frac{d}{dx}(\tan x)$ [using chain rule]

We have $\frac{d}{dx}(\tan x) = \sec^2 x$

$\therefore \frac{dy}{dx} = e^{\tan x} \sec^2 x$

Thus, $\frac{d}{dx}(e^{\tan x}) = e^{\tan x} \sec^2 x$

7. Question

Differentiate the following functions with respect to x:

$\sin^2(2x + 1)$

Answer

Let y = $\sin^2(2x + 1)$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}[\sin^2(2x+1)]$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2\sin^{2-1}(2x+1) \frac{d}{dx}[\sin(2x+1)]$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = 2\sin(2x+1) \frac{d}{dx}[\sin(2x+1)]$

We have $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = 2\sin(2x+1)\cos(2x+1) \frac{d}{dx}(2x+1)$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = \sin[2(2x+1)] \frac{d}{dx}(2x+1)$ [$\because$ sin(2θ) = 2sinθcosθ]

$\Rightarrow \frac{dy}{dx} = \sin(4x+2)\left[\frac{d}{dx}(2x) + \frac{d}{dx}(1)\right]$

$\Rightarrow \frac{dy}{dx} = \sin(4x+2)\left[2\frac{d}{dx}(x) + \frac{d}{dx}(1)\right]$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$\Rightarrow \frac{dy}{dx} = \sin(4x+2)\,[2 \times 1 + 0]$

$\therefore \frac{dy}{dx} = 2\sin(4x+2)$

Thus, $\frac{d}{dx}[\sin^2(2x+1)] = 2\sin(4x+2)$

8. Question

Differentiate the following functions with respect to x:

$\log_7(2x - 3)$

Answer

Let $y = \log_7(2x - 3)$

Recall that $\log_a b = \frac{\log b}{\log a}$.

$\Rightarrow \log_7(2x-3) = \frac{\log(2x-3)}{\log 7}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\log(2x-3)}{\log 7}\right]$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\log 7}\right)\frac{d}{dx}[\log(2x-3)]$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\log 7}\right)\left(\frac{1}{2x-3}\right)\frac{d}{dx}(2x-3)$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x-3)\log 7}\left[\frac{d}{dx}(2x) - \frac{d}{dx}(3)\right]$

$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x-3)\log 7}\left[2\frac{d}{dx}(x) - \frac{d}{dx}(3)\right]$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$\Rightarrow \frac{dy}{dx} = \frac{1}{(2x-3)\log 7}[2 \times 1 - 0]$

$\therefore \frac{dy}{dx} = \frac{2}{(2x-3)\log 7}$

Thus, $\frac{d}{dx}[\log_7(2x-3)] = \frac{2}{(2x-3)\log 7}$

9. Question

Differentiate the following functions with respect to x:

tan(5x°)

Answer

Let y = tan(5x°)

First, we will convert the angle from degrees to radians.

We have $1° = \left(\frac{\pi}{180}\right)^c \Rightarrow 5x° = 5x \times \frac{\pi}{180}^c$

$\Rightarrow\ y = \tan\left(5x \times \frac{\pi}{180}\right)$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left[\tan\left(5x \times \frac{\pi}{180}\right)\right]$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$

$\Rightarrow \frac{dy}{dx} = \sec^2\left(5x \times \frac{\pi}{180}\right)\frac{d}{dx}\left(5x \times \frac{\pi}{180}\right)$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = \sec^2(5x°)\frac{\pi}{180}\frac{d}{dx}(5x)$

$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(5x°)\left[5\frac{d}{dx}(x)\right]$

However, $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{dy}{dx} = \frac{\pi}{180}\sec^2(5x°)\,[5]$

$\therefore \frac{dy}{dx} = \frac{5\pi}{180}\sec^2(5x°)$

Thus, $\frac{d}{dx}(\tan 5x°) = \frac{5\pi}{180}\sec^2(5x°)$

10. Question

Differentiate the following functions with respect to x:

2^{x^3}

Answer

Let $y = 2^{x^3}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(2^{x^3}\right)$

We know $\frac{d}{dx}(a^x) = a^x \log a$

$\Rightarrow \frac{dy}{dx} = 2^{x^3}\log 2\frac{d}{dx}(x^3)$ [using chain rule]

We have $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2^{x^3}\log 2 \times 3x^{3-1}$

$\Rightarrow \frac{dy}{dx} = 2^{x^3} \log 2 \times 3x^2$

$\therefore \frac{dy}{dx} = 2^{x^3} 3x^2 \log 2$

Thus, $\frac{d}{dx}\left(2^{x^3}\right) = 2^{x^3} 3x^2 \log 2$

11. Question

Differentiate the following functions with respect to x:

3^{e^x}

Answer

Let $y = 3^{e^x}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(3^{e^x}\right)$

We know $\frac{d}{dx}(a^x) = a^x \log a$

$\Rightarrow \frac{dy}{dx} = 3^{e^x} \log 3 \frac{d}{dx}(e^x)$ [using chain rule]

We have $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = 3^{e^x} \log 3 \times e^x$

$\therefore \frac{dy}{dx} = 3^{e^x} e^x \log 3$

Thus, $\frac{d}{dx}\left(3^{e^x}\right) = 3^{e^x} e^x \log 3$

12. Question

Differentiate the following functions with respect to x:

$\log_x 3$

Answer

Let $y = \log_x 3$

Recall that $\log_a b = \frac{\log b}{\log a}$.

$\Rightarrow \log_x 3 = \frac{\log 3}{\log x}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\log 3}{\log x}\right)$

$\Rightarrow \frac{dy}{dx} = \log 3 \frac{d}{dx}\left(\frac{1}{\log x}\right)$

$\Rightarrow \frac{dy}{dx} = \log 3 \frac{d}{dx}(\log x)^{-1}$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \log 3\,[-1 \times (\log x)^{-1-1}]\,\frac{d}{dx}(\log x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = -\log 3\,(\log x)^{-2}\frac{d}{dx}(\log x)$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = -\log 3\,(\log x)^{-2} \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x}\frac{\log 3}{(\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x}\frac{\log 3}{(\log x)^2} \times \frac{\log 3}{\log 3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\log 3}\frac{(\log 3)^2}{(\log x)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\log 3}\left(\frac{\log 3}{\log x}\right)^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x\log 3 \times \left(\frac{\log x}{\log 3}\right)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\log 3\,(\log_3 x)^2}$$

Thus, $\frac{d}{dx}(\log_x 3) = -\frac{1}{x\log 3(\log_3 x)^2}$

13. Question

Differentiate the following functions with respect to x:

3^{x^2+2x}

Answer

Let $y = 3^{x^2+2x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(3^{x^2+2x}\right)$$

We know $\frac{d}{dx}(a^x) = a^x\log a$

$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x}\log 3\,\frac{d}{dx}(x^2 + 2x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x}\log 3\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x}\log 3\left[\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x)\right]$$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$ and $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3\, [2x + 2 \times 1]$

$\Rightarrow \frac{dy}{dx} = 3^{x^2+2x} \log 3\, (2x + 2)$

$\therefore \frac{dy}{dx} = (2x+2)3^{x^2+2x} \log 3$

Thus, $\frac{d}{dx}\left(3^{x^2+2x}\right) = (2x+2)3^{x^2+2x}\log 3$

14. Question

Differentiate the following functions with respect to x:

$$\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$$

Answer

Let $y = \sqrt{\frac{a^2-x^2}{a^2+x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2-x^2}{a^2+x^2}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{a^2-x^2}{a^2+x^2}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \left[\frac{(a^2+x^2)\frac{d}{dx}(a^2-x^2) - (a^2-x^2)\frac{d}{dx}(a^2+x^2)}{(a^2+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \left[\frac{(a^2+x^2)\left(\frac{d}{dx}(a^2) - \frac{d}{dx}(x^2)\right) - (a^2-x^2)\left(\frac{d}{dx}(a^2) + \frac{d}{dx}(x^2)\right)}{(a^2+x^2)^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}} \left[\frac{(a^2+x^2)(0-2x) - (a^2-x^2)(0+2x)}{(a^2+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(a^2 + x^2) - 2x(a^2 - x^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(a^2 + x^2 + a^2 - x^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(2a^2)}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{a^2 - x^2}{a^2 + x^2}\right)^{-\frac{1}{2}}\left[\frac{-2xa^2}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{-\frac{1}{2}}}\left[\frac{-2xa^2}{(a^2 + x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 + x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xa^2}{(a^2 + x^2)^{\frac{3}{2}}(a^2 - x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{-2xa^2}{(a^2 + x^2)^{\frac{3}{2}}\sqrt{a^2 - x^2}}$$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}\right) = \frac{-2xa^2}{(a^2+x^2)^{\frac{3}{2}}\sqrt{a^2 - x^2}}$

15. Question

Differentiate the following functions with respect to x:

$3^{x \log x}$

Answer

Let $y = 3^{x \log x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(3^{x \log x})$$

We know $\frac{d}{dx}(a^x) = a^x \log a$

$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \frac{d}{dx}(x \log x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \frac{d}{dx}(x \times \log x)$$

Recall that (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = 3^{x \log x} \log 3 \left[\log x \frac{d}{dx}(x) + x \frac{d}{dx}(\log x)\right]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \left[\log x \times 1 + x \times \frac{1}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3^{x\log x} \log 3 \,[\log x + 1]$$

$$\therefore \frac{dy}{dx} = (1 + \log x)3^{x\log x} \log 3$$

Thus, $\frac{d}{dx}\left(3^{x\log x}\right) = (1 + \log x)3^{x\log x} \log 3$

16. Question

Differentiate the following functions with respect to x:

$$\sqrt{\frac{1+\sin\ x}{1-\sin\ x}}$$

Answer

Let $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{1+\sin x}{1-\sin x}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{1+\sin x}{1-\sin x}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}} \left[\frac{(1-\sin x)\frac{d}{dx}(1+\sin x) - (1+\sin x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}} \left[\frac{(1-\sin x)\left(\frac{d}{dx}(1) + \frac{d}{dx}(\sin x)\right) - (1+\sin x)\left(\frac{d}{dx}(1) - \frac{d}{dx}(\sin x)\right)}{(1-\sin x)^2}\right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x)(0+\cos x)-(1+\sin x)(0-\cos x)}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x)\cos x+(1+\sin x)\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{(1-\sin x+1+\sin x)\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{2\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}\left[\frac{\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)^{-\frac{1}{2}}}{(1-\sin x)^{-\frac{1}{2}}}\left[\frac{\cos x}{(1-\sin x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)^{-\frac{1}{2}}\cos x}{(1-\sin x)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)^{-\frac{1}{2}}\cos x}{(1-\sin x)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)^{1+\frac{1}{2}}(1+\sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)(1-\sin x)^{\frac{1}{2}}(1+\sin x)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\sqrt{(1-\sin x)(1+\sin x)}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\sqrt{1-\sin^2 x}}$$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\sqrt{\cos^2 x}}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{(1-\sin x)\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\sin x}{1-\sin^2 x}$$

$\Rightarrow \frac{dy}{dx} = \frac{1+\sin x}{\cos^2 x}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}$$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\cos x}\right)^2 + \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$

$\Rightarrow \frac{dy}{dx} = \sec^2 x + \sec x \tan x$

$\therefore \frac{dy}{dx} = \sec x\,(\sec x + \tan x)$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right) = \sec x\,(\sec x + \tan x)$

17. Question

Differentiate the following functions with respect to x:

$\sqrt{\frac{1-x^2}{1+x^2}}$

Answer

Let $y = \sqrt{\frac{1-x^2}{1+x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{(1+x^2)\frac{d}{dx}(1-x^2) - (1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$
$$= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{(1+x^2)\left(\frac{d}{dx}(1) - \frac{d}{dx}(x^2)\right) - (1-x^2)\left(\frac{d}{dx}(1) + \frac{d}{dx}(x^2)\right)}{(1+x^2)^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{(1+x^2)(0-2x) - (1-x^2)(0+2x)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(1+x^2+1-x^2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x(2)}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-x^2}{1+x^2}\right)^{-\frac{1}{2}}\left[\frac{-2x}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{-\frac{1}{2}}}\left[\frac{-2x}{(1+x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x(1-x^2)^{-\frac{1}{2}}}{(1+x^2)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{(1+x^2)^{\frac{3}{2}}\sqrt{1-x^2}}$$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{1-x^2}{1+x^2}}\right) = \frac{-2x}{(1+x^2)^{\frac{3}{2}}\sqrt{1-x^2}}$

18. Question

Differentiate the following functions with respect to x:

$(\log \sin x)^2$

Answer

Let $y = (\log \sin x)^2$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(\log(\sin x))^2]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2(\log(\sin x))^{2-1}\frac{d}{dx}[\log(\sin x)]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = 2\log(\sin x)\frac{d}{dx}[\log(\sin x)]$$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = 2\log(\sin x)\left[\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin x}\log(\sin x)\frac{d}{dx}(\sin x)$$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin x}\log(\sin x)\cos x$$

$$\Rightarrow \frac{dy}{dx} = 2\left(\frac{\cos x}{\sin x}\right)\log(\sin x)$$

$$\therefore \frac{dy}{dx} = 2\cot x \log(\sin x)$$

Thus, $\frac{d}{dx}[(\log(\sin x))^2] = 2\cot x \log(\sin x)$

19. Question

Differentiate the following functions with respect to x:

$$\sqrt{\frac{1+x}{1-x}}$$

Answer

Let $y = \sqrt{\frac{1+x}{1-x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)\left(\frac{d}{dx}(1) + \frac{d}{dx}(x)\right) - (1+x)\left(\frac{d}{dx}(1) - \frac{d}{dx}(x)\right)}{(1-x)^2}\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)(0+1)-(1+x)(0-1)}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{(1-x)+(1+x)}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{2}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}}\left[\frac{1}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{-\frac{1}{2}}}\left[\frac{1}{(1-x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{-\frac{1}{2}+2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+x)^{-\frac{1}{2}}}{(1-x)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$$

Thus, $\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{1}{(1-x)^{\frac{3}{2}}\sqrt{1+x}}$

20. Question

Differentiate the following functions with respect to x:

$$\sin\left(\frac{1+x^2}{1-x^2}\right)$$

Answer

Let $y = \sin\left(\frac{1+x^2}{1-x^2}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\frac{1+x^2}{1-x^2}\right)\right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$ [using chain rule]

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1-x^2)\frac{d}{dx}(1+x^2)-(1+x^2)\frac{d}{dx}(1-x^2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1-x^2)\left(\frac{d}{dx}(1)+\frac{d}{dx}(x^2)\right)-(1+x^2)\left(\frac{d}{dx}(1)-\frac{d}{dx}(x^2)\right)}{(1-x^2)^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{(1-x^2)(0+2x)-(1+x^2)(0-2x)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{2x(1-x^2)+2x(1+x^2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{2x(1-x^2+1+x^2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{2x(2)}{(1-x^2)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{1+x^2}{1-x^2}\right)\left[\frac{4x}{(1-x^2)^2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{4x}{(1-x^2)^2}\cos\left(\frac{1+x^2}{1-x^2}\right)$$

Thus, $\frac{d}{dx}\left[\sin\left(\frac{1+x^2}{1-x^2}\right)\right] = \frac{4x}{(1-x^2)^2}\cos\left(\frac{1+x^2}{1-x^2}\right)$

21. Question

Differentiate the following functions with respect to x:

e^{3x} cos(2x)

Answer

Let y = e^{3x}cos(2x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{3x}\cos 2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^{3x} \times \cos 2x)$$

Recall that (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \cos 2x\frac{d}{dx}(e^{3x}) + e^{3x}\frac{d}{dx}(\cos 2x)$$

We know $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow \frac{dy}{dx} = \cos 2x\left[e^{3x}\frac{d}{dx}(3x)\right] + e^{3x}\left[-\sin 2x\frac{d}{dx}(2x)\right]$ [chain rule]

$$\Rightarrow \frac{dy}{dx} = e^{3x}\cos 2x\left[\frac{d}{dx}(3x)\right] - e^{3x}\sin 2x\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = e^{3x}\cos 2x\left[3\frac{d}{dx}(x)\right] - e^{3x}\sin 2x\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x}\cos 2x\left[\frac{d}{dx}(x)\right] - 2e^{3x}\sin 2x\left[\frac{d}{dx}(x)\right]$$

We have $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x}\cos 2x \times 1 - 2e^{3x}\sin 2x \times 1$$

$$\Rightarrow \frac{dy}{dx} = 3e^{3x}\cos 2x - 2e^{3x}\sin 2x$$

$$\therefore \frac{dy}{dx} = e^{3x}(3\cos 2x - 2\sin 2x)$$

Thus, $\frac{d}{dx}(e^{3x}\cos 2x) = e^{3x}(3\cos 2x - 2\sin 2x)$

22. Question

Differentiate the following functions with respect to x:

sin(log sin x)

Answer

Let y = sin(log sin x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log(\sin x))]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(\log(\sin x))\frac{d}{dx}[\log(\sin x)]$ [using chain rule]

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \cos(\log(\sin x))\left[\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}\cos(\log(\sin x))\frac{d}{dx}(\sin x)$$

However, $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}\cos(\log(\sin x))\cos x$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{\cos x}{\sin x}\right)\cos(\log(\sin x))$$

$$\therefore \frac{dy}{dx} = \cot x\cos(\log(\sin x))$$

Thus, $\frac{d}{dx}[\sin(\log(\sin x))] = \cot x\cos(\log(\sin x))$

23. Question

Differentiate the following functions with respect to x:

$e^{\tan 3x}$

Answer

Let $y = e^{\tan 3x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\tan 3x})$$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\tan 3x}\frac{d}{dx}(\tan 3x)$ [using chain rule]

We have $\frac{d}{dx}(\tan x) = \sec^2 x$

$\Rightarrow \frac{dy}{dx} = e^{\tan 3x}\sec^2 3x\frac{d}{dx}(3x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = 3e^{\tan 3x}\sec^2 3x\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = 3e^{\tan 3x}\sec^2 3x \times 1$$

$$\therefore \frac{dy}{dx} = 3e^{\tan 3x}\sec^2 3x$$

Thus, $\frac{d}{dx}(e^{\tan 3x}) = 3e^{\tan 3x}\sec^2 3x$

24. Question

Differentiate the following functions with respect to x:

$e^{\sqrt{\cot x}}$

Answer

Let $y = e^{\sqrt{\cot x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sqrt{\cot x}}\right)$$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}}\frac{d}{dx}\left(\sqrt{\cot x}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}}\frac{d}{dx}\left[(\cot x)^{\frac{1}{2}}\right]$$

We have $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = e^{\sqrt{\cot x}}\left[\frac{1}{2}(\cot x)^{\frac{1}{2}-1}\frac{d}{dx}(\cot x)\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}e^{\sqrt{\cot x}}(\cot x)^{-\frac{1}{2}}\frac{d}{dx}(\cot x)$$

However, $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}e^{\sqrt{\cot x}}(\cot x)^{-\frac{1}{2}}\operatorname{cosec}^2 x$$

$$\Rightarrow \frac{dy}{dx} = -\frac{e^{\sqrt{\cot x}}\operatorname{cosec}^2 x}{2(\cot x)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = -\frac{e^{\sqrt{\cot x}}\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$$

Thus, $\frac{d}{dx}\left(e^{\sqrt{\cot x}}\right) = -\frac{e^{\sqrt{\cot x}}\operatorname{cosec}^2 x}{2\sqrt{\cot x}}$

25. Question

Differentiate the following functions with respect to x:

$$\log\left(\frac{\sin x}{1+\cos x}\right)$$

Answer

Let $y = \log\left(\frac{\sin x}{1+\cos x}\right)$

$$\Rightarrow y = \log\left(\frac{\sin 2 \times \frac{x}{2}}{1 + \cos 2 \times \frac{x}{2}}\right)$$

We have sin2θ = 2sinθcosθ and 1 + cos2θ = 2cos^{2}θ.

$$\Rightarrow y = \log\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$\Rightarrow y = \log\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$\Rightarrow y = \log\left(\tan\frac{x}{2}\right)$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\tan\frac{x}{2}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1}{\tan\frac{x}{2}}\right)\frac{d}{dx}\left(\tan\frac{x}{2}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \cot\frac{x}{2}\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

We have $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \cot\frac{x}{2}\sec^2\frac{x}{2}\frac{d}{dx}\left(\frac{x}{2}\right)$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\cot\frac{x}{2}\sec^2\frac{x}{2}\frac{d}{dx}(x)$

However, $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\cot\frac{x}{2}\sec^2\frac{x}{2} \times 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \times \frac{1}{\cos^2\frac{x}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin 2\times\frac{x}{2}}$ [$\because$ sin2θ = 2sinθcosθ]

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}$

$\therefore \frac{dy}{dx} = \operatorname{cosec} x$

Thus, $\frac{d}{dx}\left[\log\left(\frac{\sin x}{1+\cos x}\right)\right] = \operatorname{cosec} x$

26. Question

Differentiate the following functions with respect to x:

$$\log\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Answer

Let $y = \log\sqrt{\frac{1-\cos x}{1+\cos x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}}\frac{d}{dx}\left[\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left[\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}}\frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}}\left(\frac{1-\cos x}{1+\cos x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\cos x}{1+\cos x}\right)^{-1}\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\frac{d}{dx}\left(\frac{1-\cos x}{1+\cos x}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)\left(\frac{d}{dx}(1)-\frac{d}{dx}(\cos x)\right)-(1-\cos x)\left(\frac{d}{dx}(1)+\frac{d}{dx}(\cos x)\right)}{(1+\cos x)^2}\right]$$

We know $\frac{d}{dx}(\cos x) = -\sin x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)(0+\sin x)-(1-\cos x)(0-\sin x)}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x)\sin x+(1-\cos x)\sin x}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{(1+\cos x+1-\cos x)\sin x}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\cos x}{1-\cos x}\right)\left[\frac{2\sin x}{(1+\cos x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{(1-\cos x)(1+\cos x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{1-\cos^2 x}$$

$\Rightarrow \frac{dy}{dx} = \frac{\sin x}{\sin^2 x}$ ($\because \sin^2\theta + \cos^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x}$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

Thus, $\frac{d}{dx}\left(\log\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \operatorname{cosec} x$

27. Question

Differentiate the following functions with respect to x:

$\tan(e^{\sin x})$

Answer

Let y = tan($e^{\sin x}$)

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\tan\left(e^{\sin x}\right)\right]$$

We know $\frac{d}{dx}(\tan x)=\sec^2 x$

$\Rightarrow \frac{dy}{dx}=\sec^2\left(e^{\sin x}\right)\frac{d}{dx}\left(e^{\sin x}\right)$ [using chain rule]

We have $\frac{d}{dx}(e^x)=e^x$

$\Rightarrow \frac{dy}{dx}=\sec^2\left(e^{\sin x}\right)e^{\sin x}\frac{d}{dx}(\sin x)$ [using chain rule]

However, $\frac{d}{dx}(\sin x)=\cos x$

$$\Rightarrow \frac{dy}{dx}=\sec^2\left(e^{\sin x}\right)e^{\sin x}\cos x$$

$$\therefore \frac{dy}{dx}=e^{\sin x}\cos x\sec^2\left(e^{\sin x}\right)$$

Thus, $\frac{d}{dx}\left[\tan\left(e^{\sin x}\right)\right]=e^{\sin x}\cos x\sec^2\left(e^{\sin x}\right)$

28. Question

Differentiate the following functions with respect to x:

$$\log\left(x+\sqrt{x^2+1}\right)$$

Answer

Let $y=\log\left(x+\sqrt{x^2+1}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(x+\sqrt{x^2+1}\right)\right]$$

We know $\frac{d}{dx}(\log x)=\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\left[\frac{d}{dx}(x)+\frac{d}{dx}\left(\sqrt{x^2+1}\right)\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\left[\frac{d}{dx}(x)+\frac{d}{dx}(x^2+1)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x)=1$ and $\frac{d}{dx}(x^n)=nx^{n-1}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}(x^2+1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+1)\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\left(\frac{d}{dx}(x^2)+\frac{d}{dx}(1)\right)\right]$$

However, $\frac{d}{dx}(x^2)=2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x+0)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1+x(x^2+1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{x}{\sqrt{x^2+1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left[\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

Thus, $\frac{d}{dx}\left[\log\left(x+\sqrt{x^2+1}\right)\right] = \frac{1}{\sqrt{x^2+1}}$

29. Question

Differentiate the following functions with respect to x:

$$\frac{e^x \log x}{x^2}$$

Answer

Let $y = \frac{e^x \log x}{x^2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x \log x}{x^2}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\frac{d}{dx}(e^x \log x) - (e^x \log x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\left[\log x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\log x)\right] - (e^x \log x)\frac{d}{dx}(x^2)}{x^4}$$

We know $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x^2) = 2x$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\left[\log x \times e^x + e^x \times \frac{1}{x}\right] - (e^x \log x) \times 2x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2)\left[e^x \log x + \frac{e^x}{x}\right] - 2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 e^x \log x + xe^x - 2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 e^x \log x}{x^4} + \frac{xe^x}{x^4} - \frac{2xe^x \log x}{x^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x \log x}{x^2} + \frac{e^x}{x^3} - \frac{2e^x \log x}{x^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{x^2}\left(\log x + \frac{1}{x} - \frac{2\log x}{x}\right)$$

$$\therefore \frac{dy}{dx} = e^x x^{-2}\left(\log x + \frac{1}{x} - \frac{2}{x}\log x\right)$$

Thus, $\frac{d}{dx}\left(\frac{e^x \log x}{x^2}\right) = e^x x^{-2}\left(\log x + \frac{1}{x} - \frac{2}{x}\log x\right)$

30. Question

Differentiate the following functions with respect to x:

log(cosec x – cot x)

Answer

Let y = log(cosec x – cot x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\operatorname{cosec} x - \cot x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}\frac{d}{dx}(\operatorname{cosec} x - \cot x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}\left[\frac{d}{dx}(\operatorname{cosec} x) - \frac{d}{dx}(\cot x)\right]$$

We know $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ and $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[-\operatorname{cosec} x \cot x - (-\operatorname{cosec}^2 x)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{cosec} x - \cot x}[(\operatorname{cosec} x - \cot x)\operatorname{cosec} x]$$

$$\therefore \frac{dy}{dx} = \operatorname{cosec} x$$

Thus, $\frac{d}{dx}[\log(\operatorname{cosec} x - \cot x)] = \operatorname{cosec} x$

31. Question

Differentiate the following functions with respect to x:

$$\frac{e^{ex} + e^{-2x}}{e^{2x} - e^{-2x}}$$

Answer

Let $y = \frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})\frac{d}{dx}(e^{2x} + e^{-2x}) - (e^{2x} + e^{-2x})\frac{d}{dx}(e^{2x} - e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})\left[\frac{d}{dx}(e^{2x}) + \frac{d}{dx}(e^{-2x})\right] - (e^{2x} + e^{-2x})\left[\frac{d}{dx}(e^{2x}) - \frac{d}{dx}(e^{-2x})\right]}{(e^{2x} - e^{-2x})^2}$$

We know $\frac{d}{dx}(e^x) = e^x$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})\left[e^{2x}\frac{d}{dx}(2x) + e^{-2x}\frac{d}{dx}(-2x)\right] - (e^{2x} + e^{-2x})\left[e^{2x}\frac{d}{dx}(2x) - e^{-2x}\frac{d}{dx}(-2x)\right]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})\left[2e^{2x}\frac{d}{dx}(x) - 2e^{-2x}\frac{d}{dx}(x)\right] - (e^{2x} + e^{-2x})\left[2e^{2x}\frac{d}{dx}(x) + 2e^{-2x}\frac{d}{dx}(x)\right]}{(e^{2x} - e^{-2x})^2}$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})[2e^{2x} \times 1 - 2e^{-2x} \times 1] - (e^{2x} + e^{-2x})[2e^{2x} \times 1 + 2e^{-2x} \times 1]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(e^{2x} - e^{-2x})[2e^{2x} - 2e^{-2x}] - (e^{2x} + e^{-2x})[2e^{2x} + 2e^{-2x}]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x})(e^{2x} - e^{-2x}) - 2(e^{2x} + e^{-2x})(e^{2x} + e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2[(e^{2x} - e^{-2x})^2 - (e^{2x} + e^{-2x})^2]}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(e^{2x} - e^{-2x} + e^{2x} + e^{-2x})(e^{2x} - e^{-2x} - e^{2x} - e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(2e^{2x})(-2e^{-2x})}{(e^{2x} - e^{-2x})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-8e^{2x+(-2x)}}{(e^{2x} - e^{-2x})^2}$$

$$\therefore \frac{dy}{dx} = \frac{-8}{(e^{2x} - e^{-2x})^2}$$

Thus, $\frac{d}{dx}\left(\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\right) = \frac{-8}{(e^{2x}-e^{-2x})^2}$

32. Question

Differentiate the following functions with respect to x:

$$\log\left(\frac{x^2+x+1}{x^2-x+1}\right)$$

Answer

Let $y=\log\left(\frac{x^2+x+1}{x^2-x+1}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\left(\frac{x^2+x+1}{x^2-x+1}\right)\right]$$

We know $\frac{d}{dx}(\log x)=\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\left(\frac{x^2+x+1}{x^2-x+1}\right)}\frac{d}{dx}\left(\frac{x^2+x+1}{x^2-x+1}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\frac{d}{dx}\left(\frac{x^2+x+1}{x^2-x+1}\right)$$

Recall that $\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{(x^2-x+1)\frac{d}{dx}(x^2+x+1)-(x^2+x+1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{(x^2-x+1)\left(\frac{d}{dx}(x^2)+\frac{d}{dx}(x)+\frac{d}{dx}(1)\right)-(x^2+x+1)\left(\frac{d}{dx}(x^2)-\frac{d}{dx}(x)+\frac{d}{dx}(1)\right)}{(x^2-x+1)^2}\right]$$

We know $\frac{d}{dx}(x^2)=2x$, $\frac{d}{dx}(x)=1$ and derivative of constant is 0.

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{(x^2-x+1)(2x+1+0)-(x^2+x+1)(2x-1+0)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2x(x^2-x+1)+(x^2-x+1)-2x(x^2+x+1)+(x^2+x+1)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2x(x^2-x+1-x^2-x-1)+(x^2-x+1+x^2+x+1)}{(x^2-x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x^2-x+1}{x^2+x+1}\right)\left[\frac{2x(-2x)+(2x^2+2)}{(x^2-x+1)^2}\right]$$

$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\left[\frac{-4x^2 + 2x^2 + 2}{(x^2 - x + 1)^2}\right]$

$\Rightarrow \frac{dy}{dx} = \left(\frac{x^2 - x + 1}{x^2 + x + 1}\right)\left[\frac{2 - 2x^2}{(x^2 - x + 1)^2}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{2 - 2x^2}{(x^2 + x + 1)(x^2 - x + 1)}$

$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{(x^2 + 1)^2 - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{(x^2 + 1)^2 - x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2(1 - x^2)}{x^4 + 2x^2 + 1 - x^2}$

$\therefore \frac{dy}{dx} = \frac{2(1 - x^2)}{x^4 + x^2 + 1}$

Thus, $\frac{d}{dx}\left[\log\left(\frac{x^2+x+1}{x^2-x+1}\right)\right] = \frac{2(1-x^2)}{x^4+x^2+1}$

33. Question

Differentiate the following functions with respect to x:

$\tan^{-1}(e^x)$

Answer

Let $y = \tan^{-1}(e^x)$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1} e^x)$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1+(e^x)^2}\frac{d}{dx}(e^x)$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + e^{2x}}\frac{d}{dx}(e^x)$

However, $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1 + e^{2x}} \times e^x$

$\therefore \frac{dy}{dx} = \frac{e^x}{1 + e^{2x}}$

Thus, $\frac{d}{dx}(\tan^{-1} e^x) = \frac{e^x}{1+e^{2x}}$

34. Question

Differentiate the following functions with respect to x:

$e^{\sin^{-1} 2x}$

Answer

Let $y = e^{\sin^{-1} 2x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sin^{-1} 2x}\right)$$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} 2x} \frac{d}{dx}(\sin^{-1} 2x)$ [using chain rule]

We have $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = e^{\sin^{-1} 2x} \frac{1}{\sqrt{1-(2x)^2}} \frac{d}{dx}(2x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times 2\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times \frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{dy}{dx} = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}}$$

Thus, $\frac{d}{dx}\left(e^{\sin^{-1} 2x}\right) = \frac{2e^{\sin^{-1} 2x}}{\sqrt{1-4x^2}}$

35. Question

Differentiate the following functions with respect to x:

$\sin(2\sin^{-1}x)$

Answer

Let $y = \sin(2\sin^{-1}x)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(2\sin^{-1} x)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(2\sin^{-1} x)\frac{d}{dx}(2\sin^{-1} x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \cos(2\sin^{-1} x) \times 2\frac{d}{dx}(\sin^{-1} x)$$

$$\Rightarrow \frac{dy}{dx} = 2\cos(2\sin^{-1} x)\frac{d}{dx}(\sin^{-1} x)$$

We have $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{dy}{dx} = 2\cos(2\sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}}$$

$\therefore \frac{dy}{dx} = \frac{2\cos(2\sin^{-1}x)}{\sqrt{1-x^2}}$

Thus, $\frac{d}{dx}[\sin(2\sin^{-1}x)] = \frac{2\cos(2\sin^{-1}x)}{\sqrt{1-x^2}}$

36. Question

Differentiate the following functions with respect to x:

$e^{\tan^{-1}\sqrt{x}}$

Answer

Let $y = e^{\tan^{-1}\sqrt{x}}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\tan^{-1}\sqrt{x}}\right)$

We know $\frac{d}{dx}(e^x) = e^x$

$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1}\sqrt{x}}\frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)$ [using chain rule]

We have $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = e^{\tan^{-1}\sqrt{x}}\frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x})$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{1+x}\left(\frac{1}{2\sqrt{x}}\right)$

$\therefore \frac{dy}{dx} = \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$

Thus, $\frac{d}{dx}\left(e^{\tan^{-1}\sqrt{x}}\right) = \frac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}(1+x)}$

37. Question

Differentiate the following functions with respect to x:

$\sqrt{\tan^{-1}\left(\frac{x}{2}\right)}$

Answer

Let $y = \sqrt{\tan^{-1}\frac{x}{2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\tan^{-1}\frac{x}{2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\left(\tan^{-1}\frac{x}{2}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right)$$

We have $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{d}{dx}\left(\frac{x}{2}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{1+\frac{x^2}{4}} \times \frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{4}{4+x^2} \times \frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{4+x^2} \times \frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{4+x^2} \times 1$$

$$\Rightarrow \frac{dy}{dx} = \left(\tan^{-1}\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{4+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(4+x^2)\left(\tan^{-1}\frac{x}{2}\right)^{\frac{1}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{(4+x^2)\sqrt{\tan^{-1}\frac{x}{2}}}$$

Thus, $\frac{d}{dx}\left(\sqrt{\tan^{-1}\frac{x}{2}}\right) = \frac{1}{(4+x^2)\sqrt{\tan^{-1}\frac{x}{2}}}$

38. Question

Differentiate the following functions with respect to x:

$\log(\tan^{-1}x)$

Answer

Let $y = \log(\tan^{-1}x)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\tan^{-1}x)]$$

We know $\frac{d}{dx}(\log x)=\frac{1}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{\tan^{-1}x}\frac{d}{dx}(\tan^{-1}x)$ [using chain rule]

We have $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\tan^{-1}x}\times\frac{1}{1+x^2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{(1+x^2)\tan^{-1}x}$$

Thus, $\frac{d}{dx}[\log(\tan^{-1}x)]=\frac{1}{(1+x^2)\tan^{-1}x}$

39. Question

Differentiate the following functions with respect to x:

$$\frac{2^x\cos x}{\left(x^2+3\right)^2}$$

Answer

Let $y=\frac{2^x\cos x}{(x^2+3)^2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{2^x\cos x}{(x^2+3)^2}\right]$$

Recall that $\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx}=\frac{(x^2+3)^2\frac{d}{dx}(2^x\cos x)-(2^x\cos x)\frac{d}{dx}[(x^2+3)^2]}{[(x^2+3)^2]^2}$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx}=\frac{(x^2+3)^2\left[\cos x\frac{d}{dx}(2^x)+2^x\frac{d}{dx}(\cos x)\right]-(2^x\cos x)\frac{d}{dx}[(x^2+3)^2]}{(x^2+3)^4}$$

We know $\frac{d}{dx}(a^x)=a^x\log a$, $\frac{d}{dx}(\cos x)=-\sin x$ and $\frac{d}{dx}(x^n)=nx^{n-1}$

$$\Rightarrow \frac{dy}{dx}$$
$$=\frac{(x^2+3)^2[\cos x\,(2^x\log 2)+2^x(-\sin x)]-(2^x\cos x)\left[2(x^2+3)^{2-1}\frac{d}{dx}(x^2+3)\right]}{(x^2+3)^4}$$

$$\Rightarrow \frac{dy}{dx}$$
$$=\frac{(x^2+3)^2[2^x\log 2\cos x-2^x\sin x]-(2^x\cos x)\left[2(x^2+3)\left\{\frac{d}{dx}(x^2)+\frac{d}{dx}(3)\right\}\right]}{(x^2+3)^4}$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+3)^2[2^x \log 2 \cos x - 2^x \sin x] - (2^x \cos x)[2(x^2+3)\{2x+0\}]}{(x^2+3)^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+3)^2 2^x(\log 2 \cos x - \sin x) - 2^x 4x(x^2+3)\cos x}{(x^2+3)^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+3)^2 2^x(\log 2 \cos x - \sin x)}{(x^2+3)^4} - \frac{2^x 4x(x^2+3)\cos x}{(x^2+3)^4}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2^x(\log 2 \cos x - \sin x)}{(x^2+3)^2} - \frac{2^x 4x \cos x}{(x^2+3)^3}$$

$$\therefore \frac{dy}{dx} = \frac{2^x}{(x^2+3)^2}\left(\log 2 \cos x - \sin x - \frac{4x \cos x}{x^2+3}\right)$$

Thus, $\frac{d}{dx}\left[\frac{2^x \cos x}{(x^2+3)^2}\right] = \frac{2^x}{(x^2+3)^2}\left(\log 2 \cos x - \sin x - \frac{4x\cos x}{x^2+3}\right)$

40. Question

Differentiate the following functions with respect to x:

$x\sin(2x) + 5^x + k^k + (\tan^2 x)^3$

Answer

Let $y = x\sin(2x) + 5^x + k^k + (\tan^2 x)^3$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[x \sin 2x + 5^x + k^k + (\tan^2 x)^3\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x \sin 2x) + \frac{d}{dx}(5^x) + \frac{d}{dx}\left(k^k\right) + \frac{d}{dx}[(\tan^2 x)^3]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x \times \sin 2x) + \frac{d}{dx}(5^x) + \frac{d}{dx}\left(k^k\right) + \frac{d}{dx}(\tan^6 x)$$

Recall that (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \sin 2x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin 2x) + \frac{d}{dx}(5^x) + \frac{d}{dx}\left(k^k\right) + \frac{d}{dx}(\tan^6 x)$$

We know $\frac{d}{dx}(a^x) = a^x \log a$, $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

Also, the derivation of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \sin 2x + x \cos 2x \frac{d}{dx}(2x) + 5^x \log 5 + 0 + 6 \tan^{6-1} x \frac{d}{dx}(\tan x)$$

$$\Rightarrow \frac{dy}{dx} = \sin 2x + 2x \cos 2x \frac{d}{dx}(x) + 5^x \log 5 + 6 \tan^5 x \frac{d}{dx}(\tan x)$$

We have $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \sin 2x + 2x \cos 2x \times 1 + 5^x \log 5 + 6 \tan^5 x \times \sec^2 x$$

$$\therefore \frac{dy}{dx} = \sin 2x + 2x \cos 2x + 5^x \log 5 + 6 \tan^5 x \sec^2 x$$

Thus, $\frac{d}{dx}\left[x\sin 2x + 5^x + k^k + (\tan^2 x)^3\right] = \sin 2x + 2x\cos 2x + 5^x\log 5 + 6\tan^5 x\sec^2 x$

41. Question

Differentiate the following functions with respect to x:

log(3x + 2) – x^2log(2x – 1)

Answer

Let y = log(3x + 2) – x^2log(2x – 1)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(3x+2) - x^2\log(2x-1)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\log(3x+2)] - \frac{d}{dx}[x^2\log(2x-1)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\log(3x+2)] - \frac{d}{dx}[x^2 \times \log(2x-1)]$$

Recall that (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\log(3x+2)] - \left[\log(2x-1)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}[\log(2x-1)]\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[\log(3x+2)] - \log(2x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}[\log(2x-1)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3x+2}\frac{d}{dx}(3x+2) - \log(2x-1)\times 2x - x^2 \times \frac{1}{2x-1}\frac{d}{dx}(2x-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3x+2}\left[\frac{d}{dx}(3x) + \frac{d}{dx}(2)\right] - 2x\log(2x-1) - \frac{x^2}{2x-1}\left[\frac{d}{dx}(2x) - \frac{d}{dx}(1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3x+2}\left[3\frac{d}{dx}(x) + \frac{d}{dx}(2)\right] - 2x\log(2x-1) - \frac{x^2}{2x-1}\left[2\frac{d}{dx}(x) - \frac{d}{dx}(1)\right]$$

We have $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3x+2}[3\times 1 + 0] - 2x\log(2x-1) - \frac{x^2}{2x-1}[2\times 1 - 0]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{3x+2}\times 3 - 2x\log(2x-1) - \frac{x^2}{2x-1}\times 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{3x+2} - 2x\log(2x-1) - \frac{2x^2}{2x-1}$$

$$\therefore \frac{dy}{dx} = \frac{3}{3x+2} - \frac{2x^2}{2x-1} - 2x\log(2x-1)$$

Thus, $\frac{d}{dx}[\log(3x+2) - x^2\log(2x-1)] = \frac{3}{3x+2} - \frac{2x^2}{2x-1} - 2x\log(2x-1)$

42. Question

Differentiate the following functions with respect to x:

$$\frac{3x^2 \sin x}{\sqrt{7-x^2}}$$

Answer

Let $y = \frac{3x^2 \sin x}{\sqrt{7-x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^2 \sin x}{\sqrt{7-x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = 3\frac{d}{dx}\left(\frac{x^2 \sin x}{\sqrt{7-x^2}}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{\sqrt{7-x^2}\frac{d}{dx}(x^2 \sin x) - (x^2 \sin x)\frac{d}{dx}\left(\sqrt{7-x^2}\right)}{\left(\sqrt{7-x^2}\right)^2}\right]$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{\sqrt{7-x^2}\left(\sin x\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(\sin x)\right) - (x^2 \sin x)\frac{d}{dx}\left((7-x^2)^{\frac{1}{2}}\right)}{7-x^2}\right]$$

We know $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{\sqrt{7-x^2}\left(\sin x(2x) + x^2(\cos x)\right) - (x^2 \sin x)\frac{1}{2}(7-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(-x^2)}{7-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{\sqrt{7-x^2}(2x\sin x + x^2 \cos x) + \frac{x^2}{2}(7-x^2)^{-\frac{1}{2}}\sin x\frac{d}{dx}(x^2)}{7-x^2}\right]$$

However, $\frac{d}{dx}(x^2) = 2x$

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{(2x\sin x + x^2 \cos x)(7-x^2)^{\frac{1}{2}} + x^3 \sin x(7-x^2)^{-\frac{1}{2}}}{7-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{(2x\sin x + x^2 \cos x)(7-x^2)^{\frac{1}{2}}}{7-x^2} + \frac{x^3 \sin x(7-x^2)^{-\frac{1}{2}}}{7-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3\left[\frac{2x\sin x + x^2 \cos x}{(7-x^2)^{\frac{1}{2}}} + \frac{x^3 \sin x}{(7-x^2)^{\frac{3}{2}}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x}{(7-x^2)^{\frac{1}{2}}}\left[2\sin x + x\cos x + \frac{x^2 \sin x}{7-x^2}\right]$$

$$\therefore \frac{dy}{dx} = \frac{3x}{\sqrt{7-x^2}}\left(2\sin x + x\cos x + \frac{x^2 \sin x}{7-x^2}\right)$$

Thus, $\frac{d}{dx}\left(\frac{3x^2 \sin x}{\sqrt{7-x^2}}\right) = \frac{3x}{\sqrt{7-x^2}}\left(2\sin x + x\cos x + \frac{x^2 \sin x}{7-x^2}\right)$

43. Question

Differentiate the following functions with respect to x:

$\sin^2\{\log(2x + 3)\}$

Answer

Let $y = \sin^2\{\log(2x + 3)\}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin^2\{\log(2x+3)\}]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2\sin^{2-1}\{\log(2x+3)\}\frac{d}{dx}[\sin\{\log(2x+3)\}]$ [chain rule]

$$\Rightarrow \frac{dy}{dx} = 2\sin\{\log(2x+3)\}\frac{d}{dx}[\sin\{\log(2x+3)\}]$$

We have $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = 2\sin\{\log(2x+3)\}\cos\{\log(2x+3)\}\frac{d}{dx}[\log(2x+3)]$$

As sin(2θ) = 2sinθcosθ, we have

$$\frac{dy}{dx} = \sin\{2 \times \log(2x+3)\}\frac{d}{dx}[\log(2x+3)]$$

$$\frac{dy}{dx} = \sin\{2\log(2x+3)\}\frac{d}{dx}[\log(2x+3)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = \sin\{2\log(2x+3)\}\left[\frac{1}{(2x+3)}\frac{d}{dx}(2x+3)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\{2\log(2x+3)\}}{2x+3}\frac{d}{dx}(2x+3)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\{2\log(2x+3)\}}{2x+3}\left[\frac{d}{dx}(2x) + \frac{d}{dx}(3)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\{2\log(2x+3)\}}{2x+3}\left[2\frac{d}{dx}(x) + \frac{d}{dx}(3)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\{2\log(2x+3)\}}{2x+3}[2 \times 1 + 0]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\{2\log(2x+3)\}}{2x+3} \times 2$$

$$\therefore \frac{dy}{dx} = \frac{2\sin\{2\log(2x+3)\}}{2x+3}$$

Thus, $\frac{d}{dx}[\sin^2\{\log(2x+3)\}] = \frac{2\sin\{2\log(2x+3)\}}{2x+3}$

44. Question

Differentiate the following functions with respect to x:

$e^x \log(\sin 2x)$

Answer

Let $y = e^x \log(\sin 2x)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[e^x \log(\sin 2x)]$$

We have (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{dy}{dx} = \log(\sin 2x)\frac{d}{dx}(e^x) + e^x \frac{d}{dx}[\log(\sin 2x)]$$

We know $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \log(\sin 2x) \times e^x + e^x\left[\frac{1}{\sin 2x}\frac{d}{dx}(\sin 2x)\right]$ [chain rule]

$$\Rightarrow \frac{dy}{dx} = e^x \log(\sin 2x) + \frac{e^x}{\sin 2x}\left[\frac{d}{dx}(\sin 2x)\right]$$

We have $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{dy}{dx} = e^x \log(\sin 2x) + \frac{e^x}{\sin 2x}\left[\cos 2x \frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{dy}{dx} = e^x \log(\sin 2x) + \frac{2e^x \cos 2x}{\sin 2x}\left[\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{dy}{dx} = e^x \log(\sin 2x) + 2e^x \cot 2x\left[\frac{d}{dx}(x)\right]$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = e^x \log(\sin 2x) + 2e^x \cot 2x \times 1$$

$$\Rightarrow \frac{dy}{dx} = e^x \log(\sin 2x) + 2e^x \cot 2x$$

$$\therefore \frac{dy}{dx} = e^x[\log(\sin 2x) + 2\cot 2x]$$

Thus, $\frac{d}{dx}[e^x \log(\sin 2x)] = e^x[\log(\sin 2x) + 2\cot 2x]$

45. Question

Differentiate the following functions with respect to x:

$$\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$$

Answer

Let $y = \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$

$$\Rightarrow y=\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}\times\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$\Rightarrow y=\frac{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)^2}{\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}$$

$$\Rightarrow y=\frac{\left(\sqrt{x^2+1}\right)^2+\left(\sqrt{x^2-1}\right)^2+2\sqrt{(x^2+1)(x^2-1)}}{\left(\sqrt{x^2+1}\right)^2-\left(\sqrt{x^2-1}\right)^2}$$

$$\Rightarrow y=\frac{x^2+1+x^2-1+2\sqrt{(x^2)^2-(1)^2}}{(x^2+1)-(x^2-1)}$$

$$\Rightarrow y=\frac{2x^2+2\sqrt{x^4-1}}{2}$$

$$\Rightarrow y=x^2+\sqrt{x^4-1}$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2+\sqrt{x^4-1}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}(x^2)+\frac{d}{dx}\left(\sqrt{x^4-1}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}(x^2)+\frac{d}{dx}\left[(x^4-1)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$

$$\Rightarrow\frac{dy}{dx}=2x+\frac{1}{2}(x^4-1)^{\frac{1}{2}-1}\frac{d}{dx}(x^4-1)$$

$$\Rightarrow\frac{dy}{dx}=2x+\frac{1}{2}(x^4-1)^{-\frac{1}{2}}\frac{d}{dx}(x^4-1)$$

$$\Rightarrow\frac{dy}{dx}=2x+\frac{1}{2\sqrt{x^4-1}}\left[\frac{d}{dx}(x^4)-\frac{d}{dx}(1)\right]$$

We have $\frac{d}{dx}(x^4)=4x^3$ and derivative of a constant is 0.

$$\Rightarrow\frac{dy}{dx}=2x+\frac{1}{2\sqrt{x^4-1}}[4x^3-0]$$

$$\Rightarrow\frac{dy}{dx}=2x+\frac{1}{2\sqrt{x^4-1}}\times 4x^3$$

$$\therefore\frac{dy}{dx}=2x+\frac{2x^3}{\sqrt{x^4-1}}$$

Thus, $\frac{d}{dx}\left(\frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}\right)=2x+\frac{2x^3}{\sqrt{x^4-1}}$

46. Question

Differentiate the following functions with respect to x:

$$\log\left\{x+2+\sqrt{x^2+4x+1}\right\}$$

Answer

Let $y = \log\left(x + 2 + \sqrt{x^2 + 4x + 1}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(x + 2 + \sqrt{x^2 + 4x + 1}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+2+\sqrt{x^2+4x+1}}\frac{d}{dx}\left(x + 2 + \sqrt{x^2 + 4x + 1}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[\frac{d}{dx}(x) + \frac{d}{dx}(2) + \frac{d}{dx}\left(\sqrt{x^2 + 4x + 1}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[\frac{d}{dx}(x) + \frac{d}{dx}(2) + \frac{d}{dx}(x^2 + 4x + 1)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

Also the derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[1 + 0 + \frac{1}{2}(x^2 + 4x + 1)^{\frac{1}{2}-1}\frac{d}{dx}(x^2 + 4x + 1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[1 + \frac{1}{2}(x^2 + 4x + 1)^{-\frac{1}{2}}\left(\frac{d}{dx}(x^2) + \frac{d}{dx}(4x) + \frac{d}{dx}(1)\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[1 + \frac{1}{2\sqrt{x^2 + 4x + 1}}\left(\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x) + \frac{d}{dx}(1)\right)\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[1 + \frac{1}{2\sqrt{x^2 + 4x + 1}}(2x + 4 \times 1 + 0)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[1 + \frac{2x + 4}{2\sqrt{x^2 + 4x + 1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[1 + \frac{x + 2}{\sqrt{x^2 + 4x + 1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x + 2 + \sqrt{x^2 + 4x + 1}}\left[\frac{\sqrt{x^2 + 4x + 1} + x + 2}{\sqrt{x^2 + 4x + 1}}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 4x + 1}}$$

Thus, $\frac{d}{dx}\left[\log\left(x + 2 + \sqrt{x^2 + 4x + 1}\right)\right] = \frac{1}{\sqrt{x^2+4x+1}}$

47. Question

Differentiate the following functions with respect to x:

$(\sin^{-1} x^4)^4$

Answer

Let y = $(\sin^{-1} x^4)^4$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(\sin^{-1} x^4)^4]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 4(\sin^{-1} x^4)^{4-1}\frac{d}{dx}(\sin^{-1} x^4)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = 4(\sin^{-1} x^4)^3\frac{d}{dx}(\sin^{-1} x^4)$$

We have $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = 4(\sin^{-1} x^4)^3\frac{1}{\sqrt{1-(x^4)^2}}\frac{d}{dx}(x^4)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{4(\sin^{-1} x^4)^3}{\sqrt{1-x^8}}\frac{d}{dx}(x^4)$$

We have $\frac{d}{dx}(x^4) = 4x^3$

$$\Rightarrow \frac{dy}{dx} = \frac{4(\sin^{-1} x^4)^3}{\sqrt{1-x^8}} \times 4x^3$$

$$\therefore \frac{dy}{dx} = \frac{16x^3(\sin^{-1} x^4)^3}{\sqrt{1-x^8}}$$

Thus, $\frac{d}{dx}[(\sin^{-1} x^4)^4] = \frac{16x^3(\sin^{-1} x^4)^3}{\sqrt{1-x^8}}$

48. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$$

Answer

Let $y = \sin^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)\right]$$

We have $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{x^2+a^2}}\right)^2}}\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{x^2}{x^2+a^2}}}\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\frac{x^2 + a^2 - x^2}{x^2 + a^2}}} \frac{d}{dx}\left(\frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{\frac{a^2}{x^2 + a^2}}} \frac{d}{dx}\left(\frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a} \frac{d}{dx}\left(\frac{x}{\sqrt{x^2 + a^2}}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2}\frac{d}{dx}(x) - x\frac{d}{dx}\left(\sqrt{x^2 + a^2}\right)}{\left(\sqrt{x^2 + a^2}\right)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2}\frac{d}{dx}(x) - x\frac{d}{dx}\left[(x^2 + a^2)\right]^{\frac{1}{2}}}{x^2 + a^2}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2} \times 1 - x\left(\frac{1}{2}(x^2 + a^2)^{\frac{1}{2}-1}\frac{d}{dx}(x^2 + a^2)\right)}{x^2 + a^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2} - x\left(\frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}}\frac{d}{dx}(x^2 + a^2)\right)}{x^2 + a^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2} - \frac{x}{2\sqrt{x^2 + a^2}}\left(\frac{d}{dx}(x^2) + \frac{d}{dx}(a^2)\right)}{x^2 + a^2}\right]$$

We have $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2} - \frac{x}{2\sqrt{x^2 + a^2}}(2x + 0)}{x^2 + a^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\sqrt{x^2 + a^2} - \frac{x^2}{\sqrt{x^2 + a^2}}}{x^2 + a^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{\frac{\left(\sqrt{x^2 + a^2}\right)^2 - x^2}{\sqrt{x^2 + a^2}}}{x^2 + a^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + a^2}}{a}\left[\frac{x^2 + a^2 - x^2}{\sqrt{x^2 + a^2}(x^2 + a^2)}\right]$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{a}\left[\frac{a^2}{x^2 + a^2}\right]$

$\therefore \frac{dy}{dx} = \frac{a}{x^2 + a^2}$

Thus, $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)\right] = \frac{a}{x^2+a^2}$

49. Question

Differentiate the following functions with respect to x:

$$\frac{e^x \sin x}{\left(x^2+2\right)^3}$$

Answer

Let $y = \frac{e^x \sin x}{(x^2+2)^3}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{e^x \sin x}{(x^2 + 2)^3}\right]$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3 \frac{d}{dx}(e^x \sin x) - (e^x \sin x)\frac{d}{dx}[(x^2+2)^3]}{[(x^2+2)^3]^2}$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3\left[\sin x \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(\sin x)\right] - (e^x \sin x)\frac{d}{dx}[(x^2+2)^3]}{(x^2+2)^6}$$

We know $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3[\sin x\,(e^x) + e^x(\cos x)] - (e^x \sin x)\left[3(x^2+2)^{3-1}\frac{d}{dx}(x^2+2)\right]}{(x^2+2)^6}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3[e^x \sin x + e^x \cos x] - (e^x \sin x)\left[3(x^2+2)^2\left\{\frac{d}{dx}(x^2) + \frac{d}{dx}(2)\right\}\right]}{(x^2+2)^6}$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3[e^x \sin x + e^x \cos x] - (e^x \sin x)[3(x^2+2)^2 \times 2x]}{(x^2+2)^6}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3 e^x(\sin x + \cos x) - 6xe^x \sin x\,(x^2+2)^2}{(x^2+2)^6}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+2)^3 e^x(\sin x + \cos x)}{(x^2+2)^6} - \frac{6xe^x \sin x\,(x^2+2)^2}{(x^2+2)^6}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x(\sin x + \cos x)}{(x^2+2)^3} - \frac{6xe^x \sin x}{(x^2+2)^4}$$

$$\therefore \frac{dy}{dx} = \frac{e^x}{(x^2+2)^3}\left(\sin x + \cos x - \frac{6x \sin x}{x^2+2}\right)$$

Thus, $\frac{d}{dx}\left[\frac{e^x \sin x}{(x^2+2)^3}\right] = \frac{e^x}{(x^2+2)^3}\left(\sin x + \cos x - \frac{6x\sin x}{x^2+2}\right)$

50. Question

Differentiate the following functions with respect to x:

$3e^{-3x}\log(1 + x)$

Answer

Let $y = 3e^{-3x}\log(1 + x)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[3e^{-3x}\log(1+x)]$$

$$\Rightarrow \frac{dy}{dx} = 3\frac{d}{dx}[e^{-3x}\log(1+x)]$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = 3\left[\log(1+x)\frac{d}{dx}(e^{-3x}) + e^{-3x}\frac{d}{dx}[\log(1+x)]\right]$$

We know $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = 3\left[\log(1+x) \times e^{-3x}\frac{d}{dx}(-3x) + e^{-3x}\left(\frac{1}{1+x}\frac{d}{dx}(1+x)\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3\left[-3e^{-3x}\log(1+x)\frac{d}{dx}(x) + \frac{e^{-3x}}{1+x}\left(\frac{d}{dx}(1) + \frac{d}{dx}(x)\right)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = 3\left[-3e^{-3x}\log(1+x) \times 1 + \frac{e^{-3x}}{1+x}(0+1)\right]$$

$$\Rightarrow \frac{dy}{dx} = 3\left[-3e^{-3x}\log(1+x) + \frac{e^{-3x}}{1+x}\right]$$

$$\Rightarrow \frac{dy}{dx} = 3e^{-3x}\left[-3\log(1+x) + \frac{1}{1+x}\right]$$

$$\therefore \frac{dy}{dx} = 3e^{-3x}\left[\frac{1}{1+x} - 3\log(1+x)\right]$$

Thus, $\frac{d}{dx}[3e^{-3x}\log(1+x)] = 3e^{-3x}\left[\frac{1}{1+x} - 3\log(1+x)\right]$

51. Question

Differentiate the following functions with respect to x:

$$\frac{x^2+2}{\sqrt{\cos x}}$$

Answer

Let $y = \frac{x^2+2}{\sqrt{\cos x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+2}{\sqrt{\cos x}}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\cos x}\frac{d}{dx}(x^2+2) - (x^2+2)\frac{d}{dx}\left(\sqrt{\cos x}\right)}{\left(\sqrt{\cos x}\right)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\cos x}\left[\frac{d}{dx}(x^2) + \frac{d}{dx}(2)\right] - (x^2+2)\frac{d}{dx}\left[(\cos x)^{\frac{1}{2}}\right]}{\cos x}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\cos x}[2x+0] - (x^2+2)\left[\frac{1}{2}(\cos x)^{\frac{1}{2}-1}\frac{d}{dx}(\cos x)\right]}{\cos x} \text{ [chain rule]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sqrt{\cos x} - \frac{(x^2+2)}{2}(\cos x)^{-\frac{1}{2}}\left[\frac{d}{dx}(\cos x)\right]}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sqrt{\cos x} - \frac{(x^2+2)}{2}(\cos x)^{-\frac{1}{2}}\left[\frac{d}{dx}(\cos x)\right]}{\cos x}$$

We know $\frac{d}{dx}(\cos x) = -\sin x$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sqrt{\cos x} - \frac{(x^2+2)}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\sqrt{\cos x} + \frac{(x^2+2)\sin x}{2\sqrt{\cos x}}}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x\left(\sqrt{\cos x}\right)^2 + (x^2+2)\sin x}{2\sqrt{\cos x}\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x\cos x + (x^2+2)\sin x}{2\sqrt{\cos x}\cos x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x\cos x}{2\sqrt{\cos x}\cos x} + \frac{(x^2+2)\sin x}{2\sqrt{\cos x}\cos x}$$

$$\therefore \frac{dy}{dx} = \frac{2x}{\sqrt{\cos x}} + \frac{(x^2+2)\sin x}{2(\cos x)^{\frac{3}{2}}}$$

Thus, $\frac{d}{dx}\left(\frac{x^2+2}{\sqrt{\cos x}}\right) = \frac{2x}{\sqrt{\cos x}} + \frac{(x^2+2)\sin x}{2(\cos x)^{\frac{3}{2}}}$

52. Question

Differentiate the following functions with respect to x:

$$\frac{x^2\left(1-x^2\right)^3}{\cos 2x}$$

Answer

Let $y=\frac{x^2(1-x^2)^3}{\cos 2x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{x^2(1-x^2)^3}{\cos 2x}\right]$$

Recall that $\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\frac{d}{dx}[x^2(1-x^2)^3]-x^2(1-x^2)^3\frac{d}{dx}(\cos 2x)}{(\cos 2x)^2}$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\left[(1-x^2)^3\frac{d}{dx}(x^2)+x^2\frac{d}{dx}\{(1-x^2)^3\}\right]-x^2(1-x^2)^3\frac{d}{dx}(\cos 2x)}{\cos^2 2x}$$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$ and $\frac{d}{dx}(\cos x)=-\sin x$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\left[(1-x^2)^3(2x)+x^2\left\{3(1-x^2)^2\frac{d}{dx}(1-x^2)\right\}\right]-x^2(1-x^2)^3\left(-\sin 2x\frac{d}{dx}(2x)\right)}{\cos^2 2x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\left[2x(1-x^2)^3+3x^2(1-x^2)^2\left\{\frac{d}{dx}(1)-\frac{d}{dx}(x^2)\right\}\right]+2x^2(1-x^2)^3\sin 2x\frac{d}{dx}(x)}{\cos^2 2x}$$

However, $\frac{d}{dx}(x^n)=nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\,[2x(1-x^2)^3+3x^2(1-x^2)^2\{0-2x\}]+2x^2(1-x^2)^3\sin 2x\times 1}{\cos^2 2x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\,[2x(1-x^2)^3+3x^2(1-x^2)^2(-2x)]+2x^2(1-x^2)^3\sin 2x}{\cos^2 2x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos 2x\,[2x(1-x^2)^3-6x^3(1-x^2)^2]+2x^2(1-x^2)^3\sin 2x}{\cos^2 2x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{2x(1-x^2)^3\cos 2x-6x^3(1-x^2)^2\cos 2x+2x^2(1-x^2)^3\sin 2x}{\cos^2 2x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{2x(1-x^2)^3\cos 2x}{\cos^2 2x}-\frac{6x^3(1-x^2)^2\cos 2x+}{\cos^2 2x}+\frac{2x^2(1-x^2)^3\sin 2x}{\cos^2 2x}$$

$$\Rightarrow\frac{dy}{dx}=\frac{2x(1-x^2)^3}{\cos 2x}-\frac{6x^3(1-x^2)^2}{\cos 2x}+\frac{2x^2(1-x^2)^3\sin 2x}{\cos 2x\times\cos 2x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(1-x^2)^3}{\cos 2x} - \frac{6x^3(1-x^2)^2}{\cos 2x} + \frac{2x^2(1-x^2)^3 \tan 2x}{\cos 2x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(1-x^2)^2}{\cos 2x}[(1-x^2) - 3x^2 + x(1-x^2)\tan 2x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x(1-x^2)^2}{\cos 2x}[1 - 4x^2 + x(1-x^2)\tan 2x]$$

$$\therefore \frac{dy}{dx} = 2x(1-x^2)^2 \sec 2x\,[1 - 4x^2 + x(1-x^2)\tan 2x]$$

Thus, $\frac{d}{dx}\left[\frac{x^2(1-x^2)^3}{\cos 2x}\right] = 2x(1-x^2)^2 \sec 2x\,[1 - 4x^2 + x(1-x^2)\tan 2x]$

53. Question

Differentiate the following functions with respect to x:

$$\log\left\{\cot\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}$$

Answer

Let $y = \log\left\{\cot\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left\{\cot\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cot\left(\frac{\pi}{4} + \frac{x}{2}\right)} \frac{d}{dx}\left[\cot\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) \frac{d}{dx}\left[\cot\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

We have $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$$\Rightarrow \frac{dy}{dx} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\left[-\operatorname{cosec}^2\left(\frac{\pi}{4} + \frac{x}{2}\right) \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = -\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\operatorname{cosec}^2\left(\frac{\pi}{4} + \frac{x}{2}\right)\left[\frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}\left(\frac{x}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = -\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\operatorname{cosec}^2\left(\frac{\pi}{4} + \frac{x}{2}\right)\left[\frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{1}{2}\frac{d}{dx}(x)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = -\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\operatorname{cosec}^2\left(\frac{\pi}{4} + \frac{x}{2}\right)\left[0 + \frac{1}{2} \times 1\right]$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\operatorname{cosec}^2\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} \times \frac{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)} \times \frac{1}{\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin\left[2\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]}$ [∵ sin2θ = 2sinθcosθ]

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\cos x}$ [∵ sin(90°+θ) = cosθ]

$\therefore \frac{dy}{dx} = -\sec x$

Thus, $\frac{d}{dx}\left[\log\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}\right] = -\sec x$

54. Question

Differentiate the following functions with respect to x:

$e^{ax}\sec(x)\tan(2x)$

Answer

Let y = $e^{ax}\sec(x)\tan(2x)$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}(e^{ax}\sec x\tan 2x)$

$\frac{dy}{dx} = \frac{d}{dx}[e^{ax} \times (\sec x\tan 2x)]$

We have (uv)' = vu' + uv' (product rule)

$\Rightarrow \frac{dy}{dx} = \sec x\tan 2x\frac{d}{dx}(e^{ax}) + e^{ax}\frac{d}{dx}(\sec x\tan 2x)$

$\Rightarrow \frac{dy}{dx} = \sec x\tan 2x\frac{d}{dx}(e^{ax}) + e^{ax}\frac{d}{dx}(\sec x \times \tan 2x)$

We will use the product rule once again.

$\Rightarrow \frac{dy}{dx} = \sec x\tan 2x\frac{d}{dx}(e^{ax}) + e^{ax}\left[\tan 2x\frac{d}{dx}(\sec x) + \sec x\frac{d}{dx}(\tan 2x)\right]$

We know $\frac{d}{dx}(e^x) = e^x$, $\frac{d}{dx}(\sec x) = \sec x\tan x$ and $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \sec x\tan 2x\left[e^{ax}\frac{d}{dx}(ax)\right] + e^{ax}\left[\tan 2x(\sec x\tan x) + \sec x\left\{\sec^2 2x\frac{d}{dx}(2x)\right\}\right]$$

$\Rightarrow \frac{dy}{dx} = ae^{ax}\sec x\tan 2x\frac{d}{dx}(x) + e^{ax}\left[\sec x\tan x\tan 2x + 2\sec x\sec^2 2x\frac{d}{dx}(x)\right]$

However, $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{dy}{dx} = ae^{ax}\sec x\tan 2x \times 1 + e^{ax}[\sec x\tan x\tan 2x + 2\sec x\sec^2 2x \times 1]$

$\Rightarrow \frac{dy}{dx} = ae^{ax}\sec x\tan 2x + e^{ax}[\sec x\tan x\tan 2x + 2\sec x\sec^2 2x]$

$\Rightarrow \frac{dy}{dx} = ae^{ax}\sec x\tan 2x + e^{ax}\sec x\,[\tan x\tan 2x + 2\sec^2 2x]$

$\therefore \frac{dy}{dx} = e^{ax}\sec x\,(a\tan 2x + \tan x\tan 2x + 2\sec^2 2x)$

Thus, $\frac{d}{dx}(e^{ax}\sec x\tan 2x) = e^{ax}\sec x\,(a\tan 2x + \tan x\tan 2x + 2\sec^2 2x)$

55. Question

Differentiate the following functions with respect to x:

log(cos x^2)

Answer

Let y = log(cos x^2)

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}[\log(\cos x^2)]$

We have $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x^2}\frac{d}{dx}(\cos x^2)$ [using chain rule]

We know $\frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos x^2}\left[-\sin x^2\frac{d}{dx}(x^2)\right]$ [using chain rule]

$\Rightarrow \frac{dy}{dx} = -\frac{\sin x^2}{\cos x^2}\frac{d}{dx}(x^2)$

$\Rightarrow \frac{dy}{dx} = -\tan x^2\frac{d}{dx}(x^2)$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = -\tan x^2 \times 2x$

$\therefore \frac{dy}{dx} = -2x\tan x^2$

Thus, $\frac{d}{dx}[\log(\cos x^2)] = -2x\tan x^2$

56. Question

Differentiate the following functions with respect to x:

cos(log x)^2

Answer

Let y = cos(log x)^2

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}[\cos(\log x)^2]$

We have $\frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow \frac{dy}{dx} = -\sin(\log x)^2 \frac{d}{dx}[(\log x)^2]$ [using chain rule]

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = -\sin(\log x)^2 \left[2(\log x)^{2-1} \frac{d}{dx}(\log x)\right]$ [chain rule]

$$\Rightarrow \frac{dy}{dx} = -\sin(\log x)^2 \left[2 \log x \frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{dy}{dx} = -2 \log x \sin(\log x)^2 \frac{d}{dx}(\log x)$$

However, $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = -2 \log x \sin(\log x)^2 \times \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = -\frac{2}{x} \log x \sin(\log x)^2$$

Thus, $\frac{d}{dx}[\cos(\log x)^2] = -\frac{2}{x} \log x \sin(\log x)^2$

57. Question

Differentiate the following functions with respect to x:

$$\log \sqrt{\frac{x-1}{x+1}}$$

Answer

Let $y = \log \sqrt{\frac{x-1}{x+1}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log \sqrt{\frac{x-1}{x+1}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\log \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}} \frac{d}{dx}\left[\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}} \frac{d}{dx}\left[\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}} \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}-1} \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}}\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x-1}{x+1}\right)^{-1} \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right) \frac{d}{dx}\left(\frac{x-1}{x+1}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{(x+1)\left(\frac{d}{dx}(x) - \frac{d}{dx}(1)\right) - (x-1)\left(\frac{d}{dx}(x) + \frac{d}{dx}(1)\right)}{(x+1)^2}\right]$$

We know $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{(x+1)(1-0) - (x-1)(1+0)}{(x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{(x+1) - (x-1)}{(x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{x+1}{x-1}\right)\left[\frac{2}{(x+1)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(x-1)(x+1)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x^2-1}$$

Thus, $\frac{d}{dx}\left(\log\sqrt{\frac{x-1}{x+1}}\right) = \frac{1}{x^2-1}$

58. Question

If $y = \log\left\{\sqrt{x-1} - \sqrt{x+1}\right\}$, show that $\frac{dy}{dx} = \frac{-1}{2\sqrt{x^2-1}}$.

Answer

Given $y = \log\left(\sqrt{x-1} - \sqrt{x+1}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\sqrt{x-1} - \sqrt{x+1}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x-1} - \sqrt{x+1}}\frac{d}{dx}\left(\sqrt{x-1} - \sqrt{x+1}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x-1} - \sqrt{x+1}}\left[\frac{d}{dx}\left(\sqrt{x-1}\right) - \frac{d}{dx}\left(\sqrt{x+1}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x-1} - \sqrt{x+1}}\left[\frac{d}{dx}(x-1)^{\frac{1}{2}} - \frac{d}{dx}(x+1)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x-1}-\sqrt{x+1}}\left[\frac{1}{2}(x-1)^{\frac{1}{2}-1}\frac{d}{dx}(x-1) - \frac{1}{2}(x+1)^{\frac{1}{2}-1}\frac{d}{dx}(x+1)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\left(\sqrt{x-1}-\sqrt{x+1}\right)}\left[(x-1)^{-\frac{1}{2}}\left\{\frac{d}{dx}(x) - \frac{d}{dx}(1)\right\}\right.$$
$$\left. - (x+1)^{-\frac{1}{2}}\left\{\frac{d}{dx}(x) + \frac{d}{dx}(1)\right\}\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\left(\sqrt{x-1}-\sqrt{x+1}\right)}\left[(x-1)^{-\frac{1}{2}}\{1-0\} - (x+1)^{-\frac{1}{2}}\{1+0\}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\left(\sqrt{x-1}-\sqrt{x+1}\right)}\left[(x-1)^{-\frac{1}{2}} - (x+1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\left(\sqrt{x-1}-\sqrt{x+1}\right)}\left[\frac{1}{\sqrt{x-1}} - \frac{1}{\sqrt{x+1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\left(\sqrt{x-1}-\sqrt{x+1}\right)}\left[\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}\sqrt{x-1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2\sqrt{x+1}\sqrt{x-1}}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2\sqrt{x^2-1}}$$

Thus, $\frac{d}{dx}\left[\log\left(\sqrt{x-1}-\sqrt{x+1}\right)\right] = -\frac{1}{2\sqrt{x^2-1}}$

59. Question

If $y = \sqrt{x+1} + \sqrt{x-1}$, prove that $\sqrt{x^2-1}\frac{dy}{dx} = \frac{1}{2}y$.

Answer

Given $y = \sqrt{x+1} + \sqrt{x-1}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x+1} + \sqrt{x-1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x+1}\right) + \frac{d}{dx}\left(\sqrt{x-1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x+1)^{\frac{1}{2}} + \frac{d}{dx}(x-1)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x+1)^{\frac{1}{2}-1}\frac{d}{dx}(x+1) + \frac{1}{2}(x-1)^{\frac{1}{2}-1}\frac{d}{dx}(x-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}\left[\frac{d}{dx}(x) + \frac{d}{dx}(1)\right] + \frac{1}{2}(x-1)^{-\frac{1}{2}}\left[\frac{d}{dx}(x) - \frac{d}{dx}(1)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}}[1+0] + \frac{1}{2}(x-1)^{-\frac{1}{2}}[1-0]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} + \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[(x+1)^{-\frac{1}{2}} + (x-1)^{-\frac{1}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x-1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{\sqrt{x-1}+\sqrt{x+1}}{\sqrt{x+1}\sqrt{x-1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x-1}+\sqrt{x+1}}{2\sqrt{x^2-1}}$$

But, $y = \sqrt{x+1} + \sqrt{x-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2\sqrt{x^2-1}}$$

$$\therefore \sqrt{x^2-1}\frac{dy}{dx} = \frac{1}{2}y$$

Thus, $\sqrt{x^2-1}\frac{dy}{dx} = \frac{1}{2}y$

60. Question

If $y = \frac{x}{x+2}$, prove that $x\frac{dy}{dx} = (1-y)y$.

Answer

Given $y = \frac{x}{x+2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x+2}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{(x+2)\frac{d}{dx}(x) - (x)\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+2)\frac{d}{dx}(x) - (x)\left[\frac{d}{dx}(x) + \frac{d}{dx}(2)\right]}{(x+2)^2}$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{(x+2)\times 1 - (x)[1+0]}{(x+2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+2-x}{(x+2)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{(x+2)^2}$$

On multiplying both sides with x, we get

$$x\frac{dy}{dx} = \frac{2x}{(x+2)^2}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{2}{x+2} \times \frac{x}{x+2}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{x+2-x}{x+2} \times \frac{x}{x+2}$$

$$\Rightarrow x\frac{dy}{dx} = \left(1 - \frac{x}{x+2}\right) \times \frac{x}{x+2}$$

But, $y = \frac{x}{x+2}$

$$\Rightarrow x\frac{dy}{dx} = (1-y) \times y$$

$$\therefore x\frac{dy}{dx} = (1-y)y$$

Thus, $x\frac{dy}{dx} = (1-y)y$

61. Question

If $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$, prove that $\frac{dy}{dx} = \frac{x-1}{2x(x+1)}$.

Answer

Given $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)} \frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{x+1}{\sqrt{x}}\right)}\left[\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}}{x+1}\left[\frac{d}{dx}(x)^{\frac{1}{2}} + \frac{d}{dx}(x)^{-\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}}{x+1}\left[\frac{1}{2}(x)^{\frac{1}{2}-1} + \left(-\frac{1}{2}\right)(x)^{-\frac{1}{2}-1}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}}{x+1}\left[\frac{1}{2}(x)^{-\frac{1}{2}} - \frac{1}{2}(x)^{-\frac{3}{2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}}{2(x+1)}\left[\frac{1}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{3}{2}}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}}{2(x+1)}\left[\frac{1}{\sqrt{x}} - \frac{1}{x\sqrt{x}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}}{2(x+1)}\left[\frac{x-1}{x\sqrt{x}}\right]$$

$$\therefore \frac{dy}{dx} = \frac{x-1}{2x(x+1)}$$

Thus, $\frac{dy}{dx} = \frac{x-1}{2x(x+1)}$

62. Question

If $y = \log\sqrt{\frac{1+\tan x}{1-\tan x}}$, prove that $\frac{dy}{dx} = \sec 2x$.

Answer

Given $y = \log\sqrt{\frac{1+\tan x}{1-\tan x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\sqrt{\frac{1+\tan x}{1-\tan x}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{2}}}\frac{d}{dx}\left[\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{2}}\right]$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \left(\frac{1+\tan x}{1-\tan x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left[\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \left(\frac{1+\tan x}{1-\tan x}\right)^{-\frac{1}{2}}\frac{1}{2}\left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{2}-1}\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\tan x}{1-\tan x}\right)^{-\frac{1}{2}}\left(\frac{1+\tan x}{1-\tan x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1+\tan x}{1-\tan x}\right)^{-1}\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\frac{d}{dx}\left(\frac{1+\tan x}{1-\tan x}\right)$$

Recall that $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\left[\frac{(1-\tan x)\frac{d}{dx}(1+\tan x) - (1+\tan x)\frac{d}{dx}(1-\tan x)}{(1-\tan x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx}$$

$$= \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\left[\frac{(1-\tan x)\left(\frac{d}{dx}(1) + \frac{d}{dx}(\tan x)\right) - (1+\tan x)\left(\frac{d}{dx}(1) - \frac{d}{dx}(\tan x)\right)}{(1-\tan x)^2}\right]$$

We know $\frac{d}{dx}(\tan x) = \sec^2 x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\left[\frac{(1-\tan x)(0+\sec^2 x) - (1+\tan x)(0-\sec^2 x)}{(1-\tan x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\left[\frac{(1-\tan x)\sec^2 x + (1+\tan x)\sec^2 x}{(1-\tan x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\left[\frac{(1-\tan x + 1 + \tan x)\sec^2 x}{(1-\tan x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1-\tan x}{1+\tan x}\right)\left[\frac{2\sec^2 x}{(1-\tan x)^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{(1+\tan x)(1-\tan x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{1-\tan^2 x}$$

$\Rightarrow \frac{dy}{dx} = \frac{1+\tan^2 x}{1-\tan^2 x}$ ($\because \sec^2\theta - \tan^2\theta = 1$)

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x - \sin^2 x}$$

But, $\cos^2\theta + \sin^2\theta = 1$ and $\cos^2\theta - \sin^2\theta = \cos(2\theta)$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos 2x}$$

$$\therefore \frac{dy}{dx} = \sec 2x$$

Thus, $\frac{dy}{dx} = \sec 2x$

63. Question

If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, prove that $2x\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$.

Answer

Given $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(\sqrt{x})+\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(x)^{\frac{1}{2}}+\frac{d}{dx}(x)^{-\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}(x)^{\frac{1}{2}-1}+\left(-\frac{1}{2}\right)(x)^{-\frac{1}{2}-1}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}(x)^{-\frac{1}{2}}-\frac{1}{2}(x)^{-\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{x^{\frac{1}{2}}}-\frac{1}{x^{\frac{3}{2}}}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{\sqrt{x}}-\frac{1}{x\sqrt{x}}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}\left[\frac{x-1}{x\sqrt{x}}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{x-1}{2x\sqrt{x}}$$

$$\Rightarrow 2x\frac{dy}{dx}=\frac{x-1}{\sqrt{x}}$$

$$\Rightarrow 2x\frac{dy}{dx}=\frac{x}{\sqrt{x}}-\frac{1}{\sqrt{x}}$$

$$\therefore 2x\frac{dy}{dx}=\sqrt{x}-\frac{1}{\sqrt{x}}$$

Thus, $2x\frac{dy}{dx}=\sqrt{x}-\frac{1}{\sqrt{x}}$

64. Question

If $y=\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$, prove that $\left(1-x^2\right)\frac{dy}{dx}=x+\frac{y}{x}$.

Answer

Given $y=\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right)$$

Recall that $\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}\frac{d}{dx}(x\sin^{-1}x)-(x\sin^{-1}x)\frac{d}{dx}\left(\sqrt{1-x^2}\right)}{\left(\sqrt{1-x^2}\right)^2}$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}\left(\sin^{-1}x\frac{d}{dx}(x)+x\frac{d}{dx}(\sin^{-1}x)\right)-(x\sin^{-1}x)\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}\right)}{1-x^2}$$

We know $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(x^n)=nx^{n-1}$

$$\Rightarrow \frac{dy}{dx}$$

$$=\frac{\sqrt{1-x^2}\left(\sin^{-1}x\times 1+x\times\frac{1}{\sqrt{1-x^2}}\right)-(x\sin^{-1}x)\frac{1}{2}(1-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1-x^2)}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}\left(\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}\right)-\frac{x\sin^{-1}x}{2}(1-x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1)-\frac{d}{dx}(x^2)\right]}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}\left(\frac{\sqrt{1-x^2}\sin^{-1}x+x}{\sqrt{1-x^2}}\right)-\frac{x\sin^{-1}x}{2\sqrt{1-x^2}}\left[\frac{d}{dx}(1)-\frac{d}{dx}(x^2)\right]}{1-x^2}$$

However, $\frac{d}{dx}(x^2)=2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}\sin^{-1}x+x-\frac{x\sin^{-1}x}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{1-x^2}\sin^{-1}x+x+\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\times x}{1-x^2}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=\sqrt{1-x^2}\sin^{-1}x+x+\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=x+\sqrt{1-x^2}\sin^{-1}x+\frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=x+\frac{\left(\sqrt{1-x^2}\right)^2\sin^{-1}x+x^2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=x+\frac{(1-x^2)\sin^{-1}x+x^2\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=x+\frac{(1-x^2+x^2)\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{dy}{dx}=x+\frac{\sin^{-1}x}{\sqrt{1-x^2}}$$

But, $y=\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\Rightarrow\frac{y}{x}=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$$\therefore (1-x^2)\frac{dy}{dx}=x+\frac{y}{x}$$

Thus, $(1-x^2)\frac{dy}{dx}=x+\frac{y}{x}$

65. Question

If $y=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$, prove that $\frac{dy}{dx}=1-y^{2}$.

Answer

Given $y=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)$$

Recall that $\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{v^2}$ (quotient rule)

$$\Rightarrow\frac{dy}{dx}=\frac{(e^{x}+e^{-x})\frac{d}{dx}(e^{x}-e^{-x})-(e^{x}-e^{-x})\frac{d}{dx}(e^{x}+e^{-x})}{(e^{x}+e^{-x})^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{(e^{x}+e^{-x})\left[\frac{d}{dx}(e^{x})-\frac{d}{dx}(e^{-x})\right]-(e^{x}-e^{-x})\left[\frac{d}{dx}(e^{x})+\frac{d}{dx}(e^{-x})\right]}{(e^{x}+e^{-x})^2}$$

We know $\frac{d}{dx}(e^{x})=e^{x}$

$$\Rightarrow\frac{dy}{dx}=\frac{(e^{x}+e^{-x})[e^{x}-(-e^{-x})]-(e^{x}-e^{-x})[e^{x}+(-e^{-x})]}{(e^{x}+e^{-x})^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{(e^{x}+e^{-x})[e^{x}+e^{-x}]-(e^{x}-e^{-x})[e^{x}-e^{-x}]}{(e^{x}+e^{-x})^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{(e^{x}+e^{-x})^2-(e^{x}-e^{-x})^2}{(e^{x}+e^{-x})^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{(e^{x}+e^{-x})^2}{(e^{x}+e^{-x})^2}-\frac{(e^{x}-e^{-x})^2}{(e^{x}+e^{-x})^2}$$

$$\Rightarrow\frac{dy}{dx}=1-\left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)^2$$

But, $y=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$$\therefore\frac{dy}{dx}=1-y^2$$

Thus, $\frac{dy}{dx}=1-y^2$

66. Question

If y = (x – 1)log (x – 1) – (x + 1) log (x +1), prove that $\frac{dy}{dx}=\log\left(\frac{x-1}{1+x}\right)$.

Answer

Given y = (x – 1)log(x – 1) – (x + 1)log(x + 1)

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[(x-1)\log(x-1)-(x+1)\log(x+1)]$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}[(x-1)\log(x-1)]-\frac{d}{dx}[(x+1)\log(x+1)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[(x-1)\times \log(x-1)] - \frac{d}{dx}[(x+1)\times \log(x+1)]$$

Recall that (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \log(x-1)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}[\log(x-1)] - \left(\log(x+1)\frac{d}{dx}(x+1) + (x+1)\frac{d}{dx}[\log(x+1)]\right)$$

$$\Rightarrow \frac{dy}{dx} = \log(x-1)\left[\frac{d}{dx}(x) - \frac{d}{dx}(1)\right] + (x-1)\frac{d}{dx}[\log(x-1)] - \left(\log(x+1)\left[\frac{d}{dx}(x) + \frac{d}{dx}(1)\right] + (x+1)\frac{d}{dx}[\log(x+1)]\right)$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$.

Also, the derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \log(x-1)[1-0] + (x-1)\times\frac{1}{x-1} - \left(\log(x+1)[1+0] + (x+1)\times\frac{1}{x+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = \log(x-1) + 1 - (\log(x+1) + 1)$$

$$\Rightarrow \frac{dy}{dx} = \log(x-1) - \log(x+1)$$

$$\therefore \frac{dy}{dx} = \log\left(\frac{x-1}{x+1}\right)$$

Thus, $\frac{dy}{dx} = \log\left(\frac{x-1}{x+1}\right)$

67. Question

If y = e^x cos x, prove that $\frac{dy}{dx} = \sqrt{2}\,e^x\cos\left(x + \frac{\pi}{4}\right)$.

Answer

Given y = e^xcos(x)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^x\cos x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^x \times \cos x)$$

Recall that (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \cos x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(\cos x)$$

We know $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(\cos x) = -\sin x$

$\Rightarrow \frac{dy}{dx} = \cos x\,(e^x) + e^x(-\sin x)$ [chain rule]

$$\Rightarrow \frac{dy}{dx} = e^x\cos x - e^x\sin x$$

$$\Rightarrow \frac{dy}{dx} = e^x(\cos x - \sin x)$$

$$\Rightarrow \frac{dy}{dx} = e^x(\cos x - \sin x) \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{2}e^x\left(\frac{\cos x - \sin x}{\sqrt{2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{2}e^x\left(\cos x \times \frac{1}{\sqrt{2}} - \sin x \times \frac{1}{\sqrt{2}}\right)$$

We know $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$\Rightarrow \frac{dy}{dx} = \sqrt{2}e^x\left(\cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}\right)$$

However, cos(A)cos(B) – sin(A)sin(B) = cos(A + B)

$$\therefore \frac{dy}{dx} = \sqrt{2}e^x \cos\left(x + \frac{\pi}{4}\right)$$

Thus, $\frac{dy}{dx} = \sqrt{2}e^x \cos\left(x + \frac{\pi}{4}\right)$

68. Question

If $y = \frac{1}{2}\log\left(\frac{1-\cos 2x}{1+\cos 2x}\right)$, prove that $\frac{dy}{dx} = 2 \operatorname{cosec} 2x$.

Answer

Given $y = \frac{1}{2}\log\left(\frac{1-\cos 2x}{1+\cos 2x}\right)$

We have 1 + cos(2θ) = $2\cos^2\theta$ and 1 + cos(2θ) = $2\sin^2\theta$.

$$\Rightarrow y = \frac{1}{2}\log\left(\frac{2\sin^2 x}{2\cos^2 x}\right)$$

$$\Rightarrow y = \frac{1}{2}\log(\tan^2 x)$$

$$\Rightarrow y = \frac{1}{2}\log(\tan x)^2$$

$\Rightarrow y = \frac{1}{2} \times 2\log(\tan x)$ [∵ $\log(a^m) = m \times \log(a)$]

$$\Rightarrow y = \log(\tan x)$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log(\tan x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\tan x}\frac{d}{dx}(\tan x)$ [using chain rule]

However, $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\tan x} \times \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\tan x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{\cos^2 x}\right)}{\left(\frac{\sin x}{\cos x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2 x} \times \frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sin x \cos x}$$

We have sin(2θ) = 2sinθcosθ

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{\sin 2x}{2}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sin 2x}$$

$$\therefore \frac{dy}{dx} = 2\operatorname{cosec} 2x$$

Thus, $\frac{dy}{dx} = 2\operatorname{cosec} 2x$

69. Question

If $y = x\sin^{-1} x + \sqrt{1 - x^2}$, prove that $\frac{dy}{dx} = \sin^{-1} x$.

Answer

Given $y = x\sin^{-1} x + \sqrt{1 - x^2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(x\sin^{-1} x + \sqrt{1 - x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x\sin^{-1} x) + \frac{d}{dx}\left(\sqrt{1 - x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x \times \sin^{-1} x) + \frac{d}{dx}\left[(1 - x^2)^{\frac{1}{2}}\right]$$

We have (uv)’ = vu’ + uv’ (product rule)

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}\left[(1 - x^2)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x \times 1 + x \times \frac{1}{\sqrt{1 - x^2}} + \frac{1}{2}(1 - x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x + \frac{x}{\sqrt{1 - x^2}} + \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(x^2)\right]$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x + \frac{x}{\sqrt{1 - x^2}} + \frac{1}{2\sqrt{1 - x^2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(x^2)\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}[0-2x]$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{2x}{2\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = \sin^{-1}x$$

Thus, $\frac{dy}{dx} = \sin^{-1}x$

70. Question

If $y = \sqrt{x^2 + a^2}$, prove that $y\frac{dy}{dx} - x = 0$.

Answer

Given $y = \sqrt{x^2 + a^2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2 + a^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[(x^2 + a^2)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x^2 + a^2)^{\frac{1}{2}-1}\frac{d}{dx}(x^2 + a^2)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(a^2)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + a^2}}\left[\frac{d}{dx}(x^2) - \frac{d}{dx}(a^2)\right]$$

However, $\frac{d}{dx}(x^2) = 2x$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + a^2}}[2x - 0]$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{2\sqrt{x^2 + a^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{x^2 + a^2}}$$

But, $y = \sqrt{x^2 + a^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow y\frac{dy}{dx} = x$$

$$\therefore y\frac{dy}{dx} - x = 0$$

Thus, $y\frac{dy}{dx}-x=0$

71. Question

If y = $e^x + e^{-x}$, prove that $\frac{dy}{dx}=\sqrt{y^2-4}$.

Answer

Given y = $e^x + e^{-x}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(e^x+e^{-x})$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}(e^x)+\frac{d}{dx}(e^{-x})$$

We know $\frac{d}{dx}(e^x)=e^x$

$\Rightarrow \frac{dy}{dx}=e^x+e^{-x}\frac{d}{dx}(-x)$ [using chain rule]

$$\Rightarrow \frac{dy}{dx}=e^x-e^{-x}\frac{d}{dx}(x)$$

We have $\frac{d}{dx}(x)=1$

$$\Rightarrow \frac{dy}{dx}=e^x-e^{-x}\times 1$$

$$\Rightarrow \frac{dy}{dx}=e^x-e^{-x}$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{(e^x-e^{-x})^2}$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{(e^x)^2+(e^{-x})^2-2(e^x)(e^{-x})}$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{(e^x)^2+(e^{-x})^2-2(e^x)(e^{-x})+2(e^x)(e^{-x})-2(e^x)(e^{-x})}$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{(e^x)^2+(e^{-x})^2+2(e^x)(e^{-x})-4(e^x)(e^{-x})}$$

$$\Rightarrow \frac{dy}{dx}=\sqrt{(e^x+e^{-x})^2-4}$$

But, y = $e^x + e^{-x}$

$$\therefore \frac{dy}{dx}=\sqrt{y^2-4}$$

Thus, $\frac{dy}{dx}=\sqrt{y^2-4}$

72. Question

If $y=\sqrt{a^2-x^2}$, prove that $y\frac{dy}{dx}+x=0$.

Answer

Given $y=\sqrt{a^2-x^2}$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)$$

$$\Rightarrow\frac{dy}{dx}=\frac{d}{dx}\left[(a^2-x^2)^{\frac{1}{2}}\right]$$

We know $\frac{d}{dx}(x^n)=nx^{n-1}$

$\Rightarrow\frac{dy}{dx}=\frac{1}{2}(a^2-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(a^2-x^2)$ [using chain rule]

$$\Rightarrow\frac{dy}{dx}=\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(a^2)-\frac{d}{dx}(x^2)\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}\left[\frac{d}{dx}(a^2)-\frac{d}{dx}(x^2)\right]$$

However, $\frac{d}{dx}(x^2)=2x$ and derivative of a constant is 0.

$$\Rightarrow\frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}[0-2x]$$

$$\Rightarrow\frac{dy}{dx}=\frac{-2x}{2\sqrt{a^2-x^2}}$$

$$\Rightarrow\frac{dy}{dx}=\frac{-x}{\sqrt{a^2-x^2}}$$

But, $y=\sqrt{a^2-x^2}$

$$\Rightarrow\frac{dy}{dx}=\frac{-x}{y}$$

$$\Rightarrow y\frac{dy}{dx}=-x$$

$$\therefore y\frac{dy}{dx}+x=0$$

Thus, $y\frac{dy}{dx}+x=0$

73. Question

If xy = 4, prove that $x\left(\frac{dy}{dx}+y^2\right)=3y$.

Answer

Given xy = 4

$$\Rightarrow y=\frac{4}{x}$$

On differentiating y with respect to x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{4}{x}\right)$$

$$\Rightarrow\frac{dy}{dx}=4\frac{d}{dx}\left(\frac{1}{x}\right)$$

$\Rightarrow \frac{dy}{dx} = 4\frac{d}{dx}(x^{-1})$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 4(-1x^{-1-1})$

$\Rightarrow \frac{dy}{dx} = -4x^{-2}$

$\therefore \frac{dy}{dx} = -\frac{4}{x^2}$

Now, we will evaluate the LHS of the given equation.

$x\left(\frac{dy}{dx} + y^2\right) = x\left(-\frac{4}{x^2} + y^2\right)$

$\Rightarrow x\left(\frac{dy}{dx} + y^2\right) = x\left(\frac{-4 + x^2y^2}{x^2}\right)$

$\Rightarrow x\left(\frac{dy}{dx} + y^2\right) = \frac{x^2y^2 - 4}{x}$

$\Rightarrow x\left(\frac{dy}{dx} + y^2\right) = \frac{(xy)^2 - 4}{x}$

However, xy = 4

$\Rightarrow x\left(\frac{dy}{dx} + y^2\right) = \frac{4^2 - 4}{x}$

$\Rightarrow x\left(\frac{dy}{dx} + y^2\right) = \frac{12}{x}$

$\Rightarrow x\left(\frac{dy}{dx} + y^2\right) = 3\left(\frac{4}{x}\right)$

$\therefore x\left(\frac{dy}{dx} + y^2\right) = 3y$ [∵ xy = 4]

Thus, $x\left(\frac{dy}{dx} + y^2\right) = 3y$

74. Question

If prove that $\frac{d}{dx}\left\{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\} = \sqrt{a^2 - x^2}$.

Answer

Let $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

On differentiating y with respect to x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{d}{dx}\left(x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{a}\right)\right]$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{d}{dx}\left(x\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(a^2\sin^{-1}\frac{x}{a}\right)\right]$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\frac{d}{dx}\left(x \times \sqrt{a^2 - x^2}\right) + a^2\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)\right]$$

We have (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2}\frac{d}{dx}(x) + x\frac{d}{dx}\left(\sqrt{a^2 - x^2}\right) + a^2\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2}\frac{d}{dx}(x) + x\frac{d}{dx}\left\{(a^2 - x^2)^{\frac{1}{2}}\right\} + a^2\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)\right]$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} \times 1 + x\left\{\frac{1}{2}(a^2 - x^2)^{\frac{1}{2}-1}\frac{d}{dx}(a^2 - x^2)\right\} + a^2\left\{\frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\frac{d}{dx}\left(\frac{x}{a}\right)\right\}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} + \frac{x}{2}(a^2 - x^2)^{-\frac{1}{2}}\left\{\frac{d}{dx}(a^2) - \frac{d}{dx}(x^2)\right\} + \frac{a^2}{\sqrt{\frac{a^2 - x^2}{a^2}}}\left\{\frac{1}{a}\frac{d}{dx}(x)\right\}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} + \frac{x}{2\sqrt{a^2 - x^2}}\left\{\frac{d}{dx}(a^2) - \frac{d}{dx}(x^2)\right\} + \frac{a^2}{\sqrt{a^2 - x^2}}\frac{d}{dx}(x)\right]$$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} + \frac{x}{2\sqrt{a^2 - x^2}}\{0 - 2x\} + \frac{a^2}{\sqrt{a^2 - x^2}} \times 1\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} + \frac{x(-2x)}{2\sqrt{a^2 - x^2}} + \frac{a^2}{\sqrt{a^2 - x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}} + \frac{a^2}{\sqrt{a^2 - x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left[\sqrt{a^2 - x^2} + \sqrt{a^2 - x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \times 2\sqrt{a^2 - x^2}$$

$$\therefore \frac{dy}{dx} = \sqrt{a^2 - x^2}$$

Thus, $\frac{d}{dx}\left\{\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\} = \sqrt{a^2 - x^2}$

Exercise 11.3

1. Question

Differentiate the following functions with respect to x:

$\cos^{-1}\left\{2x\sqrt{1-x^2}\right\}, \frac{1}{\sqrt{2}} < x < 1$

Answer

$y = \cos^{-1}\{2x\sqrt{1-x^2}\}$

let $x = \cos\theta$

Now

$y = \cos^{-1}\{2\cos\theta\sqrt{1-\cos^2\theta}\}$

$= \cos^{-1}\{2\cos\theta\sqrt{\sin^2\theta}\}$

Using $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin2\theta$

$= \cos^{-1}(2\cos\theta\sin\theta)$

$= \cos^{-1}(\sin2\theta)$

$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$

Considering the limits,

$\frac{1}{\sqrt{2}} < x < 1$

$\Rightarrow \frac{1}{\sqrt{2}} < \cos\theta < 1$

$\Rightarrow 0 < \theta < \frac{\pi}{4}$

$\Rightarrow 0 < 2\theta < \frac{\pi}{2}$

$\Rightarrow 0 > -2\theta > -\frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} > \frac{\pi}{2} - 2\theta > \frac{\pi}{2} - \frac{\pi}{2}$

$\Rightarrow 0 < \frac{\pi}{2} - 2\theta < \frac{\pi}{2}$

Therefore,

$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$

$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$

$y = \left(\frac{\pi}{2} - 2\theta\right)$

$y = \frac{\pi}{2} - 2\cos^{-1}x$

Differentiating w.r.t x,

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$

$\Rightarrow \frac{dy}{dx} = 0 - 2\left(\frac{-1}{\sqrt{1-x^2}}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

2. Question

Differentiate the following functions with respect to x:

$$\cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}, -1 < x < 1$$

Answer

$$y = \cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$$

$\text{let } x = \cos 2\theta$

Now

$$y = \cos^{-1}\left\{\sqrt{\frac{1+\cos 2\theta}{2}}\right\}$$

$$y = \cos^{-1}\left\{\sqrt{\frac{2\cos^2\theta}{2}}\right\}$$

Using $\cos 2\theta = 2\cos^2\theta - 1$

$y = \cos^{-1}(\cos\theta)$

Considering the limits,

$-1 < x < 1$

$-1 < \cos 2\theta < 1$

$0 < 2\theta < \pi$

$$0 < \theta < \frac{\pi}{2}$$

Now, $y = \cos^{-1}(\cos\theta)$

$y = \theta$

$$y = \frac{1}{2}\cos^{-1} x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

3. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left\{\sqrt{\frac{1-x}{2}}\right\}, 0 < x < 1$$

Answer

$$y = \sin^{-1}\left\{\sqrt{\frac{1-x}{2}}\right\}$$

$\text{let } x = \cos 2\theta$

Now

$$y = \sin^{-1}\left\{\sqrt{\frac{1-\cos 2\theta}{2}}\right\}$$

$$y = \sin^{-1}\left\{\sqrt{\frac{2\sin^2\theta}{2}}\right\}$$

Using $\cos 2\theta = 1 - 2\sin^2\theta$

$y = \sin^{-1}(\sin\theta)$

Considering the limits,

$0 < x < 1$

$0 < \cos 2\theta < 1$

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{4}$$

Now, $y = \sin^{-1}(\sin\theta)$

$y = \theta$

$$y = \frac{1}{2}\cos^{-1}x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

4. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left\{\sqrt{1-x^2}\right\}, 0 < x < 1$$

Answer

$$y = \sin^{-1}\left\{\sqrt{1-x^2}\right\}$$

$\text{let } x = \cos\theta$

Now

$$y = \sin^{-1}\left\{\sqrt{1-\cos^2\theta}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$y = \sin^{-1}(\sin\theta)$

Considering the limits,

$0 < x < 1$

$0 < \cos\theta < 1$

$0 < \theta < \frac{\pi}{2}$

Now, $y = \sin^{-1}(\sin\theta)$

$y = \theta$

$y = \cos^{-1}x$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

5. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, a < x < a$$

Answer

$$y = \tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}$$

Let $x = a\sin\theta$

Now

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{\cos\theta}\right\}$$

$y = \tan^{-1}(\tan\theta)$

Considering the limits,

$-a < x < a$

$-a < a\sin\theta < a$

$-1 < \sin\theta < 1$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Now, $y = \tan^{-1}(\tan\theta)$

$y = \theta$

$$y = \sin^{-1}\left(\frac{x}{a}\right)$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right)$$

$$\frac{dy}{dx} = \frac{a}{\sqrt{a^2 - x^2}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

6. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left\{\frac{x}{\sqrt{x^2 + a^2}}\right\}$$

Answer

$$y = \sin^{-1}\left\{\frac{x}{\sqrt{x^2 + a^2}}\right\}$$

Let x = a tanθ

Now

$$y = \sin^{-1}\left\{\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta + a^2}}\right\}$$

Using 1 + $\tan^2\theta = \sec^2\theta$

$$y = \sin^{-1}\left\{\frac{a\tan\theta}{a\sqrt{\tan^2\theta + 1}}\right\}$$

$$y = \sin^{-1}\left\{\frac{a\tan\theta}{a\sqrt{\sec^2\theta}}\right\}$$

$$y = \sin^{-1}\left\{\frac{\tan\theta}{\sec\theta}\right\}$$

$y = \sin^{-1}(\sin\theta)$

y = θ

$$y = \tan^{-1}\left(\frac{x}{a}\right)$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$\frac{dy}{dx} = \frac{a^2}{a^2 + x^2} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

7. Question

Differentiate the following functions with respect to x:

$\sin^{-1}(2x^2 - 1)$, 0 < x < 1

Answer

$$y = \sin^{-1}\{2x^2 - 1\}$$

$\text{let } x = \cos\theta$

Now

$$y = \sin^{-1}\left\{\sqrt{2\cos^2\theta - 1}\right\}$$

Using $2\cos^2\theta - 1 = \cos 2\theta$

$y = \sin^{-1}(\cos 2\theta)$

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

Considering the limits,

$0 < x < 1$

$0 < \cos\theta < 1$

$$0 < \theta < \frac{\pi}{2}$$

$0 < 2\theta < \pi$

$0 > -2\theta > -\pi$

$$\frac{\pi}{2} > \frac{\pi}{2} - 2\theta > -\frac{\pi}{2}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

$$y = \frac{\pi}{2} - 2\theta$$

$$y = \frac{\pi}{2} - 2\cos^{-1}x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$$

$$\frac{dy}{dx} = 0 - 2\left(-\frac{1}{\sqrt{1 - x^2}}\right)$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1 - x^2}}$$

8. Question

Differentiate the following functions with respect to x:

$\sin^{-1}(1 - 2x^2)$, $0 < x < 1$

Answer

$$y = \sin^{-1}\{1 - 2x^2\}$$

$$\text{let } x = \sin\theta$$

Now

$$y = \sin^{-1}\left\{\sqrt{1 - 2\sin^2\theta}\right\}$$

Using $1 - 2\sin^2\theta = \cos 2\theta$

$y = \sin^{-1}(\cos 2\theta)$

$$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

Considering the limits,

$0 < x < 1$

$0 < \sin\theta < 1$

$0 < \theta < \frac{\pi}{2}$

$0 < 2\theta < \pi$

$0 > -2\theta > -\pi$

$\frac{\pi}{2} > \frac{\pi}{2} - 2\theta > -\frac{\pi}{2}$

Now,

$y = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$

$y = \frac{\pi}{2} - 2\theta$

$y = \frac{\pi}{2} - 2\sin^{-1}x$

Differentiating w.r.t x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\cos^{-1}x\right)$

$\frac{dy}{dx} = 0 - 2\left(\frac{1}{\sqrt{1-x^2}}\right)$

$\frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$

9. Question

Differentiate the following functions with respect to x:

$\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

Answer

$y = \cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

Let $x = a\cot\theta$

Now

$y = \cos^{-1}\left\{\frac{a\cot\theta}{\sqrt{a^2\cot^2\theta + a^2}}\right\}$

Using $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$y = \cos^{-1}\left\{\frac{a\cot\theta}{a\sqrt{\cot^2\theta + 1}}\right\}$

$y = \cos^{-1}\left\{\frac{a\cot\theta}{a\sqrt{\operatorname{cosec}^2\theta}}\right\}$

$y = \cos^{-1}\left\{\frac{\cot\theta}{\operatorname{cosec}\theta}\right\}$

$y = \cos^{-1}(\cos\theta)$

$y = \theta$

$y = \cot^{-1}\left(\frac{x}{a}\right)$

Differentiating w.r.t x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^{-1}\left(\frac{x}{a}\right)\right)$

$\frac{dy}{dx} = \frac{-a^2}{a^2 + x^2} \times \frac{1}{a}$

$\frac{dy}{dx} = \frac{-a}{a^2 + x^2}$

10. Question

Differentiate the following functions with respect to x:

$\sin^{-1}\left\{\frac{\sin x + \cos x}{\sqrt{2}}\right\}, -\frac{3\pi}{4} < x < \frac{\pi}{4}$

Answer

$y = \sin^{-1}\left\{\frac{\sin x + \cos x}{\sqrt{2}}\right\}$

Now

$y = \sin^{-1}\left\{\sin x \frac{1}{\sqrt{2}} + \cos x \frac{1}{\sqrt{2}}\right\}$

$y = \sin^{-1}\left\{\sin x \cos\left(\frac{\pi}{4}\right) + \cos x \sin\left(\frac{\pi}{4}\right)\right\}$

Using sin(A + B) = sinA cosB + cosA sinB

$y = \sin^{-1}\left\{\sin\left(x + \frac{\pi}{4}\right)\right\}$

Considering the limits,

$-\frac{3\pi}{4} < x < \frac{\pi}{4}$

Differentiating it w.r.t x,

$y = x + \frac{\pi}{4}$

$\frac{dy}{dx} = 1$

11. Question

Differentiate the following functions with respect to x:

$\cos^{-1}\left\{\frac{\cos x + \sin x}{\sqrt{2}}\right\}, \frac{\pi}{4} < x < \frac{\pi}{4}$

Answer

$y = \cos^{-1}\left\{\frac{\cos x + \sin x}{\sqrt{2}}\right\}$

Now

$$y = \cos^{-1}\left\{\cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}}\right\}$$

$$y = \cos^{-1}\left\{\cos x \cos\left(\frac{\pi}{4}\right) + \sin x \sin\left(\frac{\pi}{4}\right)\right\}$$

Using cos(A – B) = cosA cosB + sinA sinB

$$y = \cos^{-1}\left\{\cos\left(x - \frac{\pi}{4}\right)\right\}$$

Considering the limits,

$$-\frac{\pi}{4} < x < \frac{\pi}{4}$$

$$-\frac{\pi}{2} < x - \frac{\pi}{4} < 0$$

Now,

$$y = -x + \frac{\pi}{4}$$

Differentiating it w.r.t x,

$$\frac{dy}{dx} = -1$$

12. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}, -1 < x < 1$$

Answer

$$y = \tan^{-1}\left\{\frac{x}{1 + \sqrt{1 - x^2}}\right\}$$

Let x = sinθ

Now

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1 + \sqrt{1 - \sin^2\theta}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1 + \sqrt{\cos^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1 + \cos\theta}\right\}$$

Using $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin\theta \cos\theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

Considering the limits,

–1 < x < 1

–1 < sin θ < 1

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

Now,

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2}\sin^{-1} x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\sin^{-1} x\right)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

13. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}, -a < x < a$$

Answer

$$y = \tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}$$

Let x = a sinθ

Now

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{a+\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \tan^{-1}\left\{\frac{a\sin\theta}{a+a\sqrt{\cos^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

Using $2\cos^2\theta = 1 + \cos\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

Considering the limits,

–a < x < a

–1 < sin θ < 1

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$$

Now,

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2}\sin^{-1}\frac{x}{a}$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\sin^{-1}\frac{x}{a}\right)$$

$$\frac{dy}{dx} = \frac{a}{2\sqrt{a^2 - x^2}} \times \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}}$$

14. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left\{\frac{x + \sqrt{1 - x^2}}{\sqrt{2}}\right\}, -1 < x < 1$$

Answer

$$y = \sin^{-1}\left\{\frac{x + \sqrt{1 - x^2}}{\sqrt{2}}\right\}$$

Let x = sinθ

Now

$$y = \sin^{-1}\left\{\frac{\sin\theta + \sqrt{1 - \sin^2\theta}}{\sqrt{2}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \sin^{-1}\left\{\frac{\sin\theta + \cos\theta}{\sqrt{2}}\right\}$$

Now

$$y = \sin^{-1}\left\{\sin\theta\frac{1}{\sqrt{2}} + \cos\theta\frac{1}{\sqrt{2}}\right\}$$

$$y = \sin^{-1}\left\{\sin\theta\cos\left(\frac{\pi}{4}\right) + \cos\theta\sin\left(\frac{\pi}{4}\right)\right\}$$

Using sin(A + B) = sinA cosB + cosA sinB

$$y = \sin^{-1}\left\{\sin\left(\theta + \frac{\pi}{4}\right)\right\}$$

Considering the limits,

–1 < x < 1

–1 < sin θ < 1

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$-\frac{\pi}{2} + \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{\pi}{2} + \frac{\pi}{4}$$

$$-\frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{3\pi}{4}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\theta + \frac{\pi}{4}\right)\right\}$$

$$y = \theta + \frac{\pi}{4}$$

$$y = \sin^{-1} x + \frac{\pi}{4}$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

15. Question

Differentiate the following functions with respect to x:

$$\cos^{-1}\left\{\frac{x + \sqrt{1 - x^2}}{\sqrt{2}}\right\}, -1 < x < 1$$

Answer

$$y = \cos^{-1}\left\{\frac{x + \sqrt{1 - x^2}}{\sqrt{2}}\right\}$$

Let x = sinθ

Now

$$y = \cos^{-1}\left\{\frac{\sin\theta + \sqrt{1 - \sin^2\theta}}{\sqrt{2}}\right\}$$

Using $\sin^2\theta + \cos^2\theta = 1$

$$y = \cos^{-1}\left\{\frac{\sin\theta + \cos\theta}{\sqrt{2}}\right\}$$

Now

$$y = \cos^{-1}\left\{\sin\theta \frac{1}{\sqrt{2}} + \cos\theta \frac{1}{\sqrt{2}}\right\}$$

$$y = \cos^{-1}\left\{\sin\theta \sin\left(\frac{\pi}{4}\right) + \cos\theta \cos\left(\frac{\pi}{4}\right)\right\}$$

Using cos(A – B) = cosA cosB + sinA sinB

$y = \cos^{-1}\left\{\cos\left(\theta - \frac{\pi}{4}\right)\right\}$

Considering the limits,

–1 < x < 1

–1 < sin θ < 1

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$-\frac{\pi}{2} - \frac{\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{2} - \frac{\pi}{4}$

$-\frac{3\pi}{4} < \theta - \frac{\pi}{4} < \frac{\pi}{4}$

Now,

$y = \cos^{-1}\left\{\cos\left(\theta - \frac{\pi}{4}\right)\right\}$

$y = -\left(\theta - \frac{\pi}{4}\right)$

$y = -\sin^{-1} x + \frac{\pi}{4}$

Differentiating w.r.t x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(-\sin^{-1} x + \frac{\pi}{4}\right)$

$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$

16. Question

Differentiate the following functions with respect to x:

$\tan^{-1}\left\{\frac{4x}{1 - 4x^2}\right\}, -\frac{1}{2} < x < \frac{1}{2}$

Answer

$y = \tan^{-1}\left\{\frac{4x}{1 - 4x^2}\right\}$

Let 2x = tanθ

$y = \tan^{-1}\left\{\frac{2\tan\theta}{1 - \tan^2\theta}\right\}$

Using $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$y = \tan^{-1}(\tan 2\theta)$

Considering the limits,

$-\frac{1}{2} < x < \frac{1}{2}$

–1 < 2x < 1

–1 < tanθ < 1

$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$

$-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

Now,

$y = \tan^{-1}(\tan 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(2x)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} 2x)$$

$$\frac{dy}{dx} = 2 \times \frac{2}{1 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{4}{1 + 4x^2}$$

17. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{2^{x+1}}{1-4^x}\right), -\infty < x < 0$$

Answer

$$y = \tan^{-1}\left\{\frac{2^{x+1}}{1-4^x}\right\}$$

Let $2^x = \tan\theta$

$$y = \tan^{-1}\left\{\frac{2 \times 2^x}{1-(2^x)^2}\right\}$$

$$y = \tan^{-1}\left\{\frac{2\tan\theta}{1-\tan^2\theta}\right\}$$

Using $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$y = \tan^{-1}(\tan 2\theta)$

Considering the limits,

$-\infty < x < 0$

$2^{-\infty} < 2^x < 2^0$

$0 < \tan\theta < 1$

$0 < \theta < \frac{\pi}{4}$

$0 < 2\theta < \frac{\pi}{2}$

Now,

$y = \tan^{-1}(\tan 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(2^x)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} 2^x)$$

$$\frac{dy}{dx} = 2 \times \frac{2^x \log 2}{1 + (2^x)^2}$$

$$\frac{dy}{dx} = \frac{2^{x+1} \log 2}{1 + 4^x}$$

18. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{2a^x}{1-a^{2x}}\right), a < 1, -\infty < x < 0$$

Answer

$$y = \tan^{-1}\left\{\frac{2a^x}{1 - a^{2x}}\right\}$$

Let $a^x = \tan\theta$

$$y = \tan^{-1}\left\{\frac{2\tan\theta}{1 - \tan^2\theta}\right\}$$

Using $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

$$y = \tan^{-1}(\tan 2\theta)$$

Considering the limits,

$-\infty < x < 0$

$a^{-\infty} < a^x < a^0$

$0 < \tan\theta < 1$

$$0 < \theta < \frac{\pi}{4}$$

$$0 < 2\theta < \frac{\pi}{2}$$

Now,

$y = \tan^{-1}(\tan 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(a^x)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} a^x)$$

$$\frac{dy}{dx} = 2 \times \frac{a^x \log a}{1 + (a^x)^2}$$

$$\frac{dy}{dx} = \frac{2a^x \log a}{1 + a^{2x}}$$

19. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}, 0 < x < 1$$

Answer

$$y = \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$$

Let x = cos2θ

Now

$$y = \sin^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{2}\right\}$$

Using 1 – 2sin^{2}θ = cos2θ and 2cos^{2}θ – 1 = cos2θ

$$y = \sin^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{2}\right\}$$

Now

$$y = \sin^{-1}\left\{\sin\theta\frac{1}{\sqrt{2}}+\cos\theta\frac{1}{\sqrt{2}}\right\}$$

$$y = \sin^{-1}\left\{\sin\theta\cos\left(\frac{\pi}{4}\right)+\cos\theta\sin\left(\frac{\pi}{4}\right)\right\}$$

Using sin(A + B) = sinA cosB + cosA sinB

$$y = \sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\}$$

Considering the limits,

0 < x < 1

0 < cos 2θ < 1

$$0 < 2\theta < \frac{\pi}{2}$$

$$0 < \theta < \frac{\pi}{4}$$

Now,

$$y = \sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\}$$

$$y = \theta + \frac{\pi}{4}$$

$$y = \frac{1}{2}\cos^{-1}x + \frac{\pi}{4}$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\cos^{-1}x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\times\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}}$$

20. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{\sqrt{1+a^2x^2}-1}{ax}\right\}, x \neq 0$$

Answer

$$y = \tan^{-1}\left\{\frac{\sqrt{1+a^2x^2}-1}{ax}\right\}$$

Let ax = tanθ

Now

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\}$$

Using $\sec^2\theta = 1 + \tan^2\theta$

$$y = \tan^{-1}\left\{\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right\}$$

$$y = \tan^{-1}\left\{\frac{\sec\theta-1}{\tan\theta}\right\}$$

$$y = \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\}$$

Using $2\sin^2\theta = 1 - \cos2\theta$ and $2\sin\theta\cos\theta = \sin2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$y = \frac{\theta}{2}$$

$$y = \frac{1}{2}\tan^{-1}ax$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1}ax\right)$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{a}{1+(ax)^2}$$

$$\frac{dy}{dx} = \frac{a}{2(1+a^2x^2)}$$

21. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right), -\pi < x < \pi$$

Answer

$$y = \tan^{-1}\left\{\frac{\sin x}{1 + \cos x}\right\}$$

Function y is defined for all real numbers where cosx ≠ –1

Using $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin\theta\cos\theta = \sin 2\theta$

$$y = \tan^{-1}\left\{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right\}$$

$$y = \tan^{-1}\left\{\tan\frac{x}{2}\right\}$$

$$y = \frac{x}{2}$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}$$

22. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Answer

$$y = \sin^{-1}\left\{\frac{1}{\sqrt{1 + x^2}}\right\}$$

Let x = cotθ

Now

$$y = \sin^{-1}\left\{\frac{1}{\sqrt{1 + \cot^2\theta}}\right\}$$

Using, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Now

$$y = \sin^{-1}\left\{\frac{1}{\sqrt{\operatorname{cosec}^2\theta}}\right\}$$

$$y = \sin^{-1}\left\{\frac{1}{\operatorname{cosec}\theta}\right\}$$

$y = \sin^{-1}(\sin\theta)$

$y = \theta$

$y = \cot^{-1}x$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\cot^{-1}x)$$

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

23. Question

Differentiate the following functions with respect to x:

$$\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right), 0 < x < \infty$$

Answer

$$y = \cos^{-1}\left\{\frac{1-x^{2n}}{1+x^{2n}}\right\}$$

Let $x^n = \tan\theta$

Now

$$y = \cos^{-1}\left\{\frac{1-\tan^2\theta}{1+\tan^2\theta}\right\}$$

Using $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$y = \cos^{-1}\{\cos 2\theta\}$$

Considering the limits,

$0 < x < \infty$

$0 < x^n < \infty$

$0 < \theta < \frac{\pi}{2}$

Now,

$y = \cos^{-1}(\cos 2\theta)$

$y = 2\theta$

$y = \tan^{-1}(x^n)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(x^n))$$

$$\frac{dy}{dx} = \frac{2nx^{n-1}}{1+(x^n)^2}$$

$$\frac{dy}{dx} = \frac{2nx^{n-1}}{1+x^{2n}}$$

24. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right), x \in R$$

Answer

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

Using, $\sec^{-1}x = \frac{1}{\cos^{-1}x}$

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Using, $\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$$y = \frac{\pi}{2}$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right)$$

$$\frac{dy}{dx} = 0$$

25. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{a+x}{1-ax}\right)$$

Answer

$$y = \tan^{-1}\left(\frac{a+x}{1-ax}\right)$$

Using, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$y = \tan^{-1}x + \tan^{-1}a$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x + \tan^{-1}a)$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} + 0$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

26. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{xa}}\right)$$

Answer

$$y = \tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{xa}}\right)$$

Using, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$y = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a}$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a})$$

$$\frac{dy}{dx} = \frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(\sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}(1+x^2)}$$

27. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{a+b\ \tan\ x}{b-a\ \tan\ x}\right)$$

Answer

$$y = \tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$$

Dividing numerator and denominator by b

$$y = \tan^{-1}\left(\frac{\frac{a}{b}+\tan x}{1-\frac{a}{b}\tan x}\right)$$

$$y = \tan^{-1}\left(\frac{\tan\left(\tan^{-1}\frac{a}{b}\right)+\tan x}{1-\tan\left(\tan^{-1}\frac{a}{b}\right)\tan x}\right)$$

$$\text{Using}, \tan(x+y) = \left(\frac{\tan x+\tan y}{1-\tan x\tan y}\right)$$

$$y = \tan^{-1}\left(\tan\left(\tan^{-1}\frac{a}{b}+x\right)\right)$$

$$y = \tan^{-1}\frac{a}{b}+x$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b}+x\right)$$

$$\frac{dy}{dx} = 0+1$$

$$\frac{dy}{dx} = 1$$

28. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{a+bx}{b-ax}\right)$$

Answer

$$y = \tan^{-1}\left(\frac{a+bx}{b-ax}\right)$$

Dividing numerator and denominator by b

$$y = \tan^{-1}\left(\frac{\frac{a}{b}+x}{1-\frac{a}{b}x}\right)$$

$$\text{Using}, \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$y = \tan^{-1}\frac{a}{b} + \tan^{-1}x$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}\frac{a}{b} + \tan^{-1}x\right)$$

$$\frac{dy}{dx} = 0 + \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

29. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{x-a}{x+a}\right)$$

Answer

$$y = \tan^{-1}\left(\frac{x-a}{x+a}\right)$$

Dividing numerator and denominator by x

$$y = \tan^{-1}\left(\frac{1-\frac{a}{x}}{1+1\times\frac{a}{x}}\right)$$

$$\text{Using}, \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$y = \tan^{-1}1 - \tan^{-1}\frac{a}{x}$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}1 - \tan^{-1}\frac{a}{x}\right)$$

$$\frac{dy}{dx} = 0 - \frac{1}{1+\left(\frac{a}{x}\right)^2}\frac{d}{dx}\left(\frac{a}{x}\right)$$

$$\frac{dy}{dx} = -\frac{x^2}{a^2+x^2}\left(-\frac{a}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{a}{a^2+x^2}$$

30. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left(\frac{x}{1+6x^2}\right)$$

Answer

$$y = \tan^{-1}\left(\frac{x}{1+6x^2}\right)$$

Arranging the terms in equation

$$y = \tan^{-1}\left(\frac{3x - 2x}{1 + 3x \times 2x}\right)$$

Using, $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x - y}{1 + xy}\right)$

$$y = \tan^{-1}(3x) - \tan^{-1}(2x)$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(3x) - \tan^{-1}(2x))$$

$$\frac{dy}{dx} = \frac{3}{1 + (3x)^2} - \frac{2}{1 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{3}{1 + 9x^2} - \frac{2}{1 + 4x^2}$$

31. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{5x}{1 - 6x^2}\right\}, -\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$$

Answer

$$y = \tan^{-1}\left(\frac{5x}{1 - 6x^2}\right)$$

Arranging the terms in equation

$$y = \tan^{-1}\left(\frac{3x + 2x}{1 - 3x \times 2x}\right)$$

Using, $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$

$$y = \tan^{-1}(3x) + \tan^{-1}(2x)$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(3x) + \tan^{-1}(2x))$$

$$\frac{dy}{dx} = \frac{3}{1 + (3x)^2} + \frac{2}{1 + (2x)^2}$$

$$\frac{dy}{dx} = \frac{3}{1 + 9x^2} + \frac{2}{1 + 4x^2}$$

32. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{\cos x + \sin x}{\cos x - \sin x}\right\}, -\frac{\pi}{4} < x < \frac{\pi}{4}$$

Answer

$$y = \tan^{-1}\left(\frac{\cos x + \sin x}{\cos x - \sin x}\right)$$

Dividing numerator and denominator by cosx

$$y = \tan^{-1}\left(\frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}\right)$$

$$y = \tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)$$

$$y = \tan^{-1}\left(\frac{\tan\left(\frac{\pi}{4}\right)+\tan x}{1-\tan\left(\frac{\pi}{4}\right)\tan x}\right)$$

$$\text{Using, } \tan(x+y) = \left(\frac{\tan x+\tan y}{1-\tan x\tan y}\right)$$

$$y = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+x\right)\right)$$

$$y = \frac{\pi}{4}+x$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+x\right)$$

$$\frac{dy}{dx} = 0+1$$

$$\frac{dy}{dx} = 1$$

33. Question

Differentiate the following functions with respect to x:

$$\tan^{-1}\left\{\frac{x^{1/3}+a^{1/3}}{1-(a\ x)^{1/3}}\right\}$$

Answer

$$y = \tan^{-1}\left(\frac{x^{\frac{1}{3}}+\ a^{\frac{1}{3}}}{1-(ax)^{\frac{1}{3}}}\right)$$

Arranging the terms in equation

$$y = \tan^{-1}\left(\frac{x^{\frac{1}{3}}+\ a^{\frac{1}{3}}}{1-x^{\frac{1}{3}}\times a^{\frac{1}{3}}}\right)$$

$$\text{Using, } \tan^{-1}x+\ \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$y = \tan^{-1}(x^{\frac{1}{3}})+\tan^{-1}(a^{\frac{1}{3}})$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\tan^{-1}(x^{\frac{1}{3}})+\tan^{-1}(a^{\frac{1}{3}})\right)$$

$$\frac{dy}{dx} = \frac{3}{1+\left(x^{\frac{1}{3}}\right)^{2}}\times\frac{d}{dx}\left(x^{\frac{1}{3}}\right)$$

$$\frac{dy}{dx} = \frac{3}{1+\left(x^{\frac{1}{3}}\right)^2} \times \frac{1}{3}\left(x^{-\frac{2}{3}}\right)$$

$$\frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}\left(1+\left(x^{\frac{1}{3}}\right)^2\right)}$$

34. Question

Differentiate the following functions with respect to x:

$$\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$$

Answer

$$y = \sin^{-1}\left\{\frac{2^{x+1}}{1+4^x}\right\}$$

For function to be defined

$$-1 \leq \frac{2^{x+1}}{1+4^x} \leq 1$$

Since the quantity is positive always

$$\Rightarrow 0 \leq \frac{2^{x+1}}{1+4^x} \leq 1$$

$$\Rightarrow 0 < 2^{x+1} \leq 1 + 4^x$$

$$\Rightarrow 0 < 2 \leq 2^{-x} + 2^x$$

This condition is always true, hence function is always defined.

$$y = \sin^{-1}\left\{\frac{2 \times 2^x}{1+(2^2)^x}\right\}$$

Let 2^X = tanθ

$$y = \sin^{-1}\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}$$

$$\text{Using } \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

Now,

$y = \sin^{-1}(\sin 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(2^x)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} 2^x)$$

$$\frac{dy}{dx} = 2 \times \frac{2^x \log 2}{1+(2^x)^2}$$

$$\frac{dy}{dx} = \frac{2^{x+1}\log 2}{1+4^x}$$

35. Question

If $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, 0 < x < 1, prove that $\frac{dy}{dx}=\frac{4}{1+x^2}$.

Answer

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$$

Put x = tan θ

$$\text{Using}, \sec^{-1}x=\frac{1}{\cos^{-1}x}$$

$$y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$\text{Using}, \frac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta \text{ and } \frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$$

$$y=\sin^{-1}(\sin 2\theta)+\cos^{-1}(\cos 2\theta)$$

Considering the limits

0 < x < 1

0 < tan θ < 1

$$0<\theta<\frac{\pi}{4}$$

$$0<2\theta<\frac{\pi}{2}$$

Now,

y = 2θ + 2θ

y = 4θ

y = 4$\tan^{-1}$x

Differentiating w.r.t x we get

$$\frac{dy}{dx}=\frac{d}{dx}(4\tan^{-1}x)$$

$$\frac{dy}{dx}=\frac{4}{1+x^2}$$

36. Question

If $y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, 0 < x < ∞, prove that $\frac{dy}{dx}=\frac{2}{1+x^2}$.

Answer

$$y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

Put x = tan θ

$$y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{1+\tan^2\theta}}\right)$$

Using, $\sec^2\theta = 1 + \tan^2\theta$

$$y = \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\sec^2\theta}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{\sec^2\theta}}\right)$$

$$y = \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right) + \cos^{-1}\left(\frac{1}{\sec\theta}\right)$$

$$y = \sin^{-1}(\sin\theta) + \cos^{-1}(\cos\theta)$$

Considering the limits

$0 < x < \infty$

$0 < \tan\theta < \infty$

$$0 < \theta < \frac{\pi}{2}$$

Now,

$y = \theta + \theta$

$y = 2\theta$

$y = 2\tan^{-1}x$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

37 A. Question

Differentiate the following with respect to x:

cos–1 (sin x)

Answer

$y = \cos^{-1}(\sin x)$

Function is defined for all x

$$y = \cos^{-1}\left(\cos\left(\frac{\pi}{2} - x\right)\right)$$

$$y = \frac{\pi}{2} - x$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - x\right)$$

$$\frac{dy}{dx} = -1$$

37 B. Question

Differentiate the following with respect to x:

$$\cot^{-1}\left(\frac{1-x}{1+x}\right)$$

Answer

$$y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$$

Put x = tan θ

$$y = \cot^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$y = \cot^{-1}\left(\frac{\tan\left(\frac{\pi}{4}\right)-\tan\theta}{1+\tan\left(\frac{\pi}{4}\right)\tan\theta}\right)$$

$$\text{Using}, \tan(x-y) = \left(\frac{\tan x - \tan y}{1+\tan x \tan y}\right)$$

$$y = \cot^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)$$

$$y = \cot^{-1}\left(\cot\left(\frac{\pi}{2}-\frac{\pi}{4}+\theta\right)\right)$$

$$y = \frac{\pi}{4}+\theta$$

$$y = \frac{\pi}{4}+\tan^{-1}x$$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\tan^{-1}x\right)$$

$$\frac{dy}{dx} = 0+\frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

38. Question

If $y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$, show that $\frac{dy}{dx}$ is independent of x.

Answer

$$y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$$

Multiplying numerator and denominator

$$y = \cot^{-1}\left\{\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}\right\}$$

$$y = \cot^{-1}\left\{\frac{1+\sin x+1-\sin x+2\sqrt{1+\sin x}\sqrt{1-\sin x}}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2}\right\}$$

$$y = \cot^{-1}\left\{\frac{2+2\sqrt{1-\sin^2 x}}{(1+\sin x)-(1-\sin x)}\right\}$$

$$y = \cot^{-1}\left\{\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}\right\}$$

$y = \cot^{-1}\left\{\frac{2(1+\cos x)}{2\sin x}\right\}$

Using $\sin^2\theta + \cos^2\theta = 1$

$y = \cot^{-1}\left\{\frac{1+\cos x}{\sin x}\right\}$

Using $2\sin\theta\cos\theta = \sin 2\theta$ and $2\cos^2\theta - 1 = \cos 2\theta$

$y = \cot^{-1}\left\{\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$

Now

$y = \cot^{-1}\left\{\cot\frac{x}{2}\right\}$

$y = \frac{x}{2}$

Differentiating w.r.t x, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = \frac{1}{2}$

39. Question

If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$, $x > 0$, prove that $\frac{dy}{dx} = \frac{4}{1+x^2}$.

Answer

$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \sec^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Using, $\sec^{-1}x = \frac{1}{\cos^{-1}x}$

$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

Using, $\frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$ and $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$y = \sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta)$

Considering the limits

$0 < x < \infty$

$0 < \tan\theta < \infty$

$0 < \theta < \frac{\pi}{2}$

$0 < 2\theta < \pi$

Now,

y = 2θ + 2θ

y = 4θ

y = $4\tan^{-1}x$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(4\tan^{-1}x)$$

$$\frac{dy}{dx} = \frac{4}{1+x^2}$$

40. Question

If $y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$, x > 0. Find $\frac{dy}{dx}$.

Answer

$$y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$\text{Using}, \sec^{-1}x = \frac{1}{\cos^{-1}x}$$

$$y = \cos^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$$

$$\text{Using}, \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$$

$$y = \frac{\pi}{2}$$

Now differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}\right)$$

$$\frac{dy}{dx} = 0$$

41. Question

If $y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$, find $\frac{dy}{dx}$.

Answer

$$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-x}{1+x}}\right\}\right]$$

Put x =cos 2θ

$$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}\right\}\right]$$

$$\text{Using } 2\cos^2\theta - 1 = \cos 2\theta \text{ and } 1 - 2\sin^2\theta = \cos 2\theta$$

$y = \sin\left[2\tan^{-1}\left\{\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right\}\right]$

$y = \sin[2\tan^{-1}(\tan\theta)]$

$y = \sin(2\theta)$

$y = \sin\left[\frac{2}{2} \times \cos^{-1} x\right]$

Using $\cos^{-1} x = \sin^{-1}\sqrt{1 - x^2}$

$y = \sin\left[\sin^{-1}\sqrt{1 - x^2}\right]$

$y = \sqrt{1 - x^2}$

Differentiating w.r.t x we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{1 - x^2}\right)$

$\frac{dy}{dx} = \frac{1}{2\sqrt{1 - x^2}}\frac{d}{dx}(1 - x^2)$

$\frac{dy}{dx} = -\frac{2x}{2\sqrt{1 - x^2}}$

$\frac{dy}{dx} = -\frac{x}{\sqrt{1 - x^2}}$

42. Question

If $y = \cos^{-1}(2x) + 2\cos^{-1}\sqrt{1 - 4x^2}$, $0 < x < \frac{1}{2}$, find $\frac{dy}{dx}$.

Answer

$y = \cos^{-1}(2x) + 2\cos^{-1}\sqrt{1 - 4x^2}$

Put $2x = \cos\theta$

$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}\sqrt{1 - \cos^2\theta}$

$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}(\sin\theta)$

$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}\left(\cos\left(\frac{\pi}{2} - \theta\right)\right)$

Considering the limits

$0 < x < \frac{1}{2}$

$0 < 2x < 1$

$0 < \cos\theta < 1$

$0 < \theta < \frac{\pi}{2}$

$0 > -\theta > -\frac{\pi}{2}$

$\frac{\pi}{2} > \frac{\pi}{2} - \theta > 0$

Now,

$$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}\left(\cos\left(\frac{\pi}{2} - \theta\right)\right)$$

$$y = \theta + 2\left(\frac{\pi}{2} - \theta\right)$$

$$y = \pi - \theta$$

$y = \pi - \cos^{-1}(2x)$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(\pi - \cos^{-1}(2x))$$

$$\frac{dy}{dx} = 0 - \left[\frac{-2}{\sqrt{1 - (2x)^2}}\right]$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1 - 4x^2}}$$

43. Question

If the derivative of $\tan^{-1}(a + bx)$ takes the value 1 at x = 0, prove that $1 + a^2 = b$.

Answer

$y = \tan^{-1}(a + bx)$

And y'(0) = 1

Now

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(a + bx))$$

$$\frac{dy}{dx} = \frac{b}{1 + (a + bx)^2}$$

At x = 0,

$$\frac{dy}{dx} = \frac{b}{1 + (a + b(0))^2}$$

$$\frac{b}{1 + a^2} = 1$$

$\Rightarrow b = 1 + a^2$

44. Question

If $y = \cos^{-1}(2x) + 2\cos^{-1}\sqrt{1-4x^2}$, < x < 0, find $\frac{dy}{dx}$.

Answer

$$y = \cos^{-1}(2x) + 2\cos^{-1}\sqrt{1 - 4x^2}$$

Put $2x = \cos\theta$

$$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}\sqrt{1 - \cos^2\theta}$$

$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}(\sin\theta)$

$$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}\left(\cos\left(\frac{\pi}{2} - \theta\right)\right)$$

Considering the limits

$$-\frac{1}{2} < x < 0$$

-1 < 2x < 0

-1 < cosθ < 0

$$\frac{\pi}{2} < \theta < \pi$$

$$-\frac{\pi}{2} > -\theta > -\pi$$

$$0 > \frac{\pi}{2} - \theta > -\frac{\pi}{2}$$

Now,

$$y = \cos^{-1}(\cos\theta) + 2\cos^{-1}\left(\cos\left(\frac{\pi}{2} - \theta\right)\right)$$

$$y = \theta + 2\left\{-\left(\frac{\pi}{2} - \theta\right)\right\}$$

$$y = -\pi + 3\theta$$

$y = -\pi + \cos^{-1}(2x)$

Differentiating w.r.t x we get

$$\frac{dy}{dx} = \frac{d}{dx}(-\pi + 3\cos^{-1}(2x))$$

$$\frac{dy}{dx} = 0 + 3\left[\frac{-2}{\sqrt{1-(2x)^2}}\right]$$

$$\frac{dy}{dx} = \frac{-6}{\sqrt{1-4x^2}}$$

45. Question

If $y = \tan^{-1}\left\{\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right\}$, find $\frac{dy}{dx}$.

Answer

$$y = \tan^{-1}\left\{\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right\}$$

Put x = cos 2θ

$$y = \tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}\right\}$$

Using $2\cos^2\theta - 1 = \cos 2\theta$ and $1 - 2\sin^2\theta = \cos 2\theta$

$$y = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta} - \sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}\right\}$$

$$y = \tan^{-1}\left\{\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right\}$$

Dividing by cosθ both numerator and denominator,

$$y=\tan^{-1}\left\{\frac{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}\right\}$$

$$y=\tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\}$$

$$y=\tan^{-1}\left\{\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}$$

$$y=\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\}$$

$$y=\frac{\pi}{4}-\theta$$

$$y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)$$

$$\frac{dy}{dx}=0-\frac{1}{2}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$$

46. Question

If $y=\cos^{-1}\left\{\frac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right\}$, find $\frac{dy}{dx}$.

Answer

$$y=\cos^{-1}\left\{\frac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right\}$$

Put x = cos θ

$$y=\cos^{-1}\left\{\frac{2\cos\theta-3\sqrt{1-\cos^2\theta}}{\sqrt{13}}\right\}$$

$$y=\cos^{-1}\left\{\frac{2}{\sqrt{13}}\cos\theta-\frac{3}{\sqrt{13}}\sin\theta\right\}$$

$$\text{let }\cos\phi=\frac{2}{\sqrt{13}}$$

Now,

$$\Rightarrow \sin^2\phi=1-\cos^2\phi$$

$$\Rightarrow \sin\phi=\sqrt{1-\cos^2\phi}$$

$$\Rightarrow \sin\phi=\sqrt{1-\frac{4}{13}}$$

$$\Rightarrow \sin\phi=\frac{3}{\sqrt{13}}$$

Again,

$$y = \cos^{-1}\{\cos\phi\cos\theta - \sin\phi\sin\theta\}$$

Using $\cos A\cos B - \sin A\sin B = \cos(A + B)$

$$y = \cos^{-1}\{\cos(\phi + \theta)\}$$

$$y = \phi + \theta$$

$$y = \cos^{-1}\left\{\frac{2}{\sqrt{13}}\right\} + \cos^{-1}x$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}\left\{\frac{2}{\sqrt{13}}\right\} + \cos^{-1}x\right)$$

$$\frac{dy}{dx} = 0 + \left(-\frac{1}{\sqrt{1 - x^2}}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

47. Question

Differentiate $\sin^{-1}\left\{\frac{2^{x+1}.3^x}{1+(36)^x}\right\}$ with respect to x.

Answer

$$y = \sin^{-1}\left\{\frac{2^{x+1}.3^x}{1 + (36)^x}\right\}$$

$$y = \sin^{-1}\left\{\frac{2 \times 2^x \times 3^x}{1 + (6^2)^x}\right\}$$

$$y = \sin^{-1}\left\{\frac{2 \times 6^x}{1 + (6^x)^2}\right\}$$

Put $6^x = \tan\theta$

$$y = \sin^{-1}\left\{\frac{2\tan\theta}{1 + \tan^2\theta}\right\}$$

Using $\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$

Now,

$y = \sin^{-1}(\sin 2\theta)$

$y = 2\theta$

$y = 2\tan^{-1}(6^x)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}6^x)$$

$$\frac{dy}{dx} = 2 \times \frac{6^x\log 6}{1 + (6^x)^2}$$

$$\frac{dy}{dx} = \frac{2 \times 6^x\log 6}{1 + 6^{2x}}$$

48. Question

If $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$.

Answer

$y = \sin^{-1}\left\{6x\sqrt{1-9x^2}\right\}$

$y = \sin^{-1}\left\{2 \times 3x\sqrt{1-(3x)^2}\right\}$

let $3x = \cos\theta$

$y = \sin^{-1}\left\{2 \times \sin\theta\sqrt{1-\cos^2\theta}\right\}$

Using $\sin^2\theta + \cos^2\theta = 1$

$y = \sin^{-1}\{2 \times \sin\theta\cos\theta\}$

Using $2\sin\theta\cos\theta = \sin2\theta$

$y = \sin^{-1}(\sin2\theta)$

Considering the limits,

$-\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$

$-\frac{1}{\sqrt{2}} < 3x < \frac{1}{\sqrt{2}}$

$-\frac{1}{\sqrt{2}} < \cos\theta < \frac{1}{\sqrt{2}}$

$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$

$-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

For

$0 < 2\theta < \frac{\pi}{2}$

Now, $y = \sin^{-1}(\sin2\theta)$

$y = 2\theta$

$y = 2\cos^{-1}x$

Differentiating w.r.t x, we get

$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

For

$-\frac{\pi}{2} < 2\theta < 0$

Now, $y = \sin^{-1}(\sin2\theta)$

$y = -2\theta$

$y = -2\cos^{-1}x$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Exercise 11.4

1. Question

Find $\frac{dy}{dx}$ in each of the following:

$xy = c^2$

Answer

We are given with an equation $xy = c^2$; we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

By using the product rule on the left hand side,

$$\frac{d(xy)}{dx} = \frac{dc^2}{dx}$$

$x\frac{dy}{dx} + y(1) = 0$

$$\frac{dy}{dx} = \frac{-y}{x}$$

Or we can further solve it by putting the value of y,

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

2. Question

Find $\frac{dy}{dx}$ in each of the following:

$y^3 - 3xy^2 = x^3 + 3x^2 y$

Answer

We are given with an equation $y^3 - 3xy^2 = x^3 + 3x^2y$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$3y^2 \frac{dy}{dx} - 3[y^2(1) + 2xy \frac{dy}{dx}] = 3x^2 + 3[2xy + x^2\frac{dy}{dx}]$

Taking $\frac{dy}{dx}$ terms to left hand side and taking common $\frac{dy}{dx}$, we get,

$\frac{dy}{dx}[3y^2 - 6xy - 3x^2] = 3x^2 + 6xy + 3y^2$

$$\frac{dy}{dx} = \frac{3x^2 + 3y^2 + 6xy}{3y^2 - 3x^2 - 6xy} = \frac{x^2 + y^2 + 2xy}{y^2 - x^2 - 2xy}$$

3. Question

Find $\frac{dy}{dx}$ in each of the following:

$x^{2/3} + y^{2/3} = a^{2/3}$

Answer

We are given with an equation $x^{2/3} + y^{2/3} = a^{2/3}$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{2}{3}\frac{1}{x^{1/3}} + \frac{2}{3}\frac{1}{y^{1/3}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-y^{1/3}}{x^{1/3}}$$

Or we can write it as,

$$\frac{dy}{dx} = \frac{-\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}}$$

4. Question

Find $\frac{dy}{dx}$ in each of the following:

4x + 3y = log (4x – 3y)

Answer

We are given with an equation 4x + 3y = log(4x – 3y), we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$$4 + 3\frac{dy}{dx} = \frac{1}{(4x-3y)}[4 - 3\frac{dy}{dx}]$$

$$3\frac{dy}{dx} + \frac{3}{(4x-3y)}\frac{dy}{dx} = \frac{4}{(4x-3y)} - 4$$

$$\frac{dy}{dx}[1 + \frac{1}{4x-3y}] = \frac{12y - 16x + 4}{3(4x-3y)}$$

$$\frac{dy}{dx} = \frac{\frac{12y - 16x + 4}{3(4x-3y)}}{\frac{4x - 3y + 1}{4x - 3y}} = \frac{12y - 16x + 4}{12x - 9y + 3}$$

5. Question

Find $\frac{dy}{dx}$ in each of the following:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Answer

We are given with an equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-xb^2}{ya^2}$$

6. Question

Find $\frac{dy}{dx}$ in each of the following:

$x^5 + y^5 = 5xy$

Answer

We are given with an equation $x^5 + y^5 = 5xy$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$5x^4 + 5y^4\frac{dy}{dx} = 5[y(1) + x\frac{dy}{dx}]$

$\frac{dy}{dx}[y^4 - x] = y - x^4$

$$\frac{dy}{dx} = \frac{y - x^4}{y^4 - x}$$

7. Question

Find $\frac{dy}{dx}$ in each of the following:

$(x + y)^2 = 2axy$

Answer

We are given with an equation $(x + y)^2 = 2axy$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$2(x + y)(1 + \frac{dy}{dx}) = 2a[y + x\frac{dy}{dx}]$

$x + y + \frac{dy}{dx}[x + y] = a[y + x\frac{dy}{dx}]$

$\frac{dy}{dx}[x + y - ax] = ay - x - y$

$$\frac{dy}{dx} = \frac{y(a-1) - x}{y + x(1-a)}$$

8. Question

Find $\frac{dy}{dx}$ in each of the following:

$(x^2 + y^2)^2 = xy$

Answer

We are given with an equation $(x^2 + y^2)^2 = xy$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$2(x^2 + y^2)[2x + 2y\frac{dy}{dx}] = y(1) + x\frac{dy}{dx}$

$\frac{dy}{dx}[4y(x^2 + y^2) - x] = y - 4x(x^2 + y^2)$

$$\frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$$

9. Question

Find $\frac{dy}{dx}$ in each of the following:

$\tan^{-1}(x^2 + y^2) = a$

Answer

We are given with an equation $\tan^{-1}(x^2 + y^2) = a$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{1}{x^2 + y^2}\left(2x + 2y\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

10. Question

Find $\frac{dy}{dx}$ in each of the following:

$$e^{x-y} = \log\left(\frac{x}{y}\right)$$

Answer

We are given with an equation $e^{x-y} = \log(\frac{x}{y}) = \log x - \log y$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$e^{x-y}(1 - \frac{dy}{dx}) = \frac{1}{x\ln 10} - \frac{1}{y\ln 10}\frac{dy}{dx}$

$\frac{dy}{dx}[\frac{1}{y\ln 10} - e^{x-y}] = \frac{1}{x\ln 10} - e^{x-y}$

$$\frac{dy}{dx} = \frac{\frac{1}{x\ln 10} - e^{x-y}}{\frac{1}{y\ln 10} - e^{x-y}}$$

$$\frac{dy}{dx} = \frac{\frac{1 - x\ln 10\, e^{x-y}}{x}}{\frac{1 - y\ln 10\, e^{x-y}}{y}} = \frac{y(1 - x\ln 10\, e^{x-y})}{x(1 - y\ln 10\, e^{x-y})}$$

11. Question

Find $\frac{dy}{dx}$ in each of the following:

$\sin xy + \cos(x + y) = 1$

Answer

We are given with an equation $\sin xy + \cos(x + y) = 1$, we have to find $\frac{dy}{dx}$ of it, so by differentiating the equation on both sides with respect to x, we get,

$\cos xy\,(y + x\frac{dy}{dx}) - \sin(x + y)\,(1 + \frac{dy}{dx}) = 0$

$\frac{dy}{dx}[x\cos xy - \sin(x + y)] = \sin(x + y) - y\cos xy$

$$\frac{dy}{dx} = \frac{\sin(x + y) - y\cos xy}{x\cos xy - \sin(x + y)}$$

12. Question

If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Answer

We are given with an equation $\sqrt{1-x^2} + \sqrt{1-y^2}$ = a(x – y), we have to prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

Put x = sinA and y = sinB in the given equation,

$\sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B}$ = a(sinA – sinB)

cosA + cosB = a(sinA – sinB)

$2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2}) = a2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$

By using cosA + cosB = $2\cos(\frac{A-B}{2})\cos(\frac{A+B}{2})$ and sinA – sinB = $2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2})$

$a = \cot(\frac{A-B}{2})$

$\cot^{-1}a = \frac{A-B}{2}$

$2\cot^{-1}a = A - B$

$2\cot^{-1}a = \sin^{-1}x - \sin^{-1}y$

$0 = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

13. Question

If $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$, prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Answer

We are given with an equation $y\sqrt{1-x^2} + x\sqrt{1-y^2}$ = 1, we have to prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

Put x = sinA and y = sinB in the given equation,

$\sin B\sqrt{1-\sin^2 A} + \sin A\sqrt{1-\sin^2 B} = 1$

sinB cosA + sinA cosB = 1

sin(A + B) = 1

$\sin^{-1}1 = A + B$

$\frac{\pi}{2}$ = $\sin^{-1}x + \sin^{-1}y$

Differentiating we get,

$0 = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

14. Question

If xy = 1, prove that $\frac{dy}{dx} + y^2 = 0.$

Answer

We are given with an equation xy = 1, we have to prove that $\frac{dy}{dx}$ + y^2 = 0 by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

By using product rule, we get,

$y(1) + x\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-y}{x}$$

Or we can further solve it by using the given equation,

$$\frac{dy}{dx} = \frac{-y}{\frac{1}{y}} = -y^2$$

By putting this value in the L.H.S. of the equation, we get,

$-y^2 + y^2 = 0$ = R.H.S.

15. Question

If $xy^2 = 1$, prove that $2\frac{dy}{dx} + y^3 = 0.$

Answer

We are given with an equation $xy^2 = 1$, we have to prove that $2\frac{dy}{dx} + y^3 = 0$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$y^2(1) + 2xy\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-y}{2x}$$

Or we can further solve it by using the given equation,

$$\frac{dy}{dx} = \frac{-y}{2\frac{1}{y^2}}$$

$$\frac{dy}{dx} = \frac{-y^3}{2}$$

By putting this value in the L.H.S. of the equation, we get,

$2(\frac{-y^3}{2}) + y^3 = 0$ = R.H.S.

16. Question

If $\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $(1+x)^2 \frac{dy}{dx} + 1 = 0$.

Answer

We are given with an equation $xy^2 = 1$, we have to prove that $2\frac{dy}{dx} + y^3 = 0$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove

But first we need to simplify this equation in accordance with our result, which is that in our result there is no square root and our derivative is only in the form of x.

$x\sqrt{1 + y} + y\sqrt{1 + x} = 0$

$x\sqrt{1 + y} = -y\sqrt{1 + x}$

Squaring both sides,

$x^2(1 + y) = y^2(1 + x)$

$x^2 + x^2y = y^2 + xy^2$

$x^2 - y^2 = xy^2 - x^2y$

$(x - y)(x + y) = xy(y - x)$

$x + y = - xy$

$y = \frac{-x}{1 + x}$

So, now by differentiating the equation on both sides with respect to x, we get,

By using quotient rule, we get,

$$\frac{dy}{dx} = \frac{(1 + x)(-1) - (-x)(1)}{(1 + x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{(1 + x)^2}$$

17. Question

If $\log\sqrt{x^2 + y^2} = \tan^{-1}\left(\frac{y}{x}\right)$, prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

Answer

We are given with an equation $\log\sqrt{x^2 + y^2} = \tan^{-1}(\frac{y}{x})$, we have to prove that $\frac{dy}{dx} = \frac{y + x}{y-x}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$\log(x^2 + y^2) = 2\tan^{-1}(\frac{y}{x})$

$$\frac{2x + 2y\frac{dy}{dx}}{x^2 + y^2} = \frac{2}{1 + \left(\frac{y}{x}\right)^2} \frac{x\frac{dy}{dx} - y(1)}{x^2}$$

$x + y\frac{dy}{dx} = x\frac{dy}{dx} - y$

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

18. Question

If $\sec\left(\frac{x+y}{x-y}\right) = a$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Answer

We are given with an equation $\sec(\frac{x+y}{x-y}) = a$, we have to prove that $\frac{dy}{dx} = \frac{y}{x}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$\sec(\frac{x+y}{x-y}) \tan(\frac{x+y}{x-y}) [\frac{(x-y)\left(1+\frac{dy}{dx}\right)-(x+y)(1-\frac{dy}{dx})}{(x-y)^2}] = 0$

$[\frac{(x-y)\left(1+\frac{dy}{dx}\right)-(x+y)(1-\frac{dy}{dx})}{(x-y)^2}] = 0$

$-2y + 2x\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{y}{x}$$

19. Question

If $\tan^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = a$, prove that $\frac{dy}{dx} = \frac{x}{y}\frac{(1-\tan a)}{(1+\tan a)}$.

Answer

We are given with an equation $\tan^{-1}(\frac{x^2-y^2}{x^2+y^2}) = a$, we have to prove that $\frac{dy}{dx} = \frac{x}{y}\frac{(1-\tan a)}{(1+\tan a)}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$\frac{x^2-y^2}{x^2+y^2} = \tan a$

$x^2 - y^2 = (x^2 + y^2)\tan a$

Now differentiating with respect to x, we get,

$2x - 2y\frac{dy}{dx} = (2x + 2y\frac{dy}{dx})\tan a$

$\frac{dy}{dx}[y\tan a + y] = x - x\tan x$

$$\frac{dy}{dx} = \frac{x - x\tan a}{y + y\tan a}$$

$$\frac{dy}{dx} = \frac{x}{y}\frac{(1 - \tan a)}{(1 + \tan a)}$$

20. Question

If xy log (x + y) = 1, prove that $\frac{dy}{dx}=\frac{y\left(x^{2}y+x+y\right)}{x\left(xy^{2}+x+y\right)}$.

Answer

We are given with an equation xy log(x + y) = 1, we have to prove that $\frac{dy}{dx} = \frac{y(x^2y+x+y)}{x(y^2x+x+y)}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

By using the triple product rule, which is, $\frac{d(uvw)}{dx} = uw\frac{dv}{dx} + vw\frac{du}{dx} + uv\frac{dw}{dx}$,

(1)y log(x + y) + x$\frac{dy}{dx}$ log(x + y) + xy$\frac{(1+\frac{dy}{dx})}{(x+y)}$ = 0

From the equation put log(x + y) = $\frac{1}{xy}$

$\frac{y}{xy} + \frac{x}{xy}\frac{dy}{dx} + \frac{xy}{(x+y)}(1 + \frac{dy}{dx}) = 0$

$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + \frac{xy}{(x+y)} + \frac{xy}{(x+y)}\frac{dy}{dx} = 0$

$\frac{x^2y+x+y}{(x+y)x} + \frac{dy}{dx}[\frac{y^2x+x+y}{(x+y)y}] = 0$

$$\frac{dy}{dx} = -\frac{\frac{x^2y + x + y}{(x + y)x}}{\frac{y^2x + x + y}{(x + y)y}} = \frac{-(x^2y + x + y)y}{(y^2x + x + y)x}$$

21. Question

If y = x sin (a + y), prove that $\frac{dy}{dx}=\frac{\sin^{2}(a+y)}{\sin(a+y)-y\cos(a+y)}$.

Answer

We are given with an equation y = x sin(a + y), we have to prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y)-y\cos(a+y)}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$\frac{dy}{dx}$ = (1) sin(a + y) + x cos(a + y)$\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\sin(a + y)}{1 - x\cos(a + y)}$$

We can further solve it by using the given equation,

$$\frac{dy}{dx} = \frac{\sin(a + y)}{1 - \frac{y}{\sin(a + y)}\cos(a + y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin(a + y) - y\cos(a + y)}$$

22. Question

If x sin (a + y) + sin a cos (a + y) = 0, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Answer

We are given with an equation x sin(a + y) + sina cos(a + y) = 0 , we have to prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$\tan(a + y) = \frac{-\sin a}{x}$

$\sec^2(a + y)\frac{dy}{dx} = \frac{\sin a}{x^2}$

we can further solve it by using the given equation,

$\sec^2(a + y)\frac{dy}{dx} = \frac{\tan^2(a+y)}{\sin^2 a}\sin a$

$$\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$$

23. Question

If y – x sin y, prove that $\frac{dy}{dx} = \frac{\sin y}{(1 - x\cos y)}$.

Answer

We are given with an equation y = x siny , we have to prove that $\frac{dy}{dx} = \frac{\sin y}{1-x\cos y}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$\frac{dy}{dx}$ = siny + x cosy $\frac{dy}{dx}$

$\frac{dy}{dx}$[1 – x cosy] = siny

$$\frac{dy}{dx} = \frac{\sin y}{1 - x\cos y}$$

24. Question

If $y\sqrt{x^2+1} = \log\left(\sqrt{x^2+1} - x\right)$, show that $\left(x^2+1\right)\frac{dy}{dx} + xy + 1 = 0$.

Answer

We are given with an equation $y\sqrt{x^2 + 1} = \log(\sqrt{x^2 + 1} - x)$, we have to prove that

$(x^2 + 1)\frac{dy}{dx}$ + xy + 1 = 0 by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{2x}{2\sqrt{x^2 + 1}}y + \sqrt{x^2 + 1}\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1} - x}[\frac{2x}{2\sqrt{x^2 + 1}} - 1]$$

$$\frac{x}{\sqrt{x^2 + 1}}y + \sqrt{x^2 + 1}\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1} - x}[\frac{x - \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}]$$

$$\frac{xy + (x^2 + 1)\frac{dy}{dx}}{\sqrt{x^2 + 1}} = [\frac{-1}{\sqrt{x^2 + 1}}]$$

$xy + (x^2 + 1)\frac{dy}{dx} = -1$

$xy + (x^2 + 1)\frac{dy}{dx} + 1 = 0$

25. Question

If $\sin(xy) + \frac{y}{x} = x^2 - y^2$, find $\frac{dy}{dx}$.

Answer

We are given with an equation $\sin(xy) + \frac{y}{x} = x^2 - y^2$, we have to find $\frac{dy}{dx}$ by using the given equation, so by differentiating the equation on both sides with respect to x, we get,

$\cos(xy)\ [(1)y + x\frac{dy}{dx}] + \frac{x\frac{dy}{dx} - y(1)}{x^2} = 2x - 2y\frac{dy}{dx}$

$y\cos(xy) + x\cos(xy)\frac{dy}{dx} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 2x - 2y\frac{dy}{dx}$

$\frac{dy}{dx}[x\cos(xy) + \frac{1}{x} + 2y] = 2x - y\cos(xy) + \frac{y}{x^2}$

$$\frac{dy}{dx} = \frac{2x - y\cos(xy) + \frac{y}{x^2}}{x\cos(xy) + \frac{1}{x} + 2y}$$

$$\frac{dy}{dx} = \frac{2x^3 - yx^2\cos(xy) + y}{x[x^2\cos(xy) + 1 + 2xy]}$$

26. Question

If $\tan(x + y) + \tan(x - y) = 1$, find $\frac{dy}{dx}$.

Answer

We are given with an equation $\tan(x + y) + \tan(x - y) = 1$, we have to find $\frac{dy}{dx}$ by using the given equation, so by differentiating the equation on both sides with respect to x, we get,

$\sec^2(x + y)[1 + \frac{dy}{dx}] + \sec^2(x - y)[1 - \frac{dy}{dx}] = 0$

$\frac{dy}{dx}[\sec^2(x + y) - \sec^2(x - y)] + \sec^2(x + y) + \sec^2(x - y) = 0$

$$\frac{dy}{dx} = \frac{\sec^2(x + y) + \sec^2(x - y)}{\sec^2(x - y) - \sec^2(x + y)}$$

27. Question

If $e^x + e^y = e^{x + y}$, prove that $\frac{dy}{dx} = -\frac{e^x(e^y - 1)}{e^y(e^x - 1)}$ or, $\frac{dy}{dx}, e^{y-x} = 0$

Answer

We are given with an equation $e^x + e^y = e^{x + y}$, we have to prove that $\frac{dy}{dx} = \frac{-e^x(e^y - 1)}{e^y(e^x - 1)}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$e^x + e^y\frac{dy}{dx} = e^{(x + y)}\left[1 + \frac{dy}{dx}\right]$

$\frac{dy}{dx}[e^y - e^{x+y}] = e^{x+y} - e^x$

$$\frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$$

$$\frac{dy}{dx} = \frac{-e^x(e^y - 1)}{e^y(e^x - 1)}$$

28. Question

If cos y = x cos(a + y), with cos a ≠±1, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Answer

We are given with an equation cosy = x cos(a + y) , we have to prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$-\sin y\frac{dy}{dx} = \cos(a + y) - x\sin(a + y)\frac{dy}{dx}$

$\frac{dy}{dx}[x\sin(a + y) - \sin y] = \cos(a + y)$

$$\frac{dy}{dx} = \frac{\cos(a + y)}{x\sin(a + y) - \sin y}$$

We can further solve it by using the given equation,

$$\frac{dy}{dx} = \frac{\cos(a + y)}{\frac{\cos y}{\cos(a + y)} \times \sin(a + y) - \sin y}$$

$$\frac{dy}{dx} = \frac{\cos^2(a + y)}{\cos y\sin(a + y) - \sin y\cos(a + y)}$$

By using sinA cosB – cosA sinB = sin(A – B)

$$\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin(a + y - y)} = \frac{\cos^2(a + y)}{\sin a}$$

29. Question

If $\sin^2 y$ + cosxy = k, find $\frac{dy}{dx}$ at x = 1, $y = \frac{\pi}{4}$.

Answer

We are given with an equation $\sin^2 y + \cos(xy) = k$, we have to find $\frac{dy}{dx}$ at x = 1, $y = \frac{\pi}{4}$ by using the given equation, so by differentiating the equation on both sides with respect to x, we get,

$2\sin y\cos y\frac{dy}{dx} - \sin(xy)\left[(1)y + x\frac{dy}{dx}\right] = 0$

$\frac{dy}{dx}[2\sin y\cos y - x\sin(xy)] = y\sin(xy)$

$$\frac{dy}{dx} = \frac{y\sin(xy)}{2\sin y\cos y - x\sin(xy)}$$

By putting the value of point in the derivative, which is x = 1, y = $\frac{\pi}{4}$,

$$\frac{dy}{dx}(x = 1, y = \pi/4) = \frac{\frac{\pi}{4}\sin(\frac{\pi}{4})}{2\sin\frac{\pi}{4}\cos\frac{\pi}{4} - (1)\sin\frac{\pi}{4}}$$

$$\frac{dy}{dx}(x = 1, y = \pi/4) = \frac{\frac{\pi}{4\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{\frac{\pi}{4\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2}-1)}$$

30. Question

If $y = \{\log_{\cos x}\sin x\}\{\log_{\sin x}\cos x\}^{-1} + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

Answer

We are given with an equation y = $\{\log_{\cos x}\sin x\}\ \{\log_{\sin x}\cos x\}^{-1} + \sin^{-1}(\frac{2x}{1+x^2})$, we have to find $\frac{dy}{dx}$ at

x = $\frac{\pi}{4}$ by using the given equation, so by differentiating the equation on both sides with respect to x, we get,

By using the properties of logarithms,

$$y = \{\log_{\cos x}\sin x\}^2 + \sin^{-1}(\frac{2x}{1+x^2})$$

$$y = \{\frac{\ln\sin x}{\ln\cos x}\}^2 + \sin^{-1}(\frac{2x}{1+x^2})$$

$$\frac{dy}{dx} = 2\{\frac{\ln\sin x}{\ln\cos x}\}\frac{\ln\cos x\frac{\cos x}{\sin x} - \ln\sin x\frac{-\sin x}{\cos x}}{(\ln\cos x)^2} + \frac{1}{\sqrt{1-(\frac{2x}{1+x^2})^2}}\frac{(1+x^2)2-2x(2x)}{(1+x^2)^2}$$

$$\frac{dy}{dx} = 2\{\frac{\ln\sin x}{\ln\cos x}\}\frac{\ln\cos x(\cot x) - \ln\sin x(-\tan x)}{(\ln\cos x)^2} + \frac{\sqrt{(1+x^2)^2}}{\sqrt{(1-x^2)^2}}\frac{2(1-x^2)}{(1+x^2)^2}$$

$$\frac{dy}{dx} = 2\{\frac{\ln\sin x}{\ln\cos x}\}\frac{\ln\cos x(\cot x) + \ln\sin x(\tan x)}{(\ln\cos x)^2} + \frac{2}{1+x^2}$$

Now putting the value of x = $\frac{\pi}{4}$ in the derivative solved above, we get,

$$\frac{dy}{dx}(x = \pi/4) = 2\{1\}\frac{\ln\frac{1}{\sqrt{2}}(1) + \ln\frac{1}{\sqrt{2}}(1)}{(\ln\frac{1}{\sqrt{2}})^2} + \frac{2}{1+(\frac{\pi}{4})^2}$$

$$\frac{dy}{dx}(x = \pi/4) = 2\{1\}\frac{\ln\frac{1}{2}}{(\frac{1}{2}\ln 2)^2} + \frac{2}{\frac{16+\pi^2}{16}}$$

$$\frac{dy}{dx}(x = \pi/4) = 2\{1\}\frac{-4\ln 2}{(\ln 2)^2} + \frac{32}{16+(\pi)^2}$$

$$\frac{dy}{dx}(x = \pi/4) = \frac{-8}{\ln 2} + \frac{32}{16+(\pi)^2}$$

31. Question

If $\sqrt{y+x} + \sqrt{y-x} = c$, show that $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$.

Answer

We are given with an equation $\sqrt{y+x} + \sqrt{y-x}$ = c , we have to prove that $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$ by using the given equation we will first find the value of $\frac{dy}{dx}$ and we will put this in the equation we have to prove, so by differentiating the equation on both sides with respect to x, we get,

$$\frac{(1+\frac{dy}{dx})}{2\sqrt{y+x}} + \frac{(\frac{dy}{dx}-1)}{2\sqrt{y-x}} = 0$$

$$\frac{\sqrt{y-x} + \sqrt{y-x}\frac{dy}{dx} + \sqrt{y+x}\frac{dy}{dx} - \sqrt{y+x}}{2\sqrt{y+x}\sqrt{y-x}} = 0$$

$$\sqrt{y-x} + \sqrt{y-x}\frac{dy}{dx} + \sqrt{y+x}\frac{dy}{dx} - \sqrt{y+x} = 0$$

$$\frac{dy}{dx}[\sqrt{y-x} + \sqrt{y+x}] = \sqrt{y+x} - \sqrt{y-x}$$

$$\frac{dy}{dx} = \frac{\sqrt{y+x} - \sqrt{y-x}}{\sqrt{y-x} + \sqrt{y+x}}$$

$$\frac{dy}{dx} = \frac{\sqrt{y+x} - \sqrt{y-x}}{\sqrt{y-x} + \sqrt{y+x}} \times \frac{\sqrt{y+x} - \sqrt{y-x}}{\sqrt{y+x} - \sqrt{y-x}}$$

$$\frac{dy}{dx} = \frac{2y - 2\sqrt{y+x}\sqrt{y-x}}{2x}$$

$$\frac{dy}{dx} = \frac{y - \sqrt{y^2 - x^2}}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{\sqrt{y^2 - x^2}}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{\sqrt{y^2 - x^2}}{\sqrt{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2 - x^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$$

Exercise 11.5

1. Question

Differentiate the following functions with respect to x :

$x^{1/x}$

Answer

Let $y = x^{\frac{1}{x}}$

Taking log both the sides:

$\Rightarrow \log y = \log x^{\frac{1}{x}}$

$\Rightarrow \log y = \frac{1}{x}\log x$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\frac{1}{x}\log x\right)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{1}{x} \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x^{-1})}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{x}\frac{dx}{dx} + \log x\left(\frac{-1}{x^2}\right)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^2}\log x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1-\log x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1-\log x}{x^2}\right)$$

Put the value of $y = x^{\frac{1}{x}}$:

$$\Rightarrow \frac{dy}{dx} = x^{\frac{1}{x}}\left(\frac{1-\log x}{x^2}\right)$$

2. Question

Differentiate the following functions with respect to x :

$x^{\sin x}$

Answer

Let y = $x^{\sin x}$

Taking log both the sides:

log y = log ($x^{\sin x}$)

log y = sin x log x {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\sin x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \sin x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x)}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin x \times \frac{1}{x}\frac{dx}{dx} + \log x(\cos x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \ \& \ \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin x}{x} + \log x \cos x$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \log x \cos x\right)$$

Put the value of y = $x^{\sin x}$:

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)$$

3. Question

Differentiate the following functions with respect to x :

$(1 + \cos x)^x$

Answer

Let y = $(1 + \cos x)^x$

Taking log both the sides:

⇒ log y = log $(1 + \cos x)^x$

⇒ log y = x log (1+ cos x) {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d[x \log(1 + \cos x)]}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d[\log(1 + \cos x)]}{dx} + \log(1 + \cos x) \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{(1 + \cos x)}\frac{d(1 + \cos x)}{dx} + \log(1 + \cos x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{(1 + \cos x)}(-\sin x) + \log(1 + \cos x)$$

$$\left\{\frac{d(1 + \cos x)}{dx} = \frac{d(1)}{dx} + \frac{d(\cos x)}{dx} = 0 + (-\sin x)\frac{dx}{dx} = -\sin x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{-x\sin x}{1 + \cos x} + \log(1 + \cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{-x\sin x}{1 + \cos x} + \log(1 + \cos x)\right\}$$

Put the value of y = $(1 + \cos x)^x$:

$$\Rightarrow \frac{dy}{dx} = (1 + \cos x)^x\left\{\frac{-x\sin x}{1 + \cos x} + \log(1 + \cos x)\right\}$$

4. Question

Differentiate the following functions with respect to x :

$x^{\cos^{-1} x}$

Answer

Let $y = x^{\cos^{-1} x}$

Taking log both the sides:

$\Rightarrow \log y = \log x^{\cos^{-1} x}$

$\Rightarrow \log y = \cos^{-1} x \log x$ {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos^{-1} x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \cos^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\cos^{-1} x)}{dx}$$

$$\left\{ \text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos^{-1} x}{x} + \log x \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \ \& \ \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}} \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y \left\{ \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}} \right\}$$

Put the value of $y = x^{\cos^{-1} x}$:

$$\Rightarrow \frac{dy}{dx} = x^{\cos^{-1} x} \left\{ \frac{\cos^{-1} x}{x} - \frac{\log x}{\sqrt{1-x^2}} \right\}$$

5. Question

Differentiate the following functions with respect to x :

$(\log x)^x$

Answer

Let y = $(\log x)^x$

Taking log both the sides:

$\Rightarrow$ log y = log $(\log x)^x$

$\Rightarrow$ log y = x log (log x) {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log \log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log \log x)}{dx} + \log \log x \times \frac{dx}{dx}$$

$$\left\{ \text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\log x}\frac{d(\log x)}{dx} + \log \log x$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\log x} \times \frac{1}{x} + \log\log x$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{\log x} + \log\log x\right\}$$

Put the value of $y = (\log x)^x$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x\left\{\frac{1}{\log x} + \log\log x\right\}$$

6. Question

Differentiate the following functions with respect to x :

$(\log x)^{\cos x}$

Answer

Let $y = (\log x)^{\cos x}$

Taking log both the sides:

$\Rightarrow \log y = \log (\log x)^{\cos x}$

$\Rightarrow \log y = \cos x \log \log x$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos x \log\log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \cos x \times \frac{d(\log\log x)}{dx} + \log\log x \times \frac{d(\cos x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{1}{\log x}\frac{d(\log x)}{dx} + \log\log x\ (-\sin x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\ \&\ \frac{d(\cos x)}{dx} = -\sin x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\log x} \times \frac{1}{x} - \sin x \log\log x$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\cos x}{x\log x} - \sin x \log\log x\right\}$$

Put the value of $y = (\log x)^{\cos x}$:

$$\Rightarrow \frac{dy}{dx} = (\log x)^{\cos x}\left\{\frac{\cos x}{x\log x} - \sin x \log\log x\right\}$$

7. Question

Differentiate the following functions with respect to x :

$(\sin x)^{\cos x}$

Answer

Let $y = (\sin x)^{\cos x}$

Taking log both the sides:

$\Rightarrow \log y = \log (\sin x)^{\cos x}$

$\Rightarrow \log y = \cos x \log \sin x$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\cos x \log \sin x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \cos x \times \frac{d(\log \sin x)}{dx} + \log \sin x \times \frac{d(\cos x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log \sin x\,(-\sin x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\,;\ \frac{d(\cos x)}{dx} = -\sin x\,;\ \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cot x\,(\cos x) - \sin x \log \sin x$$

$$\Rightarrow \frac{dy}{dx} = y\{\cos x \cot x - \sin x \log \sin x\}$$

Put the value of $y = (\sin x)^{\cos x}$:

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}\{\cos x \cot x - \sin x \log \sin x\}$$

8. Question

Differentiate the following functions with respect to x :

$e^{x \log x}$

Answer

Let $y = e^{x \log x}$

Taking log both the sides:

$\Rightarrow \log y = \log (e)^{x \log x}$

$\Rightarrow \log y = x \log x \log e$ $\{\log x^a = a\log x\}$

$\Rightarrow \log y = x \log x$ $\{\log e = 1\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = y\{1 + \log x\}$$

Put the value of $y = e^{x \log x}$:

$$\Rightarrow \frac{dy}{dx} = e^{x\log x}\{1 + \log x\}$$

$\Rightarrow \frac{dy}{dx} = e^{\log x^x}\{1 + \log x\}$ { $e^{\log a} = a$; $a\log x = x^a$}

$$\Rightarrow \frac{dy}{dx} = x^x\{1 + \log x\}$$

9. Question

Differentiate the following functions with respect to x :

$(\sin x)^{\log x}$

Answer

Let $y = (\sin x)^{\log x}$

Taking log both the sides:

$\Rightarrow \log y = \log (\sin x)^{\log x}$

$\Rightarrow \log y = \log x \log \sin x$ {$\log x^a = a\log x$}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log x \log \sin x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \log x \times \frac{d(\log \sin x)}{dx} + \log \sin x \times \frac{d(\log x)}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log \sin x\left(\frac{1}{x}\frac{dx}{dx}\right)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\,;\ \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log x}{\sin x}(\cos x) + \frac{\log \sin x}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\log x \cot x + \frac{\log \sin x}{x}\right\}$$

Put the value of $y = (\sin x)^{\log x}$:

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log x}\left\{\log x \cot x + \frac{\log \sin x}{x}\right\}$$

10. Question

Differentiate the following functions with respect to x :

$10^{\log \sin x}$

Answer

Let y = $10^{\log \sin x}$

Taking log both the sides:

⇒ log y = log $10^{\log \sin x}$

⇒ log y = log sin x log 10 {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log 10 \log \sin x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \log 10 \times \frac{d(\log \sin x)}{dx}$$

$$\left\{ \text{Using chain rule,} \frac{d(au)}{dx} = a \frac{du}{dx} \text{ where a is any constant and u is any variable} \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log 10 \times \frac{1}{\sin x}\frac{d(\sin x)}{dx}$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\sin x)}{dx} = \cos x \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log 10}{\sin x}(\cos x)$$

$$\Rightarrow \frac{dy}{dx} = y\{\log 10 \cot x\}$$

Put the value of y = $10^{\log \sin x}$:

$$\Rightarrow \frac{dy}{dx} = 10^{\log \sin x}\{\log 10 \cot x\}$$

11. Question

Differentiate the following functions with respect to x :

$(\log x)^{\log x}$

Answer

Let y = $(\log x)^{\log x}$

Taking log both the sides:

⇒ log y = log $(\log x)^{\log x}$

⇒ log y = log x log (log x) {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log x \log(\log x))}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \log x \times \frac{d(\log(\log x))}{dx} + \log(\log x) \times \frac{d(\log x)}{dx}$$

$$\left\{ \text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x \times \frac{1}{\log x}\frac{d(\log x)}{dx} + \log\log x \left(\frac{1}{x}\frac{dx}{dx}\right)$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \right\}$$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\log x}{\log x}\left(\frac{1}{x}\frac{dx}{dx}\right) + \frac{\log(\log x)}{x}$

$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{x} + \frac{\log(\log x)}{x}\right\}$

$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1 + \log(\log x)}{x}\right\}$

Put the value of $y = (\log x)^{\log x}$:

$\Rightarrow \frac{dy}{dx} = (\log x)^{\log x}\left\{\frac{1 + \log(\log x)}{x}\right\}$

12. Question

Differentiate the following functions with respect to x :

$10^{(10x)}$

Answer

Let $y = 10^{10x}$

Taking log both the sides:

$\Rightarrow \log y = \log 10^{10x}$

$\Rightarrow \log y = 10x \log 10$ $\{\log x^a = a\log x\}$

$\Rightarrow \log y = (10\log 10)x$

Differentiating with respect to x:

$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\{(10\log 10)x\}}{dx}$

$\Rightarrow \frac{d(\log y)}{dx} = 10 \times \log(10) \times \frac{d(x)}{dx}$ {Here 10log (10) is a constant term}

$\left\{\text{Using chain rule, } \frac{d(au)}{dx} = a\,\frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 10\log(10)$

$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\sin x)}{dx} = \cos x\right\}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 10\log(10)$

$\Rightarrow \frac{dy}{dx} = y\{10\log(10)\}$

Put the value of $y = 10^{10x}$:

$\Rightarrow \frac{dy}{dx} = 10^{10x}\{10\log(10)\}$

13. Question

Differentiate the following functions with respect to x :

$\sin(x^x)$

Answer

Let $y = \sin(x^x)$

Take sin inverse both sides:

$\Rightarrow \sin^{-1} y = \sin^{-1} (\sin x^x)$

$\Rightarrow \sin^{-1} y = x^x$

Taking log both the sides:

$\Rightarrow \log (\sin^{-1} y) = \log x^x$

$\Rightarrow \log (\sin^{-1} y) = x \log x$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log(\sin^{-1} y))}{dx} = \frac{d(x\log x)}{dx}$$

$$\Rightarrow \frac{d(\log(\sin^{-1} y))}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{\sin^{-1} y}\frac{d(\sin^{-1} y)}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{\sin^{-1} y} \times \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\left\{\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{\sin^{-1} y(\sqrt{1-y^2})}\frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} y\left(\sqrt{1-y^2}\right)(1 + \log x)$$

Put the value of $y = \sin(x^x)$:

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}(\sin x^x)\left(\sqrt{1-\sin^2(x^x)}\right)(1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x\left(\sqrt{\cos^2(x^x)}\right)(1 + \log x)$$

$\{\sin^2 x + \cos^2 x = 1\}$

$$\Rightarrow \frac{dy}{dx} = x^x \cos x^x (1 + \log x)$$

14. Question

Differentiate the following functions with respect to x :

$(\sin^{-1} x)^x$

Answer

Let $y = (\sin^{-1} x)^x$

Taking log both the sides:

$\Rightarrow \log y = \log (\sin^{-1} x)^x$

$\Rightarrow \log y = x \log(\sin^{-1} x)$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(x\log(\sin^{-1}x))}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = x \times \frac{d(\log(\sin^{-1}x))}{dx} + \log(\sin^{-1}x) \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{\sin^{-1}x}\frac{d(\sin^{-1}x)}{dx} + \log(\sin^{-1}x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^{-1}x} \times \frac{1}{\sqrt{1-x^2}}\frac{dx}{dx} + \log(\sin^{-1}x)$$

$$\left\{\frac{d(\sin^{-1}u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{\sin^{-1}x\sqrt{1-x^2}} + \log(\sin^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}} + \log(\sin^{-1}x)\right\}$$

Put the value of $y = (\sin^{-1} x)^x$:

$$\Rightarrow \frac{dy}{dx} = (\sin^{-1}x)^x\left\{\frac{x}{\sin^{-1}x\sqrt{1-x^2}} + \log(\sin^{-1}x)\right\}$$

15. Question

Differentiate the following functions with respect to x :

$x^{\sin^{-1}x}$

Answer

Let $y = x^{\sin^{-1}x}$

Taking log both the sides:

$\Rightarrow \log y = \log x^{\sin^{-1}x}$

$\Rightarrow \log y = \sin^{-1} x \log x$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\sin^{-1}x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \sin^{-1}x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin^{-1}x)}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin^{-1}x \times \frac{1}{x}\frac{dx}{dx} + \log x \times \frac{1}{\sqrt{1-x^2}}\frac{dx}{dx}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}$$

Put the value of $y = x^{\sin^{-1} x}$:

$$\Rightarrow \frac{dy}{dx} = x^{\sin^{-1} x}\left\{\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}$$

16. Question

Differentiate the following functions with respect to x :

$(\tan x)^{1/x}$

Answer

Let $y = (\tan x)^{\frac{1}{x}}$

Taking log both the sides:

$$\Rightarrow \log y = \log(\tan x)^{\frac{1}{x}}$$

$\Rightarrow \log y = \frac{1}{x}\log \tan x$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\frac{1}{x}\log \tan x\right)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{1}{x} \times \frac{d(\log \tan x)}{dx} + \log \tan x \times \frac{d(x^{-1})}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\tan x}\frac{d(\tan x)}{dx} + \log \tan x\,(-x^{-2})$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x\tan x}(\sec^2 x) - \frac{\log \tan x}{x^2}$$

$$\left\{\frac{d(\tan x)}{dx} = \sec^2 x\right\}$$

$$\frac{dy}{dx} = y\left\{\frac{\sec^2 x}{x\tan x} - \frac{\log \tan x}{x^2}\right\}$$

Put the value of $y = (\tan x)^{\frac{1}{x}}$:

$$\frac{dy}{dx} = (\tan x)^{\frac{1}{x}}\left\{\frac{\sec^2 x}{x\tan x} - \frac{\log \tan x}{x^2}\right\}$$

17. Question

Differentiate the following functions with respect to x :

$x^{\tan^{-1} x}$

Answer

Let $y = x^{\tan^{-1} x}$

Taking log both the sides:

$\Rightarrow \log y = \log x^{\tan^{-1} x}$

⇒ log y = $\tan^{-1}$ x log x{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\tan^{-1} x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \tan^{-1} x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\tan^{-1} x)}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \tan^{-1} x \times \frac{1}{x}\frac{dx}{dx} + \log x \times \frac{1}{x^2+1}\frac{dx}{dx}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\tan^{-1} u)}{dx} = \frac{1}{u^2+1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\tan^{-1} x}{x} + \frac{\log x}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{\tan^{-1} x}{x} + \frac{\log x}{x^2+1}\right\}$$

Put the value of $y = x^{\tan^{-1} x}$:

$$\Rightarrow \frac{dy}{dx} = x^{\tan^{-1} x}\left\{\frac{\tan^{-1} x}{x} + \frac{\log x}{x^2+1}\right\}$$

18 A. Question

Differentiate the following functions with respect to x :

$(x^x)\sqrt{x}$

Answer

Let $y = (x)^x \sqrt{x}$

Taking log both the sides:

$\Rightarrow \log y = \log(x)^x \sqrt{x}$

$\Rightarrow \log y = \log(x)^x + \log\sqrt{x}$ {log (ab) = log a +log b}

$\Rightarrow \log y = \log(x)^x + \log x^{\frac{1}{2}}$

$\Rightarrow \log y = x\log x + \frac{1}{2}\log x$

{log x^a = alog x}

$\Rightarrow \log y = \left(x+\frac{1}{2}\right)\log x$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d\left(\left(x+\frac{1}{2}\right)\log x\right)}{dx}$$

$$\Rightarrow \frac{d(\log y)}{dx} = \left(x+\frac{1}{2}\right)\times\frac{d(\log x)}{dx} + \log x \times \frac{d\left(x+\frac{1}{2}\right)}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \left(x+\frac{1}{2}\right)\times\frac{1}{x}\frac{dx}{dx} + \log x\frac{dx}{dx}$$

$$\left\{\begin{array}{c}\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx};\\ \text{Using chain rule}, \frac{d(u+a)}{dx} = \frac{du}{dx} \text{ where a is any constant and u is any variable}\end{array}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{(2x+1)}{2}\times\frac{1}{x} + \log x$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{(2x+1)}{2x} + \log x\right\}$$

Put the value of $y = (x)^x\sqrt{x}$:

$$\Rightarrow \frac{dy}{dx} = (x)^x\sqrt{x}\left\{\frac{(2x+1)}{2x} + \log x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (x)^x\sqrt{x}\left\{\frac{2x}{2x} + \frac{1}{2x} + \log x\right\}$$

$$\Rightarrow \frac{dy}{dx} = (x)^x\sqrt{x}\left\{1 + \frac{1}{2x} + \log x\right\}$$

18 B. Question

Differentiate the following functions with respect to x :

$$x^{(\sin x - \cos x)} + \frac{x^2-1}{x^2+1}$$

Answer

Let $y = x^{(\sin x - \cos x)} + \frac{x^2-1}{x^2+1}$

$\Rightarrow$ y = a + b

where $a = x^{(\sin x - \cos x)}$; $b = \frac{x^2-1}{x^2+1}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule}, \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$a = x^{(\sin x - \cos x)}$

Taking log both the sides:

$\Rightarrow \log a = \log x^{(\sin x - \cos x)}$

$\Rightarrow \log a = (\sin x - \cos x)\log x$

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d((\sin x - \cos x)\log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = (\sin x - \cos x) \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x - \cos x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = (\sin x - \cos x) \times \frac{1}{x}\frac{dx}{dx} + \log x\left(\frac{d(\sin x)}{dx} - \frac{d(\cos x)}{dx}\right)$$

$$\left\{\begin{matrix} \dfrac{d(\log u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}; \\ \text{Using chain rule, } \dfrac{d(u+a)}{dx} = \dfrac{du}{dx} + \dfrac{da}{dx} \text{ where a and u are any variables} \end{matrix}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{(\sin x - \cos x)}{x} + \log x(\cos x - (-\sin x))$$

$$\left\{\frac{d(\cos x)}{dx} = -\sin x; \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{(\sin x - \cos x)}{x} + \log x(\cos x + \sin x)$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x)\right\}$$

Put the value of $a = x^{(\sin x - \cos x)}$:

$$\Rightarrow \frac{da}{dx} = x^{(\sin x - \cos x)}\left\{\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x)\right\}$$

$$b = \frac{x^2 - 1}{x^2 + 1}$$

$$\Rightarrow \frac{db}{dx} = \frac{(x^2 + 1)\dfrac{d(x^2 - 1)}{dx} - (x^2 - 1)\dfrac{d(x^2 + 1)}{dx}}{(x^2 + 1)^2}$$

$$\left\{\frac{d(\frac{u}{v})}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}; \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx} = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 + 2x) - (2x^3 - 2x)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 + 2x - 2x^3 + 2x)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{4x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{(\sin x - \cos x)}\left\{\frac{\sin x - \cos x}{x} + \log x\,(\cos x + \sin x)\right\} + \frac{4x}{(x^2+1)^2}$$

18 C. Question

Differentiate the following functions with respect to x :

$$x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

Answer

$$\text{Let } y = x^{x\cos x} + \frac{x^2+1}{x^2-1}$$

$\Rightarrow$ y = a + b

$$\text{where } a = x^{x\cos x};\, b = \frac{x^2+1}{x^2-1}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$$a = x^{x\cos x}$$

Taking log both the sides:

$$\Rightarrow \log a = \log x^{x\cos x}$$

$$\Rightarrow \log a = x\cos x\log x$$

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x\cos x\log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x\cos x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x\cos x)}{dx}$$

$$\left\{\begin{aligned} &\text{Using product rule, } \frac{d(uvw)}{dx} = uv\frac{dw}{dx} + w\frac{duv}{dx} \\ &\qquad = uv\frac{dw}{dx} + w\left\{u\frac{dv}{dx} + v\frac{du}{dx}\right\} \end{aligned}\right\}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x\cos x \times \frac{d(\log x)}{dx} + \log x\left\{x\frac{d(\cos x)}{dx} + \cos x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x\cos x \times \frac{1}{x}\frac{dx}{dx} + \log x\{x(-\sin x) + \cos x\}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x\cos x}{x} + \log x(\cos x - x\sin x)$$

$$\left\{\frac{d(\cos x)}{dx} = -\sin x\,;\ \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{da}{dx} = a\{\cos x + \log x(\cos x - x\sin x)\}$$

Put the value of $a = x^{x\cos x}$:

$$\Rightarrow \frac{da}{dx} = x^{x\cos x}\{\cos x + \log x(\cos x - x\sin x)\}$$

$$\Rightarrow \frac{da}{dx} = x^{x\cos x}\{\cos x + \log x\cos x - x\sin x\log x\}$$

$$\Rightarrow \frac{da}{dx} = x^{x\cos x}\{\cos x\,(1 + \log x) - x\sin x\log x\}$$

$$b = \frac{x^2+1}{x^2-1}$$

$$\Rightarrow \frac{db}{dx} = \frac{(x^2-1)\frac{d(x^2+1)}{dx} - (x^2+1)\frac{d(x^2-1)}{dx}}{(x^2-1)^2}$$

$$\left\{\frac{d(\frac{u}{v})}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2};\ \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx} = \frac{(x^2-1)(2x) - (x^2+1)\,(2x)}{(x^2+1)^2}$$

$$\left\{\text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3-2x) - (2x^3+2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{(2x^3 - 2x - 2x^3 - 2x)}{(x^2+1)^2}$$

$$\Rightarrow \frac{db}{dx} = \frac{-4x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{x\cos x}\{\cos x\,(1+\log x) - x\sin x\log x\} - \frac{4x}{(x^2+1)^2}$$

18 D. Question

Differentiate the following functions with respect to x :

$(x \cos x)^x + (x \sin x)^{1/x}$

Answer

Let $y = (x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

$\Rightarrow$ y = a + b

where $a = (x\cos x)^x; b = (x\sin x)^{\frac{1}{x}}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$

$a = (x \cos x)^x$

Taking log both the sides:

$\Rightarrow \log a = \log(x \cos x)^x$

$\Rightarrow \log a = x \log(x \cos x)$

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x \log(x \cos x))}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log(x \cos x))}{dx} + \log(x \cos x) \times \frac{dx}{dx}$$

$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{x \cos x}\frac{d(x \cos x)}{dx} + \log(x \cos x)$$

$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x}{x \cos x}\left\{x\frac{d(\cos x)}{dx} + \cos x\right\} + \log(x \cos x)$$

$\left\{\text{Again using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{1}{\cos x}\{x(-\sin x) + \cos x\} + \log(x \cos x)$$

$\left\{\frac{d(\cos x)}{dx} = -\sin x\right\}$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{\cos x - x \sin x}{\cos x} + \log(x \cos x)\right\}$$

Put the value of $a = (x \cos x)^x$:

$$\Rightarrow \frac{da}{dx} = (x \cos x)^x\left\{\frac{\cos x - x \sin x}{\cos x} + \log(x \cos x)\right\}$$

$$\Rightarrow \frac{da}{dx} = (x \cos x)^x\{1 - x \tan x + \log(x \cos x)\}$$

$b = (x \sin x)^{\frac{1}{x}}$

Taking log both the sides:

$\Rightarrow \log b = \log(x \sin x)^{\frac{1}{x}}$

$\Rightarrow \log b = \frac{1}{x}\log(x \sin x)$ {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\frac{1}{x}\log(x \sin x)\right)}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{1}{x} \times \frac{d(\log(x\sin x))}{dx} + \log(x\sin x) \times \frac{d(x^{-1})}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x} \times \frac{1}{x\sin x}\frac{d(x\sin x)}{dx} + \log(x\sin x)\,(-x^{-2})$$

$$\left\{\text{Again using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\ ;\ \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x^2\sin x}\left(x\frac{d(\sin x)}{dx} + \sin x\frac{dx}{dx}\right) - \frac{\log(x\sin x)}{x^2}$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x\cos x + \sin x}{x^2\sin x} - \frac{\log(x\sin x)}{x^2}\right\}$$

$$\left\{\frac{d(\sin x)}{dx} = \cos x\right\}$$

Put the value of $b = (x\sin x)^{\frac{1}{x}}$:

$$\Rightarrow \frac{db}{dx} = (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cos x + \sin x}{x^2\sin x} - \frac{\log(x\sin x)}{x^2}\right\}$$

$$\Rightarrow \frac{db}{dx} = (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cot x + 1}{x^2} - \frac{\log(x\sin x)}{x^2}\right\}$$

$$\Rightarrow \frac{db}{dx} = (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (x\cos x)^x\{1 - x\tan x + \log(x\cos x)\} + (x\sin x)^{\frac{1}{x}}\left\{\frac{x\cot x + 1 - \log(x\sin x)}{x^2}\right\}$$

18 E. Question

Differentiate the following functions with respect to x :

$$\left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$$

Answer

Let $y = \left(x + \frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

$\Rightarrow y = a + b$

where $a = \left(x + \frac{1}{x}\right)^x$; $b = x^{\left(1+\frac{1}{x}\right)}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$a = \left(x+\frac{1}{x}\right)^{x}$

Taking log both the sides:

$\Rightarrow \log a = \log\left(x+\frac{1}{x}\right)^{x}$

$\Rightarrow \log a = x\log\left(x+\frac{1}{x}\right)$

$\{\log x^{a} = a\log x\}$

Differentiating with respect to x:

$\Rightarrow \frac{d(\log a)}{dx} = \frac{d\left(x\log\left(x+\frac{1}{x}\right)\right)}{dx}$

$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d\left(\log\left(x+\frac{1}{x}\right)\right)}{dx} + \log\left(x+\frac{1}{x}\right) \times \frac{dx}{dx}$

$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$

$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{x+\frac{1}{x}}\frac{d\left(x+\frac{1}{x}\right)}{dx} + \log\left(x+\frac{1}{x}\right)$

$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$

$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x}{\frac{x^2+1}{x}}\left\{\frac{dx}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx}\right\} + \log\left(x+\frac{1}{x}\right)$

$\left\{\text{Using chain rule}, \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$

$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x^2}{x^2+1}\left\{1+\left(-\frac{1}{x^2}\right)\right\} + \log\left(x+\frac{1}{x}\right)$

$\left\{\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$

$\Rightarrow \frac{da}{dx} = a\left\{\frac{x^2}{x^2+1}\left\{1-\frac{1}{x^2}\right\} + \log\left(x+\frac{1}{x}\right)\right\}$

Put the value of $a = \left(x+\frac{1}{x}\right)^{x}$:

$\Rightarrow \frac{da}{dx} = \left(x+\frac{1}{x}\right)^{x}\left\{\frac{x^2}{x^2+1}\left\{1-\frac{1}{x^2}\right\} + \log\left(x+\frac{1}{x}\right)\right\}$

$\Rightarrow \frac{da}{dx} = \left(x+\frac{1}{x}\right)^{x}\left\{\frac{x^2}{x^2+1} - \frac{1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right\}$

$\Rightarrow \frac{da}{dx} = \left(x+\frac{1}{x}\right)^{x}\left\{\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right\}$

$b = x^{\left(1+\frac{1}{x}\right)}$

Taking log both the sides:

$\Rightarrow \log b = \log x^{\left(1+\frac{1}{x}\right)}$

$\Rightarrow \log b = \left(1+\frac{1}{x}\right)\log x$ {log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\left(1+\frac{1}{x}\right)\log x\right)}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \left(1+\frac{1}{x}\right) \times \frac{d(\log x)}{dx} + \log x \times \frac{d\left(1+\frac{1}{x}\right)}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x+1}{x} \times \frac{1}{x}\frac{dx}{dx} + \log x\left(\frac{d(1)}{dx} + \frac{d\left(\frac{1}{x}\right)}{dx}\right)$$

$$\left\{\begin{array}{c} \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \\ \text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables} \end{array}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x+1}{x^2} + \log x\left(-\frac{1}{x^2}\right)$$

$$\left\{\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x+1}{x^2} - \frac{\log x}{x^2}\right\}$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x+1-\log x}{x^2}\right\}$$

Put the value of $b = x^{\left(1+\frac{1}{x}\right)}$:

$$\Rightarrow \frac{db}{dx} = x^{\left(1+\frac{1}{x}\right)}\left\{\frac{x+1-\log x}{x^2}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left\{\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right\} + x^{\left(1+\frac{1}{x}\right)}\left\{\frac{x+1-\log x}{x^2}\right\}$$

18 F. Question

Differentiate the following functions with respect to x :

$e^{\sin x} + (\tan x)^x$

Answer

let y = $e^{\sin x} + (\tan x)^x$

⇒ y = a + b

where a= $e^{\sin x}$; b = $(\tan x)^x$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$

a= $e^{\sin x}$

Taking log both the sides:

⇒ log a= log $e^{\sin x}$

⇒ log a= sin x log e

{log x^a = alog x}

⇒ log a= sin x {log e =1}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cos x$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} ; \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{da}{dx} = a\,(\cos x)$$

Put the value of a = $e^{\sin x}$

$$\Rightarrow \frac{da}{dx} = e^{\sin x}\cos x$$

b = $(\tan x)^x$

Taking log both the sides:

⇒ log b= log $(\tan x)^x$

⇒ log b= x log (tan x)

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x\log(\tan x))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = x \times \frac{d(\log(\tan x))}{dx} + \log(\tan x) \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\,\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\,\frac{db}{dx} = x \times \frac{1}{\tan x}\frac{d(\tan x)}{dx} + \log(\tan x)$$

$$\left\{\frac{d(\tan x)}{dx} = \sec^2 x\right\}$$

$$\Rightarrow \frac{1}{b}\,\frac{db}{dx} = \frac{x}{\tan x}(\sec^2 x) + \log(\tan x)$$

$$\Rightarrow \frac{1}{b}\,\frac{db}{dx} = \frac{x\cos x}{\sin x}\left(\frac{1}{\cos^2 x}\right) + \log(\tan x)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x}{\sin x}\left(\frac{1}{\cos x}\right) + \log(\tan x)$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x}{\sin x \cos x} + \log(\tan x)\right\}$$

Put the value of b = $(\tan x)^x$:

$$\Rightarrow \frac{db}{dx} = (\tan x)^x\left\{\frac{x}{\sin x \cos x} + \log(\tan x)\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^{\sin x}\cos x + (\tan x)^x\left\{\frac{x}{\sin x \cos x} + \log(\tan x)\right\}$$

18 G. Question

Differentiate the following functions with respect to x :

$(\cos x)^x + (\sin x)^{1/x}$

Answer

$$\text{Let } y = (\cos x)^x + (\sin x)^{\frac{1}{x}}$$

$\Rightarrow$ y = a + b

$$\text{where } a = (\cos x)^x ; b = (\sin x)^{\frac{1}{x}}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$$a = (\cos x)^x$$

Taking log both the sides:

$$\Rightarrow \log a = \log(\cos x)^x$$

$$\Rightarrow \log a = x\log(\cos x)$$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x\log(\cos x))}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log(\cos x))}{dx} + \log(\cos x) \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{\cos x}\frac{d(\cos x)}{dx} + \log(\cos x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{x}{\cos x}(-\sin x) + \log(\cos x)$$

$$\left\{\frac{d(\cos x)}{dx} = -\sin x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{-x\sin x}{\cos x} + \log(\cos x)$$

$$\Rightarrow \frac{da}{dx} = a\{-x\tan x + \log(\cos x)\}$$

Put the value of $a = (\cos x)^x$:

$$\Rightarrow \frac{da}{dx} = (\cos x)^x\{-x\tan x + \log(\cos x)\}$$

$$b = (\sin x)^{\frac{1}{x}}$$

Taking log both the sides:

$$\Rightarrow \log b = \log(\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \log b = \frac{1}{x}\log(\sin x)\ \{\log x^a = a\log x\}$$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d\left(\frac{1}{x}\log(\sin x)\right)}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{1}{x}\times\frac{d(\log(\sin x))}{dx} + \log(\sin x)\times\frac{d(x^{-1})}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x}\times\frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log(\sin x)\,(-x^{-2})$$

$$\left\{\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{1}{x\sin x}(\cos x) - \frac{\log(\sin x)}{x^2}$$

$$\left\{\frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{\cos x}{x\sin x} - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

Put the value of $b = (\sin x)^{\frac{1}{x}}$:

$$\Rightarrow \frac{db}{dx} = (\sin x)^{\frac{1}{x}}\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x\{-x\tan x + \log(\cos x)\} + (\sin x)^{\frac{1}{x}}\left\{\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right\}$$

18 H. Question

Differentiate the following functions with respect to x :

$$x^{x^2-3}+(x-3)^{x^2}$$

Answer

Let $y = x^{x^2-3} + (x-3)^{x^2}$

⇒ y = a + b

where $a = x^{x^2-3}; b = (x-3)^{x^2}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$a = x^{x^2-3}$

Taking log both the sides:

$$\Rightarrow \log a = \log x^{x^2-3}$$

$$\Rightarrow \log a = (x^2-3)\log x$$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d((x^2-3)\log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = (x^2-3) \times \frac{d(\log x)}{dx} + \log x \times \frac{d(x^2-3)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = (x^2-3) \times \frac{1}{x}\frac{dx}{dx} + \log x \times (2x)$$

$$\left\{\begin{array}{c} \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}; \\ \text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables} \end{array}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{(x^2-3)}{x} + 2x\log x$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{(x^2-3)}{x} + 2x\log x\right\}$$

Put the value of $a = x^{x^2-3}$:

$$\Rightarrow \frac{da}{dx} = x^{x^2-3}\left\{\frac{(x^2-3)}{x} + 2x\log x\right\}$$

$b = (x-3)^{x^2}$

Taking log both the sides:

$$\Rightarrow \log b = (x-3)^{x^2}$$

$\Rightarrow \log b = x^2\log(x-3)$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x^2 \log(x-3))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = x^2 \times \frac{d(\log\,(x-3))}{dx} + \log(x-3) \times \frac{d(x^2)}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\,\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\,\frac{db}{dx} = x^2 \times \frac{1}{(x-3)}\,\frac{d(x-3)}{dx} + \log(x-3) \times (2x)$$

$$\left\{\begin{matrix} \frac{d(\log u)}{dx} = \frac{1}{u}\,\frac{du}{dx}\,; \frac{d(u^n)}{dx} = nu^{n-1}\,\frac{du}{dx} \\ \text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables} \end{matrix}\right\}$$

$$\Rightarrow \frac{1}{b}\,\frac{db}{dx} = \frac{x^2}{(x-3)}\left(\frac{dx}{dx} - \frac{d(3)}{dx}\right) + 2x\log(x-3)$$

$$\Rightarrow \frac{1}{b}\,\frac{db}{dx} = \frac{x^2}{(x-3)}(\,1) + 2x\log\,(x-3)$$

$$\Rightarrow \frac{db}{dx} = b\left\{\frac{x^2}{(x-3)} + 2x\log(x-3)\right\}$$

Put the value of $b = (x-3)^{x^2}$:

$$\Rightarrow \frac{db}{dx} = (x-3)^{x^2}\left\{\frac{x^2}{(x-3)} + 2x\log(x-3)\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2-3}\left\{\frac{(x^2-3)}{x} + 2x\log x\right\} + (x-3)^{x^2}\left\{\frac{x^2}{(x-3)} + 2x\log(x-3)\right\}$$

19. Question

Find $\frac{dy}{dx}$, when

1y = $e^x + 10^x + x^x$

Answer

let y = $e^x + 10^x + x^x$

⇒ y = a + b + c

where a= e^x; b = 10^x ; c = x^x

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

a= e^x

Taking log both the sides:

⇒ log a= log e^x

$\Rightarrow$ log a= x log e

{log x^a = alog x}

$\Rightarrow$ log a= x {log e =1}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{dx}{dx}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = 1$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{da}{dx} = a$$

Put the value of a = e^x

$$\Rightarrow \frac{da}{dx} = e^x$$

b = 10^x

Taking log both the sides:

$\Rightarrow$ log b= log 10^x

$\Rightarrow$ log b= x log 10

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log 10)}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \log 10 \times \frac{dx}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(au)}{dx} = a\ \frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = b(\log 10)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx} = b(\log 10)$$

Put the value of b = 10^x

$$\Rightarrow \frac{db}{dx} = 10^x(\log 10)$$

c = x^x

Taking log both the sides:

$\Rightarrow$ log c= log x^x

$\Rightarrow$ log c= x log x

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log c)}{dx} = \frac{d(x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log c)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\left\{ \text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\ \frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dc}{dx} = c\{1 + \log x\}$$

Put the value of $c = x^x$

$$\Rightarrow \frac{dc}{dx} = x^x\{1 + \log x\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x + 10^x(\log 10) + x^x\{1 + \log x\}$$

20. Question

Find $\frac{dy}{dx}$, when

$y = x^n + n^x + x^x + n^n$

Answer

let $y = x^n + n^x + x^x + n^n$

$\Rightarrow y = a + b + c + m$

where $a= x^n$; $b = n^x$; $c = x^x$; $m= n^n$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx} + \frac{dm}{dx}$$

$$\left\{ \text{Using chain rule,} \frac{d(u + a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables} \right\}$$

$a= x^n$

Taking log both the sides:

$\Rightarrow \log a= \log x^n$

$\Rightarrow \log a= n \log x$

$\{\log x^a = a\log x\}$

$\Rightarrow \log a= n \log x$ $\{\log e =1\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(n \log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = n\frac{d(\log x)}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(au)}{dx} = a\ \frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = n \times \frac{1}{x}\frac{dx}{dx}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{n}{x}$$

$$\Rightarrow \frac{da}{dx} = \frac{an}{x}$$

Put the value of a = x^n

$$\frac{da}{dx} = \frac{nx^n}{x}$$

$$\frac{da}{dx} = nx^{n-1}$$

$$\left\{\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

b = n^x

Taking log both the sides:

⇒ log b= log n^x

⇒ log b= x log n

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log n)}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \log n \times \frac{dx}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(au)}{dx} = a\ \frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = b(\log n)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx} = b(\log n)$$

Put the value of b = n^x:

$$\Rightarrow \frac{db}{dx} = n^x(\log n)$$

c = x^x

Taking log both the sides:

⇒ log c= log x^x

⇒ log c= x log x

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log c)}{dx} = \frac{d(x\log x)}{dx}$$

$$\Rightarrow \frac{d(\log c)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\,\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\,\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{c}\frac{dc}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dc}{dx} = c\{1 + \log x\}$$

Put the value of c = x^x

$$\Rightarrow \frac{dc}{dx} = x^x\{1 + \log x\}$$

m = n^n

$$\Rightarrow \frac{dm}{dx} = \frac{d(n^n)}{dx}$$

$$\Rightarrow \frac{dm}{dx} = 0$$

$$\left\{\frac{du}{dx} = 0 \text{ if u is any contant}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx} + \frac{dc}{dx} + \frac{dm}{dx}$$

$$\Rightarrow \frac{dy}{dx} = nx^{n-1} + n^x(\log n) + x^x\{1 + \log x\} + 0$$

$$\Rightarrow \frac{dy}{dx} = nx^{n-1} + n^x(\log n) + x^x\{1 + \log x\}$$

21. Question

Find $\frac{dy}{dx}$, when

$$y = \frac{\left(x^2 - 1\right)^3\left(2x - 1\right)}{\sqrt{\left(x - 3\right)\left(4x - 1\right)}}$$

Answer

Let $y = \dfrac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}$

$$\Rightarrow y = \frac{(x^2-1)^3(2x-1)}{((x-3)(4x-1))^{\frac{1}{2}}}$$

$$\Rightarrow y = \frac{(x^2-1)^3(2x-1)}{(x-3)^{\frac{1}{2}}(4x-1)^{\frac{1}{2}}}$$

Take log both sides:

$$\Rightarrow \log y = \log\left\{\frac{(x^2-1)^3(2x-1)}{(x-3)^{\frac{1}{2}}(4x-1)^{\frac{1}{2}}}\right\}$$

$$\left\{\log(ab) = \log a + \log b\,;\log\left(\frac{a}{b}\right) = \log a - \log b\right\}$$

$$\Rightarrow \log y = \{\log(x^2-1)^3 + \log(2x-1)\} - \{\log(x-3)^{\frac{1}{2}} + \log(4x-1)^{\frac{1}{2}}\}$$

$$\Rightarrow \log y = \log(x^2-1)^3 + \log(2x-1) - \log(x-3)^{\frac{1}{2}} - \log(4x-1)^{\frac{1}{2}}$$

$$\Rightarrow \log y = 3\log(x^2-1) + \log(2x-1) - \tfrac{1}{2}\log(x-3) - \tfrac{1}{2}\log(4x-1)$$ $\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = 3\frac{d(\log(x^2-1))}{dx} + \frac{d(\log(2x-1))}{dx} - \frac{1}{2}\frac{d(\log(x-3))}{dx} - \frac{1}{2}\frac{d(\log(4x-1))}{dx}$$

$$\left\{\text{Using chain rule, }\frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx}\text{ where a and u are any variables}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{x^2-1}\frac{d(x^2-1)}{dx} + \frac{1}{(2x-1)}\frac{d(2x-1)}{dx} - \frac{1}{2(x-3)}\frac{d(x-3)}{dx} - \frac{1}{2(4x-1)}\frac{d(4x-1)}{dx}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{x^2-1}(2x) + \frac{1}{(2x-1)}(2) - \frac{1}{2(x-3)}(1) - \frac{1}{2(4x-1)}(4)$$

$$\left\{\begin{array}{l}\text{Using chain rule, }\dfrac{d(a+u)}{dx} = \dfrac{du}{dx}\text{ where a is any constant and u is any variable;}\\ \qquad\qquad\qquad\qquad\qquad\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}\end{array}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{6x}{x^2-1} + \frac{2}{(2x-1)} - \frac{1}{2(x-3)} - \frac{4}{2(4x-1)}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{6x}{x^2-1} + \frac{2}{(2x-1)} - \frac{1}{2(x-3)} - \frac{2}{(4x-1)}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{6x}{x^2-1} + \frac{2}{(2x-1)} - \frac{1}{2(x-3)} - \frac{2}{(4x-1)}\right\}$$

Put the value of $y = \dfrac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}$:

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}\left\{\frac{6x}{x^2-1} + \frac{2}{(2x-1)} - \frac{1}{2(x-3)} - \frac{2}{(4x-1)}\right\}$$

22. Question

Find $\frac{dy}{dx}$, when

$$y = \frac{e^{ax}\text{sex } x \log x}{\sqrt{1-2x}}$$

Answer

$$\text{Let } y = \frac{e^{ax}\sec^x x \log x}{\sqrt{1-2x}}$$

$$\Rightarrow y = \frac{e^{ax}\sec^x x \log x}{(1-2x)^{\frac{1}{2}}}$$

Take log both sides:

$$\Rightarrow \log y = \log\left(\frac{e^{ax}\sec^x x \log x}{(1-2x)^{\frac{1}{2}}}\right)$$

$$\Rightarrow \log y = \log e^{ax} + \log \sec^x x + \log\log x - \log(1-2x)^{\frac{1}{2}}$$

$$\left\{\log(ab) = \log a + \log b\,;\log\left(\frac{a}{b}\right) = \log a - \log b\right\}$$

$\Rightarrow \log y = ax\log e + x\log\sec x + \log\log x - \frac{1}{2}\log(1-2x)$ {log x^a = alog x}

$\Rightarrow \log y = ax + x\log\sec x + \log\log x - \frac{1}{2}\log(1-2x)$ {log e = 1}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(ax)}{dx} + \frac{d(x\log\sec x)}{dx} + \frac{d(\log\log x)}{dx} - \frac{1}{2}\frac{d(\log(1-2x))}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a\frac{dx}{dx} + \left\{x\frac{d(\log\sec x)}{dx} + \log\sec x\frac{dx}{dx}\right\} + \frac{1}{\log x}\frac{d(\log x)}{dx} - \frac{1}{2(1-2x)}\frac{d(1-2x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\,;\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a + \left\{x \times \frac{1}{\sec x}\frac{d(\sec x)}{dx} + \log\sec x\right\} + \frac{1}{\log x} \times \frac{1}{x}\frac{dx}{dx} - \frac{1}{2(1-2x)}(-2)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\,;\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a + \left\{\frac{x}{\sec x}(\sec x\tan x) + \log\sec x\right\} + \frac{1}{x\log x} - \frac{(-2)}{2(1-2x)}$$

$$\left\{\frac{d(\sec x)}{dx} = \sec x\tan x\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a + \left\{\frac{x\sec x\tan x}{\sec x} + \log\sec x\right\} + \frac{1}{x\ \log x} + \frac{1}{(1-2x)}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = a + \{x\tan x + \log\sec x\} + \frac{1}{x\ \log x} + \frac{1}{(1-2x)}$$

$$\Rightarrow \frac{dy}{dx} = y\left\{a + x\tan x + \log\sec x + \frac{1}{x\ \log x} + \frac{1}{(1-2x)}\right\}$$

Put the value of $y = \frac{e^{ax}\sec^x x\log x}{\sqrt{1-2x}}$:

$$\Rightarrow \frac{dy}{dx} = \frac{e^{ax}\sec^x x\log x}{\sqrt{1-2x}}\left\{a + x\tan x + \log\sec x + \frac{1}{x\ \log x} + \frac{1}{(1-2x)}\right\}$$

23. Question

Find $\frac{dy}{dx}$, when

$y = e^{3x} \sin 4x\ 2^x$

Answer

Let $y = e^{3x} \sin 4x\ 2^x$

Take log both sides:

$\Rightarrow \log y = \log (e^{3x} \sin 4x\ 2^x)$

$\Rightarrow \log y = \log (e^{3x}) + \log (\sin 4x) + \log (2^x)$

$$\left\{\log(ab) = \log a + \log b\ ; \log\left(\frac{a}{b}\right) = \log a - \log b\right\}$$

$\Rightarrow \log y = 3x \log e + \log (\sin 4x) + x \log 2 \{\log x^a = a\log x\}$

$\Rightarrow \log y = 3x + \log (\sin 4x) + x \log 2\ \{\log e = 1\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(3x)}{dx} + \frac{d(\log\sin 4x)}{dx} + \frac{d(x\log 2)}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 3\frac{dx}{dx} + \frac{1}{\sin 4x}\frac{d(\sin 4x)}{dx} + \log 2\frac{dx}{dx}$$

$$\left\{\begin{array}{c}\text{Using chain rule,} \frac{d(au)}{dx} = a\ \frac{du}{dx} \text{ where a is any constant and u is any variable ;} \\ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\end{array}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 3 + \frac{\cos 4x}{\sin 4x}\frac{d(4x)}{dx} + \log 2$$

$$\left\{\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 3 + \cot 4x \times 4\frac{dx}{dx} + \log 2$$

$$\Rightarrow \frac{dy}{dx} = y\{3 + 4\cot 4x + \log 2\}$$

Put the value of $y = e^{3x}\sin 4x\, 2^x$:

$$\Rightarrow \frac{dy}{dx} = e^{3x}\sin 4x\, 2^x\{3 + 4\cot 4x + \log 2\}$$

24. Question

Find $\frac{dy}{dx}$, when

y = sin x sin 2x sin 3x sin 4x

Answer

Let y = sin x sin 2x sin 3x sin 4x

Take log both sides:

⇒ log y = log (sin x sin 2x sin 3x sin 4x)

⇒ log y = log (sin x) + log (sin 2x) + log (sin 3x) + log (sin 4x)

$$\left\{\log(ab) = \log a + \log b\,;\log\left(\frac{a}{b}\right) = \log a - \log b\right\}$$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log y)}{dx} = \frac{d(\log(\sin x))}{dx} + \frac{d(\log(\sin 2x))}{dx} + \frac{d(\log(\sin 3x))}{dx} + \frac{d(\log(\sin 4x))}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \frac{1}{\sin 2x}\frac{d(\sin 2x)}{dx} + \frac{1}{\sin 3x}\frac{d(\sin 3x)}{dx} + \frac{1}{\sin 4x}\frac{d(\sin 4x)}{dx}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\sin x}\frac{dx}{dx} + \frac{\cos 2x}{\sin 2x}\frac{d(2x)}{dx} + \frac{\cos 3x}{\sin 3x}\frac{d(3x)}{dx} + \frac{\cos 4x}{\sin 4x}\frac{d(4x)}{dx}$$

$$\left\{\frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cot x + \cot 2x \times 2\frac{dx}{dx} + \cot 3x \times 3\frac{dx}{dx} + \cot 4x \times 4\frac{dx}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(au)}{dx} = a\frac{du}{dx} \text{ where a is any constant and u is any variable}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cot x + 2\cot 2x + 3\cot 3x + 4\cot 4x$$

$$\Rightarrow \frac{dy}{dx} = y\{\cot x + 2\cot 2x + 3\cot 3x + 4\cot 4x\}$$

Put the value of $y = \sin x \sin 2x \sin 3x \sin 4x$:

$$\Rightarrow \frac{dy}{dx} = \sin x \sin 2x \sin 3x \sin 4x\{\cot x + 2\cot 2x + 3\cot 3x + 4\cot 4x\}$$

25. Question

Find $\frac{dy}{dx}$, when

$y = x^{\sin x} + (\sin x)^x$

Answer

let $y = x^{\sin x} + (\sin x)^x$

$\Rightarrow y = a + b$

where $a = x^{\sin x}$; $b = (\sin x)^x$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$a = x^{\sin x}$

Taking log both the sides:

$\Rightarrow \log a = \log x^{\sin x}$

$\Rightarrow \log a = \sin x \log x$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\sin x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = \sin x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\sin x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \sin x \times \frac{1}{x}\frac{dx}{dx} + \log x\,(\cos x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \ \&\ \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{\sin x}{x} + \log x \cos x$$

$$\Rightarrow \frac{da}{dx} = a\left(\frac{\sin x}{x} + \log x \cos x\right)$$

Put the value of $a = x^{\sin x}$:

$$\Rightarrow \frac{da}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right)$$

$b = (\sin x)^x$

Taking log both the sides:

$\Rightarrow \log b = \log (\sin x)^x$

$\Rightarrow \log b = x \log (\sin x)$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x \log (\sin x))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = x \times \frac{d(\log(\sin x))}{dx} + \log(\sin x) \times \frac{dx}{dx}$$

$$\left\{ \text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log(\sin x)$$

$$\left\{\frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x}{\sin x}(\cos x) + \log(\sin x)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x\cos x}{\sin x} + \log(\sin x)$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = x\cot x + \log(\sin x)$$

$$\Rightarrow \frac{db}{dx} = b\{x\cot x + \log(\sin x)\}$$

Put the value of b = $(\sin x)^x$:

$$\Rightarrow \frac{db}{dx} = (\sin x)^x\{x\cot x + \log(\sin x)\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x\cos x\right) + (\sin x)^x\{x\cot x + \log(\sin x)\}$$

26. Question

Find $\frac{dy}{dx}$, when

y = $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

Answer

let y = $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

⇒ y = a + b

where a= $(\sin x)^{\cos x}$; b = $(\cos x)^{\sin x}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

a= $(\sin x)^{\cos x}$

Taking log both the sides:

⇒ log a= log $(\sin x)^{\cos x}$

⇒ log a= cos x log (sin x)

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\cos x \log(\sin x))}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = \cos x \times \frac{d(\log(\sin x))}{dx} + \log(\sin x) \times \frac{d(\cos x)}{dx}$$

$$\left\{ \text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cos x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log(\sin x)\,(-\sin x)$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} ; \frac{d(\cos x)}{dx} = -\sin x ; \frac{d(\sin x)}{dx} = \cos x \right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cot x\,(\cos x) - \sin x \log(\sin x)$$

$$\Rightarrow \frac{da}{dx} = a\{\cos x \cot x - \sin x \log(\sin x)\}$$

Put the value of $a = (\sin x)^{\cos x}$:

$$\Rightarrow \frac{da}{dx} = (\sin x)^{\cos x}\{\cos x \cot x - \sin x \log(\sin x)\}$$

$b = (\cos x)^{\sin x}$

Taking log both the sides:

$\Rightarrow \log b = \log (\cos x)^{\sin x}$

$\Rightarrow \log b = \sin x \log (\cos x)$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(\sin x \log(\cos x))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \sin x \times \frac{d(\log(\cos x))}{dx} + \log(\cos x) \times \frac{d(\sin x)}{dx}$$

$$\left\{ \text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \sin x \times \frac{1}{\cos x}\frac{d(\cos x)}{dx} + \log(\cos x)\{\cos x\}$$

$$\left\{ \frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} ; \frac{d(\cos x)}{dx} = -\sin x ; \frac{d(\sin x)}{dx} = \cos x \right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \tan x\,(-\sin x) + \cos x \log(\cos x)$$

$$\Rightarrow \frac{db}{dx} = b\{-\sin x \tan x + \cos x \log(\cos x)\}$$

Put the value of $b = (\cos x)^{\sin x}$:

$$\Rightarrow \frac{db}{dx} = (\cos x)^{\sin x}\{\cos x \log(\cos x) - \sin x \tan x\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x}\{\cos x \cot x - \sin x \log(\sin x)\} + (\cos x)^{\sin x}\{\cos x \log(\cos x) - \sin x \tan x\}$$

27. Question

Find $\frac{dy}{dx}$, when

$y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

Answer

let $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

$\Rightarrow y = a + b$

where $a= (\tan x)^{\cot x}$; $b = (\cot x)^{\tan x}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$a= (\tan x)^{\cot x}$

Taking log both the sides:

$\Rightarrow \log a= \log (\tan x)^{\cot x}$

$\Rightarrow \log a= \cot x \log (\tan x)$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\cot x \log(\tan x))}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = \cot x \times \frac{d(\log(\tan x))}{dx} + \log(\tan x) \times \frac{d(\cot x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cot x \times \frac{1}{\tan x}\frac{d(\tan x)}{dx} + \log(\tan x)\,(-\operatorname{cosec}^2 x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} \;;\; \frac{d(\tan x)}{dx} = \sec^2 x \;;\; \frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cot^2 x(\sec^2 x) - \operatorname{cosec}^2 x \log(\tan x)$$

$$\left\{\tan x = \frac{1}{\cot x}\right\}$$

$$\Rightarrow \frac{da}{dx} = a\{\cot^2 x \sec^2 x - \operatorname{cosec}^2 x \log(\tan x)\}$$

Put the value of $a = (\tan x)^{\cot x}$:

$$\Rightarrow \frac{da}{dx} = (\tan x)^{\cot x}\{\cot^2 x \sec^2 x - \operatorname{cosec}^2 x \log(\tan x)\}$$

$b = (\cot x)^{\tan x}$

Taking log both the sides:

⇒ log b= log (cot x)$^{\tan x}$

⇒ log b= tan x log (cot x)

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(\tan x \log(\cot x))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \tan x \times \frac{d(\log(\cot x))}{dx} + \log(\cot x) \times \frac{d(\tan x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\,\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \tan x \times \frac{1}{\cot x}\,\frac{d(\cot x)}{dx} + \log(\cot x)\{\sec^2 x\}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\,;\,\frac{d(\tan x)}{dx} = \sec^2 x\,;\,\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \tan^2 x\,(-\operatorname{cosec}^2 x) + \sec^2 x \log(\cot x)$$

$$\left\{\cot x = \frac{1}{\tan x}\right\}$$

$$\Rightarrow \frac{db}{dx} = b\{-\tan^2 x \operatorname{cosec}^2 x + \sec^2 x \log(\cot x)\}$$

Put the value of b $= (\cot x)^{\tan x}$:

$$\Rightarrow \frac{db}{dx} = (\cot x)^{\tan x}\{\sec^2 x \log(\cot x) - \tan^2 x \operatorname{cosec}^2 x\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\tan x)^{\cot x}\{\cot^2 x \sec^2 x - \operatorname{cosec}^2 x \log(\tan x)\} + (\cot x)^{\tan x}\{\sec^2 x \log(\cot x) - \tan^2 x \operatorname{cosec}^2 x\}$$

28. Question

Find $\frac{dy}{dx}$, when

$$y = (\sin x)^x + \sin^{-1}\sqrt{x}$$

Answer

Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

⇒ y = a + b

where, $a = (\sin x)^x$; $b = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

a = $(\sin x)^x$

Taking log both the sides:

⇒ log a= log $(\sin x)^x$

⇒ log a= x log (sin x)

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx}=\frac{d(x\log(\sin x))}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx}= x\times\frac{d(\log(\sin x))}{dx}+\log(\sin x)\times\frac{dx}{dx}$$

$$\left\{\text{Using product rule,}\frac{d(uv)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx}=x\times\frac{1}{\sin x}\frac{d(\sin x)}{dx}+\log(\sin x)$$

$$\left\{\frac{d(\sin x)}{dx}=\cos x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx}=\frac{x}{\sin x}(\cos x)+\log(\sin x)$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx}=\frac{x\cos x}{\sin x}+\log(\sin x)$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx}= x\cot x+\log(\sin x)$$

$$\Rightarrow \frac{da}{dx}=a\{x\cot x+\log(\sin x)\}$$

Put the value of a= $(\sin x)^x$:

$$\Rightarrow \frac{da}{dx}=(\sin x)^x\{x\cot x+\log(\sin x)\}$$

$$b = \sin^{-1}\sqrt{x}$$

$$\Rightarrow b=\sin^{-1}(x)^{\frac{1}{2}}$$

Differentiating with respect to x:

$$\frac{db}{dx}=\frac{d\left(\sin^{-1}(x)^{\frac{1}{2}}\right)}{dx}$$

$$\Rightarrow \frac{db}{dx}=\frac{1}{\sqrt{1-\left(x^{\frac{1}{2}}\right)^2}}\frac{d(x)^{\frac{1}{2}}}{dx}$$

$$\left\{\frac{d(\sin^{-1}u)}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx}=\frac{1}{\sqrt{1-x}}\left(\frac{1}{2}x^{\left(\frac{1}{2}-1\right)}\right)$$

$$\left\{\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{db}{dx} = \frac{1}{2\sqrt{1-x}}\left(x^{\left(-\frac{1}{2}\right)}\right)$$

$$\Rightarrow \frac{db}{dx} = \frac{1}{2\sqrt{1-x}}\left(\frac{1}{\sqrt{x}}\right)$$

$$\Rightarrow \frac{db}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\Rightarrow \frac{db}{dx} = \frac{1}{2\sqrt{x(1-x)}}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x\{x\cot x + \log(\sin x)\} + \frac{1}{2\sqrt{x(1-x)}}$$

29 A. Question

Find $\frac{dy}{dx}$, when

$y = x^{\cos x} + (\sin x)^{\tan x}$

Answer

let $y = x^{\cos x} + (\sin x)^{\tan x}$

$\Rightarrow y = a + b$

where $a= x^{\cos x}$; $b = (\sin x)^{\tan x}$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

$a= x^{\cos x}$

Taking log both the sides:

$\Rightarrow \log a= \log (x)^{\cos x}$

$\Rightarrow \log a= \cos x \log x$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\cos x \log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = \cos x \times \frac{d(\log x)}{dx} + \log x \times \frac{d(\cos x)}{dx}$$

$$\left\{\text{Using product rule, } \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \cos x \times \frac{1}{x}\frac{dx}{dx} + \log x\,(-\sin x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\ ;\ \frac{d(\cos x)}{dx} = -\sin x\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{\cos x}{x} - \sin x \log x$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{\cos x}{x} - \sin x \log x\right\}$$

Put the value of $a = x^{\cos x}$:

$$\Rightarrow \frac{da}{dx} = x^{\cos x}\left\{\frac{\cos x}{x} - \sin x \log x\right\}$$

$b = (\sin x)^{\tan x}$

Taking log both the sides:

$\Rightarrow \log b = \log (\sin x)^{\tan x}$

$\Rightarrow \log b = \tan x \log (\sin x)$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(\tan x \log(\sin x))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = \tan x \times \frac{d(\log(\sin x))}{dx} + \log(\sin x) \times \frac{d(\tan x)}{dx}$$

$$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \tan x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log(\sin x)\{\sec^2 x\}$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\ ;\ \frac{d(\tan x)}{dx} = \sec^2 x\ ;\ \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{\sin x}{\cos x} \times \frac{1}{\sin x}(\cos x) + \sec^2 x \log(\sin x)$$

$$\Rightarrow \frac{db}{dx} = b\{1 + \sec^2 x \log(\sin x)\}$$

Put the value of $b = (\sin x)^{\tan x}$:

$$\Rightarrow \frac{db}{dx} = (\sin x)^{\tan x}\{1 + \sec^2 x \log(\sin x)\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^{\cos x}\left\{\frac{\cos x}{x} - \sin x \log x\right\} + (\sin x)^{\tan x}\{1 + \sec^2 x \log(\sin x)\}$$

29 B. Question

Find $\frac{dy}{dx}$, when

$y = x^x + (\sin x)^x$

Answer

let $y = x^x + (\sin x)^x$

$\Rightarrow$ y = a + b

where a= x^x ; b = $(\sin x)^x$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule,} \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

a= x^x

Taking log both the sides:

$\Rightarrow$ log a= log $(x)^x$

$\Rightarrow$ log a= x log x

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(x\log x)}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = x \times \frac{d(\log x)}{dx} + \log x \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = x \times \frac{1}{x}\frac{dx}{dx} + \log x$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = 1 + \log x$$

$$\Rightarrow \frac{da}{dx} = a\{1 + \log x\}$$

Put the value of $a = x^x$:

$$\Rightarrow \frac{da}{dx} = x^x\{1 + \log x\}$$

b = $(\sin x)^x$

Taking log both the sides:

$\Rightarrow$ log b= log $(\sin x)^x$

$\Rightarrow$ log b= x log (sin x)

{log x^a = alog x}

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log b)}{dx} = \frac{d(x\log(\sin x))}{dx}$$

$$\Rightarrow \frac{d(\log b)}{dx} = x \times \frac{d(\log(\sin x))}{dx} + \log(\sin x) \times \frac{dx}{dx}$$

$$\left\{\text{Using product rule,} \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = x \times \frac{1}{\sin x}\frac{d(\sin x)}{dx} + \log(\sin x)$$

$$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx} ; \frac{d(\sin x)}{dx} = \cos x\right\}$$

$$\Rightarrow \frac{1}{b}\frac{db}{dx} = \frac{x}{\sin x}(\cos x) + \log(\sin x)$$

$$\left\{\cot x = \frac{\cos x}{\sin x}\right\}$$

$$\Rightarrow \frac{db}{dx} = b\{x\cot x + \log(\sin x)\}$$

Put the value of $b = (\sin x)^x$:

$$\Rightarrow \frac{db}{dx} = (\sin x)^x\{x\cot x + \log(\sin x)\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = x^x\{1 + \log x\} + (\sin x)^x\{x\cot x + \log(\sin x)\}$$

30. Question

Find $\frac{dy}{dx}$, when

$y = (\tan x)^{\log x} + \cos^2\left(\frac{\pi}{4}\right)$

Answer

Let $y = (\tan x)^{\log x} + \cos^2\left(\frac{\pi}{4}\right)$

$\Rightarrow y = a + b$

where, $a = (\tan x)^{\log x}$; $b = \cos^2\left(\frac{\pi}{4}\right)$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\left\{\text{Using chain rule, } \frac{d(u+a)}{dx} = \frac{du}{dx} + \frac{da}{dx} \text{ where a and u are any variables}\right\}$$

a= $(\tan x)^{\log x}$

Taking log both the sides:

$\Rightarrow \log a = \log (\tan x)^{\log x}$

$\Rightarrow \log a = \log x \, . \log (\tan x)$

$\{\log x^a = a\log x\}$

Differentiating with respect to x:

$$\Rightarrow \frac{d(\log a)}{dx} = \frac{d(\log x \log(\tan x))}{dx}$$

$$\Rightarrow \frac{d(\log a)}{dx} = \log x \times \frac{d(\log(\tan x))}{dx} + \log(\tan x) \times \frac{d(\log x)}{dx}$$

$\left\{\text{Using product rule}, \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}\right\}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \log x \times \frac{1}{\tan x}\frac{d(\tan x)}{dx} + \log(\tan x)\left(\frac{1}{x}\frac{dx}{dx}\right)$$

$\left\{\frac{d(\log u)}{dx} = \frac{1}{u}\frac{du}{dx}; \frac{d(\tan x)}{dx} = \sec^2 x\right\}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{\log x}{\tan x}(\sec^2 x) + \frac{\log(\tan x)}{x}$$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{\log x \cos x}{\sin x}\left(\frac{1}{\cos^2 x}\right) + \frac{\log(\tan x)}{x}$$

$\left\{\tan x = \frac{\sin x}{\cos x}; \sec x = \frac{1}{\cos x}\right\}$

$$\Rightarrow \frac{1}{a}\frac{da}{dx} = \frac{\log x}{\sin x \cos x} + \frac{\log(\tan x)}{x}$$

$$\Rightarrow \frac{da}{dx} = a\left\{\frac{\log x}{\sin x \cos x} + \frac{\log(\tan x)}{x}\right\}$$

Put the value of $a = (\tan x)^{\log x}$:

$$\Rightarrow \frac{da}{dx} = (\tan x)^{\log x}\left\{\frac{\log x}{\sin x \cos x} + \frac{\log(\tan x)}{x}\right\}$$

$$b = \cos^2\left(\frac{\pi}{4}\right)$$

Differentiating with respect to x:

$$\Rightarrow \frac{db}{dx} = \frac{d\left(\cos^2\left(\frac{\pi}{4}\right)\right)}{dx}$$

$\left\{\frac{du}{dx} = 0, \text{where u is any constant}\right\}$

$$\Rightarrow \frac{db}{dx} = 0$$

$$\left\{\begin{array}{c}\text{Here}, \cos^2\left(\frac{\pi}{4}\right) \text{ is a constant value} \\ \text{As}, \frac{\pi}{4} = 45° \\ \cos^2\left(\frac{\pi}{4}\right) = \cos^2 45° = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}\end{array}\right\}$$

$$\frac{dy}{dx} = \frac{da}{dx} + \frac{db}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\tan x)^{\log x}\left\{\frac{\log x}{\sin x \cos x} + \frac{\log(\tan x)}{x}\right\} + 0$$

$$\Rightarrow \frac{dy}{dx} = (\tan x)^{\log x}\left\{\frac{\log x}{\sin x \cos x} + \frac{\log(\tan x)}{x}\right\}$$

31. Question

Find $\frac{dy}{dx}$, when

$y = x^x + x^{1/x}$

Answer

Here,

y = $x^x + x^{1/x}$

$= e^{\log x^x} + e^{\log x^{\frac{1}{x}}}$

y = $e^{x \log x} + e^{(\frac{1}{x}\log x)}$

[Since $\log a^b = b \log a$]

Differentiating it with respect to x using the chain rule and product rule,

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x\log x}\right) + \frac{d}{dx}\left(e^{\frac{1}{x}\log x}\right)$$

$$= e^{x\log x} + \frac{d}{dx}(x\log x) + e^{\frac{1}{x}\log x}\frac{d}{dx}\left(\frac{1}{x}\log x\right)$$

$$= e^{\log x^x}\left[x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)\right] + e^{\log x^{\frac{1}{x}}}\left[\frac{1}{x}\frac{d}{dx}(\log x) + \log x\frac{d}{dx}\left(\frac{1}{x}\right)\right]$$

$$= x^x\left[x\left(\frac{1}{x}\right) + \log x(1)\right] + x^{\frac{1}{x}}\left[\left(\frac{1}{x}\right)\left(\frac{1}{x}\right) + \log x\left(-\frac{1}{x^2}\right)\right]$$

$$= x^x[1 + \log x] + x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{1}{x^2}\log x\right)$$

$$\frac{dy}{dx} = x^x[1 + \log x] + x^{\frac{1}{x}}\frac{(1 - \log x)}{x^2}$$

32. Question

Find $\frac{dy}{dx}$, when

y = $x^{\log x} + (\log x)^x$

Answer

Here,

y = $x^{\log x} + (\log x)^x$

Let

$u = (\log x)^x$, and $v = x^{\log x}$

∴ y=u+v

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ ......(i)

$u = (\log x)^x$

$\log u = \log[(\log x)^x]$

$\log u = x\log(\log x)$

Differentiating both sides with respect to x, we get

$$\frac{1}{u}.\frac{du}{dx} = \frac{d}{dx}(x) \times \log(\log x) + x\frac{d}{dx}[\log(\log x)]$$

$$\frac{du}{dx} = u\left[1 \times \log(\log x) + x.\frac{1}{\log x}.\frac{d}{dx}(\log x)\right]$$

$$\frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{x}{\log x}.\frac{1}{x}\right]$$

$$\frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]$$

$$\frac{du}{dx} = (\log x)^x \left[\frac{\{\log(\log x) \times \log x\} + 1}{\log x}\right]$$

$$\frac{du}{dx} = (\log x)^{x-1}[1 + \{\log(\log x) \times \log x\}] \dots\dots (ii)$$

$$\frac{1}{v}.\frac{dv}{dx} = \frac{d}{dx}[(\log x)^2]$$

$$\frac{1}{v}.\frac{dv}{dx} = 2(\log x).\frac{d}{dx}(\log x)$$

$$\frac{dv}{dx} = 2v(\log x).\frac{1}{x}$$

$$\frac{dv}{dx} = 2x^{\log x}\frac{\log x}{x}$$

$$\frac{dv}{dx} = 2x^{\log x - 1}.\log x \ \dots\dots (iii)$$

Therefore from (i), (ii), (iii), we get

$$\frac{dy}{dx} = (\log x)^{x-1}[1 + \{\log(\log x) \times \log x\}] + 2x^{\log x - 1}.\log x$$

33. Question

If $x^{13}\ y^7 = (x + y)^{20}$, prove that $\frac{dy}{dx} = \frac{y}{x}$

Answer

Here,

$x^{13}\ y^7 = (x + y)^{20}$

Taking log on both sides,

$\log(x^{13}\ y^7) = \log(x + y)^{20}$

13 log x+7log y = 20 log(x+y)

[Since, log (AB)=logA+logB ; log a^b=b log a]

Differentiating it with respect to x using the chain rule,

$$13\frac{d}{dx}(\log x) + 7\frac{d}{dx}(\log y) = 20\frac{d}{dx}\log(x + y)$$

$$\frac{13}{x} + \frac{7}{y}\frac{dy}{dx} = \frac{20}{x + y}\frac{d}{dx}(x + y)$$

$$\frac{7}{y}\frac{dy}{dx} - \frac{20}{x + y} = \frac{20}{x + y} - \frac{13}{x}$$

$$\frac{dy}{dx}\left[\frac{7}{y} - \frac{20}{x + y}\right] = \frac{20}{x + y} - \frac{13}{x}$$

$$\frac{dy}{dx}\left[\frac{7(x + y) - 20y}{y(x + y)}\right] = \frac{20x - 13(x + y)}{(x + y)x}$$

$$\frac{dy}{dx}=\left[\frac{20x-13(x+y)}{(x+y)x}\right]\times\left[\frac{y(x+y)}{7(x+y)-20y}\right]$$

$$\frac{dy}{dx}=\left[\frac{20x-13x-13y}{(x+y)x}\right]\times\left[\frac{y(x+y)}{7x+7y-20y}\right]$$

$$\frac{dy}{dx}=\frac{y}{x}\left[\frac{7x-13y}{7x-13y}\right]$$

$$\frac{dy}{dx}=\frac{y}{x}$$

Hence, Proved.

34. Question

If $x^{16} y^9 = (x^2 + y)^{17}$, prove that $x\frac{dy}{dx}=2\,y$

Answer

Here,

$x^{16} y^9 = (x^2 + y)^{17}$

Taking log on both sides,

$$\log(x^{16}y^{9})=\log(x^{2}+\ y)^{17}$$

$16 \log x + 9 \log y = 17 \log(x^2+y)$

[Since, log (AB)=logA+logB ; $\log a^b = b \log a$]

Differentiating it with respect to x using the chain rule,

$$16\frac{d}{dx}(\log x)+9\frac{d}{dx}(\log y)=17\frac{d}{dx}\log(x^{2}+y)$$

$$\frac{16}{x}+\frac{9}{y}\frac{dy}{dx}=17.\frac{1}{(x^{2}+y)}\frac{d}{dx}(x^{2}+y)$$

$$\frac{16}{x}+\frac{9}{y}\frac{dy}{dx}=\frac{17}{(x^{2}+y)}\left[2x+\frac{dy}{dx}\right]$$

$$\frac{16}{x}+\frac{9}{y}\frac{dy}{dx}=\left[\frac{17}{(x^{2}+y)}\right]\frac{dy}{dx}+\left[\frac{34x}{(x^{2}+y)}\right]$$

$$\frac{9}{y}\frac{dy}{dx}-\frac{17}{(x^{2}+y)}\frac{dy}{dx}=\left(\frac{34x}{x^{2}+y}\right)-\frac{16}{x}$$

$$\frac{dy}{dx}\left[\frac{9}{y}-\frac{17}{(x^{2}+y)}\right]=\frac{34x^{2}-16(x^{2}+y)}{(x^{2}+y)x}$$

$$\frac{dy}{dx}\left[\frac{9(x^{2}+y)-17y}{y(x^{2}+y)}\right]=\frac{34x^{2}-16x^{2}-16y}{(x^{2}+y)x}$$

$$\frac{dy}{dx}\left[\frac{9x^{2}+9y-17y}{y(x^{2}+y)}\right]=\frac{18x^{2}-16y}{(x^{2}+y)x}$$

$$\frac{dy}{dx}\left[\frac{9x^{2}+9y-17y}{y(x^{2}+y)}\right]=\frac{2(9x^{2}-8y)}{(x^{2}+y)x}$$

$$\frac{dy}{dx}\left[\frac{9x^{2}-8y}{y(x^{2}+y)}\right]=\frac{2(9x^{2}-8y)}{(x^{2}+y)x}$$

$$\frac{dy}{dx}=\left[\frac{2(9x^2-8y)}{(x^2+y)x}\right]\left[\frac{y(x^2+y)}{9x^2-8y}\right]$$

$$\frac{dy}{dx}=\frac{2y}{x}$$

Hence, Proved.

35. Question

If y = sin (x^x), prove that $\frac{dy}{dx}=\cos\left(x^x\right).x^x\left(1+\log\ x\right)$

Answer

Here,

y = sin (x^x) ......(i)

Let $u=x^x$ ......(ii)

Taking log on both sides,

log u = log x^x

log u = x log x

Differentiating both sides with respect to x,

$$\frac{1}{u}\frac{du}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(x)$$

$$=x\left(\frac{1}{x}\right)+\log x(1)$$

$$\frac{1}{u}\frac{du}{dx}=1+\log x$$

$$\frac{du}{dx}=u(1+\log x)$$

$\frac{du}{dx}=x^x(1+\log x)$ ...... (iii) [from (ii)]

Now, using equation (ii) in (i)

y = sin u

Differentiating both sides with respect to x,

$$\frac{dy}{dx}=\frac{d}{dx}(\sin u)$$

$$=\cos u\frac{du}{dx}$$

Using equation (ii) and (iii),

$$\frac{dy}{dx}=\cos x^x.\ x^x(1+\log x)$$

Hence Proved.

36. Question

If $x^x + y^x = 1$, prove that $\frac{dy}{dx} = -\left\{\frac{x^x(1+\log x)+y^x.\log y}{x.y^{(x-1)}}\right\}$

Answer

Here

$x^x + y^x = 1$

$e^{\log x^x} + e^{\log y^x} = 1$

$e^{x\log x} + e^{x\log y} = 1$

[Since $e^{\log a} = a, \log a^b = b\log a$]

Differentiating it with respect to x using chain rule and product rule,

$$\frac{d}{dx}e^{x\log x} + \frac{d}{dx}e^{x\log y} = \frac{d}{dx}(1)$$

$$e^{x\log x}\frac{d}{dx}(x\log x) + e^{x\log y}\frac{d}{dx}(x\log y) = 0$$

$$e^{\log x^x}\left[x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x)\right] + e^{\log y^x}\left[x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)\right] = 0$$

$$x^x\left[x\left(\frac{1}{x}\right) + \log x(1)\right] + y^x\left[x\left(\frac{1}{y}\right) + \log y(1)\right] = 0$$

$$x^x[1+\log x] + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$$

$$y^x \times \frac{x}{y}\frac{dy}{dx} = -[x^x(+\log x) + y^x\log y] = 0$$

$$(xy^{x-1})\frac{dy}{dx} = -[x^x(1+\log x) + y^x\log y] = 0$$

$$\frac{dy}{dx} = -\left[\frac{x^x(1+\log x) + y^x\log y}{xy^{x-1}}\right]$$

Hence Proved.

37. Question

If $x^x + y^x = 1$, find $\frac{dy}{dx} = -\frac{y(y+x\log y)}{x(y\log x+x)}$

Answer

Let $x^x = u$ and $y^x = v$

Taking log on both sides we get,

x log x = log u ......(1),

x log y = log v ......(2)

Using $\log a^b = b\log a$

Differentiating both sides of equation (1) we get,

$$x \times \frac{1}{x} + \log x = \frac{1}{u}\frac{du}{dx}$$

$$\frac{du}{dx} = x^x\ (\ 1 + \log x)\ \ \ldots.(3)$$

Differentiating both sides of equation (2) we get,

$$x\ \times \frac{1}{y}\frac{dy}{dx} + \log y\ = \frac{1}{v}\frac{dv}{dx}$$

$$\frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \ldots\ldots\ldots\ldots.(4)$$

We know that, from question,

u + v = 1

Differentiating both sides we get,

$$\frac{du}{dx}\ +\ \frac{dv}{dx} = 0$$

Putinng the value of eq(4) and eq(5) in equation above we get,

$$x^x\ (1 + \log x\ ) + y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$

$$y^x\frac{x}{y}\frac{dy}{dx} = \frac{-x^x(1 + \log x)}{y^x\ (\log y)}$$

$$\frac{dy}{dx} = \frac{-\ x^{x-1}\ (1 + \log x)}{y^{2x-1}\log y}$$

38. Question

If $x^y + y^x = (x + y)^{x + y}$, find $\frac{dy}{dx}$

Answer

Here,

$$x^y + y^x = (x + y)^{x+y}$$

$$e^{\log x^y} +\ e^{\log y^x} = e^{\log(x+y)^{x+y}}$$

$$e^{y\log x} + e^{x\log y} = e^{(x+y)\log(x+y)}$$

Differentiating it with respect to x using chain rule, product rule,

$$\frac{d}{dx}e^{y\log x} + \frac{d}{dx}e^{x\log y} = \frac{d}{dx}e^{(x+y)\log(x+y)}$$

$$e^{y\log x}\left[y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}\right] + e^{x\log y}\left[x\frac{d}{dx}(\log y) + \log y\frac{dx}{dx}\right]$$
$$= e^{(x+y)\log(x+y)}\frac{d}{dx}[(x + y)\log(x + y)]$$

$$e^{\log x^y}\left[y\left(\frac{1}{x}\right) + \log x\frac{dy}{dx}\right] + e^{\log y^x}\left[\frac{x}{y}\frac{dy}{dx} + \log y(1)\right]$$
$$= e^{\log(x+y)^{x+y}}\left[(x + y)\frac{d}{dx}\log(x + y) + \log(x + y)\frac{d}{dx}(x + y)\right]$$

$$x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] + y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$
$$= (x + y)^{x+y}\left[(x + y).\frac{1}{(x + y)}.\frac{d}{dx}.(x + y) + \log(x + y)\left(1 + \frac{dy}{dx}\right)\right]$$

$$x^y\frac{y}{x}+x^y.\log x\frac{dy}{dx}+y^x.\frac{x}{y}.\frac{dy}{dx}$$
$$+y^x\log y\ =(x+y)^{x+y}\left[1\times\left(1+\frac{dy}{dx}\right)+\log(x+y)\left(1+\frac{dy}{dx}\right)\right]$$

$$x^{y-1}\times y+x^y.\log x\frac{dy}{dx}+y^x.\frac{x}{y}.\frac{dy}{dx}+y^x\log y=(x+y)^{x+y}+(x+y)^{x+y}\frac{dy}{dx}$$
$$+(x+y)^{x+y}\log(x+y)+(x+y)^{x+y}\log(x+y)\frac{dy}{dx}$$

$$\frac{dy}{dx}[x^y\log x+x\,y^{x-1}-(x+y)^{x+y}(1+\log(x+y))]$$
$$=(x+y)^{x+y}(1+\log(x+y))-x^{y-1}\times y-\ y^x\log y$$

$$\frac{dy}{dx}=\left[\frac{(x+y)^{x+y}(1+\log(x+y))-x^{y-1}\times y-\ y^x\log y}{x^y\log x+xy^{x-1}-(x+y)^{x+y}(1+\log(x+y))}\right]$$

39. Question

If $x^m y^n = 1$, prove that $\frac{dy}{dx}=-\frac{my}{nx}$

Answer

Here,

$x^m y^n = 1$

Taking log on both sides,

$\log(x^m y^n) = \log 1$

m logx + n logy=log 1 [Since, log (AB)=logA+logB ; log a^b=b log a]

Differentiating with respect to x

$$\frac{d}{dx}(m\log x)+\frac{d}{dx}(n\log y)=\frac{d}{dx}(\log(1))$$

$$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=0$$

$$\frac{n}{y}\frac{dy}{dx}=-\frac{m}{x}$$

$$\frac{dy}{dx}=-\frac{m}{x}\times\frac{y}{n}$$

$$\frac{dy}{dx}=-\frac{my}{nx}$$

Hence Proved.

40. Question

If $y^x = e^{y-x}$ prove that $\frac{dy}{dx}=\frac{(1+\log\ y)^2}{\log\ y}$

Answer

Here, $y^x=e^{y-x}$

Taking log on both sides,

$\log y^x=\log e^{y-x}$

$\mathrm{xlogy} = (\mathrm{y}-\mathrm{x})\log e$ [Since, log (AB)=logA+logB ; log a^b=b log a]

$\mathrm{xlogy} = (\mathrm{y}-\mathrm{x})$ ......(i)

Differentiating with respect to x using product rule,

$$\frac{d}{dx}(x\log y) = \frac{d}{dx}(y-x)$$

$$\left[x\frac{d}{dx}(\log y) + \log y\frac{d}{dx}(x)\right] = \frac{dy}{dx} - 1$$

$$x\left(\frac{1}{y}\right)\frac{dy}{dx} + \log y(1) = \frac{dy}{dx} - 1$$

$$\frac{dy}{dx}\left[\frac{y}{(1+\log y)y}\right] = -(1+\log y)$$

$$\frac{dy}{dx}\left[\frac{1-1-\log y}{(1+\log y)}\right] = -(1+\log y)$$

$$\frac{dy}{dx} = -\frac{(1+\log y)^2}{-\log y}$$

$$\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

Hence Proved.

41. Question

If $(\sin x)^y = (\cos y)^x$, prove that $\frac{dy}{dx} = \frac{\log\cos y - y\cot x}{\log\sin x + x\tan y}$

Answer

Here,

$$(\sin x)^{y} = (\cos y)^{x}$$

Taking log on both sides,

$$\log(\sin x)^{y} = \log(\cos y)^{x}$$

$\mathrm{y}\log(\sin x) = x\log(\cos y)$ [Using log a^b=b log a]

Differentiating it with respect to x using product rule and chain rule,

$$\frac{d}{dx}[y\log\sin x] = \frac{d}{dx}[x\log\cos y]$$

$$y\frac{d}{dx}(\log\sin x) + \log\sin x\frac{dy}{dx} = x\frac{dy}{dx}\log\cos y + \log\cos y\frac{d}{dx}(x)$$

$$y\left(\frac{1}{\sin x}\right)\frac{d}{dx}(\sin x) + \log\sin x\frac{dy}{dx} = \frac{x}{\cos y}\frac{d}{dx}(\cos y) + \log\cos y\,(1)$$

$$\frac{y}{\sin x}(\cos x) + \log\sin x\,\frac{dy}{dx} = \frac{x}{\cos y}(-\sin y)\frac{dy}{dx} + \log\cos y$$

$$y\cot x + \log\sin x\frac{dy}{dx} = -x\tan y\frac{dy}{dx} + \log\cos y$$

$$\log\sin x\frac{dy}{dx} + x\tan y\frac{dy}{dx} = \log\cos y - y\cot x$$

$$\frac{dy}{dx}(\log \sin x + x \tan y) = \log \cos y - y \cot x$$

$$\frac{dy}{dx} = \frac{\log \cos y - y \cot x}{\log \sin x + x \tan y}$$

Hence Proved.

42. Question

If $(\cos x)^y = (\tan y)^x$, prove that $\frac{dy}{dx} = \frac{\log\ \tan\ y + y\ \tan\ x}{\log\ \cos\ x - x\ \sec\ y\ \operatorname{cosec}\ y}$

Answer

Here,

$$(\cos x)^{\ y} = (\tan y)^{\ x}$$

Taking log on both sides,

$$\log\ (\cos x)^{\ y} = \log(\tan y)^{\ x}$$

$y \log\ (\cos x) = x \log\ \ (\tan y)$ [Using log a^b=b log a]

Differentiating it with respect to x using product rule and chain rule,

$$\frac{d}{dx}[y \log \cos x] = \frac{d}{dx}[x \log \tan y]$$

$$y\frac{d}{dx}(\log \cos x) + \log \cos x\frac{dy}{dx} = x\frac{d}{dx}\log \tan y + \log \tan y\frac{d}{dx}(x)$$

$$y\left(\frac{1}{\cos x}\right)\frac{d}{dx}(\cos x) + \log \cos x\frac{dy}{dx} = \frac{x}{\tan\ y}\frac{d}{dx}(\tan y) + \log \tan y\ (1)$$

$$\left(\frac{y}{\cos x}(-\sin x) + \log \cos x\frac{dy}{dx}\right) = \left(\frac{x}{\tan\ y}(\sec^2 x)\right)\frac{dy}{dx} + \log \tan y$$

$$-y \tan x + \log \cos x\frac{dy}{dx} = \left(\sec y \operatorname{cosec} y\ \times x\frac{dy}{dx} + \log \tan y\right)$$

$$\frac{dy}{dx}[\log \cos x - x \sec y \operatorname{cosec} y\,] = \log \tan y + y \tan x$$

$$\frac{dy}{dx} = \frac{\log \tan y + y \tan x}{\log \cos x - x \sec y \operatorname{cosec} y}$$

43. Question

If $e^x + e^y = e^{x + y}$, prove that $\frac{dy}{dx} + e^{y - x} = 0$

Answer

Here,

$e^x + e^y = e^{x+y}$ ……(i)

Differentiating both the sides using chain rule,

$$\frac{d}{dx}e^x + \frac{d}{dx}e^y = \frac{d}{dx}(e^{x+y})$$

$$e^x + e^y\frac{dy}{dx} = e^{x+y}\frac{d}{dx}(x+y)$$

$$e^x + e^y\frac{dy}{dx} = e^{x+y}\left[1+\frac{dy}{dx}\right]$$

$$e^y\frac{dy}{dx} - e^{x+y}\frac{dy}{dx} = e^{x+y} - e^x$$

$$\frac{dy}{dx}(e^y - e^{x+y}) = e^{x+y} - e^x$$

$$\frac{dy}{dx} = \frac{e^{x+y} - e^x}{e^y - e^{x+y}}$$

$$\frac{dy}{dx} = \frac{e^x + e^y - e^x}{e^y - (e^x + e^y)}$$

$$\frac{dy}{dx} = \frac{e^x + e^y - e^x}{e^y - e^x - e^y}$$

$$\frac{dy}{dx} = \frac{e^y}{-e^x}$$

$$\frac{dy}{dx} = -e^{y-x}$$

$$\frac{dy}{dx} + e^{y-x} = 0$$

Hence Proved.

44. Question

If $e^y = y^x$, prove that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$

Answer

Here

$$e^y = y^x$$

Taking log on both sides,

$$\log e^y = \log y^x$$

$$y\log e = x\log y$$

[Using $\log a^b = b\log a$]

$$y = x\log y \text{ --------- (i)}$$

Differentiating it with respect to x using product rule,

$$\frac{dy}{dx} = \frac{d}{dx}(x\log y)$$

$$= x\frac{dy}{dx}(\log y) + \log y\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y(1)$$

$$\frac{dy}{dx}\left(1 - \frac{x}{y}\right) = \log y$$

$$\frac{dy}{dx}\left(\frac{y-x}{y}\right) = \log y$$

$$\frac{dy}{dx} = \left(\frac{y\log y}{y-x}\right)$$

$$\frac{dy}{dx} = \frac{y\log y}{y - \frac{y}{\log y}} \text{ [Using (i)]}$$

$$= \frac{y\log y \times \log y}{y\log y - y}$$

$$= \frac{y(\log y)^2}{y(\log y - 1)}$$

$$\frac{dy}{dx} = \frac{(\log y)^2}{(\log y - 1)}$$

Hence Proved.

45. Question

If $e^{x+y} - x = 0$, prove that $\frac{dy}{dx} = \frac{1-x}{x}$

Answer

Here,

$e^{x+y} - x = 0$

$e^{x+y} = x$ ...... (i)

Differentiating it with respect to x using chain rule,

$$\frac{d}{dx}(e^{x+y}) = \frac{d}{dx}(x)$$

$$e^{x+y}\frac{d}{dx}(x+y) = 1$$

$$x\left[1 + \frac{dy}{dx}\right] = 1 \text{ [Using (i)]}$$

$$1 + \frac{dy}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x} - 1$$

$$\frac{dy}{dx} = \frac{1-x}{x}$$

Hence Proved.

46. Question

If y = x sin(a + y), prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y) - y\cos(a+y)}$

Answer

Here

y = x sin(a+y)

Differentiating it with respect to x using the chain rule and product rule,

$$\frac{dy}{dx} = x\frac{d}{dx}\sin(a+y) + \sin(a+y)\frac{dx}{dx}$$

$$\frac{dy}{dx} = x\cos(a+y)\frac{dy}{dx} + \sin(a+y)$$

$$\left(1 - x\cos(a+y)\right)\frac{dy}{dx} = \sin(a+y)$$

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\left(1 - x\cos(a+y)\right)}$$

$\frac{dy}{dx} = \frac{\sin(a+y)}{\left(1-\frac{y}{\sin(a+y)}\cos(a+y)\right)}$ [Since, $\frac{y}{\sin(a+y)} = x$]

$$\frac{dy}{dx} = \frac{\sin(a+y)}{\left(\frac{\sin(a+y) - \cos(a+y).y}{\sin(a+y)}\right)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y) - y\cos(a+y)}$$

Hence Proved.

47. Question

If x sin (a + y) + sin a cos (a + y) = 0, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Answer

Here, x sin(a+y) + sin a cos(a+y) = 0

x=$\frac{-\sin a\cos(a+y)}{x\sin(a+y)}$

Differentiating it with respect to x using the chain rule and product rule,

$$\frac{d}{dx}[x\sin(a+y) + \sin a\cos(a+y)] = 0$$

$$x\frac{d}{dx}\sin(a+y) + \sin(a+y)\frac{dx}{dx} + \sin a\frac{d}{dx}\cos(a+y) + \cos(a+y)\frac{d}{dx}\sin a = 0$$

$$x\cos(a+y)\left(0+\frac{dy}{dx}\right) + \sin(a+y) = \sin a\left(-\sin(a+y)\frac{dy}{dx}\right) + 0 = 0$$

$$[x\cos(a+y) - \sin a\sin(a+y)]\frac{dy}{dx} + \sin(a+y) = 0$$

$$\frac{dy}{dx} = -\frac{\sin(a+y)}{x\cos(a+y) - \sin a\sin(a+y)}$$

$\frac{dy}{dx} = -\frac{\sin(a+y)}{\left(\frac{-\sin a\cos(a+y)}{\sin(a+y)}\right)\cos(a+y) - \sin a\sin(a+y)}$ [Since x=$\frac{-\sin a\cos(a+y)}{x\sin(a+y)}$]

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{(\sin a)\cos^2(a+y) + (\sin a)\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{(\sin a)[\cos^2(a+y) + \sin^2(a+y)]}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

[Since $\cos^2 a + \sin^2 a = 1$]

Hence Proved.

48. Question

If $(\sin x)^y = x + y$, prove that $\frac{dy}{dx} = \frac{1-(x+y)y \cot x}{(x+y)\log \sin x - 1}$

Answer

Here

$(\sin x)^y = x + y$

Taking log both sides,

$\log (\sin x)^y = \log(x + y)$

$y \log(\sin x) = \log(x+y)$ [Using $\log a^b = b \log a$]

Differentiating it with respect to x using the chain rule and product rule,

$$\frac{d}{dx}(y \log(\sin x)) = \frac{d}{dx}\log(x+y)$$

$$y\frac{d}{dx}\log(\sin x) + \log \sin x \frac{dy}{dx} = \frac{1}{(x+y)}\frac{d}{dx}(x+y)$$

$$\frac{y}{\sin x}\frac{d}{dx}(\sin x) + \log \sin x \frac{dy}{dx} = \frac{1}{(x+y)}\frac{d}{dx}(x+y)$$

$$\frac{y \cos x}{\sin x} + \log \sin x \frac{dy}{dx} = \frac{1}{(x+y)} + \frac{1}{(x+y)}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\log \sin x - \frac{1}{(x+y)}\right) = \frac{1}{(x+y)} - y \cot x$$

$$\frac{dy}{dx}\left(\frac{(x+y)\log \sin x - 1}{x+y}\right) = \frac{1 - y(x+y)\cot x}{(x+y)}$$

$$\frac{dy}{dx} = \frac{1 - y(x+y)\cot x}{(x+y)} \times \frac{(x+y)}{(x+y)\log \sin x - 1}$$

$$\frac{dy}{dx} = \frac{1 - y(x+y)\cot x}{(x+y)\log \sin x - 1}$$

Hence Proved.

49. Question

If $xy \log(x + y) = 1$, prove that $\frac{dy}{dx} = \frac{y\left(x^2 y + x + y\right)}{x\left(xy^2 + x + y\right)}$

Answer

Here,

$xy \log(x + y) = 1$ ......(i)

Differentiating it with respect to x using the chain rule and product rule,

$$\frac{dy}{dx}(xy \log(x + y)) = \frac{d}{dx}(1)$$

$$xy\frac{d}{dx}\log(x+y)+x\log(x+y)\frac{dy}{dx}+y\log(x+y)\frac{d}{dx}(x)=0$$

$$\left(\frac{xy}{(x+y)}\right)(1+)+x\log(x+y)\frac{dy}{dx}+y\log(x+y)=0$$

$\left(\frac{xy}{(x+y)}\right)\frac{dy}{dx}+\frac{xy}{(x+y)}+x\left(\frac{1}{xy}\right)\frac{dy}{dx}+y\left(\frac{1}{xy}\right)=0$ [Using (i)]

$$\frac{dy}{dx}\left[\frac{xy}{(x+y)}+\frac{1}{y}\right]=-\left[\frac{xy}{(x+y)}+\frac{1}{x}\right]$$

$$\frac{dy}{dx}\left[\frac{xy^2+x+y}{(x+y)y}\right]=-\left[\frac{x^2y+x+y}{(x+y)x}\right]$$

$$\frac{dy}{dx}=-\left[\frac{x^2y+x+y}{(x+y)x}\right]\times\left[\frac{(x+y)x}{xy^2+x+y}\right]$$

$$\frac{dy}{dx}=-\frac{y}{x}\left(\frac{x^2y+x+y}{xy^2+x+y}\right)$$

Hence Proved.

50. Question

If y = x sin y, prove that $\frac{dy}{dx}=\frac{y}{x(1-x\cos y)}$

Answer

Here,

y = x sin y

$\sin y=\frac{y}{x}$ ......(i)

Differentiating it with respect to x using product rule,

$$\frac{dy}{dx}=\frac{d}{dx}(x\sin y)$$

$$\frac{dy}{dx}=x\frac{d}{dx}(\sin y)+\sin y\frac{d}{dx}(x)$$

$$\frac{dy}{dx}=x\cos y\frac{dy}{dx}+\sin y(1)$$

$$\frac{dy}{dx}-x\cos y\frac{dy}{dx}=\sin y$$

$$\frac{dy}{dx}(1-x\cos y)=\sin y$$

$$\frac{dy}{dx}=\frac{\sin y}{1-x\cos y}$$

$\frac{dy}{dx}=\frac{y}{x(1-x\cos y)}$ [From (i)]

Hence Proved.

51. Question

Find the derivative of the function f(x) given by

f(x) = (1 + x)$(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find f’(1)

Answer

Here,

$f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

$f(1) = (2)(2)(2)(2) = 16$

Taking log on both sides we get,

$\text{Log}\,(f(x)) = \log(1 + x) + \log(1 + x^2) + \log(1 + x^4) + \log(1 + x^8)$

Differentiating it with respect to x we get,

$$\frac{1}{f(x)}\frac{d\big(f(x)\big)}{dx} = \frac{1}{x+1} + \frac{1}{1+x^2}\,2x + \frac{1}{1+x^4}\,4x^3 + \frac{1}{1+x^8}\,8x^7$$

$$f'(x) = f(x)\left[\frac{1}{x+1} + \frac{1}{1+x^2}\,2x + \frac{1}{1+x^4}\,4x^3 + \frac{1}{1+x^8}\,8x^7\right]$$

$$f'(1) = f(1)\left[\frac{1}{2} + \frac{2}{1+1} + \frac{4}{1+1} + \frac{8}{1+1}\right]$$

$$f'(1) = 16\left[7 + \frac{1}{2}\right]$$

$$f'(1) = 16 \times \frac{15}{2}$$

$F'(1) = 120$

52. Question

If $y = \log\dfrac{x^2+x+1}{x^2-x+1} + \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{\sqrt{3}\,x}{1-x^2}\right)$, find $\dfrac{dy}{dx}$.

Answer

Here, $y = \log\frac{x^2+x+1}{x^2-x+1} + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{1-x^2}\right)$

Differentiating it with respect to x using chain and quotient rule,

$$\frac{dy}{dx} = \frac{d}{dx}\log\frac{x^2+x+1}{x^2-x+1} + \frac{2}{\sqrt{3}}\frac{d}{dx}\tan^{-1}\left(\frac{\sqrt{3}x}{1-x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+x+1}{x^2-x+1}\right)}\frac{d}{dx}\left(\frac{x^2+x+1}{x^2-x+1}\right) + \frac{2}{\sqrt{3}}\left\{\frac{1}{1+\left(\frac{\sqrt{3}x}{1-x^2}\right)}\right\}\frac{d}{dx}\left(\frac{\sqrt{3}x}{1-x^2}\right)$$

$$\frac{dy}{dx} = \left(\frac{x^2-x+1}{x^2+x+1}\right)\left(\frac{(x^2-x+1)\frac{d}{dx}(x^2+x+1) - (x^2+x+1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}\right)$$
$$+ + \frac{2}{\sqrt{3}}\left\{\frac{(1-x)^2}{1+x^4-2x^2+3x^2}\right\}\left\{\frac{(1-x^2)^2\frac{d}{dx}(\sqrt{3}x) - \sqrt{3}x\frac{d}{dx}(1-x)^2}{(1-x^2)^2}\right\}$$

$$\frac{dy}{dx} = \left(\frac{1}{x^2+x+1}\right)\left(\frac{(x^2-x+1)\,(2x+1) - (x^2+x+1)(2x-1)}{x^2-x+1}\right)$$
$$+ \frac{2}{\sqrt{3}}\left(\frac{(1-x^2)^2}{1+x^2+x^4}\right)\left(\frac{(1-x^2)(\sqrt{3}) - \sqrt{3}x(-2x)}{(1-x^2)^2}\right)$$

$$\frac{dy}{dx}=\left(\frac{2x^3-2x^2+2x+x^2-x+1-2x^3-2x^2+2x+x^2+x+1}{x^2-x+1}\right)$$
$$+\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}-\sqrt{3}x^2+2\sqrt{3}x^2}{1+x^2+x^4}\right)$$

$$=\left(\frac{-2x^2+2}{x^4+x^2+1}\right)+\frac{2\sqrt{3}(x^2+1)}{\sqrt{3}(1+x^2+x^4)}$$

$$=\left(\frac{2(1-x^2)}{x^4+x^2+1}\right)+\frac{2(x^2+1)}{(1+x^2+x^4)}$$

$$=\frac{2(1-x^2+x^2+1)}{(1+x^2+x^4)}$$

Hence,

$$\frac{dy}{dx}=\frac{4}{(1+x^2+x^4)}$$

53. Question

If $y=\left(\sin x-\cos x\right)^{\sin x-\cos x}, \frac{\pi}{4}<x<\frac{3\pi}{4}$, find $\frac{dy}{dx}$.

Answer

Here, y = $(\sin x-\cos x)^{(\sin x-\cos x)}$ ......(i)

Taking log on both sides,

log y = log $(\sin x-\cos x)^{(\sin x-\cos x)}$

log y = $(\sin x-\cos x)$ log $(\sin x-\cos x)$

Differentiating it with respect to x using product rule, chain rule,

$$\frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\frac{d}{dx}(\sin x-\cos x)+(\sin x-\cos x)\frac{d}{dx}\log(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=\log(\sin x-\cos x)\times(\cos x+\sin x)+\frac{(\sin x-\cos x)}{(\sin x-\cos x)}\frac{d}{dx}(\sin x-\cos x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\cos x+\sin x)\log(\sin x-\cos x)+(\cos x+\sin x)$$

$$\frac{1}{y}\frac{dy}{dx}=(\cos x+\sin x)(1+\log(\sin x-\cos x))$$

$$\frac{dy}{dx}=y[(\cos x+\sin x)(1+\log(\sin x-\cos x))]$$

Using (i),

$$\frac{dy}{dx}=(\sin x-\cos x)^{(\sin x-\cos x)}[(\cos x+\sin x)(1+\log(\sin x-\cos x))]$$

54. Question

If xy = e^{x-y}, find $\frac{dy}{dx}$.

Answer

The given function is xy = e^{x-y}

Taking log on both sides, we obtain

log (xy)= log (e^{x-y})

log x + log y = (x-y) log e

log x + log y = (x-y)$\times$1

log x + log y = x-y

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(\log x)+\frac{d}{dx}(\log y)=\frac{d}{dx}(x)-\frac{dy}{dx}$$

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\left(1+\frac{1}{y}\right)\frac{dy}{dx}=1-\frac{1}{x}$$

$$\left(\frac{y+1}{y}\right)\frac{dy}{dx}=\frac{x-1}{x}$$

$$\therefore\ \frac{dy}{dx}=\frac{y(x-1)}{x(y+1)}$$

55. Question

If $y^x + x^y + x^x = a^b$, find $\frac{dy}{dx}$.

Answer

Given that, $y^x + x^y + x^x = a^b$

Putting, $u=y^x$, $v=x^y$, $w=x^x$,we get

$u+v+w=a^b$

Therefore, $\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}=0$ ……(i)

Now, $u=y^x$,

Taking log on both sides, we have

log u = x log y

Differentiating both sides with respect to x, we have

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}(\log y)+\log y\frac{dx}{dx}$$

$$=x\frac{1}{y}.\frac{dy}{dx}+\log y.1$$

So, $\frac{du}{dx}=u\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)$

$=y^x\left[\frac{x}{y}\frac{dy}{dx}+\log y\right]$ ……(ii)

Also, $v=x^y$,

Taking log on both sides, we have

log v = y log x

Differentiating both sides with respect to x, we have

$$\frac{1}{v}\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}$$

$$= y\frac{1}{x} + \log x\frac{dy}{dx}$$

So, $\frac{dv}{dx} = v\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$

$= x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$ ......(iii)

Again, w=x^x,

Taking log on both sides, we have

log w = x log x

Differentiating both sides with respect to x, we have

$$\frac{1}{w}\frac{dw}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{dx}{dx}$$

$$= x.\frac{1}{x} + \log x.1$$

So, $\frac{dw}{dx} = w(1 + \log x)$

$= x^x[1 + \log x]$ ......(iv)

From (i), (ii), (iii), (iv)

$$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$$

$$y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] + x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] + x^x[1 + \log x] = 0$$

$$(xy^{x-1} + x^y.\log x)\frac{dy}{dx} = -x^x(1 + \log x) - y.x^{y-1} - y^x\log y$$

Therefore,

$$\frac{dy}{dx} = \frac{-[x^x(1 + \log x) + y.x^{y-1} + y^x\log y]}{(xy^{x-1} + x^y.\log x)}$$

56. Question

If (cos x)^y = (cos y)^x find $\frac{dy}{dx}$.

Answer

Here, $(\cos x)^y = (\cos y)^x$

Taking log on both sides,

$$\log(\cos x)^y = \log(\cos y)^x$$

$$y\log(\cos x) = x\log(\cos y)$$

Differentiating it with respect to x using the chain rule and product rule,

$$\frac{d}{dx}(y\log\cos x) = \frac{d}{dx}(x\log\cos y)$$

$$y\frac{d}{dx}\log\cos x + \log\cos x\frac{dy}{dx} = x\frac{d}{dx}\log\cos y + \log\cos y\frac{dx}{dx}$$

$$y\frac{1}{\cos x}(-\sin x)+\log\cos x\frac{dy}{dx}=x\frac{1}{\cos y}(-\sin y)\frac{dy}{dx}+\log\cos y$$

$$\left(\log\cos x+\frac{x\sin y}{\cos y}\right)\frac{dy}{dx}=\log\cos y+y\frac{\sin y}{\cos y}$$

$$(\log\cos x+x\tan y)\frac{dy}{dx}=\log\cos y+y\tan y$$

$$\frac{dy}{dx}=\frac{\log\cos y+y\tan y}{\log\cos x+x\tan y}$$

57. Question

If cos y = x cos (a + y), where $\cos a \neq \pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$.

Answer

Here,

cos y = x cos (a + y), where cos a$\neq \pm 1$

Differentiating both sides with respect to x, we get

$$-\sin y\frac{dy}{dx}=x\left(-\sin(a+y)\frac{dy}{dx}\right)+\cos(a+y)$$

$$\frac{dy}{dx}[x\sin(a+y)-\sin y]=\cos(a+y)$$

$$\frac{dy}{dx}=\frac{\cos(a+y)}{x\sin(a+y)-\sin y}$$

Multiplying the numerator and the denominator by cos(a+y) on th RHS we have,

$$\frac{dy}{dx}=\frac{\cos^2(a+y)}{x\cos(a+y)\sin(a+y)-\cos(a+y)\sin y}$$

$\frac{dy}{dx}=\frac{\cos^2(a+y)}{\cos y\sin(a+y)-\cos(a+y)\sin y}$ [Given cos y = x cos (a + y)]

$\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin[(a+y)-y]}$ [$\because$ sin(a-b)=sina cosb - cosa sinb]

$$\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

Hence Proved.

58. Question

If $(x-y)e^{\frac{x}{x-y}}=a$, prove that: $\frac{dy}{dx}=\frac{2y-3x}{2x-1}$

Answer

Given:

$$(x-y)e^{\frac{x}{x-y}}=a$$

Taking log on both sides we get,

log (x – y) + $\frac{x}{x-y}\log(e)=\log a$

(Using log a^b = b log a and log (e) = 1)

Differentiating both sides we get,

$$\frac{1}{x-y}\left[1-\frac{dy}{dx}\right]+\frac{(x-y)\frac{d}{dx}(x)+x\left(1-\frac{dy}{dx}\right)}{(x-y)^2}=0$$

Taking L.C.M and solving the equation we get,

$$(x-y)\left[1-\frac{dy}{dx}\right]+(x-y)+x-x\frac{dy}{dx}=0$$

$$x-y-x\frac{dy}{dx}+y\frac{dy}{dx}+x-y+x-x\frac{dy}{dx}=0$$

$$3x-2y-(2x-1)\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{2y-3x}{2x-1}$$

59. Question

If $x=e^{x/y}$, prove that $\frac{dy}{dx}=\frac{x-y}{x\log x}$

Answer

$x = e^{x/y}$

Taking logon both sides,

Log x = log $e^{x/y}$

log x = $\frac{x}{y}$ ......(i) [Since log e^a = a]

or, y = $\frac{x}{\log x}$ ......(ii)

Differentiating the given equation with respect to x,

$$\frac{dy}{dx}=\frac{\log x\frac{d}{dx}(x)-x\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$\frac{dy}{dx}=\frac{\log x-x\times\frac{1}{x}}{(\log x)^2}$$

$$\frac{dy}{dx}=\frac{\log x-1}{(\log x)^2}$$

$\frac{dy}{dx}=\frac{\frac{x}{y}-1}{(\log x)^2}$ [From (i)]

$$\frac{dy}{dx}=\frac{x-y}{y(\log x)^2}$$

$\frac{dy}{dx}=\frac{x-y}{\frac{x}{\log x}(\log x)^2}$ [From (ii)]

Therefore, $\frac{dy}{dx}=\frac{x-y}{x\log x}$

60. Question

If $y=x^{\tan x}+\sqrt{\frac{x^2+1}{2}}$, find $\frac{dy}{dx}$

Answer

Given $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$

$y = e^{\tan x \log x} + e^{\frac{1}{2}\log\frac{x^2+1}{2}}$

$$\frac{dy}{dx} = e^{\tan x \log x}\frac{d}{dx}(\tan x \log x) + e^{\frac{1}{2}\log\frac{x^2+1}{2}}\frac{d}{dx}\left(\frac{1}{2}\log\frac{x^2+1}{2}\right)$$

$$\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x \log x\right] + \sqrt{\frac{x^2+1}{2}}\left(\frac{1}{2} \times \frac{2}{x^2+1} \times x\right)$$

$$\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x \log x\right] + \sqrt{\frac{x^2+1}{2}}\left(\frac{x}{x^2+1}\right)$$

$$\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x \log x\right] + \frac{x}{\sqrt{2(x^2+1)}}$$

61. Question

If $y = 1 + \frac{\alpha}{\left(\frac{1}{x} - \alpha\right)} + \frac{\beta / x}{\left(\frac{1}{x} - \alpha\right)\left(\frac{1}{x} - \beta\right)} + \frac{\gamma / x^2}{\left(\frac{1}{x} - \alpha\right)\left(\frac{1}{x} - \beta\right)\left(\frac{1}{x} - \gamma\right)}$, find $\frac{dy}{dx}$

Answer

Given,

$$y = 1 + \frac{\alpha}{\left(\frac{1}{x} - \alpha\right)} + \frac{\beta/x}{\left(\frac{1}{x} - \alpha\right)\left(\frac{1}{x} - \beta\right)} + \frac{\gamma/x^2}{\left(\frac{1}{x} - \alpha\right)\left(\frac{1}{x} - \beta\right)\left(\frac{1}{x} - \gamma\right)}$$

Using the theorem,

$$\text{If } y = 1 + \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{(x-c)}, \text{ then, } \frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right\}$$

Here we have $\frac{1}{x}$ instead of x.

Hence, using the above theorem, we get,

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{\alpha}{\frac{1}{x} - \alpha} + \frac{\beta}{\frac{1}{x} - \beta} + \frac{\gamma}{\frac{1}{x} - \gamma}\right]$$

Exercise 11.6

1. Question

If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{to} \infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{2y - 1}$.

Answer

Here,

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \cdots \text{to } \infty}}}$$

y = $\sqrt{x+y}$

On squaring both sides,

y^2 = x + y

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-1) = 1$$

$$\frac{dy}{dx} = \frac{1}{2y-1}$$

Hence proved.

2. Question

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \text{ to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sin x}{1-2y}$.

Answer

Here,

y = $\sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \cdots \text{ to } \infty}}}$

y = $\sqrt{\cos x + y}$

On squaring both sides,

y^2 = cos x + y

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-1) = -\sin x$$

$$\frac{dy}{dx} = -\frac{\sin x}{2y-1}$$

$$\frac{dy}{dx} = \frac{\sin x}{1-2y}$$

Hence proved.

3. Question

If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \text{ to } \infty}}}$, prove that $(2y-1)\frac{dy}{dx} = \frac{1}{x}$.

Answer

y = $\sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \cdots \text{ to } \infty}}}$

y = $\sqrt{\log x + y}$

On squaring both sides,

$y^2 = \log x + y$

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x(2y - 1)}$$

Hence proved.

4. Question

If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \text{ to } \infty}}}$, prove that $\frac{dy}{dx} = \frac{\sec^2 x}{2y - 1}$.

Answer

$y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots \text{ to } \infty}}}$

$y = \sqrt{\tan x + y}$

On squaring both sides,

$y^2 = \tan x + y$

Differentiating both sides with respect to x,

$$2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\frac{dy}{dx}(2y - 1) = \sec^2 x$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{(2y - 1)}$$

Hence proved.

5. Question

If $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdots\infty}}}$, prove that $\frac{dy}{dx} = \frac{y^2 \cot x}{(1 - y \log \sin x)}$

Answer

Here,

$y = (\sin x)^{(\sin x)^{(\sin x)^{\cdots\infty}}}$

$y = (\sin x)^y$

By taking log on both sides ,

$\log y = \log(\sin x)^y$

$\log y = y(\log \sin x)$

Differentiating both sides with respect to x by using product rule,

$$\frac{1}{y}\frac{dy}{dx} = y\frac{d(\log\sin x)}{dx} + \log\sin x\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{\sin x}\frac{d(\sin x)}{dx} + \log\sin x\frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log\sin x\right)\frac{dy}{dx} = \frac{y}{\sin x}(\cos x)$$

$$\left(\frac{1 - y\log\sin x}{y}\right)\frac{dy}{dx} = y\cot x$$

$$\frac{dy}{dx} = \frac{y^2\cot x}{1 - y\log\sin x}$$

Hence proved.

6. Question

If $y = (\tan x)^{(\tan x)^{(\tan x)^{\cdots\infty}}}$, prove that $\frac{dy}{dx} = 2$ at $x = \frac{\pi}{4}$.

Answer

Here,

$y = (\tan x)^{(\tan x)^{(\tan x)^{\cdots\infty}}}$

$y = (\tan x)^y$

By taking log on both sides,

log y = log$(\tan x)^y$

log y = y(log tan x)

Differentiating both sides with respect to x using the product rule and chain rule,

$$\frac{1}{y}\frac{dy}{dx} = y\frac{d(\log\tan x)}{dx} + \log\tan x\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{\tan x}\frac{d(\tan x)}{dx} + \log\tan x\frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log\tan x\right)\frac{dy}{dx} = \frac{y}{\tan x}(\sec^2 x)$$

$$\left(\frac{1 - y\log\tan x}{y}\right)\frac{dy}{dx} = \frac{y\sec^2 x}{\tan x}$$

$$\frac{dy}{dx} = \frac{y^2\sec^2 x}{\tan x\,(1 - y\log\tan x)}$$

$$\frac{dy}{dx}_{\left(x=\frac{\pi}{4}\right)} = \frac{y^2\sec^2\left(\frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4}\right)\left(1 - y\log\tan\left(\frac{\pi}{4}\right)\right)}$$

$$\frac{dy}{dx}_{\left(x=\frac{\pi}{4}\right)} = \frac{2y^2}{1(1 - y\log 1)}$$

Since $\{(y)_{\frac{\pi}{4}} = \left(\tan\frac{\pi}{4}\right)^{\left(\tan\frac{\pi}{4}\right)^{\left(\tan\frac{\pi}{4}\right)^{\cdots\infty}}} = (1)^\infty = 1\}$

$$\frac{dy}{dx}_{\left(x=\frac{\pi}{4}\right)} = \frac{2}{1(1-0)}$$

$$\frac{dy}{dx}_{\left(x=\frac{\pi}{4}\right)} = 2$$

Hence proved.

7. Question

If $y = e^{x^{e^x}} + x^{e^{e^x}} + e^{x^{x^e}}$, prove that

$$\frac{dy}{dx} = e^{x^{e^x}} .x^{e^x}\left\{\frac{e^x}{x} + e^x .\log x\right\} + x^{e^{e^x}} .e^{e^x}$$

$$\left\{\frac{1}{x} + e^x .\log x\right\} + e^{x^{x^e}} x^{x^e} .x^{e-1}\{1 + e \log x\}$$

Answer

Here,

y $= e^{x^{e^x}} + x^{e^{e^x}} + e^{x^{x^e}}$

y = U + V + W

$\frac{dy}{dx} = \frac{dU}{dx} + \frac{dV}{dx} + \frac{dZ}{dx}$ ……(1)

Where, u $= e^{x^{e^x}}$,v $= x^{e^{e^x}}$,w $= e^{x^{x^e}}$

u $= e^{x^{e^x}}$

Taking log on both sides,

log u = log $e^{x^{e^x}}$

log u $= x^{e^x} \log e$

log u $= x^{e^x}$

Again, Taking log on both sides,

log log u = log x^{e^x}

loglog u $= e^x \log x$

Differentiating both sides with respect to x by using the product rule,

$$\frac{1}{\log u}\frac{d(\log u)}{dx} = e^x\frac{d(\log x)}{dx} + \log x\frac{d(e^x)}{dx}$$

$$\frac{1}{u}\frac{1}{\log u}\frac{du}{dx} = e^x\frac{1}{x} + e^x \log x$$

$$\frac{du}{dx} = u * \log u\left(\frac{e^x}{x} + e^x \log x\right)$$

Put value of u and log u,

$\frac{du}{dx} = e^{x^{e^x}} * x^{e^x}\left(\frac{e^x}{x} + e^x \log x\right)$ ……(A)

Now,

v $= x^{e^{e^x}}$

taking log on both sides,

log v = log $x^{e^{e^x}}$

log v $= e^{e^x} \log x$

Differentiating both sides with respect to x by using the product rule,

$$\frac{1}{v}\frac{dv}{dx} = e^{e^x}\frac{d(\log x)}{dx} + \log x\frac{d\left(e^{e^x}\right)}{dx}$$

$$\frac{1}{v}\frac{dv}{dx} = e^{e^x}\frac{1}{x} + \log x\, e^{e^x}\frac{d(e^x)}{dx}$$

$$\frac{dv}{dx} = v\left[e^{e^x}\frac{1}{x} + e^x \log x\, e^{e^x}\right]$$

Put value of v,

$\frac{dv}{dx} = x^{e^{e^x}}\left[e^{e^x}\frac{1}{x} + e^x \log x\, e^{e^x}\right]$ ......(B)

Now ,

w = $e^{x^{x^e}}$

taking log on both sides,

log w = log $e^{x^{x^e}}$

log w $= x^{x^e} \log e$

log w $= x^{x^e}$

taking log both sides,

log log w $= x^e \log x$

Differentiating both sides with respect to x by using the product rule,

$$\frac{1}{\log w}\frac{d(\log w)}{dx} = x^e\frac{d(\log x)}{dx} + \log x\frac{d(x^e)}{dx}$$

$$\frac{1}{w}\frac{1}{\log w}\frac{dw}{dx} = x^e\frac{1}{x} + x^{e-1}\log e$$

$$\frac{dw}{dx} = w * \log w\,(x^{e-1} + e\log x\ x^{e-1})$$

Put the value of w and log w,

$$\frac{dw}{dx} = e^{x^{x^e}} * x^{x^e}(x^{e-1} + e\log x\ x^{e-1})\ \ldots\ldots (C)$$

Using equation A, B and C in equation (1),

$$\frac{dy}{dx} = e^{x^{e^x}} * x^{e^x}\left(\frac{e^x}{x} + e^x \log x\right) + x^{e^{e^x}}\left[e^{e^x}\frac{1}{x} + e^x \log x\, e^{e^x}\right] + e^{x^{x^e}} * x^{x^e}(x^{e-1} + e\log x\ x^{e-1})$$

Hence, proved.

8. Question

If $y\left(\cos x\right)^{\left(\cos x\right)^{\left(\cos x\right)\ldots\infty}}$, prove that $\frac{dy}{dx}=\frac{y^2\tan x}{\left(1-y\log\cos x\right)}$.

Answer

Here,

$y = (\cos x)^{(\cos x)^{(\cos x)\cdots\infty}}$

$y = (\cos x)^y$

By taking log on both sides,

$\log y = \log(\cos x)^y$

$\log y = y(\log\cos x)$

Differentiating both sides with respect to x by using the product rule,

$$\frac{1}{y}\frac{dy}{dx} = y\frac{d(\log\cos x)}{dx} + \log\cos x\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{\cos x}\frac{d(\cos x)}{dx} + \log\cos x\frac{dy}{dx}$$

$$\left(\frac{1}{y} - \log\cos x\right)\frac{dy}{dx} = \frac{y}{\cos x}(-\sin x)$$

$$\left(\frac{1 - y\log\cos x}{y}\right)\frac{dy}{dx} = -y\tan x$$

$$\frac{dy}{dx} = -\frac{y^2\cot x}{1 - y\log\cos x}$$

Exercise 11.7

1. Question

Find $\frac{dy}{dx}$, when

$x = at^2$ and $y = 2at$

Answer

Given that $x = at^2$, $y = 2at$

So, $\frac{dx}{dt} = \frac{d(at^2)}{dt} = 2at$

$$\frac{dy}{dt} = \frac{d(2at)}{dt} = 2a$$

Therefore, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

2. Question

Find $\frac{dy}{dx}$, when

$x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

Answer

$x = a(\theta + \sin\theta)$

Differentiating it with respect to θ,

$\frac{dx}{d\theta} = a(1 + \cos\theta)$ ......(1)

And ,

y = a(1- cosθ)

Differentiating it with respect to θ ,

$$\frac{dy}{d\theta} = a(0 + \sin\theta)$$

$\frac{dy}{d\theta} = a\sin\theta$ ......(2)

Using equation (1) and (2) ,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{a\sin\theta}{a(1 - \cos\theta)}$$

$$= \frac{\frac{2\sin\theta}{2}\frac{(\cos\theta)}{2}}{\frac{2\sin^2\theta}{2}},$$

{since, $1 - \cos\theta = \frac{2\sin^2\theta}{2}$}

$$= \frac{dy}{dx} = \frac{\tan\theta}{2}$$

3. Question

Find $\frac{dy}{dx}$, when

x = acosθ and y = bsinθ

Answer

as x = acosθ and y = bsinθ

Then,

$$\frac{dx}{d\theta} = \frac{d(a\cos\theta)}{d\theta} = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d(b\sin\theta)}{d\theta} = b\cos\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

4. Question

Find $\frac{dy}{dx}$, when

$x = ae^{\theta}$ (sin θ – cos θ),$y = ae^{\theta}$(sinθ + cosθ)

Answer

as $x = ae^{\theta}$ (sin θ – cos θ)

Differentiating it with respect to θ

$$\frac{dx}{d\theta} = a[e^{\theta}\frac{d(\sin\theta - \cos\theta)}{d\theta} + (\sin\theta - \cos\theta)\frac{d\left(e^{\theta}\right)}{d\theta}]$$

$= a[e^{\theta}(\cos\theta + \sin\theta) + (\sin\theta - \cos\theta)e^{\theta}]$

$\frac{dx}{d\theta} = a[2e^{\theta}\sin\theta]$ ......(1)

And , $y = ae^{\theta}(\sin\theta + \cos\theta)$

Differentiating it with respect to θ,

$$\frac{dy}{d\theta} = a[e^{\theta}\frac{d(\sin\theta + \cos\theta)}{d\theta} + (\sin\theta + \cos\theta)\frac{d\left(e^{\theta}\right)}{d\theta}]$$

$= a[e^{\theta}(\cos\theta - \sin\theta) + (\sin\theta + \cos\theta)\, e^{\theta}]$

$\frac{dy}{d\theta} = a[2e^{\theta}\cos\theta]$ ......(2)

Dividing equation (2) by equation (1),

$$\frac{dy}{dx} = \frac{a\left(2e^{\theta}\cos\theta\right)}{a(2e^{\theta}\sin\theta)}$$

$$\frac{dy}{dx} = \cot\theta$$

5. Question

Find $\frac{dy}{dx}$, when

$x = b\sin^2\theta$ and $y = a\cos^2\theta$

Answer

as $x = b\sin^2\theta$

Then

$$\frac{dx}{d\theta} = \frac{d(b\sin^2\theta)}{d\theta} = 2b\sin\theta\cos\theta$$

And $y = a\cos^2\theta$

$$\frac{dy}{d\theta} = d(a\cos^2\theta) = -2a\cos\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{2a\cos\theta\sin\theta}{2b\sin\theta\cos\theta} = -\frac{a}{b}$$

6. Question

Find $\frac{dy}{dx}$, when

$x = a(1 - \cos\theta)$ and $y = a(\theta + \sin\theta)$ at $\theta = \frac{\pi}{2}$

Answer

as x = a(1 – cos θ)

$$\frac{dx}{d\theta} = \frac{d[a(1-\cos\theta)]}{d\theta} = a(\sin\theta)$$

And y = a(θ + sin θ)

$$\frac{dy}{d\theta} = \frac{d(\theta+\sin\theta)}{d\theta} = a(1+\cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(1+\cos\theta)}{a(\sin\theta)}\Big|\left(\theta = \frac{\pi}{2}\right)$$

$$= \frac{a(1+0)}{a} = 1$$

7. Question

Find $\frac{dy}{dx}$, when

$x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$

Answer

as $x = \frac{e^\theta + e^\theta}{2}$

Differentiating it with respect to t

$$\frac{dx}{dt} = \frac{1}{2}\left[\frac{d(e^t)}{dt} + \frac{d(e^{-t})}{dt}\right]$$

$$= \frac{1}{2}\left[e^t + e^{-t}\frac{d(-t)}{dt}\right]$$

$\frac{dx}{dt} = \frac{1}{2}(e^t - e^{-t}) = y$ ……(1)

And $y = \frac{e^\theta - e^\theta}{2}$

Differentiating it with respect to t,

$$\frac{dy}{dt} = \frac{1}{2}\left[\frac{d(e^t)}{dt} - \frac{d(e^{-t})}{dt}\right]$$

$$= \frac{1}{2}\left[e^{t-} - e^{-t}\,\frac{d(-t)}{dt}\right]$$

$$= \frac{1}{2}\left(e^t - e^{-t}(-1)\right)$$

$\frac{dy}{dt} = \frac{e^\theta + e^\theta}{2} = x$ ……(2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

8. Question

Find $\frac{dy}{dx}$, when

$x=\frac{3\,at}{1+t^2}$ and $y=\frac{3\,at^2}{1+t^2}$

Answer

as $x=\frac{3at}{1+t^2}$

Differentiating it with respect to t using quotient rule,

$$\frac{dx}{dt}=\left[\frac{\left((1+t^2)\frac{d(3at)}{dt}-3at\frac{d(1+t^2)}{dt}\right)}{(1+t^2)^2}\right]$$

$$=\left[\frac{(1+t^2)(3a)-3at(2t)}{(1+t^2)^2}\right]$$

$$=\left[\frac{(3a)+3at^2-6at^2}{(1+t^2)^2}\right]$$

$$=\left[\frac{3a-3at^2}{(1+t^2)^2}\right]$$

$\frac{dx}{dt}=\frac{3a(1-t^2)}{(1+t^2)^2}$ ......(1)

And $y=\frac{3at^2}{1+t^2}$

Differentiating it with respect to t using quotion rule

$$\frac{dy}{dx}=\left[\frac{(1+t^2)\frac{d(3at^2)}{dt}-3at^2\frac{d(1+t^2)}{dt}}{(1+t^2)^2}\right]$$

$$\frac{dy}{dt}=\left[\frac{(1+t^2)(6at)-(3at^2)(2t)}{(1+t^2)^2}\right]$$

$$=\left[\frac{6at+6at^3-6at^3}{(1+t^2)^2}\right]$$

$\frac{dy}{dt}=\frac{6at}{(1+t^2)^2}$ ----(2)

Dividing equation (2) by (1),

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{6at}{(1+t^2)^2}\times\frac{3a(1-t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dx}=\frac{2t}{1-t^2}$$

9. Question

Find $\frac{dy}{dx}$, when

x = a(cosθ + θ sinθ) and y = a(sinθ – cosθ)

Answer

the given equation are x = a(cosθ + θ sinθ)

Then $\frac{dx}{d\theta} = a\left[\frac{d}{d\theta}\cos\theta + \frac{d}{d\theta}(\theta\sin\theta)\right]$

$= a\left[-\sin\theta + \frac{\theta d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta)\right]$

$= a[-\sin\theta + \theta\cos\theta + \sin\theta] = a\theta\cos\theta$

And y = a(sinθ – cosθ) so,

$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(\sin\theta) - \frac{d}{d\theta}(\theta\cos\theta)\right]$

$= a\left[\cos\theta - \left\{\frac{\theta d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(\theta)\right\}\right]$

$= a[\cos\theta + \theta\sin\theta - \cos\theta]$

$= a\theta\sin\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

10. Question

Find $\frac{dy}{dx}$, when

$x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$.

Answer

as $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$

Differentiating it with respect to θ using the product rule,

$\frac{dx}{d\theta} = e^{\theta}\frac{d}{d\theta}\left(\theta + \frac{1}{\theta}\right) + \left(\theta + \frac{1}{\theta}\right)\frac{d}{d\theta}(e^{\theta})$

$= e^{\theta}\left(1 - \frac{1}{\theta^2}\right) + \frac{\theta^2 + 1}{\theta}(e^{\theta})$

$= e^{\theta}\left(1 - \frac{1}{\theta^2} + \frac{\theta^2 + 1}{\theta}\right)$

$= e^{\theta}\left(\frac{\theta^2 - 1 + \theta^3 + \theta}{\theta^2}\right)$

$\frac{dx}{d\theta} = e^{\theta}\left(\frac{\theta^3 + \theta^2 + \theta - 1}{\theta^2}\right)$ ......(1)

And, $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$

Differentiating it with respect to θ using the product rule,

$\frac{dy}{d\theta} = e^{-\theta}\frac{d}{d\theta}\left(\theta - \frac{1}{\theta}\right) + \left(\theta - \frac{1}{\theta}\right)\frac{d}{d\theta}(e^{-\theta})$

$$= e^{-\theta}\left(1+\frac{1}{\theta^2}\right)+\left(\theta-\frac{1}{\theta}\right)e^{-\theta}\frac{d}{d\theta}(-\theta)$$

$$= e^{-\theta}\left(1+\frac{1}{\theta^2}\right)+\left(\theta-\frac{1}{\theta}\right)e^{-\theta}(-1)$$

$$\frac{dy}{d\theta}= e^{-\theta}\left(1+\frac{1}{\theta^2}-\theta+\frac{1}{\theta}\right)$$

$$= e^{-\theta}\left(\frac{\theta^2+1-\theta^3+\theta}{\theta^2}\right)$$

$\frac{dy}{d\theta}= e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)$ ......(2)

divide equation (2)by (1)

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}= e^{-\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^2}\right)\times\frac{1}{e^{\theta}\left(\frac{\theta^3+\theta^2+\theta-1}{\theta^2}\right)}$$

$$= e^{-2\theta}\left(\frac{-\theta^3+\theta^2+\theta+1}{\theta^3+\theta^2+\theta-1}\right)$$

11. Question

Find $\frac{dy}{dx}$, when

$x=\frac{2t}{1+t^2}$ and $y=\frac{1-t^2}{1+t^2}$.

Answer

as, $x=\frac{2t}{1+t^2}$

Differentiating it with respect to t using quotient rule,

$$\frac{dx}{dt}=\left[\frac{(1+t^2)\frac{d}{dt}(2t)-2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$=\left[\frac{(1+t^2)(2)-2t(2t)}{(1+t^2)^2}\right]$$

$$=\left[\frac{2+2t^2-4t^2}{(1+t^2)^2}\right]$$

$$=\left[\frac{2-2t^2}{(1+t^2)^2}\right]$$

$\frac{dx}{dt}=\left[\frac{2-2t^2}{(1+t^2)^2}\right]$ ......(1)

And,$y=\frac{1-t^2}{1+t^2}$

Differentiating it with respect to t using quotient rule,

$$\frac{dy}{dt}=\left[\frac{(1+t^2)\frac{d}{dt}(1-t^2)-(1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(-2t)-(1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}\right]$$

$\frac{dy}{dt} = \left[\frac{-4t}{(1+t^2)^2}\right]$ ----(2)

dividing equation (2)by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left[\frac{-4t}{(1+t^2)^2}\right] \times \frac{1}{\left[\frac{2-2t^2}{(1+t^2)^2}\right]}$$

$$= -\frac{2t}{1-t^2}$$

$$\frac{dy}{dx} = -\frac{x}{y}\ \left[\text{since}, \frac{x}{y} = \frac{2t}{1+t^2} \times \frac{1+t^2}{1-t^2} = \frac{2t}{1-t^2}\ \right]$$

12. Question

Find $\frac{dy}{dx}$, when

$x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}, t \in R$

Answer

as $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$

Differentiating it with respect to t using chain rule ,

$$\frac{dx}{dt} = -\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+t^2}}\right)^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1}{1+t^2}}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}\frac{d}{dt}(1+t^2)$$

$$= -\frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}(2t)$$

$$= -\frac{t}{\sqrt{t^2} \times (1+t^2)}$$

$\frac{dx}{dt} = -\frac{1}{1+t^2}$ ......(1)

Now , $y = \sin^{-1}\frac{1}{\sqrt{1+t^2}}$

Differentiating it with respect to t using chain rule ,

$$\frac{dy}{dt} = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1+t^2}}\right)^2}}\frac{d}{dt}\left(\frac{1}{\sqrt{1+t^2}}\right)$$

$$= \frac{1}{\sqrt{1 - \frac{1}{1+t^2}}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}\frac{d}{dt}(1+t^2)$$

$$= \frac{(1+t^2)^{\frac{1}{2}}}{\sqrt{(1+t^2-1)}}\left\{-\frac{1}{2(1+t^2)^{\frac{3}{2}}}\right\}(2t)$$

$$= \frac{t}{\sqrt{t^2} \times (1+t^2)}$$

$\frac{dy}{dt} = -\frac{1}{1+t^2}$ ......(2)

dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{1}{1+t^2} \times -\frac{1+t^2}{1}$$

$$\frac{dy}{dx} = 1$$

13. Question

Find $\frac{dy}{dx}$, when

$x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

Answer

as $x = \frac{1-t^2}{1+t^2}$

Differentiating it with respect to t using quotient rule,

$$\frac{dx}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{-2t - 2t^3 - 2t + 2t^3}{(1+t^2)^2}\right]$$

$\frac{dx}{dt} = \left[\frac{-4t}{(1+t^2)^2}\right]$ ......(1)

And, $y = \frac{2t}{1+t^2}$

Differentiating it with respect to t using quotient rule,

$$\frac{dy}{dt} = \left[\frac{(1+t^2)\frac{d}{dt}(2t) - (2t)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$= \left[\frac{(1+t^2)(2) - (2t)(2t)}{(1+t^2)^2}\right]$$

$$= \left[\frac{2 + 2t^2 - 4t^2}{(1+t^2)^2}\right]$$

$\frac{dy}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$ ......(2)

divided equation (2)by (1) so,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2(1-t^2)}{(1+t^2)^2} \times \frac{1}{\frac{-4t}{(1+t^2)^2}}$$

$$\frac{dy}{dx} = \frac{2(1-t^2)}{-4t}$$

14. Question

Find $\frac{dy}{dx}$, when

If x = 2cos θ – cos 2θ and y = 2sin θ – sin 2θ, prove that $\frac{dy}{dx} = \tan\left(\frac{3\,\theta}{2}\right)$.

Answer

as x = 2cos θ – cos 2θ

Differentiating it with respect to θ using chain rule ,

$$\frac{dx}{d\theta} = 2(-\sin\theta) - (-\sin 2\theta)\frac{d}{d\theta}(2\theta)$$

$$= -2\sin\theta + 2\sin 2\theta$$

$\frac{dx}{d\theta} = 2(\sin 2\theta - \sin\theta)$ ......(1)

And, y = 2sin θ – sin 2θ

Differentiating it with respect to θ using chain rule ,

$$\frac{dy}{d\theta} = 2\cos\theta - \cos 2\theta\frac{d}{d\theta}(2\theta)$$

$$= 2\cos\theta - \cos 2\theta(2)$$

$$= 2\cos\theta - 2\cos 2\theta$$

$\frac{dy}{d\theta} = 2(\cos\theta - \cos 2\theta)$ ......(2)

dividing equation (2)by equation (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2(\cos\theta - \cos 2\theta)}{2(\sin 2\theta - \sin\theta)}$$

$$= \frac{(\cos\theta - \cos 2\theta)}{(\sin 2\theta - \sin\theta)}$$

$$\frac{dy}{dx} = \frac{-2\sin\left(\frac{\theta+2\theta}{2}\right)\sin\left(\frac{\theta-2\theta}{2}\right)}{2\cos\left(\frac{\theta+2\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)} \left[\text{since } \sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right]$$

$$\left[\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right]$$

$$= -\frac{\sin\left(\frac{3\theta}{2}\right)\left(\sin\left(-\frac{\theta}{2}\right)\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\sin\left(\frac{3\theta}{2}\right)\left(-\sin\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)}$$

$$\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$$

15. Question

Find $\frac{dy}{dx}$, when

If x = $e^{\cos 2t}$ and y = $e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$

Answer

Here , $x = e^{\cos 2t}$

Differentiating it with respect to θ using chain rule ,

$$\frac{dx}{dt} = \frac{d}{dt}\left(e^{\cos 2t}\right)$$

$$= e^{\cos 2t}\frac{d}{dt}(\cos 2t)$$

$$= e^{\cos 2t}(-\sin 2t)\frac{d}{dt}(2t)$$

$$= e^{\cos 2t}(-\sin 2t)(2)$$

$\frac{dx}{dt} = -2\sin 2t e^{\cos 2t}$ ......(1)

And, $y = e^{\sin 2t}$

Differentiating it with respect to θ using chain rule ,

$$\frac{dy}{dt} = \frac{d}{dt}\left(e^{\sin 2t}\right)$$

$$= e^{\sin 2t}\frac{d}{dt}(\sin 2t)$$

$$= e^{\sin 2t}\cos 2t\frac{d}{dt}(2t)$$

$$= e^{\sin 2t}\cos 2t(2)$$

$\frac{dy}{dt} = 2\cos 2t e^{\sin 2t}$ ......(2)

dividing equation (2)by (1),

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos 2t e^{\sin 2t}}{-2\sin 2t e^{\cos 2t}}$

$\frac{dy}{dx} = -\frac{y\log x}{x\log y}$ [since $x = e^{\cos 2t} \Rightarrow \log x = \cos 2t$]

[$y = e^{\sin 2t} \Rightarrow \log y = \sin 2t$]

16. Question

Find $\frac{dy}{dx}$, when

If x = cos t and y = sin t, prove that $\frac{dy}{dx} = \frac{1}{\sqrt{3}}$ at $1 = \frac{2\pi}{3}$

Answer

as x = cost

Differentiating it with respect to t ,

$\frac{dx}{dt} = \frac{d}{dt}(\cos t)$

$\frac{dx}{dt} = -\sin t$ ......(1)

And, y = sint

Differentiating it with respect to t,

$\frac{dy}{dt} = \frac{d}{dt}(\sin t)$

$\frac{dy}{dt} = \cos t$ ......(2)

Dividing equation (2) by (1),

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t}$

$\frac{dy}{dx} = -\cot t$

$\left(\frac{dy}{dx}\right) = -\cot\left(\frac{2\pi}{3}\right)$

$\left(\frac{dy}{dx}\right) = -\cot\left(\pi - \frac{\pi}{3}\right)$

$= -[-\cot\left(\frac{\pi}{3}\right)]$

$= \cot\left(\frac{\pi}{3}\right)$

$\frac{dy}{dx} = \frac{1}{\sqrt{3}}$

17. Question

Find $\frac{dy}{dx}$, when

If $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$, prove that $\frac{dy}{dx} = \frac{x}{y}$

Answer

as $x = a(t + \frac{1}{t})$

Differentiating it with respect to t,

$$\frac{dx}{dt} = \frac{ad}{dt}(t + \frac{1}{t})$$

$$= a(1 - \frac{1}{t^2})$$

$\frac{dx}{dt} = a(\frac{t^2-1}{t^2})$ ......(1)

And $y = a\left(t - \frac{1}{t}\right)$

Differentiating it with respect to t,

$$\frac{dy}{dt} = \frac{ad}{dt}\left(t - \frac{1}{t}\right)$$

$$= a\left(1 + \frac{1}{t^2}\right)$$

$\frac{dy}{dt} = a\left(\frac{t^2+1}{t^2}\right)$ ......(2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = a\left(\frac{t^2+1}{t^2}\right) \times \frac{t^2}{a(t^2-1)}$$

$$\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$$

$$\frac{dy}{dx} = \frac{x}{y} \text{ [since, } \frac{x}{y} = a\left(\frac{t^2+1}{t^2}\right) \times \frac{t^2}{a(t^2-1)} = \left(\frac{t^2+1}{t^2-1}\right)\text{]}$$

18. Question

Find $\frac{dy}{dx}$, when

If $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, -1 < t < 1, prove that $\frac{dy}{dx} = 1$

Answer

as $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$

Put t = tan θ

$$x = \sin^{-1}\left(\frac{2\tan\theta}{1 + \tan^2\theta}\right)$$

$$= \sin^{-1}\sin 2\theta$$

$= 2\,\theta$ $\left[\text{since}, \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right]$

$x = 2(\tan^{-1} t)$ [since, t = sin θ]

differentiating it with respect to t,

$\frac{dx}{dt} = \frac{2}{1+t^2}$ ......(1)

Now ,

$y = \tan^{-1}\frac{2}{1+t^2}$

Put t = tanθ

$y = \tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}$

$= \tan^{-1}\tan 2\theta \left[\text{since } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$

= 2θ

$y = 2\tan^{-1} t$ [since t = tan θ]

differentiating it with respect to t,

$\frac{dy}{dt} = \frac{2}{1+t^2}$ ......(2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{1+t^2} \times \frac{1+t^2}{2}$$

$$\frac{dy}{dx} = 1$$

19. Question

Find $\frac{dy}{dx}$, when

If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}, y = \frac{\cos^3 t}{\sqrt{\cos 2\,t}}$, find $\frac{dy}{dx}$

Answer

as $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$

Then $\frac{dx}{dt} = \frac{d}{dt}\left[\frac{\sin^3 t}{\sqrt{\cos 2t}}\right]$

$$= \frac{\sqrt{\cos 2t}\,.\frac{d}{dt}(\sin^3 t) - \sin^3 t\,.\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}\,.3\sin^2 t\frac{d}{dt}(\sin t) - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}}\frac{d}{dt}\cos 2t}{\cos 2t}$$

$$= \frac{3\sqrt{\cos 2t}\sin^2 t\cos t - \sin^3 t \times \frac{1}{2\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$= \frac{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}{\cos 2t\sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{\cos^3 t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{\sqrt{\cos 2t}.\frac{d}{dt}(\cos^3 t) - \cos^3 t.\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\cos^2 t\frac{d}{dt}(\cos t) - \cos^3 t.\frac{1}{2\sqrt{\cos 2t}}\frac{d}{dt}\sqrt{\cos 2t}}{\cos 2t}$$

$$= \frac{\sqrt{\cos 2t}.3\cos^2 t(-\sin t) - \cos^3 t.\frac{1}{2\sqrt{\cos 2t}}(-2\sin 2t)}{\cos 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t\sin t + \cos^3 t.\sin 2t}{\cos 2t.\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{-3\cos 2t.\cos^2 t\sin t - \cos^3 t.\sin 2t}{3\cos 2t\sin^2 t\cos t + \sin^3 t\sin 2t}$$

$$= \frac{-3\cos 2t.\cos^2 t\sin t - \cos^3 t.(2\sin t\cos t)}{3\cos 2t\sin^2 t\cos t + \sin^3 t(2\sin t\cos t)}$$

$$= \frac{\sin t\cos t[-3\cos 2t.\cos t - 2\cos^3 t]}{\sin t\cos t[3\cos 2t\sin t + 2\sin^3 t]}$$

$$= \frac{-3\cos 2t.\cos t - 2\cos^3 t}{[3\cos 2t\sin t + 2\sin^3 t]}\begin{bmatrix}\cos 2t = (2\cos^2 t - 1),\\ \cos 2t = (1 - 2\sin^2 t)\end{bmatrix}$$

$$\frac{-4\cos^3 t + 3\cos t}{3\sin t - 4\sin^3 t}$$

$$= -\frac{\cos 3t}{\sin 3t}\ [\cos 3t = 4\cos^3 t - 3\cos t$$

$$\sin 3t = 3\sin t - 4\sin^3 t]$$

$$= \cot 3t$$

20. Question

Find $\frac{dy}{dx}$, when

If $x = \left(t + \frac{1}{t}\right)^a, y = a^{t+\frac{1}{t}}$, find $\frac{dy}{dx}$

Answer

as $x = \left(t + \frac{1}{t}\right)^a$

Differentiating it with respect to t using chain rule,

$$\frac{dx}{dt} = \frac{d}{dt}\left(\left(t + \frac{1}{t}\right)^a\right)$$

$$= a\left(\left(t + \frac{1}{t}\right)^{a-1}\right)\frac{d}{dt}\left(t + \frac{1}{t}\right)$$

$$\frac{dx}{dt} = a\left(\left(t + \frac{1}{t}\right)^{a-1}\right)\left(1 - \frac{1}{t^2}\right) \text{......(1)}$$

And , $y = a^{(t+\frac{1}{t})}$

Differentiating it with respect to t using chain rule,

$$\frac{dy}{dt} = \frac{d}{dt}(a^{(t+\frac{1}{t})})$$

$$= a^{(t+\frac{1}{t})} \times \log a \frac{d}{dt}(t + \frac{1}{t})$$

$\frac{dy}{dt} = a^{(t+\frac{1}{t})} \times \log a\,(1 - \frac{1}{t^2})$ ......(2)

Dividing equation (2) by (1),

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a^{\left(t+\frac{1}{t}\right)} \log a \left(1 - \frac{1}{t^2}\right)}{a\left(\left(t + \frac{1}{t}\right)^{a-1}\right)\left(1 - \frac{1}{t^2}\right)}$$

$$\frac{dy}{dx} = \frac{a^{\left(t+\frac{1}{t}\right)} \log a}{a\left(t + \frac{1}{t}\right)^{a-1}}$$

21. Question

Find $\frac{dy}{dx}$, when

If $x = a\left(\frac{1+t^2}{1-t^2}\right)$ and $y = \frac{2t}{1-t^2}$, find $\frac{dy}{dx}$

Answer

Here,

x = $a(\frac{1+t^2}{1-t^2})$

differentiating bove function with respect to t, we have,

$$\frac{dx}{dt} = a\left[\frac{(1-t^2)\frac{d(1+t^2)}{dt} - (1+t^2)\frac{d(1-t^2)}{dt}}{(1-t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left[\frac{(1-t^2)(2t) - (1+t^2)(-2t)}{(1-t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left[\frac{2t - 2t^2 + 2t + 2t^3}{(1-t^2)^2}\right]$$

$$\frac{dx}{dt} = \left[\frac{4at}{(1-t^2)^2}\right] \dots\dots(1)$$

And

$$y = \frac{2t}{1-t^2}$$

differentiating bove function with respect to t, we have,

$$\frac{dy}{dt} = 2\left[\frac{(1-t^2)\frac{d(t)}{dt} - (t)\frac{d(1-t^2)}{dt}}{(1-t^2)^2}\right]$$

$$\frac{dy}{dt} = 2\left[\frac{(1-t^2)-(t)(-2t)}{(1-t^2)^2}\right]$$

$$= 2\left[\frac{1-t^2+2t^2}{(1-t^2)^2}\right]$$

$$= 2\left[\frac{1+t^2}{1-t^2}\right] \ldots\ldots (2)$$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\left[\frac{1+t^2}{1-t^2}\right]}{\frac{4at}{(1-t^2)^2}}$ | from equation 1 and 2

$$\frac{dy}{dx} = \frac{1-t^4}{2at}$$

22. Question

Find $\frac{dy}{dx}$, when

If x = 10 (t – sin t), y = 12 (1 – cos t), find $\frac{dy}{dx}$.

Answer

Here, x = 10(t - sin t) y = 12(1-cos t)

$\frac{dx}{dt} = 10(1-\cos t)$ ……(1)

$\frac{dy}{dt} = 12(\sin t)$ ……(2)

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12(\sin t)}{10(1-\cos t)}$ | from equation 1 and 2

$$\frac{dy}{dx} = \frac{12\sin\frac{t}{2}.\cos t/2}{10\ \sin^2 t/2}$$

$$\frac{dy}{dx} = \frac{6}{5}\cot\frac{t}{2}$$

23. Question

Find $\frac{dy}{dx}$, when

If x = a(θ – sin θ) and y = a (1 + cos θ), find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

Answer

Here,

x = ($\theta - \sin\theta$) and y = a(1 + cos θ)

then ,

$$\frac{dx}{d\theta} = a(1-\cos\theta)$$

$$\frac{dy}{d\theta} = a(-\sin\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(-\sin\theta)}{a(1-\cos\theta)} = \frac{(-\sin\theta)}{(1-\cos\theta)}$$

At $x = \frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{a(-\sin\frac{\pi}{3})}{a\left(1-\cos\frac{\pi}{3}\right)} = \frac{\sqrt{3}/2}{1-1/2}$$

$= \sqrt{3}$

24. Question

Find $\frac{dy}{dx}$, when

If x = a sin 2t (1 + cos 2t) and y = b cos 2t (1 – cos 2t), show that at $t = \frac{\pi}{4}, \frac{dy}{dx} = \frac{b}{a}.$

Answer

considering the given functions,

x = asin 2t(1 + cos 2t) and y = b cos 2t(1-cos2t)

rewriting the above equations,

x = a sin 2t $+ \frac{a}{2}$sin 4t

differentiating bove function with respect to t, we have,

$\frac{dx}{dt} = 2a\cos 2t + 2a\cos 4t$ ......(1)

y = b cos 2t - b$\cos^2 2t$

differentiating above function with respect to t, we have,

$\frac{dy}{dt} = -2b\sin 2t + 2b\cos 2t\sin 2t == -2b\sin 2t + 2b\sin 4t$ ......(2)

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2b\sin 2t+2b\sin 4t}{2a\cos 2t+2a\cos 4t}$ | from equation 1 and 2

At $t = \frac{\pi}{4}$

$$\frac{dy}{dx} = \frac{b}{a}$$

25. Question

Find $\frac{dy}{dx}$, when

If x = cos t (3 – 2 cos^2 t) and y = sin t (3 – 2 sin^2 t) find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}.$

Answer

considering the given functions,

x = cost(3-2$\cos^2 t$)

x = 3cos t - $2\cos^3 t$

$\frac{dx}{dt} = -3\sin t + 6\cos^2 t \sin t$ ......(1)

$\frac{dy}{dt} = 3\cos t + 6\sin^2 t \cos t$ ......(2)

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t+6\sin^2 t\cos t}{-3\sin t+6\cos^2 t\sin t}$ | from equation 1 and 2

$= \frac{3\cos t\,(1 + 2\sin^2 t)}{3\sin t\,(-1 + 2\cos^2 t)}$

$= \frac{\cot t\,(1 - 2(1 - \cos^2 t))}{(2\cos^2 t - 1)} = \cot t$

When $t = \frac{\pi}{4}$

$\frac{dy}{dx} = \cot\frac{\pi}{4} = 1$

26. Question

Find $\frac{dy}{dx}$, when

If $x = \frac{1+\log t}{t^2}, y = \frac{3+2\log t}{t}$, find $\frac{dy}{dx}$.

Answer

: $x = \frac{1+\log t}{t^2}$, $y = \frac{3+2\log t}{t}$

$\frac{dx}{dt} = \frac{t^2\left(\frac{1}{t}\right) - (1+\log t)(2t)}{t^4} = \frac{t - 2t - 2t\log t}{t^4} = \frac{-2\log t - 1}{t^3}$

$\frac{dy}{dt} = \frac{t\left(\frac{2}{t}\right) - (3+2\log t)(1)}{t^2} = \frac{2 - 3 - 2\log t}{t^2} = \frac{-2\log t - 1}{t^2}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{-2\log t - 1}{t^2}}{\frac{-2\log t - 1}{t^3}} = t$

27. Question

Find $\frac{dy}{dx}$, when

If x = 3 sin t – sin 3t, y = 3 cost – cos 3t, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$.

Answer

x = 3sin t – sin 3t ,y = 3 cos t – cos 3t

$\frac{dx}{dt} = 3\cos t - 3\cos 3t$

$\frac{dy}{dt} = -3\sin t + 3\sin 3t$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3\sin t + 3\sin 3t}{3\cos t - 3\cos 3t}$

When t = $\frac{\pi}{3}$

$$\frac{dy}{dx} = \frac{-3\sin\left(\frac{\pi}{3}\right) + 3\sin 3\left(\frac{\pi}{3}\right)}{3\cos\left(\frac{\pi}{3}\right) - 3\cos 3\left(\frac{\pi}{3}\right)}$$

$$\frac{dy}{dx} = \frac{-3 \times \frac{\sqrt{3}}{2} + 0}{\frac{3}{2} - 3(-1)} = \frac{1}{\sqrt{3}}$$

28. Question

Find $\frac{dy}{dx}$, when

If $\sin x = \frac{2t}{1+t^2}, \tan y = \frac{2t}{1-t^2}$, find $\frac{dy}{dx}$.

Answer

$\sin x = \frac{2t}{1+t^2}$,$\tan y = \frac{2t}{1-t^2}$

$x = \sin^{-1}\frac{2t}{1+t^2}$ and $y = \tan^{-1}\frac{2t}{1-t^2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-\left(\frac{2t}{1+t^2}\right)^2}} \times \frac{2(1+t^2)-(2t)(2t)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{2}{1+t^2}$$

$$\frac{dy}{dt} = \frac{1}{1+\left(\frac{2t}{1+t^2}\right)^2} \times \frac{2(1-t^2)-(2t)(-2t)}{(1-t^2)^2}$$

$$\frac{dy}{dt} = \frac{2}{1+t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1$$

Exercise 11.8

1. Question

Differentiate x^2 with respect to x^3.

Answer

Let u = x^2 and v = x^3.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(x^2)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{du}{dx} = 2x^{2-1}$

$\therefore \frac{du}{dx} = 2x$

Now, on differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(x^3)$

$\Rightarrow \frac{dv}{dx} = 3x^{3-1}$ (using the same formula)

$\therefore \frac{dv}{dx} = 3x^2$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{2x}{3x^2}$

$\therefore \frac{du}{dv} = \frac{2}{3x}$

Thus, $\frac{du}{dv} = \frac{2}{3x}$

2. Question

Differentiate log(1 + x^2) with respect to $\tan^{-1}x$.

Answer

Let u = log(1 + x^2) and v = $\tan^{-1}x$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}[\log(1 + x^2)]$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$

$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}\frac{d}{dx}(1 + x^2)$ [using chain rule]

$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}\left[\frac{d}{dx}(1) + \frac{d}{dx}(x^2)\right]$

However, $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}[0 + 2x^{2-1}]$

$\Rightarrow \frac{du}{dx} = \frac{1}{1 + x^2}[2x]$

$\therefore \frac{du}{dx} = \frac{2x}{1 + x^2}$

Now, on differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(\tan^{-1}x)$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\therefore \frac{dv}{dx} = \frac{1}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2x}{1+x^2}}{\frac{1}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2x}{1+x^2} \times (1+x^2)$$

$$\therefore \frac{du}{dv} = 2x$$

Thus, $\frac{du}{dv} = 2x$

3. Question

Differentiate $(\log x)^x$ with respect to log x.

Answer

Let u = $(\log x)^x$ and v = log x.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = $(\log x)^x$

Taking log on both sides, we get

log u = $\log(\log x)^x$

⇒ log u = x × log(log x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[x \times \log(\log x)]$$

Recall that (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\log x)\frac{d}{dx}(x) + x\frac{d}{dx}[\log(\log x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\log x) \times 1 + x\left[\frac{1}{\log x}\frac{d}{dx}(\log x)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log(\log x) + \frac{x}{\log x}\frac{d}{dx}(\log x)$$

But, u = $(\log x)^x$ and $\frac{d}{dx}(\log x) = \frac{1}{x}$

$$\Rightarrow \frac{1}{(\log x)^x}\frac{du}{dx} = \log(\log x) + \frac{x}{\log x} \times \frac{1}{x}$$

$$\Rightarrow \frac{1}{(\log x)^x}\frac{du}{dx} = \log(\log x) + \frac{1}{\log x}$$

$\therefore \frac{du}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]$

Now, on differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(\log x)$

$\therefore \frac{dv}{dx} = \frac{1}{x}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{(\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]}{\frac{1}{x}}$

$\Rightarrow \frac{du}{dv} = x(\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]$

$\Rightarrow \frac{du}{dv} = x(\log x)^x \left[\frac{\log(\log x)\log x + 1}{\log x}\right]$

$\Rightarrow \frac{du}{dv} = \frac{x(\log x)^x}{\log x}[\log(\log x)\log x + 1]$

$\therefore \frac{du}{dv} = x(\log x)^{x-1}[1 + \log x \log(\log x)]$

Thus, $\frac{du}{dv} = x(\log x)^{x-1}[1 + \log x \log(\log x)]$

4 A. Question

Differentiate $\sin^{-1}\sqrt{1-x^2}$ with respect to $\cos^{-1}x$, if

$x \in (0, 1)$

Answer

Let $u = \sin^{-1}\sqrt{1-x^2}$ and $v = \cos^{-1}x$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\sqrt{1-x^2}$

By substituting $x = \cos\theta$, we have

$u = \sin^{-1}\sqrt{1-(\cos\theta)^2}$

$\Rightarrow u = \sin^{-1}\sqrt{1-\cos^2\theta}$

$\Rightarrow u = \sin^{-1}\sqrt{\sin^2\theta}$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(\sin\theta)$

(i) Given $x \in (0, 1)$

However, $x = \cos\theta$.

$\Rightarrow \cos\theta \in (0, 1)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

Hence, u = $\sin^{-1}(\sin\theta) = \theta$.

$\Rightarrow u = \cos^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have, $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right)$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

4 B. Question

Differentiate $\sin^{-1}\sqrt{1-x^2}$ with respect to $\cos^{-1}x$, if

$x \in (-1, 0)$

Answer

Given $x \in (-1, 0)$

However, $x = \cos\theta$.

$\Rightarrow \cos\theta \in (-1, 0)$

$\Rightarrow \theta \in \left(\frac{\pi}{2}, \pi\right)$

Hence, u = $\sin^{-1}(\sin\theta) = \pi - \theta$.

$\Rightarrow u = \pi - \cos^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - \cos^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - \left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Now, on differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \left(-\sqrt{1-x^2}\right)$$

$$\therefore \frac{du}{dv} = -1$$

Thus, $\frac{du}{dv} = -1$

5 A. Question

Differentiate $\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ with respect to $\sqrt{1-4x^2}$, if

$$x \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$$

Answer

Let $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ and $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$$

By substituting 2x = cos θ, we have

$$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

However, $2x = \cos\theta \Rightarrow x = \frac{\cos\theta}{2}$

$$\Rightarrow \frac{\cos\theta}{2} \in \left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \cos\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$\Rightarrow u = \pi - 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

Now, we have $v = \sqrt{1-4x^2}$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1-4x^2}\right)$$

$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1-4x^2)^{\frac{1}{2}}$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-4x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1-4x^2)$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1-4x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(4x^2)\right]$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}\left[\frac{d}{dx}(1) - 4\frac{d}{dx}(x^2)\right]$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}[0-4(2x^{2-1})]$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1-4x^2}}[-8x]$

$\therefore \frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$

$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$

$\therefore \frac{du}{dv} = -\frac{1}{x}$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

5 B. Question

Differentiate $\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ with respect to $\sqrt{1-4x^2}$, if

$x \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

Answer

Let $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ and $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$

By substituting 2x = cos θ, we have

$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

However, $2x = \cos\theta \Rightarrow x = \frac{\cos\theta}{2}$

$\Rightarrow \frac{\cos\theta}{2} \in \left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right)$

$\Rightarrow \cos\theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = -\frac{4}{\sqrt{1-4x^2}}$$

In part (i), we found $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = \frac{1}{x}$$

Thus, $\frac{du}{dv} = \frac{1}{x}$

5 C. Question

Differentiate $\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ with respect to $\sqrt{1-4x^2}$, if

$$x \in \left(-\frac{1}{2}, \frac{1}{2\sqrt{2}}\right)$$

Answer

Let $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$ and $v = \sqrt{1-4x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$$\Rightarrow u = \sin^{-1}\left(4x\sqrt{1-(2x)^2}\right)$$

By substituting 2x = cos θ, we have

$$u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{1-(\cos\theta)^2}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\cos\theta\sqrt{\sin^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2 \cos \theta \sin \theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $x \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$

However, $2x = \cos \theta \Rightarrow x = \frac{\cos\theta}{2}$

$$\Rightarrow \frac{\cos\theta}{2} \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow \cos\theta \in \left(-1, -\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{4}, \pi\right)$$

$\Rightarrow 2\theta \in \left(\frac{3\pi}{2}, 2\pi\right)$

Hence, u = $\sin^{-1}(\sin 2\theta) = 2\pi - 2\theta$.

$\Rightarrow u = 2\pi - 2\cos^{-1}(2x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}[2\pi - 2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(2\pi) - \frac{d}{dx}[2\cos^{-1}(2x)]$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\pi) - 2\frac{d}{dx}[\cos^{-1}(2x)]$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2\left[-\frac{1}{\sqrt{1-(2x)^2}}\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[\frac{d}{dx}(2x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}\left[2\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}\frac{d}{dx}(x)$$

However, $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \times 1$$

$$\therefore \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}}$$

In part (i), we found $\frac{dv}{dx} = -\frac{4x}{\sqrt{1-4x^2}}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{4}{\sqrt{1-4x^2}}}{-\frac{4x}{\sqrt{1-4x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{4}{\sqrt{1-4x^2}} \times \left(-\frac{\sqrt{1-x^2}}{4x}\right)$$

$$\therefore \frac{du}{dv} = -\frac{1}{x}$$

Thus, $\frac{du}{dv} = -\frac{1}{x}$

6. Question

Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, if –1<x<1, x ≠ 0.

Answer

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

By substituting x = tan θ, we have

$$u = \tan^{-1}\left(\frac{\sqrt{1+(\tan\theta)^2}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$\Rightarrow u = \tan^{-1}\left(\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}\right)$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1-\cos\left(2\times\frac{\theta}{2}\right)}{\sin\left(2\times\frac{\theta}{2}\right)}\right)$$

But, $\cos2\theta = 1 - 2\sin^2\theta$ and $\sin2\theta = 2\sin\theta\cos\theta$.

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

Given –1 < x < 1 ⇒ x ϵ (–1, 1)

However, x = tan θ

⇒ tan θ ϵ (–1, 1)

$$\Rightarrow \theta \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\theta}{2} \in \left(-\frac{\pi}{8},\frac{\pi}{8}\right)$$

Hence, $u = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$

$\Rightarrow u = \frac{1}{2}\tan^{-1}x$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x\right)$

$\Rightarrow \frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1}x)$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\Rightarrow \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2}$

$\therefore \frac{du}{dx} = \frac{1}{2(1+x^2)}$

Now, we have $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$v = \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$

$\Rightarrow v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow v = \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$\Rightarrow v = \sin^{-1}\left(\frac{2 \times \frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$

$\Rightarrow v = \sin^{-1}\left(2 \times \frac{\sin\theta}{\cos\theta} \times \cos^2\theta\right)$

$\Rightarrow v = \sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow v = \sin^{-1}(\sin 2\theta)$

However, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $v = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow v = 2\tan^{-1}x$

On differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$

$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{2(1+x^2)} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = \frac{1}{4}$$

Thus, $\frac{du}{dv} = \frac{1}{4}$

7 A. Question

Differentiate $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, if

$x \in \left(0, 1/\sqrt{2}\right)$

Answer

Let $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ and $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting x = sin θ, we have

$$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

⇒ u = $\sin^{-1}$(2sinθcosθ)

⇒ u = $\sin^{-1}$(sin2θ)

Now, we have $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$$v = \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$$

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$

$\Rightarrow v = \sec^{-1}(\sec\theta)$

Given $x \in \left(0, \frac{1}{\sqrt{2}}\right)$

However, x = sin θ

$\Rightarrow \sin\theta \in \left(0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}(2\sin^{-1}x)$

$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\sin^{-1}x)$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{du}{dx} = \frac{2}{\sqrt{1-x^2}}$

We have $\theta \in \left(0, \frac{\pi}{4}\right)$

Hence, $v = \sec^{-1}(\sec\theta) = \theta$

$\Rightarrow v = \sin^{-1}x$

On differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$

$$\Rightarrow \frac{du}{dv} = \frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = 2$$

Thus, $\frac{du}{dv} = 2$

7 B. Question

Differentiate $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, if $x \in \left(1/\sqrt{2}, 1\right)$

Answer

Let $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ and $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting x = sin θ, we have

$$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Now, we have $v = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$$v = \sec^{-1}\left(\frac{1}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$$

$\Rightarrow v = \sec^{-1}\left(\frac{1}{\sqrt{\cos^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow v = \sec^{-1}\left(\frac{1}{\cos\theta}\right)$$

$\Rightarrow v = \sec^{-1}(\sec\theta)$

Given $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$

However, x = sin θ

$$\Rightarrow \sin\theta \in \left(\frac{1}{\sqrt{2}}, 1\right)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$\Rightarrow 2\theta \in \left(\frac{\pi}{2}, \pi\right)$

Hence, u = $\sin^{-1}(\sin 2\theta) = \pi - 2\theta$.

$\Rightarrow u = \pi - 2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\pi - 2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - \frac{d}{dx}(2\sin^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\pi) - 2\frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

We have $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Hence, v = $\sec^{-1}(\sec\theta) = \theta$

$\Rightarrow v = \sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = -2$$

Thus, $\frac{du}{dv} = -2$

8. Question

Differentiate $(\cos x)^{\sin x}$ with respect to $(\sin x)^{\cos x}$.

Answer

Let u = $(\cos x)^{\sin x}$ and v = $(\sin x)^{\cos x}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = $(\cos x)^{\sin x}$

Taking log on both sides, we get

log u = $\log(\cos x)^{\sin x}$

⇒ log u = (sin x) × log(cos x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{du}(\log u) \times \frac{du}{dx} = \frac{d}{dx}[\sin x \times \log(\cos x)]$$

Recall that (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{du}{dx} = \log(\cos x)\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}[\log(\cos x)]$$

We know $\frac{d}{dx}(\log x) = \frac{1}{x}$ and $\frac{d}{dx}(\sin x) = \cos x$

$$\Rightarrow \frac{1}{u} \times \frac{du}{dx} = \log(\cos x) \times \cos x + \sin x\left[\frac{1}{\cos x}\frac{d}{dx}(\cos x)\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \frac{\sin x}{\cos x}\frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \tan x\frac{d}{dx}(\cos x)$$

We know $\frac{d}{dx}(\cos x) = -\sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) + \tan x\,(-\sin x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

But, u = $(\cos x)^{\sin x}$

$$\Rightarrow \frac{1}{(\cos x)^{\sin x}}\frac{du}{dx} = \cos x \log(\cos x) - \tan x \sin x$$

$$\therefore \frac{du}{dx} = (\cos x)^{\sin x}[\cos x \log(\cos x) - \tan x \sin x]$$

Now, we have v = $(\sin x)^{\cos x}$

Taking log on both sides, we get

log v = $\log(\sin x)^{\cos x}$

⇒ log v = (cos x) × log(sin x) [∵ log a^m = m × log a]

On differentiating both the sides with respect to x, we get

$$\frac{d}{dv}(\log v) \times \frac{dv}{dx} = \frac{d}{dx}[\cos x \times \log(\sin x)]$$

Recall that (uv)' = vu' + uv' (product rule)

$$\Rightarrow \frac{d}{du}(\log u) \times \frac{dv}{dx} = \log(\sin x)\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}[\log(\sin x)]$$

We know $\frac{d}{dx}(\log x)=\frac{1}{x}$ and $\frac{d}{dx}(\cos x)=-\sin x$

$$\Rightarrow \frac{1}{v}\times\frac{dv}{dx}=\log(\sin x)\times(-\sin x)+\cos x\left[\frac{1}{\sin x}\frac{d}{dx}(\sin x)\right]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=-\sin x\log(\sin x)+\frac{\cos x}{\sin x}\frac{d}{dx}(\sin x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=-\sin x\log(\sin x)+\cot x\frac{d}{dx}(\sin x)$$

We know $\frac{d}{dx}(\sin x)=\cos x$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=-\sin x\log(\sin x)+\cot x\times(\cos x)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx}=-\sin x\log(\sin x)+\cot x\cos x$$

But, v = $(\sin x)^{\cos x}$

$$\Rightarrow \frac{1}{(\sin x)^{\cos x}}\frac{dv}{dx}=-\sin x\log(\sin x)+\cot x\cos x$$

$$\therefore \frac{dv}{dx}=(\sin x)^{\cos x}[-\sin x\log(\sin x)+\cot x\cos x]$$

We have $\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv}=\frac{(\cos x)^{\sin x}[\cos x\log(\cos x)-\tan x\sin x]}{(\sin x)^{\cos x}[-\sin x\log(\sin x)+\cot x\cos x]}$$

$$\therefore \frac{du}{dv}=\frac{(\cos x)^{\sin x}[\cos x\log(\cos x)-\tan x\sin x]}{(\sin x)^{\cos x}[\cot x\cos x-\sin x\log(\sin x)]}$$

Thus, $\frac{du}{dv}=\frac{(\cos x)^{\sin x}[\cos x\log(\cos x)-\tan x\sin x]}{(\sin x)^{\cos x}[\cot x\cos x-\sin x\log(\sin x)]}$

9. Question

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, if 0 < x < 1.

Answer

Let $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$$u=\sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow u=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow u=\sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$\Rightarrow u = \sin^{-1}\left(\dfrac{2 \times \dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos^2\theta}}\right)$

$\Rightarrow u = \sin^{-1}\left(2 \times \dfrac{\sin\theta}{\cos\theta} \times \cos^2\theta\right)$

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin2\theta = 2\sin\theta\cos\theta$

$\Rightarrow u = \sin^{-1}(\sin2\theta)$

Given $0 < x < 1 \Rightarrow x \in (0, 1)$

However, $x = \tan\theta$

$\Rightarrow \tan\theta \in (0, 1)$

$\Rightarrow \theta \in \left(0, \dfrac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(0, \dfrac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin2\theta) = 2\theta$

$\Rightarrow u = 2\tan^{-1}x$

On differentiating u with respect to x, we get

$\dfrac{du}{dx} = \dfrac{d}{dx}(2\tan^{-1}x)$

$\Rightarrow \dfrac{du}{dx} = 2\dfrac{d}{dx}(\tan^{-1}x)$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\Rightarrow \dfrac{du}{dx} = 2 \times \dfrac{1}{1+x^2}$

$\therefore \dfrac{du}{dx} = \dfrac{2}{1+x^2}$

Now, we have $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

By substituting $x = \tan\theta$, we have

$v = \cos^{-1}\left(\dfrac{1-(\tan\theta)^2}{1+(\tan\theta)^2}\right)$

$\Rightarrow v = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{\sec^2\theta}\right)$ [$\because \sec^2\theta - \tan^2\theta = 1$]

$\Rightarrow v = \cos^{-1}\left(\dfrac{1}{\sec^2\theta} - \dfrac{\tan^2\theta}{\sec^2\theta}\right)$

$\Rightarrow v = \cos^{-1}\left(\dfrac{1}{\dfrac{1}{\cos^2\theta}} - \dfrac{\dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{1}{\cos^2\theta}}\right)$

$\Rightarrow$ v = $\cos^{-1}(\cos^2\theta - \sin^2\theta)$

But, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow$ v = $\cos^{-1}(\cos 2\theta)$

However, $\theta \in \left(0, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, v = $\cos^{-1}(\cos 2\theta) = 2\theta$

$\Rightarrow$ v = $2\tan^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{1+x^2} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

10. Question

Differentiate $\tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ with respect to $\sqrt{1+a^2x^2}$.

Answer

Let $u = \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$ and $v = \sqrt{1+a^2x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{1+ax}{1-ax}\right)$

By substituting ax = tan θ, we have

$$u = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \theta\right)\right) \left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right]$$

$$\Rightarrow u = \frac{\pi}{4} + \theta$$

$$\Rightarrow u = \frac{\pi}{4} + \tan^{-1}(ax)$$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{\pi}{4} + \tan^{-1}(ax)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}[\tan^{-1}(ax)]$$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 + \frac{1}{1 + (ax)^2}\frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{1 + a^2x^2}\left[a\frac{d}{dx}(x)\right]$$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1 + a^2x^2}\frac{d}{dx}(x)$$

We know $\frac{d}{dx}(x) = 1$

$$\Rightarrow \frac{du}{dx} = \frac{a}{1 + a^2x^2} \times 1$$

$$\therefore \frac{du}{dx} = \frac{a}{1 + a^2x^2}$$

Now, we have $v = \sqrt{1 + a^2x^2}$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1 + a^2x^2}\right)$$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1 + a^2x^2)^{\frac{1}{2}}$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 + a^2x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1 + a^2x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 + a^2x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) + \frac{d}{dx}(a^2x^2)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 + a^2x^2}}\left[\frac{d}{dx}(1) + a^2\frac{d}{dx}(x^2)\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}[0 + a^2(2x^{2-1})]$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1+a^2x^2}}[2a^2x]$

$\therefore \frac{dv}{dx} = \frac{a^2x}{\sqrt{1+a^2x^2}}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{\frac{a}{1+a^2x^2}}{\frac{a^2x}{\sqrt{1+a^2x^2}}}$

$\Rightarrow \frac{du}{dv} = \frac{a}{1+a^2x^2} \times \frac{\sqrt{1+a^2x^2}}{a^2x}$

$\therefore \frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$

Thus, $\frac{du}{dv} = \frac{1}{ax\sqrt{1+a^2x^2}}$

11. Question

Differentiate $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

Answer

Let $u = \sin^{-1}(2x\sqrt{1-x^2})$ and $v = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}(2x\sqrt{1-x^2})$

By substituting x = sin θ, we have

$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [$\because$ $\sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \Rightarrow x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

However, x = sin θ

$\Rightarrow \sin\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, u = $\sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow$ u = $2\sin^{-1}(x)$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(2\sin^{-1} x)$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\sin^{-1} x)$$

We know $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{du}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Now, we have $v = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$$v = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-(\sin\theta)^2}}\right)$$

$$\Rightarrow v = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$$

$\Rightarrow v = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow v = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$\Rightarrow$ v = $\tan^{-1}(\tan\theta)$

We have $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Hence, v = $\tan^{-1}(\tan\theta) = \theta$

$\Rightarrow$ v = $\sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sin^{-1} x)$$

We know $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{\sqrt{1-x^2}}}{\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{\sqrt{1-x^2}} \times \sqrt{1-x^2}$$

$$\therefore \frac{du}{dv} = 2$$

Thus, $\frac{du}{dv} = 2$

12. Question

Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, if $0 < x < 1$.

Answer

Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

By substituting $x = \tan\theta$, we have

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-(\tan\theta)^2}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

But, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow u = \tan^{-1}(\tan 2\theta)$

Given $0 < x < 1 \Rightarrow x \in (0, 1)$

However, $x = \tan\theta$

$\Rightarrow \tan\theta \in (0, 1)$

$$\Rightarrow \theta \in \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$$

Hence, $u = \tan^{-1}(\tan 2\theta) = 2\theta$

$\Rightarrow u = 2\tan^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{2}{1+x^2}$$

Now, we have $v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

By substituting x = tan θ, we have

$$v = \cos^{-1}\left(\frac{1-(\tan\theta)^2}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow v = \cos^{-1}\left(\frac{1-\tan^2\theta}{\sec^2\theta}\right)$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\right)$$

$$\Rightarrow v = \cos^{-1}\left(\frac{1}{\frac{1}{\cos^2\theta}} - \frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$\Rightarrow v = \cos^{-1}(\cos^2\theta - \sin^2\theta)$

But, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\Rightarrow v = \cos^{-1}(\cos 2\theta)$

However, $\theta \in \left(0, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $v = \cos^{-1}(\cos 2\theta) = 2\theta$

$\Rightarrow v = 2\tan^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$\Rightarrow \frac{du}{dv} = \frac{2}{1+x^2} \times \frac{1+x^2}{2}$

$\therefore \frac{du}{dv} = 1$

Thus, $\frac{du}{dv} = 1$

13. Question

Differentiate $\tan^{-1}\left(\frac{x-1}{x+1}\right)$ with respect to $\sin^{-1}\left(3x-4x^3\right)$, if $-\frac{1}{2} < x < \frac{1}{2}$.

Answer

Let $u = \tan^{-1}\left(\frac{x-1}{x+1}\right)$ and v = $\sin^{-1}(3x - 4x^3)$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{x-1}{x+1}\right)$

By substituting x = tan θ, we have

$u = \tan^{-1}\left(\frac{\tan\theta - 1}{\tan\theta + 1}\right)$

$\Rightarrow u = \tan^{-1}\left(\frac{\tan\theta - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{4}\tan\theta}\right)$

$\Rightarrow u = \tan^{-1}\left(\tan\left(\theta - \frac{\pi}{4}\right)\right)$ $\left[\because \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$

Given, $-\frac{1}{2} < x < \frac{1}{2} \Rightarrow x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

However, x = tan θ

$\Rightarrow \tan\theta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow \theta \in \left(\tan^{-1}\left(-\frac{1}{2}\right), \tan^{-1}\left(\frac{1}{2}\right)\right)$

$\Rightarrow \theta \in \left(-\tan^{-1}\left(\frac{1}{2}\right), \tan^{-1}\left(\frac{1}{2}\right)\right)$

As tan 0 = 0 and $\tan\frac{\pi}{4}$ = 1, we have $\tan^{-1}\left(\frac{1}{2}\right) \in \left(0, \frac{\pi}{4}\right)$.

Thus, $\theta - \frac{\pi}{4}$ lies in the range of $\tan^{-1}x$.

Hence, $u = \tan^{-1}\left(\tan\left(\theta - \frac{\pi}{4}\right)\right) = \theta - \frac{\pi}{4}$

$\Rightarrow u = \tan^{-1}x - \frac{\pi}{4}$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}\left(\tan^{-1}x - \frac{\pi}{4}\right)$

$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(\tan^{-1}x) - \frac{d}{dx}\left(\frac{\pi}{4}\right)$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} + 0$$

$$\therefore \frac{du}{dx} = \frac{1}{1+x^2}$$

Now, we have $v = \sin^{-1}(3x - 4x^3)$

By substituting $x = \sin\theta$, we have

$v = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

But, $\sin3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow v = \sin^{-1}(\sin3\theta)$

Given, $-\frac{1}{2} < x < \frac{1}{2} \Rightarrow x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

However, $x = \sin\theta$

$$\Rightarrow \sin\theta \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$\Rightarrow \theta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$$

$$\Rightarrow 3\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence, $v = \sin^{-1}(\sin3\theta) = 3\theta$

$\Rightarrow v = 3\sin^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(3\sin^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 3\frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{dv}{dx} = 3 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{3}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{1+x^2}}{\frac{3}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{1}{1+x^2} \times \frac{\sqrt{1-x^2}}{3}$$

$$\therefore \frac{du}{dv} = \frac{\sqrt{1-x^2}}{3(1+x^2)}$$

Thus, $\frac{du}{dv} = \frac{\sqrt{1-x^2}}{3(1+x^2)}$

14. Question

Differentiate $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ with respect to $\sec^{-1} x$.

Answer

Let $u = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ and $v = \sec^{-1}x$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$\Rightarrow u = \tan^{-1}\left(\frac{\cos\left(2\times\frac{x}{2}\right)}{1+\sin\left(2\times\frac{x}{2}\right)}\right)$$

But, $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\Rightarrow u = \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$\Rightarrow u = \tan^{-1}\left(\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$$\Rightarrow u = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2}\right)^2 - \left(\sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2}\right)^2 + \left(\sin\frac{x}{2}\right)^2 + 2\left(\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}\right)}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}\right)$$

Dividing the numerator and denominator with $\cos\frac{x}{2}$, we get

$$\Rightarrow u = \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2}}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} - \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\frac{x}{2}}{1 + \tan\frac{\pi}{4}\tan\frac{x}{2}}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) \left[\because \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$\Rightarrow u = \frac{\pi}{4} - \frac{x}{2}$$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{1}{2}\frac{d}{dx}(x)$$

We know $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{du}{dx} = 0 - \frac{1}{2} \times 1$$

$$\therefore \frac{du}{dx} = -\frac{1}{2}$$

Now, we have v = $\sec^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\sec^{-1}x)$$

We know $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

$$\therefore \frac{dv}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{1}{2}}{\frac{1}{x\sqrt{x^2 - 1}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{2} \times x\sqrt{x^2 - 1}$$

$$\therefore \frac{du}{dv} = -\frac{x\sqrt{x^2 - 1}}{2}$$

Thus, $\frac{du}{dv} = -\frac{x\sqrt{x^2-1}}{2}$

15. Question

Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, if -1 < x < 1.

Answer

Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

By substituting x = tan θ, we have

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+(\tan\theta)^2}\right)$$

$$\Rightarrow u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow u = \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$ [∵ $\sec^2\theta - \tan^2\theta = 1$]

$$\Rightarrow u = \sin^{-1}\left(\frac{2\times\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$$

$$\Rightarrow u = \sin^{-1}\left(2\times\frac{\sin\theta}{\cos\theta}\times\cos^2\theta\right)$$

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

But, $\sin2\theta = 2\sin\theta\cos\theta$

$\Rightarrow u = \sin^{-1}(\sin2\theta)$

Given $-1 < x < 1 \Rightarrow x \in (-1, 1)$

However, x = tan θ

$\Rightarrow \tan\theta \in (-1, 1)$

$$\Rightarrow \theta \in \left(-\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$\Rightarrow 2\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

Hence, $u = \sin^{-1}(\sin2\theta) = 2\theta$

$\Rightarrow u = 2\tan^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{du}{dx} = 2\times\frac{1}{1+x^2}$$

$$\therefore \frac{du}{dx} = \frac{2}{1+x^2}$$

Now, we have $v = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

By substituting x = tan θ, we have

$$v = \tan^{-1}\left(\frac{2\tan\theta}{1-(\tan\theta)^2}\right)$$

$$\Rightarrow v = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

But, $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

⇒ v = $\tan^{-1}(\tan 2\theta)$

However, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, v = $\tan^{-1}(\tan 2\theta) = 2\theta$

⇒ v = $2\tan^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\tan^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x)$$

We know $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{1+x^2}$$

$$\therefore \frac{dv}{dx} = \frac{2}{1+x^2}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2}{1+x^2} \times \frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

16. Question

Differentiate $\cos^{-1}(4x^3 - 3x)$ with respect to $\tan^{-1}\left(\frac{1-x^2}{x}\right)$, if $\frac{1}{2} < x < 1$.

Answer

Let u = $\cos^{-1}(4x^3 - 3x)$ and $v = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have u = $\cos^{-1}(4x^3 - 3x)$

By substituting x = cos θ, we have

$u = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

But, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow u = \cos^{-1}(\cos 3\theta)$

Given, $\frac{1}{2} < x < 1 \Rightarrow x \in \left(\frac{1}{2}, 1\right)$

However, $x = \cos\theta$

$\Rightarrow \cos\theta \in \left(\frac{1}{2}, 1\right)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{3}\right)$

$\Rightarrow 3\theta \in (0, \pi)$

Hence, $u = \cos^{-1}(\cos 3\theta) = 3\theta$

$\Rightarrow u = 3\cos^{-1}x$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}(3\cos^{-1}x)$

$\Rightarrow \frac{du}{dx} = 3\frac{d}{dx}(\cos^{-1}x)$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{du}{dx} = 3\left(-\frac{1}{\sqrt{1-x^2}}\right)$

$\therefore \frac{du}{dx} = -\frac{3}{\sqrt{1-x^2}}$

Now, we have $v = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

By substituting $x = \cos\theta$, we have

$v = \tan^{-1}\left(\frac{\sqrt{1-(\cos\theta)^2}}{\cos\theta}\right)$

$\Rightarrow v = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$

$\Rightarrow v = \tan^{-1}\left(\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow v = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$\Rightarrow v = \tan^{-1}(\tan\theta)$

However, $\theta \in \left(0, \frac{\pi}{3}\right)$

Hence, $v = \tan^{-1}(\tan\theta) = \theta$

$\Rightarrow v = \cos^{-1}x$

On differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1} x)$

We know $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$\Rightarrow \frac{du}{dv} = \frac{-\frac{3}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$

$\Rightarrow \frac{du}{dv} = -\frac{3}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-1}$

$\therefore \frac{du}{dv} = 3$

Thus, $\frac{du}{dv} = 3$

17. Question

Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$, if $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

Answer

Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and $v = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

By substituting x = sin θ, we have

$u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-(\sin\theta)^2}}\right)$

$\Rightarrow u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$\Rightarrow u = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$\Rightarrow u = \tan^{-1}(\tan\theta)$

Given $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \Rightarrow x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

However, x = sin θ

$\Rightarrow \sin\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Hence, $u = \tan^{-1}(\tan\theta) = \theta$

$\Rightarrow u = \sin^{-1}x$

On differentiating u with respect to x, we get

$$\frac{du}{dx} = \frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Now, we have $v = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

By substituting $x = \sin\theta$, we have

$$v = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$$

$$\Rightarrow v = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$\Rightarrow v = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow v = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow v = \sin^{-1}(\sin 2\theta)$

However, $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $v = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow v = 2\sin^{-1}(x)$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(2\sin^{-1}x)$$

$$\Rightarrow \frac{dv}{dx} = 2\frac{d}{dx}(\sin^{-1}x)$$

We know $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \frac{dv}{dx} = 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}}$$

$\Rightarrow \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2}$

$\therefore \frac{du}{dv} = \frac{1}{2}$

Thus, $\frac{du}{dv} = \frac{1}{2}$

18. Question

Differentiate $\sin^{-1}\sqrt{1-x^2}$ with respect to $\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if 0 < x < 1.

Answer

Let $u = \sin^{-1}\sqrt{1-x^2}$ and $v = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\sqrt{1-x^2}$

By substituting x = cos θ, we have

$u = \sin^{-1}\sqrt{1-(\cos\theta)^2}$

$\Rightarrow u = \sin^{-1}\sqrt{1-\cos^2\theta}$

$\Rightarrow u = \sin^{-1}\sqrt{\sin^2\theta}$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(\sin\theta)$

Given, $0 < x < 1 \Rightarrow x \in (0, 1)$

However, x = cos θ

$\Rightarrow \cos\theta \in (0, 1)$

$\Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin\theta) = \theta$

$\Rightarrow u = \cos^{-1}x$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1}x)$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$\therefore \frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$

Now, we have $v = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

By substituting x = cos θ, we have

$v = \cot^{-1}\left(\frac{\cos\theta}{\sqrt{1-(\cos\theta)^2}}\right)$

$\Rightarrow v = \cot^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)$

$\Rightarrow v = \cot^{-1}\left(\frac{\cos\theta}{\sqrt{\sin^2\theta}}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow v = \cot^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$

$\Rightarrow v = \cot^{-1}(\cot\theta)$

However, $\theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $v = \cot^{-1}(\cot\theta) = \theta$

$\Rightarrow v = \cos^{-1}x$

On differentiating v with respect to x, we get

$$\frac{dv}{dx} = \frac{d}{dx}(\cos^{-1}x)$$

We know $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv} = -\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-1}$$

$$\therefore \frac{du}{dv} = 1$$

Thus, $\frac{du}{dv} = 1$

19. Question

Differentiate $\sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$ with respect to $\sqrt{1-a^2x^2}$, if $-\frac{1}{\sqrt{2}} < ax < \frac{1}{\sqrt{2}}$.

Answer

Let $u = \sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$ and $v = \sqrt{1-a^2x^2}$.

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2ax\sqrt{1-(ax)^2}\right)$

By substituting ax = sin θ, we have

$u = \sin^{-1}\left(2\sin\theta\sqrt{1-(\sin\theta)^2}\right)$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow u = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$ [$\because \sin^2\theta + \cos^2\theta = 1$]

$\Rightarrow u = \sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow u = \sin^{-1}(\sin 2\theta)$

Given $-\frac{1}{\sqrt{2}} < ax < \frac{1}{\sqrt{2}} \Rightarrow ax \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

However, $ax = \sin\theta$

$\Rightarrow \sin\theta \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\Rightarrow 2\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $u = \sin^{-1}(\sin 2\theta) = 2\theta$.

$\Rightarrow u = 2\sin^{-1}(ax)$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}(2\sin^{-1} ax)$

$\Rightarrow \frac{du}{dx} = 2\frac{d}{dx}(\sin^{-1} ax)$

We know $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \frac{du}{dx} = 2 \times \frac{1}{\sqrt{1-(ax)^2}}\frac{d}{dx}(ax)$

$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-a^2x^2}}\left[a\frac{d}{dx}(x)\right]$

$\Rightarrow \frac{du}{dx} = \frac{2a}{\sqrt{1-a^2x^2}}\frac{d}{dx}(x)$

We know $\frac{d}{dx}(x) = 1$

$\Rightarrow \frac{du}{dx} = \frac{2a}{\sqrt{1-a^2x^2}} \times 1$

$\therefore \frac{du}{dx} = \frac{2a}{\sqrt{1-a^2x^2}}$

Now, we have $v = \sqrt{1-a^2x^2}$

On differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1-a^2x^2}\right)$

$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1-a^2x^2)^{\frac{1}{2}}$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 - a^2x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1 - a^2x^2)$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 - a^2x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(a^2x^2)\right]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 - a^2x^2}}\left[\frac{d}{dx}(1) - a^2\frac{d}{dx}(x^2)\right]$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 - a^2x^2}}[0 - a^2(2x^{2-1})]$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 - a^2x^2}}[-2a^2x]$$

$$\therefore \frac{dv}{dx} = -\frac{a^2x}{\sqrt{1 - a^2x^2}}$$

We have $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{2a}{\sqrt{1 - a^2x^2}}}{-\frac{a^2x}{\sqrt{1 - a^2x^2}}}$$

$$\Rightarrow \frac{du}{dv} = \frac{2a}{\sqrt{1 - a^2x^2}} \times \frac{\sqrt{1 - a^2x^2}}{-a^2x}$$

$$\therefore \frac{du}{dv} = -\frac{2}{ax}$$

Thus, $\frac{du}{dv} = -\frac{2}{ax}$

20. Question

Differentiate $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ with respect to $\sqrt{1-x^2}$, if $-1 < x < 1$.

Answer

Let $u = \tan^{-1}\left(\frac{1-x}{1+x}\right)$ and $v = \sqrt{1 - x^2}$

We need to differentiate u with respect to v that is find $\frac{du}{dv}$.

We have $u = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

By substituting x = tan θ, we have

$$u = \tan^{-1}\left(\frac{1 - \tan\theta}{1 + \tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) \left[\because \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

Given, $-1 < x < 1 \Rightarrow x \in (-1, 1)$

However, $x = \tan\theta$

$\Rightarrow \tan\theta \in (-1, 1)$

$\Rightarrow \theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$\Rightarrow \frac{\pi}{4} - \theta \in \left(0, \frac{\pi}{2}\right)$

Hence, $u = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \frac{\pi}{4} - \theta$

$\Rightarrow u = \frac{\pi}{4} - \tan^{-1} x$

On differentiating u with respect to x, we get

$\frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \tan^{-1} x\right)$

$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{d}{dx}(\tan^{-1} x)$

We know $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ and derivative of a constant is 0.

$\Rightarrow \frac{du}{dx} = 0 - \frac{1}{1 + x^2}$

$\therefore \frac{du}{dx} = -\frac{1}{1 + x^2}$

Now, we have $v = \sqrt{1 - x^2}$

On differentiating v with respect to x, we get

$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1 - x^2}\right)$

$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}(1 - x^2)^{\frac{1}{2}}$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 - x^2)^{\frac{1}{2}-1}\frac{d}{dx}(1 - x^2)$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(x^2)\right]$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 - x^2}}\left[\frac{d}{dx}(1) - \frac{d}{dx}(x^2)\right]$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$ and derivative of a constant is 0.

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 - x^2}}[0 - 2x^{2-1}]$

$\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{1 - x^2}}[-2x]$

$\therefore \frac{dv}{dx} = -\frac{x}{\sqrt{1 - x^2}}$

We have $\frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}$

$$\Rightarrow \frac{du}{dv}=\frac{-\frac{1}{1+x^2}}{-\frac{x}{\sqrt{1-x^2}}}$$

$$\Rightarrow \frac{du}{dv}=-\frac{1}{1+x^2}\times\frac{\sqrt{1-x^2}}{-x}$$

$$\therefore \frac{du}{dv}=\frac{\sqrt{1-x^2}}{x(1+x^2)}$$

Thus, $\frac{du}{dv}=\frac{\sqrt{1-x^2}}{x(1+x^2)}$

MCQ

1. Question

Choose the correct alternative in the following:

If f(x) = $\log_x 2$ (log x), then f′(x) at x = e is

A. 0

B. 1

C. 1/e

D 1/2e

Answer

f(x) = $\log_x 2$ (log x)

Changing the base, we get

$$\Longrightarrow f(x)=\frac{\log(\log x)}{\log x^2}$$

$$\because\ \log_b a=\frac{\log a}{\log b}$$

$$\Longrightarrow f(x)=\frac{\log(\log x)}{2.\log x}$$

$$\text{So}, f'(x)=\frac{1}{2}\left\{\frac{1}{\log x}\left[\frac{d}{dx}\{\log(\log x)\}\right]+\log(\log x)\left[\frac{d}{dx}\left\{\frac{1}{\log x}\right\}\right]\right\}$$

$$\Longrightarrow f'(x)=\frac{1}{2}\left\{\frac{1}{\log x}\left[\frac{1}{\log x}.\frac{1}{x}\right]+\log(\log x)\left[-\left(\frac{1}{\log x}\right)^2.\frac{1}{x}\right]\right\}$$

Putting x = e, we get

$$\Longrightarrow f'(e)=\frac{1}{2}\left\{\frac{1}{\log e}\left[\frac{1}{\log e}.\frac{1}{e}\right]+\log(\log e)\left[-\left(\frac{1}{\log e}\right)^2.\frac{1}{e}\right]\right\}$$

$$\Longrightarrow f'(e)=\frac{1}{2}\left\{\left[\frac{1}{(\log e)^2}.\frac{1}{e}\right]+\log(\log e)\left[-\left(\frac{1}{\log e}\right)^2.\frac{1}{e}\right]\right\}$$

$\Longrightarrow f'(e)=\frac{1}{2}\left\{\left[\frac{1}{1^2}.\frac{1}{e}\right]+\log(1)\left[-\left(\frac{1}{1}\right)^2.\frac{1}{e}\right]\right\}$ ($\because$ log e = 1)

$\Rightarrow f'(e) = \frac{1}{2}\left\{\left[\frac{1}{1^2}\cdot\frac{1}{e}\right] + 0.\left[-\left(\frac{1}{1}\right)^2\cdot\frac{1}{e}\right]\right\}$ ($\because$ log 1 = 0

$\therefore f'(e) = \frac{1}{2e}$

2. Question

Choose the correct alternative in the following:

The differential coefficient of f(log x) with respect to x, where f(x) = log x is

A. $\frac{x}{\log x}$

B. $\frac{\log x}{x}$

C. $(x \log x)^{-1}$

D. none of these

Answer

Given: f(x) = log x

$\therefore$ f(log x) = log(log x)

$f'(\log x) = \frac{d}{dx}\log(\log x)$

$f'(\log x) = \frac{1}{\log x}\cdot\frac{1}{x} = \frac{1}{x\log x}$

$\therefore f'(\log x) = (x \log x)^{-1}$

3. Question

Choose the correct alternative in the following:

The derivative of the function $\cos^{-1}\left\{(\cos 2x)^{1/2}\right\}$ at x = π/6 is

A. $(2/3)^{1/2}$

B. $(1/3)^{1/2}$

C. $3^{1/2}$

D. $6^{1/2}$

Answer

$$f'(x) = -\frac{1}{\sqrt{1-\left[(\cos 2x)^{\frac{1}{2}}\right]^2}}\cdot\frac{1}{2\sqrt{\cos 2x}}\cdot(-\sin 2x)\cdot 2$$

$$= \frac{1}{\sqrt{1-\cos 2x}}\cdot\frac{1}{\sqrt{\cos 2x}}\cdot(\sin 2x)$$

Putting x = π/6, we get

$$= \frac{1}{\sqrt{1-\left(\cos\frac{2\pi}{6}\right)}}\cdot\frac{1}{\sqrt{\cos\frac{2\pi}{6}}}\cdot\left(\sin\frac{2\pi}{6}\right)$$

$$= \frac{1}{\sqrt{1-\cos\frac{\pi}{3}}} \cdot \frac{1}{\sqrt{\cos\frac{\pi}{3}}} \cdot \left(\sin\frac{\pi}{3}\right)$$

$$= \frac{1}{\sqrt{1-\frac{1}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{2}}} \cdot \left(\frac{\sqrt{3}}{2}\right)$$

Simplifying above we get

$$= \sqrt{2} \,.\, \sqrt{2} \,.\left(\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{2} \,.\, \sqrt{2} \,.\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$\therefore f'(x) = \sqrt{3} = (3)^{1/2}$

4. Question

Choose the correct alternative in the following:

Differential coefficient of sec ($\tan^{-1}$ x) is

A. $\frac{x}{1+x^2}$

B. $x\sqrt{1+x^2}$

C. $\frac{1}{\sqrt{1+x^2}}$

D. $\frac{x}{\sqrt{1+x^2}}$

Answer

Let f(x) = sec ($\tan^{-1}$ x)

Let θ = $\tan^{-1}$x

$\frac{d\theta}{dx} = \frac{1}{1+x^2}$ --(1)

$$f'(x) = \frac{d}{dx}(\sec\theta).\frac{d\theta}{dx}$$

$= \sec\theta.\tan\theta.\frac{1}{1+x^2}$ -- From (1)

Now θ = $\tan^{-1}$x

= x = tan θ

$= \sqrt{1+x^2} = \sec\theta$ ∵ $\sec^2\theta - \tan^2\theta = 1$

Putting values, we get

$$= \sec\theta.\tan\theta.\frac{1}{1+x^2}$$

$$= \sqrt{1+x^2}.x.\frac{1}{(1+x^2)}$$

$$\therefore f'(x) = \frac{x}{\sqrt{1+x^2}}$$

5. Question

Choose the correct alternative in the following:

If f(x) = $\tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}$, 0 ≤ x ≤ π/2, then f' (π/6) is

A. –1/4

B. –1/2

C. 1/4

D. 1/2

Answer

$$f(x) = \tan^{-1}\sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$= \tan^{-1}\sqrt{\frac{1+2.\sin\frac{x}{2}\cos\frac{x}{2}}{1-2.\sin\frac{x}{2}\cos\frac{x}{2}}}$$

∵ sin2x = 2 sin x cos x

⇒ sin x = 2 sin x/2 cos x/2

$$= \tan^{-1}\sqrt{\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2.\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2.\sin\frac{x}{2}\cos\frac{x}{2}}}$$

∵ $\sin^2 x/2 + \cos^2 x/2 = 1$

$$= \tan^{-1}\sqrt{\frac{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}-\cos\frac{x}{2}}\right)$$

Dividing by cos x/2 we get

$$= \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}-1}\right) = -\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{1-\tan\frac{x}{2}}\right)$$ Taking – common

$$= -\tan^{-1}\left(\frac{\tan\frac{x}{2}+\tan\frac{\pi}{4}}{1-\tan\frac{x}{2}.\tan\frac{\pi}{4}}\right)$$

$$= -\tan^{-1}\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$$

$$\because \tan(A-B) = \frac{\tan A + \tan B}{1-\tan A.\tan B}$$

$$= \tan^{-1}\left[\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right]$$ ∵ 0 ≤ x ≤ π/2

$$\therefore f(x) = \left(\frac{x}{2}+\frac{\pi}{4}\right)$$

$f'\left(\frac{\pi}{6}\right) = \frac{1}{2} = (D)$

6. Question

Choose the correct alternative in the following:

If $y = \left(1+\frac{1}{x}\right)^x$, then $\frac{dy}{dx} =$

A. $\left(1+\frac{1}{x}\right)^x \left\{\log\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right\}$

B. $\left(1+\frac{1}{x}\right)^x \log\left(1+\frac{1}{x}\right)$

C. $\left(1+\frac{1}{x}\right)^x \left\{\log(x+1) - \frac{x}{x+1}\right\}$

D. $\left(1+\frac{1}{x}\right)^x \left\{\log\left(x+\frac{1}{x}\right) + \frac{1}{x+1}\right\}$

Answer

Given $y = \left(1+\frac{1}{x}\right)^x$

Taking log both sides we get

$\Rightarrow \log y = \log\left(1+\frac{1}{x}\right)^x$

$\Rightarrow \log y = x.\log\left(1+\frac{1}{x}\right)$

Differentiating w.r.t x we get,

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1.\log\left(1+\frac{1}{x}\right) + \frac{1}{1+\frac{1}{x}}.\left(-\frac{1}{x^2}\right).x$

$\Rightarrow \frac{dy}{dx} = y\left(\log\left(1+\frac{1}{x}\right) + \frac{x}{x+1}.\left(-\frac{1}{x}\right)\right)$

Putting value of y, we get

$\Rightarrow \frac{dy}{dx} = \left(1+\frac{1}{x}\right)^x \left(\log\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right)$

7. Question

Choose the correct alternative in the following:

If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is

A. $\frac{1+x}{1+\log x}$

B. $\frac{1-\log x}{1+\log x}$

C. not defined

D. $\frac{\log x}{(1+\log x)^2}$

Answer

$x^y = e^{x-y}$

Taking log both sides we get

$\log x^y = \log e^{x-y}$

y log x = (x-y) loge

y log x = (x-y) ∵ loge = 1

$y = \frac{x}{\log x + 1}$

Differentiating w.r.t x we get,

$$\frac{dy}{dx} = \frac{1.(\log x + 1) - \frac{1}{x}.x}{(\log x + 1)^2}$$

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$$

8. Question

Choose the correct alternative in the following:

Given $f(x) = 4x^8$, then

A. $f'\left(\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)$

B. $f\left(\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)$

C. $f\left(-\frac{1}{2}\right) = f\left(-\frac{1}{2}\right)$

D. $f\left(\frac{1}{2}\right) = f'\left(-\frac{1}{2}\right)$

Answer

$f(x) = 4x^8$

$f'(x) = 32x^7$

Consider option (A)

$$f'\left(\frac{1}{2}\right) = 32\left(\frac{1}{2}\right)^7 = 32\left(\frac{1}{128}\right) = 4$$

$$f'\left(-\frac{1}{2}\right) = 32\left(-\frac{1}{2}\right)^7 = 32\left(-\frac{1}{128}\right) = -4$$

$$f'\left(\frac{1}{2}\right) \neq f'\left(-\frac{1}{2}\right)$$

Consider option (B)

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^8 = 4\left(\frac{1}{256}\right) = 64$$

$$f'\left(-\frac{1}{2}\right) = 32\left(-\frac{1}{2}\right)^7 = 32\left(-\frac{1}{128}\right) = -4$$

$$f\left(\frac{1}{2}\right) \neq f'\left(-\frac{1}{2}\right)$$

Consider option (C)

$$f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^8 = 4\left(\frac{1}{256}\right) = 64$$

$$f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^8 = 4\left(\frac{1}{256}\right) = 64$$

$$\therefore f\left(-\frac{1}{2}\right) = f\left(-\frac{1}{2}\right) = (C)$$

Consider option (D)

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^8 = 4\left(\frac{1}{256}\right) = 64$$

$$f'\left(-\frac{1}{2}\right) = 32\left(-\frac{1}{2}\right)^7 = 32\left(-\frac{1}{128}\right) = -4$$

$$f\left(\frac{1}{2}\right) \neq f'\left(-\frac{1}{2}\right)$$

9. Question

Choose the correct alternative in the following:

If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then $\sqrt{1+\left(\frac{dy}{dx}\right)^2} =$

A. $\tan^2\theta$

B. $\sec^2\theta$

C. $\sec\theta$

D. $|\sec\theta|$

Answer

We are given that

$$x = a.\cos^3\theta\ , y = a.\sin^3\theta$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = ?$$

Now, we know

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Now,

$$\frac{dx}{d\theta} = \frac{d}{d\theta} a.\cos^3\theta$$

$= -3a\cos^2\theta\sin\theta$ (Using Chain Rule)

Again

$$\frac{dy}{d\theta} = \frac{d}{d\theta} a.\sin^3\theta$$

$= 3a\sin^2\theta\cos\theta$ (Using Chain Rule)

Now, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$

By Simplifying we get,

$$\frac{dy}{dx} = -\tan\theta$$

$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+(-\tan\theta)^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta}$$

$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = |\sec\theta| = (D)$$

10. Question

Choose the correct alternative in the following:

If

A. $-\frac{2}{1+x^2}$

B. $\frac{2}{1+x^2}$

C. $\frac{1}{2-x^2}$

D. $\frac{2}{2-x^2}$

Answer

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put x = tan θ ⇒ θ = $\tan^{-1}$x

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$y = \sin^{-1}(\cos 2\theta) \quad \because \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

$$y = \sin^{-1}\left(\sin\left(\frac{\pi}{2}-2\theta\right)\right)$$

$y=\frac{\pi}{2}-2\theta$

Putting value of θ we get,

$y=\frac{\pi}{2}-2\tan^{-1}x$

Differentiating w.r.t x we get,

$\frac{dy}{dx}=0-2\left(\frac{1}{1+x^2}\right)$

$\because \frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}$

$\therefore \frac{dy}{dx}=-\frac{2}{1+x^2}=(A)$

11. Question

Choose the correct alternative in the following:

The derivative of $\sec^{-1}\left(\frac{1}{2x^2+1}\right)$ with respect to $\sqrt{1+3x}$ at x = -1/3

A. does not exist

B. 0

C. 1/2

D. 1/3

Answer

$\text{Let } u=\sec^{-1}\left(\frac{1}{2x^2+1}\right) \text{ and } v=\sqrt{1+3x}$

$\left(\frac{du}{dv}\right)_{x=-\frac{1}{3}}=?$

Considering u,

$u=\sec^{-1}\left(\frac{1}{2x^2+1}\right)$

Put x = cos θ

θ = $\cos^{-1}$x ----(1)

$u=\sec^{-1}\left(\frac{1}{2\cos^2\theta+1}\right)=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right) \because 2\cos^2\theta+1=\cos 2\theta$

$=\sec^{-1}(\sec 2\theta)=2\theta$

$\Rightarrow u=2\cos^{-1}x$ From (1)

Differentiating w.r.t x

$\Rightarrow \frac{du}{dx}=-\frac{2}{\sqrt{1-x^2}}$

Considering v,

$v=\sqrt{1+3x}$

Differentiating w.r.t x

$$\Rightarrow \frac{dv}{dx} = \frac{3}{2\sqrt{1+3x}}$$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{du}{dx} \cdot \frac{dx}{dv}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2}{\sqrt{1-x^2}} \cdot \left(\frac{2\sqrt{1+3x}}{3}\right)$$

$$\Rightarrow \frac{du}{dv} = -\frac{4}{3} \cdot \left(\sqrt{\frac{1+3x}{1-x^2}}\right)$$

$$\Rightarrow \left(\frac{du}{dv}\right)_{x=-\frac{1}{3}} = -\frac{4}{3} \cdot \left(\sqrt{\frac{1+3\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)^2}}\right)$$

$$\Rightarrow \left(\frac{du}{dv}\right)_{x=-\frac{1}{3}} = 0 = (B)$$

12. Question

Choose the correct alternative in the following:

For the curve . $\sqrt{x} + \sqrt{y} = 1, \frac{dy}{dx}$. at (1/4, 1/4) is

A. 1/2

B. 1

C. –1

D. 2

Answer

$\sqrt{x} + \sqrt{y} = 1$

Differentiating w.r.t x we get,

$$\Rightarrow \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(\sqrt{y}) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0 \ \because \frac{d}{dx}(x^n) = n.x^{n-1}$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x,y)=\left(\frac{1}{4},\frac{1}{4}\right)} = -\sqrt{\frac{\frac{1}{4}}{\frac{1}{4}}} = -1$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x,y)=\left(\frac{1}{4},\frac{1}{4}\right)} = -1 = (C)$$

13. Question

Choose the correct alternative in the following:

If sin (x + y) = log (x + y), then $\frac{dy}{dx}=$

A. 2

B. –2

C. 1

D. –1

Answer

sin (x + y) = log (x + y)

Differentiating w.r.t x we get,

$$\Rightarrow \cos(x+y).\left(1+\frac{dy}{dx}\right)=\frac{1}{x+y}.\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \cos(x+y).\left(1+\frac{dy}{dx}\right)-\frac{1}{x+y}.\left(1+\frac{dy}{dx}\right)=0$$

$$\Rightarrow \left(1+\frac{dy}{dx}\right)\left(\cos(x+y)-\frac{1}{x+y}\right)=0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)\left(\cos(x+y)-\frac{1}{x+y}\right)+\left(\cos(x+y)-\frac{1}{x+y}\right)=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\left(\cos(x+y)-\frac{1}{x+y}\right)}{\left(\cos(x+y)-\frac{1}{x+y}\right)}=-1$$

$$\Rightarrow \frac{dy}{dx}=-1$$

14. Question

Choose the correct alternative in the following:

Let $U=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $V=\tan^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dU}{dV}=$

A. 1/2

B. x

C. $\frac{1-x^2}{1+x^2}$

D. 1

Answer

We are given that

$$U=\sin^{-1}\left(\frac{2x}{1+x^2}\right),V=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$\frac{dU}{dV}=?$$

Now, we know

$$\frac{dU}{dV} = \frac{\frac{dU}{dx}}{\frac{dV}{dx}}$$

Now,

$$\frac{dU}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Put x = tan θ

θ = $\tan^{-1}$x ----(1)

$$\Rightarrow \frac{dU}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \frac{d}{dx}\sin^{-1}(\sin 2\theta) \ \because \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$$

$$= \frac{d}{dx}2\theta$$

$$= \frac{d}{dx}2\tan^{-1}x = \frac{2}{1+x^2}$$

Again

$$\frac{dV}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Put x = tan θ

θ = $\tan^{-1}$x ----(1)

$$\Rightarrow \frac{dV}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \frac{d}{dx}\tan^{-1}(\tan 2\theta) \ \because \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta$$

$$= \frac{d}{dx}2\theta$$

$= \frac{d}{dx}2\tan^{-1}x = \frac{2}{1+x^2}$ -- From (1)

Now, $\frac{dU}{dV} = \frac{\frac{dU}{dx}}{\frac{dV}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1$

$$\therefore \frac{dU}{dV} = 1 = (D)$$

15. Question

Choose the correct alternative in the following:

$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right\}$ equals

A. 1/2

B. –1/2

C. 1

D. –1

Answer

$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right\}$

$\Rightarrow \frac{d}{dx}\left\{\tan^{-1}\left(\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}\right)\right\} \because \sin\left(\frac{\pi}{2}-x\right) = \cos x$ and $\cos\left(\frac{\pi}{2}-x\right)$
$= \sin x$

Put $\frac{\pi}{2} - x = 2t \Rightarrow t = \frac{\pi}{4} - \frac{x}{2}$ --(1)

$\Rightarrow \frac{d}{dx}\left\{\tan^{-1}\left(\frac{\sin 2t}{1+\cos 2t}\right)\right\}$

$\Rightarrow \frac{d}{dx}\left\{\tan^{-1}\left(\frac{2\sin t.\cos t}{2\cos^2 t}\right)\right\}$ ∵ sin2t = 2sint.cost and 1+cos2t = 2cos^{2}t

$\Rightarrow \frac{d}{dx}\{\tan^{-1}(\tan t)\}$

$\Rightarrow \frac{d}{dx}\{t\} \Rightarrow \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right) = -\frac{1}{2}$ --From (1)

$\therefore \frac{d}{dx}\left\{\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right\} = -\frac{1}{2}$

16. Question

Choose the correct alternative in the following:

$\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ equals

A. $\frac{x^2-1}{x^2-4}$

B. 1

C. $\frac{x^2+1}{x^2-4}$

D. $e^x \frac{x^2-1}{x^2-4}$

Answer

$\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}\right]$

Let $u = \frac{x-2}{x+2} \Rightarrow \frac{du}{dx} = \frac{1.(x+2)-(x-2).1}{(x+2)^2} = \frac{4}{(x+2)^2}$ -----(1)

$\Rightarrow \frac{d}{dx}\left[\log\left\{e^x(u)^{\frac{3}{4}}\right\}\right]$

$\Rightarrow \frac{d}{dx}\left[\log e^x + \log(u)^{\frac{3}{4}}\right]$

$\Rightarrow \frac{d}{dx}\left[x.\log e + \frac{3}{4}\log(u)\right]$

$\Rightarrow \frac{d}{dx}\left[x + \frac{3}{4}\log(u)\right]$ ∵ log e = 1

$\Rightarrow 1+\frac{3}{4}.\frac{1}{u}.\frac{du}{dx}$

$\Rightarrow 1+\frac{3}{4}.\frac{(x+2)}{x-2}.\frac{4}{(x+2)^2}$ --From (1)

$\Rightarrow 1+\frac{3}{(x^2-2^2)}$

$\Rightarrow \frac{(x^2-4)+3}{(x^2-4)}$

$\therefore \frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}\right] = \frac{x^2-1}{x^2-4} = (A)$

17. Question

Choose the correct alternative in the following:

If $y=\sqrt{\sin x+y}$, then $\frac{dy}{dx} =$

A. $\frac{\sin x}{2y-1}$

B. $\frac{\sin x}{1-2y}$

C. $\frac{\cos x}{1-2y}$

D. $\frac{\cos x}{2y-1}$

Answer

$y=\sqrt{\sin x+y}$

Squaring both sides

$\Rightarrow y^2 = \sin x + y$

Differentiating w.r.t x we get,

$\Rightarrow 2y.\frac{dy}{dx} = \cos x + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(2y-1) = \cos x$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1} = D$

18. Question

Choose the correct alternative in the following:

If 3 sin(xy) + 4cos (xy) = 5, then $\frac{dy}{dx} =$

A. $-\frac{y}{x}$

B. $\dfrac{3\sin(xy)+4\cos(xy)}{3\cos(xy)-4\sin(xy)}$

C. $\dfrac{3\cos(xy)+4\sin(xy)}{4\cos(xy)-3\sin(xy)}$

D. none of these

Answer

3 sin(xy) + 4cos (xy) = 5

Differentiating w.r.t x we get,

$$\Rightarrow 3\left[\cos(xy).\left(1.y+x.\frac{dy}{dx}\right)\right]+4\left[-\sin(xy).\left(1.y+x.\frac{dy}{dx}\right)\right]=0$$

(Using Chain Rule)

$$\Rightarrow \left[3y\cos(xy)+3x\cos(xy).\frac{dy}{dx}\right]+\left[-4y\sin(xy)-4x\sin(xy).\frac{dy}{dx}\right]=0$$

$$\Rightarrow \frac{dy}{dx}[3x\cos(xy)-4x\sin(xy)]=4y\sin(xy)-3y\cos(xy)$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y[-4\sin(xy)+3\cos(xy)]}{x[3\cos(xy)-4\sin(xy)]}=-\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{y}{x}=(A)$$

19. Question

Choose the correct alternative in the following:

If sin y = x sin (a + y), then $\dfrac{dy}{dx}$ is

A. $\dfrac{\sin a}{\sin a\sin^2(a+y)}$

B. $\dfrac{\sin^2(a+y)}{\sin a}$

C. sin a $\sin^2$ (a + y)

D. $\dfrac{\sin^2(a-y)}{\sin a}$

Answer

sin y = x sin (a + y)

$$\Rightarrow \frac{\sin y}{\sin(a+y)}=x$$

Differentiating w.r.t y we get,

$$\Rightarrow \frac{dx}{dy}=\frac{d}{dy}\left(\frac{\sin y}{\sin(a+y)}\right)$$

$$=\frac{\cos y\,(\sin(a+y))-\cos(a+y)\,.\sin y}{[\sin(a+y)]^2}$$

$$=\frac{\cos y(\sin a\cos y+\cos a\sin y)-(\cos a\cos y-\sin a\sin y)\sin y}{[\sin(a+y)]^2}$$

$$=\frac{\sin a\cos^2 y+\cos a\cos y\sin y-\sin y\cos a\cos y+\sin a\sin^2 y}{[\sin(a+y)]^2}$$

$$=\frac{\sin a(\cos^2 y+\sin^2 y)+\cos a\cos y\sin y-\sin y\cos a\cos y}{[\sin(a+y)]^2}$$

$$\frac{dx}{dy}=\frac{\sin a}{[\sin(a+y)]^2}\ \because \cos^2 y+\sin^2 y=1$$

$$\therefore \frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}=(B)$$

20. Question

Choose the correct alternative in the following:

The derivative of $\cos^{-1}(2x^2 - 1)$ with respect to $\cos^{-1} x$ is

A. 2

B. $\frac{1}{2\sqrt{1-x^2}}$

C. 2/x

D. $1 - x^2$

Answer

Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1} x$

$$\frac{du}{dv}=?$$

Considering $u = \cos^{-1}(2x^2 - 1)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$ ---(1)

$u = \cos^{-1}(2\cos^2\theta - 1)$

$u = \cos^{-1}(\cos 2\theta)$ $\because 2\cos^2\theta - 1 = \cos 2\theta$

$u = 2\theta$

$u = 2\cos^{-1}x$ -- From(1)

Differentiating w.r.t x we get,

$$\Rightarrow \frac{du}{dx}=-\frac{2}{\sqrt{1-x^2}}$$

Considering $v = \cos^{-1} x$

Differentiating w.r.t x we get,

$$\Rightarrow \frac{dv}{dx}=-\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{du}{dv}=\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{du}{dx}\cdot\frac{dx}{dv}$$

$$\Rightarrow \frac{du}{dv} = -\frac{2}{\sqrt{1-x^2}}\left(-\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{du}{dv} = 2$$

21. Question

Choose the correct alternative in the following:

If $f(x) = \sqrt{x^2 + 6x + 9}$, then f'(x) is equal to

A. 1 for x < –3

B. –1 for x < –3

C. 1 for all xϵ R

D. none of these

Answer

$$f(x) = \sqrt{x^2 + 6x + 9}$$

$$\Rightarrow f(x) = \sqrt{(x+3)^2}$$

$$\Rightarrow f(x) = |x+3|$$

$$\Rightarrow f(x) = \begin{cases} (x+3),x+3 \geq 0 \Leftrightarrow x \geq -3 \\ -(x+3),x+3 < 0 \Leftrightarrow x < -3 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1,x \geq -3 \\ -1,x < -3 \end{cases}$$

22. Question

Choose the correct alternative in the following:

If f(x) = |x^2 – 9x + 20|, then f'(x) is equal to

A. –2x + 9 for all xϵ R

B. 2x – 9 if 4 < x < 5

C. –2x + 9 if 4 < x < 5

D. none of these

Answer

f(x) = |x^2 – 9x + 20|

= |x^2 – 4x – 5x + 20|

= |x(x – 4) – 5(x – 4)|

f(x) = | (x –5) (x – 4) |

$$\Rightarrow f(x) = \begin{cases} (x-5)(x-4),x \geq 5 \text{ and } x \geq 4 \\ -(x-5)(x-4),4 < x < 5 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} (2x-9),x \geq 5 \text{ and } x \geq 4 \\ -2x+9,4 < x < 5 \end{cases}$$

23. Question

Choose the correct alternative in the following:

If $f(x) = \sqrt{x^2 - 10x + 25}$, then the derivative of f(x) in the interval [0, 7] is

A. 1

B. –1

C. 0

D. none of these

Answer

$f(x) = \sqrt{x^2 - 10x + 25}$

$\Rightarrow f(x) = \sqrt{x^2 - (2)(5)x + 5^2}$

$\Rightarrow f(x) = \sqrt{(x-5)^2}$

$\Rightarrow f(x) = |x - 5|$

$\Rightarrow f(x) = \begin{cases} (x-5)\,, x - 5 \geq 0 \Leftrightarrow x \geq 5 \\ -(x-5)\,, x - 5 < 0 \Leftrightarrow x < 5 \end{cases}$

$\Rightarrow f'(x) = \begin{cases} 1\,, x \geq 5 \\ -1\,, x < 5 \end{cases}$

Since there is no fixed value of f’(x) in the interval [0,7], so the answer is (D) none of these

24. Question

Choose the correct alternative in the following:

If f(x) = |x – 3| and g(x) = fof(x), then for x > 10, g’(x) is equal to

A. 1

B. –1

C. 0

D. none of these

Answer

g(x) = fof(x) = f(f(x)) = |f(x) – 3| ∵ f(x) = |x – 3|

= ||x – 3| – 3|

$\because |x - 3| = \begin{cases} (x-3)\,, x > 3 \\ -(x-3)\,, x < 3 \end{cases}$

Since we have given x > 10 then |x – 3| = (x – 3)

∴ g(x) = |(x – 3) – 3| = |x – 6|

$\because |x - 6| = \begin{cases} (x-6)\,, x > 6 \\ -(x-6)\,, x < 6 \end{cases}$

Since we have given x > 10 then |x – 6| = (x – 6)

∴ g(x) = (x – 6)

$g'(x) = \frac{d}{dx}(x-6) = 1 = (A)$

25. Question

Choose the correct alternative in the following:

If $f(x) = \left(\frac{x^l}{x^m}\right)^{l+m}\left(\frac{x^m}{x^n}\right)^{m+n}\left(\frac{x^n}{x^l}\right)^{n+l}$, then f’(x) is equal to

A. 1

B. 0

C. $x^{\ell+m+n}$

D. none of these

Answer

$$f(x)=\left(\frac{x^l}{x^m}\right)^{l+m}\left(\frac{x^m}{x^n}\right)^{m+n}\left(\frac{x^n}{x^l}\right)^{n+l}$$

$$f(x)=\frac{\left(x^l\right)^{l+m}.\left(x^m\right)^{m+n}.\left(x^n\right)^{n+l}}{\left(x^m\right)^{l+m}.\left(x^n\right)^{m+n}.\left(x^l\right)^{n+l}}$$

$$=\frac{(x)^{l^2+m}.(x)^{m^2+n}.(x)^{n^2+l}}{(x)^{l+m^2}.(x)^{m+n^2}.(x)^{n+l^2}}$$

$$\Rightarrow f(x)=\frac{(x)^{l^2+m^2+n^2+m+n+l}}{(x)^{l^2+m^2+n^2+m+n+l}}=1$$

Differentiating w.r.t x

� �$\Rightarrow \frac{dy}{dx}=0$

26. Question

Choose the correct alternative in the following:

If, $y=\frac{1}{1+x^{a-b}+x^{c-b}}+\frac{1}{1+x^{b-c}+x^{a-c}}\cdot+\frac{1}{1+x^{b-a}+x^{c-a}}$,. then $\frac{dy}{dx}$ is equal to

A. 1

B. $(a+b-c)^{x^{a+b+c-1}}$

C. 0

D. none of these

Answer

$$y=\frac{1}{1+x^{a-b}+x^{c-b}}+\frac{1}{1+x^{b-c}+x^{a-c}}+\frac{1}{1+x^{b-a}+x^{c-a}}$$

$$\Rightarrow y=\frac{1}{1+\frac{x^a}{x^b}+\frac{x^c}{x^b}}+\frac{1}{1+\frac{x^b}{x^c}+\frac{x^a}{x^c}}+\frac{1}{1+\frac{x^b}{x^a}+\frac{x^c}{x^a}}$$

$$\Rightarrow y=\frac{x^b}{x^b+x^a+x^c}+\frac{x^c}{x^c+x^b+x^a}+\frac{x^a}{x^a+x^b+x^c}$$

$$\Rightarrow y=\frac{x^a+x^b+x^c}{x^a+x^b+x^c}=1$$

Differentiating w.r.t x

$$\Rightarrow \frac{dy}{dx}=0$$

27. Question

Choose the correct alternative in the following:

If $\sqrt{1-x^6}+\sqrt{1-y^6}=a^3\left(x^3-y^3\right)$, then $\frac{dy}{dx}$ is equal to

A. $\frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$

B. $\frac{y^2}{x^2}\sqrt{\frac{1-y^6}{1-x^6}}$

C. $\frac{x^2}{y^2}\sqrt{\frac{1-x^6}{1-y^6}}$

D. none of these

Answer

$\sqrt{1-x^6}+\sqrt{1-y^6}=a^3(x^3-y^3)$

Let x^3 = cos p and y^3 = cos q

$\cos^{-1}x^3$ = p and $\cos^{-1}y^3$ = q --- (1)

$\Rightarrow \sqrt{1-\cos^2 p}+\sqrt{1-\cos^2 q}=a^3(\cos p-\cos q)$

$\Rightarrow \sin p+\sin q=a(\cos p-\cos q)$

$\Rightarrow 2\sin\left(\frac{p+q}{2}\right).\cos\left(\frac{p-q}{2}\right)=-2a^3\sin\left(\frac{p-q}{2}\right).\sin\left(\frac{p+q}{2}\right)$

Comparing L.H.S and R.H.S we get,

$\Rightarrow \cos\left(\frac{p-q}{2}\right)=-a^3\sin\left(\frac{p-q}{2}\right)$

$\Rightarrow \frac{\sin\left(\frac{p-q}{2}\right)}{\cos\left(\frac{p-q}{2}\right)}=-\frac{1}{a^3}$

$\Rightarrow \tan\left(\frac{p-q}{2}\right)=-\frac{1}{a^3}$

$\Rightarrow \frac{p-q}{2}=\tan^{-1}\left(-\frac{1}{a^3}\right)$

$\Rightarrow p-q=2.\tan^{-1}\left(-\frac{1}{a^3}\right)$

Substituting value of p and q from (1)

$\Rightarrow \cos^{-1}(x^3)-\cos^{-1}(y^3)=2.\tan^{-1}\left(-\frac{1}{a^3}\right)$

Differentiating w.r.t x we get,

$\Rightarrow -\frac{3x^2}{\sqrt{1-x^6}}-\left(-\frac{3y^2}{\sqrt{1-y^6}}\right).\frac{dy}{dx}=0$

$\Rightarrow \left(\frac{3y^2}{\sqrt{1-y^6}}\right).\frac{dy}{dx}=\frac{3x^2}{\sqrt{1-x^6}}$

Comparing L.H.S and R.H.S we get

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

28. Question

Choose the correct alternative in the following:

If $y = \log\sqrt{\tan x}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is given by

A. ∞

B. 1

C. 0

D. 1/2

Answer

$y = \log\sqrt{\tan x}$

$\Rightarrow y = \log(\tan x)^{\frac{1}{2}}$

$\Rightarrow y = \frac{1}{2}\log(\tan x)$

Differentiating w.r.t x we get,

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}.\frac{1}{\tan x}.(\sec^2 x)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \frac{1}{2}.\frac{1}{\tan\frac{\pi}{4}}.\left(\sec^2\frac{\pi}{4}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \frac{1}{2}.\frac{1}{1}.\left(\sqrt{2}\right)^2$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = 1$$

29. Question

Choose the correct alternative in the following:

If $\sin^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \log a$ then $\frac{dy}{dx}$ is equal to

A. $\frac{x^2-y^2}{x^2+y^2}$

B. $\frac{y}{x}$

C. $\frac{x}{y}$

D. none of these

Answer

$\sin^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \log a$

$\frac{x^2 - y^2}{x^2 + y^2} = \sin(\log a)$

Put y = x tanθ

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$----(1)

$\Rightarrow \frac{x^2 - x^2\tan^2\theta}{x^2 + x^2\tan^2\theta} = \sin(\log a)$

$\Rightarrow \frac{x^2 - x^2\tan^2\theta}{x^2 + x^2\tan^2\theta} = \sin(\log a)$

$\Rightarrow \frac{x^2(1 - \tan^2\theta)}{x^2(1 + \tan^2\theta)} = \sin(\log a)$

$\Rightarrow \cos 2\theta = \sin(\log a)\ \because \frac{(1 - \tan^2\theta)}{(1 + \tan^2\theta)} = \cos 2\theta$

$\Rightarrow 2\theta = \cos^{-1}[\sin(\log a)]$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\cos^{-1}[\sin(\log a)]$

Taking tan on both sides

$\Rightarrow \tan\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \tan\left[\frac{1}{2}\cos^{-1}[\sin(\log a)]\right]$

$\Rightarrow \frac{y}{x} = \tan\left[\frac{1}{2}\cos^{-1}[\sin(\log a)]\right]$

Differentiating w.r.t x we get,

$\Rightarrow \frac{\frac{dy}{dx}.x - y.\frac{dx}{dx}}{x^2} = 0\ \because \tan\left[\frac{1}{2}\cos^{-1}[\sin(\log a)]\right]$ is a constant

$\Rightarrow x.\frac{dy}{dx} - y = 0$

$\therefore \frac{dy}{dx} = \frac{y}{x}$

30. Question

Choose the correct alternative in the following:

If sin y = x cos(a + y), then $\frac{dy}{dx}$ is equal to

A. $\frac{\cos^2(a+y)}{\cos a}$

B. $\frac{\cos a}{\cos^2(a+y)}$

C. $\frac{\sin^2 y}{\cos a}$

D. none of these

Answer

sin y = x cos(a + y)

$x = \frac{\sin y}{\cos(a+y)}$

Differentiating w.r.t y we get,

$\frac{dx}{dy} = \frac{\frac{d\sin y}{dx}.\cos(a+y) - \frac{d\cos(a+y)}{dx}.(\sin y)}{\cos^2(a+y)}$ (Using quotient rule)

$\frac{dx}{dy} = \frac{\cos y.\cos(a+y) - [-\sin(a+y)].(\sin y)}{\cos^2(a+y)}$

$\frac{dx}{dy} = \frac{\cos(a+y).\cos y + \sin(a+y).(\sin y)}{\cos^2(a+y)}$

$\frac{dx}{dy} = \frac{\cos[(a+y)-y]}{\cos^2(a+y)}$ Using cos(a-b) = cos a.cos b + sin a.sin b

$\frac{dx}{dy} = \frac{\cos a}{\cos^2(a+y)}$

$\therefore \frac{dy}{dx} = \frac{\cos^2(a+y)}{\cos a} = (A)$

31. Question

Choose the correct alternative in the following:

If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx} =$

A. $\frac{4x^3}{1-x^4}$

B. $-\frac{4x}{1-x^4}$

C. $\frac{1}{4-x^4}$

D. $\frac{4x^3}{1-x^4}$

Answer

$y = \log\left(\frac{1-x^2}{1+x^2}\right)$

$\frac{dy}{dx} = \frac{1}{\frac{1-x^2}{1+x^2}}\left[\frac{\frac{d(1-x^2)}{dx}.(1+x^2) - \frac{d(1+x^2)}{dx}.(1-x^2)}{(1+x^2)^2}\right]$ (Using quotient rule)

$\frac{dy}{dx} = \frac{1+x^2}{1-x^2}\left[\frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2}\right]$

$\frac{dy}{dx} = \frac{1}{1-x^2}\left[\frac{-2x(1+x^2+1-x^2)}{(1+x^2)}\right]$

$\frac{dy}{dx} = \left[\frac{-4x}{1^2-(x^2)^2}\right]$

$\therefore \frac{dy}{dx} = \left[\frac{-4x}{1-x^4}\right]$

32. Question

Choose the correct alternative in the following:

If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ equals.

A. $\frac{\cos x}{2y-1}$

B. $\frac{\cos x}{1-2y}$

C. $\frac{\sin x}{1-2y}$

D. $\frac{\sin x}{2y-1}$

Answer

$y = \sqrt{\sin x + y}$

Squaring both sides, we get

$y^2 = \sin x + y$

Differentiating w.r.t y we get

$2y = \cos x \frac{dx}{dy} + 1$

$\frac{dx}{dy} = \frac{2y-1}{\cos x}$

$\therefore \frac{dy}{dx} = \frac{\cos x}{2y-1}$

33. Question

Choose the correct alternative in the following:

If $y = \tan^{-1}\left(\frac{\sin x + \cos x}{\cos x - \sin x}\right)$, then $\frac{dy}{dx}$ is equal to

A. $\frac{1}{2}$

B. 0

C. 1

D. none of these

Answer

$y = \tan^{-1}\left(\frac{\sin x+\cos x}{\cos x-\sin x}\right)$

Dividing Numerator and denominator by cos x we get,

$$y = \tan^{-1}\left(\frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\cos x}}{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}\right)$$

$$y = \tan^{-1}\left(\frac{\tan x + 1}{1 - 1.\tan x}\right) = \tan^{-1}\left(\frac{1 + \tan x}{1 - 1.\tan x}\right)$$

$$y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}.\tan x}\right)$$

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + x\right)\right] \because \frac{\tan a + \tan b}{1 - \tan a.\tan b} = \tan(a + b)$$

$$y = \frac{\pi}{4} + x$$

Differentiating w.r.t x we get,

$$\frac{dy}{dx} = 1$$

Very short answer

1. Question

If f(x) = $\log_e(\log_e x)$, then write the value of f’(e).

Answer

f(x) = $\log_e(\log_e x)$

Using the Chain Rule of Differentiation,

$$f'(x) = \frac{1}{\log_e x} \cdot \frac{1}{x}$$

So, $f'(e) = \frac{1}{\log_e e} \cdot \frac{1}{e}$

$= \frac{1}{e}$ (Ans)

2. Question

If f(x) = x + 1, then write the value of

Answer

f(x) = x + 1

$\Longrightarrow$ (fof)(x) = f(x) + 1

= (x + 1) + 1

= x + 2

So, $\frac{d}{dx}(fof)(x) = \frac{d}{dx}(x + 2)$

=1 (Ans)

3. Question

If f’ (1) = 2 and y = f($\log_e x$), find .$\frac{dy}{dx}$. at x = e.

Answer

y = f($\log_e x$)

Using the Chain Rule of Differentiation,

$$\frac{dy}{dx} = f'(\log_e x) \cdot \frac{1}{x}$$

So, at x = e

$$\frac{dy}{dx} = f'(\log_e e) \cdot \frac{1}{e}$$

$$= f'(1) \cdot \frac{1}{e}$$

$= \frac{2}{e}$ (Ans)

4. Question

If f(1) = 4, f'(1) = 2, find the value of the derivative of log (f(e^x)) with respect to x at the point x = 0.

Answer

Using the Chain Rule of Differentiation, derivative of log(f(e^x)) w.r.t. x is $\frac{1}{f(e^x)} \cdot f'(e^x)$

So, the value of the derivative at x = 0 is

$$\frac{1}{f(e^0)} \cdot f'(e^0) = \frac{1}{f(1)} \cdot f'(1)$$

$$= \frac{1}{4} \cdot 2$$

$$= \frac{1}{2}$$

So, the value of the derivative at x = 0 is 0.5 (Ans)

5. Question

If $f'(x) = \sqrt{2x^2 - 1}$ and y = f(x^2), then find at x = 1.

Answer

y = f(x^2)

$$\therefore \frac{dy}{dx} = f'(x^2) \cdot 2x$$

$$= 2x\sqrt{2(x^2)^2 - 1}$$

$$= 2x\sqrt{2x^4 - 1}$$

Putting x = 1,

$$\frac{dy}{dx} = 2 \cdot 1 \cdot \sqrt{2 \cdot 1^4 - 1}$$

$$= 2\sqrt{2 - 1}$$

=2√1

=2

i.e., $\frac{dy}{dx} = 2$ at x = 1. (Ans)

6. Question

Let g(x) be the inverse of an invertible function f(x) which is derivable at x = 3. If f(3) = 9 and f'(3) = 9, write the value of g'(9).

Answer

From the definition of invertible function,

g(f(x)) = x ...(i)

So, g(f(3)) = 3, i.e., g(9) = 3

Now, differentiating both sides of equation (i) w.r.t. x using the Chain Rule of Differentiation, we get –

g'(f(x)). f'(x) = 1 ...(ii)

Plugging in x = 3 in equation (ii) gives us –

g'(f(3)).f'(3) = 1

or, g'(9).9 = 1

i.e., g'(9) = 1/9 (Ans)

7. Question

If y = $\sin^{-1}$ (sin x), $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Then write the value of $\frac{dy}{dx}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Answer

For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$y = \sin^{-1}(\sin x)$

=x

So, $\frac{dy}{dx} = 1$ (Ans)

8. Question

If $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ and y = $\sin^{-1}$ (sin x), find $\frac{dy}{dx}$.

Answer

For $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$,

y=$\sin^{-1}$ (sin x)

= $\sin^{-1}$ (sin (π –(π -x))

(to get y in principal range of $\sin^{-1}$ x)

i.e.,

y = π - x

$\therefore \frac{dy}{dx} = -1$

From the last problem we see that $\frac{dy}{dx}_{x \to \frac{\pi}{2}^-} = 1$ and $\frac{dy}{dx}_{x \to \frac{\pi}{2}^+} = -1$

So, y is not differentiable at $x = \frac{\pi}{2}$.

Extending this, we can say that y is not differentiable at x = $(2n+1)\frac{\pi}{2}$

So, for $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\frac{dy}{dx} = \begin{cases} -1, x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \text{does not exist at } x = \frac{\pi}{2}, \frac{3\pi}{2} \end{cases} \text{(Ans)}$$

9. Question

If $\pi \le x \le 2\pi$ and $y = \cos^{-1}(\cos x)$, find $\frac{dy}{dx}$.

Answer

$y = \cos^{-1}(\cos x)$

for $x \in (\pi, 2\pi)$

$y = \cos^{-1}(\cos x)$

$= \cos^{-1}(\cos(\pi + (x - \pi)))$

$= \cos^{-1}(-\cos(x-\pi))$

$= \pi - (x - \pi)$

$= 2\pi - x$

[Since, $\cos(\pi+x) = -\cos x$ and $\cos^{-1}(-x) = \pi-x$]

So, $\frac{dy}{dx} = -1$

For $\cos^{-1}(\cos x)$, $x = n\pi$ are the 'sharp corners' where slope changes from 1 to -1 or vice versa, i.e., the points where the curves are not differentiable.

So, for $x \in [\pi, 2\pi]$

$$\frac{dy}{dx} = \begin{cases} -1, x \in (\pi, 2\pi) \\ \text{does not exist for } x = \pi, 2\pi \end{cases} \text{(Ans)}$$

10. Question

If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, write the value of $\frac{dy}{dx}$ for $x > 1$.

Answer

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$

So,

$\frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2}$

$= \frac{2}{1+x^2}$

So, answer is $\frac{dy}{dx} = \frac{2}{1+x^2}$ (Ans)

11. Question

If $f(0) = f(1) = 0$, $f'(1) = 2$ and $y = f(e^x)\, e^{f(x)}$, write the value of $\frac{dy}{dx}$ at $x = 0$.

Answer

$$y = \underbrace{\boxed{f(e^x)}}_{u}\ \underbrace{\boxed{e^{f(x)}}}_{v}$$

Using the Chain Rule of Differentiation,

$\frac{dy}{dx} = u \cdot v' + u' \cdot v$

$= f(e^x).\ e^{f(x)}\ f'(x) + f'(e^x)e^x.\ e^{f(x)}$

At x = 0,

$\frac{dy}{dx} = f(e^0) \cdot e^{f(0)} f'(0) + f'(e^0)e^0 \cdot e^{f(0)}$

$= f(1).\ e^{f(0)}\ f'(0) + f'(1).\ e^{f(0)}$

$= 0.\ e^0\ f'(0) + 2.e^0$

$= 0 + 2.1$

$= 2$

12. Question

If y = x|x|, find $\frac{dy}{dx}$ for x < 0.

Answer

y = x|x|

or, $y = \begin{cases} x^2, \text{when } x \geq 0 \\ -x^2, \text{when } x < 0 \end{cases}$

So, for x < 0

$\frac{dy}{dx} = \frac{d}{dx}(-x^2)$

=-2x (Ans)

13. Question

If y = $\sin^{-1} x + \cos^{-1} x$, find $\frac{dy}{dx}$.

Answer

We know that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

So, here y = $\sin^{-1} x + \cos^{-1} x$

$= \frac{\pi}{2}$ which is a constant.

Also, $\sin^{-1} x$ and $\cos^{-1} x$ exist only when $-1 \leq x \leq 1$

So, $\frac{dy}{dx} = 0$ when $x \in [-1, 1]$ and does not exist for all other values of x.

14. Question

If x = a(θ + sin θ), y = a (1 + cos θ), find $\frac{dy}{dx}$.

Answer

$\frac{dx}{d\theta} = a(1+\cos\theta)$ and $\frac{dy}{d\theta} = a(-\sin\theta)$

Using Chain Rule of Differentiation,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= a(-\sin\theta) \cdot \frac{1}{a(1+\cos\theta)}$$

$$= -\frac{\sin\theta}{1+\cos\theta}$$

$$= -\frac{\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta}$$

$$= -\frac{\sin\theta\,(1-\cos\theta)}{1-\cos^2\theta}$$

$$= -\frac{\sin\theta\,(1-\cos\theta)}{\sin^2\theta}$$

$$= -\frac{1-\cos\theta}{\sin\theta}$$

=cot θ-cosec θ (Ans)

15. Question

If $-\frac{\pi}{2} < x < 0$ and $y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, find $\frac{dy}{dx}$.

Answer

$$y = \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$$

$$= \tan^{-1}\sqrt{\frac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)}}$$

$$= \tan^{-1}\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}$$

$$= \tan^{-1}\sqrt{\tan^2 x}$$

When $-\frac{\pi}{2} < x < 0$, tan x is negative. So, square root of $\tan^2 x$ in this condition is –tan x.

So, $y = \tan^{-1}\sqrt{\tan^2 x}$

$=\tan^{-1}(-\tan x)$

$=-\tan^{-1}(\tan x)$

=-x

And so $\frac{dy}{dx} = \frac{d}{dx}(-x)$

$= -1$, for $x \in (-\frac{\pi}{2}, 0)$ (Ans)

16. Question

If $y = x^x$, find $\frac{dy}{dx}$ at x = e.

Answer

$y = x^x$

Taking logarithm on both sides,

log y = x log x

Differentiating w.r.t. x on both sides,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + 1 \cdot \log x$$

=1+log x

$$\Rightarrow \frac{dy}{dx} = y(1 + \log x)$$

$=x^x$ (1+log x)

So, at x = e,

$$\frac{dy}{dx} = e^e(1 + \log e)$$

$=e^e$ (1+1)

$=2e^e$ (Ans)

17. Question

If $y = \tan^{-1}\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$.

Answer

$$y = \tan^{-1}\left(\frac{1-x}{1+x}\right)$$

Using the Chain Rule of Differentiation,

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{1-x}{1+x}\right)^2} \cdot \frac{(1+x) \cdot (1-x)' - (1+x)' \cdot (1-x)}{(1+x)^2}$$

$$= \frac{(1+x)^2}{(1+x)^2 + (1-x)^2} \cdot \frac{(1+x)(-1) - (1)(1-x)}{(1+x)^2}$$

$$= -\frac{2}{(1+x)^2 + (1-x)^2}$$

$= -\frac{1}{1+x^2}$ (Ans)

18. Question

if $y = \log_a x$, find $\frac{dy}{dx}$.

Answer

$y = \log_a x = \frac{\log_e x}{\log_e a}$

$\therefore \frac{dy}{dx} = \frac{1}{\log_e a} \cdot \frac{1}{x}$

$= \frac{1}{x \log_e a}$ (Ans)

19. Question

If $y = \log \sqrt{\tan x}$, write $\frac{dy}{dx}$.

Answer

This particular problem is a perfect way to demonstrate how simple but powerful the Chain Rule of Differentiation is.

It is important to identify and break the problem into the individual functions with respect to which successive differentiation shall be done.

In this case, this is the way to break down the problem –

$$\frac{dy}{dx} = \frac{dy}{d(\sqrt{\tan x})} \cdot \frac{d(\sqrt{\tan x})}{d(\tan x)} \cdot \frac{d(\tan x)}{dx}$$

i.e., $\frac{dy}{dx} = \frac{d(\log\sqrt{\tan x})}{d(\sqrt{\tan x})} \cdot \frac{d(\sqrt{\tan x})}{d(\tan x)} \cdot \frac{d(\tan x)}{dx}$

$= \frac{1}{\sqrt{\tan x}} \cdot \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x$

$= \frac{\sec^2 x}{2 \tan x}$

$= \frac{1 + \tan^2 x}{2 \tan x}$

$= \frac{1}{2}(\tan x + \cot x)$ (Ans)

20. Question

If $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, find $\frac{dy}{dx}$.

Answer

$-1 < \frac{1-x^2}{1+x^2} \leq 1$ holds for all $x \in \mathbb{R}$.

So, $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\pi}{2}$, for all $x \in \mathbb{R}$

($\because \sin^{-1} m + \cos^{-1} m = \frac{\pi}{2}, m \in [-1, 1]$)

Hence, $\frac{dy}{dx} = 0,$ for all $x \in \mathbb{R}$.

21. Question

If $y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$, then write the value of $\frac{dy}{dx}$.

Answer

$y = \sec^{-1}\left(\frac{x+1}{x-1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$

$= \cos^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right)$

Which exists for $-1 \leq \frac{x-1}{x+1} \leq 1$ and is equal to $\frac{\pi}{2}$

Now, $\frac{x-1}{x+1} \leq 1$

$\Rightarrow \frac{x-1}{x+1} - 1 \leq 0$

$\Rightarrow \frac{x-1}{x+1} - \frac{x+1}{x+1} \leq 0$

$\Rightarrow -\frac{2}{x+1} \leq 0$

$\Rightarrow \frac{2}{x+1} \geq 0$

$\Rightarrow x+1>0$

$\Rightarrow x>-1$ ...(i)

Also, $\frac{x-1}{x+1} \geq -1$

$\Rightarrow \frac{x-1}{x+1} + 1 \geq 0$

$\Rightarrow \frac{x-1}{x+1} + \frac{x+1}{x+1} \geq 0$

$\Rightarrow \frac{2x}{x+1} \geq 0$

$\Rightarrow x\geq0$ or $x<-1$ ...(ii)

Comparing equations (i) and (ii), we understand that the condition satisfying both inequalities is $x \geq 0$.

So, for $x\geq0$,

$y = \cos^{-1}\left(\frac{x-1}{x+1}\right) + \sin^{-1}\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$, which is a constant

So, $\frac{dy}{dx} = \begin{cases} 0, x \geq 0 \\ \text{does not exist for } x < 0 \end{cases}$ (Ans)

22. Question

If |x| < 1 and y = 1 + x + x^2 + ... to ∞, then find the value of $\frac{dy}{dx}$.

Answer

Since |x| < 1,

y = 1 + x + x^2 + ... to ∞

$= \frac{1}{1-x}$

$\therefore \frac{dy}{dx} = -\frac{1}{(1-x)^2} \cdot -1$

$= \frac{1}{(1-x)^2}$ (Ans)

23. Question

If $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v=\tan^{-1}\left(\frac{2x}{1+x^2}\right)$, where –1 < x < 1, then write the value of $\frac{du}{dv}$.

Answer

$u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v=\tan^{-1}\left(\frac{2x}{1+x^2}\right)$

We know, $\frac{du}{dx}=\frac{2}{1+x^2}$

Using the chain rule of differentiation,

$$\frac{dv}{dx}=\frac{1}{1+\left(\frac{2x}{1+x^2}\right)^2}\cdot\frac{(1+x^2)\cdot(2x)'-(1+x^2)'\cdot(2x)}{(1+x^2)^2}$$

$$=\frac{(1+x^2)^2}{(1+x^2)^2+(2x)^2}\cdot\frac{2(1+x^2)-(2x)(2x)}{(1+x^2)^2}$$

$$=\frac{2(1-x^2)}{(1+x^2)^2+(2x)^2}$$

Using Chain Rule of Differentiation,

$$\frac{du}{dv}=\frac{du}{dx}\cdot\frac{dx}{dv}$$

$$=\frac{2}{1+x^2}\cdot\frac{(1+x^2)^2+(2x)^2}{2(1-x^2)}$$

$$=\frac{(1+x^2)^2+(2x)^2}{(1+x^2)(1-x^2)}$$

Dividing numerator and denominator by $(1+x^2)^2$,

$$\frac{du}{dv}=\frac{1+\left(\frac{2x}{1+x^2}\right)^2}{\frac{1-x^2}{1+x^2}}$$

$$=\frac{1+\sin^2 u}{\cos u}$$

=sec u (1+tan u) (Ans)

24. Question

If $f(x)=\log\left\{\frac{u(x)}{v(x)}\right\}$, $u(1)=v$ (1) and u’(1) = v’(1) = 2, then find the value of f’(1).

Answer

Using the Chain Rule of Differentiation,

$$f'(x)=\frac{1}{\frac{u(x)}{v(x)}}\cdot\frac{v(x)\cdot u'(x)-v'(x)\cdot u(x)}{\left(v(x)\right)^2}$$

$$=\frac{v(x)\cdot u'(x)-v'(x)\cdot u(x)}{u(x)\cdot v(x)}$$

Putting x = 1,

$$f'(1)=\frac{v(1)\cdot u'(1)-v'(1)\cdot u(1)}{u(1)\cdot v(1)}$$

$= \frac{2v(1) - 2u(1)}{u(1) \cdot v(1)}$

Since, u(1) = v(1),

2v(1) – 2u(1) = 0

i.e., f'(1) = 0 (Ans)

25. Question

If y = log |3x|, x ≠ 0, find $\frac{dy}{dx}$.

Answer

y = log |3x|

So, $\frac{dy}{dx} = \frac{1}{3x} \cdot 3$

$= \frac{1}{x}, x \neq 0$

i.e.,; $\frac{dy}{dx} = \frac{1}{x}, x \neq 0$ (Ans)

26. Question

If f(x) is an even function, then write whether f' (x) is even or odd.

Answer

f(x) is an even function.

This means that f(-x) = f(x).

If we differentiate this equation on both sides w.r.t. x, we get –

f'(-x).(-1) = f'(x)

or, -f'(-x) = f'(x)

i.e., f'(x) is an odd function. (Ans)

27. Question

If f(x) is an odd function, then write whether f'(x) is even or odd.

Answer

f(x) is an odd function.

This means that f(-x) = -f(x).

If we differentiate this equation on both sides w.r.t. x, we get –

f'(-x).(-1) = -f'(x)

or, f'(-x) = f'(x)

i.e., f'(x) is an even function. (Ans)

28. Question

Write the derivative of sin x with respect to cos x.

Answer

We have to find $\frac{d}{d(\cos x)}(\sin x)$

So, we use the Chain Rule of Differentiation to evaluate this.

$$\frac{d}{d(\cos x)}(\sin x) = \frac{d(\sin x)}{dx} \cdot \frac{dx}{d(\cos x)}$$

$$= \cos x \cdot \frac{1}{-\sin x}$$

=-cot x (Ans)