18. Maxima and Minima

Exercise 18.1

1. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = 4x^2 - 4x + 4$ on R

Answer

$f(x) = 4x^2 - 4x + 4$ on R

$= 4x^2 - 4x + 1 + 3$

$= (2x - 1)^2 + 3$

Since, $(2x - 1)^2 \geq 0$

$= (2x - 1)^2 + 3 \geq 3$

$= f(x) \geq f\left(\frac{1}{2}\right)$

Thus, the minimum value of f(x) is 3 at $x = \frac{1}{2}$

Since, f(x) can be made large. Therefore maximum value does not exist.

2. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = -(x - 1)^2 + 2$ on R

Answer

We have $f(x) = -(x - 1)^2 + 2$

It can be observed that $(x - 1)^2 \geq 0$ for every $x \in R$

Therefore, $f(x) = -(x - 1)^2 + 2 \leq 2$ for every $x \in R$

The maximum value of f is attained when $(x - 1) = 0$

$(x - 1)=0, x=1$

Since, Maximum value of $f = f(1) = -(1 - 1)^2 + 2 = 2$

Hence, function f does not have minimum value.

3. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = |x + 2|$ on R

Answer

$|x + 2| \geq 0$ for $x \in R$

$= f(x) \geq 0$ for all $x \in R$

So the minimum value of f(x) is 0, which attains at x =2

Hence, $f(x) = |x + 2|$ does not have the maximum value.

4. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = \sin 2x + 5$ on R

Answer

We know that $-1 \leq \sin 2x \leq 1$

$= -1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$= 4 \leq \sin 2x + 5 \leq 6$

Hence, the maximum and minimum value of h are 4 and 6 respectively.

5. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = |\sin 4x + 3|$ on R

Answer

We know that $-1 \leq \sin 4x \leq 1$

$= 2 \leq \sin 4x + 3 \leq 4$

$= 2 \leq |\sin 4x + 3| \leq 4$

Hence, the maximum and minimum value of f are 4 and 2 respectively.

6. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = 2x^3 + 5$ on R

Answer

We have $f(x) = 2x^3 + 5$ on R

Here, we observe that the values of f(x) increase when the values of x are increased and f(x) can be made large,

So, f(x) does not have the maximum value

Similarly, f(x) can be made as small as we want by giving smaller values to x.

So, f(x) does not have the minimum value.

7. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = -|x + 1| + 3$ on R

Answer

We know that $-|x + 1| \leq 0$ for every $x \in R$.

Therefore, $g(x) = -|x + 1| + 3 \leq 3$ for every $x \in R$.

The maximum value of g is attained when $|x + 1| = 0$

$|x + 1| = 0$

$x = -1$

Since, Maximum Value of $g = g(-1) = -|-1 + 1| + 3 = 3$

Hence, function g does not have minimum value.

8. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = 16x^2 - 16x + 28$ on R

Answer

We have $f(x) = 16x^2 – 16x + 28$ on R

$= 16x^2 – 16x + 4 + 24$

$= (4x – 2)^2 + 24$

Now, $(4x – 2)^2 \geq 0$ for all $x \in R$

$= (4x – 2)^2 + 24 \geq 24$ for all $x \in R$

$= f(x) \geq f\left(\frac{1}{2}\right)$

Thus, the minimum value of f(x) is 24 at $x = \left(\frac{1}{2}\right)$

Hence, f(x) can be made large as possibly by giving difference value to x.

Thus, maximum values does not exist.

9. Question

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

$f(x) = x^3 – 1$ on R

Answer

We have $f(x) = x^3 – 1$ on R

Here, we observe that the values of f(x) increase when the values of x are increased, and f(x) can be made large, by giving large value.

So, f(x) does not have the maximum value

Similarly, f(x) can be made as small as we want by giving smaller values to x.

So, f(x) does not have the minimum value.

Exercise 18.2

1. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x) = (x – 5)^4$

Answer

$f(x) = (x – 5)^4$

Differentiate w.r.t x

$f'(x) = 4(x – 5)^3$

for local maxima and minima

$f'(x) = 0$

$= 4(x – 5)^3 = 0$

$= x – 5 = 0$

$x = 5$

f ‘ (x) changes from –ve to + ve as passes through 5.

So, x = 5 is the point of local minima

Thus, local minima value is f(5) = 0

2. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x) = x^3 – 3x$

Answer

We have, $g(x) = x^3 – 3x$

Differentiate w.r.t x then we get,

$g'(x) = 3x^2 – 3$

Now, $g'(x) = 0$

$= 3x^2 = 3 \Rightarrow x = \pm 1$

Again differentiate $g'(x) = 3x^2 – 3$

$g''(x) = 6x$

$g''(1) = 6 > 0$

$g''(–1) = –6 > 0$

By second derivative test, x=1 is a point of local minima and local minimum value of g at

$x = 1$ is $g(1) = 1^3 – 3 = 1 – 3 = –2$

However, x = – 1 is a point of local maxima and local maxima value of g at

$x = –1$ is $g(–1) = (–1)^3 – 3(–1)$

$= –1 + 3$

$= 2$

Hence, The value of Minima is – 2 and Maxima is 2.

3. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x) = x^3 (x – 1)^2$

Answer

We have, $f(x) = x^3(x – 1)^2$

Differentiate w.r.t x, we get,

$f'(x) = 3x^2(x – 1)^2 + 2x^3(x – 1)$

$= (x – 1)(3x^2(x – 1) + 2x^3)$

$= (x – 1)(3x^3 – 3x^2 + 2x^3)$

$= (x – 1)(5x^3 – 3x^2)$

$= x^2 (x – 1)(5x – 3)$

For all maxima and minima,

$f'(x) = 0$

$= x^2(x – 1)(5x – 3) = 0$

$= x = 0, 1, \frac{3}{5}$

At $x = \frac{3}{5}$ f ’(x) changes from –ve to + ve

Since, $x = \frac{3}{5}$ is a point of Minima

At x =1 f ‘ (x) changes from –ve to + ve

Since, x =1 is point of maxima.

4. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x) = (x - 1)(x + 2)^2$

Answer

We have, $f(x) = (x - 1)(x + 2)^2$

Differentiate w.r.t x, we get,

$f'(x) = (x + 2)^2 + 2(x - 1)(x + 2)$

$= (x + 2)(x + 2 + 2x - 2)$

$= (x + 2)(3x)$

For all maxima and minima,

$f'(x) = 0$

$= (x + 2)(3x) = 0$

$= x = 0, -2$

At x = – 2 f ’(x) changes from –ve to + ve

Since, x = – 2 is a point of Maxima

At x =0 f ‘ (x) changes from –ve to + ve

Since, x =0 is point of Minima.

Hence, local min value = f(0) = – 4

local max value = f(– 2) = 0.

5. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$$f(x) = \frac{1}{x^2 + 2}$$

Answer

We have, $f(x) = (x - 1)^3(x + 1)^2$

Differentiate w.r.t x, we get,

$f'(x) = 3(x - 1)^2(x + 1)^2 + 2(x - 1)^3(x + 1)$

$= (x - 1)^2(x + 1)\,(3(x + 1) + 2(x - 1)$

$= (x - 1)^2(x + 1)(5x + 1)$

For the point of local maxima and minima,

f ’(x) = 0

= $(x - 1)^2(x + 1)(5x + 1) = 0$

= x = 1, – 1, $-\frac{1}{5}$

At x = – 1 f ’(x) changes from –ve to + ve

Since, x = – 1 is a point of Maxima

At x =$-\frac{1}{5}$ f ‘ (x) changes from –ve to + ve

Since, x =$-\frac{1}{5}$ is point of Minima.

Hence, local min value = $-\frac{3456}{3125}$

local max value = 0.

6. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x) = x^3 - 6x^2 + 9x + 15$

Answer

We have, $f(x) = x^3 - 6x^2 + 9x + 15$

Differentiate w.r.t x, we get,

f ‘(x) = $3x^2 - 12x + 9$

= $3(x^2 - 4x + 3)$

= 3(x – 3)(x – 1)

For all maxima and minima,

f ’(x) = 0

= 3(x – 3)(x – 1) = 0

= x = 3, 1

At x = 1 f ’(x) changes from –ve to + ve

Since, x = – 1 is a point of Maxima

At x =3 f ‘ (x) changes from –ve to + ve

Since, x =3 is point of Minima.

Hence, local max value f(1)= $(1)^3 - 6(1)^2 + 9(1) + 15 = 19$

local min value f(3) = $(3)^3 - 6(3)^2 + 9(3) + 15 = 15$

7. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

f(x) = sin 2x, $0 < x < \pi$

Answer

We have, f(x) = sin 2x

Differentiate w.r.t x, we get,

f '(x) = 2cos 2x, 0 < x,π

For, the point of local maxima and minima,

f '(x) = 0

= 2x = $\frac{\pi}{2}, \frac{3\pi}{2}$

= x = $\frac{\pi}{4}, \frac{3\pi}{4}$

At x = $\frac{\pi}{4}$ f '(x) changes from –ve to + ve

Since, x = $\frac{\pi}{4}$ is a point of Maxima

At x = $\frac{3\pi}{4}$ f ' (x) changes from –ve to + ve

Since, x = $\frac{3\pi}{4}$ is point of Minima.

Hence, local max value f $\left(\frac{\pi}{4}\right)$ = 1

local min value f$\left(\frac{3\pi}{4}\right)$ = – 1

8. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

f(x) = sin x – cos x, 0 < x < 2π

Answer

We have, f(x) = sin x – cos x

Differentiate w.r.t x, we get,

f '(x) = cos x + sin x

For, the point of local maxima and minima,

f '(x) = 0

= cos x = – sin x => tan x = – 1 = x = $\frac{3\pi}{4}, \frac{7\pi}{4} \in (0,2\pi)$

Again differentiate w.r.t x

f ''(x) = – sin x + cos x

f ''$\left(\frac{3\pi}{4}\right)$= – sin$\frac{3\pi}{4}$ + cos $\frac{3\pi}{4}$ = - $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

f ''$\left(\frac{7\pi}{4}\right)$= – sin$\frac{7\pi}{4}$ + cos $\frac{7\pi}{4}$ = - $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

Therefore, by second derivative test, x = $\frac{3\pi}{4}$ is a point of local maxima and the local maximum of f at x = $\frac{3\pi}{4}$ is

F'' $\left(\frac{3\pi}{4}\right)$= sin $\frac{3\pi}{4}$ – cos $\frac{3\pi}{4}$ = $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$

F'' $\left(\frac{7\pi}{4}\right)$= sin $\frac{7\pi}{4}$ – cos $\frac{7\pi}{4}$ = $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

9. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

f(x) = cos x, 0 < x < π

Answer

f(x) = cos x

Differentiate w.r.t x

f ’(x) = – sin x

for the point of local maxima and minima,

f ‘ (x) = 0

= – sin x = 0

x = 0, and π

But, these two interval lies outside the interval (0,π)

So, no local maxima and minima will exist in the interval (0, π)

10. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$$f(x) = \sin 2x - x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Answer

We have, f(x) = sin 2x – x

Differentiate w.r.t x, we get,

f ‘(x) = 2cos 2x – 1,

For, the point of local maxima and minima,

f ’(x) = 0

2cos2x – 1 = 0

= cos 2x = $\frac{1}{2}$ = cos $\frac{\pi}{3}$

= 2x = $\frac{\pi}{3}, -\frac{\pi}{3}$

= x = $\frac{\pi}{6}, -\frac{\pi}{6}$

At x = $-\frac{\pi}{6}$ f ’(x) changes from –ve to + ve

Since, x =$-\frac{\pi}{6}$ is a point of Maxima

At x = $\frac{\pi}{6}$ f ‘ (x) changes from –ve to + ve

Since, x =$\frac{\pi}{6}$ is point of local maxima

Hence, local max value f $\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

local min value f$\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

11. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$$f(x) = 2\sin x - x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Answer

We have, f(x) = 2 sin x – x

Differentiate w.r.t x, we get,

f '(x) = 2cos x – 1 = 0

For, the point of local maxima and minima,

f '(x) = 0

cos x $= \frac{1}{2} = \cos\frac{\pi}{3}$

$= x = -\frac{\pi}{3}, \frac{\pi}{3}$

At x $= -\frac{\pi}{3}$ f '(x) changes from –ve to + ve

Since, x $=-\frac{\pi}{3}$ is a point of Minima with value $= -\sqrt{3} - \frac{\pi}{3}$

At x $= \frac{\pi}{3}$ f ' (x) changes from –ve to + ve

Since, x $=\frac{\pi}{3}$ is point of local maxima with value $= \sqrt{3} - \frac{\pi}{3}$

Hence, local max value f $\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}$

local min value f$\left(-\frac{\pi}{3}\right) = -\sqrt{3} - \frac{\pi}{3}$

12. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$$f(x) = x\sqrt{1-x},\ x > 0$$

Answer

$$f'(x) = \sqrt{(1-x)} + x.\frac{1}{2\sqrt{1-x}}(-1)$$

$$=\sqrt{1-x} - \frac{x}{2\sqrt{(1-x)}}$$

$$= \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

For f '(0) $= \frac{2-3x}{2\sqrt{1-x}} = 0$

So, 2 – 3x = 0 => x $= \frac{2}{3}$

$$f''(x) = \frac{1}{2}\left[\frac{\left[\sqrt{(1-x)}(-3) - (2-3x)\left(-\frac{1}{2\sqrt{1-x}}\right)\right]}{1-x}\right]$$

$=\left[\frac{\left[\sqrt{(1-x)}(-3)+(2-3x)\left(\frac{1}{2\sqrt{1-x}}\right)\right]}{2(1-x)}\right]$

$=\left[\frac{[-6(1-x)+(2-3x)]}{4(1-x)^{\frac{3}{2}}}\right]$

$=\frac{3x-4}{4(1-x)^{\frac{3}{2}}}$

$f''\left(\frac{2}{3}\right)=\frac{\left[3\left(\frac{2}{3}\right)-4\right]}{4\left(1-\frac{2}{3}\right)^{\frac{3}{2}}}=\frac{2-4}{4\left(\frac{1}{3}\right)^{\frac{3}{2}}}=-\frac{1}{2\left(\frac{1}{3}\right)^{\frac{3}{2}}}<0$

Therefore, by second derivative test, x $=\frac{2}{3}$ is a point of local maxima and the local maxima value of F at x $=\frac{2}{3}$ is

Hence, $f\left(\frac{2}{3}\right)=\frac{2}{3}\sqrt{\left(1-\frac{2}{3}\right)}=\frac{2}{3}\frac{\sqrt{1}}{3}=\frac{2}{3\sqrt{3}}=\frac{2\sqrt{3}}{9}$

13. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x) = x^3(2x - 1)^3$

Answer

We have, $f(x) = x^3(2x - 1)^3$

Differentiate w.r.t x, we get,

$f'(x) = 3x^2(2x - 1)^3 + 3x^3(2x - 1)^{2}\cdot 2$

$= 3x^2(2x - 1)^2(2x - 1 + 2x)$

$= 3x^2(4x - 1)$

For the point of local maxima and minima,

$f'(x) = 0$

$= 3x^2(4x - 1)= 0$

$= x = 0, \frac{1}{4}$

At x $=\frac{1}{4}$ f '(x) changes from –ve to + ve

Since, x $=\frac{1}{4}$ is a point of Minima

Hence, local min value $f\left(\frac{1}{4}\right)=-\frac{1}{512}$

14. Question

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

$f(x)=\frac{x}{2}+\frac{2}{x},\ x>0$

Answer

We have, f(x) = $\frac{x}{2} + \frac{2}{x}, x > 0$

Differentiate w.r.t x, we get,

f '(x) = $\frac{1}{2} - \frac{2}{x^2}, x > 0$

For the point of local maxima and minima,

f '(x) = 0

= $\frac{1}{2} - \frac{2}{x^2} = 0$

= $x^2 - 4 = 0$

= x = $\sqrt{4}, -\sqrt{4}$

= x = 2, – 2

At x = 2 f '(x) changes from –ve to + ve

Since, x = 2 is a point of Minima

Hence, local min value f (2) = 2

Exercise 18.3

1 A. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$f(x) = x^4 - 62x^2 + 120x + 9$

Answer

$f(x) = x^4 - 62x^{2+} 120x + 9$

$\therefore f'(x) = 4x^3 - 124x + 120 = 4(x^3 - 31x + 30)$

$f''(x) = 12x^2 - 124 = 4(3x^2 - 31)$

for maxima and minima,

f'(x) = 0

$4(x^3 - 31x + 30) = 0$

So roots will be x = 5, 1, – 6

Now,

f''(5) = 176 > 0

x = 5 is point of local minima

f''(1) = – 112 < 0

x = 1 is point of local maxima

f''(– 6) = 308 > 0

x = – 6 is point of local minima

local max value = f(1) = 68

local min value = f(5) = – 316

and f(– 6) = – 1647

1 B. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$f(x) = x^3 – 6x^2 + 9x + 15$

Answer

$f(x) = x^3 – 6x^2 + 9x + 15$

$\therefore f'(x) = 3x^2 – 12x + 9 = 3(x^2 – 4x + 3)$

$f''(x) = 6x – 12 = 6(x – 2)$

for maxima and minima,

$f'(x) = 0$

$3(x^2 – 4x + 3) = 0$

So roots will be x = 3, 1

Now,

$f''(3) = 6 > 0$

x = 3 is point of local minima

$f''(1) = – 6 < 0$

x = 1 is point of local maxima

local max value = f(1) = 19

local min value = f(3) = 15

1 C. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$f(x) = (x – 1) (x + 2)^2$

Answer

$f(x) = (x – 1) (x + 2)^2$

$\therefore f'(x) = (x + 2)^2 + 2(x – 1)(x + 2)$

$= (x + 2)(x + 2 + 2x – 2)$

$= (x + 2)(3x)$

And $f''(x) = 3(x + 2) + 3x$

$= 6x + 6$

for maxima and minima,

$f'(x) = 0$

$(x + 2)(3x) = 0$

So roots will be x = 0, – 2

Now,

$f''(0) = 6 > 0$

x = 0 is point of local minima

$f''(– 2) = – 6 < 0$

x = 1 is point of local maxima

local max value = f(– 2) = 0

local min value = f(0) = – 4

1 D. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

f(x) = 2/x – 1/x^2, x > 0

Answer

we have f(x) = $\frac{2}{x} - \frac{1}{x^2}$, x > 0

$$\therefore f'(x) = -\frac{2}{x^2} + \frac{4}{x^3}$$

And, $f''(x) = +\frac{4}{x^3} - \frac{12}{x^4}$

for maxima and minima,

f'(x) = 0

$$-\frac{2}{x^2} + \frac{4}{x^3} = 0$$

$$-\frac{2(x-2)}{x^3} = 0$$

x = 2

now,

f''(2) = $\frac{4}{8} - \frac{12}{6} = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4} < 0$

x = 2 is point of local maxima

local max value = f(2) = 1/2

1 E. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

f(x) = x e^x

Answer

we have f(x) = x e^x

f'(x) = e^x + xe^x = e^x(x + 1)

f''(x) = e^x(x + 1) + e^x

= e^x(x + 2)

For maxima and minima,

f'(x) = 0

e^x(x + 1) = 0

x = – 1

now f''(– 1) = e^{-1} = 1/e > 0

x = – 1 is point of local minima

hence, local min = f(– 1) = – 1/e

1 F. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

f(x) = x/2 + 2/x, x > 0

Answer

$f(x) = \frac{x}{2} + \frac{2}{x}$, x > 0

$f'(x) = \frac{1}{2} - \frac{2}{x^2}$

And $f''(x) = \frac{4}{x^3}$

For maxima and minima,

f'(x) = 0

$\frac{1}{2} - \frac{2}{x^2} = 0$

$$\frac{x^2 - 4}{2x^2} = 0$$

x = 2, – 2

now,

f’’(2) = 1/2 > 0

x = 2 is point of minima

we will not consider x = – 2 as x > 0

local min value = f(2) = 2

1 G. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$f(x) = (x + 1)(x + 2)^{1/3}$, x ≥ –2

Answer

we have $f(x) = (x + 1)(x + 2)^{1/3}$, x ≥ –2

$f'(x) = (x + 2)^{1/3} + 1/3(x + 1)(x + 2)^{-2/3}$

$= (x + 2)^{-2/3}(x + 2 + 1/3(x + 1))$

$= 1/3(x + 2)^{-2/3}(4x + 7)$

And $f''(x) = -\frac{2}{9}(x + 2)^{-\frac{5}{3}}(4x + 7) + \frac{1}{3}(x + 2)^{-\frac{2}{3}} \times 4$

For maxima and minima,

f'(x) = 0

$1/3(x + 2)^{-2/3}(4x + 7) = 0$

x = – 7/4

Now

f''(– 7/4) = $\frac{4}{3}\left(-\frac{7}{4}+2\right)^{-\frac{2}{3}}$

x = – 7/4 is point of minima

local min value = f(– 7/4) = $-\frac{3}{4^{\frac{4}{3}}}$

1 H. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$$f(x)=x\sqrt{32-x^{2}},\,-5\leq x\leq 5$$

Answer

we have,

$$f(x)=x\sqrt{32-x^2}$$

$$\therefore f'(x)\ =\ \sqrt{32-x^2}\ +\ \frac{x}{2\sqrt{32-x^2}}x(-2x)$$

$$=\frac{2(32-x^2)-2x^2}{2\sqrt{32-x^2}}$$

$$=\frac{64-4x^2}{2\sqrt{32-x^2}}$$

And f''(x) = $\frac{2\sqrt{32-x^2}\times(-8)-2\left(\frac{64-4x^2}{2\sqrt{32-x^2}}\right)\times(-2x)}{4(32-x^2)}$

$$=\frac{-4(32-x^2)\times(8x)\ +\ 4x(64-4x^2)}{8(32-x^2)^{\frac{3}{2}}}$$

For maxima and minima,

f'(x) = 0

$$\frac{64-4x^2}{2\sqrt{32-x^2}}\ =\ 0$$

X = ±4

Now

f''(4) = $\frac{4\times4(64-16-8\times32+8\times16)}{8(32-x^2)^{\frac{3}{2}}}\ <\ 0$

x = 4 is point of maxima

1 I. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

f(x) = x^3 – $2ax^2$ + a^2 x, a > 0, x ∈ R

Answer

local maximum value = f(4)

= 4√(32 – 4^2)

= 4√(32 – 16)

= 4√16

= 16

Local minimum at x = – 4

Local minimum value = f(– 4)

= – 4√(32 – (– 4)^2)

= – 4√(32 – 16)

= – 4√16

= – 16

1 J. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$$f(x) = x + \frac{a^2}{x},\ a > 0,\ x \neq 0$$

Answer

: f(x) = $x + \frac{a^2}{x}$, a > 0 x≠0

f'(x) = $1 - \frac{a^2}{x^2}$

and f’’(x) = $\frac{2a^2}{x^3}$

For maxima and minima,

f'(x) = 0

$1 - \frac{a^2}{x^2}$ = 0

$x^2 - a^2 = 0$

x = ±a

now,

f’’(a) = 2/a > 0 as a > 0

x = a is point of minima

f’’(– a) = – 2/a < 0 as a > 0

x = – a is point of maxima

hence

local max value = f(– a) = – 2a

local min value = f(a) = 2a

1 K. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$$f(x) = x\sqrt{2 - x^2} - \sqrt{2} \leq x \leq \sqrt{2}$$

Answer

$f(x) = x\sqrt{2-x^2}$

$f'(x) = \sqrt{2-x^2} - \frac{2x^2}{2\sqrt{2-x^2}}$

$$= \frac{2(2-x^2) - 2x^2}{2\sqrt{2-x^2}}$$

$$= \frac{2-2x^2}{\sqrt{2-x^2}}$$

$f''(x) = \frac{\sqrt{2-x^2}(-4x) + \frac{(2-2x^2)2x}{\sqrt{2-x^2}}}{\left(\sqrt{2-x^2}\right)^2}$

$$= \frac{-(2-x^2)4x + 4x - 4x^3}{(2-x^2)^{\frac{3}{2}}}$$

For maxima and minima,

$f'(x) = 0$

$$\frac{2-2x^2}{\sqrt{2-x^2}} = 0$$

$X = \pm 1$

Now

$f''(1) < 0$

x = 1 is point of local maxima

$f''(-1) > 0$

x = – 1 is point of minima

hence

local max value = f(1) = 1

local min value = f(– 1) = – 1

1 L. Question

Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:

$$f(x) = x + \sqrt{1-x}, x \le 1$$

Answer

$f(x) = x + \sqrt{1-x}$

$f'(x) = 1 - \frac{1}{2\sqrt{1-x}} = \frac{2\sqrt{1-x}-1}{2\sqrt{1-x}}$

$f''(x) = \frac{\left(2\sqrt{(1-x)}\left(-\frac{1}{\sqrt{1-x}}\right) + \frac{2\sqrt{1-x}-1}{\sqrt{1-x}}\right)}{4(1-x)}$

For maxima and minima,

$f'(x) = 0$

$$\frac{2\sqrt{1-x}-1}{2\sqrt{1-x}} = 0$$

$\sqrt{1-x} = \frac{1}{2}$

X = 1 – 1/4 = 3/4

Now

f’’(3/4) < 0

x = 3/4 is point of local maxima

hence

local max value = f(3/4) = 5/4

2 A. Question

Find the local extremum values of the following functions:

$f(x) = (x - 1)(x - 2)^2$

Answer

f(x) = (x – 1)(x – 2)2

$f'(x) = (x - 2)^2 + 2(x - 1)(x - 2)$

= (x – 2)(x – 2 + 2x – 2)

= (x – 2)(3x – 4)

f’’(x) = (3x – 4) + 3(x – 2)

For maxima and minima,

f'(x) = 0

(x – 2)(3x – 4) = 0

x = 2, 4/3

Now

f’’(2) > 0

x = 2 is point of local minima

f’’(4/3) = – 2 < 0

x = 4/3 is point of local maxima

hence

local max value = f(4/3) = 4/27

local min value = f(2) = 0

2 B. Question

Find the local extremum values of the following functions:

$f(x) = x\sqrt{1-x},\ x \le 1$

Answer

$f(x) = x\sqrt{1-x}$

$\therefore f'(x) = \sqrt{1-x} + \frac{x}{2\sqrt{1-x}}(-1)$

$= \frac{2(1-x)-x}{2\sqrt{1-x}}$

$= \frac{2-3x}{2\sqrt{1-x}}$

f''(x) = $\frac{2\sqrt{(1-x)}(-3)+\frac{2-3x}{\sqrt{1-x}}}{4(1-x)}$

For maxima and minima,

f'(x) = 0

$\frac{2-3x}{2\sqrt{1-x}} = 0$

x = 2/3

Now

f''(2/3) < 0

x = 2/3 is point of maxima

hence

local max value = f(2/3) = $\frac{2}{3\sqrt{3}}$

2 C. Question

Find the local extremum values of the following functions:

f(x) = – (x – 1)^3(x + 1)^2

Answer

f(x) = – (x – 1)^3(x + 1)^2

f'(x) = – 3(x – 1)^2(x + 1)^2 – 2(x – 1)^3(x + 1)

= – (x – 1)^2(x + 1)(3x + 3 + 2x – 2)

= – (x – 1)^2(x + 1)(5x + 1)

f''(x) = – 2(x – 1)(x + 1)(5x + 1) – (x – 1)^2(5x + 1) – 5(x – 1)^2(x – 1)

For maxima and minima,

f'(x) = 0

– (x – 1)^2(x + 1)(5x + 1) = 0

x = 1, – 1, – 1/5

Now

f''(1) = 0

x = 1 is inflection point

f''(– 1) = – 4× – 4 = 16 > 0

x = – 1 is point of minima

f''(– 1/5) = – 5(36/25)*4/5 = – 144/25 < 0

x = – 1/5 is point of maxima

hence

local max value = f(– 1/5) = $\frac{3456}{3125}$

local min value = f(– 1) = 0

3. Question

The function y = a log x + bx^2 + x has extreme values at x = 1 and x = 2. Find a and b.

Answer

we have

y = a log x + bx^2 + x

$\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

And $\frac{d^2y}{dx^2} = -\frac{a}{x^2} + 2b$

For maxima and minima,

$\frac{dy}{dx} = 0$

$\frac{a}{x} + 2bx + 1 = 0$

Given that extreme value exist at x = 1, 2

a + 2b = – 1 ......(1)

$\frac{a}{2} + 4b = -1$

a + 8b = – 2 ...... (2)

solving (1) and (2) we get

a = – 2/3 b = – 1/6

4. Question

Show that $\frac{\log x}{x}$ has a maximum value at x = e

Answer

the given function is $f(x) = \frac{\log x}{x}$

$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$

Now f'(x) = 0

1 – log x = 0

log x = 1

log x = log e

x = e

now $f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$

$= \frac{-x - 2x(1 - \log x)}{x^4}$

$= \frac{-3 + 2\log x}{x^3}$

Now f''(x) = $\frac{-3 + 2\log e}{e^3} = \frac{-3 + 2}{e^3} = -\frac{1}{e^3} < 0$

Therefore, by second derivation test f is the maximum at x = e.

5. Question

Find the maximum and minimum values of the function $f(x) = \frac{4}{x+2} + x$.

Answer

$f(x) = \frac{4}{x+2} + x$

$f'(x) = -\frac{4}{(x+2)^2} + 1$

$f''(x) = \frac{8}{(x+2)^3}$

For maxima and minima,

$f'(x) = 0$

$-\frac{4}{(x + 2)^2} + 1 = 0$

$(x + 2)^2 = 4$

$x^2 + 4x = 0$

$x(x + 4) = 0$

$x = 0, -4$

Now $f''(0) = 1 > 0$

x = 0 is point of minima

$f''(-4) = -1 < 0$

x = – 4 is point of maxima

Local max value = f(– 4) = – 6

Local min value = f(0) = 2

6. Question

Find the maximum and minimum values of f(x) = tan x – 2x.

Answer

we have

$y = \tan x - 2x$

$y' = \sec^2 x - 2$

$y'' = 2\sec^2 x \tan x$

For maxima and minima,

$y' = 0$

sec2x = 2

$\sec x = \pm\sqrt{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}$

$y''(\frac{\pi}{4}) = 4 > 0$

$x = \frac{\pi}{4}$ is the point of minima

$y''(\frac{3\pi}{4}) = -4 < 0$

$x = \frac{3\pi}{4}$ is the point of maxima

hence

max value = $f(\frac{3\pi}{4}) = -1 - \frac{3\pi}{2}$

min value = $f(\frac{\pi}{4}) = 1 - \frac{\pi}{2}$

7. Question

If $f(x) = x^3 + ax^2 + bx + c$ has a maximum at x = – 1 and minimum at x = 3. Determine a, b and c.

Answer

consider the function $f(x) = x^3 + ax^2 + bx + c$

Then $f'(x) = 3x^2 + 2ax + b$

It is given that f(x) is maximum at x = – 1

f'(– 1) = 3(– 1)2 + 2a(– 1) + b = 0

f'(– 1) = 3 – 3a + b = 0……(1)

it is given that f(x) is minimum at x = 3

f'(x) = 3(3)2 + 2a(3) + b = 0

f'(3) = 27 + 6a + b = 0 …… (2)

solving equation (1) and (2) we have

a = – 3 and b = – 9

since f'(x) is independent of constant c, it can be any real number

8. Question

Prove that f(x) = sin x + √3 cos x has maximum value at x = π/6.

Answer

f(x) = sin x + √3 cos x

f'(x) = cos x – √3 sin x

Now,

f'(x) = 0

cos x – √3 sin x = 0

cos x = √3 sin x

cot x = √3

$x = \frac{\pi}{6}$

Differentiate f''(x), we get

f''(x) = – sin x –√3 cos x

$f''(\frac{\pi}{6}) = -\sin\left(\frac{\pi}{6}\right) - \sqrt{3}\cos\left(\frac{\pi}{6}\right) < 0$

Hence, at x = $\frac{\pi}{6}$ is the point of local maxima.

Exercise 18.4

1 A. Question

Find the absolute maximum and the absolute minimum values of the following functions in the given intervals:

f(x) = 4x – x^2/2 in [–2, 45]

Answer

given function is f(x) = $4x - \frac{x^2}{2}$

∴f'(x) = 4 – x

Now,

f'(x) = 0

4 – x = 0

x = 4

Then, we evaluate of f at critical points x = 4 and at the interval [– 2, $\frac{9}{2}$]

f(4) = $4(4) - \frac{(4)^2}{2}$ = 8

f(– 2) = $4(-2) - \frac{(-2)^2}{2} = -10$

f($\frac{9}{2}$) = $4\left(\frac{9}{2}\right) - \frac{\left(\frac{9}{2}\right)^2}{2} = 18 - \frac{81}{8} = 7.875$

Hence, we can conclude that the absolute maximum value of f on [– 2, 9/2] is 8 occurring at x = 4 and the absolute minimum value of f on [– 2, 9/2] is – 10 occurring at x = – 2

1 B. Question

Find the absolute maximum and the absolute minimum values of the following functions in the given intervals:

f(x) = $(x – 1)^2$ + 3 in [–3, 1]

Answer

given function is f(x) = $(x - 1)^2 + 3$

∴f'(x) = 2(x – 1)

Now,

f'(x) = 0

2(x – 1) = 0

x = 1

Then, we evaluate of f at critical points x = 1 and at the interval [– 3, 1]

f(1) = $(1 – 1)^2$ + 3 = 3

f(– 3) = $(– 3 – 1)^2$ + 3 = 19

Hence, we can conclude that the absolute maximum value of f on [– 3, 1] is 19 occurring at x = – 3 and the minimum value of f on [– 3, 1] is 3 occurring at x = 1

1 C. Question

Find the absolute maximum and the absolute minimum values of the following functions in the given intervals:

$f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$ in [0, 3]

Answer

given function is $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$

$\therefore f'(x) = 12x^3 - 24x^2 + 24x - 48$

$f'(x) = 12(x^3 - 2x^2 + 2x - 4)$

$f'(x) = 12(x-2)(x^2 + 2)$

Now,

$f'(x) = 0$

$x = 2$ or $x^2 + 2 = 0$ for which there are no real roots.

Therefore, we consider only $x = 2 \in [0, 3]$.

Then, we evaluate of f at critical points x = 2 and at the interval [0, 3]

$f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 25$

f(2) = 48 – 64 + 48 – 96 + 25 = – 39

$f(0) = 3(0)^4 - 8(0)^3 + 12(0)^2 - 48(0) + 25 = 25$

$f(3) = 3(3)^4 - 8(3)^3 + 12(3)^2 - 48(3) + 25 = 16$

Hence, we can conclude that the absolute maximum value of f on [0, 3] is 25 occurring at x = 0 and the minimum value of f on [0, 3] is – 39 occurring at x = 2

1 D. Question

Find the absolute maximum and the absolute minimum values of the following functions in the given intervals:

$f(x) = (x-2)\sqrt{x-1}$ in [1, 9]

Answer

$f(x) = (x-2)\sqrt{x-1}$

$f'(x) = \sqrt{x-1} + \frac{(x-2)}{2\sqrt{x-1}}$

put

$f'(x) = 0$

$\Rightarrow \sqrt{x-1} + \frac{(x-2)}{2\sqrt{x-1}} = 0$

$\Rightarrow \frac{2(x-1) + (x-2)}{2\sqrt{x-1}} = 0$

$\Rightarrow \frac{3x-4}{2\sqrt{x-1}} = 0$

$\Rightarrow x = \frac{4}{3}$

Now,

$f(1) = 0$

$f(4/3) = \left(\frac{4}{3} - 2\right)\sqrt{\frac{4}{3} - 1} = -\frac{2}{3\sqrt{3}} = -\frac{2\sqrt{3}}{9}$

Hence, we can conclude that the absolute maximum value of f is $14\sqrt{2}$ occurring at x = 9 and the minimum value of is $-\frac{2\sqrt{3}}{9}$ occurring at x = 4/3

2. Question

Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [–3, –1].

Answer

let $f(x) = 2x^3 - 24x + 107$

$\therefore f'(x) = 6x^2 - 24 = 6(x^2 - 4)$

Now,

$f'(x) = 0$

$\Rightarrow 6(x^2 - 4) = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

We first consider the interval [1, 3].

Then, we evaluate the value of f at the critical point $x = 2 \in [1, 3]$ and at the end points of the interval [1, 3].

$f(2) = 2(2^3) - 24(2) + 107 = 75$

$f(1) = 2(1)^3 - 24(1) + 107 = 85$

$f(3) = 2(3)^3 - 24(3) + 107 = 89$

Hence, the absolute maximum value of f(x) in the interval [1, 3] is 89 occurring at x = 3,

Next, we consider te interval [– 3, – 1].

Evaluate the value of f at the critical point $x = -2 \in [1, 3]$

3. Question

Find the absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$.

Answer

$f(x) = \cos^2 x + \sin x$

$f'(x) = 2\cos x(-\sin x) + \cos x$

$= 2\sin x \cos x + \cos x$

now, $f'(x) = 0$

$\Rightarrow 2\sin x \cos x = \cos x$

$\Rightarrow \cos x(2\sin x - 1) = 0$

$\Rightarrow \sin x = 1/2$ or $\cos x = 0$

$\Rightarrow x = \frac{\pi}{6}$ or $\frac{\pi}{2}$ as $x \in [0, \pi]$

So, the critical points are $x = \frac{\pi}{2}$ and $x = \frac{\pi}{6}$ and at the end point of the interval $[0, \pi]$ we have,

$f(\frac{\pi}{6}) = \cos^2\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = 5/4$

f(0) = $\cos^2 0 + \sin 0$ = 1 + 0 = 1

f(π) = $\cos^2 \pi + \sin \pi$ = (–1^2) + 0 = 1

f($\frac{\pi}{2}$) = $\cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2}$ = 0 + 1 = 1

Thus, we conclude that the absolute maximum value of f is 5/4 at x = $\frac{\pi}{6}$, and absolute minimum value of f is 1 which occurs at x = 0, $\frac{\pi}{2}$ and π.

4. Question

Find absolute maximum and minimum values of a function f given by f(x) = $12x^{4/3} – 6x^{1/3}$, x ∈ [–1, 1].

Answer

We have,

f(x) = $12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$

f'(x) = $16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x-1)}{x^{\frac{2}{3}}}$

Thus, f'(x) = 0

⇒ x = $\frac{1}{8}$

Further note that f'(x) is not define at x = 0.

So, the critical points are x = 0 and x = $\frac{1}{8}$ and at the end point of the interval x = – 1 and x = 1

f(– 1) = $12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}}$

f(0) = $12(0)^{\frac{4}{3}} - 6(0)^{\frac{1}{3}}$

f($\frac{1}{8}$) = $12(\frac{1}{8})^{\frac{4}{3}} - 6(\frac{1}{8})^{\frac{1}{3}}$

f(1) = $12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}}$

Thus, we conclude that the absolute maximum value of f is 18 at x = 1, and absolute minimum value of f is $-\frac{9}{4}$ which occurs at x = $\frac{1}{8}$.

5. Question

Find the absolute maximum and minimum values of a function f given by f(x) = $2x^3 – 15x^2 + 36x + 1$ on the interval [1, 5].

Answer

Given,

f (x) = $2x^3 – 15x^2 + 36x + 1$

f'(x) = $6x^2 – 30x + 36$

f'(x) = $6(x^2 – 5x + 6)$ = 6 (x – 2)(x – 3)

Note that f'(x) = 0 gives x = 2 and x = 3

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5],

i.e. at x = 1, 2, 3 and 5

At x = 1, f(1) = $2(1^3) – 15(1)^2 + 36(1) + 1$ = 24

At x = 2, f(2) = $2(2^3) – 15(2)^2 + 36(2) + 1 = 29$

At x = 3, f(3) = $2(3)^3 – 15(3)^2 + 36(3) + 1 = 28$

At x = 5, f(5) = $2(5)^3 – 15(5)^2 + 36(5) + 1 = 56$

Thus, we conclude that the absolute maximum value of f on [1, 5] is 56, occurring at x = 5, and absolute value of f on [1, 5] is 24 which occurs at x = 1.

Exercise 18.5

1. Question

Determine two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Answer

Let the two positive numbers be a and b.

Given: a + b = 15 … 1

Also, $a^2 + b^2$ is minima

Assume, $S = a^2 + b^2$

(from equation 1)

$\Rightarrow S = a^2 + (15 – a)^2$

$\Rightarrow S = a^2 + 225 + a^2 – 30a = 2a^2 – 30a + 225$

$\Rightarrow \frac{dS}{da} = 4a - 30$

$\Rightarrow \frac{d^2S}{da^2} = 4$

Since, $\frac{d^2S}{da^2} > 0 \Rightarrow \frac{dS}{da} = 0$ will give minimum value of S.

4a – 30 = 0

$\Rightarrow$ a = 7.5

Hence, two numbers will be 7.5 and 7.5.

2. Question

Divide 64 into two parts such that the sum of the cubes of two parts is minimum.

Answer

Let the two positive numbers be a and b.

Given: a + b = 64 … 1

Also, $a^3 + b^3$ is minima

Assume, $S = a^3 + b^3$

(from equation 1)

$\Rightarrow S = a^3 + (64 – a)^3$

$\Rightarrow \frac{dS}{da} = 3a^2 + 3(64 – a)^2 \times (- 1)$

$\Rightarrow \frac{dS}{da} = 0$ (condition for maxima and minima)

$\Rightarrow 3a^2 + 3(64 – a)^2 \times (- 1) = 0$

$\Rightarrow 3a^2 + 3(4096 + a^2 – 128a) \times (- 1) = 0$

$\Rightarrow 3a^2 – 3 \times 4096 - 3a^2 + 424a = 0$

$\Rightarrow a = 32$

$$\frac{d^2S}{da^2} = 6a + 6(64 - a) = 384$$

Since, $\frac{d^2S}{da^2} > 0 \Rightarrow a = 32$ will give minimum value

Hence, two numbers will be 32 and 32.

3. Question

How should we choose two numbers, each greater than or equal to –2, whose sum is 1/2 so that the sum of the first and the cube of the second is minimum?

Answer

Let a and b be two numbers such that a, b ≥ - 2

Given: $a + b = \frac{1}{2}$

Assume, $S = a + b^3$

$\Rightarrow S = a + (\frac{1}{2} - a)^3$

$\Rightarrow \frac{dS}{da} = 1 + 3(\frac{1}{2} - a)^2(- 1)$

Condition maxima and minima is

$$\Rightarrow \frac{dS}{da} = 0$$

$$\Rightarrow 1 + 3\left(\frac{1}{2} - a\right)^2 - 1 = 0$$

$$\Rightarrow 3\left(\frac{1}{2} - a\right)^2 = 1$$

$$\Rightarrow \left(\frac{1}{2} - a\right)^2 = \frac{1}{3}$$

$$\Rightarrow \left(\frac{1}{2} - a\right) = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow a = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$$

$$\frac{d^2S}{da^2} = 6\left(\frac{1}{2} - a\right)$$

For S to minimum, $\frac{d^2S}{da^2} > 0$

$$\Rightarrow a = \frac{1}{2} - \frac{1}{\sqrt{3}}$$

Hence, $a = \frac{1}{2} - \frac{1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{3}}$

4. Question

Divide 15 into two parts such that the square of one multiplied with the cube of the other is minimum.

Answer

Let the two numbers be a and b.

Given: a + b = 15

Assume, $S = a^2b^3$

$\Rightarrow S = a^2(15 - a)^3$

$\frac{dS}{da} = 2a(15 - a)^3 - 3a^2(15 - a)^2$

Condition maxima and minima is

$\Rightarrow \frac{dS}{da} = 0$

$\Rightarrow 2a(15 - a)^3 - 3a^2(15 - a)^2 = 0$

$\Rightarrow a(15 - a)^2 \{2(15 – a) – 3a\} = 0$

$\Rightarrow a(15 - a)^2 \{30 – 5a\} = 0$

$\Rightarrow a = 0, 15, 6$

$\frac{d^2S}{da^2} = 2(15 - a)^3 - 6a(15 - a)^2 - 6a(15 - a)^2 + 3a^2(15 - a)$

$\Rightarrow \frac{d^2S}{da^2} = (15 - a)\{2(15 - a)^2 - 12a(15 - a) + 3a^2\}$

$\Rightarrow \frac{d^2S}{da^2} = (15 - a)\{2(15 - a)^2 - 12a(15 - a) + 3a^2\}$

For S to minimum, $\frac{d^2S}{da^2} > 0$

$a = 0 \Rightarrow \frac{d^2S}{da^2} = 450 \Rightarrow$ minima

$a = 6 \Rightarrow \frac{d^2S}{da^2} = -378 \Rightarrow$ maxima

$a = 15 \Rightarrow \frac{d^2S}{da^2} = 0 \Rightarrow$ point of inflection

Hence, two numbers are 0 and 15

5. Question

Of all the closed cylindrical cans (right circular), which enclose a given volume of 100 cm^3, which has the minimum surface area?

Answer

Let the radius and height of right circular cylinder be r and h respectively.

Given: Volume of Cylindrical can = 100 cm^3

Volume of a cylinder = $\pi r^2 h$

$\Rightarrow \pi r^2 h = 100$

$\Rightarrow h = \frac{100}{\pi r^2}$ … 1

Surface of a cylinder, $S = 2\pi rh + 2\pi r^2$

From equation we get,

$\Rightarrow\ S = 2\pi r(\frac{100}{\pi r^2}) + 2\pi r^2$

$\Rightarrow\ S = \frac{200}{r} + 2\pi r^2$

Condition for maxima and minima

$\Rightarrow \frac{dS}{dr} = 0$

$\Rightarrow\ -\frac{200}{r^2} + 4\pi r = 0$

$\Rightarrow\ 4\pi r = \frac{200}{r^2}$

$\Rightarrow r^3 = \frac{200}{4\pi}$

$\Rightarrow\ r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

$\frac{d^2S}{dr^2} = \frac{400}{r^3} + 4\pi$

So, for $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

$\Rightarrow \frac{d^2S}{dr^2} = 12\pi > 0$

This is the condition for minima

From equation 1, $h = \frac{100}{\pi r^2}$

$\Rightarrow\ h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

Hence, required dimensions of cylinders are radius = $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ and height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$

6. Question

A beam is supported at the two ends and is uniformly loaded. The bending moment M at a distance x from one end is given by

(i) $M = \frac{WL}{2}x - \frac{W}{2}x^2$ (ii) $M = \frac{Wx}{3} - \frac{W}{3}\frac{x^3}{L^2}$

Find the point at which M is maximum in each case.

Answer

Condition for maxima and minima is $\frac{dM}{dx} = 0$

And for M to maximum $\frac{d^2M}{dx^2} < 0$.

(i) M **=** $\frac{WL}{2}x - \frac{W}{2}x^2$

$\frac{dM}{dx} = 0$

$\Rightarrow \frac{WL}{2} - Wx = 0$

$\Rightarrow \frac{WL}{2} = Wx$

$\Rightarrow x = \frac{L}{2}$

$\frac{d^2M}{dx^2} = -W < 0$

Hence, for $x = \frac{L}{2}$, M will be maximum.

(ii) $M = \frac{Wx}{3} - \frac{W}{3}\frac{x^3}{L^2}$

$\frac{dM}{dx} = \frac{W}{3} - W\frac{x^2}{L^2} = 0$

$\Rightarrow \frac{W}{3} - W\frac{x^2}{L^2} = 0$

$\Rightarrow \frac{W}{3} = W\frac{x^2}{L^2}$

$\Rightarrow x^2 = \frac{L^2}{3}$

$\Rightarrow x = \pm\frac{L}{\sqrt{3}}$

$\frac{d^2M}{dx^2} = -2W\frac{x}{L^2}$

So,

For $x = \frac{L}{\sqrt{3}} \Rightarrow \frac{d^2M}{dx^2} = -\frac{2W}{\sqrt{3}L}$

$\Rightarrow \frac{d^2M}{dx^2} < 0$ (condition for maximum value)

For $x = -\frac{L}{\sqrt{3}} \Rightarrow \frac{d^2M}{dx^2} = \frac{2W}{\sqrt{3}L}$

$\Rightarrow \frac{d^2M}{dx^2} > 0$ (condition for minimum value)

Therefore, for $x = \frac{L}{\sqrt{3}}$, M will have maximum value.

7. Question

A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the circle and the square is minimum?

Answer

Let the length of side of square be a and radius of circle be r.

It is given that wire is cut into two parts to form a square and a circle

Therefore, perimeter of square + circumference of circle = length of wire

$\Rightarrow$ 4a + 2πr = 28

$\Rightarrow$ a = $\frac{28-2\pi r}{4} = \frac{14-\pi r}{2}$ ...1

Let us assume area of square + area of circle = S

$\Rightarrow$ S = $a^2 + \pi r^2$

$\Rightarrow$ S = $\left(\frac{14-\pi r}{2}\right)^2 + \pi r^2$ (from equation 1)

Condition for maxima and minima

$\Rightarrow \frac{dS}{dr} = 0$

$\Rightarrow 2\left(-\frac{\pi}{2}\right)\left(\frac{14-\pi r}{2}\right) + 2\pi r = 0$

$\Rightarrow 2(\frac{\pi}{2})\left(\frac{14-\pi r}{2}\right) = 2\pi r$

$\Rightarrow (14-\pi r) = 4r$

$\Rightarrow r = \frac{14}{4+\pi}$ ...2

$\frac{d^2S}{dr^2} = \frac{\pi^2}{2} + 2\pi$

So, for $r = \frac{14}{4+\pi} \Rightarrow \frac{d^2S}{dr^2} = \frac{\pi^2}{2} + 2\pi > 0$

This is the condition for minima

From equation 1

a = $\frac{14-\pi r}{2}$

Substituting from equation 2

$\Rightarrow$ a = $\frac{14-\pi\frac{14}{4+\pi}}{2}$

$\Rightarrow$ a = $7 - \frac{7\pi}{4+\pi}$

$\Rightarrow$ a = $\frac{28+7\pi-7\pi}{4+\pi}$

$\Rightarrow a = \frac{28}{4+\pi}$

Hence, radius of circle and length of square be $\frac{14}{4+\pi}$ and $\frac{28}{4+\pi}$ respectively.

8. Question

A wire of length 20 m is to be cut into two pieces. One of the pieces will be bent into shape of a square and the other into shape of an equilateral triangle. Where the wire should be cut so that the sum of the areas of the square and triangle is minimum?

Answer

Let the length of side of square and equilateral triangle be a and r respectively.

It is given that wire is cut into two parts to form a square and a equilateral triangle

Therefore, perimeter of square + perimeter of equilateral triangle = length of wire

$\Rightarrow$ 4a + 3r = 20

$\Rightarrow a = \frac{20-3r}{4}$ ...1

Let us assume area of square + area of circle = S

$\Rightarrow S = a^2 + \frac{\sqrt{3}}{4}r^2$

$\Rightarrow S = \left(\frac{20-3r}{4}\right)^2 + \frac{\sqrt{3}}{4}r^2$ (from equation 1)

Condition for maxima and minima

$$\Rightarrow \frac{dS}{dr} = 0$$

$\Rightarrow 2(-\frac{3}{4})\left(\frac{20-3r}{4}\right) + \frac{\sqrt{3}}{2}r = 0$

$$\Rightarrow 2\left(\frac{3}{4}\right)\left(\frac{20-3r}{4}\right) = \frac{\sqrt{3}}{2}r$$

$\Rightarrow 60 - 9r = 4\sqrt{3}r$

$\Rightarrow r = \frac{60}{9+4\sqrt{3}}$ ...2

$$\frac{d^2S}{dr^2} = \left(\frac{9}{8}\right) + \frac{\sqrt{3}}{2}$$

So, for $r = \frac{60}{9+4\sqrt{3}} \Rightarrow \frac{d^2S}{dr^2} = \left(\frac{9}{8}\right) + \frac{\sqrt{3}}{2} > 0$

This is the condition for minima

From equation 1

$$a = \frac{20-3r}{4}$$

Substituting from equation 2

$$\Rightarrow a = \frac{20 - 3\frac{60}{9+4\sqrt{3}}}{4}$$

$$\Rightarrow a = 5 - \frac{45}{9+4\sqrt{3}}$$

$$\Rightarrow a = \frac{45 + 20\sqrt{3} - 45}{9+4\sqrt{3}}$$

$$\Rightarrow a = \frac{20\sqrt{3}}{9+\sqrt{3}}$$

Hence, side of equilateral triangle and length of square be $\frac{60}{9+4\sqrt{3}}$ and $\frac{20\sqrt{3}}{9+\sqrt{3}}$ respectively.

9. Question

Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when one side of the square is equal to diameter of the circle.

Answer

Let us say the sum of perimeter of square and circumference of circle be L

Given: Sum of the perimeters of a square and a circle.

Assuming, side of square = a and radius of circle = r

Then, $L = 4a + 2\pi r$ ...1

Let the sum of area of square and circle be S

So, $S = a^2 + \pi r^2$

$\Rightarrow S = \left(\frac{L-2\pi r}{4}\right)^2 + \pi r^2$

Condition for maxima and minima

$\Rightarrow \frac{dS}{dr} = 0$

$\Rightarrow (2)(-2\pi)\left(\frac{L-2\pi r}{16}\right) + 2\pi r = 0$

$\Rightarrow (2)(2\pi)\left(\frac{L-2\pi r}{16}\right) = 2\pi r$

$\Rightarrow L - 2\pi r = 8r$

$\Rightarrow (8 + 2\pi)r = L$

$\Rightarrow r = \frac{L}{8 + 2\pi}$

$\frac{d^2S}{dr^2} = \frac{\pi^2}{2} + 2\pi$

So, for $r = \frac{L}{8+2\pi} \Rightarrow \frac{d^2S}{dr^2} = \frac{\pi^2}{2} + 2\pi > 0$

This is the condition for minima

From equation 1

$a = \frac{L - 2\pi r}{4}$

Substituting from equation 2

$\Rightarrow a = \frac{L - 2\pi \frac{L}{8 + 2\pi}}{4}$

$\Rightarrow a = \frac{8L + 2\pi L - 2\pi L}{4(8 + 2\pi)}$

$\Rightarrow a = \frac{8L}{4(8 + 2\pi)}$

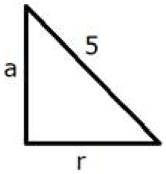
$\Rightarrow a = 2r$

Hence, proved.

10. Question

Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long.

Answer



Given: Hypotenuse = 5 cm.

Let the two sides of right angle triangle be a and r.

According to the Pythagoras theorem, $a^2 + r^2 = 25$ ...1

Assuming, area of triangle be, $A = \frac{1}{2}ar$

$\Rightarrow A = \frac{1}{2}\sqrt{25 - r^2}\,.r$

Condition for maxima and minima

$\Rightarrow \frac{dA}{dr} = 0$

$\Rightarrow \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(-2r)(\frac{1}{25-r^2})^{\frac{1}{2}} \cdot r + \left(\frac{1}{2}\right)(25 - r^2)^{\frac{1}{2}} = 0$

$\Rightarrow (-r^2)(\frac{1}{25-r^2})^{\frac{1}{2}} + (25 - r^2)^{\frac{1}{2}} = 0$

$\Rightarrow (r^2)(\frac{1}{25 - r^2})^{\frac{1}{2}} = (25 - r^2)^{\frac{1}{2}}$

$\Rightarrow r^2 = 25 - r^2$

$\Rightarrow 2r^2 = 25$

$\Rightarrow r = \pm\frac{5\sqrt{2}}{2}$

Since, r cannot be negative

Therefore, $r = \frac{5\sqrt{2}}{2}$

$\frac{d^2A}{dr^2} = (r^3)(-\frac{1}{2})(\frac{1}{25 - r^2})^{\frac{3}{2}} + (-r)(\frac{1}{25 - r^2})^{\frac{1}{2}} + (\frac{1}{2})(-r)(\frac{1}{25 - r^2})^{\frac{1}{2}}$

So, for $r = \frac{5\sqrt{2}}{2}$,

$\frac{d^2A}{dr^2} < 0 \Rightarrow$ Area will be maximum

From equation 1,

$a^2 + r^2 = 25$

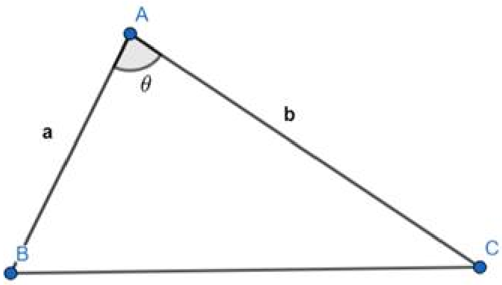
$\Rightarrow a = \frac{5\sqrt{2}}{2}$

$A = \frac{1}{2}ar = \frac{50}{4}\,\text{sq. units}$

11. Question

Two sides of a triangle have lengths ‘a’ and ‘b’ and the angle between them is θ. What value of θ will maximize the area of the triangle? Find the maximum area of the triangle also.

Answer



It is given that Two sides of a triangle have lengths 'a' and 'b' and the angle between them is θ.

Let the area of triangle be A

Then, $A = \frac{1}{2}ab\operatorname{Sin}\theta$

$\Rightarrow \frac{dA}{d\theta} = \frac{1}{2}ab\operatorname{Cos}\theta$

Condition for maxima and minima is

$\frac{dA}{d\theta} = 0$

$\Rightarrow \frac{1}{2}ab\operatorname{Cos}\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

$\frac{d^2A}{d\theta^2} = -\frac{1}{2}ab\operatorname{Sin}\theta$

So, for A to be maximum,

$\frac{d^2A}{d\theta^2} < 0$

For $\theta = \frac{\pi}{2} \Rightarrow \frac{d^2A}{d\theta^2} < 0$

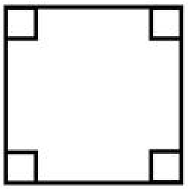
Hence, $\theta = \frac{\pi}{2}$ will give maximum area.

And maximum area will be $A = \frac{1}{2}ab$

12. Question

A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume

Answer



Side length of big square is 18 cm

Let the side length of each small square be a.

If by cutting a square from each corner and folding up the flaps we will get a cuboidal box with

Length, L = 18 – 2a

Breadth, B = 18 – 2a and

Height, H = a

Assuming,

Volume of box, V = LBH = $a(18 - 2a)^2$

Condition for maxima and minima is

$\frac{dV}{da} = 0$

$\Rightarrow$ (18 - 2a)^2 + (a)(- 2)(2)(18 - 2a) = 0

$\Rightarrow$ (18 - 2a)[(18 - 2a) - 4a] = 0

$\Rightarrow$ (18 - 2a)[18 – 6a] = 0

$\Rightarrow$ a = 3, 9

$\frac{d^2V}{da^2}$ = (- 2)(18 – 6a) + (- 6)(18 – 2a)

$\Rightarrow \frac{d^2V}{da^2} = 24a - 144$

For a = 3, $\frac{d^2V}{da^2}$ = - 72, $\Rightarrow \frac{d^2A}{d\theta^2} < 0$

For a = 9, $\frac{d^2V}{da^2}$ = 72, $\Rightarrow \frac{d^2A}{d\theta^2} > 0$

So for A to maximum

$\frac{d^2A}{d\theta^2} < 0$

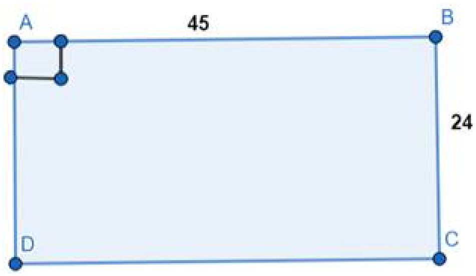
Hence, a = 3 will give maximum volume.

And maximum volume, V = a(18 - 2a)^2 = 432 cm^3

13. Question

A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum possible?

Answer



Length of rectangle sheet = 45 cm

Breath of rectangle sheet = 24 cm

Let the side length of each small square be a.

If by cutting a square from each corner and folding up the flaps we will get a cuboidal box with

Length, L = 45 – 2a

Breadth, B = 24 – 2a and

Height, H = a

Assuming,

Volume of box, V = LBH = (45 - 2a)(24 - 2a)(a)

Condition for maxima and minima is

$\frac{dV}{da} = 0$

$\Rightarrow$ (45 - 2a)(24 - 2a) + (- 2)(24 - 2a)(a) + (45 - 2a)(- 2)(a) = 0

$\Rightarrow$ $4a^2 - 138a + 1080 + 4a^2 - 48a + 4a^2 - 90a = 0$

$\Rightarrow$ $12a^2 - 276a + 1080 = 0$

$\Rightarrow$ $a^2 - 23a + 90 = 0$

$\Rightarrow$ a = 5, 18

$\frac{d^2V}{da^2} = 24a - 276$

For a = 5, $\frac{d^2V}{da^2} = -156$, $\Rightarrow \frac{d^2A}{d\theta^2} < 0$

For a = 18, $\frac{d^2V}{da^2} = +156$, $\Rightarrow \frac{d^2A}{d\theta^2} > 0$

So for A to maximum

$\frac{d^2A}{d\theta^2} < 0$

Hence, a = 5 will give maximum volume.

And maximum volume V = (45 - 2a)(24 - 2a)(a) = 2450 cm^3

14. Question

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank cost Rs 70 per square metre for the base and Rs 45 per square matre for sides, what is the cost of least expensive tank?

Answer

Let the length, breath and height of tank be l, b and h respectively.

Also, assume volume of tank as V

h = 2 m (given)

V = 8 cm^3

$\Rightarrow$ lbh = 8

$\Rightarrow$ 2lb = 8 (given)

$\Rightarrow$ lb = 4

$\Rightarrow$ $b = \frac{4}{l}$ ...1

Cost for building base = Rs 70/m^2

Cost for building sides = Rs 45/m^2

Cost for building the tank, C = Cost for base + cost for sides

$\Rightarrow$ C = lb × 70 + 2(l + b) h × 45

$\Rightarrow$ $C = l \times \frac{4}{l} \times 70 + 2(l+\frac{4}{l}) \times 2 \times 45$

$\Rightarrow$ $C = 280 + 180(l+\frac{4}{l})$ ...2

Condition for maxima and minima

$\Rightarrow \frac{dC}{dl} = 0$

$\Rightarrow 180(1 - \frac{4}{l^2}) = 0$

$\Rightarrow \frac{4}{l^2} = 1$

$\Rightarrow l^2 = 4$

$\Rightarrow l = \pm 2$ cm

Since, l cannot be negative

So, l = 2 cm

$\frac{d^2C}{dl^2} = 180\left(\frac{8}{l^3}\right)$

For l = 2 $\frac{d^2C}{dl^2} = 180 \Rightarrow \frac{d^2C}{dl^2} > 0$

Therefore, cost will be minimum for l =2

From equation 2

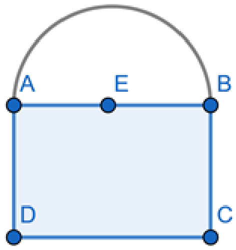
$C = 280 + 180(l+\frac{4}{l})$

$\Rightarrow$ C = Rs 1000

15. Question

A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.

Answer



Let the radius of semicircle, length and breadth of rectangle be r, x and y respectively

AE = r

AB = x = 2r (semicircle is mounted over rectangle) …1

AD = y

Given: Perimeter of window = 10 m

$x + 2y + \pi r = 10$

$\Rightarrow 2r + 2y + \pi r = 10$

$\Rightarrow 2y = 10 - (\pi + 2).r$

$\Rightarrow y = \frac{10 - (\pi + 2)r}{2}$ … 2

To admit maximum amount of light, area of window should be maximum

Assuming area of window as A

$A = xy + \frac{\pi r^2}{2}$

$\Rightarrow A = (2r)\left(\frac{10-(\pi+2)r}{2}\right) + \frac{\pi r^2}{2}$

$\Rightarrow A = 10r - \pi r^2 - 2r^2 + \frac{\pi r^2}{2}$

$\Rightarrow A = 10r - 2r^2 - \frac{\pi r^2}{2}$

Condition for maxima and minima is

$$\frac{dA}{dr} = 0$$

$\Rightarrow 10 - 4r - \pi r = 0$

$$\Rightarrow r = \frac{10}{4+\pi}$$

$\frac{d^2A}{dr^2} = -4 - \pi < 0$

For $r = \frac{10}{4+\pi}$ A will be maximum.

Length of rectangular part $= \frac{20}{4+\pi}\,m$ (from equation 1)

Breath of rectangular part $= \frac{10-(\pi+2)r}{2}\,m$ (from equation 2)

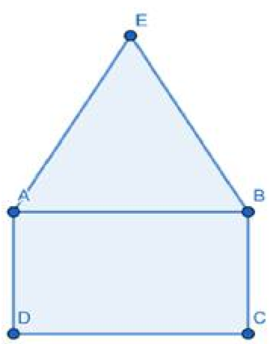
$$\Rightarrow y = \frac{10 - \frac{(\pi+2)10}{4+\pi}}{2}$$

$$\Rightarrow y = \frac{10}{4+\pi}$$

16. Question

A large window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 metres find the dimensions of the rectangle that will produce the largest area of the window.

Answer



Let the side of equilateral triangle, length and breadth of rectangle be a, x and y respectively

AE = AB = a (ABE is equilateral triangle)

AB = x = a (triangle is mounted over rectangle) ...1

AD = y

Perimeter of window = 12 m (given)

$\Rightarrow$ AE + EB + BC + CD + DA = 12

⇒ a + a + y + x + y = 10

⇒ 2a + 2y + x = 10

⇒ 3x + 2y = 12 (from equation 1)

⇒ $y = \frac{12-3x}{2}$ … 2

To admit maximum amount of light, area of window should be maximum

Assuming area of window as A

$A = xy + \frac{\sqrt{3}}{4}a^2$

$\Rightarrow A = (x)\left(\frac{12-3x}{2}\right) + \frac{\sqrt{3}}{4}x^2$ (from equation 1 & 2)

$$\Rightarrow A = 6x + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2$$

Condition for maxima and minima is

$$\frac{dA}{dx} = 0$$

$$\Rightarrow 6 + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)(2)x = 0$$

$$\Rightarrow x = \frac{6}{\left(\frac{3}{2} - \frac{\sqrt{3}}{4}\right)(2)}$$

$$\Rightarrow x = \frac{12}{(6-\sqrt{3})}$$

$$\Rightarrow x = \frac{12}{(6-\sqrt{3})} \times \frac{(6+\sqrt{3})}{(6+\sqrt{3})} = \frac{4(6+\sqrt{3})}{11}$$

$$\frac{d^2A}{dr^2} = \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)(2) = -2.14 < 0$$

For $x = \frac{4(6+\sqrt{3})}{11}$ A will be maximum.

Length of rectangular part = $\frac{4(6+\sqrt{3})}{11}$ m (from equation 1)

Breath of rectangular part = $\frac{12-3x}{2}$ m (from equation 2)

$$\Rightarrow y = \frac{12 - 3.\frac{4(6+\sqrt{3})}{11}}{2}$$

$$y = \frac{132 - 72 - 12\sqrt{3}}{22}$$

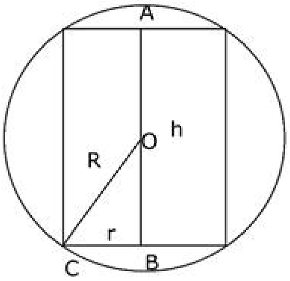
$$y = \frac{30 - 6\sqrt{3}}{11}$$

17. Question

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$

.

Answer



Let the radius, height and volume of cylinder be r, h and V respectively

Radius of sphere = R (Given)

Volume of cylinder, $V = \pi r^2 h$ ...1

$OB = \frac{h}{2}$

OC = R

BC = r

In triangle OBC,

$\left(\frac{h}{2}\right)^2 + r^2 = R^2$

$\Rightarrow r^2 = R^2 - \left(\frac{h}{2}\right)^2$ ...2

Replacing equation 2 in equation 1, we get

$V = \pi\left(R^2 - \left(\frac{h}{2}\right)^2\right)(h) = \pi R^2 h - \pi\frac{h^3}{4}$

Condition for maxima and minima is

$\frac{dV}{dh} = 0$

$\Rightarrow \pi R^2 - \pi\frac{3h^2}{4} = 0$

$\Rightarrow \pi R^2 = \pi\frac{3h^2}{4}$

$\Rightarrow h^2 = \frac{4}{3}R^2$

$\Rightarrow h = \pm\frac{2}{\sqrt{3}}R$

Since, h cannot be negative

Hence, $h = \frac{2}{\sqrt{3}}R$

$\frac{d^2V}{dh^2} = -\pi\frac{6h}{4}$

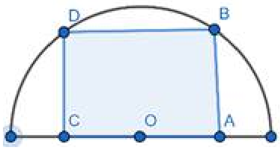
For $h = \frac{2}{\sqrt{3}}R$ $\frac{d^2V}{dh^2} < 0$

$\Rightarrow$ V will be maximum for $h = \frac{2}{\sqrt{3}}R$

18. Question

A rectangle is inscribed in a semi-circle of radius r with one of its sides on diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find also the area.

Answer



Let the length and breadth of rectangle ABCD be 2x and y respectively

Radius of semicircle = r (given)

In triangle OBA

$r^2 = x^2 + y^2$ (Pythagoras theorem)

$\Rightarrow y^2 = r^2 - x^2$

$\Rightarrow y = \sqrt{r^2 - x^2}$ ...1

Let us say, area of rectangle = A =xy

$\Rightarrow$ A = x ($\sqrt{r^2 - x^2}$) (from equation 1)

Condition for maxima and minima is

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \sqrt{r^2 - x^2} + x\left(\frac{1}{\sqrt{r^2 - x^2}}\right)\left(\frac{1}{2}\cdot(-2x)\right) = 0$$

$$\Rightarrow \sqrt{r^2 - x^2} - \left(\frac{x^2}{\sqrt{r^2 - x^2}}\right) = 0$$

$$\Rightarrow \sqrt{r^2 - x^2} = \frac{x^2}{\sqrt{r^2 - x^2}}$$

$\Rightarrow r^2 - x^2 = x^2$

$\Rightarrow 2x^2 = r^2$

$\Rightarrow x = \pm\frac{r}{\sqrt{2}}$

Since, x cannot be negative

Hence, $x = \frac{r}{\sqrt{2}}$

$$\frac{d^2A}{dx^2} = \frac{-2x}{\sqrt{r^2 - x^2}} - \left(\frac{2x\sqrt{r^2 - x^2} - x^2\frac{-2x}{\sqrt{r^2 - x^2}}}{\left(\sqrt{r^2 - x^2}\right)^2}\right)$$

For $x = \frac{r}{\sqrt{2}}$, $\frac{d^2A}{dx^2} < 0$

$\Rightarrow$ A will be maximum for $x = \frac{r}{\sqrt{2}}$

From equation 1

$y = \sqrt{r^2 - x^2} = \frac{r}{\sqrt{2}}$

Length of rectangle = $\sqrt{2}r$

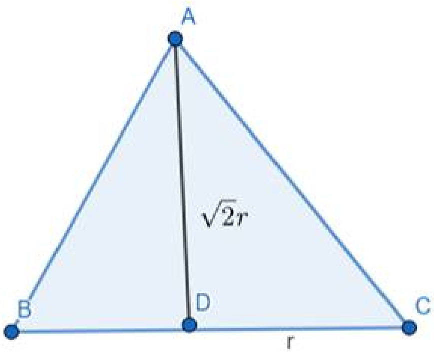
Breadth of rectangle = $\frac{r}{\sqrt{2}}$

Area of rectangle = r^2

19. Question

Prove that a conical tent of given capacity will require the least amount of canvas when the height is √2 times the radius of the base.

Answer



Let the radius and height of cone be r and h respectively

It is given that volume of cone is fixed.

Volume of cone, V = $\frac{1}{3}\pi r^2 h$

$\Rightarrow$ h = $\frac{3V}{\pi r^2}$ ...1

Curved surface area of cone, S = πrl (l is slant height)

Since, $l = \sqrt{r^2 + h^2}$

$$\Rightarrow\ l = \sqrt{r^2 + \left(\frac{3V}{\pi r^2}\right)^2}$$

$$\Rightarrow\ l = \sqrt{\frac{\pi^2 r^6 + 9V^2}{\pi^2 r^4}}$$

$$\Rightarrow\ l = \frac{\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2}$$

So, $S = \pi r \frac{\sqrt{\pi^2 r^6 + 9V^2}}{\pi r^2}$

$$\Rightarrow\ S = \frac{\sqrt{\pi^2 r^6 + 9V^2}}{r}$$

Condition for maxima and minima is

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \frac{\frac{3\pi^2 r^5}{\sqrt{\pi^2 r^6 + 9V^2}} \cdot r - \sqrt{\pi^2 r^6 + 9V^2}}{r^2} = 0$$

$$\Rightarrow \frac{3\pi^2 r^6 - \pi^2 r^6 - 9V^2}{r^2\sqrt{\pi^2 r^6 + 9V^2}} = 0$$

$$\Rightarrow\ 2\pi^2 r^6 - 9V^2 = 0$$

$\Rightarrow\ 2\pi^2 r^6 = 9V^2$ ...2

$\Rightarrow \quad r = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}$

For $r = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}, \frac{d^2S}{dr^2} > 0$

$\Rightarrow$ S will be minimum for $r = \left(\frac{9V^2}{2\pi^2}\right)^{\frac{1}{6}}$

From equation 1

$h = \frac{3}{\pi r^2} \cdot \frac{\sqrt{2}\pi r^3}{3}$ (from equation 3)

$\Rightarrow h = \sqrt{2}\, r$

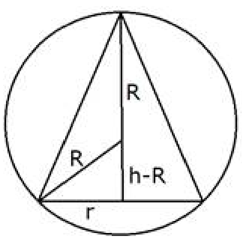
20. Question

Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to 2/3 of the diameter of the sphere.

Answer

Let the radius and height of cone be r and h respectively

Radius of sphere = R



$R^2 = r^2 + (h - R)^2$

$\Rightarrow R^2 = r^2 + h^2 + R^2 - 2hR$

$\Rightarrow r^2 = 2hR - h^2$ ...1

Assuming volume of cone be V

Volume of cone, $V = \frac{1}{3}\pi(2hR - h^2)h$ (from equation 1)

$\Rightarrow \quad V = \frac{1}{3}\pi(2h^2R - h^3)$

Condition for maxima and minima is

$\frac{dV}{dh} = 0$

$\Rightarrow \frac{1}{3}\pi(4hR - 3h^2) = 0$

$\Rightarrow 4hR - 3h^2 = 0$

$\Rightarrow \quad h = \frac{4R}{3}$

For $h = \frac{4R}{3}, \frac{d^2V}{dh^2} < 0$

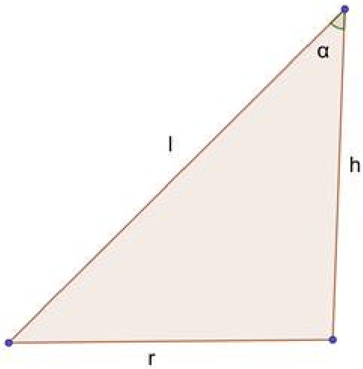
$\Rightarrow$ V will be maximum for $h = \frac{4R}{3}$

$h = \frac{2}{3}(2R)$

21. Question

Prove that the semi - vertical angle of the right circular cone of given volume and least curved surface is $\cot^{-1}(\sqrt{2})$.

Answer



Let ‘r’ be the radius of the base circle of the cone and ‘l’ be the slant length and ‘h’ be the height of the cone:

Let us assume ‘α’ be the semi - vertical angle of the cone.

We know that Volume of a right circular cone is given by:

$\Rightarrow V = \frac{\pi r^2 h}{3}$

Let us assume r^2h = k(constant) ...... (1)

$\Rightarrow V = \frac{\pi k}{3}$

$\Rightarrow h = \frac{k}{r^2}$ ...... (2)

We know that surface area of a cone is

$\Rightarrow S = \pi r l$ ...... (3)

From the cross - section of cone we see that,

$\Rightarrow l^2 = r^2 + h^2$

$\Rightarrow l = \sqrt{r^2 + h^2}$ ...... (4)

Substituting (4) in (3), we get

$\Rightarrow S = \pi r\left(\sqrt{r^2 + h^2}\right)$

From (2)

$\Rightarrow S = \pi r\left(\sqrt{r^2 + \left(\frac{k}{r^2}\right)^2}\right)$

$\Rightarrow S = \pi r\left(\sqrt{r^2 + \left(\frac{k^2}{r^4}\right)}\right)$

$\Rightarrow S = \pi r\left(\sqrt{\frac{r^6 + k^2}{r^4}}\right)$

$\Rightarrow S = \pi r\left(\frac{\sqrt{r^6 + k^2}}{r^2}\right)$

$\Rightarrow S = \left(\frac{\pi \times \sqrt{r^6 + k^2}}{r}\right)$

Let us consider S as a function of R and We find the value of 'r' for its extremum,

Differentiating S w.r.t r we get

$\Rightarrow \frac{dS}{dr} = \frac{d}{dr}\left(\frac{\pi\sqrt{r^6 + k^2}}{r}\right)$

Differentiating using U/V rule

$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(r \times \frac{d\left(\sqrt{r^6 + k^2}\right)}{dr} - \left(\sqrt{r^6 + k^2}\right)\frac{dr}{dr}\right)}{r^2}$

$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(r \times \frac{1}{2\sqrt{r^6 + k^2}} \times \frac{d(r^6 + k^2)}{dr} - \left(\sqrt{r^6 + k^2} \times 1\right)\right)}{r^2}$

$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(\frac{r \times 6r^5}{2\sqrt{r^6 + k^2}} - \sqrt{r^6 + k^2}\right)}{r^2}$

$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(\frac{3r^6}{\sqrt{r^6 + k^2}} - \sqrt{r^6 + k^2}\right)}{r^2}$

$\Rightarrow \frac{dS}{dr} = \frac{\pi\left(\frac{3r^6 - (r^6 + k^2)}{\sqrt{r^6 + k^2}}\right)}{r^2}$

$\Rightarrow \frac{dS}{dr} = \frac{\pi(2r^6 - k^2)}{r^2\sqrt{r^6 + k^2}}$

Equating the differentiate to zero to get the relation between h and r.

$\Rightarrow \frac{dS}{dr} = 0$

$\Rightarrow \frac{\pi(2r^6 - k^2)}{r^2 \times \sqrt{r^6 + k^2}} = 0$

Since the remainder is greater than zero only the remainder gets equal to zero

$\Rightarrow 2r^6 = k^2$

From(1)

$\Rightarrow 2r^6 = (r^2h)^2$

$\Rightarrow 2r^6 = r^4h^2$

$\Rightarrow 2r^2 = h^2$

Since height and radius cannot be negative,

$\Rightarrow h = \sqrt{2}r$ …… (5)

From the figure

$\Rightarrow \cot\alpha = \frac{h}{r}$

From(5)

$\Rightarrow \cot\alpha = \sqrt{2}$

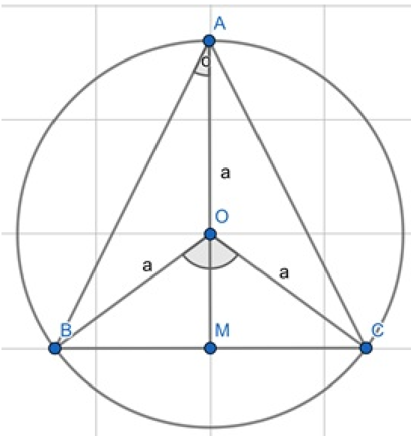
$\Rightarrow \alpha = \cot^{-1}\sqrt{2}$

∴ Thus proved.

22. Question

An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of the triangle is maximum when $\theta = \frac{\pi}{6}$.

Answer



Δ ABC is an isosceles triangle such that AB = AC.

The vertical angle BAC = 2θ

Triangle is inscribed in the circle with center O and radius a.

Draw AM perpendicular to BC.

Since, Δ ABC is an isosceles triangle, the circumcenter of the circle will lie on the perpendicular from A to BC.

Let O be the circumcenter.

BOC = 2×2θ = 4θ (Using central angle theorem)

COM = 2θ (Since, Δ OMB and Δ OMC are congruent triangles)

OA = OB = OC = a (radius of the cicle)

In Δ OMC,

CM = asin2θ

OM = acos2θ

BC = 2CM (Perpendicular from the center bisects the chord)

BC = 2asin2θ

Height of Δ ABC = AM = AO + OM

AM = a + acos2θ

$$\text{Area of } \Delta \text{ ABC } = \frac{1}{2} \times \text{AM} \times \text{BC}$$

Differentiation this equation with respect to θ

$$\frac{dA}{d\theta} = \frac{d\left[\frac{1}{2} \times (a + a\cos 2\theta) \times (2a\sin 2\theta)\right]}{d\theta}$$

$$\frac{dA}{d\theta} = (2a\sin 2\theta)(-2a\sin 2\theta) + (a + a\cos 2\theta)(2a\cos 2\theta)$$

$$\frac{dA}{d\theta} = (-2a^2\sin^2 2\theta) + (2a^2\cos 2\theta + 2a^2\cos^2 2\theta)$$

$\Rightarrow \frac{dA}{d\theta} = 2a^2(\cos^2 2\theta - \sin^2 2\theta) + 2a^2\cos 2\theta$

$\Rightarrow \frac{dA}{d\theta} = 2a^2(\cos 4\theta) + 2a^2\cos 2\theta \ (\cos^2 x - \sin^2 x = \cos 2x)$

$\Rightarrow \frac{d^2A}{d\theta^2} = -2 \times 4 \times a^2(\sin 4\theta) + (-4a^2\sin 2\theta)$

Maxima or minima exists when:

$\frac{dA}{d\theta} = 0$

Therefore,

$2a^2(\cos 4\theta) + 2a^2\cos 2\theta = 0$

$\Rightarrow \cos 4\theta + \cos 2\theta = 0$

$\Rightarrow 2\cos^2\theta - 1 + \cos 2\theta = 0$

$\Rightarrow (2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$

Therefore,$\Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3}$

and $\cos 2\theta = -1 \Rightarrow 2\theta = \pi$

$\theta = \frac{\pi}{6}, \frac{\pi}{2}$

To check whether which point has a maxima, we have to check the double differentiate.

Therefore, at $\theta = \frac{\pi}{6}$:

$\frac{d^2A}{d\theta^2} = -2 \times 4 \times a^2\left(\sin 4 \times \frac{\pi}{6}\right) + (-4a^2 \sin 2 \times \frac{\pi}{6})$

$\frac{d^2A}{d\theta^2} = -2 \times 4 \times a^2\left(\sin\frac{2\pi}{3}\right) + (-4a^2\sin\frac{\pi}{3})$

Both the sin values are positive. So the entire expression is negative. Hence there is a maxima at this point.

$\theta = \frac{\pi}{2}$ will not form a triangle. Hence it is discarded.

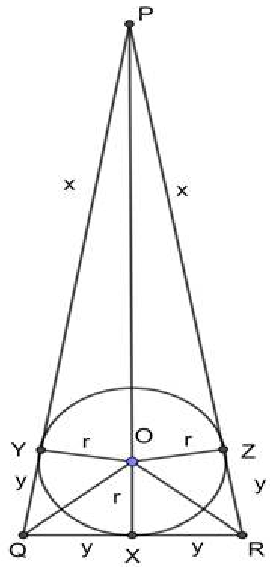
There fore the maxima exits at:

$\theta = \frac{\pi}{6}$

23. Question

Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is 6√3r.

Answer



Let PQR is the triangle with inscribed circle of radius ‘r’, touching sides PQ at Y, QR at X and PR at Z.

OZ, OX, OY are perpendicular to the sides PR,QR,PQ.

Here PQR is an isosceles triangle with sides PQ = PR and also from the figure,

⇒ PY = PZ = x

⇒ YQ = QX = XR = RZ = y

From the figure we can see that,

⇒ Area(ΔPQR) = Area(ΔPOR) + Area(ΔPOQ) + Area(ΔQOR)

We know that area of a triangle = $\frac{1}{2}$×base×height

$$\Rightarrow \frac{1}{2} \times QR \times PX = \left(\frac{1}{2} \times OZ \times PR\right) + \left(\frac{1}{2} \times OY \times PQ\right) + \left(\frac{1}{2} \times QR \times OX\right)$$

$$\Rightarrow \frac{1}{2} \times 2y\left(r + \sqrt{x^2 + r^2}\right) = \left(\frac{1}{2} \times r \times (x + y)\right) + \left(\frac{1}{2} \times r \times (x + y)\right) + \left(\frac{1}{2} \times 2y \times r\right)$$

$$\Rightarrow y\left(r + \sqrt{x^2 + r^2}\right) = r(x + y) + yr$$

$$\Rightarrow y\left(\sqrt{x^2 + r^2}\right) = r(x + y)$$

$$\Rightarrow \sqrt{x^2 + r^2} = \frac{r(x+y)}{y}$$

$$\Rightarrow x^2 + r^2 = \frac{r^2(x+y)^2}{y^2}$$

$$\Rightarrow x^2 + r^2 = r^2 + \frac{r^2x^2}{y^2} + \frac{r^2(2xy)}{y^2}$$

$$\Rightarrow x^2\left(1 - \frac{r^2}{y^2}\right) - \frac{r^2(2xy)}{y^2} = 0$$

$$\Rightarrow x\left(x\left(1 - \frac{r^2}{y^2}\right) - \frac{r^2(2y)}{y^2}\right) = 0$$

$$\Rightarrow x = \frac{2r^2y}{y^2 - r^2} \text{ ...... (1)}$$

We know that perimeter of the triangle is Per = PQ + QR + RP

⇒ PER = (x + y) + (x + y) + 2y

⇒ PER = 2x + 4y …… (2)

From(1)

$$\Rightarrow \text{PER} = \frac{4r^2y}{y^2-r^2} + 4y$$

$$\Rightarrow \text{PER} = \frac{4y(r^2+y^2-r^2)}{y^2-r^2}$$

$$\Rightarrow \text{PER} = \frac{4y^3}{y^2-r^2}$$

We need perimeter to be minimum and let us PER as the function of y,

We know that for maxima and minima $\frac{d(\text{PER})}{dy} = 0$,

$$\Rightarrow \frac{d(\text{PER})}{dy} = \frac{d\left(\frac{4y^3}{y^2-r^2}\right)}{dr}$$

$$\Rightarrow \frac{d(\text{PER})}{dy} = \frac{(y^2-r^2)\frac{d(4y^3)}{dy}-(4y^3)\frac{d(y^2-r^2)}{dy}}{(y^2-r^2)^2}$$

$$\Rightarrow \frac{d(\text{PER})}{dy} = \frac{((y^2-r^2)(12y^2))-((4y^3)(2y))}{(y^2-r^2)^2}$$

$$\Rightarrow \frac{d(\text{PER})}{dy} = \frac{4y^4-12y^2r^2}{(y^2-r^2)^2}$$

$\Rightarrow 4y^4 - 12y^2r^2 = 0$

$\Rightarrow 4y^2(y^2 - 3r^2) = 0$

$$\Rightarrow y = \sqrt{3}r$$

Differentiating PER again,

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2} = \frac{d}{dy}\left(\frac{6y^4-12y^2r^2}{(y^2-r^2)^2}\right)$$

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2} = \frac{(y^2-r^2)^2\frac{d(6y^4-12y^2r^2)}{dy}-(6y^4-12y^2r^2)\frac{d}{dy}\left((y^2-r^2)^2\right)}{(y^2-r^2)^4}$$

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2} = \frac{((y^2-r^2)^2(24y^3-24yr^2))-\left((6y^4-12y^2r^2)\left(2(y^2-r^2)(2y)\right)\right)}{(y^2-r^2)^4}$$

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{\left((3r^2-r^2)^2(72\sqrt{3}r^3-24\sqrt{3}r^3)\right)-\left((54r^4-36r^4)\left(2(3r^2-r^2)(2\sqrt{3}r)\right)\right)}{(3r^2-r^2)^4}$$

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{\left((4r^4)(48\sqrt{3}r^3)\right)-\left((18r^4)(8\sqrt{3}r^3)\right)}{16r^8}$$

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{48\sqrt{3}r^7}{16r^8}$$

$$\Rightarrow \frac{d^2(\text{PER})}{dy^2}\Big|_{y=\sqrt{3}r} = \frac{3\sqrt{3}}{r} > 0 \text{(minima)}$$

We got minima at $y = \sqrt{3}r$.

Let's find the value of x,

$$\Rightarrow x = \frac{2r^2(\sqrt{3}r)}{(\sqrt{3}r)^2-r^2}$$

$\Rightarrow x = \sqrt{3}r$

⇒ PER = $2(\sqrt{3}r) + 4(\sqrt{3}r)$

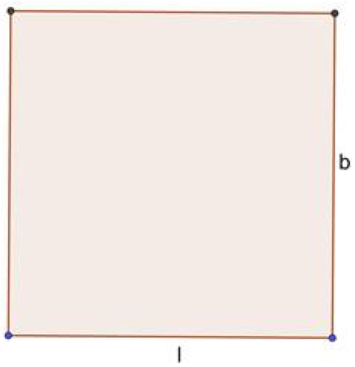
⇒ PER = $6\sqrt{3}r$

∴ Thus proved

24. Question

Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved about one of its sides.

Answer



The perimeter of the rectangle with length L and breadth b is 2(l + b)

Therefore,

2(L + b) = 36

L + b = 18

b = 18 - L

Let the rectangle be rotated about its breadth. Then the resulting cylinder formed will be of radius L and height b.

Volume of cylinder formed V = $\pi L^2 b = \pi(18L^2 - L^3)$

To find the dimensions that will result in the maximum volume:

$$\frac{dV}{dL} = \pi(18 \times 2 \times L - 3 \times (L^2) = 0$$

$$36L = 3 \times (L^2)$$

$$L = 12{,}0$$

L cannot be 0. L is taken as 12 cm.

Therefore b = 24.

$$\frac{d^2V}{dL^2} = \pi(18 \times 2 - 3 \times 2 \times (L))$$

At L = 12,

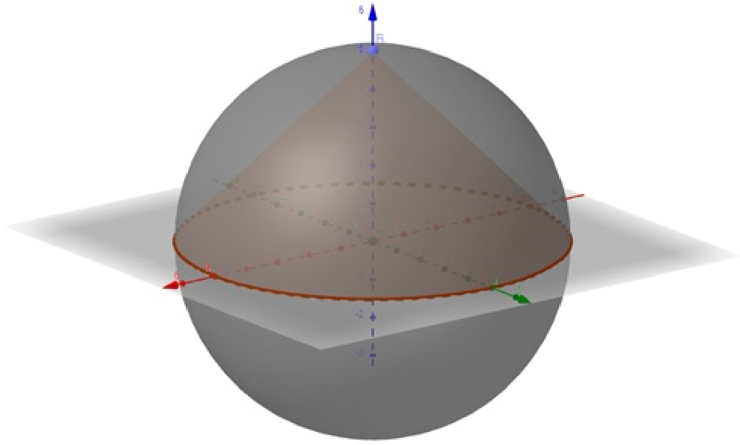
$$\frac{d^2V}{dL^2} = -36\pi = a - \text{ve value}$$

Therefore a maxima exists at L = 12, meaning the volume of the constructed cylinder will be maximum at L = 12 cm.

25. Question

Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm.

Answer



Volume of cone $= \frac{1}{3}\pi r^2 h$, r is the radius of the cone

Let h and r be height and radius of the required cone.

R be the radius of the sphere.

Now, it must be understood that for the cone to have maximum volume, the axis of cone and sphere must be the same.

Let OD = x

In Δ BOD,

$BD = \sqrt{R^2 - x^2} = \sqrt{144 - x^2}$

AD = AO + OD = 12 + x

Volume of cone $= V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi BD^2 AD = \frac{1}{3}\pi \sqrt{144 - x^2}^2 (12 + x)$

$$\frac{dV}{dx} = \frac{d\left(\frac{1}{3}\pi(144 - x^2)(12 + x)\right)}{dx}$$

$$\frac{dV}{dx} = \frac{d\left(\frac{1}{3}\pi(1728 + 144x - 12x^2 - x^3\right)}{dx}$$

$$\Rightarrow \frac{dV}{dx} = \frac{\pi}{3}(144 - 24x - 3x^2) = 0$$

The roots of this quadratic equation is - 12 and 4. As - 12 is not possible, we have x = 4.

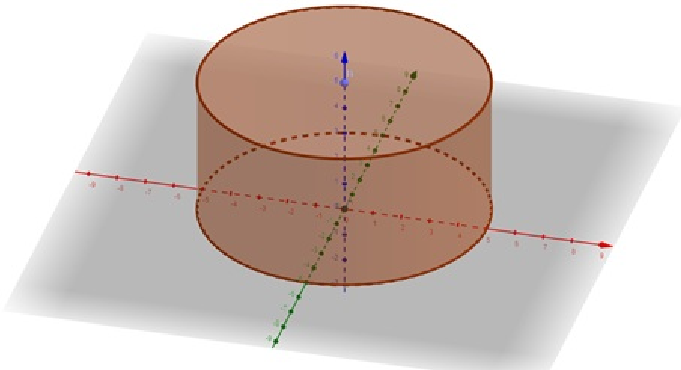
Therefore, the volume is maximum when the x = 4.

Therefore the height of the cone = R + x = 12 + 4 = 16cm.

26. Question

A closed cylinder has volume 2156 cm^3. What will be the radius of its base so that its total surface area is minimum?

Answer



Volume of the cylinder = 2156 cm^3

Formula for volume: $V = \pi r^2 h$

Therefore,

$$h = \frac{2156}{\pi r^2}$$

We need to find the value of r for which the total surface area will be minimum. Hence we rewrite the value of h in terms of r.

Total surface area = $2\pi r^2 + 2\pi rh$

Therefore,

$$TSA = S = 2\pi r^2 + \frac{2\pi r \times 2156}{\pi r^2} = 2\pi r^2 + \frac{2 \times 2156}{r}$$

Hence,

$$\frac{dS}{dr} = 2\pi(2r) + \left[\frac{4312}{-r^2}\right] = 0 \text{ (For maxima and minima)}$$

$$r^3 = \frac{4312}{4\pi}$$

$$r = \sqrt[3]{\frac{4312}{4\pi}} \approx 7\text{cm}$$

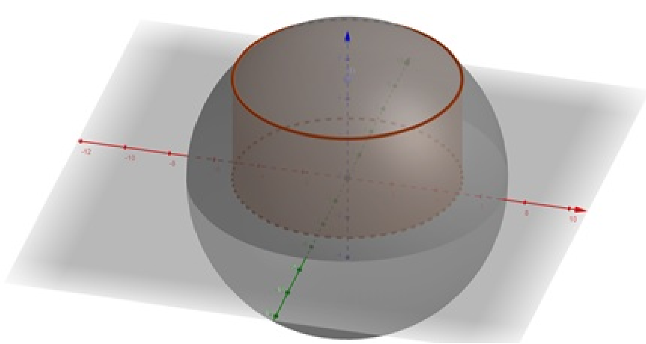
$$\frac{d^2S}{dr^2} = 2\pi(2) + \left[\frac{4312 \times 2}{r^3}\right] \text{ at 7cm is a positive value}$$

Therefore, r = 7cm is the answer

27. Question

Show what the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is 500 π cm^3 .

Answer



The maximum volume cylinder will be carved when the diameter of sphere and the axis of cylinder coincide.

Let h be the height of our cylinder.

r be the radius of the cylinder.

R = 53 radius of the sphere

OL = x

H = 2x

In Δ AOL,

$AL = \sqrt{AO^2 - OL^2} AL = \sqrt{75 - x^2}$Volume of cylinder V = $\pi r^2 h$

$V = \pi(AL^2)LM$

$V = \pi(75 - x^2)(2x)$

$\Rightarrow V = \pi(150x - 2x^3)$

Therefore,

$\Rightarrow \frac{dV}{dx} = \pi(150 - 6x^2) = 0$ (Since we need maximum volume)

$6x^2 = 150x = \pm 5$

x = - 5 cannot be taken as the length cannot be negative.

At x = 5, we have to check whether maxima exits or not.

Therefore,

$\frac{dV^2}{dx^2} = \pi(-12x)$

At x = 5, $\frac{dV^2}{dx^2}$ is negative. Hence, maxima exits at x = 5.

Therefore, at x = 5

Volume of Cylinder = $\pi(75 - x^2)(2x)$

Volume = π(75 - 25)(10) = 500π

28. Question

Show that among all positive numbers x and y with $x^2 + y^2 = r^2$, the sum x + y is largest when $x = y = \frac{r}{\sqrt{2}}$.

Answer

x,y are positive numbers

$x^2 + y^2 = r^2$ is the given condition

Therefore,

$y = \sqrt{r^2 - x^2}$ (Note that the $-\sqrt{r^2 - x^2}$ is not considered as the x and y are + ve)

Let

$f = x + y = x + \sqrt{r^2 - x^2}$

$\frac{df}{dx} = 1 + \frac{1}{2\sqrt{r^2 - x^2}} \times (-2x) = 0$

$$\frac{2x}{2\sqrt{r^2 - x^2}} = 1$$

$$\frac{2x}{1} = 2\sqrt{r^2 - x^2}$$

$$x^2 = (r^2 - x^2)$$

$$x = \frac{r}{\sqrt{2}}$$

And,

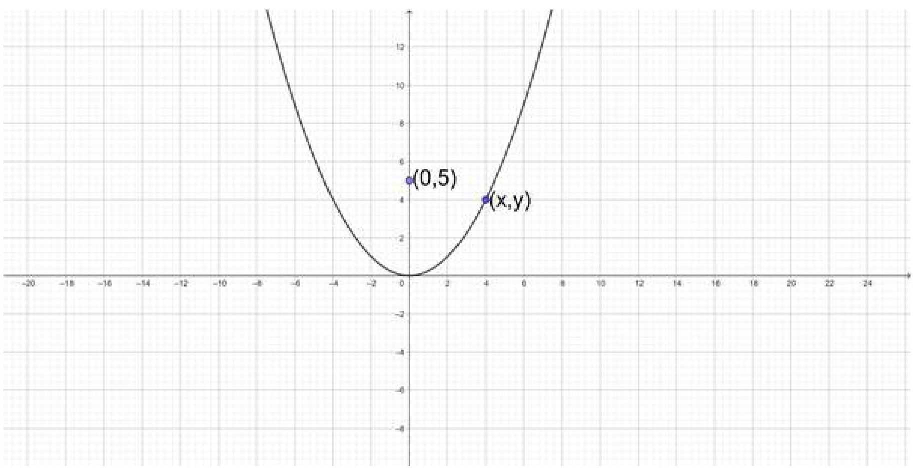
$$\frac{d^2f}{dx^2} = \frac{-\left(\sqrt{r^2 - x^2} + \frac{x^2}{\sqrt{r^2 - x^2}}\right)}{r^2 - x^2} \text{at } x = \frac{r}{\sqrt{2}}$$

$$\frac{d^2f}{dx^2} = \frac{-\left(\sqrt{r^2 - \frac{r^2}{2}} + \frac{\frac{r^2}{2}}{\sqrt{r^2 - \frac{r^2}{2}}}\right)}{r^2 - \frac{r^2}{2}} < 0 \text{ (Maxima)}$$

29. Question

Determine the points on the curve $x^2 = 4y$ which are nearest to the point (0, 5).

Answer



Given Curve is $x^2 = 4y$ ...... (1)

Let us assume the point on the curve which is nearest to the point (0, 5) be (x, y)

The (x, y) satisfies the relation(1)

Let us find the distance(S) between the points (x, y) and (0, 5)

We know that distance between two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$\Rightarrow S = \sqrt{(x - 0)^2 + (y - 5)^2}$

$\Rightarrow S = \sqrt{x^2 + y^2 - 10y + 25}$

Squaring on both sides we get,

$\Rightarrow S^2 = x^2 + y^2 - 10y + 25$

From (1)

$\Rightarrow S^2 = x^2 + \left(\frac{x^2}{4}\right)^2 - 10\left(\frac{x^2}{4}\right) + 25$

$\Rightarrow S^2 = \frac{x^4}{16} - \left(\frac{6x^2}{4}\right) + 25$

We know that distance is an positive number so, for a minimum distance S, S^2 will also be minimum

Let us S^2 as the function of x.

For maxima and minima,

$\Rightarrow \frac{dS^2}{dx} = 0$

$\Rightarrow \frac{d\left(\frac{x^4}{16} - \frac{6x^2}{4} + 25\right)}{dx} = 0$

$\Rightarrow \frac{4x^3}{16} - \frac{12x}{4} = 0$

$\Rightarrow \frac{x^3}{4} - \frac{12x}{4} = 0$

$\Rightarrow x^3 - 12x = 0$

$\Rightarrow x(x^2 - 12) = 0$

On solving we get

$\Rightarrow x = 0, x = 2\sqrt{3}, x = -2\sqrt{3}$

Now differentiating again

$\Rightarrow \frac{d^2S^2}{dx^2} = \frac{d}{dx}\left(\frac{x^3}{4} - \frac{12x}{4}\right)$

$\Rightarrow \frac{d^2S^2}{dx^2} = \frac{3x^2}{4} - \frac{12}{4}$

$\Rightarrow \frac{d^2S^2}{dx^2} = \frac{3x^2}{4} - 3$

At x = 0

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=0} = \frac{3(0)^2}{4} - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=0} = -3 < 0(\text{maxima})$

At x = $2\sqrt{3}$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{3}} = \frac{3\left(2\sqrt{3}\right)^2}{4} - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{3}} = \frac{3(12)}{4} - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{3}} = 9 - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{3}} = 6 > 0(\text{Minima})$

At x = - $2\sqrt{3}$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=-2\sqrt{3}} = \frac{3\left(-2\sqrt{3}\right)^2}{4} - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=-2\sqrt{3}} = \frac{3(12)}{4} - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}|_{x=-2\sqrt{3}} = 9 - 3$

$\Rightarrow \frac{d^2S^2}{dx^2}|_{x=-2\sqrt{3}} = 6 > 0 (\text{Minima})$

We get minimum distance at x = $\pm 2\sqrt{3}$

Let find the value of y at these x values

$\Rightarrow y = \frac{(2\sqrt{3})^2}{4}$

⇒ y = 3

$\Rightarrow y = \frac{(-2\sqrt{3})^2}{4}$

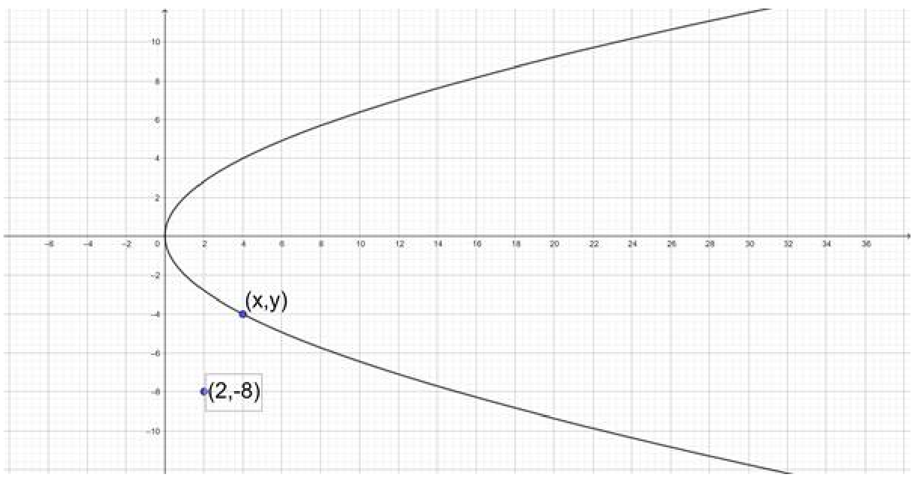
⇒ y = 3

∴ The nearest points to the point (0,5) on the curve are $(\pm 2\sqrt{3}, 3)$.

30. Question

Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, –8).

Answer



Given Curve is $y^2 = 4x$ ...... (1)

Let us assume the point on the curve which is nearest to the point (2, - 8) be (x, y)

The (x, y) satisfies the relation(1)

Let us find the distance(S) between the points (x, y) and (2, - 8)

We know that distance between two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$\Rightarrow S = \sqrt{(x-2)^2 + (y+8)^2}$

$\Rightarrow S = \sqrt{x^2 - 4x + 4 + y^2 + 16y + 64}$

$\Rightarrow S = \sqrt{x^2 - 4x + y^2 + 16y + 68}$

Squaring on both sides we get,

$\Rightarrow S^2 = x^2 + y^2 - 4x + 16y + 68$

From (1)

$\Rightarrow S^2 = \left(\frac{y^2}{4}\right)^2 + y^2 - 4\left(\frac{y^2}{4}\right) + 16y + 68$

$\Rightarrow S^2 = \frac{y^4}{16} + 16y + 68$

We know that distance is an positive number so, for a minimum distance S, S^2 will also be minimum

Let us S^2 as the function of y.

For maxima and minima,

$\Rightarrow \frac{dS^2}{dy} = 0$

$\Rightarrow \frac{d\left(\frac{y^4}{16} + 16y + 68\right)}{dy} = 0$

$\Rightarrow \frac{4y^3}{16} - 16 = 0$

$\Rightarrow \frac{y^3}{4} - 16 = 0$

$\Rightarrow y^3 - 64 = 0$

$\Rightarrow y = (64)^{\frac{1}{3}}$

On solving we get

$\Rightarrow y = 4$

Now differentiating again

$\Rightarrow \frac{d^2S^2}{dy^2} = \frac{d}{dy}\left(\frac{y^3}{4} - 16\right)$

$\Rightarrow \frac{d^2S^2}{dy^2} = \frac{3y^2}{4}$

At y = 4

$\Rightarrow \frac{d^2S^2}{dy^2}\Big|_{y=4} = \frac{3(4)^2}{4} - 3$

$\Rightarrow \frac{d^2S^2}{dy^2}\Big|_{y=4} = 9 > 0(\text{minima})$

We get minimum distance at y = 4

Let find the value of x at these y values

$\Rightarrow x = \frac{(4)^2}{4}$

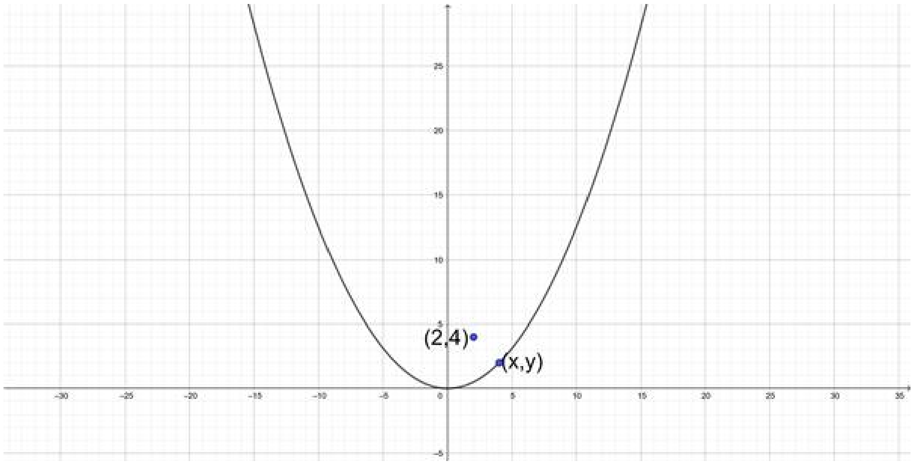
$\Rightarrow x = 4$

∴ The nearest point to the point (2, - 8) on the curve is (4,4).

31. Question

Find the point on the curve $x^2 = 8y$ which is nearest to the point (2, 4).

Answer



Given Curve is $x^2 = 8y$ ...... (1)

Let us assume the point on the curve which is nearest to the point (2, 4) be (x, y)

The (x, y) satisfies the relation(1)

Let us find the distance(S) between the points (x, y) and (2, 4)

We know that distance between two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$\Rightarrow S = \sqrt{(x-2)^2 + (y-4)^2}$

$\Rightarrow S = \sqrt{x^2 - 4x + 4 + y^2 - 8y + 16}$

$\Rightarrow S = \sqrt{x^2 + y^2 - 4x - 8y + 20}$

Squaring on both sides we get,

$\Rightarrow S^2 = x^2 + y^2 - 4x - 8y + 20$

From (1)

$\Rightarrow S^2 = x^2 + \left(\frac{x^2}{8}\right)^2 - 4x - 8\left(\frac{x^2}{8}\right) + 25$

$\Rightarrow S^2 = \frac{x^4}{64} - 4x + 20$

We know that distance is an positive number so, for a minimum distance S, S^2 will also be minimum

Let us S^2 as the function of x.

For maxima and minima,

$\Rightarrow \frac{dS^2}{dx} = 0$

$\Rightarrow \frac{d\left(\frac{x^4}{64} - 4x + 20\right)}{dx} = 0$

$\Rightarrow \frac{4x^3}{64} - 4 = 0$

$\Rightarrow \frac{x^3}{16} - 4 = 0$

$\Rightarrow x^3 - 64 = 0$

$\Rightarrow x = (64)^{\frac{1}{3}}$

On solving we get

$\Rightarrow x = 4$

Now differentiating again

$\Rightarrow \frac{d^2S^2}{dx^2} = \frac{d}{dx}\left(\frac{x^3}{16} - 4\right)$

$\Rightarrow \frac{d^2S^2}{dx^2} = \frac{3x^2}{16}$

At x = 4

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=4} = \frac{3(4)^2}{16}$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=4} = 3 > 0(\text{minima})$

We get minimum distance at x = 4

Let find the value of y at these x values

$\Rightarrow y = \frac{(4)^2}{8}$

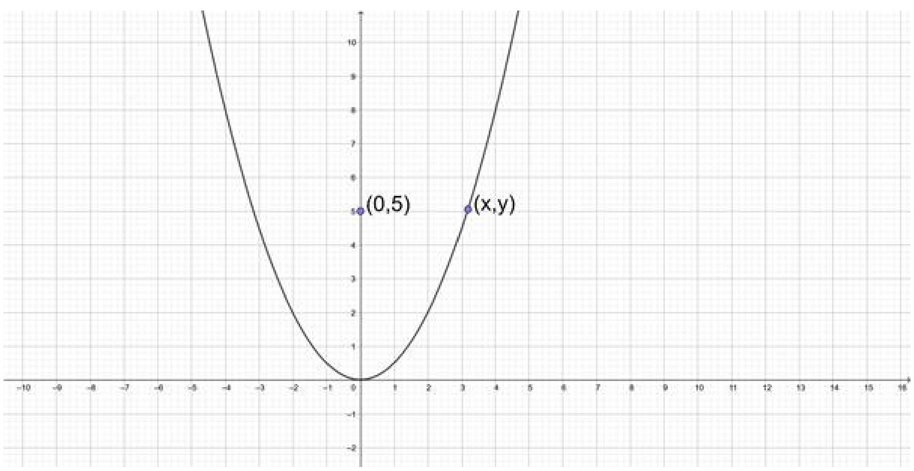
$\Rightarrow y = 2$

∴ The nearest point to the point (2,4) on the curve is (4,2).

32. Question

Find the point on the parabolas $x^2 = 2y$ which is closest to the point (0, 5).

Answer



Given Curve is $x^2 = 2y$ ...... (1)

Let us assume the point on the curve which is nearest to the point (0, 5) be (x, y)

The (x, y) satisfies the relation(1)

Let us find the distance(S) between the points (x, y) and (0, 5)

We know that distance between two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

$\Rightarrow S = \sqrt{(x-0)^2 + (y-5)^2}$

$\Rightarrow S = \sqrt{x^2 + y^2 - 10y + 25}$

Squaring on both sides we get,

$\Rightarrow S^2 = x^2 + y^2 - 10y + 25$

From (1)

$\Rightarrow S^2 = x^2 + \left(\frac{x^2}{2}\right)^2 - 10\left(\frac{x^2}{2}\right) + 25$

$\Rightarrow S^2 = \frac{x^4}{4} - 4x^2 + 25$

We know that distance is an positive number so, for a minimum distance S, S^2 will also be minimum

Let us S^2 as the function of x.

For maxima and minima,

$\Rightarrow \frac{dS^2}{dx} = 0$

$\Rightarrow \frac{d\left(\frac{x^4}{4} - 4x^2 + 25\right)}{dx} = 0$

$\Rightarrow \frac{4x^3}{4} - 8x = 0$

$\Rightarrow x^3 - 8x = 0$

$\Rightarrow x(x^2 - 8) = 0$

On solving we get

$\Rightarrow x = 0, x = 2\sqrt{2}, x = -2\sqrt{2}$

Now differentiating again

$\Rightarrow \frac{d^2S^2}{dx^2} = \frac{d}{dx}(x^3 - 8x)$

$\Rightarrow \frac{d^2S^2}{dx^2} = 3x^2 - 8$

At x = 0

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=0} = 3(0)^2 - 8$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=0} = -8 < 0(\text{maxima})$

At x = $2\sqrt{2}$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{2}} = 3\left(2\sqrt{2}\right)^2 - 8$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{2}} = 3(8) - 8$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=2\sqrt{2}} = 16 > 0(\text{Minima})$

At x = - $2\sqrt{2}$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=-2\sqrt{2}} = 3\left(-2\sqrt{2}\right)^2 - 8$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=-2\sqrt{2}} = 3(8) - 8$

$\Rightarrow \frac{d^2S^2}{dx^2}\Big|_{x=-2\sqrt{2}} = 16 > 0(\text{Minima})$

We get minimum distance at x = $\pm 2\sqrt{2}$

Let find the value of y at these x values

$\Rightarrow y = \frac{(2\sqrt{2})^2}{2}$

$\Rightarrow$ y = 4

$\Rightarrow y = \frac{(-2\sqrt{2})^2}{2}$

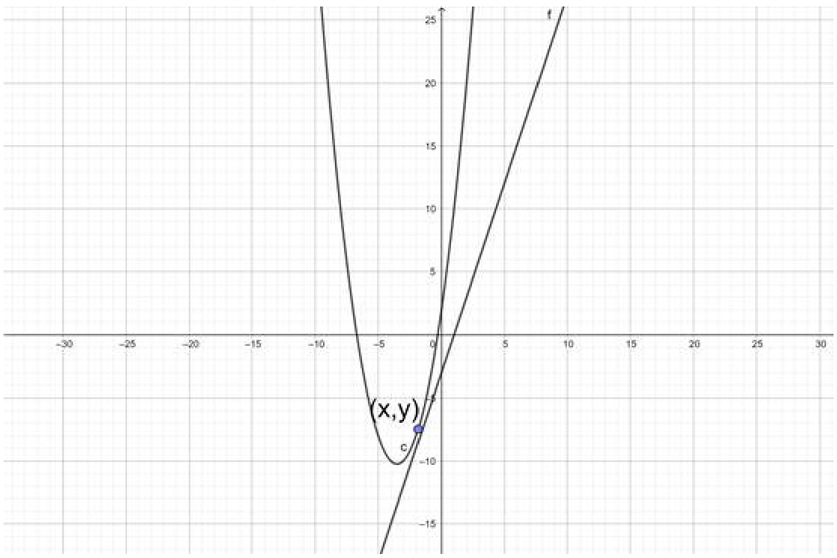
$\Rightarrow$ y = 4

$\therefore$ The nearest points to the point (0,5) on the curve are $(\pm 2\sqrt{2}, 4)$.

33. Question

Find the coordinates of a point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight line y = 3x – 3.

Answer



Given equation of the parabola is $y = x^2 + 7x + 2$ and straight line is y = 3x - 3 or 3x - y - 3 = 0

Equation of parabola $y = x^2 + 7x + 2$ …… (1)

Let us assume the point on parabola which is closest to the line be (x, y)

We know that distance between the point (x_0, y_0) and the line ax + by + c = 0 is

$\Rightarrow S = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$

Now we find the distance between the line and point on parabola

$\Rightarrow S = \frac{|3x - y - 3|}{\sqrt{(3)^2 + (-1)^2}}$

From(1)

$\Rightarrow S = \frac{|3x - (x^2 + 7x) + 2|}{\sqrt{9 + 1}}$

$\Rightarrow S = \frac{|x^2 + 4x - 2|}{\sqrt{10}}$

$\Rightarrow S = \frac{x^2 + 4x - 2}{\sqrt{10}}$

Let us assume S be the function of x

$\Rightarrow \frac{dS}{dx} = \frac{d\left(\frac{x^2 + 4x - 2}{\sqrt{10}}\right)}{dx}$

$\Rightarrow \frac{dS}{dx} = \frac{2x+4}{\sqrt{10}}$

For maxima and minima

$\Rightarrow \frac{dS}{dx} = 0$

$\Rightarrow \frac{2x+4}{\sqrt{10}} = 0$

⇒ 2x + 4 = 0

⇒ 2x = - 4

⇒ x = - 2

Now differentiating again

$\Rightarrow \frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{2x+4}{\sqrt{10}}\right)$

$\Rightarrow \frac{d^2S}{dx^2} = \frac{2}{\sqrt{10}}$

At x = - 2

$\Rightarrow \frac{d^2S}{dx^2}|_{x=-2} = \frac{2}{\sqrt{10}} > 0(\text{minima})$

We get the minimum distance at x = - 2

Let’s find the value of y at this x value

⇒ y = (- 2)^2 + 7(- 2) + 2

⇒ y = 4 - 14 + 2

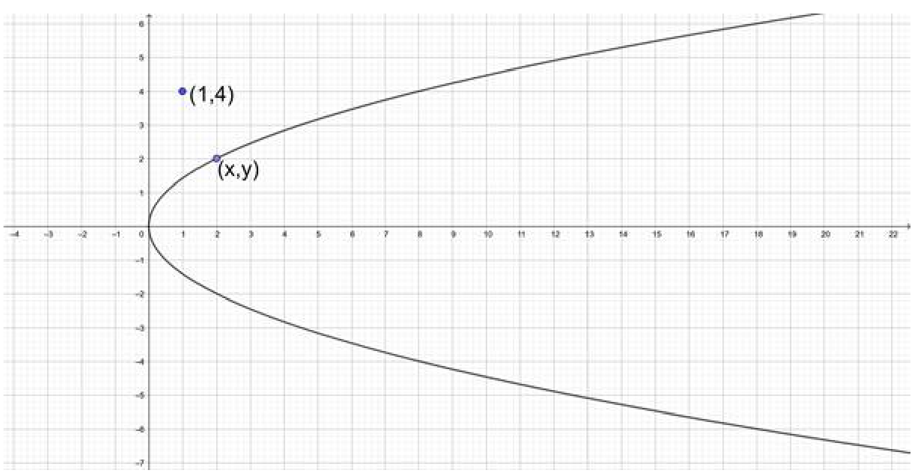
⇒ y = - 8

∴ The point (- 2, - 8) is the nearest point on the parabola to the line y = 3x - 3.

34. Question

Find the point on the curve y^2 = 2x which is at a minimum distance from the point (1, 4).

Answer



Given Curve is y^2 = 2x …… (1)

Let us assume the point on the curve which is nearest to the point (1, 4) be (x, y)

The (x, y) satisfies the relation(1)

Let us find the distance(S) between the points (x, y) and (1,4)

We know that distance between two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

$\Rightarrow S = \sqrt{(x-1)^2+(y-4)^2}$

$\Rightarrow S = \sqrt{x^2-2x+1+y^2-8y+16}$

$\Rightarrow S = \sqrt{x^2-2x+y^2-8y+17}$

Squaring on both sides we get,

$\Rightarrow S^2 = x^2 + y^2 - 2x - 8y + 17$

From (1)

$\Rightarrow S^2 = \left(\frac{y^2}{2}\right)^2 + y^2 - 2\left(\frac{y^2}{2}\right) - 8y + 17$

$\Rightarrow S^2 = \frac{y^4}{4} - 8y + 17$

We know that distance is an positive number so, for a minimum distance S, S^2 will also be minimum

Let us S^2 as the function of y.

For maxima and minima,

$\Rightarrow \frac{dS^2}{dy} = 0$

$\Rightarrow \frac{d\left(\frac{y^4}{4}-8y+17\right)}{dy} = 0$

$\Rightarrow \frac{4y^3}{4} - 8 = 0$

$\Rightarrow y^3 - 8 = 0$

$\Rightarrow y = (8)^{\frac{1}{3}}$

On solving we get

$\Rightarrow y = 2$

Now differentiating again

$\Rightarrow \frac{d^2S^2}{dy^2} = \frac{d}{dy}(y^3-8)$

$\Rightarrow \frac{d^2S^2}{dy^2} = 3y^2$

At y = 2

$\Rightarrow \frac{d^2S^2}{dy^2}\Big|_{y=2} = 3(2)^2$

$\Rightarrow \frac{d^2S^2}{dy^2}\Big|_{y=2} = 12 > 0(\text{minima})$

We get minimum distance at y = 2

Let find the value of x at these y values

$\Rightarrow x = \frac{(2)^2}{2}$

$\Rightarrow x = 2$

∴ The nearest point to the point (2, 2) on the curve is (4,4).

35. Question

Find the maximum slope of the curve y = –x^3 + $3x^2$ + 2x – 27

Answer

$$\frac{dy}{dx} = -3x^2 + 6x + 2 = \text{slope}$$

We need to find the maximum value of the slope, meaning we have to double differentiate the value of slope to find the maximum value of slope

$$\frac{d(\text{slope})}{dx} = -6x + 6 = 0x = 1$$

$$\frac{d^2(\text{slope})}{dx^2} = -6 = \text{Maxima exists at } x = 1.\text{Maximum slope:}$$

Slope (max) = - 3(1) + 6(1) + 2 = 5

36. Question

The total cost of producing x radio sets per days is Rs $\left(\frac{x^2}{4}+35x+25\right)$ and the price per set at which they may be solid is Rs $\left(50-\frac{x}{2}\right)$. Find the daily output to maximize the total profit.

Answer

Cost Price (CP) for producing x amount $= \frac{x^2}{4} + 35x + 25$

Selling Price (SP) per producing x amount $= 50 - \frac{x}{2}$

Selling Price (SP) for producing x amount $= 50x - \frac{x^2}{2}$

Profit P = SP - CP

$$P = 50x - \frac{x^2}{2} - \left(\frac{x^2}{4} + 35x + 25\right)$$

$$\frac{dP}{dx} = 50 - x - \frac{x}{2} - 35 = 15 - \frac{3}{2}x = 0$$

$$x = 10$$

Therefore,

$$\frac{d^2P}{dx^2} = -\frac{3}{2}$$

Maxima exits, therefore we will get maximum profit at x = 10.

37. Question

Manufactures can sell x items at a price of Rs $\left(5-\frac{x}{100}\right)$ each. The cost price is Rs $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

Answer

$$\text{Cost Price (CP) for producing x amount} = 500 + \frac{x}{5}$$

$$\text{Selling Price (SP) per producing x amount} = 5 - \frac{x}{100}$$

$$\text{Selling Price (SP) for producing x amount} = 5x - \frac{x^2}{100}$$

Profit P = SP - CP

$$P = 5x - \frac{x^2}{100} - \left(\frac{x}{5} + 500\right)$$

$$\frac{dP}{dx} = 5 - \frac{x}{50} - \frac{1}{5} = \frac{24}{5} - \frac{x}{50} = 0 \text{ (to find maxima)}$$

x = 240

Therefore,

$$\frac{d^2P}{dx^2} = -\frac{1}{50} < 0$$

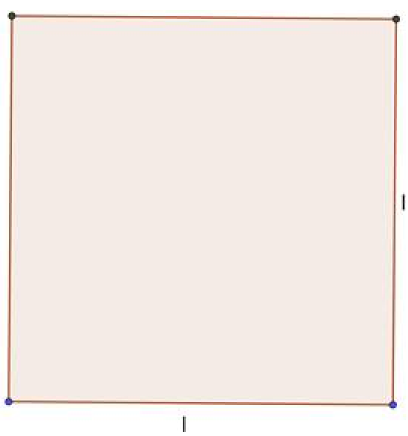
Thus x = 240 is a point of maxima.

Thus for maximum profit, x should be 240.

38. Question

An open tank is to be constructed with a square base and vertical sides so as to contain a given quantity of water. Show that the expenses of lining with lead will be least, if depth is made half of width.

Answer



Let L be the length of the square base and h be the height/depth of the tank.

Expenses of lining implies the cost for lining the entire inner surface area of the tank; a base and four vertical sides.

If we have minimum area to cover, we will have minimum costs incurred.

Internal area of the tank = $L^2 + 4Lh$

Volume of the tank = $L^2h = V$

Therefore,

$$A = L^2 + 4 \times \frac{V}{L}$$

$$\frac{dA}{dL} = 2L - 4 \times \frac{V}{L^2} = 0 \text{ (For maxima and minima)}$$

$$2L^3 = 4V = 4L^2h$$

We get two outcomes here.

L = 0,2h

We discard L = 0, as it makes no sense.

So;

L = 2h.

Now to check whether a maxima or a minima exists

$$\frac{d^2A}{dL^2} = 2 + 4 \times 2 \times \frac{V}{L^3} > 0$$

Therefore a minima exists for all non zero values of L.

Hence for the tank lining costs to be minimum, h = L/2.

39. Question

A box of constant volume c is to be twice as long as it is wide. The material on the top and four sides cost three times as much per square metre as that in the bottom. What are the most economic dimensions?

Answer

Let,

Width = b

Length L = 2b

Height = h

Cost of material on top and 4 sides = 3 x cost of material at bottom.

Here we do not have individual cost per area of different materials. But we can use the total surface area of individual parts as replacement for the cost.

bx2b + 2(hxb) + 2(hx2b) = 3(bx2b)

$2b^2 + 2hb + 4hb = 6b^2$

$6hb = 4b^2$

$$h = \frac{2}{3}b \text{ (Neglect } b = 0)$$

Volume of the box = c = bx2bxh = $2b^2h$

Therefore,

$$h = \frac{c}{2b^2}$$

We want the optimum dimensions of the box. We find this by optimizing the total surface area to be minimum. This is because the majority of the surface costs more.

Surface area(total) = S = $2(bh + 2bh + 2b^2) = 2(2b^2 + 3bh)$

$$S = 2\left(2b^2 + 3b \times \frac{c}{2b^2}\right)$$

$$S = 2\left(2b^2 + 3 \times \frac{c}{2b}\right)$$

$$\frac{dS}{dx} = 2\left(4b - \frac{3c}{2b^2}\right) = 0$$

$$b^3 = \frac{3c}{8}b = \left(\frac{3c}{8}\right)^{\frac{1}{3}}$$

$\frac{d^2S}{dx^2} = 2\left(4 + \frac{3c}{4b^3}\right) > 0$ for all values of b. Therefore minima.

A minima exists at $b = \left(\frac{3c}{8}\right)^{\frac{1}{3}}$

Therefore:

$h = \frac{2}{3}b = \frac{2}{3}\left(\frac{3c}{8}\right)^{\frac{1}{3}}$

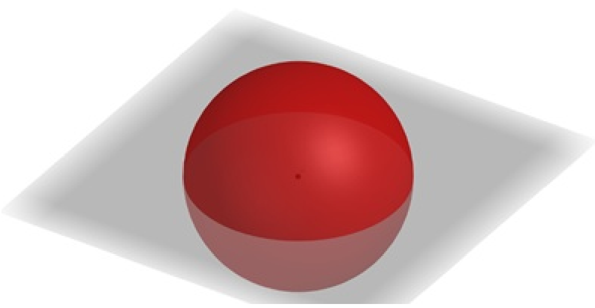
And Length = 2b

$L = 2b = 2\left(\frac{3c}{8}\right)^{\frac{1}{3}}$

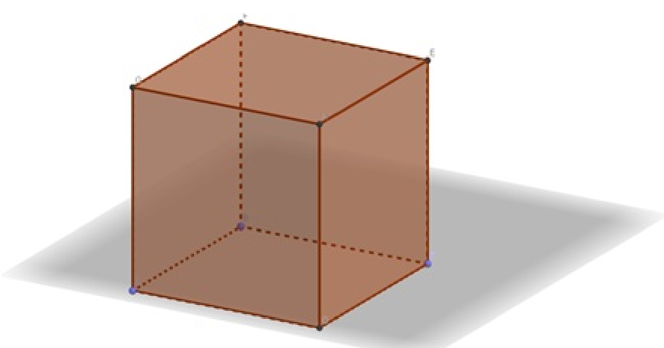
40. Question

The sum of the surface areas of a sphere and a cube is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

Answer



Let us assume radius of sphere be ‘r’ and length of side of cube is ‘l’



We know that,

⇒ Surface area of sphere = $4\pi r^2$

⇒ Surface area of cube = $6l^2$

According the problem, the sum of surface areas of a sphere and cube is known. Let us assume the sum be S

⇒ $S = 4\pi r^2 + 6l^2$ ...... (1)

We also know that,

⇒ Volume of sphere = $\frac{4}{3}\pi r^3$

⇒ Volume of cube = l^3

We need the sum of volumes to be least. Let us assume the sum of volumes be V

$\Rightarrow V = \frac{4}{3}\pi r^3 + l^3$

From (1)

$\Rightarrow V = \frac{4}{3}\pi r^3 + \left(\frac{S-4\pi r^2}{6}\right)^{\frac{3}{2}}$

We assume V as a function of r.

For maxima and minima,

$\Rightarrow \frac{dV}{dr} = 0$

$\Rightarrow \frac{d}{dr}\left(\frac{4}{3}\pi r^3 + \left(\frac{S-4\pi r^2}{6}\right)^{\frac{3}{2}}\right) = 0$

$\Rightarrow 4\pi r^2 + \frac{3}{2}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}} \frac{d\left(\frac{S-4\pi r^2}{6}\right)}{dr} = 0$

$\Rightarrow 4\pi r^2 + \frac{3}{2}\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}}\left(\frac{-8\pi r}{6}\right) = 0$

$\Rightarrow 4\pi r^2 - (2\pi r)\left(\frac{s-4\pi r^2}{6}\right)^{\frac{1}{2}} = 0$

$\Rightarrow 4\pi r^2 - (2\pi r)\left(\frac{4\pi r^2 + 6l^2 - 4\pi r^2}{6}\right)^{\frac{1}{2}} = 0$

$\Rightarrow 4\pi r^2 - (2\pi r l) = 0$

$\Rightarrow 2\pi r(2r - l) = 0$

$\Rightarrow r = 0, r = \frac{l}{2}$

Differentiating V again

$\Rightarrow \frac{d^2V}{dx^2} = \frac{d}{dx}\left(4\pi r^2 - (2\pi r)\left(\frac{s-4\pi r^2}{6}\right)^{\frac{1}{2}}\right)$

$\Rightarrow \frac{d^2V}{dx^2} = 8\pi r - 2\pi\left(\frac{S-4\pi r^2}{6}\right)^{\frac{1}{2}} - (2\pi r) \times \frac{1}{2} \times \left(\frac{S-4\pi r^2}{6}\right)^{\frac{-1}{2}} \times \frac{d}{dx}\left(\frac{S-4\pi r^2}{6}\right)$

$\Rightarrow \frac{d^2V}{dx^2} = 8\pi r - 2\pi\left(\frac{4\pi r^2 + 6l^2 - 4\pi r^2}{6}\right)^{\frac{1}{2}} - (\pi r)\left(\frac{4\pi r^2 + 6l^2 - 4\pi r^2}{6}\right)^{-\frac{1}{2}} \times (-8\pi r)$

$\Rightarrow \frac{d^2V}{dx^2} = 8\pi r - 2\pi l + \frac{8\pi r^2}{l}$

At r = 0

$\Rightarrow \frac{d^2V}{dx^2}\Big|_{r=0} = 8\pi(0) - 2\pi l + \frac{8\pi(0)^2}{l}$

$\Rightarrow \frac{d^2V}{dx^2}\Big|_{r=0} = -2\pi l < 0(\text{Maxima})$

At $r = \frac{l}{2}$

$\Rightarrow \frac{d^2V}{dx^2}\Big|_{r=\frac{l}{2}} = 8\pi\left(\frac{l}{2}\right) - 2\pi l + \frac{8\pi\left(\frac{l}{2}\right)^2}{l}$

$\Rightarrow \frac{d^2V}{dx^2}\Big|_{r=\frac{l}{2}} = 4\pi l - 2\pi l + \frac{8\pi\left(\frac{l^2}{4}\right)}{l}$

$\Rightarrow \frac{d^2V}{dx^2}|_{r=\frac{1}{2}} = 2\pi l + 2\pi l$

$\Rightarrow \frac{d^2V}{dx^2}|_{r=\frac{1}{2}} = 4\pi l > 0(\text{Minima})$

We get the sum of values least for $r = \frac{1}{2}$.

We know that diameter(d) is twice of radius. So,

$\Rightarrow d = 2 \times \frac{1}{2}$

⇒ d = l

∴ Thus proved.

41. Question

A given quantity of metal is to be cast into a half cylinder with a rectangular base and semi - circular ends. Show that in order that the total surface area may be minimum the ratio of the length of the cylinder to the diameter of its semi - circular ends is π : (π + 2).

Answer

Let 'h' be the height 'or' length of half cylinder, 'r' be the radius of half cylinder and 'd' be the diameter.

We know that,

⇒ Volume of half cylinder (V) $= \frac{1}{2}\pi r^2 h$

$\Rightarrow h = \frac{2V}{\pi r^2}$ ...... (1)

Now we find the Total surface area (TSA) of the half cylinder,

⇒ TSA = Lateral surface area of the half cylinder + Area of two semi - circular ends + Area of the rectangular base

$\Rightarrow \text{TSA} = \pi rh + \frac{\pi r^2}{2} + \frac{\pi r^2}{2} + (h \times d)$

From (1)

$\Rightarrow \text{TSA} = \pi r \times \frac{2V}{\pi r^2} + \pi r^2 + \frac{2V}{\pi r^2} \times 2r$

$\Rightarrow \text{TSA} = \frac{2V}{\pi r}(\pi + 2) + \pi r^2$

We need total surface area to be minimum and let us take the TSA as the function of r,

For maxima and minima,

$\Rightarrow \frac{d(\text{TSA})}{dr} = 0$

$\Rightarrow \frac{d}{dr}\left(\frac{2V}{\pi r}(\pi + 2) + \pi r^2\right) = 0$

$\Rightarrow \frac{-2V}{\pi r^2}(\pi + 2) + 2\pi r = 0$

$\Rightarrow \frac{-2\left(\frac{1}{2}\pi r^2 h\right)(\pi + 2)}{\pi r^2} + 2\pi r = 0$

$\Rightarrow -(\pi + 2)h + 2\pi r = 0$

$\Rightarrow r = \frac{(\pi + 2)h}{2\pi}$

Differentiating TSA again,

$\Rightarrow \frac{d^2TSA}{dr^2} = \frac{d}{dr}\left(\frac{-2V(\pi+2)}{\pi r^2} + 2\pi r\right)$

$\Rightarrow \frac{d^2TSA}{dr^2} = \frac{4V(\pi+2)}{\pi r^3} + 2\pi$

$\Rightarrow \frac{d^2TSA}{dr^2} = \frac{4\left(\frac{1}{2}\pi r^2 h\right)(\pi+2)}{\pi r^3} + 2\pi$

$\Rightarrow \frac{d^2TSA}{dr^2} = \frac{2h(\pi+2)}{r} + 2\pi$

At $r = \frac{(\pi+2)h}{2\pi}$

$\Rightarrow \frac{d^2TSA}{dr^2} = \frac{2h(\pi+2)}{\frac{(\pi+2)h}{2\pi}} + 2\pi$

$\Rightarrow \frac{d^2TSA}{dr^2} = 4\pi + 2\pi$

$\Rightarrow \frac{d^2TSA}{dr^2} = 6\pi > 0$(Minima)

We have got Total surface area minimum for $r = \frac{(\pi+2)h}{2\pi}$

We know that diameter is twice of radius

$\Rightarrow d = 2r = 2 \times \frac{(\pi+2)h}{2\pi}$

$\Rightarrow d = \frac{(\pi+2)h}{\pi}$

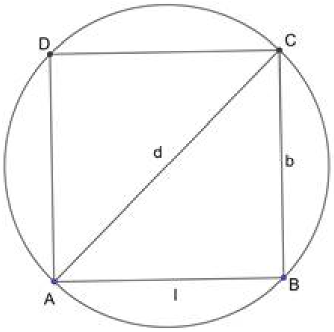
$\Rightarrow \frac{h}{d} = \frac{\pi}{\pi+2}$

∴ Thus proved.

42. Question

The strength of a beam varies as the product of its breadth and square of its depth. Find the dimensions of the strongest beam which can be cut from a circular log of radius a.

Answer



Let us assume ABCD be the cross section of beam that need to cut from the circular log of radius ‘a’.

Let us assume ‘b’ be the depth of the rectangle, ‘l’ be the length of the rectangle and ‘d’ be the diameter of circular log.

$\Rightarrow d = 2a$ …… (1)

According to the problem, strength of the beam is given by

$\Rightarrow S = lb^2$ …… (2)

From the ΔABC,

$\Rightarrow d^2 = l^2 + b^2$

From (1)

⇒ $(2a)^2 = l^2 + b^2$

⇒ $b^2 = 4a^2 - l^2$ ...... (3)

From(2) and (3)

⇒ $S = l(4a^2 - l^2)$

⇒ $S = 4a^2l - l^3$

We need strength of the beam to be maximum, let us take S as a function of l

For maxima and minima,

⇒ $\frac{dS}{dl} = 0$

⇒ $\frac{d(4a^2l - l^3)}{dl} = 0$

⇒ $4a^2 - 3l^2 = 0$

⇒ $3l^2 = 4a^2$

⇒ $l = \sqrt{\frac{4a^2}{3}}$

⇒ $l = \frac{2a}{\sqrt{3}}$

Differentiating S again,

⇒ $\frac{d^2S}{dl^2} = \frac{d}{dl}(4a^2 - 3l^2)$

⇒ $\frac{d^2S}{dl^2} = -6l$

At $l = \frac{2a}{\sqrt{3}}$,

⇒ $\frac{d^2S}{dl^2}\Big|_{l=\frac{2a}{\sqrt{3}}} = -6\left(\frac{2a}{\sqrt{3}}\right)$

⇒ $\frac{d^2S}{dl^2}\Big|_{l=\frac{2a}{\sqrt{3}}} = -4\sqrt{3}a$<0(Maxima)

We get maximum strength for length $l = \frac{2a}{\sqrt{3}}$, lets find the depth 'b' for this l:

⇒ $b^2 = (2a)^2 - \left(\frac{2a}{\sqrt{3}}\right)^2$

⇒ $b^2 = 4a^2 - \frac{4a^2}{3}$

⇒ $b^2 = \frac{8a^2}{3}$

⇒ $b = \frac{2\sqrt{2}a}{\sqrt{3}}$

Let's find the strength of beam for $l = \frac{2a}{\sqrt{3}}$ and $b = \frac{2\sqrt{2}a}{\sqrt{3}}$

⇒ $S = \left(\frac{2a}{\sqrt{3}}\right)\left(\frac{2\sqrt{2}a}{\sqrt{3}}\right)^2$

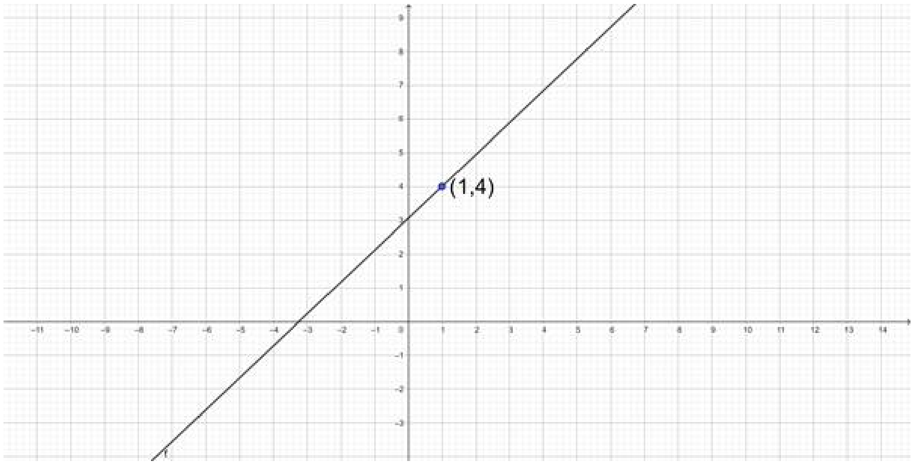
⇒ $S = \frac{16a^3}{3\sqrt{3}}$

The dimensions of beam of maximum strength is $\left(\frac{2a}{\sqrt{3}}, \frac{2\sqrt{2}a}{\sqrt{3}}\right)$.

43. Question

A straight line is drawn through a given point P (1, 4). Determine the least value of the sum of the intercepts on the coordinate axes.

Answer



Let us the slope of the line passing through the point P(1,4) be m.

We know that equation of a straight line passing through the point (x_1,y_1) and having slope m is given by:

$\Rightarrow y - y_1 = m(x - x_1)$

The equation of the straight line is:

$\Rightarrow y - 4 = m(x - 1)$

$\Rightarrow mx - y = m - 4$

$\Rightarrow \frac{mx}{m-4} - \frac{y}{m-4} = 1$

$\Rightarrow \frac{x}{\frac{m-4}{m}} + \frac{y}{4-m} = 1$

This resembles the standard form $\frac{x}{a} + \frac{y}{b} = 1$, where a is x - intercept and b is y - intercept.

Here x - intercept $a = \frac{m-4}{m}$ and b = 4 - m

According to the problem, we need sum of intercepts to be minimum,

Let us take the sum of intercepts to be S,

$\Rightarrow S = a + b$

$\Rightarrow S = \frac{m-4}{m} + (4 - m)$

$\Rightarrow S = 5 - \frac{4}{m} - m$

Let us assume S is the function of m,

We know that for maxima and minima,

$\Rightarrow \frac{dS}{dm} = 0$

$\Rightarrow \frac{d\left(5 - \frac{4}{m} - m\right)}{dm} = 0$

$\Rightarrow \frac{4}{m^2} - 1 = 0$

$\Rightarrow 4 - m^2 = 0$ ($\because m^2 > 0$)

$\Rightarrow m = \pm 2$

Differentiating S again

$\Rightarrow \frac{d^2S}{dm^2} = \frac{d}{dm}\left(\frac{4}{m^2} - 1\right)$

$\Rightarrow \frac{d^2S}{dm^2} = \frac{-8}{m^3}$

At m = - 2

$\Rightarrow \frac{d^2S}{dm^2} = \frac{-8}{(-2)^3}$

$\Rightarrow \frac{d^2S}{dm^2} = \frac{-8}{-8}$

$\Rightarrow \frac{d^2S}{dm^2} = 1 > 0$(Minima)

At m = + 2

$\Rightarrow \frac{d^2S}{dm^2} = \frac{-8}{2^3}$

$\Rightarrow \frac{d^2S}{dm^2} = \frac{-8}{8}$

$\Rightarrow \frac{d^2S}{dm^2} = -1 < 0$(Maxima)

We have got minima for m = - 2

Using this value we find the sum of intercepts:

$\Rightarrow S_{min} = \frac{-2-4}{-2} + \left(4 - (-2)\right)$

$\Rightarrow S_{min} = \frac{-6}{-2} + 6$

$\Rightarrow S_{min} = 3 + 6$

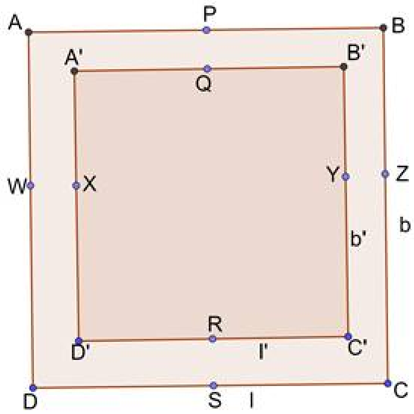
$\Rightarrow S_{min} = 9$

∴ The least value of sum of intercepts is 9.

44. Question

The total area of a page is 150 cm^2. The combined width of the margin at the top and bottom is 3 cm and the side 2 cm. What must be the dimensions of the page in order that the area of the printed matter may be maximum?

Answer



Let ABCD be the total page and A’B’C’D’ be the area in which the matter is printed.

Let ‘l’ and ‘b’ be the length and breadth of the total page.

We know that area of the rectangle is l×b

From the problem,

⇒ lb = 150cm^2 ...... (1)

⇒ PQ + RS = 3cm

⇒ WX + YZ = 2cm

Let ‘l’’ and ‘b’’ be the length and breadth of the area in which the matter is printed,

From the figure,

⇒ l’ = (l - 2)cm

⇒ b’ = (b - 3)cm

⇒ Area of the printed matter (A) = (l - 2)(b - 3)cm^2

From (1)

⇒ $A = (l-2)\left(\frac{150}{l}-3\right)$

⇒ $A = 150-\frac{300}{l}-3l + 6$

⇒ $A = 156-\frac{300}{l}-3l$

We need Area of the printed matter maximum and let us take A as the function of l

We know that for maxima and minima,

⇒ $\frac{dA}{dl} = 0$

⇒ $\frac{d\left(156-\frac{300}{l}-3l\right)}{dl} = 0$

⇒ $\frac{300}{l^2}-3 = 0$

⇒ $300-3l^2 = 0$ (∵ l^2>0)

⇒ $l^2 = 100$

⇒ l = 10cm (since length is an positive quantity)

Differentiating A again,

$\Rightarrow \frac{d^2A}{dl^2} = \frac{d\left(\frac{300}{l^2}-3\right)}{dl}$

$\Rightarrow \frac{d^2A}{dl^2} = \frac{-600}{l^3}$

At l = 10

$\Rightarrow \frac{d^2A}{dl^2}\Big|_{l=10} = \frac{-600}{1000}$

$\Rightarrow \frac{d^2A}{dl^2}\Big|_{l=10} = -0.6$<0(Maxima)

We get the area of the printed matter maximum for l = 10cm

Let's find the corresponding breadth using eq(1)

⇒ 10b = 150

⇒ b = 15cm

∴ The dimensions required for the printed area to be maximum is l = 10cm and b = 15cm.

45. Question

The space s described in time t by a particle moving in a straight line is given by s = t^5 – $40t^3$ + $30t^2$ + 80t – 250 Find the minimum value of acceleration.

Answer

Given:

The Distance(S) covered by a particle in time t is given by

⇒ S = t^5 - $40t^3$ + $30t^2$ + 80t - 250

We know that acceleration of a particle is given by $\frac{d^2S}{dt^2}$.

$\Rightarrow \frac{dS}{dt} = \frac{d(t^5-40t^3+30t^2+80t-250)}{dt}$

$\Rightarrow \frac{dS}{dt} = 5t^4 - 120t^2 + 60t + 80$

$\Rightarrow \frac{d^2S}{dt^2} = \frac{d(5t^4-120t^2+60t+80)}{dt}$

$\Rightarrow a = \frac{d^2S}{dt^2} = 20t^3 - 240t + 60$

We need acceleration to be minimum,

We know that for maxima and minima,

$\Rightarrow \frac{da}{dt} = 0$

$\Rightarrow \frac{d(20t^3-240t+60)}{dt} = 0$

$\Rightarrow 60t^2 - 240 = 0$

⇒ t^2 = 4

⇒ t = 2 (∵ Time cannot be negative)

Differentiating 'a' again,

$\Rightarrow \frac{d^2a}{dt^2} = \frac{d(60t^2-240)}{dt}$

$\Rightarrow \frac{d^2a}{dt^2} = 120t$

At t = 2

$\Rightarrow \frac{d^2a}{dt^2}|_{t=2} = 120 \times 2$

$\Rightarrow \frac{d^2a}{dt^2}|_{t=2} = 240 > 0$(Minima)

We get minimum for t = 2 sec,

The corresponding acceleration at t = 2 sec is,

$\Rightarrow a = 20(2)^3 - 240(2) + 60$

⇒ a = 160 - 480 + 60

⇒ a = - 260

∴ The minimum acceleration is - 260.

46. Question

A particle is moving in a straight line such that its distance s at any time t is given by $s = \frac{t^4}{4} - 2t^3 + 4t^2 - 7$.

Find when its velocity is maximum and acceleration minimum.

Answer

Given:

The distance covered by a particle at time 't' is given by,

$\Rightarrow S = \frac{t^4}{4} - 2t^3 + 4t^2 - 7$

We know that velocity of a particle is given by $\frac{dS}{dt}$ and acceleration of a particle is given by $\frac{d^2S}{dt^2}$.

$\Rightarrow v = \frac{dS}{dt} = \frac{d\left(\frac{t^4}{4} - 2t^3 + 4t^2 - 7\right)}{dt}$

$\Rightarrow v = t^3 - 6t^2 + 8t$

We need velocity to be maximum,

We know that for maxima and minima,

$\Rightarrow \frac{dv}{dt} = 0$

$\Rightarrow \frac{d(t^3 - 6t^2 + 8t)}{dt} = 0$

$\Rightarrow 3t^2 - 12t + 8 = 0$

$\Rightarrow t = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(8)}}{2 \times 3}$

$\Rightarrow t = \frac{12 \pm \sqrt{144 - 96}}{6}$

$\Rightarrow t = \frac{12 \pm \sqrt{48}}{6}$

$\Rightarrow t = \frac{12 \pm 4\sqrt{3}}{6}$

$\Rightarrow t = 2 + \frac{2}{\sqrt{3}}, t = 2 - \frac{2}{\sqrt{3}}$

Differentiating ‘v’ again,

$\Rightarrow \frac{d^2v}{dt^2} = \frac{d(3t^2-12t+8)}{dt}$

$\Rightarrow \frac{d^2v}{dt^2} = 6t - 12$

At $t = 2 + \frac{2}{\sqrt{3}}$

$\Rightarrow \frac{d^2v}{dt^2}\Big|_{t=2+\frac{2}{\sqrt{3}}} = 6\left(2 + \frac{2}{\sqrt{3}}\right) - 12$

$\Rightarrow \frac{d^2v}{dt^2}\Big|_{t=2+\frac{2}{\sqrt{3}}} = 12 + 4\sqrt{3} - 12$

$\Rightarrow \frac{d^2v}{dt^2}\Big|_{t=2+\frac{2}{\sqrt{3}}} = 4\sqrt{3} > 0$(Minima)

At $t = 2 - \frac{2}{\sqrt{3}}$

$\Rightarrow \frac{d^2v}{dt^2}\Big|_{t=2-\frac{2}{\sqrt{3}}} = 6\left(2 - \frac{2}{\sqrt{3}}\right) - 12$

$\Rightarrow \frac{d^2v}{dt^2}\Big|_{t=2-\frac{2}{\sqrt{3}}} = 12 - 4\sqrt{3} - 12$

$\Rightarrow \frac{d^2v}{dt^2}\Big|_{t=2-\frac{2}{\sqrt{3}}} = -4\sqrt{3} < 0$(Maxima)

We get the velocity maximum at $t = 2 - \frac{2}{\sqrt{3}}$

Now, we find the acceleration:

$\Rightarrow a = \frac{d^2S}{dt^2} = \frac{d(t^3-6t^2+8t)}{dt}$

$\Rightarrow a = 3t^2 - 12t + 8$

We need acceleration to be minimum,

We know that for maxima and minima,

$\Rightarrow \frac{da}{dt} = 0$

$\Rightarrow \frac{d(3t^2-12t+8)}{dt} = 0$

$\Rightarrow 6t - 12 = 0$

$\Rightarrow t = 2$

Differentiating ‘a’ again,

$\Rightarrow \frac{d^2a}{dt^2} = \frac{d(6t-12)}{dt}$

$\Rightarrow \frac{d^2a}{dt^2} = 6$

At $t = 2$

$\Rightarrow \frac{d^2a}{dt^2}\Big|_{t=2} = 6 > 0$(Minima)

We get minimum for $t = 2$,

$\therefore$ we get maximum velocity at $t = 2 - \frac{2}{\sqrt{3}}$ and minimum acceleration at $t = 2$

MCQ

1. Question

#Mark the correct alternative in each of the following

The maximum value of $x^{1/x}$, x > 0 is

A. $e^{1/e}$

B. $\left(\frac{1}{e}\right)^e$

C. 1

D. None of these

Answer

$f(x) = x^{\frac{1}{x}}$.

Let $y = x^{\frac{1}{x}}$

therefore,$logy = \frac{1}{x}\log_e x$.

Differentiating w.r.t x

so,

$$\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = \left(\frac{1}{x^2} - \frac{logx}{x^2}\right)$$

$$\frac{dy}{dx} = x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{logx}{x^2}\right) = y'$$

Now, lets put y'=0

$$x^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{logx}{x^2}\right) = 0$$

$$\frac{x^{\frac{1}{x}}}{x^2}(1 - logx) = 0$$

so $x^{\frac{1-2x}{x}} = 0$ OR (1-log x) =0

therefore,x=1/e and x=0

Hence by second derivative test

f''(1/e)<0 so it's a point of maximum

and maximum value is $\left(\frac{1}{e}\right)^e$.

2. Question

#Mark the correct alternative in each of the following

If $ax + \frac{b}{x} \geq c$ for all positive x where a, b, > 0, then

A. $ab < \frac{c^2}{4}$

B. $ab \geq \frac{c^2}{4}$

C. $ab \geq \frac{c}{4}$

D. none of these

Answer

let f(x)=ax+(b/x)-c;

x>0 where a,b>0

$f'(x) = a - \frac{b}{x^2}$

f'(x)=0

$f'(x) = a - \frac{b}{x^2} = 0$

$x^2 = \frac{b}{a}$

$x = \sqrt{\frac{b}{a}}$

$f\left(\sqrt{\frac{b}{a}}\right) \geq 0 \ldots .(Given)$

$a\sqrt{\frac{b}{a}} + b\sqrt{\frac{a}{b}} - c \geq 0$

$2(ab)^{\frac{1}{2}} \geq c = ab \geq \frac{c^2}{4}$

3. Question

#Mark the correct alternative in each of the following

The minimum value of $\frac{x}{\log_e x}$ is

A. e

B. 1/e

C. 1

D. none of these

Answer

$f(x) = \frac{x}{logx}$

$f'(x) = \frac{logx - 1}{(logx)^2}$

f'(x)=0

$$\frac{logx-1}{(logx)^2}=0$$

logx-1=0

⇒ x=e

for second derivative we find f’(x)

$$f''(x)=\frac{-1}{x(logx)^2}+\frac{2}{x(logx)^3}$$

Hence by second derivative test

f’’(x)>0 so it’s a point of minimum.

therefore, $f''(e)=\frac{-1}{e(loge)^2}+\frac{2}{e(loge)^3}$

$$=-\left(\frac{1}{e}\right)+\left(\frac{2}{e}\right)$$

$$\Rightarrow \frac{1}{e}>0$$

x=e is a point of minimum

so minimum value is f(e)=e

4. Question

#Mark the correct alternative in each of the following

For the function $f(x)=x+\frac{1}{x}$

A. x = 1 is a point of maximum

B. x = –1 is a point of minimum

C maximum value > minimum value

D. maximum value < minimum value

Answer

In such type of questions

find both the maximum and minimum value to compare the options.

so first,$f(x)=x+\frac{1}{x}$

so,$f'(x)=1-\frac{1}{x^2}$

put f’(x)=0;

$$1-\frac{1}{x^2}=0$$

$$1=\frac{1}{x^2}$$

⇒ x=±1

Hence by second derivative test

f’’(x)>0 or f”(x)<0

so it’s a point of minimum or maximum respecctively.

$f''(x) = \frac{2}{x^3};$

f”(-1)=-2<0;

f”(1)=2>0

so x=1 is a point of minimum

and x=-1 is a point of maximum

f(1)=2 is minimum value.

f(-1)=-2 is maximum value.

therefore,

<u>maximum value<minimum value.</u>

5. Question

#Mark the correct alternative in each of the following

Let f(x) = x^3 + $3x^2$ – 9x + 2. Then f(x) has

A. a maximum at x = 1

B. a minimum at x = 1

C. neither a maximum nor a minimum at x = 3

D. none of these

Answer

f(x)=x^3+3x^2-9x+2

f’(x)=$3x^2+6x-9$

f’(x)=0

$3x^2+6x-9=0$

by solving the quadratic we get roots as follows

x=-3 and x=1

Hence by second derivative test

f’’(x)>0 or f”(x)<0 so it’s a point of minimum or maximum respecctively.

f”(x)= 6x+6

f”(-3)=6(-3)+6

=-12<0

so At x=-3 maximum value.

f”(1)=6+6

=12>0

so At x=1 minimum value.

Option=(B)

6. Question

#Mark the correct alternative in each of the following

The minimum value of f(x) = x^4 – x^2 – 2x + 6 is

A. 6

B. 4

C. 8

D. none of these

Answer

$f(x)=x^4-x^2-2x+6$

$f'(x)=4x^3-2x-2$

so here put f'(x)=0

$4x^3-2x-2=0$

$2(x-1)(2x^2+2x+1)=0$

x=1 and other roots are complex.

so we will consider x=1

Hence by second derivative test

f''(x)>0 so it's a point of minimum.

$f''(x)=12x^2-1$

f" (1) =12(1)-2

=10>0

At x=1 minimum

So, f (1) =1-1-2(1) +6

=4

Option(B)

7. Question

#Mark the correct alternative in each of the following

The number which exceeds its square by the greatest possible quantity is

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. $\frac{3}{4}$

D. none of these

Answer

Let the function be according to the question,

$f(x)=x-x^2$

Here,it is asked "greatest possible quantity" so indirectly they are asking for the maximum value at which point x=?

so,f'(x)=1-2x

f'(x)=0

=>1-2x=0

x= �

f"(x)=-2<0

So for x= � maximum value of the function.

Option(A).

8. Question

#Mark the correct alternative in each of the following

Let f(x) = $(x – a)^2 + (x – b)^2 + (x – c)^2$. Then, f(x) has a minimum at x =

A. $\frac{a+b+c}{3}$

B. $3\sqrt{abc}$

C. $\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$

D. none of these

Answer

f(x)=$(x-a)^2 +(x-b)^2 +(x - c)^2$

f'(x)=2[x – a + x – b + x - c]

f'(x)=2[3x-a-b-c]

f'(x)=0

2[3x-a-b-c] =0

3x-a-b-c=0

3x=a + b + c

$x = \frac{a+b+c}{3}$

Hence by second derivative test

f''(x)>0 so it's a point of minimum.

f" (x)=2(3(1))

=6>0

so $x = \frac{a+b+c}{3}$ point of minimum.

Option(A).

9. Question

#Mark the correct alternative in each of the following

The sum of two non-zero numbers is 8, the minimum value of the sum of their reciprocals is

A. $\frac{1}{4}$

B. $\frac{1}{2}$

C. $\frac{1}{8}$

D. none of these

Answer

Let the two non zero numbers be x and y.

so,x+y=8

y=8-x

and,$f(x,y) = \frac{1}{x} + \frac{1}{y}$

put y=8-x in f(x,y)

therefore it will become a function in $f(x) = \frac{1}{x} + \frac{1}{(8-x)}$

$f'(x) = \frac{-1}{x^2} + \frac{1}{(8-x)^2}$

f’(x)=0

$\Rightarrow \frac{1}{x^2} = \frac{1}{(8-x)^2}$

⇒ x=8-x

⇒ x=4

$\Rightarrow f''(x)\ \frac{2}{x^3} + \frac{2}{(8-x)^3}$

Hence by second derivative test

f’’(x)>0 so it’s a point of minimum.

$f''(4) = \frac{2}{4^3} + \frac{2}{(8-4)^3} > 0$

Hence at x=4 minimum value

$f(4) = \frac{1}{4} + \frac{1}{(8-4)}$

$= \frac{2}{4}$

$= \frac{1}{2}$

Option(B)

10. Question

#Mark the correct alternative in each of the following

The function $f(x) = \sum_{r=1}^{5} (x-r)^2$ assumes minimum value at x =

A. 5

B. $\frac{5}{2}$

C. 3

D. 2

Answer

$$\sum_{r=1}^{5}(x-r)^2$$

$f(x)=(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2$

f'(x)=2[5x-15]

f'(x)=0;x=3

Hence by second derivative test

f''(x)>0 so it's a point of minimum.

f"(x)=1>0 so At x=3 minimum value.Option(C)

11. Question

#Mark the correct alternative in each of the following

At $x = \frac{5\pi}{6}$, f(x) = 2sin 3x + 3 cos 3x is

A. 0

B. maximum

C. minimum

D. none of these

Answer

$f(x) = 2sin3x + 3cos3x$

$f'(x) = 6cos3x - 9sin3x$

f'(x)=0

⇒6 cos 3x – 9 sin 3x = 0

⇒6 cos 3x = 9 sin 3x

$\Rightarrow \frac{6}{9} = tan3x$

$\Rightarrow \frac{2}{3} = tan3x$

$\Rightarrow x = \frac{1}{3}tan^{-1}\frac{2}{3}$

So, at $x = \frac{5\pi}{6}$, neither is maximum nor minimum

Option(D)

12. Question

#Mark the correct alternative in each of the following

If x lies in the interval [0, 1], then the least value of $x^2 + x + 1$ is

A. 3

B. $\frac{3}{4}$.

C. 1

D. none of these

Answer

f(x)=$x^2 + x + 1$

f’(x)= $2x + 1$

f’(x)=0

⇒2x+1=0

⇒ x= - �

At extreme points,

f(0)=0

f(1)=1+1+1>0

so,x=1 is a least value.

Option(C)

13. Question

#Mark the correct alternative in each of the following

The least value of the function f(x) = x^3 – $18x^2$ + 96x in the interval [0, 9] is

A. 126

B. 135

C 160

D. 0

Answer

f(x)=x^3-$18x^2$+96x

f(x)=$3x^2$-36x+96

f’(x)=0

$3x^2$-36x+96=0

x^2-12x+32=0

x^2-8x-4x+32=0

x(x-8)-4(x-8)=0

Now we have f(0)=0,f(4)=160,f(9)=135

Hence in the given interval least value is f(0)=0

Option(D)

14. Question

#Mark the correct alternative in each of the following

The least value of the function $f(x)=\frac{x}{4-x+x^2}$ on [–1, 1] is

A. $-\frac{1}{4}$

B. $-\frac{1}{3}$

C. -$\frac{1}{6}$

D. $\frac{1}{5}$

Answer

$f(x)=\frac{x}{4-x+x^2}$

$f'(x)=\frac{4-x+x^2+x-2x}{4-x+x^2}$

$f'(x)=\frac{4-x^2}{(4-x+x^2)^2}$=0

$\Rightarrow 4-x^2=0$

$=>$ x=± 2 but [-1,1]

=>f(-1)=-1/6 Option(C)

15. Question

#Mark the correct alternative in each of the following

The point on the curve $y^2 = 4x$ which is nearest to the point (2, 1) is

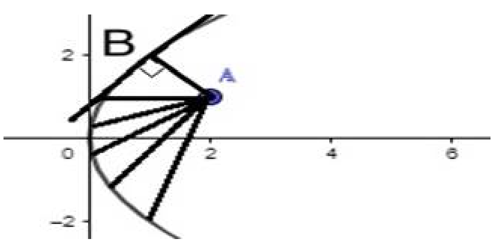
A. $\left(1, 2\sqrt{2}\right)$

B. (1, 2)

C. (1, –2)

D. (–2, 1)

Answer



If we consider the above figure as for getting it,their will be many points on the curve.

If the normal to the point B passes through the point(2,1) then point B will be the point having nearest distance from point (2,1).

Let B(x,y)

$\frac{dy}{dx}=\frac{2}{y}$

slope at the point B is (2/y) and normal's slope will be m=(–y/2) so by point slope formula.

⇒ $(y-y_1)=m(x-x_1)$;$(x_1=2,y_1=1)$

⇒ (y-1)=(-y/2)(x-2)

⇒2y-2=-xy+2y

⇒xy=2; $y^2 = 4x$

⇒from above to equations $y^3 = 8$

⇒y=2 and x=1

so the nearest point is (1,2).

Option(B)

16. Question

#Mark the correct alternative in each of the following

If x + y = 8, then the maximum value of xy is

A. 8

B. 16

C. 20

D. 24

Answer

x+y=8 ⇒ y=8-x

xy=x(8-x)

Let $f(x)=8x-x^2$

Differentiating f(x) with respect to x, we get

f'(x)=8-2x

Differentiating f'(x) with respect to x, we get

f''(x)=-2<0

For maxima at x=c, f'(c)=0 and f''(c)<0

f'(x)=0 ⇒ x=4

Also f''(4)=-2<0

Hence, x=4 is a point of maxima for f(x) and f(4)=16 is the maximum value of f(x).

17. Question

#Mark the correct alternative in each of the following

The least and greatest values of $f(x) = x^3 – 6x^2 + 9x$ in [0, 6], are

A. 3, 4

B. 0, 6

C. 0, 3

D. 3, 6

Answer

$f(x) = x^3 – 6x^2 + 9x, x\in[0,6]$

Differentiating f(x) with respect to x, we get

f'(x)= $3x^2$ - 12x + 9=3(x-3)(x-1)

For extreme points, f'(x)=0 ⇒ x=1 or x=3

For least and greatest value of f(x) in [0,6], we will have to check at extreme points as well as interval extremes

f(1)=4

f(3)=0

f(0)=0

f(6)=54

Hence the least value of f(x) in [0,6] is 0 and it's greatest value is 54.

18. Question

#Mark the correct alternative in each of the following

$f(x) = \sin + \sqrt{3}\cos x$ is maximum when x =

A. $\frac{\pi}{3}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{6}$

D. 0

Answer

$f(x) = sinx + \sqrt{3}cosx$

Differentiating f(x) with respect to x, we get

$f'(x) = cosx - \sqrt{3}sinx$

Differentiating f'(x) with respect to x, we get

$f''(x) = -sinx - \sqrt{3}cosx$

For maxima at x=c, f'(c)=0 and f''(c)<0

f'(x)=0⇒ $tanx = \frac{1}{\sqrt{3}}$ or $x = \frac{\pi}{6}\ or\ \frac{7\pi}{6}$

$f''\left(\frac{\pi}{6}\right) = -2 < 0$ and $f''\left(\frac{7\pi}{6}\right) = 2$>0

Hence, $x = \frac{\pi}{6}$ is a point of maxima for f(x).

19. Question

#Mark the correct alternative in each of the following

If a cone of maximum volume is inscribed in a given sphere, then the ratio of the he the cone to the diameter of the sphere is

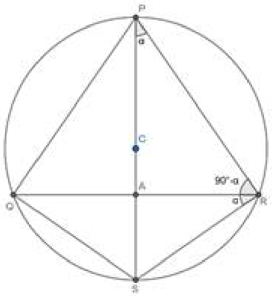
A. $\frac{3}{4}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{2}{3}$

Answer



In the figure, ΔPQR represents the 2D view of the cone and the circle represents the sphere. PA is perpendicular to QR and PS is diameter of circle. C will lie on PA due to symmetry.

Let the radius and height of cone be r and h and the radius of sphere be R. Also, the semi vertical angle of cone is α.

In ΔPAR

$tanP = tan\alpha = \frac{AR}{PA} = \frac{r}{h}$ -(1)

∠APR=α

∠PAR=90°

Hence, ∠ PRA=180° -90° -α =90° -α

Also ∠PRS=90° (Angle in a semicircle)

Hence ∠ARS=∠PRS-∠PRA=α

In ΔRAS

AS=PS-PA=2R-h

$tanR = tan\alpha = \frac{2R-h}{r}$ -(2)

From (1) and (2), we get

$r^2=2Rh-h^2$

The volume of cone will be

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h(2Rh - h^2)$$

Differentiating V with respect to h, we get

$$V' = \frac{1}{3}\pi(4Rh - 3h^2)$$

Differentiating V’ with respect to h, we get

$$V'' = \frac{1}{3}\pi(4R - 6h)$$

For maxima at h=c, V’(c)=0 and v’’(c)<0

V’=0 ⇒ $\frac{h}{2R} = \frac{2}{3}$

$$V''\left(h = \frac{4R}{3}\right) = -\frac{4}{3}\pi R < 0$$

Hence, the ratio of height of cone to diameter of sphere is $\frac{2}{3}$.

20. Question

#Mark the correct alternative in each of the following

The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is

A. 75

B. 50

C. 25

D. 55

Answer

$f(x) = x^2 + \frac{250}{x}$

Differentiating f(x) with respect to x, we get

$$f'(x) = 2x - \frac{250}{x^2}$$

Differentiating f'(x) with respect to x, we get

$$f''(x) = 2 + \frac{500}{x^3}$$

for minima at x=c, f'(c)=0 and f''(c)>0

f'(x)=0 ⇒ x^3=125 or x=5

f''(5)=7>0

Hence, x=5 is a point of minima for f(x) and f(5)=75 is the minimum value of (x).

21. Question

#Mark the correct alternative in each of the following

If $f(x) = x + \frac{1}{x}, x > 0$, then its greatest value is

A. –2

B. 0

C. 3

D. none of these

Answer

$f(x) = x + \frac{1}{x}, x > 0$

Differentiating f(x) with respect to x, we get

$$f'(x) = 1 - \frac{1}{x^2}$$

Also, differentiating f'(x) with respect to x, we get

$$f''(x) = \frac{2}{x^3}$$

For maxima at x=c, f'(c)=0 and f''(c)<0

⇒ $1-\frac{1}{x^2}=0$ or $x=1$ (Since x>0)

f''(1)=2>0

Since f''(1)>0 therefore x=1 is not a point of maxima and hence no maximum value of f(x) exists for x>0.

122. Question

#Mark the correct alternative in each of the following

If $f(x)=\frac{1}{4x^2+2x+1}$, then its maximum value is

A. $\frac{4}{3}$

B. $\frac{2}{3}$

C. 1

D. $\frac{3}{4}$

Answer

$f(x)=\frac{1}{4x^2+2x+1}$ achieves it's maximum value when g(x)=$4x^2$+2x+1 achieves it's minimum value.

Differentiating g(x) with respect to x, we get

g'(x)=8x+2

Differentiating g'(x) with respect to x, we get

g''(x)=8

For minima at x=c, g'(c)=0 and g''(c)>0

g'(x)=0 ⇒ $x=-\frac{1}{4}$

$g''\left(-\frac{1}{4}\right)=8>0$

Hence $x=-\frac{1}{4}$ is a point of minima for g(x) and $g\left(-\frac{1}{4}\right)=\frac{3}{4}$ is the minimum value of g(x).

Hence the maximum value of $f(x)=\frac{1}{g(x)}=\frac{4}{3}$

23. Question

#Mark the correct alternative in each of the following

Let x, y be two variables and x > 0, xy = 1, then minimum value of x + y is

A. 1

B. 2

C. $2\frac{1}{2}$

D. $3\frac{1}{3}$

Answer

xy=1, x>0, y>0

⇒ $y=\frac{1}{x}$

x+y= $x+\frac{1}{x}$

Let $f(x)=x+\frac{1}{x}, x>0$

Differentiating f(x) with respect to x, we get

$$f'(x)=1-\frac{1}{x^2}$$

Also, differentiating f'(x) with respect to x, we get

$$f''(x)=\frac{2}{x^3}$$

For minima at x=c, f'(c)=0 and f''(c)<0

⇒ $1-\frac{1}{x^2}=0$ or $x=1$ (Since x>0)

f''(1)=2>0

Hence, x=1 is a point of minima for f(x) and f(1)=2 is the minimum value of f(x) for x>0.

24. Question

#Mark the correct alternative in each of the following

f(x) = 1 + 2 sinx + 3cos^2 x, 0 ≤ x ≤ $\frac{2\pi}{3}$ is

A. Minimum at $x=\frac{\pi}{2}$

B. Maximum at x = $x=\sin^{-1}\left(1/\sqrt{3}\right)$

C. Minimum at $x=\frac{\pi}{6}$

D. Maximum at $\sin^{-1}(1/6)$

Answer

f(x)=1+2sinx+3cos^{2}x, $x\in[0,\frac{2\pi}{3}]$

Differentiating f(x) with respect to x, we get

f'(x)=2cosx-6cosxsinx

Also, differentiating f'(x) with respect to x, we get

f''(x)=-2sinx-6cos^{2}x+6sin^{2}x=-2sinx+12sin^{2}x-6 (Since sin^{2}x+cos^{2}x=1)

For extreme points, f'(x)=0

⇒ 2cosx(1-3sinx)=0 or $x=\frac{\pi}{2}$ or $sin^{-1}\left(\frac{1}{3}\right)$

$f''\left(\frac{\pi}{2}\right)=4>0$ and $f''\left(sin^{-1}\left(\frac{1}{3}\right)\right)=-\frac{16}{3}<0$

Hence, $x=\frac{\pi}{2}$ is a point of minima and $x=sin^{-1}\left(\frac{1}{3}\right)$ is a point of maxima.

25. Question

#Mark the correct alternative in each of the following

The function f(x) = 2x^3 – 15x^2 + 36x + 4 is maximum at x =

A. 3

B. 0

C. 4

D. 2

Answer

$f(x) = 2x^3 – 15x^2 + 36x + 4$

Differentiating f(x) with respect to x, we get

$f'(x)= 6x^2 - 30x + 36=6(x-2)(x-3)$

Differentiating f'(x) with respect to x, we get

$f''(x)=12x – 30$

for maxima at x=c, f'(c)=0 and f''(c)<0

$f'(x)=0 \Rightarrow x=2$ or $x=3$

$f''(2)=-6<0$ and $f''(3)=6>0$

Hence x=2 is a point of maxima.

26. Question

#Mark the correct alternative in each of the following

The maximum value of $f(x) = \frac{x}{4 + x + x^2}$ on [–1, 1] is

A. $-\frac{1}{4}$

B. $-\frac{1}{3}$

C. $\frac{1}{6}$

D. $\frac{1}{5}$

Answer

$f(x) = \frac{x}{4+x+x^2}, x \in [-1,1]$

Differentiating f(x) with respect to x, we get

$$f'(x) = \frac{(4+x+x^2)\times 1 - x\times(1+2x)}{(4+x+x^2)^2} = \frac{4-x^2}{(4+x+x^2)^2}$$

Since $4-x^2>0\ \forall\ x\in[-1,1]$ and $(4+x+x^2)^2>0\ \forall\ x\in R$

Therefore, $f'(x)>0\ \forall\ x\in[-1,1]$

Hence, f(x) is increasing in [-1,1] and therefore the maximum value of f(x) occurs at x=1 and $f(1) = \frac{1}{6}$

27. Question

#Mark the correct alternative in each of the following

Let $f(x) = 2x^3 – 3x^2 – 12x + 5$ on [–2, 4]. The relative maximum occurs at x =

A. –2

B. –1

C. 2

D. 4

Answer

$f(x) = 2x^3 – 3x^2 – 12x + 5, x\in[-2,4]$

Differentiating f(x) with respect to x, we get

$f'(x)= 6x^2 – 6x – 12=6(x+1)(x-2)$

Differentiating f'(x) with respect to x, we get

$f''(x)=12x-6$

For maxima at x=c, f'(c)=0 and f''(c)<0

$f'(x)=0 \Rightarrow x=-1$ or 2

$f''(-1)=-18<0$ and $f''(2)=18>0$

Hence, x=-1 is the point of local maxima.

28. Question

#Mark the correct alternative in each of the following

The minimum value of x $\log_e$ x is equal to

A. e

B. $\frac{1}{e}$

C. $-\frac{1}{e}$

D. 2e

Answer

$f(x) = x \log_e x$

Differentiating f(x) with respect to x, we get

$f'(x) = x \times \frac{1}{x} + \log_e x \times 1 = 1 + \log_e x$

Differentiating f'(x) with respect to x, we get

$f''(x) = \frac{1}{x}$

for minima at x=c, f'(c)=0 and f''(c)>0

$f'(x)=0 \Rightarrow x = \frac{1}{e}$

$f''\left(\frac{1}{e}\right) = e > 0$

Hence, $x = \frac{1}{e}$ is a point of minima for f(x) and $f\left(\frac{1}{e}\right) = -\frac{1}{e}$ is the minimum value of f(x).

29. Question

#Mark the correct alternative in each of the following

The minimum value of the function $f(x) = 2x^3 – 21x^2 + 36x – 20$ is

A. –128

B. –126

C. –120

D. none of these

Answer

$f(x) = 2x^3 – 21x^2 + 36x – 20$

Differentiating f(x) with respect to x, we get

$f'(x)= 6x^2 - 42x + 36=6(x-1)(x-6)$

Differentiating f'(x) with respect to x, we get

$f''(x)=12x-42$

for minima at x=c, f'(c)=0 and f''(c)>0

$f'(x)=0 \Rightarrow x=1$ or $x=6$

$f''(1)=-30<0$ and $f''(6)=30>0$

Hence, x=6 is the point of minima for f(x) and f(6)=-128 is the local minimum value of f(x).

Very short answer

1. Question

Write necessary condition for a point x = c to be an extreme point of the function $f(x)$.

Answer

Condition 1:- If f (x) is continuous and f'(x) and $f''(x)$ exists then all points where ($f'(x)=0$ and $f''(x)>0$) or ($f'(x)=0$ and $f''(x)<0$) are extreme points.

Condition 2:- If f(x) is continuous and defined in [a,b] and condition 1 is not satisfied, then points a and b are extreme points.

2. Question

Write sufficient conditions for a point x = c to be a point of local maximum.

Answer

For x=c to be a local maximum of f (x), f'(x)=0 & f''(x)<0 (when f(x) is defined at c).If f(x) is not defined at c, we need to check values of f(x) at all extreme points.

3. Question

If f(x) attains a local minimum at x = c, then write the values of $f'(c)$ and $f''(c)$.

Answer

If f (x) has a local minima at x =c means f'(c)=0 and f''(c)>0

4. Question

Write the minimum value of $f(x) = x + \frac{1}{x}, x > 0$.

Answer

Since x >0. By AM>=GM property;

$$\left(x + \frac{1}{x}\right) \geq \sqrt{x} * \frac{1}{\sqrt{x}}$$

So minimum value of $\left(x + \frac{1}{x}\right)$ is 1 which occurs when $x = \frac{1}{x}$ i.e x=1.

5. Question

Write the maximum value of $f(x) = x + \frac{1}{x}, x < 0.$

Answer

Since x< Let $f(x) = \left(x + \frac{1}{x}\right).$

$f'(x) = 1 - \left(\frac{1}{x^2}\right).$

$f''(x) = \frac{2}{x^3}$

For maximum value $f'(x)$=0

⇒ x^2 =1

⇒ x = -1 (as x<0)

Since f'(x)=0 and f''(x)<0 for x=-1. Hence maximum value occurs at x =-1. f (-1)= -2.

6. Question

Write the point where $f(x) = x \log_e x$ attains minimum value.

Answer

Let f (x)= x $\log_e$ x. Clearly f (x) is only defined for x >0.

For minimum value; f'(x)=0 and f''(x)>0.

f'(x)= 1+lnx

⇒ x=1/e

f''(x)= 1/x, Clearly f''(x)>0 for all x. So only minima is defined.

So, minima of f (x) is at x = 1/e. $f\left(\frac{1}{e}\right) = -\frac{1}{e}$

7. Question

Find the least value of $f(x) = ax + \frac{b}{x}$, where a>0, b>0 and x >0.

Answer

Since x >0, a>0,b>0. By AM>=GM property; $\left(ax + \frac{b}{x}\right) \geq \sqrt{a} \times \sqrt{b}.$

So minimum value of $f(x)$ is $\sqrt{a} \times \sqrt{b}$which occurs when $ax = \frac{b}{x}$

i.e. $x = \frac{\sqrt{b}}{\sqrt{a}}$

8. Question

Write the minimum value of $f(x) = x^x$.

Answer

Let y= x^x. Take antilog on both sides

ln y =x × ln x. Let us differentiate and find $f'(x)$=0

⇒ $\frac{1}{y} \times \frac{dy}{dx} = lnx + 1$

$\Rightarrow \frac{dy}{dx} = y \times (lnx + 1)$

$\Rightarrow f'(x)=x^{x} \times (\ln x + 1)$

$f'(x) = 0$

$\Rightarrow x = 0, x = \frac{1}{e}$

But ln x is not defined at x =0

Therefore, minima occur at $x = \frac{1}{e}$. So, $f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}}$

9. Question

Write the minimum value of $f(x) = x^{1/x}$.

Answer

Let $y = x^{\left(\frac{1}{x}\right)}$. Take antilog on both sides

$\Rightarrow lny = \frac{1}{x} \times lnx$

Let us differentiate and find $f'(x)=0$

$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = \frac{1}{x^2} - \left(lnx \times \left(\frac{1}{x^2}\right)\right)$

$=> \frac{dy}{dx} = \frac{(y*(1-lnx))}{x^2}$

$=> f'(x) = 0 => x = 0, e$

But lnx is not defined at x =0 $f(e) = e^{\frac{1}{e}}$

Therefore, minima occur at x = e.

So, $f(e) = e^{\frac{1}{e}}$ is the minimum value of this function.

10. Question

Write the maximum value of $f(x) = \frac{\log x}{x}$, if it exists.

Answer

Let $f(x) = \frac{logx}{x}$ Clearly $f(x)$ only defined for $x>0$. $f'(x) = \frac{(1-logx)}{x^2}$ and $f''(x) = \frac{(2xlogx-3x)}{x^4}$.($f''(x) < 0\ for\ all\ x$)

So, $f''(x) = 0$

$\Rightarrow$ x=e and f'' (e)<0

So, x=e is a point of maxima. Therefore, $f(e) = \frac{1}{e}$