

## 2. Functions

### Exercise 2.1

#### 1 A. Question

Give an example of a function

Which is one - one but not onto.

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, Let,  $f: \mathbf{N} \rightarrow \mathbf{N}$  given by  $f(x) = x^2$

Check for Injectivity:

Let x,y be elements belongs to N i.e  $x, y \in \mathbf{N}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow (x - y)(x + y) = 0$$

As  $x, y \in \mathbf{N}$  therefore  $x + y > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence f is One - One function

Check for Surjectivity:

Let y be element belongs to N i.e  $y \in \mathbf{N}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \sqrt{y}$$

$\Rightarrow \sqrt{y}$  not belongs to N for non-perfect square value of y.

Therefore no non - perfect square value of y has a pre image in domain N.

Hence,  $f: \mathbf{N} \rightarrow \mathbf{N}$  given by  $f(x) = x^2$  is One - One but not onto.

## 1 B. Question

Give an example of a function

Which is not one - one but onto.

### Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

**Onto Function:** - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, Let,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3 - x$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e  $x, y \in \mathbb{R}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 - x = y^3 - y$$

$$\Rightarrow x^3 - y^3 - (x - y) = 0$$

$$\Rightarrow (x - y)(x^2 + xy + y^2 - 1) = 0$$

$$\text{As } x^2 + xy + y^2 \geq 0$$

$$\Rightarrow \text{therefore } x^2 + xy + y^2 - 1 \geq -1$$

$$\Rightarrow x - y \neq 0$$

$$\Rightarrow x \neq y \text{ for some } x, y \in \mathbb{R}$$

Hence f is not One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 - x = y$$

$$\Rightarrow x^3 - x - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let  $x = \alpha$  be that root

$$\Rightarrow \alpha^3 - \alpha = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly  $y \in \mathbb{R}$ , there exist  $\alpha \in \mathbb{R}$  such that  $f(x) = y$

Therefore  $f$  is onto

$\Rightarrow$  Hence,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3 - x$  is not One - One but onto

### 1 C. Question

Give an example of a function

Which is neither one - one nor onto.

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, Let,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 5$

As we know

A constant function is neither one - one nor onto.

So, here  $f(x) = 5$  is constant function

Therefore

$f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 5$  is neither one - one nor onto function.

### 2 A. Question

Which of the following functions from  $A$  to  $B$  are one - one and onto?

$f_1 = \{(1, 3), (2, 5), (3, 7)\}; A = \{1, 2, 3\}, B = \{3, 5, 7\}$

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, As given,

$$f_1 = \{(1, 3), (2, 5), (3, 7)\}$$

$$A = \{1, 2, 3\}, B = \{3, 5, 7\}$$

Thus we can see that,

Check for Injectivity:

Every element of A has a different image from B

Hence f is a One - One function

Check for Surjectivity:

Also, each element of B is an image of some element of A

Hence f is Onto.

## 2 B. Question

Which of the following functions from A to B are one - one and onto?

$$f_2 = \{(2, a), (3, b), (4, c)\}; A = \{2, 3, 4\}, B = \{a, b, c\}$$

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of B i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, As given,

$$f_2 = \{(2, a), (3, b), (4, c)\}$$

$$A = \{2, 3, 4\}, B = \{a, b, c\}$$

Thus we can see that

Check for Injectivity:

Every element of A has a different image from B

Hence f is a One - One function

Check for Surjectivity:

Also, each element of B is an image of some element of A

Hence f is Onto.

## 2 C. Question

Which of the following functions from A to B are one - one and onto?

$$f_3 = \{(a, x), (b, x), (c, z), (d, z)\}; A = \{a, b, c, d\}, B = \{x, y, z\}$$

## Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

**Onto Function:** - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, As given,

$$f_3 = \{(a, x), (b, x), (c, z), (d, z)\}$$

$$A = \{a, b, c, d\}, B = \{x, y, z\}$$

Thus we can clearly see that

Check for Injectivity:

Every element of A does not have different image from B

Since,

$$f_3(a) = x = f_3(b) \text{ and } f_3(c) = z = f_3(d)$$

Therefore f is not One - One function

Check for Surjectivity:

Also each element of B is not image of any element of A

Hence f is not Onto.

### 3. Question

Prove that the function  $f: \mathbb{N} \rightarrow \mathbb{N}$ , defined by  $f(x) = x^2 + x + 1$  is one - one but not onto.

## Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

**Onto Function:** - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now,  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^2 + x + 1$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{N}$  i.e  $x, y \in \mathbb{N}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 + x + 1 = y^2 + y + 1$$

$$\Rightarrow x^2 - y^2 + x - y = 0$$

$$\Rightarrow (x - y)(x + y + 1) = 0$$

As  $x, y \in \mathbb{N}$  therefore  $x + y + 1 > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Check for Surjectivity:

$y$  be element belongs to  $\mathbb{N}$  i.e  $y \in \mathbb{N}$  be arbitrary

Since for  $y > 1$ , we do not have any pre image in domain  $\mathbb{N}$ .

Hence,  $f$  is not Onto function.

#### 4. Question

Let  $A = \{-1, 0, 1\}$  and  $f = \{(x, x^2) : x \in A\}$ . Show that  $f : A \rightarrow A$  is neither one - one nor onto.

**Answer**

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now, We have,  $A = \{-1, 0, 1\}$  and  $f = \{(x, x^2) : x \in A\}$ .

To Prove: -  $f : A \rightarrow A$  is neither One - One nor onto function

Check for Injectivity:

We can clearly see that

$$f(1) = 1$$

$$\text{and } f(-1) = 1$$

Therefore

$$f(1) = f(-1)$$

$\Rightarrow$  Every element of  $A$  does not have different image from  $A$

Hence  $f$  is not One - One function

Check for Surjectivity:

Since,  $y = -1$  be element belongs to  $A$

i.e  $-1 \in A$  in co - domain does not have any pre image in domain  $A$ .

Hence,  $f$  is not Onto function.

### 5 A. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^2$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^2$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{N}$  i.e  $x, y \in \mathbb{N}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow (x - y)(x + y) = 0$$

As  $x, y \in \mathbb{N}$  therefore  $x + y > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{N}$  i.e  $y \in \mathbb{N}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \sqrt{y}$$

$\Rightarrow \sqrt{y}$  not belongs to  $\mathbb{N}$  for non-perfect square value of  $y$ .

Therefore no non - perfect square value of  $y$  has a pre-image in domain  $\mathbb{N}$ .

Hence,  $f$  is not Onto function.

Thus, Not Bijective also.

### 5 B. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = x^2$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e. if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  given by  $f(x) = x^2$

Check for Injectivity:

Let  $x_1, -x_1$  be elements belongs to  $\mathbb{Z}$  i.e  $x_1, -x_1 \in \mathbb{Z}$  such that

So, from definition

$$\Rightarrow x_1 \neq -x_1$$

$$\Rightarrow (x_1)^2 = (-x_1)^2$$

$$\Rightarrow f(x_1)^2 = f(-x_1)^2$$

Hence  $f$  is not One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{Z}$  i.e  $y \in \mathbb{Z}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y}$$

$\Rightarrow \sqrt{y}$  not belongs to  $\mathbb{Z}$  for non-perfect square value of  $y$ .

Therefore no non - perfect square value of  $y$  has a pre-image in domain  $\mathbb{Z}$ .



Hence,  $f$  is not Onto function.

Thus, Not Bijective also

### 5 C. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^3$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = x^3$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{N}$  i.e  $x, y \in \mathbb{N}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x^3 - y^3 = 0$$

$$\Rightarrow (x - y)(x^2 + y^2 + xy) = 0$$

As  $x, y \in \mathbb{N}$  therefore  $x^2 + y^2 + xy > 0$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{N}$  i.e  $y \in \mathbb{N}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 = y$$

$$\Rightarrow x = \sqrt[3]{y}$$

$\Rightarrow \sqrt[3]{y}$  not belongs to  $\mathbb{N}$  for non-perfect cube value of  $y$ .

Since  $f$  attain only cubic number like 1,8,27.....,

Therefore no non - perfect cubic values of  $y$  in  $N$  (co - domain) has a pre-image in domain  $N$ .

Hence,  $f$  is not onto function

Thus, Not Bijective also

### 5 D. Question

Classify the following functions as injection, surjection or bijection:

$f : Z \rightarrow Z$  given by  $f(x) = x^3$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : Z \rightarrow Z$  given by  $f(x) = x^3$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $Z$  i.e  $x, y \in Z$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x^3 - y^3 = 0$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $Z$  i.e  $y \in Z$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 = y$$

$$\Rightarrow x = \sqrt[3]{y}$$

$\Rightarrow \sqrt[3]{y}$  not belongs to  $Z$  for non - perfect cube value of  $y$ .

Since  $f$  attain only cubic number like 1,8,27,....

Therefore no non - perfect cubic values of  $y$  in  $Z$  (co - domain) have a pre-image in domain  $Z$ .

Hence,  $f$  is not onto function

Thus, Not Bijective also

### 5 E. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = |x|$

#### Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

**Onto Function:** - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

**Bijection Function:** - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = |x|$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e  $x, y \in \mathbb{R}$  such that

Case i

$\Rightarrow x = y$

$\Rightarrow |x| = |y|$

Case ii

$\Rightarrow -x = y$

$\Rightarrow |-x| = |y|$

$\Rightarrow x = |y|$

Hence from case i and case ii f is not One - One function

Check for Surjectivity:

Since f attain only positive values, for negative real numbers in  $\mathbb{R}$

(co - domain) there is no pre-image in domain  $\mathbb{R}$ .

Hence, f is not onto function

Thus, Not Bijective also

### 5 F. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{Z} \rightarrow \mathbb{Z}$ , defined by  $f(x) = x^2 + x$

#### Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: Z \rightarrow Z$  given by  $f(x) = x^2 + x$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $Z$  i.e  $x, y \in Z$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 + x = y^2 + y$$

$$\Rightarrow x^2 - y^2 + x - y = 0$$

$$\Rightarrow (x - y)(x + y + 1) = 0$$

Either  $(x - y) = 0$  or  $(x + y + 1) = 0$

Case i :

$$\text{If } x - y = 0$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Case ii :

$$\text{If } x + y + 1 = 0$$

$$\Rightarrow x + y = -1$$

$$\Rightarrow x \neq y$$

Hence  $f$  is not One - One function

Thus from case i and case ii  $f$  is not One - One function

Check for Surjectivity:

As  $1 \in Z$

Let  $x$  be element belongs to  $Z$  i.e  $y \in Z$  be arbitrary, then

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^2 + x = 1$$

$$\Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+4}}{2}$$

Above value of  $x$  does not belong to  $Z$

Therefore no values of  $x$  in  $Z$  (co - domain) have a pre-image in domain  $Z$ .

Hence,  $f$  is not onto function

Thus, Not Bijective also

### 5 G. Question

Classify the following functions as injection, surjection or bijection:

$f : Z \rightarrow Z$ , defined by  $f(x) = x - 5$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : Z \rightarrow Z$  given by  $f(x) = x - 5$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $Z$  i.e  $x, y \in Z$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x - 5 = y - 5$$

$$\Rightarrow x = y$$

Hence,  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $Z$  i.e  $y \in Z$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x - 5 = y$$

$$\Rightarrow x = y + 5$$

Above value of  $x$  belongs to  $Z$

Therefore for each element in  $Z$  (co - domain) there exists an element in domain  $Z$ .

Hence,  $f$  is onto function

Thus, Bijective function

### 5 H. Question

Classify the following functions as injection, surjection or bijection:

$f : R \rightarrow R$ , defined by  $f(x) = \sin x$

## Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

**Onto Function:** - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

**Bijection Function:** - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = \sin x$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e  $x, y \in \mathbb{R}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \sin x = \sin y$$

$$\Rightarrow x = n\pi + (-1)^n y$$

$$\Rightarrow x \neq y$$

Hence, f is not One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \sin x = y$$

$$\Rightarrow x = \sin^{-1} y$$

Now, for  $y > 1$   $x$  not belongs to  $\mathbb{R}$  (Domain)

Hence, f is not onto function

Thus, It is also not Bijective function

## 5 I. Question

Classify the following functions as injection, surjection or bijection:

$f: \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = x^3 + 1$

## Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Onto Function: - A function  $f: A \rightarrow B$  is said to be an onto function or surjection if every element of  $B$  i.e. if  $f(A) = B$  or range of  $f$  is the co-domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one-to-one as well as onto function.

Now, Let,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3 + 1$

Check for Injectivity:

Let  $x, y$  be elements belong to  $\mathbb{R}$  i.e.  $x, y \in \mathbb{R}$  such that

So, from definition

$\Rightarrow f(x) = f(y)$

$\Rightarrow x^3 + 1 = y^3 + 1$

$\Rightarrow x^3 = y^3$

$\Rightarrow x = y$

Hence  $f$  is One-to-One function

Check for Surjectivity:

Let  $y$  be element belong to  $\mathbb{R}$  i.e.  $y \in \mathbb{R}$  be arbitrary, then

$\Rightarrow f(x) = y$

$\Rightarrow x^3 + 1 = y$

Now, we know that for 3 degree equation has a real root

So, let  $x = \alpha$  be that root

$\Rightarrow \alpha^3 + 1 = y$

$\Rightarrow f(\alpha) = y$

Thus for clearly  $y \in \mathbb{R}$ , there exist  $\alpha \in \mathbb{R}$  such that  $f(x) = y$

Therefore  $f$  is onto

Thus, It is also Bijective function

## 5 J. Question

Classify the following functions as injection, surjection or bijection:

$f: \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = x^3 - x$

**Answer**

**TIP**: - One-to-One Function: - A function  $f: A \rightarrow B$  is said to be a one-to-one function or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One-to-One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be an onto function or surjection if every element of  $B$  i.e. if  $f(A) = B$  or range of  $f$  is the co-domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one-to-one as well as onto function.

Now, Let,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^3 + x$

Check for Injectivity:

Let  $x, y$  be elements belonging to  $\mathbb{R}$  i.e.  $x, y \in \mathbb{R}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 + x = y^3 + y$$

$$\Rightarrow x^3 - y^3 - (x - y) = 0$$

$$\Rightarrow (x - y)(x^2 + xy + y^2 - 1) = 0$$

Hence  $f$  is not One-to-One function

Check for Surjectivity:

Let  $y$  be an element belonging to  $\mathbb{R}$  i.e.  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow x^3 + x = y$$

$$\Rightarrow x^3 + x - y = 0$$

Now, we know that for a 3-degree equation has a real root

So, let  $x = \alpha$  be that root

$$\Rightarrow \alpha^3 + \alpha = y$$

$$f(\alpha) = y$$

Thus for clearly  $y \in \mathbb{R}$ , there exist  $\alpha \in \mathbb{R}$  such that  $f(x) = y$

Therefore  $f$  is onto

Thus, It is not Bijective function

## 5 K. Question

Classify the following functions as injection, surjection or bijection:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \text{ defined by } f(x) = \sin^2 x + \cos^2 x$$

### Answer

**TIP**: - One-to-One Function: - A function  $f: A \rightarrow B$  is said to be a one-to-one function or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One-to-One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$



$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = \sin^2x + \cos^2x$

Check for Injectivity and Check for Surjectivity

Let  $x$  be element belongs to  $\mathbb{R}$  i.e  $x \in \mathbb{R}$  such that

So, from definition

$$\Rightarrow f(x) = \sin^2x + \cos^2x$$

$$\Rightarrow f(x) = \sin^2x + \cos^2x$$

$$\Rightarrow f(x) = 1$$

$$\Rightarrow f(x) = \text{constant}$$

We know that a constant function is neither One - One function nor onto function.

Thus, It is not Bijective function

### 5 L. Question

Classify the following functions as injection, surjection or bijection:

$$f: \mathbb{Q} - \{3\} \rightarrow \mathbb{Q}, \text{ defined by } f(x) = \frac{2x+3}{x-3}$$

### Answer

**TIP**: - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

$$\text{Now, } f: \mathbb{R} \rightarrow \mathbb{R} \text{ given by } f(x) = \frac{2x+3}{x-3}$$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{Q}$  i.e  $x, y \in \mathbb{Q}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{2x+3}{x-3} = \frac{2y+3}{y-3}$$

$$\Rightarrow (2x+3)(y-3) = (2y+3)(x-3)$$

$$\Rightarrow 2xy - 6x + 3y - 9 = 2xy - 6y + 3x - 9$$

$$\Rightarrow -6x + 3y = -6y + 3x$$

$$\Rightarrow -6x + 3y + 6y - 3x = 0$$

$$\Rightarrow -9x + 9y = 0$$

$$\Rightarrow x = y$$

Thus, f is One - One function

Check for Surjectivity:

Let y be element belongs to Q i.e  $y \in \mathbb{Q}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{2x+3}{x-3} = y$$

$$\Rightarrow 2x + 3 = y(x - 3)$$

$$\Rightarrow 2x + 3 = xy - 3y$$

$$\Rightarrow 2x - xy = -3(y + 1)$$

$$\Rightarrow x = \frac{-3(y+1)}{2-y}$$

Above value of x belongs to  $\mathbb{Q} - [3]$  for  $y = 2$

Therefore for each element in  $\mathbb{Q} - [3]$  (co - domain), there does not exist an element in domain  $\mathbb{Q}$ .

Hence, f is not onto function

Thus, Not Bijective function

### 5 M. Question

Classify the following functions as injection, surjection or bijection:

$$f : \mathbb{Q} \rightarrow \mathbb{Q}, \text{ defined by } f(x) = x^3 + 1$$

**Answer**

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of B i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : \mathbb{Q} \rightarrow \mathbb{Q}$ , defined by  $f(x) = x^3 + 1$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{Q}$  i.e  $x, y \in \mathbb{Q}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^3 + 1 = y^3 + 1$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence,  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{Q}$  i.e  $y \in \mathbb{Q}$  be arbitrary, then

$$\Rightarrow x^3 + 1 = y$$

$$\Rightarrow x^3 + 1 - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let  $x = \alpha$  be that root

$$\Rightarrow \alpha^3 + 1 = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly  $y \in \mathbb{Q}$ , there exist  $\alpha \in \mathbb{Q}$  such that  $f(x) = y$

Therefore  $f$  is onto

Thus, It is a Bijective function

## 5 N. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = 5x^3 + 4$

**Answer**

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = 5x^3 + 4$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e  $x, y \in \mathbb{R}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow 5x^3 + 4 = 5y^3 + 4$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence,  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow 5x^3 + 4 = y$$

$$\Rightarrow 5x^3 + 4 - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let  $x = \alpha$  be that root

$$\Rightarrow 5\alpha^3 + 4 = y$$

$$f(\alpha) = y$$

Thus for clearly  $y \in \mathbb{R}$ , there exist  $\alpha \in \mathbb{R}$  such that  $f(x) = y$

Therefore  $f$  is onto

Thus, It is a Bijective function

### 5 O. Question

Classify the following functions as injection, surjection or bijection:

$f : \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = 3 - 4x$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 3 - 4x$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e  $x, y \in \mathbb{R}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow 3 - 4x = 3 - 4y$$

$$\Rightarrow x = y$$

Hence,  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e.  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow 3 - 4x = y$$

$$\Rightarrow x = \frac{3-y}{4}$$

Above value of  $x$  belongs to  $\mathbb{R}$

Therefore for each element in  $\mathbb{R}$  (co - domain), there exists an element in domain  $\mathbb{R}$ .

Hence,  $f$  is onto function

Thus, Bijective function

### 5 P. Question

Classify the following functions as injection, surjection or bijection:

$$f : \mathbb{R} \rightarrow \mathbb{R}, \text{ defined by } f(x) = 1 + x^2$$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e. if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 1 + x^2$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e.  $x, y \in \mathbb{R}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow x^2 + 1 = y^2 + 1$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow \pm x = \pm y$$

Therefore, either  $x = y$  or  $x = -y$  or  $x \neq y$

Hence  $f$  is not One - One function

Check for Surjectivity:

1 be element belongs to  $\mathbb{R}$  i.e  $1 \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^2 + x = 1$$

$$\Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \pm \sqrt{y-1}$$

Above value of  $x$  not belongs to  $\mathbb{R}$  for  $y < 1$

Therefore  $f$  is not onto

Thus, It is also not Bijective function

### 5 Q. Question

Classify the following functions as injection, surjection or bijection:

$$f: \mathbb{R} \rightarrow \mathbb{R}, \text{ defined by } f(x) = \frac{x}{x^2+1}$$

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

$$\text{Now, } f: \mathbb{R} \rightarrow \mathbb{R} \text{ given by } f(x) = \frac{x}{x^2+1}$$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e.  $x, y \in \mathbb{R}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{x}{x^2+1} = \frac{y}{y^2+1}$$

$$\Rightarrow xy^2 + x = yx^2 + y$$

$$\Rightarrow xy^2 + x - yx^2 - y = 0$$

$$\Rightarrow xy(y-x) + (x-y) = 0$$

$$\Rightarrow (x-y)(1-xy) = 0$$

Case i :

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

f is One - One function

Case ii :

$$\Rightarrow 1 - xy = 0$$

$$\Rightarrow xy = 1$$

Thus from case i and case ii f is One - One function

Check for Surjectivity:

Let y be element belongs to R i.e  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x}{x^2 + 1} = y$$

$$\Rightarrow x = x^2y + y$$

$$\Rightarrow x - x^2y = y$$

Above value of x belongs to R

Therefore for each element in R (co - domain) there exists an element in domain R.

Hence, f is onto function

Thus, Bijjective function

## 6. Question

If  $f: A \rightarrow B$  is an injection such that range of  $f = \{a\}$ . Determine the number of elements in A.

**Answer**

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Here, Range  $\{f\} = \{a\}$

Since it is injective map, different elements have different images.

Thus A has only one element

## 7. Question

Show that the function  $f: \mathbb{R} - \{3\} \rightarrow \mathbb{R} - \{1\}$  given by  $f(x) = \frac{x-2}{x-3}$  is a bijection.

**Answer**

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = \frac{x-2}{x-3}$

To Prove: -  $f(x) = \frac{x-2}{x-3}$  is a bijection

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e.  $x, y \in \mathbb{R}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (x-3)(y-2)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$$

$$\Rightarrow -3x - 2y + 2x + 3y = 0$$

$$\Rightarrow -x + y = 0$$

$$\Rightarrow x = y$$

Hence,  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e.  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x - xy = 2 - 3y$$

$$\Rightarrow x = \frac{2-3y}{1-y}$$

$x = \frac{2-3y}{1-y}$  is a real number for all  $y \neq 1$ .

Also,  $\frac{2-3y}{1-y} \neq 2$  for any  $y$

Therefore for each element in  $\mathbb{R}$  (co - domain), there exists an element in domain  $\mathbb{R}$ .

Hence,  $f$  is onto function

Thus, Bijective function



## 8 A. Question

Let  $A = [-1, 1]$ , Then, discuss whether the following functions from  $A$  to itself are one - one, onto or bijective:

$$f(x) = \frac{x}{2}$$

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now, here  $f: A \rightarrow A: A = [-1, 1]$  given by function is  $f(x) = \frac{x}{2}$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $A$  i.e.  $x, y \in A$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \frac{x}{2} = \frac{y}{2}$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y$$

1 belongs to  $A$  then

$$f(1) = \frac{1}{2}$$

Not element of  $A$  co - domain

Hence,  $f$  is not One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $A$  i.e  $y \in A$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x}{2} = y$$

$$\Rightarrow x = 2y$$

Now,

1 belongs to  $A$

$\Rightarrow x = 2$ , which not belong to  $A$  co - domain

Hence,  $f$  is not onto function

Thus, It is not Bijective function

### 8 B. Question

Let  $A = [-1, 1]$ , Then, discuss whether the following functions from  $A$  to itself are one - one, onto or bijective:

$$g(x) = |x|$$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now, here  $f: A \rightarrow A: A = [-1, 1]$  given by function is  $g(x) = |x|$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $A$  i.e  $x, y \in A$  such that

$$\Rightarrow g(x) = g(y)$$

$$\Rightarrow |x| = |y|$$

$$\Rightarrow x = y$$

1 belongs to  $A$  then

$$\Rightarrow g(1) = 1 = g(-1)$$

Since, it has many element of  $A$  co - domain

Hence,  $g$  is not One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $A$  i.e  $y \in A$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \frac{x}{2} = y$$

$$\Rightarrow x = 2y$$

Now,

1 belongs to  $A$

$$\Rightarrow x = 2, \text{ which not belong to } A \text{ co - domain}$$

Since  $g$  attain only positive values, for negative - 1 in  $A$  (co - domain) there is no pre-image in domain  $A$ .

Hence,  $g$  is not onto function

Thus, It is not Bijective function

### 8 C. Question

Let  $A = [-1, 1]$ , Then, discuss whether the following functions from  $A$  to itself are one - one, onto or bijective:

$$h(x) = x^2$$

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now, here  $f: A \rightarrow A: A = [-1, 1]$  given by function is  $h(x) = x^2$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $A$  i.e.  $x, y \in A$  such that

$$\Rightarrow h(x) = h(y)$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow \pm x = \pm y$$

Since it has many elements of  $A$  co - domain

Hence,  $h$  is not One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $A$  i.e.  $y \in A$  be arbitrary, then

$$\Rightarrow h(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y}$$

Since  $h$  have no pre-image in domain  $A$ .

Hence,  $h$  is not onto function

Thus, It is not Bijective function

### 9 A. Question

Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective:

$$\{(x, y): x \text{ is a person, } y \text{ is the mother of } x\}$$

## Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Here, It is given (x, y): x is a person, y is the mother of x

As we know each person "x" has only one biological mother

Thus,

Given relation is a function

Since more than one person may have the same mother

Function, not One - One (injective) but Onto (Surjective)

## 9 B. Question

Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective:

$\{(a, b) : a \text{ is a person, } b \text{ is an ancestor of } a\}$

## Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of A i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Here, It is given (a, b): a is a person, b is an ancestor of a

As we know any person "a" has more than one ancestor

Thus,

Given relation is not a function

## 10. Question

Let  $A = \{1, 2, 3\}$ . Write all one - one from A to itself.

## Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

We have  $A = \{1, 2, 3\}$

So all one - one functions from  $A = \{1, 2, 3\}$  to itself are obtained by re - arranging elements of A.

Thus all possible one - one functions are:

$$f(1) = 1, f(2) = 2, f(3) = 3$$

$$f(1) = 2, f(2) = 3, f(3) = 1$$

$$f(1) = 3, f(2) = 1, f(3) = 2$$

$$f(1) = 1, f(2) = 3, f(3) = 2$$

$$f(1) = 3, f(2) = 2, f(3) = 1$$

$$f(1) = 2, f(2) = 1, f(3) = 3$$

### 11. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = 4x^3 + 7$ , show that f is a bijection.

### Answer

**TIP: - One - One Function:** - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

**Onto Function:** - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of B i.e, if  $f(A) = B$  or range of f is the co - domain of f.

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

**Bijection Function:** - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $f(x) = 4x^3 + 7$

To Prove : -  $f: \mathbb{R} \rightarrow \mathbb{R}$  is bijective defined by  $f(x) = 4x^3 + 7$

Check for Injectivity:

Let x,y be elements belongs to  $\mathbb{R}$  i.e  $x, y \in \mathbb{R}$  such that

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow 4x^3 + 7 = 4y^3 + 7$$

$$\Rightarrow x^3 = y^3$$

$$\Rightarrow x = y$$

Hence,  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e.  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow 4x^3 + 7 = y$$

$$\Rightarrow 4x^3 + 7 - y = 0$$

Now, we know that for 3 degree equation has a real root

So, let  $x = \alpha$  be that root

$$\Rightarrow 4\alpha^3 + 7 = y$$

$$\Rightarrow f(\alpha) = y$$

Thus for clearly  $y \in \mathbb{R}$ , there exist  $\alpha \in \mathbb{R}$  such that  $f(x) = y$

Therefore  $f$  is onto

Thus, It is Bijective function

Hence Proved

## 12. Question

Show that the exponential function  $f: \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = e^x$ , is one - one but not onto. What happens if the co - domain is replaced by  $\mathbb{R}_0^+$  (set of all positive real numbers).

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e., if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = e^x$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}$  i.e.  $x, y \in \mathbb{R}$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow e^x = e^y$$

$$\Rightarrow \frac{e^x}{e^y} = 1$$

$$\Rightarrow e^{x-y} = 1$$

$$\Rightarrow e^{x-y} = e^0$$

$$\Rightarrow x - y = 0$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Check for Surjectivity:

Here range of  $f = (0, \infty) \neq \mathbb{R}$

Therefore  $f$  is not onto

Now if co - domain is replaced by  $\mathbb{R}_0^+$  (set of all positive real numbers) i.e  $(0, \infty)$  then  $f$  becomes an onto function.

### 13. Question

Show that the logarithmic function  $f : \mathbb{R}_+^0 \rightarrow \mathbb{R}$  given by  $f(x) = \log_a x$ ,  $a > 0$  is a bijection.

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

To Prove : - Logarithmic function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  given by  $f(x) = \log_a x$ ,  $a > 0$  is a bijection.

Now,  $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}$  given by  $f(x) = \log_a x$ ,  $a > 0$

Check for Injectivity:

Let  $x, y$  be elements belongs to  $\mathbb{R}_0^+$  i.e  $x, y \in \mathbb{R}_0^+$  such that

So, from definition

$$\Rightarrow f(x) = f(y)$$

$$\Rightarrow \log_a x = \log_a y$$

$$\Rightarrow \log_a x - \log_a y = 0$$

$$\Rightarrow \log_a \left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \frac{x}{y} = 1$$

$$\Rightarrow x = y$$

Hence  $f$  is One - One function

Check for Surjectivity:

Let  $y$  be element belongs to  $\mathbb{R}$  i.e.  $y \in \mathbb{R}$  be arbitrary, then

$$\Rightarrow f(x) = y$$

$$\Rightarrow \log_a x = y$$

$$\Rightarrow x = a^y$$

Above value of  $x$  belongs to  $\mathbb{R}_0^+$

Therefore, for all  $y \in \mathbb{R}$  there exist  $x = a^y$  such that  $f(x) = y$ .

Hence,  $f$  is Onto function.

Thus, it is Bijective also

#### 14. Question

If  $A = \{1, 2, 3\}$ , show that a one - one function  $f : A \rightarrow A$  must be onto.

**Answer**

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $A$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now,  $f: A \rightarrow A$  where  $A = \{1, 2, 3\}$  and its a One - One function

To Prove: -  $A$  is Onto function

Since it is given that  $f$  is a One - One function,

Three elements of  $A = \{1, 2, 3\}$  must be taken to 3 different elements of co - domain  $A = \{1, 2, 3\}$  under  $f$ .

Thus by definition of Onto Function

$f$  has to be Onto function.

Hence Proved

#### 15. Question

If  $A = \{1, 2, 3\}$ , show that an onto function  $f : A \rightarrow A$  must be one - one.

**Answer**

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function



$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Onto Function: - A function  $f: A \rightarrow B$  is said to be an onto function or surjection if every element of  $B$  i.e. if  $f(A) = B$  or range of  $f$  is the co-domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now,  $f: A \rightarrow A$  where  $A = \{1, 2, 3\}$  and it is an Onto function

To Prove: -  $A$  is a One - One function

Let's assume  $f$  is not Onto function,

Then,

There must be two elements let it be 1 and 2 in Domain  $A = \{1, 2, 3\}$  whose images in co-domain  $A = \{1, 2, 3\}$  is same.

Also, Image of 3 under  $f$  can be only one element.

Therefore,

Range set can have at most two elements in co-domain  $A = \{1, 2, 3\}$

$\Rightarrow f$  is not an onto function

Hence it contradicts

$\Rightarrow f$  must be One - One function

Hence Proved

### 16. Question

Find the number of all onto functions from the set  $A = \{1, 2, 3, \dots, n\}$  to itself.

### Answer

**TIP:** -

Onto Function: - A function  $f: A \rightarrow B$  is said to be an onto function or surjection if every element of  $B$  i.e. if  $f(A) = B$  or range of  $f$  is the co-domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now,  $f: A \rightarrow A$  where  $A = \{1, 2, 3, \dots, n\}$

All onto function

It's a permutation of  $n$  symbols  $1, 2, 3, \dots, n$

Thus,

Total number of Onto maps from  $A = \{1, 2, 3, \dots, n\}$  to itself =

Total number of permutations of  $n$  symbols  $1, 2, 3, \dots, n$ .

### 17. Question

Give examples of two one - one functions  $f_1$  and  $f_2$  from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $f_1 + f_2: \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$  is not one - one.

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one function or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$a = b \text{ for all } a, b \in A$$

Let,  $f_1: R \rightarrow R$  and  $f_2: R \rightarrow R$  be two functions given by (Examples)

$$f_1(x) = x$$

$$f_2(x) = -x$$

From above function it is clear that both are One - One functions

Now,

$$\Rightarrow (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\Rightarrow (f_1 + f_2)(x) = x - x$$

$$\Rightarrow (f_1 + f_2)(x) = 0$$

Therefore,

$f_1 + f_2: R \rightarrow R$  is a function given by

$$(f_1 + f_2)(x) = 0$$

Since  $f_1 + f_2$  is a constant function,

Hence it is not an One - One function.

### 18. Question

Give examples of two surjective function  $f_1$  and  $f_2$  from  $Z$  to  $Z$  such that  $f_1 + f_2$  is not surjective.

**Answer**

**TIP: -**

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Let,  $f_1: Z \rightarrow Z$  and  $f_2: Z \rightarrow Z$  be two functions given by (Examples)

$$f_1(x) = x$$

$$f_2(x) = -x$$

From above function it is clear that both are Onto or Surjective functions

Now,

$$f_1 + f_2: Z \rightarrow Z$$

$$\Rightarrow (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$\Rightarrow (f_1 + f_2)(x) = x - x$$

$$\Rightarrow (f_1 + f_2)(x) = 0$$

Therefore,

$f_1 + f_2: Z \rightarrow Z$  is a function given by

$$(f_1 + f_2)(x) = 0$$

Since  $f_1 + f_2$  is a constant function,

Hence it is not an Onto/Surjective function.

### 19. Question

Show that if  $f_1$  and  $f_2$  are one - one maps from  $\mathbb{R}$  to  $\mathbb{R}$ , then the product  $f_1 \times f_2 : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $(f_1 \times f_2)(x) = f_1(x)f_2(x)$  need not be one - one.

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$a = b \text{ for all } a, b \in A$$

Let,  $f_1: \mathbb{R} \rightarrow \mathbb{R}$  and  $f_2: \mathbb{R} \rightarrow \mathbb{R}$  are two functions given by

$$f_1(x) = x$$

$$f_2(x) = x$$

From above function it is clear that both are One - One functions

Now,  $f_1 \times f_2 : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$\Rightarrow (f_1 \times f_2)(x) = f_1(x) \times f_2(x) = x^2$$

$$\Rightarrow (f_1 \times f_2)(x) = x^2$$

Also,

$$f(1) = 1 = f(-1)$$

Therefore,

f is not One - One

$\Rightarrow f_1 \times f_2 : \mathbb{R} \rightarrow \mathbb{R}$  is not One - One function.

Hence Proved

### 20. Question

Suppose  $f_1$  and  $f_2$  are non - zero one - one functions from  $\mathbb{R}$  to  $\mathbb{R}$ . Is  $\frac{f_1}{f_2}$  necessarily one - one? Justify your

answer. Here,  $\frac{f_1}{f_2} : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $\left(\frac{f_1}{f_2}\right)(x) = \frac{f_1(x)}{f_2(x)}$  for all  $x \in \mathbb{R}$ .

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$a = b$  for all  $a, b \in A$

Let,  $f_1: \mathbb{R} \rightarrow \mathbb{R}$  and  $f_2: \mathbb{R} \rightarrow \mathbb{R}$  are two non-zero functions given by

$$f_1(x) = x^3$$

$$f_2(x) = x$$

From above function it is clear that both are One - One functions

Now,  $\frac{f_1}{f_2}: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$\Rightarrow \frac{f_1}{f_2}(x) = \frac{f_1(x)}{f_2(x)}$$

$$\Rightarrow \frac{f_1}{f_2}(x) = x^2 \text{ for all } x \in \mathbb{R}$$

Again,

$\frac{f_1}{f_2} = f(\text{let}): \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = x^2$$

Now,

$$\Rightarrow f(1) = 1 = f(-1)$$

Therefore,

$f$  is not One - One

$\Rightarrow \frac{f_1}{f_2}: \mathbb{R} \rightarrow \mathbb{R}$  is not One - One function.

Hence it is not necessarily to  $\frac{f_1}{f_2}$  be one - one function.

## 21 A. Question

Given  $A = \{2, 3, 4\}$ ,  $B = \{2, 5, 6, 7\}$ . Construct an example of each of the following:

an injective map from  $A$  to  $B$

### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Now,  $f: A \rightarrow B$ , denotes a mapping such that

$$\Rightarrow f = \{(x, y): y = x + 3\}$$

It can be written as follows in roster form

$$f = \{(2, 5), (3, 6), (4, 7)\}$$

Hence this is injective mapping

### 21 B. Question

Given  $A = \{2, 3, 4\}$ ,  $B = \{2, 5, 6, 7\}$ . Construct an example of each of the following:

a mapping from A to B which is not injective

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Now,  $f: A \rightarrow B$ , denotes a mapping such that

$$f = \{(2,2), (3,5), (4,5)\}$$

Hence this is not injective mapping

### 21 C. Question

Given  $A = \{2, 3, 4\}$ ,  $B = \{2, 5, 6, 7\}$ . Construct an example of each of the following:

a mapping from A to B.

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$$\Rightarrow f(a) \neq f(b) \text{ for all } a, b \in A$$

$$\Leftrightarrow f(a) = f(b)$$

$$\Rightarrow a = b \text{ for all } a, b \in A$$

Now,  $f: A \rightarrow B$ , denotes a mapping such that

$$f = \{(2,2), (5,3), (6,4), (7,4)\}$$

Here it is clear that every first component is from B and second component is from A

Hence this is mapping from B to A

### 22. Question

Show that  $f: \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = x - [x]$ , is neither one - one nor onto.

#### Answer

**TIP:** - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of A have different images in B.

So,  $f: A \rightarrow B$  is One - One function

$$\Leftrightarrow a \neq b$$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Now,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x - [x]$

To Prove: -  $f(x) = x - [x]$ , is neither one - one nor onto

Check for Injectivity:

Let  $x$  be element belongs to  $\mathbb{Z}$  i.e  $x \in \mathbb{Z}$  such that

So, from definition

$\Rightarrow f(x) = x - [x]$

$\Rightarrow f(x) = 0$  for  $x \in \mathbb{Z}$

Therefore,

Range of  $f = [0,1] \neq \mathbb{R}$

Hence  $f$  is not One - One function

Check for Surjectivity:

Since Range of  $f = [0,1] \neq \mathbb{R}$

Hence,  $f$  is not Onto function.

Thus, it is neither One - One nor Onto function

Hence Proved

### 23. Question

Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  be defined by

$$f(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

Show that  $f$  is a bijection.

### Answer

**TIP**: - One - One Function: - A function  $f: A \rightarrow B$  is said to be a one - one functions or an injection if different elements of  $A$  have different images in  $B$ .

So,  $f: A \rightarrow B$  is One - One function

$\Leftrightarrow a \neq b$

$\Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

$\Leftrightarrow f(a) = f(b)$

$\Rightarrow a = b$  for all  $a, b \in A$

Onto Function: - A function  $f: A \rightarrow B$  is said to be a onto function or surjection if every element of  $B$  i.e, if  $f(A) = B$  or range of  $f$  is the co - domain of  $f$ .

So,  $f: A \rightarrow B$  is Surjection iff for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$

Bijection Function: - A function  $f: A \rightarrow B$  is said to be a bijection function if it is one - one as well as onto function.

Now, suppose

$$f(n_1) = f(n_2)$$

If  $n_1$  is odd and  $n_2$  is even, then we have

$$\Rightarrow n_1 + 1 = n_2 - 2$$

$$\Rightarrow n_2 - n_1 = 2$$

Not possible

Suppose both  $n_1$  even and  $n_2$  is odd.

$$\text{Then, } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 - 1 = n_2 + 1$$

$$\Rightarrow n_1 - n_2 = 2$$

Not possible

Therefore, both  $n_1$  and  $n_2$  must be either odd or even

Suppose both  $n_1$  and  $n_2$  are odd.

$$\text{Then, } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 + 1 = n_2 + 1$$

$$\Rightarrow n_1 = n_2$$

Suppose both  $n_1$  and  $n_2$  are even.

$$\text{Then, } f(n_1) = f(n_2)$$

$$\Rightarrow n_1 - 1 = n_2 - 1$$

$$\Rightarrow n_1 = n_2$$

Then,  $f$  is One - One

Also, any odd number  $2r + 1$  in the co - domain  $N$  will have an even number as image in domain  $N$  which is

$$\Rightarrow f(n) = 2r + 1$$

$$\Rightarrow n - 1 = 2r + 1$$

$$\Rightarrow n = 2r + 2$$

Any even number  $2r$  in the co - domain  $N$  will have an odd number as image in domain  $N$  which is

$$\Rightarrow f(n) = 2r$$

$$\Rightarrow n + 1 = 2r$$

$$\Rightarrow n = 2r - 1$$

Thus  $f$  is Onto function.

## **Exercise 2.2**

### **1 A. Question**

Find  $\text{gof}$  and  $\text{fog}$  when  $f: R \rightarrow R$  and  $g: R \rightarrow R$  is defined by

$$f(x) = 2x + 3 \text{ and } g(x) = x^2 + 5$$

**Answer**

Since,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$

$f \circ g: \mathbb{R} \rightarrow \mathbb{R}$  and  $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

Now,  $f(x) = 2x + 3$  and  $g(x) = x^2 + 5$

$$g \circ f(x) = g(2x + 3) = (2x + 3)^2 + 5$$

$$\Rightarrow g \circ f(x) = 4x^2 + 12x + 9 + 5 = 4x^2 + 12x + 14$$

$$f \circ g(x) = f(g(x)) = f(x^2 + 5) = 2(x^2 + 5) + 3$$

$$\Rightarrow f \circ g(x) = 2x^2 + 10 + 3 = 2x^2 + 13$$

Hence,  $g \circ f(x) = 4x^2 + 12x + 14$  and  $f \circ g(x) = 2x^2 + 13$

**1 B. Question**

Find  $g \circ f$  and  $f \circ g$  when  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$f(x) = 2x + x^2 \text{ and } g(x) = x^3$$

**Answer**

Since,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$

$f \circ g: \mathbb{R} \rightarrow \mathbb{R}$  and  $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 2x + x^2 \text{ and } g(x) = x^3$$

Now,  $g \circ f(x) = g(f(x)) = g(2x + x^2)$

$$g \circ f(x) = (2x + x^2)^3 = x^6 + 8x^3 + 6x^5 + 12x^4$$

$$\text{and } f \circ g(x) = f(g(x)) = f(x^3)$$

$$\Rightarrow f \circ g(x) = 2x^3 + x^6$$

So,  $g \circ f(x) = x^6 + 6x^5 + 12x^4 + 8x^3$  and  $f \circ g(x) = 2x^3 + x^6$

**1 C. Question**

Find  $g \circ f$  and  $f \circ g$  when  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$f(x) = x^2 + 8 \text{ and } g(x) = 3x^3 + 1$$

**Answer**

Since,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$

$f \circ g: \mathbb{R} \rightarrow \mathbb{R}$  and  $g \circ f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + 8 \text{ and } g(x) = 3x^3 + 1$$

So,  $g \circ f(x) = g(f(x))$

$$g \circ f(x) = g(x^2 + 8)$$

$$g \circ f(x) = 3(x^2 + 8)^3 + 1$$

$$\Rightarrow g \circ f(x) = 3(x^6 + 512 + 24x^4 + 192x^2) + 1$$

$$\Rightarrow g \circ f(x) = 3x^6 + 72x^4 + 576x^2 + 1537$$

Similarly,  $f \circ g(x) = f(g(x))$

$$\Rightarrow f \circ g(x) = f(3x^3 + 1)$$

$$\Rightarrow f \circ g(x) = (3x^3 + 1)^2 + 8$$



$$\Rightarrow \text{fog}(x) = (9x^6 + 1 + 6x^3) + 8$$

$$\Rightarrow \text{fog}(x) = 9x^6 + 6x^3 + 9$$

$$\text{So, } \text{gof}(x) = 3x^6 + 72x^4 + 576x^2 + 1537 \text{ and } \text{fog}(x) = 9x^6 + 6x^3 + 9$$

### 1 D. Question

Find gof and fog when  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$f(x) = x \text{ and } g(x) = |x|$$

#### Answer

Since,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$

$\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$  and  $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x \text{ and } g(x) = |x|$$

Now,  $\text{gof}(x) = g(f(x)) = g(x)$

$$\Rightarrow \text{gof}(x) = |x|$$

$$\text{and, } \text{fog}(x) = f(g(x)) = f(|x|) \Rightarrow \text{fog}(x) = |x|$$

$$\text{Hence, } \text{gof}(x) = \text{fog}(x) = |x|$$

### 1 E. Question

Find gof and fog when  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$f(x) = x^2 + 2x - 3 \text{ and } g(x) = 3x - 4$$

#### Answer

Since,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$

$\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$  and  $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = x^2 + 2x - 3 \text{ and } g(x) = 3x - 4$$

Now,  $\text{gof}(x) = g(f(x)) = g(x^2 + 2x - 3)$

$$\text{gof}(x) = 3(x^2 + 2x - 3) - 4$$

$$\Rightarrow \text{gof}(x) = 3x^2 + 6x - 9 - 4$$

$$\Rightarrow \text{gof}(x) = 3x^2 + 6x - 13$$

$$\text{and, } \text{fog} = f(g(x)) = f(3x - 4)$$

$$\text{fog}(x) = (3x - 4)^2 + 2(3x - 4) - 3$$

$$= 9x^2 + 16 - 24x + 6x - 8 - 3$$

$$\therefore \text{fog}(x) = 9x^2 - 18x + 5$$

$$\text{Thus, } \text{gof}(x) = 3x^2 + 6x - 13 \text{ and } \text{fog}(x) = 9x^2 - 18x + 5$$

### 1 F. Question

Find gof and fog when  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  is defined by

$$f(x) = 8x^3 \text{ and } g(x) = x^{1/3}$$

#### Answer

Since,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$

$\text{fog}: \mathbb{R} \rightarrow \mathbb{R}$  and  $\text{gof}: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = 8x^3 \text{ and } g(x) = x^{\frac{1}{3}}$$

$$\text{Now, } \text{gof}(x) = g(f(x)) = g(8x^3)$$

$$\Rightarrow \text{gof}(x) = (8x^3)^{\frac{1}{3}}$$

$$\text{gof}(x) = 2x$$

$$\text{and, } \text{fog}(x) = f(g(x)) = f(x^{\frac{1}{3}})$$

$$= 8\left(x^{\frac{1}{3}}\right)^3$$

$$\text{fog}(x) = 8x$$

$$\text{Thus, } \text{gof}(x) = 2x \text{ and } \text{fog}(x) = 8x$$

## 2. Question

Let  $f = \{(3, 1), (9, 3), (12, 4)\}$  and  $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$ . Show that  $\text{gof}$  and  $\text{fog}$  are both defined, Also, find  $\text{fog}$  and  $\text{gof}$ .

### Answer

$$\text{Let } f = \{(3,1), (9,3), (12,4)\} \text{ and}$$

$$g = \{(1,3), (3,3), (4,9), (5,9)\}$$

Now,

$$\text{range of } f = \{1, 3, 4\}$$

$$\text{domain of } f = \{3, 9, 12\}$$

$$\text{range of } g = \{3,9\}$$

$$\text{domain of } g = \{1, 3, 4, 5\}$$

since,  $\text{range of } f \subset \text{domain of } g$

$\therefore \text{gof}$  is well defined.

Again, the  $\text{range of } g \subseteq \text{domain of } f$

$\therefore \text{fog}$  is well defined.

$$\text{Finally, } \text{gof} = \{(3,3), (9,3), (12,9)\}$$

$$\text{fog} = \{(1,1), (3,1), (4,3), (5,3)\}$$

## 3. Question

Let  $f = \{(1, -1), (4, -2), (9, -3), (16, 4)\}$  and  $g = \{(-1, -2), (-2, -4), (-3, -6), (4, 8)\}$ . Show that  $\text{gof}$  is defined while  $\text{fog}$  is not defined. Also, find  $\text{gof}$ .

### Answer

We have,

$$f = \{(1, -1), (4, -2), (9, -3), (16,4)\} \text{ and}$$

$$g = \{(-1, -2), (-2, -4), (-3, -6), (4,8)\}$$

Now,

$$\text{Domain of } f = \{1,4,9,16\}$$

$$\text{Range of } f = \{-1, -2, -3, 4\}$$

$$\text{Domain of } g = \{-1, -2, -3,4\}$$

Range of  $g = \{-2, -4, -6, 8\}$

Clearly range of  $f =$  domain of  $g$

$\therefore$   $g \circ f$  is defined.

but, range of  $g \neq$  domain of  $f$  So,  $f \circ g$  is not defined.

Now,

$$g \circ f(1) = g(-1) = -2$$

$$g \circ f(4) = g(-2) = -4$$

$$g \circ f(9) = g(-3) = -6$$

$$g \circ f(16) = g(4) = 8$$

So,  $g \circ f = \{(1, -2), (4, -4), (9, -6), (16, 8)\}$

#### 4. Question

Let  $A = \{a, b, c\}$ ,  $B = \{u, v, w\}$  and let  $f$  and  $g$  be two functions from  $A$  to  $B$  and from  $B$  to  $A$  respectively defined as:  $f = \{(a, v), (b, u), (c, w)\}$ ,  $g = \{(u, b), (v, a), (w, c)\}$ .

Show that  $f$  and  $g$  both are bijections and find  $f \circ g$  and  $g \circ f$ .

#### Answer

Given,  $A = \{a, b, c\}$ ,  $B = \{u, v, w\}$  and

$f = A \rightarrow B$  and  $g: B \rightarrow A$  defined by

$f = \{(a, v), (b, u), (c, w)\}$  and

$g = \{(u, b), (v, a), (w, c)\}$

For both  $f$  and  $g$ , different elements of domain have different images

$\therefore$   $f$  and  $g$  are one - one

Again, for each element in co - domain of  $f$  and  $g$ , there is a pre - image in the domain

$\therefore$   $f$  and  $g$  are onto

Thus,  $f$  and  $g$  are bijective.

Now,

$g \circ f = \{(a, a), (b, b), (c, c)\}$  and

$f \circ g = \{(u, u), (v, v), (w, w)\}$

#### 5. Question

Find  $f \circ g(2)$  and  $g \circ f(1)$  when:  $f: \mathbb{R} \rightarrow \mathbb{R}; f(x) = x^2 + 8$  and  $g: \mathbb{R} \rightarrow \mathbb{R}; g(x) = 3x^3 + 1$ .

#### Answer

We have,  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2 + 8$  and

$g: \mathbb{R} \rightarrow \mathbb{R}$  given by  $g(x) = 3x^3 + 1$

$$f \circ g(x) = f(g(x)) = f(3x^3 + 1)$$

$$= (3x^3 + 1)^2 + 8$$

$$f \circ g(2) = (3 \times 8 + 1)^2 + 8 = 625 + 8 = 633$$

Again,

$$\text{gof}(x) = g(f(x)) = g(x^2 + 8)$$

$$= 3(x^2 + 8)^3 + 1$$

$$\text{gof}(1) = 3(1 + 8)^3 + 1 = 2188$$

### 6. Question

Let  $\mathbb{R}^+$  be the set of all non-negative real numbers. If  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are defined as  $f(x) = x^2$  and  $g(x) = \sqrt{x}$ . Find  $\text{fog}$  and  $\text{gof}$ . Are they equal functions.

### Answer

We have,  $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  given by

$$f(x) = x^2$$

$g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  given by

$$g(x) = \sqrt{x}$$

$$\text{fog}(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

Also,

$$\text{gof}(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = x$$

Thus,

$$\text{fog}(x) = \text{gof}(x)$$

They are equal functions as their domain and range are also equal.

### 7. Question

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^2$  and  $g(x) = x + 1$ . Show that  $\text{fog} \neq \text{gof}$ .

### Answer

We have,  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  are two functions defined by

$$f(x) = x^2 \text{ and } g(x) = x + 1$$

Now,

$$\text{fog}(x) = f(g(x)) = f(x + 1) = (x + 1)^2$$

$$\Rightarrow \text{fog}(x) = x^2 + 2x + 1 \dots\dots(i)$$

$$\text{gof}(x) = g(f(x)) = g(x^2) = x^2 + 1 \dots\dots(ii)$$

from (i) & (ii)

$$\text{fog} \neq \text{gof}$$

### 8. Question

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x + 1$  and  $g(x) = x - 1$ . Show that  $\text{fog} = \text{gof} = I_{\mathbb{R}}$ .

### Answer

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  are defined as

$$f(x) = x + 1 \text{ and } g(x) = x - 1$$

Now,

$$\text{fog}(x) = f(g(x)) = f(x - 1) = x - 1 + 1$$

$$= x = I_{\mathbb{R}} \dots\dots(i)$$

Again,

$$f \circ g(x) = f(g(x)) = g(x + 1) = x + 1 - 1$$

$$= x = I_{\mathbb{R}} \dots \dots (ii)$$

from (i) & (ii)

$$f \circ g = g \circ f = I_{\mathbb{R}}$$

### 9. Question

Verify associativity for the following three mappings:  $f: \mathbb{N} \rightarrow \mathbb{Z}_0$  (the set of non-zero integers),  $g: \mathbb{Z}_0 \rightarrow \mathbb{Q}$  and  $h: \mathbb{Q} \rightarrow \mathbb{R}$  given by  $f(x) = 2x$ ,  $g(x) = 1/x$  and  $h(x) = e^x$ .

### Answer

We have,  $f: \mathbb{N} \rightarrow \mathbb{Z}_0$ ,  $g: \mathbb{Z}_0 \rightarrow \mathbb{Q}$  and  $h: \mathbb{Q} \rightarrow \mathbb{R}$

$$\text{Also, } f(x) = 2x, \quad g(x) = \frac{1}{x} \text{ and } h(x) = e^x$$

Now,  $f: \mathbb{N} \rightarrow \mathbb{Z}_0$  and  $h \circ g: \mathbb{Z}_0 \rightarrow \mathbb{R}$

$$\therefore (h \circ g) \circ f: \mathbb{N} \rightarrow \mathbb{R}$$

Also,  $g \circ f: \mathbb{N} \rightarrow \mathbb{Q}$  and  $h: \mathbb{Q} \rightarrow \mathbb{R}$

$$\therefore h \circ (g \circ f): \mathbb{N} \rightarrow \mathbb{R}$$

Thus,  $(h \circ g) \circ f$  and  $h \circ (g \circ f)$  exist and are function from  $\mathbb{N}$  to set  $\mathbb{R}$ .

$$\text{Finally, } (h \circ g) \circ f(x) = (h \circ g)(f(x)) = (h \circ g)(2x)$$

$$= h\left(\frac{1}{2}\right) = e^{\frac{1}{2x}}$$

$$\text{Now, } h \circ (g \circ f)(x) = h \circ (g(2x)) = h\left(\frac{1}{2x}\right)$$

$$= e^{\frac{1}{2x}}$$

Hence, associativity verified.

### 10. Question

Consider  $f: \mathbb{N} \rightarrow \mathbb{N}$ ,  $g: \mathbb{N} \rightarrow \mathbb{N}$  and  $h: \mathbb{N} \rightarrow \mathbb{R}$  defined as  $f(x) = 2x$ ,  $g(y) = 3y + 4$  and  $h(z) = \sin z$  for all  $x, y, z \in \mathbb{N}$ . Show that  $h \circ (g \circ f) = (h \circ g) \circ f$ .

### Answer

We have,

$$h \circ (g \circ f)(x) = h(g(f(x)))$$

$$= h(g(2x)) = h(3(2x) + 4)$$

$$= h(6x + 4) = \sin(6x + 4) \quad \forall x \in \mathbb{N}$$

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = (h \circ g)(2x)$$

$$= h(g(2x)) = h(3(2x) + 4)$$

$$= h(6x + 4) = \sin(6x + 4) \quad \forall x \in \mathbb{N}$$

This shows,  $h \circ (g \circ f) = (h \circ g) \circ f$

### 11. Question

Give examples of two functions  $f: \mathbb{N} \rightarrow \mathbb{N}$  and  $g: \mathbb{N} \rightarrow \mathbb{N}$  such that  $g \circ f$  is onto, but  $f$  is not onto.

### Answer

Define  $f: \mathbb{N} \rightarrow \mathbb{N}$  by,  $f(x) = x + 1$  And,  $g: \mathbb{N} \rightarrow \mathbb{N}$  by,

$$g(x) = \begin{cases} x - 1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We first show that  $f$  is not onto.

For this, consider element 1 in co - domain  $\mathbb{N}$ . It is clear that this element is not an image of any of the elements in domain  $\mathbb{N}$ .

Therefore,  $f$  is not onto.

## 12. Question

Give examples of two functions  $f: \mathbb{N} \rightarrow \mathbb{Z}$  and  $g: \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $g \circ f$  is injective, but  $g$  is not injective.

### Answer

Define  $f: \mathbb{N} \rightarrow \mathbb{Z}$  as  $f(x) = x$  and  $g: \mathbb{N} \rightarrow \mathbb{N}$  as  $g(x) = |x|$ .

We first show that  $g$  is not injective.

It can be observed that:

$$g(-1) = |-1| = 1$$

$$g(1) = |1| = 1$$

Therefore,  $g(-1) = g(1)$ , but  $-1 \neq 1$ .

Therefore,  $g$  is not injective.

Now,  $g \circ f: \mathbb{N} \rightarrow \mathbb{Z}$  is defined as  $g \circ f(x) = g(f(x)) = g(x) = |x|$ .

Let  $x, y \in \mathbb{N}$  such that  $g \circ f(x) = g \circ f(y)$ .

$$\Rightarrow |x| = |y|$$

Since  $x$  and  $y \in \mathbb{N}$  both are positive.

$$\therefore |x| = |y| \Rightarrow x = y$$

Hence,  $g \circ f$  is injective

## 13. Question

If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are one - one functions show that  $g \circ f$  is a one - one function.

### Answer

We have,  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are one - one functions.

Now we have to prove :  $g \circ f: A \rightarrow C$  in one - one

let  $x, y \in A$  such that

$$g \circ f(x) = g \circ f(y)$$

$$g(f(x)) = g(f(y))$$

$$f(x) = f(y) \text{ [As, } g \text{ in one - one]}$$

$$x = y \text{ [As, } f \text{ in one - one]}$$

$g \circ f$  is one - one function

## 14. Question

If  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are onto functions show that  $g \circ f$  is an onto function.

### Answer

We have,  $f: A \rightarrow B$  and  $g: B \rightarrow C$  are onto functions.

Now, we need to prove:  $\text{gof}: A \rightarrow C$  is onto.

let  $y \in C$ , then

$$\text{gof}(x) = y$$

$$g(f(x)) = y \dots\dots(i)$$

Since  $g$  is onto, for each element in  $C$ , there exists a preimage in  $B$ .

$$g(x)=y \dots\dots(ii)$$

From (i) & (ii)

$$f(x)=x$$

Since  $f$  is onto, for each element in  $B$  there exists a preimage in  $A$ .

$$f(x)=x \dots\dots(iii)$$

From (ii)and(iii) we can conclude that for each  $y \in C$ , there exists a preimage in  $A$  such that  $\text{gof}(x) = y$

$\therefore$   $\text{gof}$  is onto.

### Exercise 2.3

#### 1 A. Question

Find  $\text{fog}$  and  $\text{gof}$ , if

$$f(x) = e^x, g(x) = \log_e x$$

#### Answer

$$f(x) = e^x \text{ and } g(x) = \log_e x$$

$$\text{Now, } \text{fog}(x) = f(g(x)) = f(\log_e x) = e^{\log_e x} = x$$

$$\Rightarrow \text{fog}(x) = x$$

$$\text{gof}(x) = g(f(x)) = g(e^x) = \log_e e^x = x$$

$$\Rightarrow \text{gof}(x)=x$$

Hence,  $\text{fog}(x) = x$  and  $\text{gof}(x) = x$

#### 1 B. Question

Find  $\text{fog}$  and  $\text{gof}$ , if

$$f(x) = x^2, g(x) = \cos x$$

#### Answer

$$f(x)= x, g(x) = \cos x$$

Domain of  $f$  and Domain of  $g = \mathbb{R}$

Range of  $f = (0, \infty)$

Range of  $g = (-1,1)$

$\therefore$  Range of  $f \subset$  domain of  $g \Rightarrow \text{gof}$  exist

Also, Range of  $g \subset$  domain of  $f \Rightarrow \text{fog}$  exist

Now,

$$\text{gof}(x) = g(f(x)) = g(x^2) = \cos x^2$$

And

$$f \circ g(x) = f(g(x)) = f(\cos x) = \cos^2 x$$

$$\text{Hence, } f \circ g(x) = \cos^2 x \text{ and } g \circ f(x) = \cos^2 x$$

### 1 C. Question

Find  $f \circ g$  and  $g \circ f$ , if

$$f(x) = |x|, g(x) = \sin x$$

#### Answer

$$f(x) = |x| \text{ and } g(x) = \sin x$$

Range of  $f = (0, \infty) \subset \text{Domain } g(\mathbb{R}) \Rightarrow g \circ f \text{ exist}$

Range of  $g = [-1, 1] \subset \text{Domain } f(\mathbb{R}) \Rightarrow f \circ g \text{ exist}$

$$\text{Now, } f \circ g(x) = f(g(x)) = f(\sin x) = |\sin x| \text{ and}$$

$$g \circ f(x) = g(f(x)) = g(|x|) = \sin |x|$$

$$\text{Hence, } f \circ g(x) = |\sin x| \text{ and } g \circ f(x) = \sin |x|$$

### 1 D. Question

Find  $f \circ g$  and  $g \circ f$ , if

$$f(x) = x + 1, g(x) = e^x$$

#### Answer

$$f(x) = x + 1 \text{ and } g(x) = e^x$$

Range of  $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow g \circ f \text{ exist}$

Range of  $g = (0, \infty) \subset \text{Domain of } f = \mathbb{R} \Rightarrow f \circ g \text{ exist}$

Now,

$$g \circ f(x) = g(f(x)) = g(x + 1) = e^{x+1}$$

And

$$f \circ g(x) = f(g(x)) = f(e^x) = e^x + 1$$

$$\text{Hence, } f \circ g(x) = e^x + 1 \text{ and } g \circ f(x) = e^{x+1}$$

### 1 E. Question

Find  $f \circ g$  and  $g \circ f$ , if

$$f(x) = \sin^{-1} x, g(x) = x^2$$

#### Answer

$$f(x) = \sin^{-1} x \text{ and } g(x) = x^2$$

Range of  $f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \subset \text{Domain of } g = \mathbb{R} \Rightarrow g \circ f \text{ exist}$

Range of  $g = (0, \infty) \subset \text{Domain of } f = \mathbb{R} \Rightarrow f \circ g \text{ exist}$

Now,

$$f \circ g(x) = f(g(x)) = f(x^2) = \sin^{-1} x^2 \text{ and}$$

$$g \circ f(x) = g(f(x)) = g(\sin^{-1} x) = (\sin^{-1} x)^2$$

$$\text{Hence, } f \circ g(x) = \sin^{-1} x^2 \text{ and } g \circ f(x) = (\sin^{-1} x)^2$$

### 1 F. Question



Find fog and gof, if

$$f(x) = x + 1, g(x) = \sin x$$

**Answer**

$$f(x) = x + 1 \text{ and } g(x) = \sin x$$

Range of  $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of  $g = [-1, 1] \subset \text{Domain of } f \Rightarrow \text{fog exists}$

Now,

$$\text{fog}(x) = f(g(x)) = f(\sin x) = \sin x + 1$$

And

$$\text{gof}(x) = g(f(x)) = g(x + 1) = \sin(x + 1)$$

Hence,  $\text{fog}(x) = \sin x + 1$  and  $\text{gof}(x) = \sin(x + 1)$

### 1 G. Question

Find fog and gof, if

$$f(x) = x + 1, g(x) = 2x + 3$$

**Answer**

$$f(x) = x + 1 \text{ and } g(x) = 2x + 3$$

Range of  $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of  $g = \mathbb{R} \subset \text{Domain of } f \Rightarrow \text{fog exists}$

Now,

$$\text{fog}(x) = f(g(x)) = f(2x + 3) = (2x + 3) + 1 = 2x + 4 \text{ and}$$

$$\text{gof}(x) = g(f(x)) = g(x + 1) = 2(x + 1) + 3 = 2x + 5$$

So,  $\text{fog}(x) = 2x + 4$  and  $\text{gof}(x) = 2x + 5$

### 1 H. Question

Find fog and gof, if

$$f(x) = c, c \in \mathbb{R}, g(x) = \sin x^2$$

**Answer**

$$f(x) = c, c \in \mathbb{R} \text{ and}$$

$$g(x) = \sin x^2$$

Range of  $f = \mathbb{R} \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of  $g = [-1, 1] \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exists}$

Now,

$$\text{gof}(x) = g(f(x)) = g(c) = \sin c^2 \text{ and}$$

$$\text{fog}(x) = f(g(x)) = f(\sin x^2) = c$$

Thus,  $\text{gof}(x) = \sin c^2$  and  $\text{fog}(x) = c$

### 1 I. Question

Find fog and gof, if

$$f(x) = x^2 + 2, \quad g(x) = 1 - \frac{1}{1-x}$$

**Answer**

$$f(x) = x^2 + 2 \text{ and } g(x) = 1 - \frac{1}{1-x}$$

Range of  $f = (2, \infty) \subset \text{Domain of } g = \mathbb{R} \Rightarrow \text{gof exists}$

Range of  $g = \mathbb{R} - \{-1\} \subset \text{Domain of } f = \mathbb{R} \Rightarrow \text{fog exists}$

Now,

$$\text{fog}(x) = f(g(x)) = f\left(-\frac{x}{1-x}\right) = \frac{x^2}{(1-x)^2} + 2 \text{ and}$$

$$\text{gof}(x) = g(f(x)) = g(x^2 + 2) = -\frac{x^2 + 2}{1 - (x^2 + 2)}$$

$$\text{gof}(x) = \frac{x^2 + 2}{(x^2 + 1)}$$

$$\text{Hence, } \text{fog}(x) = \frac{x^2}{(1-x)^2} + 2 \text{ and } \text{gof}(x) = -\frac{x^2 + 2}{1 - (x^2 + 2)}$$

## 2. Question

Let  $f(x) = x^2 + x + 1$  and  $g(x) = \sin x$ . Show that  $\text{fog} \neq \text{gof}$ .

**Answer**

We have,  $f(x) = x^2 + x + 1$  and  $g(x) = \sin x$

Now,

$$\text{fog}(x) = f(g(x)) = f(\sin x)$$

$$\Rightarrow \text{fog}(x) = \sin^2 x + \sin x + 1$$

$$\text{Again, } \text{gof}(x) = g(f(x)) = g(x^2 + x + 1)$$

$$\Rightarrow \text{gof}(x) = \sin(x^2 + x + 1)$$

Clearly,

$$\text{fog} \neq \text{gof}$$

## 3. Question

If  $f(x) = |x|$ , prove that  $\text{fof} = f$ .

**Answer**

We have,  $f(x) = |x|$

We assume the domain of  $f = \mathbb{R}$  and range of  $f = (0, \infty)$

Range of  $f \subset \text{domain of } f$

$\therefore \text{fof exists,}$

Now,

$$\text{fof}(x) = f(f(x)) = f(|x|) = ||x|| = f(x)$$

$\therefore \text{fof} = f$

Hence proved.

## 4. Question

If  $f(x) = 2x + 5$  and  $g(x) = x^2 + 1$  be two real functions, then describe each of the following functions:

(i) fog

(ii) gof

(iii) fof

(iv)  $f^2$

Also, show that  $\text{fof} \neq f^2$ .

### Answer

$$f(x) = 2x + 5 \text{ and } g(x) = x^2 + 1$$

The range of  $f = \mathbb{R}$  and range of  $g = [1, \infty]$

The range of  $f \subset \text{Domain of } g (\mathbb{R})$  and range of  $g \subset \text{domain of } f (\mathbb{R})$

$\therefore$  both fog and gof exist.

$$(i) \text{ fog}(x) = f(g(x)) = f(x^2 + 1)$$

$$= 2(x^2 + 1) + 5$$

$$\Rightarrow \text{fog}(x) = 2x^2 + 7$$

$$\text{Hence fog}(x) = 2x^2 + 7$$

$$(ii) \text{ gof}(x) = g(f(x)) = g(2x + 5)$$

$$= (2x + 5)^2 + 1$$

$$\text{gof}(x) = 4x^2 + 20x + 26$$

$$\text{Hence gof}(x) = 4x^2 + 20x + 26$$

$$(iii) \text{ fof}(x) = f(f(x)) = f(2x + 5)$$

$$= 2(2x + 5) + 5$$

$$\text{fof}(x) = 4x + 15$$

$$\text{Hence fof}(x) = 4x + 15$$

$$(iv) f^2(x) = [f(x)]^2 = (2x + 5)^2$$

$$= 4x^2 + 20x + 25$$

$\therefore$  from (iii) and (iv)

$$\text{fof} \neq f^2$$

### 5. Question

If  $f(x) = \sin x$  and  $g(x) = 2x$  be two real functions, then describe gof and fog. Are these equal functions?

### Answer

We have,  $f(x) = \sin x$  and  $g(x) = 2x$ .

Domain of  $f$  and  $g$  is  $\mathbb{R}$

Range of  $f = [-1, 1]$ , Range of  $g = \mathbb{R}$

$\therefore$  Range of  $f \subset \text{Domain } g$  and Range of  $g \subset \text{Domain } f$

fog and gof both exist.

$$\text{gof}(x) = g(f(x)) = g(\sin x)$$

$$\Rightarrow \text{gof}(x) = 2\sin x$$

$$\text{fog}(x) = f(g(x)) = f(2x) = \sin 2x$$

$$\therefore \text{gof} \neq \text{fog}$$

### 6. Question

Let  $f, g, h$  be real functions given by  $f(x) = \sin x$ ,  $g(x) = 2x$  and  $h(x) = \cos x$ . Prove that  $\text{fog} = \text{go(fh)}$ .

### Answer

$f, g$  and  $h$  are real functions given by  $f(x) = \sin x$ ,  $g(x) = 2x$  and

$$h(x) = \cos x$$

To prove:  $\text{fog} = \text{go(fh)}$

L.H.S

$$\text{fog}(x) = f(g(x))$$

$$= f(2x) = \sin 2x$$

$$\Rightarrow \text{fog}(x) = 2\sin x \cos x \dots\dots(A)$$

R.H.S

$$\text{go(fh)}(x) = \text{go}(f(x).h(x))$$

$$= g(\sin x \cos x) = 2\sin x \cos x$$

$$\text{go(fh)}(x) = 2 \sin x \cos x \dots\dots(B)$$

from A and B

$$\text{fog}(x) = \text{go(fh)}(x)$$

Hence proved

### 7. Question

Let  $f$  be any real function and let  $g$  be a function given by  $g(x) = 2x$ . Prove that  $\text{gof} = f + f$ .

### Answer

We are given that  $f$  is a real function and  $g$  is a function given by

$$g(x) = 2x$$

To prove;  $\text{gof} = f + f$ .

L.H.S

$$\text{gof}(x) = g(f(x)) = 2f(x)$$

$$= f + f = \text{R.H.S}$$

$$\text{gof} = f + f$$

Hence proved

### 8. Question

If  $f(x) = \sqrt{1-x}$  and  $g(x) = \log_e x$  are two real functions, then describe functions  $\text{fog}$  and  $\text{gof}$ .

### Answer

$$f(x) = \sqrt{1-x}, g(x) = \log_e x$$

Domain of  $f$  and  $g$  are  $\mathbb{R}$ .

Range of  $f = (-\infty, 1)$  Range of  $g = (0, e)$

Range of  $f \subset$  Domain of  $g \Rightarrow \text{gof}$  exists

Range of  $g \subset$  Domain  $f \Rightarrow \text{fog}$  exists

$$\therefore \text{gof}(x) = g(f(x)) = g(\sqrt{1-x})$$

$$\therefore \text{gof}(x) = \log_e \sqrt{1-x}$$

Again

$$\text{fog}(x) = f(g(x)) = f(\log_e x)$$

$$\text{fog}(x) = \sqrt{1 - \log_e x}$$

### 9. Question

If  $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$  and  $g: [-1, 1] \rightarrow \mathbb{R}$  be defined as  $f(x) = \tan x$  and  $g(x) = \sqrt{1-x^2}$  respectively. Describe fog and gof.

### Answer

$f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$  and  $g: [-1, 1] \rightarrow \mathbb{R}$  defined as  $f(x) = \tan x$  and  $g(x) = \sqrt{1-x^2}$

Range of  $f$ : let  $y = f(x)$

$$\Rightarrow y = \tan x$$

$$\Rightarrow x = \tan^{-1} y$$

Since,  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $y \in (-\infty, \infty)$

As Range of  $f \subset$  Domain of  $g$

$\therefore$  gof exists.

Similarly, let  $y = g(x)$

$$\Rightarrow y = \sqrt{1-x^2}$$

$$\Rightarrow x = \sqrt{1-y^2}$$

$\therefore$  Range of  $g$  is  $[-1, 1]$

As, Range of  $g \subset$  Domain of  $f$

Hence, fog also exists

Now,

$$\text{fog}(x) = f(g(x)) = f(\sqrt{1-x^2})$$

$$\Rightarrow \text{fog}(x) = \tan \sqrt{1-x^2}$$

Again,

$$\text{gof}(x) = g(f(x)) = g(\tan x)$$

$$\Rightarrow \text{gof}(x) = \sqrt{1 - \tan^2 x}$$

### 10. Question

If  $f(x) = \sqrt{x+3}$  and  $g(x) = x^2 + 1$  be two real functions, then find fog and gof.

### Answer

$$f(x) = \sqrt{x+3}, g(x) = x^2 + 1$$

Now,

Domain of  $f = [-3, \infty)$ , domain of  $g = (-\infty, \infty)$

Range of  $f = [0, \infty)$ , range of  $g = [1, \infty)$

Then, range of  $f \subset$  Domain of  $g$  and range of  $g \subset$  Domain of  $f$

Hence,  $f \circ g$  and  $g \circ f$  exists

Now,

$$f \circ g(x) = f(g(x)) = f(x^2 + 1)$$

$$\Rightarrow f \circ g(x) = \sqrt{x^2 + 4}$$

Again,

$$g \circ f(x) = g(f(x)) = g(\sqrt{x+3})$$

$$\Rightarrow g \circ f(x) = (\sqrt{x+3})^2 + 1$$

$$\Rightarrow g \circ f(x) = x + 4$$

### 11 A. Question

Let  $f$  be a real function given by  $f(x) = \sqrt{x-2}$ . Find each of the following:

$f \circ f$

#### Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of  $f = [2, \infty)$  and range of  $f = [0, \infty)$

We observe that range of  $f$  is not a subset of domain of  $f$

$$\therefore \text{Domain of } (f \circ f) = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Now,

$$f \circ f(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

### 11 B. Question

Let  $f$  be a real function given by  $f(x) = \sqrt{x-2}$ . Find each of the following:

$f \circ f \circ f$

#### Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of  $f = [2, \infty)$  and range of  $f = [0, \infty)$

We observe that range of  $f$  is not a subset of domain of  $f$

$$\begin{aligned}
\therefore \text{Domain of (fof)} &= \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\} \\
&= \{x: x \in [2, \infty) \text{ and } x-2 \geq 4\} \\
&= \{x: x \in [2, \infty) \text{ and } x \geq 6\} \\
&= [6, \infty)
\end{aligned}$$

Clearly, range of  $f = [0, \infty) \not\subset \text{Domain of (fof)}$

$$\begin{aligned}
\therefore \text{Domain of ((fof)of)} &= \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of (fof)}\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [6, \infty)\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 6\} \\
&= \{x: x \in [2, \infty) \text{ and } x-2 \geq 36\} \\
&= \{x: x \in [2, \infty) \text{ and } x \geq 38\} \\
&= [38, \infty)
\end{aligned}$$

Now,

$$(\text{fof})(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$$(\text{fofof})(x) = (\text{fof})(f(x)) = (\text{fof})(\sqrt{x-2}) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$\therefore \text{fofof} : [38, \infty) \rightarrow \mathbb{R}$  defined as

$$(\text{fofof})(x) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

### 11 C. Question

Let  $f$  be a real function given by  $f(x) = \sqrt{x-2}$ . Find each of the following:

$$(\text{fofof})(38)$$

### Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of  $f = [2, \infty)$  and range of  $f = [0, \infty)$

We observe that range of  $f$  is not a subset of domain of  $f$

$$\begin{aligned}
\therefore \text{Domain of (fof)} &= \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\} \\
&= \{x: x \in [2, \infty) \text{ and } x-2 \geq 4\} \\
&= \{x: x \in [2, \infty) \text{ and } x \geq 6\} \\
&= [6, \infty)
\end{aligned}$$

Clearly, range of  $f = [0, \infty) \not\subset \text{Domain of (fof)}$

$$\begin{aligned}
\therefore \text{Domain of ((fof)of)} &= \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of (fof)}\} \\
&= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [6, \infty)\}
\end{aligned}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 6\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 36\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 38\}$$

$$= [38, \infty)$$

Now,

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$$(f \circ f \circ f)(x) = (f \circ f)(f(x)) = (f \circ f)(\sqrt{x-2}) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$\therefore$  fof :  $[38, \infty) \rightarrow \mathbb{R}$  defined as

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$$(f \circ f \circ f)(x) = (f \circ f)(f(x)) = (f \circ f)(\sqrt{x-2}) = \sqrt{\sqrt{\sqrt{x-2}-2}-2}$$

$\therefore$  fof :  $[38, \infty) \rightarrow \mathbb{R}$  defined as

$$(f \circ f \circ f)(x) = \sqrt{\sqrt{\sqrt{\sqrt{x-2}-2}-2}-2}$$

$$(f \circ f \circ f)(38) = \sqrt{\sqrt{\sqrt{\sqrt{38-2}-2}-2}-2} = \sqrt{\sqrt{\sqrt{36}-2}-2}$$

$$= \sqrt{\sqrt{6-2}-2} = \sqrt{\sqrt{4}-2} = \sqrt{2-2} = 0$$

### 11 D. Question

Let  $f$  be a real function given by  $f(x) = \sqrt{x-2}$ . Find each of the following:

$$f^2$$

Also, show that  $f \circ f \neq f^2$ .

### Answer

$$\text{We have, } f(x) = \sqrt{x-2}$$

Clearly, domain of  $f = [2, \infty]$  and range of  $f = [0, \infty)$

We observe that range of  $f$  is not a subset of domain of  $f$

$\therefore$  Domain of  $(f \circ f) = \{x: x \in \text{Domain of } f \text{ and } f(x) \in \text{Domain of } f\}$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \in [2, \infty)\}$$

$$= \{x: x \in [2, \infty) \text{ and } \sqrt{x-2} \geq 2\}$$

$$= \{x: x \in [2, \infty) \text{ and } x - 2 \geq 4\}$$

$$= \{x: x \in [2, \infty) \text{ and } x \geq 6\}$$

$$= [6, \infty)$$

Now,

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x-2}) = \sqrt{\sqrt{x-2}-2}$$

$\therefore$  fof:  $[6, \infty) \rightarrow \mathbb{R}$  defined as



$$(f \circ f)(x) = \sqrt{\sqrt{x-2}-2}$$

$$f^2(x) = [f(x)]^2 = [\sqrt{x-2}]^2 = x-2$$

$\therefore f^2: [2, \infty) \rightarrow \mathbb{R}$  defined as

$$f^2(x) = x-2$$

$\therefore f \circ f \neq f^2$

## 12. Question

Let  $f(x) = \begin{cases} 1+x, & 0 \leq x \leq 2 \\ 3-x, & 2 < x \leq 3 \end{cases}$ . Find  $f \circ f$ .

## Answer

$$f(x) = \begin{cases} 1+x & 0 \leq x \leq 2 \\ 3-x & 2 < x \leq 3 \end{cases}$$

Range of  $f = [0, 3] \subset$  Domain of  $f$

$$\therefore f \circ f(x) = f(f(x)) = f\left(\begin{cases} 1+x & 0 \leq x \leq 2 \\ 3-x & 2 < x \leq 3 \end{cases}\right) = f\left(\begin{cases} 1+(1+x) & 0 \leq x \leq 1 \\ 3-(1+x) & 1 < x \leq 2 \\ 1+(3-x) & 2 < x \leq 3 \end{cases}\right)$$

$$\text{So, } f \circ f(x) = \begin{cases} 2+x & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 4-x & 2 < x \leq 3 \end{cases}$$

## 13. Question

If  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be two functions defined as  $f(x) = |x| + x$  and

$g(x) = |x|-x$  for all  $x \in \mathbb{R}$ . Then, find  $f \circ g$  and  $g \circ f$ . Hence, find  $f \circ g(-3)$ ,

$f \circ g(5)$  and  $g \circ f(-2)$ .

## Answer

Domain of  $f(x)$  and  $g(x)$  is  $\mathbb{R}$ .

Range of  $f(x) = [0, \infty)$  and range of  $g(x) = [0, \infty)$

As, range of  $f \subset$  Domain of  $g$  and range of  $g \subset$  Domain of  $f$

So,  $g \circ f$  and  $f \circ g$  exists

Now,

$$f \circ g(x) = f(g(x)) = f(|x|-x)$$

$$\Rightarrow f \circ g(x) = ||x|-x| + |x|-x$$

As, range of  $g(x) \geq 0$  so,  $||x|-x| = |x|-x$

$$\text{So, } f \circ g(x) = ||x|-x| + |x|-x = |x|-x + |x|-x$$

$$\Rightarrow f \circ g(x) = 2(|x|-x)$$

Also,

$$g \circ f(x) = g(f(x)) = g(|x| + x) = ||x| + x| - (|x| + x)$$

As, range of  $f(x) \geq 0$  so,  $||x| + x| = |x| + x$

$$\text{So, } g \circ f(x) = ||x| + x| - (|x| + x) = |x| + x - (|x| + x) = 0$$

Thus,  $g \circ f(x) = 0$

$$\text{Now, } \text{fog}(-3) = 2(|-3| - (-3)) = 2(3 + 3) = 6,$$

$$\text{fog}(5) = 2(|5| - 5) = 0, \text{gof}(-2) = 0$$

## Exercise 2.4

### 1. Question

State with reasons whether the following functions have inverse:

(i)  $f : [1, 2, 3, 4] \rightarrow \{10\}$  with  $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(ii)  $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$  with  $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(iii)  $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$  with  $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

### Answer

(i)  $f : [1, 2, 3, 4] \rightarrow \{10\}$  with  $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, we have  $f(1) = 10 = f(2) = f(3) = f(4)$

Hence,  $f$  is not one-one.

Thus, the function  $f$  does not have an inverse.

(ii)  $g : \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$  with  $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, we have  $g(5) = 4 = g(7)$

Hence,  $g$  is not one-one.

Thus, the function  $g$  does not have an inverse.

(iii)  $h : \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$  with  $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain  $\{2, 3, 4, 5\}$  are mapped to distinct elements of the co-domain  $\{7, 9, 11, 13\}$ .

Hence,  $h$  is one-one.

Also, each element of the range  $\{7, 9, 11, 13\}$  is the image of some element of  $\{2, 3, 4, 5\}$ .

Hence,  $h$  is also onto.

Thus, the function  $h$  has an inverse.

### 2. Question

Find  $f^{-1}$  if it exists for  $f: A \rightarrow B$  where

(i)  $A = \{0, -1, -3, 2\}$ ;  $B = \{-9, -3, 0, 6\}$  &  $f(x) = 3x$

(ii)  $A = \{1, 3, 5, 7, 9\}$ ;  $B = \{0, 1, 9, 25, 49, 81\}$  &  $f(x) = x^2$

### Answer

(i)  $A = \{0, -1, -3, 2\}$ ;  $B = \{-9, -3, 0, 6\}$  &  $f(x) = 3x$

We have  $f : A \rightarrow B$  and  $f(x) = 3x$ .

$$\Rightarrow f = \{(0, 3 \times 0), (-1, 3 \times (-1)), (-3, 3 \times (-3)), (2, 3 \times 2)\}$$

$$\therefore f = \{(0, 0), (-1, -3), (-3, -9), (2, 6)\}$$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain  $\{0, -1, -3, 2\}$  are mapped to distinct elements of the co-domain  $\{0, -3, -9, 6\}$ .

Hence,  $f$  is one-one.

Also, each element of the range  $\{-9, -3, 0, 6\}$  is the image of some element of  $\{0, -1, -3, 2\}$ .

Hence,  $f$  is also onto.

Thus, the function  $f$  has an inverse.

We have  $f^{-1} = \{(0, 0), (-3, -1), (-9, -3), (6, 2)\}$

(ii)  $A = \{1, 3, 5, 7, 9\}$ ;  $B = \{0, 1, 9, 25, 49, 81\}$  &  $f(x) = x^2$

We have  $f : A \rightarrow B$  and  $f(x) = x^2$ .

$\Rightarrow f = \{(1, 1^2), (3, 3^2), (5, 5^2), (7, 7^2), (9, 9^2)\}$

$\therefore f = \{(1, 1), (3, 9), (5, 25), (7, 49), (9, 81)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain  $\{1, 3, 5, 7, 9\}$  are mapped to distinct elements of the co-domain  $\{1, 9, 25, 49, 81\}$ .

Hence,  $f$  is one-one.

However, the element 0 of the range  $\{0, 1, 9, 25, 49, 81\}$  is not the image of any element of  $\{1, 3, 5, 7, 9\}$ .

Hence,  $f$  is not onto.

Thus, the function  $f$  does not have an inverse.

### 3. Question

Consider  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  and  $g : \{a, b, c\} \rightarrow \{\text{apple, ball, cat}\}$  defined as  $f(1) = a$ ,  $f(2) = b$ ,  $f(3) = c$ ,  $g(a) = \text{apple}$ ,  $g(b) = \text{ball}$  and  $g(c) = \text{cat}$ . Show that  $f$ ,  $g$  and  $g \circ f$  are invertible. Find  $f^{-1}$ ,  $g^{-1}$ ,  $(g \circ f)^{-1}$  and show that  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .

### Answer

$f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  and  $f(1) = a$ ,  $f(2) = b$ ,  $f(3) = c$

$\Rightarrow f = \{(1, a), (2, b), (3, c)\}$

Recall that a function is invertible only when it is both one-one and onto.

Here, observe that distinct elements of the domain  $\{1, 2, 3\}$  are mapped to distinct elements of the co-domain  $\{a, b, c\}$ .

Hence,  $f$  is one-one.

Also, each element of the range  $\{a, b, c\}$  is the image of some element of  $\{1, 2, 3\}$ .

Hence,  $f$  is also onto.

Thus, the function  $f$  has an inverse.

We have  $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$

$g : \{a, b, c\} \rightarrow \{\text{apple, ball, cat}\}$  and  $g(a) = \text{apple}$ ,  $g(b) = \text{ball}$ ,  $g(c) = \text{cat}$

$\Rightarrow g = \{(a, \text{apple}), (b, \text{ball}), (c, \text{cat})\}$

Similar to the function  $f$ ,  $g$  is also one-one and onto.

Thus, the function  $g$  has an inverse.

We have  $g^{-1} = \{(\text{apple}, a), (\text{ball}, b), (\text{cat}, c)\}$

We know  $(g \circ f)(x) = g(f(x))$

Thus,  $\text{gof} : \{1, 2, 3\} \rightarrow \{\text{apple, ball, cat}\}$  and

$$(\text{gof})(1) = g(f(1)) = g(a) = \text{apple}$$

$$(\text{gof})(2) = g(f(2)) = g(b) = \text{ball}$$

$$(\text{gof})(3) = g(f(3)) = g(c) = \text{cat}$$

$$\Rightarrow \text{gof} = \{(1, \text{apple}), (2, \text{ball}), (3, \text{cat})\}$$

As the functions  $f$  and  $g$ ,  $\text{gof}$  is also both one-one and onto.

Thus, the function  $\text{gof}$  has an inverse.

$$\text{We have } (\text{gof})^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$$

Now, let us consider  $f^{-1}og^{-1}$ .

$$\text{We know } (f^{-1}og^{-1})(x) = f^{-1}(g^{-1}(x))$$

Thus,  $f^{-1}og^{-1} : \{\text{apple, ball, cat}\} \rightarrow \{1, 2, 3\}$  and

$$(f^{-1}og^{-1})(\text{apple}) = f^{-1}(g^{-1}(\text{apple})) = f^{-1}(a) = 1$$

$$(f^{-1}og^{-1})(\text{ball}) = f^{-1}(g^{-1}(\text{ball})) = f^{-1}(b) = 2$$

$$(f^{-1}og^{-1})(\text{cat}) = f^{-1}(g^{-1}(\text{cat})) = f^{-1}(c) = 3$$

$$\Rightarrow f^{-1}og^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$$

Therefore, we have  $(\text{gof})^{-1} = f^{-1}og^{-1}$ .

#### 4. Question

Let  $A = \{1, 2, 3, 4\}$ ;  $B = \{3, 5, 7, 9\}$ ;  $C = \{7, 23, 47, 79\}$  and  $f : A \rightarrow B$ ,  $g : B \rightarrow C$  be defined as  $f(x) = 2x + 1$  and  $g(x) = x^2 - 2$ . Express  $(\text{gof})^{-1}$  and  $f^{-1}og^{-1}$  as the sets of ordered pairs and verify  $(\text{gof})^{-1} = f^{-1}og^{-1}$ .

#### Answer

We have  $f : A \rightarrow B$  &  $f(x) = 2x + 1$

$$\Rightarrow f = \{(1, 2 \times 1 + 1), (2, 2 \times 2 + 1), (3, 2 \times 3 + 1), (4, 2 \times 4 + 1)\}$$

$$\therefore f = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$$

Function  $f$  is clearly one-one and onto.

Thus,  $f^{-1}$  exists and  $f^{-1} = \{(3, 1), (5, 2), (7, 3), (9, 4)\}$

We have  $g : B \rightarrow C$  &  $g(x) = x^2 - 2$

$$\Rightarrow g = \{(3, 3^2 - 2), (5, 5^2 - 2), (7, 7^2 - 2), (9, 9^2 - 2)\}$$

$$\therefore g = \{(3, 7), (5, 23), (7, 47), (9, 79)\}$$

Function  $g$  is clearly one-one and onto.

Thus,  $g^{-1}$  exists and  $g^{-1} = \{(7, 3), (23, 5), (47, 7), (79, 9)\}$

We know  $(\text{gof})(x) = g(f(x))$

Thus,  $\text{gof} : A \rightarrow C$  and

$$(\text{gof})(1) = g(f(1)) = g(3) = 7$$

$$(\text{gof})(2) = g(f(2)) = g(5) = 23$$

$$(\text{gof})(3) = g(f(3)) = g(7) = 47$$

$$(\text{gof})(4) = g(f(4)) = g(9) = 79$$

$$\Rightarrow \text{gof} = \{(1, 7), (2, 23), (3, 47), (4, 79)\}$$

Clearly,  $\text{gof}$  is also both one-one and onto.

Thus, the function  $\text{gof}$  has an inverse.

We have  $(\text{gof})^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$

Now, let us consider  $f^{-1} \circ g^{-1}$ .

We know  $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$

Thus,  $f^{-1} \circ g^{-1} : C \rightarrow A$  and

$$(f^{-1} \circ g^{-1})(7) = f^{-1}(g^{-1}(7)) = f^{-1}(3) = 1$$

$$(f^{-1} \circ g^{-1})(23) = f^{-1}(g^{-1}(23)) = f^{-1}(5) = 2$$

$$(f^{-1} \circ g^{-1})(47) = f^{-1}(g^{-1}(47)) = f^{-1}(7) = 3$$

$$(f^{-1} \circ g^{-1})(79) = f^{-1}(g^{-1}(79)) = f^{-1}(9) = 4$$

$$\Rightarrow f^{-1} \circ g^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$$

Therefore, we have  $(\text{gof})^{-1} = f^{-1} \circ g^{-1}$ .

### 5. Question

Show that the function  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  defined by  $f(x) = 3x + 5$  is invertible. Also, find  $f^{-1}$ .

### Answer

We have  $f : \mathbb{Q} \rightarrow \mathbb{Q}$  and  $f(x) = 3x + 5$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{Q}$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow 3x_1 + 5 = 3x_2 + 5$$

$$\Rightarrow 3x_1 = 3x_2$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in \mathbb{Q}$  (co-domain) such that  $f(x) = y$

$$\Rightarrow 3x + 5 = y$$

$$\Rightarrow 3x = y - 5$$

$$\therefore x = \frac{y-5}{3}$$

Clearly, for every  $y \in \mathbb{Q}$ , there exists  $x \in \mathbb{Q}$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \frac{y-5}{3}$

Hence,  $f^{-1}(y) = \frac{y-5}{3}$

Thus,  $f(x)$  is invertible and  $f^{-1}(x) = \frac{x-5}{3}$

## 6. Question

Show that the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 4x + 3$  is invertible. Find the inverse of  $f$ .

### Answer

We have  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $f(x) = 4x + 3$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow 4x_1 + 3 = 4x_2 + 3$$

$$\Rightarrow 4x_1 = 4x_2$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in \mathbb{R}$  (co-domain) such that  $f(x) = y$

$$\Rightarrow 4x + 3 = y$$

$$\Rightarrow 4x = y - 3$$

$$\therefore x = \frac{y-3}{4}$$

Clearly, for every  $y \in \mathbb{R}$ , there exists  $x \in \mathbb{R}$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \frac{y-3}{4}$

Hence,  $f^{-1}(y) = \frac{y-3}{4}$

Thus,  $f(x)$  is invertible and  $f^{-1}(x) = \frac{x-3}{4}$

## 7. Question

Consider  $f : \mathbb{R}^+ \rightarrow [4, \infty)$  given by  $f(x) = x^2 + 4$ . Show that  $f$  is invertible with  $f^{-1}$  of  $f$  given by  $f^{-1}(x) = \sqrt{x-4}$ , where  $\mathbb{R}^+$  is the set of all non-negative real numbers.

### Answer

We have  $f : \mathbb{R}^+ \rightarrow [4, \infty)$  and  $f(x) = x^2 + 4$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}^+$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow x_1^2 + 4 = x_2^2 + 4$$

$$\Rightarrow x_1^2 = x_2^2$$

$$\therefore x_1 = x_2 \quad (x_1 \neq -x_2 \text{ as } x_1, x_2 \in \mathbb{R}^+)$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in [4, \infty)$  (co-domain) such that  $f(x) = y$

$$\Rightarrow x^2 + 4 = y$$

$$\Rightarrow x^2 = y - 4$$

$$\therefore x = \sqrt{y - 4}$$

Clearly, for every  $y \in [4, \infty)$ , there exists  $x \in \mathbb{R}^+$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

$$\text{We have } f(x) = y \Rightarrow x = f^{-1}(y)$$

$$\text{But, we found } f(x) = y \Rightarrow x = \sqrt{y - 4}$$

$$\text{Hence, } f^{-1}(y) = \sqrt{y - 4}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \sqrt{x - 4}$$

### 8. Question

If  $f(x) = \frac{4x+3}{6x-4}$ ,  $x \neq \frac{2}{3}$ , show that  $(f \circ f)(x) = x$  for all  $x \neq \frac{2}{3}$ . What is the inverse of  $f$ ?

### Answer

$$\text{We have } f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$$

$$\text{We know } (f \circ f)(x) = f(f(x))$$

$$\Rightarrow (f \circ f)(x) = f\left(\frac{4x+3}{6x-4}\right)$$

$$\Rightarrow (f \circ f)(x) = \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4}$$

$$\Rightarrow (f \circ f)(x) = \frac{4(4x+3) + 3(6x-4)}{6(4x+3) - 4(6x-4)}$$

$$\Rightarrow (f \circ f)(x) = \frac{16x + 12 + 18x - 12}{24x + 18 - 24x + 16}$$

$$\Rightarrow (f \circ f)(x) = \frac{34x}{34}$$

$$\therefore (f \circ f)(x) = x$$

As  $(f \circ f)(x) = x = I_x$  (the identity function),  $f(x) = f^{-1}(x)$ .

$$\text{Thus, } f^{-1}(x) = \frac{4x+3}{6x-4}$$

### 9. Question

Consider  $f: \mathbb{R}^+ \rightarrow [-5, \infty)$  given by  $f(x) = 9x^2 + 6x - 5$ . Show that  $f$  is invertible with  $f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$ .

## Answer

We have  $f: \mathbb{R}^+ \rightarrow [-5, \infty)$  and  $f(x) = 9x^2 + 6x - 5$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}^+$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow 9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$$

$$\Rightarrow 9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in [-5, \infty)$  (co-domain) such that  $f(x) = y$

$$\Rightarrow 9x^2 + 6x - 5 = y$$

Adding 6 to both sides, we get

$$9x^2 + 6x - 5 + 6 = y + 6$$

$$\Rightarrow 9x^2 + 6x + 1 = y + 6$$

$$\Rightarrow (3x + 1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y + 6}$$

$$\Rightarrow 3x = \sqrt{y + 6} - 1$$

$$\therefore x = \frac{\sqrt{y + 6} - 1}{3}$$

Clearly, for every  $y \in [4, \infty)$ , there exists  $x \in \mathbb{R}^+$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

$$\text{Hence, } f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$

Thus,  $f(x)$  is invertible and  $f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$

## 10. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^3 - 3$ , then prove that  $f^{-1}$  exists and find a formula for  $f^{-1}$ . Hence, find  $f^{-1}(24)$



and  $f^{-1}(5)$ .

### Answer

We have  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $f(x) = x^3 - 3$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow x_1^3 - 3 = x_2^3 - 3$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in \mathbb{R}$  (co-domain) such that  $f(x) = y$

$$\Rightarrow x^3 - 3 = y$$

$$\Rightarrow x^3 = y + 3$$

$$\therefore x = \sqrt[3]{y + 3}$$

Clearly, for every  $y \in \mathbb{R}$ , there exists  $x \in \mathbb{R}$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \sqrt[3]{y + 3}$

Hence,  $f^{-1}(y) = \sqrt[3]{y + 3}$

Thus,  $f(x)$  is invertible and  $f^{-1}(x) = \sqrt[3]{x + 3}$

Hence, we have

$$f^{-1}(24) = \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$$

$$f^{-1}(5) = \sqrt[3]{5 + 3} = \sqrt[3]{8} = 2$$

Thus,  $f^{-1}(24) = 3$  and  $f^{-1}(5) = 2$ .

### 11. Question

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined as  $f(x) = x^3 + 4$ . Is it a bijection or not? In case it is a bijection, find  $f^{-1}(3)$ .

### Answer

We have  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $f(x) = x^3 + 4$ .

Recall that a function is a bijection only if it is both one-one and onto.

First, we will check if  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow x_1^3 + 4 = x_2^3 + 4$$

$$\Rightarrow x_1^3 = x_2^3$$

$$\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$$

As  $x_1, x_2 \in \mathbb{R}$  and the second factor has no real roots,

$$x_1 - x_2 = 0$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will check if  $f$  is onto.

Let  $y \in \mathbb{R}$  (co-domain) such that  $f(x) = y$

$$\Rightarrow x^3 + 4 = y$$

$$\Rightarrow x^3 = y - 4$$

$$\therefore x = \sqrt[3]{y - 4}$$

Clearly, for every  $y \in \mathbb{R}$ , there exists  $x \in \mathbb{R}$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  is a bijection and has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \sqrt[3]{y - 4}$

Hence,  $f^{-1}(y) = \sqrt[3]{y - 4}$

Thus,  $f(x)$  is invertible and  $f^{-1}(x) = \sqrt[3]{x - 4}$

Hence, we have

$$f^{-1}(3) = \sqrt[3]{3 - 4} = \sqrt[3]{-1} = -1$$

Thus,  $f^{-1}(3) = -1$ .

## 12. Question

If  $f : Q \rightarrow Q$ ,  $g : Q \rightarrow Q$  are two functions defined by  $f(x) = 2x$  and  $g(x) = x + 2$ , show that  $f$  and  $g$  are bijective maps. Verify that  $(gof)^{-1} = f^{-1}og^{-1}$ .

### Answer

We have  $f : Q \rightarrow Q$  and  $f(x) = 2x$ .

Recall that a function is a bijection only if it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in Q$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow 2x_1 = 2x_2$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in Q$  (co-domain) such that  $f(x) = y$

$$\Rightarrow 2x = y$$

$$\therefore x = \frac{y}{2}$$

Clearly, for every  $y \in Q$ , there exists  $x \in Q$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  is a bijection and has an inverse.

$$\text{We have } f(x) = y \Rightarrow x = f^{-1}(y)$$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{y}{2}$$

$$\text{Hence, } f^{-1}(y) = \frac{y}{2}$$

$$\text{Thus, } f^{-1}(x) = \frac{x}{2}$$

Now, we have  $g : Q \rightarrow Q$  and  $g(x) = x + 2$ .

First, we will prove that  $g$  is one-one.

Let  $x_1, x_2 \in Q$  (domain) such that  $g(x_1) = g(x_2)$

$$\Rightarrow x_1 + 2 = x_2 + 2$$

$$\therefore x_1 = x_2$$

So, we have  $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $g$  is one-one.

Now, we will prove that  $g$  is onto.

Let  $y \in Q$  (co-domain) such that  $g(x) = y$

$$\Rightarrow x + 2 = y$$

$$\therefore x = y - 2$$

Clearly, for every  $y \in Q$ , there exists  $x \in Q$  (domain) such that  $g(x) = y$  and hence, function  $g$  is onto.

Thus, the function  $g$  is a bijection and has an inverse.

$$\text{We have } g(x) = y \Rightarrow x = g^{-1}(y)$$

$$\text{But, we found } g(x) = y \Rightarrow x = y - 2$$

$$\text{Hence, } g^{-1}(y) = y - 2$$

$$\text{Thus, } g^{-1}(x) = x - 2$$

$$\text{We have } (f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$$

$$\text{We found } f^{-1}(x) = \frac{x}{2} \text{ and } g^{-1}(x) = x - 2$$

$$\Rightarrow (f^{-1} \circ g^{-1})(x) = f^{-1}(x - 2)$$

$$\therefore (f^{-1} \circ g^{-1})(x) = \frac{x - 2}{2}$$

We know  $(g \circ f)(x) = g(f(x))$  and  $g \circ f : Q \rightarrow Q$

$$\Rightarrow (g \circ f)(x) = g(2x)$$

$$\therefore (g \circ f)(x) = 2x + 2$$

Clearly,  $g \circ f$  is a bijection and has an inverse.

Let  $y \in Q$  (co-domain) such that  $(gof)(x) = y$

$$\Rightarrow 2x + 2 = y$$

$$\Rightarrow 2x = y - 2$$

$$\therefore x = \frac{y-2}{2}$$

We have  $(gof)(x) = y \Rightarrow x = (gof)^{-1}(y)$

$$\text{But, we found } (gof)(x) = y \Rightarrow x = \frac{y-2}{2}$$

$$\text{Hence, } (gof)^{-1}(y) = \frac{y-2}{2}$$

$$\text{Thus, } (gof)^{-1}(x) = \frac{x-2}{2}$$

So, it is verified that  $(gof)^{-1} = f^{-1}og^{-1}$ .

### 13. Question

Let  $A = R - \{3\}$  and  $B = R - \{1\}$ . Consider the function  $f : A \rightarrow B$  defined by  $f(x) = \frac{x-2}{x-3}$ . Show that  $f$  is one-one and onto and hence find  $f^{-1}$ .

### Answer

We have  $f : A \rightarrow B$  where  $A = R - \{3\}$  and  $B = R - \{1\}$

$$f(x) = \frac{x-2}{x-3}$$

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in A$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow (x_1-2)(x_2-3) = (x_1-3)(x_2-2)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$

$$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2$$

$$\Rightarrow -3x_1 + 2x_1 = 2x_2 - 3x_2$$

$$\Rightarrow -x_1 = -x_2$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in B$  (co-domain) such that  $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow \frac{(x-3)+1}{x-3} = y$$

$$\Rightarrow 1 + \frac{1}{x-3} = y$$

$$\Rightarrow \frac{1}{x-3} = y-1$$

$$\Rightarrow \frac{1}{y-1} = x-3$$

$$\Rightarrow x = 3 + \frac{1}{y-1}$$

$$\therefore x = \frac{3y-2}{y-1}$$

Clearly, for every  $y \in B$ , there exists  $x \in A$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \frac{3y-2}{y-1}$

Hence,  $f^{-1}(y) = \frac{3y-2}{y-1}$

Thus,  $f(x)$  is invertible and  $f^{-1}(x) = \frac{3x-2}{x-1}$

#### 14. Question

Consider the function  $f : \mathbb{R}^+ \rightarrow [-9, \infty)$  given by  $f(x) = 5x^2 + 6x - 9$ . Prove that  $f$  is invertible with

$$f^{-1}(y) = \frac{\sqrt{54+5y}-3}{4}.$$

#### Answer

We have  $f : \mathbb{R}^+ \rightarrow [-9, \infty)$  and  $f(x) = 5x^2 + 6x - 9$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}^+$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1^2 + 6x_1 - 9 = 5x_2^2 + 6x_2 - 9$$

$$\Rightarrow 5x_1^2 + 6x_1 = 5x_2^2 + 6x_2$$

$$\Rightarrow 5x_1^2 - 5x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow 5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 5(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[5(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in [-9, \infty)$  (co-domain) such that  $f(x) = y$

$$\Rightarrow 5x^2 + 6x - 9 = y$$

$$\Rightarrow 5\left(x^2 + \frac{6}{5}x - \frac{9}{5}\right) = y$$

$$\Rightarrow x^2 + \frac{6}{5}x - \frac{9}{5} = \frac{y}{5}$$

$$\Rightarrow x^2 + \frac{6}{5}x = \frac{y+9}{5}$$

Adding  $\frac{9}{25}$  to both sides, we get

$$\Rightarrow x^2 + \frac{6}{5}x + \frac{9}{25} = \frac{y+9}{5} + \frac{9}{25}$$

$$\Rightarrow \left(x + \frac{3}{5}\right)^2 = \frac{(5y+45)+9}{25}$$

$$\Rightarrow \left(x + \frac{3}{5}\right)^2 = \frac{5y+54}{25}$$

$$\Rightarrow x + \frac{3}{5} = \sqrt{\frac{5y+54}{25}}$$

$$\Rightarrow x + \frac{3}{5} = \frac{\sqrt{5y+54}}{5}$$

$$\Rightarrow x = \frac{\sqrt{5y+54}}{5} - \frac{3}{5}$$

$$\therefore x = \frac{\sqrt{5y+54}-3}{5}$$

Clearly, for every  $y \in [-9, \infty)$ , there exists  $x \in \mathbb{R}^+$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \frac{\sqrt{5y+54}-3}{5}$

Hence,  $f^{-1}(y) = \frac{\sqrt{5y+54}-3}{5}$

### 15. Question

Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a function defined as  $f(x) = 9x^2 + 6x - 5$ . Show that  $f : \mathbb{N} \rightarrow S$ , where  $S$  is the range of  $f$ , is invertible. Find the inverse of  $f$  and hence find  $f^{-1}(43)$  and  $f^{-1}(163)$ .

### Answer

We have  $f : \mathbb{N} \rightarrow \mathbb{N}$  and  $f(x) = 9x^2 + 6x - 5$ .

We need to prove  $f : \mathbb{N} \rightarrow S$  is invertible.

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{N}$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow 9x_1^2 + 6x_1 = 9x_2^2 + 6x_2$$

$$\Rightarrow 9x_1^2 - 9x_2^2 + 6x_1 - 6x_2 = 0$$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in S$  (co-domain) such that  $f(x) = y$

$$\Rightarrow 9x^2 + 6x - 5 = y$$

Adding 6 to both sides, we get

$$9x^2 + 6x - 5 + 6 = y + 6$$

$$\Rightarrow 9x^2 + 6x + 1 = y + 6$$

$$\Rightarrow (3x + 1)^2 = y + 6$$

$$\Rightarrow 3x + 1 = \sqrt{y + 6}$$

$$\Rightarrow 3x = \sqrt{y + 6} - 1$$

$$\therefore x = \frac{\sqrt{y + 6} - 1}{3}$$

Clearly, for every  $y \in S$ , there exists  $x \in N$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

$$\text{Hence, } f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}$$

Hence, we have

$$f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = \frac{\sqrt{49}-1}{3} = \frac{7-1}{3} = 2$$

$$f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = \frac{\sqrt{169}-1}{3} = \frac{13-1}{3} = 4$$

Thus,  $f^{-1}(43) = 2$  and  $f^{-1}(163) = 4$ .

## 16. Question

Let  $f : \mathbb{R} - \left\{ -\frac{4}{3} \right\} \rightarrow \mathbb{R}$  be a function defined as  $f(x) = \frac{4x}{3x+4}$ . Show that  $f : \mathbb{R} - \left\{ -\frac{4}{3} \right\} \rightarrow \text{range}(f)$  is one-one and onto. Hence, find  $f^{-1}$ .

**Answer**

We have  $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R}$  and  $f(x) = \frac{4x}{3x+4}$

We need to prove  $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \text{range}(f)$  is invertible.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in A$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow \frac{4x_1}{3x_1+4} = \frac{4x_2}{3x_2+4}$$

$$\Rightarrow (4x_1)(3x_2+4) = (3x_1+4)(4x_2)$$

$$\Rightarrow 12x_1x_2 + 16x_1 = 12x_1x_2 + 16x_2$$

$$\Rightarrow 16x_1 = 16x_2$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in \text{range}(f)$  (co-domain) such that  $f(x) = y$

$$\Rightarrow \frac{4x}{3x+4} = y$$

$$\Rightarrow 4x = 3xy + 4y$$

$$\Rightarrow 4x - 3xy = 4y$$

$$\Rightarrow x(4 - 3y) = 4y$$

$$\therefore x = \frac{4y}{4 - 3y}$$

Clearly, for every  $y \in \text{range}(f)$ , there exists  $x \in A$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\text{But, we found } f(x) = y \Rightarrow x = \frac{4y}{4-3y}$$

$$\text{Hence, } f^{-1}(y) = \frac{4y}{4-3y}$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \frac{4x}{4-3x}$$

## 17. Question

If  $f: \mathbb{R} \rightarrow (-1, 1)$  defined by  $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$  is invertible, find  $f^{-1}$ .

## Answer

We have  $f: \mathbb{R} \rightarrow (-1, 1)$  and  $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Given that  $f^{-1}$  exists.

Let  $y \in (-1, 1)$  such that  $f(x) = y$

$$\Rightarrow \frac{10^x - 10^{-x}}{10^x + 10^{-x}} = y$$



$$\Rightarrow \frac{10^x - \frac{1}{10^x}}{10^x + \frac{1}{10^x}} = y$$

$$\Rightarrow \frac{10^{2x} - 1}{10^{2x} + 1} = y$$

$$\Rightarrow 10^{2x} - 1 = y(10^{2x} + 1)$$

$$\Rightarrow 10^{2x} - 1 = 10^{2x}y + y$$

$$\Rightarrow 10^{2x} - 10^{2x}y = 1 + y$$

$$\Rightarrow 10^{2x}(1 - y) = 1 + y$$

$$\Rightarrow 10^{2x} = \frac{1 + y}{1 - y}$$

Taking  $\log_{10}$  on both sides, we get

$$\log_{10} 10^{2x} = \log_{10} \left( \frac{1 + y}{1 - y} \right)$$

$$\Rightarrow 2x \log_{10} 10 = \log_{10} \left( \frac{1 + y}{1 - y} \right)$$

$$\Rightarrow 2x = \log_{10} \left( \frac{1 + y}{1 - y} \right)$$

$$\therefore x = \frac{1}{2} \log_{10} \left( \frac{1 + y}{1 - y} \right)$$

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \frac{1}{2} \log_{10} \left( \frac{1 + y}{1 - y} \right)$

Hence,  $f^{-1}(y) = \frac{1}{2} \log_{10} \left( \frac{1 + y}{1 - y} \right)$

Thus,  $f^{-1}(x) = \frac{1}{2} \log_{10} \left( \frac{1 + x}{1 - x} \right)$

### 18. Question

If  $f: \mathbb{R} \rightarrow (0, 2)$  defined by  $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1$  is invertible, find  $f^{-1}$ .

### Answer

We have  $f: \mathbb{R} \rightarrow (0, 2)$  and  $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1$

Given that  $f^{-1}$  exists.

Let  $y \in (0, 2)$  such that  $f(x) = y$

$$\Rightarrow \frac{e^x - e^{-x}}{e^x + e^{-x}} + 1 = y$$

$$\Rightarrow \frac{e^x - e^{-x} + (e^x + e^{-x})}{e^x + e^{-x}} = y$$

$$\Rightarrow \frac{2e^x}{e^x + e^{-x}} = y$$

$$\Rightarrow \frac{2e^x}{e^x + \frac{1}{e^x}} = y$$

$$\Rightarrow \frac{2e^{2x}}{e^{2x} + 1} = y$$

$$\Rightarrow 2e^{2x} = y(e^{2x} + 1)$$

$$\Rightarrow 2e^{2x} = e^{2x}y + y$$

$$\Rightarrow 2e^{2x} - e^{2x}y = y$$

$$\Rightarrow e^{2x}(2 - y) = y$$

$$\Rightarrow e^{2x} = \frac{y}{2 - y}$$

Taking ln on both sides, we get

$$\ln e^{2x} = \ln\left(\frac{y}{2 - y}\right)$$

$$\Rightarrow 2x \ln e = \ln\left(\frac{y}{2 - y}\right)$$

$$\Rightarrow 2x = \ln\left(\frac{y}{2 - y}\right)$$

$$\therefore x = \frac{1}{2} \ln\left(\frac{y}{2 - y}\right)$$

We have  $f(x) = y \Rightarrow x = f^{-1}(y)$

But, we found  $f(x) = y \Rightarrow x = \frac{1}{2} \ln\left(\frac{y}{2 - y}\right)$

Hence,  $f^{-1}(y) = \frac{1}{2} \ln\left(\frac{y}{2 - y}\right)$

Thus,  $f^{-1}(x) = \frac{1}{2} \ln\left(\frac{x}{2 - x}\right)$

### 19. Question

Let  $f: [-1, \infty) \rightarrow [-1, \infty)$  is given by  $f(x) = (x + 1)^2 - 1$ . Show that  $f$  is invertible. Also, find the set  $S = \{x: f(x) = f^{-1}(x)\}$

### Answer

We have  $f: [-1, \infty) \rightarrow [-1, \infty)$  and  $f(x) = (x + 1)^2 - 1$

Recall that a function is invertible only when it is both one-one and onto.

First, we will prove that  $f$  is one-one.

Let  $x_1, x_2 \in [-1, \infty)$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow (x_1 + 1)^2 - 1 = (x_2 + 1)^2 - 1$$

$$\Rightarrow (x_1 + 1)^2 = (x_2 + 1)^2$$

$$\Rightarrow x_1^2 + 2x_1 + 1 = x_2^2 + 2x_2 + 1$$

$$\Rightarrow x_1^2 + 2x_1 = x_2^2 + 2x_2$$

$$\Rightarrow x_1^2 - x_2^2 + 2x_1 - 2x_2 = 0$$

$$\Rightarrow (x_1^2 - x_2^2) + 2(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2) + 2(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)[x_1 + x_2 + 2] = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ (as } x_1, x_2 \in \mathbb{R}^+)$$

$$\therefore x_1 = x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f$  is one-one.

Now, we will prove that  $f$  is onto.

Let  $y \in [-1, \infty)$  (co-domain) such that  $f(x) = y$

$$\Rightarrow (x + 1)^2 - 1 = y$$

$$\Rightarrow (x + 1)^2 = y + 1$$

$$\Rightarrow x + 1 = \sqrt{y + 1}$$

$$\therefore x = \sqrt{y + 1} - 1$$

Clearly, for every  $y \in [-1, \infty)$ , there exists  $x \in [-1, \infty)$  (domain) such that  $f(x) = y$  and hence, function  $f$  is onto.

Thus, the function  $f$  has an inverse.

$$\text{We have } f(x) = y \Rightarrow x = f^{-1}(y)$$

$$\text{But, we found } f(x) = y \Rightarrow x = \sqrt{y + 1} - 1$$

$$\text{Hence, } f^{-1}(y) = \sqrt{y + 1} - 1$$

$$\text{Thus, } f(x) \text{ is invertible and } f^{-1}(x) = \sqrt{x + 1} - 1$$

Now, we need to find the values of  $x$  for which  $f(x) = f^{-1}(x)$ .

$$\text{We have } f(x) = f^{-1}(x)$$

$$\Rightarrow (x + 1)^2 - 1 = \sqrt{x + 1} - 1$$

$$\Rightarrow (x + 1)^2 = \sqrt{x + 1}$$

$$\text{We can write } (x + 1)^2 = (\sqrt{x + 1})^4$$

$$\Rightarrow (\sqrt{x + 1})^4 = \sqrt{x + 1}$$

On substituting  $t = \sqrt{x + 1}$ , we get

$$t^4 = t$$

$$\Rightarrow t^4 - t = 0$$

$$\Rightarrow t(t^3 - 1) = 0$$

$$\Rightarrow t(t - 1)(t^2 + t + 1) = 0$$

$t^2 + t + 1 \neq 0$  because this equation has no real root  $t$ .

$$\Rightarrow t = 0 \text{ or } t - 1 = 0$$

$$\Rightarrow t = 0 \text{ or } t = 1$$

Case - I:  $t = 0$

$$\Rightarrow \sqrt{x+1} = 0$$

$$\Rightarrow x + 1 = 0$$

$$\therefore x = -1$$

Case - II:  $t = 1$

$$\Rightarrow \sqrt{x+1} = 1$$

$$\Rightarrow x + 1 = 1$$

$$\therefore x = 0$$

Thus,  $S = \{0, -1\}$

## 20. Question

Let  $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$  and let  $f : A \rightarrow A$ ,  $g : A \rightarrow A$  be two functions defined by  $f(x) = x^2$  and  $g(x) = \sin \pi x/2$ . Show that  $g^{-1}$  exists but  $f^{-1}$  does not exist. Also, find  $g^{-1}$ .

## Answer

We have  $f : A \rightarrow A$  where  $A = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$  defined by  $f(x) = x^2$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will check if  $f$  is one-one.

Let  $x_1, x_2 \in A$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1^2 - x_2^2 = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \text{ or } x_1 + x_2 = 0$$

$$\therefore x_1 = \pm x_2$$

So, we have  $f(x_1) = f(x_2) \Rightarrow x_1 = \pm x_2$ .

This means that two different elements of the domain are mapped to the same element by the function  $f$ .

For example, consider  $f(-1)$  and  $f(1)$ .

We have  $f(-1) = (-1)^2 = 1$  and  $f(1) = 1^2 = 1 = f(-1)$

Thus,  $f$  is not one-one and hence  $f^{-1}$  doesn't exist.

Now, let us consider  $g : A \rightarrow A$  defined by  $g(x) = \sin \frac{\pi x}{2}$

First, we will prove that  $g$  is one-one.

Let  $x_1, x_2 \in A$  (domain) such that  $g(x_1) = g(x_2)$

$$\Rightarrow \sin \frac{\pi x_1}{2} = \sin \frac{\pi x_2}{2}$$

$$\Rightarrow \frac{\pi x_1}{2} = \frac{\pi x_2}{2} \text{ (in the given range)}$$

$$\therefore x_1 = x_2$$

So, we have  $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $g$  is one-one.

Let  $y \in A$  (co-domain) such that  $g(x) = y$

$$\Rightarrow \sin \frac{\pi x}{2} = y$$

$$\Rightarrow \frac{\pi x}{2} = \sin^{-1} y$$

$$\Rightarrow \pi x = 2 \sin^{-1} y$$

$$\therefore x = \frac{2}{\pi} \sin^{-1} y$$

Clearly, for every  $y \in A$ , there exists  $x \in A$  (domain) such that  $g(x) = y$  and hence, function  $g$  is onto.

Thus, the function  $g$  has an inverse.

$$\text{We have } g(x) = y \Rightarrow x = g^{-1}(y)$$

$$\text{But, we found } g(x) = y \Rightarrow x = \frac{2}{\pi} \sin^{-1} y$$

$$\text{Hence, } g^{-1}(y) = \frac{2}{\pi} \sin^{-1} y$$

$$\text{Thus, } g(x) \text{ is invertible and } g^{-1}(x) = \frac{2}{\pi} \sin^{-1} x$$

## 21. Question

Let  $f$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$  such that  $f(x) = \cos(x + 2)$ . Is  $f$  invertible? Justify your answer.

### Answer

We have  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $f(x) = \cos(x + 2)$ .

Recall that a function is invertible only when it is both one-one and onto.

First, we will check if  $f$  is one-one.

Let  $x_1, x_2 \in \mathbb{R}$  (domain) such that  $f(x_1) = f(x_2)$

$$\Rightarrow \cos(x_1 + 2) = \cos(x_2 + 2)$$

As the cosine function repeats itself with a period  $2\pi$ , we have

$$x_1 + 2 = x_2 + 2 \text{ or } x_1 + 2 = 2\pi + (x_2 + 2)$$

$$\therefore x_1 = x_2 \text{ or } x_1 = 2\pi + x_2$$

$$\text{So, we have } f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \text{ or } 2\pi + x_2$$

This means that two different elements of the domain are mapped to the same element by the function  $f$ .

For example, consider  $f(0)$  and  $f(2\pi)$ .

$$\text{We have } f(0) = \cos(0 + 2) = \cos 2 \text{ and}$$

$$f(2\pi) = \cos(2\pi + 2) = \cos 2 = f(0)$$

Thus,  $f$  is not one-one.

Hence,  $f$  is not invertible and  $f^{-1}$  does not exist.

## 22. Question

If  $A = \{1, 2, 3, 4\}$  and  $B = \{a, b, c, d\}$ , define any four bijections from  $A$  to  $B$ . Also, give their inverse function.

### Answer

Given  $A = \{1, 2, 3, 4\}$  and  $B = \{a, b, c, d\}$ .

We need to define bijections  $f_1, f_2, f_3$  and  $f_4$  from  $A$  to  $B$ .

Consider  $f_1 = \{(1, a), (2, b), (3, c), (4, d)\}$

(1)  $f_1$  is one-one because no two elements of the domain are mapped to the same element.

$f_1$  is also onto because each element in the co-domain has a pre-image in the domain.

Thus,  $f_1$  is a bijection from A to B.

We have  $f_1^{-1} = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$

Using similar explanation, we also have the following bijections defined from A to B -

(2)  $f_2 = \{(1, b), (2, c), (3, d), (4, a)\}$

We have  $f_2^{-1} = \{(b, 1), (c, 2), (d, 3), (a, 4)\}$

(3)  $f_3 = \{(1, c), (2, d), (3, a), (4, b)\}$

We have  $f_3^{-1} = \{(c, 1), (d, 2), (a, 3), (b, 4)\}$

(4)  $f_4 = \{(1, d), (2, a), (3, b), (4, c)\}$

We have  $f_4^{-1} = \{(d, 1), (a, 2), (b, 3), (c, 4)\}$

### 23. Question

Let A and B be two sets each with finite number of elements. Assume that there is an injective map from A to B and that there is an injective map from B to A. Prove that there is a bijection from A to B.

### Answer

Given A and B are two finite sets. There are injective maps from both A to B and B to A.

Let f be the injective map defined from A to B.

Thus, we have f is one-one.

We also know that there is a one-one mapping from B to A.

This means that each element of B is mapped to a distinct element of A.

But, B is the co-domain of f and A is the domain of f.

So, every element of the co-domain of the function f has a pre-image in the domain of the function f.

Thus, f is also onto.

Therefore, f is a bijection as it is both one-one and onto.

Hence, there exists a bijection defined from A to B.

### 24. Question

If  $f : A \rightarrow A$  and  $g : A \rightarrow A$  are two bijections, then prove that

(i) fog is an injection

(ii) fog is a surjection

### Answer

Given  $f : A \rightarrow A$  and  $g : A \rightarrow A$  are two bijections. So, both f and g are one-one and onto functions.

We know  $(fog)(x) = f(g(x))$

Thus, fog is also defined from A to A.

(i) First, we will prove that fog is an interjection.

Let  $x_1, x_2 \in A$  (domain) such that  $(fog)(x_1) = (fog)(x_2)$

$$\Rightarrow f(g(x_1)) = f(g(x_2))$$

$$\Rightarrow g(x_1) = g(x_2) \text{ [since } f \text{ is one-one]}$$

$$\therefore x_1 = x_2 \text{ [since } g \text{ is one-one]}$$

So, we have  $(f \circ g)(x_1) = (f \circ g)(x_2) \Rightarrow x_1 = x_2$ .

Thus, function  $f \circ g$  is an injection.

(ii) Now, we will prove that  $f \circ g$  is a surjection.

Let  $z \in A$ , the co-domain of  $f \circ g$ .

As  $f$  is onto, we have  $y \in A$  (domain of  $f$ ) such that  $f(y) = z$ .

However, as  $g$  is also onto and  $y$  belongs to the co-domain of  $g$ , we have  $x \in A$  (domain of  $g$ ) such that  $g(x) = y$ .

$$\text{Hence, } (f \circ g)(x) = f(g(x)) = f(y) = z.$$

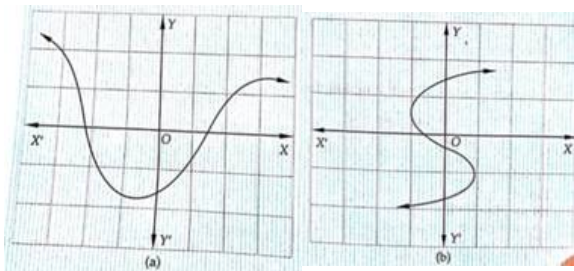
Here,  $x$  belongs to the domain of  $f \circ g$  ( $A$ ) and  $z$  belongs to the co-domain of  $f \circ g$  ( $A$ ).

Thus, function  $f \circ g$  is a surjection.

## Very short answer

### 1. Question

Which one of the following graphs represent a function?



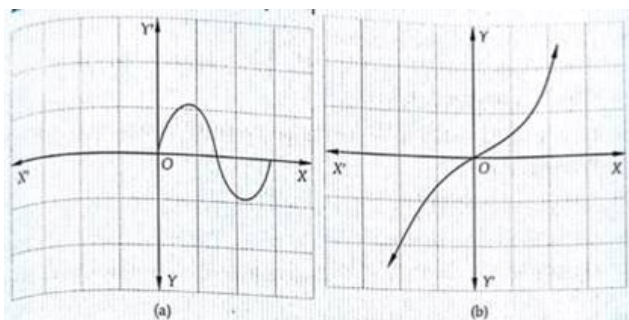
### Answer

(a) It has a unique image therefore a function

(b) It has more than one image

### 2. Question

Which one of the following graphs represent a one-one function?



### Answer

Formula:-

(i) A function  $f: A \rightarrow B$  is one-one function or an injection if

$$f(x) \neq f(y)$$

$\Rightarrow x=y$  for all  $x, y \in A$

or  $f(x) \neq f(y)$

$\Rightarrow x \neq y$  for all  $x, y \in A$

(a) It is not one-one function as it has same image on x axis

(b) It is one-one function as it has unique image

### 3. Question

If  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ , write total number of functions from A to B.

#### Answer

Formula:-if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B =  $n^m$

(ii) Number of one-one function from A to B =  $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B =  $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B =  $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$ , if  $m \geq n$

given: -

$A = \{1, 2, 3\}$  and  $B = \{a, b\}$

$n(A)=3$ , and  $n(B)=2$

total number of functions =  $2^3 = 8$

### 4. Question

If  $A = \{a, b, c\}$  and  $B = \{-2, -1, 0, 1, 2\}$ , write total number of one-one functions from A to B.

#### Answer

Formula:-

(I) A function  $f: A \rightarrow B$  is one-one function or an injection if

$f(x) = f(y)$

$\Rightarrow x=y$  for all  $x, y \in A$

or  $f(x) \neq f(y)$

$\Rightarrow x \neq y$  for all  $x, y \in A$

(II) if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B =  $n^m$

(ii) Number of one-one function from A to B =  $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B =  $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B =  $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$ , if  $m \geq n$

Let  $f: A \rightarrow B$  be one-one function

$n(A)=3$  and  $n(B)=5$

Using formula



Number of one-one function from A to B =  $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

$$\Rightarrow {}^3C_5 \cdot 5! = 60$$

### 5. Question

Write total number of one-one functions from set A = {1, 2, 3, 4} to set B = {a, b, c}.

#### Answer

Formula:-

(I) A function  $f: A \rightarrow B$  is one-one function or an injection if

$$f(x) = f(y)$$

$$\Rightarrow x = y \text{ for all } x, y \in A$$

$$\text{or } f(x) \neq f(y)$$

$$\Rightarrow x \neq y \text{ for all } x, y \in A$$

(II) if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B =  $n^m$

(ii) Number of one-one function from A to B =  $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B =  $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B =  $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$ , if  $m \geq n$

$$F(A) = 4 \text{ and } f(B) = 3$$

Using formula

Number of one-one function from A to B =  $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

Number of one-one function from A to B = 0

### 6. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2$ , write  $f^{-1}(25)$ .

#### Answer

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow x^2 = 25$$

$$\Rightarrow x = -5, 5$$

$$\Rightarrow f^{-1}(25) = \{-5, 5\}$$

### 7. Question

If  $f: \mathbb{C} \rightarrow \mathbb{C}$  is defined by  $f(x) = x^2$ , write  $f^{-1}(-4)$ . Here,  $\mathbb{C}$  denotes the set of all complex numbers.

**Answer**

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$ such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$ 

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow f(x) = -4$$

$$\Rightarrow x^2 = -4$$

$$\Rightarrow x = 2i, -2i$$

**8. Question**If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = x^3$ , write  $f^{-1}(1)$ .**Answer**

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$ such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$ 

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow f^{-1}(1) = x$$

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^3 = 1$$

$$\Rightarrow x^3 - 1 = 0$$

$$\Rightarrow (x-1)(x^2+x+1) = 0$$

$$\Rightarrow x = 1$$

**9. Question**Let  $C$  denote the set of all complex numbers. A function  $f: C \rightarrow C$  is defined by  $f(x) = x^3$ .Write  $f^{-1}(1)$ .**Answer**

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$ such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$ 

$$f(x) = y$$

$$\Rightarrow f^{-1}(y) = x$$

$$\Rightarrow f^{-1}(1) = x$$

$$\Rightarrow f(x) = 1$$

$$\Rightarrow x^3=1$$

$$\Rightarrow x^3-1=0$$

$$\Rightarrow (x-1)(x^2+x+1)=0$$

$$\Rightarrow x=1, w, w^2$$

### 10. Question

Let  $f$  be a function from  $C$  (set of all complex numbers) to itself given by  $f(x) = x^3$ . Write  $f^{-1}(-1)$ .

### Answer

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow f(x)=-1$$

$$\Rightarrow f^{-1}(-1)=x$$

$$\Rightarrow x^3=-1$$

$$\Rightarrow x^3+1=0$$

$$\Rightarrow (x+1)(x^2-x+1)=0$$

$$\Rightarrow x=-1, -w, -w^2$$

### 11. Question

Let  $f: R \rightarrow R$  be defined by  $f(x) = x^4$ , write  $f^{-1}(1)$ .

### Answer

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow f(x)=1$$

$$\Rightarrow f^{-1}(1)=x$$

$$\Rightarrow x^4=1$$

$$\Rightarrow x^4-1=0$$

$$\Rightarrow (x-1)(x^2+1)=0$$

$$\Rightarrow x=-1, 1$$

$$\Rightarrow f^{-1}(1) = \{-1, 1\}$$

### 12. Question

If  $f : C \rightarrow C$  is defined by  $f(x) = x^4$ ,  $f^{-1}(1)$ .

**Answer**

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $gof = I_x$  and  $fog = I_y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow f(x)=1$$

$$\Rightarrow f^{-1}(1)=x$$

$$\Rightarrow x^4=1$$

$$\Rightarrow x^4-1=0$$

$$\Rightarrow (x-1)(x^2+1)=0$$

$$\Rightarrow x=-1, 1, i, -i$$

$$\Rightarrow f^{-1}(1) = \{-1, -i, 1, i\}$$

**13. Question**

If  $f : R \rightarrow R$  is defined by  $f(x) = x^2$ ,  $f^{-1}(-25)$ .

**Answer**

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $gof = I_x$  and  $fog = I_y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f(x)=y$$

$$\Rightarrow f^{-1}(y)=x$$

$$\Rightarrow x^2=-25$$

but  $x$  should be Real number

$$f^{-1}(-25) = \emptyset$$

**14. Question**

If  $f : C \rightarrow C$  is defined by  $f(x) = (x - 2)^3$ , write  $f^{-1}(-1)$ .

**Answer**

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $gof = I_x$  and  $fog = I_y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f(x)=y$$

$$f^{-1}(y)=x$$

$$\Rightarrow (x - 2)^3 = -1$$

$$\Rightarrow x-2=-1, x-2=w \text{ and } x-2=-w^2$$

$$\Rightarrow x=1, -w+2, 2-w^2$$

$$\Rightarrow f^{-1}(25) = \{1, 2-w, 2-w^2\}$$

### 15. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = 10x - 7$ , then write  $f^{-1}(x)$ .

#### Answer

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$f^{-1}(x) = y$$

$$\Rightarrow f(y) = x$$

$$\Rightarrow 10y - 7 = x$$

$$\Rightarrow y = \frac{x+7}{10}$$

$$\Rightarrow f^{-1}(x) = \frac{x+7}{10}$$

### 16. Question

Let  $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}$  be a function defined by  $f(x) = \cos[x]$ . Write range (f).

#### Answer

Given:-

$$(i) f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$(ii) f(x) = \cos[x]$$

$$\text{Domain} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\text{For } f(x) = \cos[x]$$

$$\text{Range} = \{1, \cos 1, \cos 2\}$$

### 17. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 3x - 4$  is invertible then write  $f^{-1}(x)$ .

#### Answer

Given:- (i)  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$(ii) f(x) = 3x - 4$$

Formula:-

(i) A function  $f: X \rightarrow Y$  is defined to be invertible, if there exists a function  $g: Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

$$\text{For } f^{-1}(x) = y$$

$$\Rightarrow f(y) = x$$

$$\Rightarrow 3y - 4 = x$$

$$\Rightarrow y = \frac{x+4}{3}$$

$$\Rightarrow f^{-1}(x) = \frac{x+4}{3}$$

### 18. Question

If  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$  are given by  $f(x) = (x + 1)^2$  and  $g(x) = x^2 + 1$ , then write the value of  $f \circ g(-3)$ .

### Answer

Formula:-

(I) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f : A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

(i)  $f : \mathbb{R} \rightarrow \mathbb{R}$

(ii)  $g : \mathbb{R} \rightarrow \mathbb{R}$

(iii)  $f(x) = (x + 1)^2$

(iv)  $g(x) = x^2 + 1$

$f \circ g(-3) = f(g(-3))$

$\Rightarrow f \circ g(-3) = f((-3)^2 + 1)$

$\Rightarrow f \circ g(-3) = f(10)$

$\Rightarrow f \circ g(-3) = (10 + 1)^2$

$\Rightarrow f \circ g(-3) = 121$

### 19. Question

Let  $A = \{x \in \mathbb{R} : -4 \leq x \leq 4 \text{ and } x \neq 0\}$  and  $f : A \rightarrow \mathbb{R}$  be defined by  $f(x) = \frac{|x|}{x}$ . Write the range of  $f$ .

### Answer

Given:-

(i)  $A = \{x \in \mathbb{R} : -4 \leq x \leq 4 \text{ and } x \neq 0\}$

(ii)  $f : A \rightarrow \mathbb{R}$

(iii)  $f(x) = \frac{|x|}{x}$

For  $f(x) = \frac{|x|}{x}$

Range =  $\{-1, 1\}$

### 20. Question

Let  $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow A$  be defined by  $f(x) = \sin x$ . If  $f$  is a bijection, write set  $A$ .

### Answer

Formula:-

(i) A function  $f: A \rightarrow B$  is a bijection if it is one-one as well as onto

(ii) A function  $f: A \rightarrow B$  is onto function or surjection if

Range (f) = co-domain(f)

Given:-

$$(i) f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$(ii) f(x) = \sin x$$

(ii) f is bijection

For  $f(x) = \sin x$

Codomain = range

Set  $A = [-1, 1]$

### 21. Question

Let  $f: \mathbb{R} \rightarrow \mathbb{R}^+$  be defined by  $f(x) = a^x$ ,  $a > 0$  and  $a \neq 1$ . Write  $f^{-1}(x)$ .

#### Answer

Given:-

$$(i) f: \mathbb{R} \rightarrow \mathbb{R}^+$$

$$(ii) f(x) = a^x, a > 0 \text{ and } a \neq 1$$

Let

$$f(y) = x$$

$$a^y = x$$

$$\Rightarrow y = \log_a x$$

$$\Rightarrow f^{-1}(x) = \log_a x$$

### 22. Question

Let  $f: \mathbb{R} - \{-1\} \rightarrow \mathbb{R} - \{1\}$  be given by  $f(x) = \frac{x}{x+1}$ . Write  $f^{-1}(x)$ .

#### Answer

Given:-

$$(i) f: \mathbb{R} - \{-1\} \rightarrow \mathbb{R} - \{1\}$$

$$(ii) f(x) = \frac{x}{x+1}$$

$$F(y) = x$$

$$\Rightarrow \frac{y}{y+1} = x$$

$$\Rightarrow Y = xy + x$$

$$\Rightarrow y = \frac{x}{1-x}$$

$$\Rightarrow f^{-1} = \frac{x}{1-x}$$

### 23. Question

Let  $f : \mathbb{R} - \left\{ -\frac{3}{5} \right\} \rightarrow \mathbb{R}$  be a function defined as  $f(x) = \frac{2x}{5x+3}$ .

**Answer**

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

Given:-

(i)  $f : \mathbb{R} - \left\{ -\frac{3}{5} \right\} \rightarrow \mathbb{R}$

(ii)  $f(x) = \frac{2x}{5x+3}$

$F(y) = x$

$\Rightarrow \frac{2y}{5x+3} = x$

$\Rightarrow 2y - 3x - 5xy = 0$

$\Rightarrow y = \frac{3x}{2-5x}$

$\Rightarrow f^{-1}(x) = \frac{3x}{2-5x}$

**24. Question**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$  be two functions defined by  $f(x) = x^2 + x + 1$  and  $g(x) = 1 - x^2$ . Write  $f \circ g(-2)$ .

**Answer**

Formula :- (I) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f : A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

(i)  $f : \mathbb{R} \rightarrow \mathbb{R}$

(ii)  $g : \mathbb{R} \rightarrow \mathbb{R}$

(iii)  $f(x) = x^2 + x + 1$

(iv)  $g(x) = 1 - x^2$

$F \circ g(-2) = f(g(-2))$

$\Rightarrow F \circ g(-2) = f(1 - (-2)^2)$

$\Rightarrow F \circ g(-2) = f(-3)$

$\Rightarrow F \circ g(-2) = (-3)^2 - 3 + 1 = 7$

**25. Question**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x) = \frac{2x-3}{4}$ . Write  $f \circ f^{-1}(1)$ .

**Answer**



Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

(II) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f : A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

(i)  $f : \mathbb{R} \rightarrow \mathbb{R}$

$$(ii) f(x) = \frac{2x - 3}{4}$$

$F(y) = x$

$$\Rightarrow \frac{2y - 3}{4} = x$$

$$\Rightarrow 2y - 3 - 4x = 0$$

$$\Rightarrow y = \frac{4x + 3}{2}$$

Now

$$\Rightarrow f \circ f^{-1}(1) = f\left(\frac{7}{2}\right)$$

$$\Rightarrow f \circ f^{-1}(1) = \frac{7 - 3}{4} = 1$$

## 26. Question

Let  $f$  be an invertible real function. Write  $(f^{-1} \text{ of } (1)) + (f^{-1} \text{ of } (2)) + \dots + (f^{-1} \text{ of } (100))$ .

**Answer**

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

(II) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f : A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

(i)  $f$  be an invertible real function

$(f^{-1} \text{ of } (1)) + (f^{-1} \text{ of } (2)) + \dots + (f^{-1} \text{ of } (100))$

$$= 1 + 2 + 3 + \dots + 100$$

$$= \frac{100(100 + 1)}{2} = 5050$$

## 27. Question

Let  $A = \{1, 2, 3, 4\}$  and  $B = \{a, b\}$  be two sets. Write total number of onto functions from  $A$  to  $B$ .

**Answer**

Formula:-

(I) A function  $f: A \rightarrow B$  is onto function or surjection if

Range (f) = co-domain (f)

(II) if A and B are two non-empty finite sets containing m and n

(i) Number of function from A to B =  $n^m$

(ii) Number of one-one function from A to B =  $\begin{cases} C_m^n \cdot m!, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(iii) Number of one-one and onto function from A to B =  $\begin{cases} n!, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases}$

(iv) Number of onto function from A to B =  $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$ , if  $m \geq n$

Given:-

(i)  $A = \{1, 2, 3, 4\} = 4$

(ii)  $B = \{a, b\} = 2$

Using formula (iv)

Number of onto function from A to B =  $\sum_{r=1}^n (-1)^{n-r} C_r^n r^m$ , if  $m \geq n$

Where  $m=4, n=2$

$$\sum_{r=1}^n (-1)^{n-r} C_r^n r^m = (-1)^{2-1} C_1^2 (1)^4 + (-1)^{2-2} C_2^2 (2)^4$$

$$= -2 + 16 = 14$$

### 28. Question

Write the domain of the real function  $f(x) = \sqrt{x - [x]}$ .

**Answer**

$f(x) = \sqrt{x - [x]}$  where x is for all real number

Then,

domain = R

### 129. Question

Write the domain of the real function  $f(x) = \sqrt{[x] - x}$ .

**Answer**

$f(x) = \sqrt{[x] - x}$  where x is not for real number

Domain =  $\emptyset$

### 30. Question

Write the domain of the real function  $f(x) = \frac{1}{\sqrt{|x| - x}}$

**Answer**

$$f(x) = \frac{1}{\sqrt{|x| - x}}$$

When  $x < 0$  negative

$$\frac{1}{\sqrt{|x|} - x} = \frac{1}{\sqrt{-x} - x}$$
$$= \frac{1}{\sqrt{-2x}}$$

When  $x > 0$

$$\frac{1}{\sqrt{|x|} - x} = \frac{1}{\sqrt{x} - x} = \infty$$

Domain =  $(-\infty, 0)$

### 31. Question

Write whether  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x + \sqrt{x^2}$  is one-one, many-one, onto or into.

#### Answer

(I) A function  $f: A \rightarrow B$  is one-one function or an injection if

$$f(x) = f(y) \Rightarrow x = y \text{ for all } x, y \in A$$

$$\text{or } f(x) \neq f(y) \Rightarrow x \neq y \text{ for all } x, y \in A$$

(II) A function  $f: A \rightarrow B$  is onto function or surjection if

$$\text{Range}(f) = \text{co-domain}(f)$$

(III) A function  $f: A \rightarrow B$  is not onto function, then

$f: A \rightarrow A$  is always an onto function

Given:-

$$(i) f : \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) f(x) = x + \sqrt{x^2}$$

$$f(x) = x + \sqrt{x^2}$$

$$= x \pm x$$

$$= 0, 2x$$

Now putting  $x=0$

$$F(0) = 0 + \sqrt{0^2} = 0$$

Again putting  $x=-1$

$$F(-1) = -1 + \sqrt{-1^2} = 0$$

Hence  $f$  is many one

### 32. Question

If  $f(x) = x + 7$  and  $g(x) = x - 7$ ,  $x \in \mathbb{R}$ , write  $f \circ g(7)$ .

#### Answer

Formula:-

(i) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f : A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

$$(i) f(x) = x + 7$$

$$(ii) g(x) = x - 7, x \in \mathbb{R}$$

$$Fog(7) = f(g(7))$$

$$\Rightarrow Fog(7) = f(7-7)$$

$$\Rightarrow Fog(7) = f(0)$$

$$\Rightarrow Fog(7) = 0+7$$

$$\Rightarrow Fog(7) = 7$$

### 33. Question

What is the range of the function  $f(x) = \frac{|x-1|}{x-1}$ ?

**Answer**

$$f(x) = \frac{|x-1|}{x-1}$$

$$= \pm 1$$

Range of  $f = \{-1, 1\}$

### 34. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = (3 - x^3)^{1/3}$ , then find  $f \circ f(x)$ .

**Answer**

Formula:-

(i) Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f: A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

$$(i) f: \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) f(x) = (3 - x^3)^{1/3}$$

$$Fof(x) = f(f(x))$$

$$\Rightarrow fof(x) = f((3 - x^3)^{1/3})$$

$$\Rightarrow fof(x) = (3 - (3 - x^3)^{1/3})^{1/3}$$

$$\Rightarrow fof(x) = (x^3)^{1/3} = x$$

### 35. Question

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = 3x + 2$ , find  $f(f(x))$ .

**Answer**

Given:-

$$(i) f: \mathbb{R} \rightarrow \mathbb{R}$$

$$F(f(x))=f(3x+2)$$

$$\Rightarrow F(f(x))=3(3x+2)+2$$

$$\Rightarrow F(f(x))=9x+8$$

### 36. Question

Let  $A = \{1, 2, 3\}$ ,  $B = \{4, 5, 6, 7\}$  and let  $f = \{(1, 4), (2, 5), (3, 6)\}$  be a function from  $A$  to  $B$ . State whether  $f$  is one-one or not.

### Answer

Given:-

$$(i) A = \{1, 2, 3\}$$

$$(ii) B = \{4, 5, 6, 7\}$$

$$(iii) f = \{(1, 4), (2, 5), (3, 6)\}$$

each element has a unique image

hence,  $f$  is one-one

### 37. Question

If  $f : \{5, 6\} \rightarrow \{2, 3\}$  and  $g : \{2, 3\} \rightarrow \{5, 6\}$  are given by  $f = \{(5, 2), (6, 3)\}$  and  $g = \{(2, 5), (3, 6)\}$ , find  $f \circ g$ .

### Answer

Formula:-

(i) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Then, the composition of  $f$  and  $g$ , denoted by  $g \circ f$ , is defined as the function  $g \circ f : A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

$$(i) f : \{5, 6\} \rightarrow \{2, 3\}$$

$$(ii) g : \{2, 3\} \rightarrow \{5, 6\}$$

$$(iv) f = \{(5, 2), (6, 3)\}$$

$$(v) g = \{(2, 5), (3, 6)\}$$

for  $f \circ g(2) = f(g(2))$

$$\Rightarrow f \circ g(2) = f(5)$$

$$\Rightarrow f \circ g(2) = 2$$

### 38. Question

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by  $f(x) = 4x - 3$  for all  $x \in \mathbb{R}$ . Then write  $f^{-1}$ .

### Answer

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

Given:-

$$(i) f : \mathbb{R} \rightarrow \mathbb{R}$$

$$(ii) f(x) = 4x - 3 \text{ for all } x \in \mathbb{R}.$$

$$f(x)=y$$

$$\Rightarrow 4x-3=y$$

$$\Rightarrow x = \frac{y+3}{4}$$

$$f^{-1}(y) = x = \frac{y+3}{4}$$

$$f^{-1}(x) = \frac{x+3}{4}$$

### 39. Question

Which one the following relations on  $A = \{1, 2, 3\}$  is a function?

$$f = \{(1, 3), (2, 3), (3, 2)\}, g = \{(1, 2), (1, 3), (3, 1)\}.$$

### Answer

Given:-

$$(i) A = \{1, 2, 3\}$$

$$(ii) f = \{(1, 3), (2, 3), (3, 2)\}$$

$$(iii) g = \{(1, 2), (1, 3), (3, 1)\}.$$

In case of set A and f

Every element in A has a unique image in f

So, f is a function

In case of set A and g

Only one element has image in g

So, g is not a function

### 40. Question

Write the domain of the real function f defined by  $f(x) = \sqrt{25 - x^2}$ .

### Answer

$$f(x) = \sqrt{25 - x^2}$$

$$\Rightarrow 25 - x^2 \geq 0$$

$$\Rightarrow -(x+5)(x-5) \geq 0$$

$$\Rightarrow (x+5)(x-5) \leq 0$$

$$\Rightarrow x \leq -5 \text{ or } 5$$

$$\text{Domain} = [-5, 5]$$

### 41. Question

Let  $A = \{a, b, c, d\}$  and  $f : A \rightarrow A$  be given by  $f = \{(a, b), (b, d), (c, a), (d, c)\}$ , write  $f^{-1}$ .

### Answer

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function g is called the inverse of f and is denoted by  $f^{-1}$

(ii) A function  $f: A \rightarrow B$  is onto function or surjection if

Range (f) = co-domain(f)

Given:-

(i)  $A = \{a, b, c, d\}$

(ii)  $f: A \rightarrow A$

(iii)  $f = \{(a, b), (b, d), (c, a), (d, c)\}$

f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since  $f(A) = A$ .

$f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$ .

#### 42. Question

Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = 2x + 1$  and  $g(x) = x^2 - 2$  for all  $x \in \mathbb{R}$ , respectively. Then, find gof.

#### Answer

Formula:-

(i) Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions.

Then, the composition of f and g, denoted by  $g \circ f$ , is defined as the function  $g \circ f: A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

(i)  $f, g: \mathbb{R} \rightarrow \mathbb{R}$

(ii)  $f(x) = 2x + 1$

(ii)  $g(x) = x^2 - 2$  for all  $x \in \mathbb{R}$

$g \circ f(x) = g(f(x))$

$\Rightarrow g \circ f(x) = g(2x + 1)$

$\Rightarrow g \circ f(x) = (2x + 1)^2 - 2$

$\Rightarrow g \circ f(x) = 4x^2 + 4x - 1$

#### 43. Question

If the mapping  $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$  and  $g: \{1, 2, 5\} \rightarrow \{1, 3\}$ , given by

$f = \{(1, 2), (3, 5), (4, 1)\}$  and  $g = \{(2, 3), (5, 1), (1, 3)\}$ , write fog.

#### Answer

Formula:-

(i) Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions.

Then, the composition of f and g, denoted by  $g \circ f$ , is defined as the function  $g \circ f: A \rightarrow C$

given by  $g \circ f(x) = g(f(x))$

Given:-

(i)  $f: \{1, 3, 4\} \rightarrow \{1, 2, 5\}$

(ii)  $g: \{1, 2, 5\} \rightarrow \{1, 3\}$

(iii)  $f = \{(1, 2), (3, 5), (4, 1)\}$

(iv)  $g = \{(2, 3), (5, 1), (1, 3)\}$

$$f \circ g(1) = f(g(1)) = f(3) = 5$$

$$f \circ g(2) = f(g(2)) = f(3) = 5$$

$$f \circ g(5) = f(g(5)) = f(1) = 2$$

$$\Rightarrow f \circ g = \{(1, 5), (2, 5), (5, 2)\}$$

#### 44. Question

If a function  $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$  is described by  $g(x) = \alpha x + \beta$ , find the values of  $\alpha$  and  $\beta$ .

#### Answer

Given:-

$$(i) g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$$

$$(ii) g(x) = \alpha x + \beta$$

For  $x=1$  and  $\alpha x + \beta$

$$g(1) = \alpha(1) + \beta = 1$$

$$\Rightarrow \alpha + \beta = 1$$

For  $x=2$

$$g(2) = \alpha(2) + \beta = 3$$

$$\Rightarrow 2\alpha + \beta = 3$$

Similarly with  $g(3)$  and  $g(4)$

Using above value

$$\alpha = 2$$

$$\beta = 1$$

#### 45. Question

If  $f(x) = 4 - (x - 7)^3$ , write  $f^{-1}(x)$ .

#### Answer

Formula:-

(i) A function  $f : X \rightarrow Y$  is defined to be invertible, if there exists a function  $g : Y \rightarrow X$

such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . The function  $g$  is called the inverse of  $f$  and is denoted by  $f^{-1}$

Given:-

$$(i) f(x) = 4 - (x - 7)^3$$

Let  $f(x) = y$

$$y = 4 - (x - 7)^3$$

$$x = 7 + \sqrt[3]{4 - y}$$

$$f^{-1}(x) = 7 + \sqrt[3]{4 - x}$$

## MCQ

### 1. Question

Mark the correct alternative in each of the following:

Let  $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$ ,  $B = \mathbb{R}$  and  $C = \{x \in \mathbb{R} : x \geq 0\}$  and let  $S = \{(x, y) \in A \times B : x^2 + y^2 = 1\}$  and  $S_0 =$



$\{(x, y) \in A \times C : x^2 + y^2 = 1\}$  Then

- A. S defines a function from A to B
- B.  $S_0$  defines a function from A to C
- C.  $S_0$  defines a function from A to B
- D. S defines a function from A to C

**Answer**

Given that

$$A = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = B$$

$$C = \{x \in \mathbb{R} : x \geq 0\}$$

$$S = \{(x, y) \in A \times B : x^2 + y^2 = 1\}$$

$$S_0 = \{(x, y) \in A \times C : x^2 + y^2 = 1\}$$

$$x^2 + y^2 = 1$$

$$\Rightarrow y^2 = 1 - x^2$$

$$\Rightarrow y = \sqrt{1 - x^2}$$

$$\therefore y \in B$$

Hence, S defines a function from A to B.

**2. Question**

Mark the correct alternative in each of the following:

$f: \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x + \sqrt{x^2}$  is

- A. injective B. surjective
- C. bijective D. none of these

**Answer**

Given function is  $f: \mathbb{R} \rightarrow \mathbb{R}$  given

$$f(x) = x + \sqrt{x^2}$$

For this function if we take  $x = 2$ ,

$$f(x) = 2 + \sqrt{4}$$

$$\Rightarrow f(x) = 4$$

For this function if we take  $x = -2$ ,

$$f(x) = -2 + \sqrt{4}$$

$$\Rightarrow f(x) = 0$$

So, in general for every negative  $x$ ,  $f(x)$  will be always 0. There is no  $x \in \mathbb{R}$  for which  $f(x) \in (-\infty, 0)$ .

Hence, it is neither injective nor surjective and so it is not bijective either.

**3. Question**

Mark the correct alternative in each of the following:

If  $f: A \rightarrow B$  given by  $3^{f(x)} + 2^{-x} = 4$  is a bijection, then

- A.  $A = \{x \in \mathbb{R} : -1 < x < \infty\}$ ,  $B = \{x \in \mathbb{R} : 2 < x < 4\}$

B.  $A = \{x \in \mathbb{R} : -3 < x < \infty\}$ ,  $B = \{x \in \mathbb{R} : 0 < x < 4\}$

C.  $A = \{x \in \mathbb{R} : -2 < x < \infty\}$ ,  $B = \{x \in \mathbb{R} : 0 < x < 4\}$

D. none of these

**Answer**

Given that  $f: A \rightarrow B$  given by  $3^{f(x)} + 2^{-x} = 4$  is a bijection.

$$3^{f(x)} + 2^{-x} = 4$$

$$\Rightarrow 3^{f(x)} = 4 - 2^{-x}$$

$$\Rightarrow 4 - 2^{-x} \geq 0$$

$$\Rightarrow 4 \geq 2^{-x}$$

$$\Rightarrow 2 \geq -x$$

$$\Rightarrow x \geq -2$$

So,  $x \in (-2, \infty)$

But, for  $x = 0$ ,  $f(x) = 1$ .

Hence, the correct option is none of these.

**4. Question**

Mark the correct alternative in each of the following:

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 2^x + 2^{|x|}$  is

A. one-one and onto

B. many-one and onto

C. one-one and into

D. many-one and into

**Answer**

Given that  $f: \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = 2^x + 2^{|x|}$

Here, for each value of  $x$  we will get different value of  $f(x)$ .

So, it is one-one.

Also,  $f(x)$  is always positive for  $x \in \mathbb{R}$ .

There is no  $x \in \mathbb{R}$  for which  $f(x) \in (-\infty, 0)$ .

So, it is into.

Hence, the given function is one-one and into.

**5. Question**

Mark the correct alternative in each of the following:

Let the function  $f: \mathbb{R} - \{-b\} \rightarrow \mathbb{R} - \{1\}$  be defined by  $f(x) = \frac{x+a}{x+b}$ ,  $a \neq b$ , then

A.  $f$  is one-one but not onto

B.  $f$  is onto but not one-one

C.  $f$  is both one-one and onto

D. none of these

**Answer**

Given that  $f: \mathbb{R} - \{-b\} \rightarrow \mathbb{R} - \{1\}$  where

$$f(x) = \frac{x + a}{x + b}, a \neq b.$$

Here,  $f(x) = f(y)$  only when  $x=y$ .

Hence, it is one-one.

Now,  $f(x) = y$

$$\Rightarrow \frac{x + a}{x + b} = y$$

$$\Rightarrow x + a = y(x + b)$$

$$\Rightarrow x - yx = yb - a$$

$$\Rightarrow x = \frac{yb - a}{1 - y}, y \neq 1$$

So,  $x \in \mathbb{R} - \{1\}$

Hence, it is onto.

**6. Question**

Mark the correct alternative in each of the following:

The function  $f: A \rightarrow B$  defined by  $f(x) = -x^2 + 6x - 8$  is a bijection, if

A.  $A = (-\infty, 5]$  and  $B = (-\infty, 1]$

B.  $A = [-3, \infty]$  and  $B = (-\infty, 1]$

C.  $A = (-\infty, 3]$  and  $B = [1, \infty)$

D.  $A = [3, \infty)$  and  $B = [1, \infty)$

**Answer**

Given that  $f: A \rightarrow B$  defined by  $f(x) = -x^2 + 6x - 8$  is a bijection.

$$f(x) = -x^2 + 6x - 8$$

$$\Rightarrow f(x) = -(x^2 - 6x + 8)$$

$$\Rightarrow f(x) = -(x^2 - 6x + 8 + 1 - 1)$$

$$\Rightarrow f(x) = -(x^2 - 6x + 9 - 1)$$

$$\Rightarrow f(x) = -[(x - 3)^2 - 1]$$

Hence,  $x \in (-\infty, 5]$  and  $f(x) \in (-\infty, 1]$

**7. Question**

Mark the correct alternative in each of the following:

Let  $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = B$ . Then, the mapping  $f: A \rightarrow B$  given by  $f(x) = x|x|$  is

A. injective but not surjective

B. surjective but not injective

C. bijective

D. none of these

**Answer**

Given that  $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\} = B$ . Then, the mapping  $f: A \rightarrow B$  given by  $f(x) = x|x|$ .

For  $x < 0$ ,  $f(x) < 0$

$$\Rightarrow y = -x^2$$

$\Rightarrow x = \sqrt{-y}$ , which is not possible for  $x > 0$ .

Hence,  $f$  is one-one and onto.

$\therefore$  the given function is bijective.

### 8. Question

Mark the correct alternative in each of the following:

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = [x]^2 + [x + 1] - 3$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ . Then,  $f(x)$  is

- A. many-one and onto
- B. many-one and into
- C. one-one and into
- D. one-one and onto

### Answer

Given that  $f: \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = [x]^2 + [x + 1] - 3$

As  $[x]$  is the greatest integer so for different values of  $x$ , we will get same value of  $f(x)$ .

$[x]^2 + [x + 1]$  will always be an integer.

So,  $f$  is many-one.

Similarly, in this function co domain is mapped with at most one element of domain because for every  $x \in \mathbb{R}$ ,  $f(x) \in \mathbb{Z}$ .

So,  $f$  is into.

### 9. Question

Mark the correct alternative in each of the following:

Let  $M$  be the set of all  $2 \times 2$  matrices with entries from the set  $\mathbb{R}$  of real numbers. Then the function  $f: M \rightarrow \mathbb{R}$  defined by  $f(A) = |A|$  for every  $A \in M$ , is

- A. one-one and onto
- B. neither one-one nor onto
- C. one-one not one-one
- D. onto but not one-one

### Answer

Given that  $M$  is the set of all  $2 \times 2$  matrices with entries from the set  $\mathbb{R}$  of real numbers. Then the function  $f: M \rightarrow \mathbb{R}$  defined by  $f(A) = |A|$  for every  $A \in M$ .

If  $f(a) = f(b)$

$$\Rightarrow |a| = |b|$$

But this does not mean that  $a=b$ .

So,  $f$  is not one-one.

As  $a \neq b$  but  $|a|=|b|$

So,  $f$  is onto.

### 10. Question

Mark the correct alternative in each of the following:

The function  $f : [0, \infty) \rightarrow \mathbb{R}$  given by  $f(x) = \frac{x}{x+1}$  is

- A. one-one and onto
- B. one-one but not onto
- C. onto but not one-one
- D. neither one-one nor onto

### Answer

Given that  $f : [0, \infty) \rightarrow \mathbb{R}$  where  $f(x) = \frac{x}{x+1}$

Let  $f(x) = f(y)$

$$\Rightarrow \frac{x}{x+1} = \frac{y}{y+1}$$

$$\Rightarrow xy + x = xy + y$$

$$\Rightarrow x = y$$

So,  $f$  is one-one.

Now,  $y = f(x)$

$$\Rightarrow y = \frac{x}{x+1}$$

$$\Rightarrow xy + y = x$$

$$\Rightarrow y = x - xy$$

$$\Rightarrow \frac{y}{1-y} = x$$

Here,  $y \neq 1$  i.e.  $y \in \mathbb{R}$ .

So,  $f$  is not onto.

### 11. Question

Mark the correct alternative in each of the following:

The range of the function  $f(x) = {}^{7-x}P_{x-3}$  is

- A. {1, 2, 3, 4, 5}
- B. {1, 2, 3, 4, 5, 6}
- C. {1, 2, 3, 4}
- D. {1, 2, 3}

### Answer

Given that  $f(x) = {}^{7-x}P_{x-3}$

Here,  $7-x \geq x-3$

$$\Rightarrow 10 \geq 2x$$

$$\Rightarrow 5 \geq x$$

So, domain = {3, 4, 5}

$$\text{Range} = \{ {}^4P_0, {}^3P_1, {}^2P_2 \} = \{ 1, 3, 2 \}$$

## 12. Question

Mark the correct alternative in each of the following:

A function  $f$  from the set on natural numbers to integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases} \text{ is}$$

- A. neither one-one nor onto
- B. one-one but not onto
- C. onto but not one-one
- D. one-one and onto both

## Answer

Given that a function  $f$  from the set on natural numbers to integers where

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

For  $n$  is odd

Let  $f(n) = f(m)$

$$\Rightarrow \frac{n-1}{2} = \frac{m-1}{2}$$

$$\Rightarrow n = m$$

For  $n$  is even

Let  $f(n) = f(m)$

$$\Rightarrow \frac{-n}{2} = \frac{-m}{2}$$

$$\Rightarrow n = m$$

So,  $f$  is one-one.

Also, each element of  $y$  is associated with at least one element of  $x$ , so  $f$  is onto.

Hence,  $f$  is one-one and onto.

## 13. Question

Mark the correct alternative in each of the following:

Let  $f$  be an injective map with domain  $\{x, y, z\}$  and range  $\{1, 2, 3\}$  such that exactly one of the following statements is correct and the remaining are false.

$$f(x) = 1, f(y) \neq 1, f(z) \neq 2.$$

The value of  $f^{-1}(1)$  is

- A.  $x$
- B.  $y$
- C.  $z$

D. none of these

**Answer**

Given that  $f$  is an injective map with domain  $\{x, y, z\}$  and range  $\{1, 2, 3\}$ .

Case-1

Let us assume that  $f(x) = 1$  is true and  $f(y) \neq 1, f(z) \neq 2$  is false.

Then  $f(x) = 1, f(y) = 1$  and  $f(z) = 2$ .

This violates the injectivity of  $f$  because it is one-one.

Case-2

Let us assume that  $f(y) \neq 1$  is true and  $f(x) = 1, f(z) \neq 2$  is false.

Then  $f(x) \neq 1, f(y) \neq 1$  and  $f(z) = 2$ .

This means there is no pre image of 1 which contradicts the fact that the range of  $f$  is  $\{1, 2, 3\}$ .

Case-3

Let us assume that  $f(z) \neq 2$  is true and  $f(x) = 1, f(y) \neq 1$  is false.

Then  $f(z) \neq 2, f(y) = 1$  and  $f(x) \neq 1$ .

$$\Rightarrow f^{-1}(1) = y$$

**14. Question**

Mark the correct alternative in each of the following:

Which of the following functions from  $Z$  to itself are bijections?

A.  $f(x) = x^3$

B.  $f(x) = x + 2$

C.  $f(x) = 2x + 1$

D.  $f(x) = x^2 + x$

**Answer**

a.  $f(x) = x^3$

$\Rightarrow$  For no value of  $x \in Z, f(x) = 2$ .

Hence, it is not bijection.

b.  $f(x) = x + 2$

If  $f(x) = f(y)$

$$\Rightarrow x + 2 = y + 2$$

$$\Rightarrow x = y$$

So,  $f$  is one-one.

Also,  $y = x + 2$

$$\Rightarrow x = y - 2 \in Z$$

So,  $f$  is onto.

Hence, this function is bijection.

c.  $f(x) = 2x + 1$

If  $f(x) = f(y)$

$$\Rightarrow 2x + 1 = 2y + 1$$

$$\Rightarrow x = y$$

So,  $f$  is one-one.

$$\text{Also, } y = 2x + 1$$

$$\Rightarrow 2x = y - 1$$

$$\Rightarrow x = \frac{y-1}{2}$$

So,  $f$  is into because  $x$  can never be odd for any value of  $y$ .

$$\text{d. } f(x) = x^2 + x$$

For this function if we take  $x = 2$ ,

$$f(x) = 4 + 2$$

$$\Rightarrow f(x) = 6$$

For this function if we take  $x = -2$ ,

$$f(x) = 4 - 2$$

$$\Rightarrow f(x) = 2$$

So, in general for every negative  $x$ ,  $f(x)$  will be always 0. There is no  $x \in \mathbb{R}$  for which  $f(x) \in (-\infty, 0)$ .

It is not bijection.

### 15. Question

Mark the correct alternative in each of the following:

Which of the following functions from  $A = \{x : -1 \leq x \leq 1\}$  to itself are bijections?

A.  $f(x) = \frac{x}{2}$

B.  $g(x) = \sin\left(\frac{\pi x}{2}\right)$

C.  $h(x) = |x|$

D.  $k(x) = x^2$

### Answer

Given that  $A = \{x : -1 \leq x \leq 1\}$

a.  $f(x) = \frac{x}{2}$

It is one-one but not onto.

b.  $g(x) = \sin\left(\frac{\pi x}{2}\right)$

It is bijective as it is one-one and onto with range  $[-1, 1]$ .

c.  $h(x) = |x|$

It is not one-one because  $h(-1)=1$  and  $h(1)=1$ .

d.  $k(x) = x^2$

It is not one-one because  $k(-1)=1$  and  $k(1)=1$ .

### 16. Question

Mark the correct alternative in each of the following:



Let  $A = \{x : -1 \leq x \leq 1\}$  and  $f : A \rightarrow A$  such that  $f(x) = x|x|$ , then  $f$  is

- A. a bijection
- B. injective but not surjective
- C. surjective but not injective
- D. neither injective nor surjective

**Answer**

Given that  $A = \{x : -1 \leq x \leq 1\}$  and  $f : A \rightarrow A$  such that  $f(x) = x|x|$ .

For  $x < 0$ ,  $f(x) < 0$

$$\Rightarrow y = -x^2$$

$$\Rightarrow x = \sqrt{-y}, \text{ which is not possible for } x > 0.$$

Hence,  $f$  is one-one and onto.

$\therefore$  the given function is bijective.

**17. Question**

Mark the correct alternative in each of the following:

If the function  $f : \mathbb{R} \rightarrow A$  given by  $f(x) = \frac{x^2}{x^2 + 1}$  is a surjection, then  $A =$

- A.  $\mathbb{R}$
- B.  $[0, 1]$
- C.  $(0, 1]$
- D.  $[0, 1)$

**Answer**

Given that  $f : \mathbb{R} \rightarrow A$  such that  $f(x) = \frac{x^2}{x^2 + 1}$  is a surjection.

$$f(x) = y$$

$$\Rightarrow y = \frac{x^2}{x^2 + 1}$$

$$\Rightarrow y(x^2 + 1) = x^2$$

$$\Rightarrow yx^2 + y = x^2$$

$$\Rightarrow yx^2 - x^2 = -y$$

$$\Rightarrow x^2 = \frac{y}{1 - y}$$

$$\Rightarrow x = \sqrt{\frac{y}{1 - y}}$$

Here,  $\frac{y}{1 - y} \geq 0$

So,  $y \in [0, 1)$

**18. Question**

Mark the correct alternative in each of the following:

If a function  $f : [2, \infty) \rightarrow B$  defined by  $f(x) = x^2 - 4x + 5$  is a bijection, then  $B =$

- A.  $\mathbb{R}$
- B.  $[1, \infty)$
- C.  $[4, \infty)$
- D.  $[5, \infty)$

**Answer**

Given that a function  $f : [2, \infty) \rightarrow B$  defined by  $f(x) = x^2 - 4x + 5$  is a bijection.

Put  $x = 2$  in  $f(x)$ ,

$$f(x) = 2^2 - 4 \times 2 + 5$$

$$\Rightarrow f(x=2) = 4 - 8 + 5$$

$$\Rightarrow f(x=2) = 1$$

So,  $B \in [1, \infty)$

**19. Question**

Mark the correct alternative in each of the following:

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = (x - 1)(x - 2)(x - 3)$  is

- A. one-one but not onto
- B. onto but not one-one
- C. both one and onto
- D. neither one-one nor onto

**Answer**

Given that function  $f : \mathbb{R} \rightarrow \mathbb{R}$  where  $f(x) = (x - 1)(x - 2)(x - 3)$

If  $f(x) = f(y)$

Then

$$(x - 1)(x - 2)(x - 3) = (y - 1)(y - 2)(y - 3)$$

$$\Rightarrow f(1) = f(2) = f(3) = 0$$

So,  $f$  is not one-one.

$$y = f(x)$$

$\therefore x \in \mathbb{R}$  also  $y \in \mathbb{R}$  so  $f$  is onto.

**20. Question**

Mark the correct alternative in each of the following:

The function  $f : [-1/2, 1/2] \rightarrow [\pi/2, \pi/2]$  defined by  $f(x) = \sin^{-1}(3x - 4x^3)$  is

- A. bijection
- B. injection but not a surjection
- C. surjection but not an injection
- D. neither an injection nor a surjection

**Answer**

Given that  $f : [-1/2, 1/2] \rightarrow [\pi/2, \pi/2]$  where  $f(x) = \sin^{-1}(3x - 4x^3)$

Put  $x = \sin\theta$  in  $f(x) = \sin^{-1}(3x - 4x^3)$

$$\Rightarrow f(x=\sin\theta) = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$$

$$\Rightarrow f(x) = \sin^{-1}(\sin 3\theta)$$

$$\Rightarrow f(x) = 3\theta$$

$$\Rightarrow f(x) = 3 \sin^{-1}x$$

If  $f(x) = f(y)$

Then

$$3 \sin^{-1}x = 3 \sin^{-1}y$$

$$\Rightarrow x = y$$

So,  $f$  is one-one.

$$y = 3 \sin^{-1}x$$

$$\Rightarrow x = \sin \frac{y}{3}$$

$\therefore x \in \mathbb{R}$  also  $y \in \mathbb{R}$  so  $f$  is onto.

Hence,  $f$  is bijection.

### 21. Question

Mark the correct alternative in each of the following:

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by  $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$ . Then,

- A.  $f$  is a bijection
- B.  $f$  is an injection only
- C.  $f$  is surjection on only
- D.  $f$  is neither an injection nor a surjection

### Answer

Given that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a function defined as

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$$

Here,  $e^{|x|}$  is always positive whether  $x$  is negative or positive. So, we will get same values of  $f(x)$  for different values of  $x$ .

Hence, it is not one-one and onto.

$\therefore f$  is neither an injection nor a surjection

### 22. Question

Mark the correct alternative in each of the following:

Let  $f: \mathbb{R} - \{n\} \rightarrow \mathbb{R}$  be a function defined by  $f(x) = \frac{x - m}{x - n}$ , where  $m \neq n$ . Then,

- A.  $f$  is one-one onto
- B.  $f$  is one-one into
- C.  $f$  is many one onto

D.  $f$  is many one into

**Answer**

Given that  $f : \mathbb{R} - \{n\} \rightarrow \mathbb{R}$  where

$$f(x) = \frac{x - m}{x - n}, \text{ such that } m \neq n$$

Let  $f(x) = f(y)$

$$\Rightarrow \frac{x - m}{x - n} = \frac{y - m}{y - n}$$

$$\Rightarrow (x - m)(y - n) = (x - n)(y - m)$$

$$\Rightarrow xy - xn - my + mn = xy - xm - ny + mn$$

$$\Rightarrow x = y$$

So,  $f$  is one-one.

$$f(x) = \frac{x - m}{x - n}$$

$$\Rightarrow y = \frac{x - m}{x - n}$$

$$\Rightarrow y(x - n) = (x - m)$$

$$\Rightarrow xy - ny = x - m$$

$$\Rightarrow x(y - 1) = ny - m$$

$$\Rightarrow x = \frac{ny - m}{y - 1}, y \neq 1$$

For  $y = 1$ , no  $x$  is defined.

So,  $f$  is into.

**23. Question**

Mark the correct alternative in each of the following:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by  $f(x) = \frac{x^2 - 8}{x^2 + 2}$ . Then,  $f$  is

A. one-one but not onto

B. one-one and onto

C. onto but not one-one

D. neither one-one nor onto

**Answer**

Given that  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function where

$$f(x) = \frac{x^2 - 8}{x^2 + 2}$$

Here, we can see that for negative as well as positive  $x$  we will get same value.

So, it is not one-one.

$$y = f(x)$$

$$\Rightarrow y = \frac{x^2 - 8}{x^2 + 2}$$

$$\Rightarrow y(x^2 + 2) = (x^2 - 8)$$

$$\Rightarrow x^2(y-1) = -2y - 8$$

$$\Rightarrow x = \sqrt{\frac{2y + 8}{1 - y}}$$

For  $y = 1$ , no  $x$  is defined.

So,  $f$  is not onto.

#### 24. Question

Mark the correct alternative in each of the following:

$$f : \mathbb{R} \rightarrow \mathbb{R} \text{ is defined by } f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}} \text{ is}$$

- A. one-one but not onto
- B. one-one and onto
- C. onto but not one-one
- D. neither one-one nor onto

#### Answer

Given that  $f : \mathbb{R} \rightarrow \mathbb{R}$  where

$$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$$

Here, we can see that for negative as well as positive  $x$  we will get same value.

So, it is not one-one.

$$f(x) = y$$

By definition of onto, each element of  $y$  is not mapped to at least one element of  $x$ .

So, it is not onto.

#### 25. Question

Mark the correct alternative in each of the following:

The function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$  is

- A. injective but not surjective
- B. surjective but not injective
- C. injective as well as surjective
- D. neither injective nor surjective

#### Answer

Given that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$

Let  $f(x) = y(x)$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = \pm y$$

So, it is not one-one.

$$f(x) = y$$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y}$$

But co domain is R.

Hence, f is neither injective nor surjective.

### 26. Question

Mark the correct alternative in each of the following:

A function f from the set of natural, numbers to the set of integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

- A. neither one-one nor onto
- B. one-one but not onto
- C. onto but not one-one
- D. one-one and onto both

### Answer

Given that a function f from the set on natural numbers to integers where

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

For n is odd

$$\text{Let } f(n) = f(m)$$

$$\Rightarrow \frac{n-1}{2} = \frac{m-1}{2}$$

$$\Rightarrow n = m$$

For n is even

$$\text{Let } f(n) = f(m)$$

$$\Rightarrow \frac{-n}{2} = \frac{-m}{2}$$

$$\Rightarrow n = m$$

So, f is one-one.

Also, each element of y is associated with at least one element of x, so f is onto.

Hence, f is one-one and onto.

### 27. Question

Mark the correct alternative in each of the following:

Which of the following functions from  $A = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$  to itself are bijections?

A.  $f(x) = |x|$

B.  $f(x) = \sin \frac{\pi x}{2}$

C.  $f(x) = \sin \frac{\pi x}{4}$

D. none of these

**Answer**

Given that  $A = \{x : -1 \leq x \leq 1\}$

a.  $f(x) = |x|$

It is not one-one because  $f(-1) = 1$  and  $f(1) = 1$ .

b.  $f(x) = \sin\left(\frac{\pi x}{2}\right)$

It is bijective as it is one-one and onto with range  $[-1, 1]$ .

**28. Question**

Mark the correct alternative in each of the following:

Let  $f: Z \rightarrow Z$  be given by  $f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even} \\ 0, & \text{if } x \text{ is odd} \end{cases}$ . Then,  $f$  is

- A. onto but not one-one
- B. one-one but not onto
- C. one-one and onto
- D. neither one-one nor onto

**Answer**

Given function  $f: Z \rightarrow Z$  defined as

$$f(x) = \begin{cases} \frac{x}{2}, & \text{when } x \text{ is even} \\ 0, & \text{when } x \text{ is odd} \end{cases}$$

For  $x = 3$ ,  $f(x) = 0$

For  $x = 5$ ,  $f(x) = 0$

But  $3 \neq 5$

So,  $f$  is not one-one.

$Y = f(x)$

$\therefore x \in R \Rightarrow y \in R$

$\therefore \text{Domain} = \text{Range}$

Hence,  $f$  is not one-one but onto.

**29. Question**

Mark the correct alternative in each of the following:

The function  $f: R \rightarrow R$  defined by  $f(x) = 6^x + 6^{|x|}$  is

- A. one-one and onto
- B. many one and onto

- C. one-one and into
- D. many one and into

**Answer**

Given that function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 6^x + 6^{|x|}$

Let  $f(x) = f(y)$

$$\Rightarrow 6^x + 6^{|x|} = 6^y + 6^{|y|}$$

Only when  $x = y$

So,  $f$  is one-one.

Now for  $y=f(x)$

$y$  can never be negative which means for no  $x \in \mathbb{R}$   $y$  is negative.

So,  $f$  is not onto but into.

**30. Question**

Mark the correct alternative in each of the following:

Let  $f(x) = x^2$  and  $g(x) = 2^x$ . Then the solution set of the equation  $f \circ g(x) = g \circ f(x)$  is

- A.  $\mathbb{R}$
- B.  $\{0\}$
- C.  $\{0, 2\}$
- D. none of these

**Answer**

Given that  $f(x) = x^2$  and  $g(x) = 2^x$ .

Also,  $f \circ g(x) = g \circ f(x)$

$$\Rightarrow f(2^x) = g(x^2)$$

$$\Rightarrow 2^{2^x} = 2^{x^2}$$

$$\Rightarrow 2x = x^2$$

$$\Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2$$

**31. Question**

Mark the correct alternative in each of the following:

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = 3x - 5$ , then  $f^{-1}(x)$

A. is given by  $\frac{1}{3x-5}$

B. is given by  $\frac{x+5}{3}$

C. does not exist because  $f$  is not one-one

D. does not exist because  $f$  is not onto

**Answer**



Given that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = 3x - 5$

To find  $f^{-1}(x)$ :

$$y = f(x)$$

$$\Rightarrow y = 3x - 5$$

$$\Rightarrow y + 5 = 3x$$

$$\Rightarrow x = \frac{y + 5}{3}$$

$$\Rightarrow f^{-1}(x) = \frac{x + 5}{3}$$

### 32. Question

Mark the correct alternative in each of the following:

If  $g(f(x)) = |\sin x|$  and  $f(g(x)) = (\sin \sqrt{x})^2$ , then

A.  $f(x) = \sin^2 x, g(x) = \sqrt{x}$

B.  $f(x) = \sin x, g(x) = |x|$

C.  $f(x) = x^2, g(x) = \sin \sqrt{x}$

D.  $f$  and  $g$  cannot be determined

### Answer

Given that  $g(f(x)) = |\sin x|$  and  $f(g(x)) = (\sin \sqrt{x})^2$

a. For  $f(x) = \sin^2 x, g(x) = \sqrt{x}$

$$f(g(x)) = f(\sqrt{x}) = (\sin \sqrt{x})^2$$

$$g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$$

Correct

b. For  $f(x) = \sin x, g(x) = |x|$

$$f(g(x)) = f(|x|) = \sin |x|$$

$$g(f(x)) = g(\sin x) = |\sin x|$$

Incorrect

c.  $f(x) = x^2, g(x) = \sin \sqrt{x}$

$$f(g(x)) = f(\sin \sqrt{x}) = (\sin \sqrt{x})^2$$

$$g(f(x)) = g(x^2) = \sin |x|$$

Incorrect

### 33. Question

Mark the correct alternative in each of the following:

The inverse of the function  $f: \mathbb{R} \rightarrow [x \in \mathbb{R} : x < 1]$  given by  $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ , is

A.  $\frac{1}{2} \log \frac{1+x}{1-x}$

B.  $\frac{1}{2} \log \frac{2+x}{2-x}$

C.  $\frac{1}{2} \log \frac{1-x}{1+x}$

D. none of these

**Answer**

Given that  $f: \mathbb{R} \rightarrow [x \in \mathbb{R} : x < 1]$  defined by

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Put  $y = f(x)$

$$\Rightarrow y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\Rightarrow y = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow y(e^{2x} + 1) = e^{2x} - 1$$

$$\Rightarrow e^{2x}(y - 1) = -y - 1$$

$$\Rightarrow e^{2x} = \frac{y + 1}{1 - y}$$

$$\Rightarrow 2x = \log\left(\frac{y+1}{1-y}\right)$$

$$\Rightarrow x = \frac{1}{2} \log\left(\frac{y+1}{1-y}\right)$$

So,  $f^{-1}(x) = \frac{1}{2} \log\left(\frac{x+1}{1-x}\right)$

**34. Question**

Mark the correct alternative in each of the following:

Let  $A = \{x \in \mathbb{R} : x \geq 1\}$ . The inverse of the function  $f: A \rightarrow A$  given by  $f(x) = 2^{x(x-1)}$ , is

A.  $\left(\frac{1}{2}\right)^{x(x-1)}$

B.  $\frac{1}{2} \left\{ 1 + \sqrt{1 + 4 \log_2 x} \right\}$

C.  $\frac{1}{2} \left\{ 1 - \sqrt{1 + 4 \log_2 x} \right\}$

D. not defined

**Answer**

Given that  $A = \{x \in \mathbb{R} : x \geq 1\}$ . The function  $f: A \rightarrow A$  given by  $f(x) = 2^{x(x-1)}$

Put  $y = f(x)$

$$\Rightarrow y = 2^{x(x-1)}$$

$$\Rightarrow \log_2 y = x(x-1)$$

$$\Rightarrow \log_2 y = x^2 - x$$

$$\Rightarrow \log_2 y + \frac{1}{4} = x^2 - x + \frac{1}{4}$$

$$\Rightarrow \log_2 y + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2$$

$$\Rightarrow \sqrt{\frac{4\log_2 y + 1}{4}} + \frac{1}{2} = x$$

$$\Rightarrow \frac{1 + \sqrt{4\log_2 y + 1}}{2} = x$$

$$f^{-1}(x) = \frac{1 + \sqrt{4\log_2 x + 1}}{2}$$

### 35. Question

Mark the correct alternative in each of the following:

Let  $A = \{x \in \mathbb{R} : x \leq 1\}$  and  $f : A \rightarrow A$  given by  $f(x) = x(2 - x)$ . Then,  $f^{-1}(x)$  is

A.  $1 + \sqrt{1 - x}$

B.  $1 - \sqrt{1 - x}$

C.  $\sqrt{1 - x}$

D.  $1 \pm \sqrt{1 - x}$

### Answer

Given that  $A = \{x \in \mathbb{R} : x \leq 1\}$  and  $f : A \rightarrow A$  given by  $f(x) = x(2 - x)$ .

$$y = f(x)$$

$$\Rightarrow y = x(2 - x)$$

$$\Rightarrow y = 2x - x^2$$

$$\Rightarrow y - 1 = 2x - x^2 - 1$$

$$\Rightarrow y - 1 = -(x^2 + 1 - 2x)$$

$$\Rightarrow (x - 1)^2 = 1 - y$$

$$\Rightarrow x = 1 - \sqrt{1 - y}$$

$$f^{-1}(x) = 1 - \sqrt{1 - x}$$

### 36. Question

Mark the correct alternative in each of the following:

Let  $f(x) = \frac{1}{1-x}$ . Then,  $\{f \circ (f \circ f)\}(x)$

A.  $x$  for all  $x \in \mathbb{R}$

B.  $x$  for all  $x \in \mathbb{R} - \{1\}$

C.  $x$  for all  $x \in \mathbb{R} - \{0, 1\}$

D. none of these

### Answer

Given that  $f(x) = \frac{1}{1-x}$

$fof(x) = f\left(\frac{1}{1-x}\right)$ , for  $x \neq 1$

$$\Rightarrow fof = \frac{1}{1 - \frac{1}{1-x}}$$

$$\Rightarrow fof = \frac{1-x}{1-x-1}$$

$$\Rightarrow fof = \frac{x-1}{x}$$

$fofof(x) = f\left(\frac{x-1}{x}\right)$ , for  $x \neq 0$

$$\Rightarrow fofof = \frac{1}{1 - \frac{x-1}{x}}$$

$$\Rightarrow fofof = \frac{x}{x-x+1}$$

$\Rightarrow fofof = x$  for all  $x \in \mathbb{R} - \{0, 1\}$

### 37. Question

Mark the correct alternative in each of the following:

If the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be such that  $f(x) = x - [x]$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ , then  $f^{-1}(x)$  is

A.  $\frac{1}{x - [x]}$

B.  $[x] - x$

C. not defined

D. none of these

### Answer

Given that  $f: \mathbb{R} \rightarrow \mathbb{R}$  be such that  $f(x) = x - [x]$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$

We will have same value of  $f$  for different values of  $x$ .

So, the function is not one-one.

$\therefore f$  is not bijective

$\therefore f$  does not have inverse.

### 38. Question

Mark the correct alternative in each of the following:

If  $F: [1, \infty) \rightarrow [2, \infty)$  is given by  $f(x) = x + \frac{1}{x}$ , then  $f^{-1}(x)$  equals.

A.  $\frac{x + \sqrt{x^2 - 4}}{2}$

B.  $\frac{x}{1+x^2}$

$$C. \frac{x - \sqrt{x^2 - 4}}{2}$$

$$D. 1 + \sqrt{x^2 - 4}$$

**Answer**

Given that  $f: [1, \infty) \rightarrow [2, \infty)$  defined as

$$f(x) = x + \frac{1}{x}$$

$$y = f(x)$$

$$\Rightarrow y = x + \frac{1}{x}$$

$$\Rightarrow y = \frac{x^2 + 1}{x}$$

$$\Rightarrow xy = x^2 + 1$$

$$\Rightarrow x^2 - xy + \frac{y^2}{4} = \frac{y^2}{4} - 1$$

$$\Rightarrow \left(x - \frac{y}{2}\right)^2 = \frac{y^2}{4} - 1$$

$$\Rightarrow x = \frac{y}{2} + \sqrt{\frac{y^2 - 4}{4}}$$

$$\Rightarrow x = \frac{y}{2} + \frac{1}{2}\sqrt{y^2 - 4}$$

$$f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

**39. Question**

Mark the correct alternative in each of the following:

$$\text{Let } g(x) = 1 + x - [x] \text{ and } f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases} \text{, where } [x] \text{ denotes the greatest integer less than or equal to } x.$$

Then for all  $x$ ,  $f(g(x))$  is equal to

A.  $x$

B.  $1$

C.  $f(x)$

D.  $g(x)$

**Answer**

Given that  $g(x) = 1 + x - [x]$  and

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

where  $[x]$  denotes the greatest integer less than or equal to  $x$ .

(i)  $-1 < x < 0$

$$g(x) = 1 + x - [x]$$

$$\Rightarrow g(x) = 1 + x + 1 \{ \because [x] = -1 \}$$

$$\Rightarrow g(x) = 2 + x$$

$$f(g(x)) = f(2 + x)$$

$$\Rightarrow f(g(x)) = 1 + 2 + x - [2 + x]$$

$$\Rightarrow f(g(x)) = 3 + x - 2 - x$$

$$\Rightarrow f(g(x)) = 1$$

$$(ii) x = 0$$

$$f(g(x)) = f(1 + x - [x])$$

$$\Rightarrow f(g(x)) = 1 + 1 + x - [x] - [1 + x + [x]]$$

$$\Rightarrow f(g(x)) = 2 + 0 - 1$$

$$\Rightarrow f(g(x)) = 1$$

$$(iii) x > 1$$

$$f(g(x)) = f(1 + x - [x])$$

$$\Rightarrow f(g(x)) = f(x > 0) = 1$$

Hence,  $f(g(x)) = 1$  for all cases.

#### 40. Question

Mark the correct alternative in each of the following:

Let  $f(x) = \frac{\alpha x}{x+1}$ ,  $x \neq -1$ . Then, for what value of  $\alpha$  is  $f(f(x)) = x$ ?

A.  $\sqrt{2}$

B.  $-\sqrt{2}$

C. 1

D. -1

#### Answer

Given that  $f(x) = \frac{\alpha x}{x+1}$ ,  $x \neq -1$  and  $f(f(x)) = x$

$$\Rightarrow f\left(\frac{\alpha x}{x+1}\right) = x$$

$$\Rightarrow \frac{\alpha \frac{\alpha x}{x+1}}{\frac{\alpha x}{x+1} + 1} = x$$

$$\Rightarrow \frac{\alpha^2 x}{x+1} = x \left( \frac{\alpha x}{x+1} + 1 \right)$$

$$\Rightarrow \alpha^2 x = x(\alpha x + x + 1)$$

$$\Rightarrow \alpha^2 = \alpha x + x + 1$$

$$\Rightarrow \alpha^2 - \alpha x = x + 1$$

On comparing  $-\alpha x$  with  $x$ ,

We get  $\alpha = -1$

**41. Question**

Mark the correct alternative in each of the following:

The distinct linear functions which map  $[-1, 1]$  onto  $[0, 2]$  are

A.  $f(x) = x + 1, g(x) = -x + 1$

B.  $f(x) = x - 1, g(x) = x + 1$

C.  $f(x) = -x - 1, g(x) = x - 1$

D. none of these

**Answer**

a.  $f(x) = x + 1, g(x) = -x + 1$

$$f(-1) = -1 + 1 = 0$$

$$f(1) = 1 + 1 = 2$$

Also,  $g(-1) = 1 + 1 = 2$

$$g(1) = -1 + 1 = 0$$

These functions map  $[-1, 1]$  onto  $[0, 2]$ .

b.  $f(x) = x - 1, g(x) = x + 1$

$$f(-1) = -1 - 1 = -2$$

$$f(1) = 1 - 1 = 0$$

Also,  $g(-1) = -1 + 1 = 0$

$$g(1) = 1 + 1 = 2$$

These functions do not map  $[-1, 1]$  onto  $[0, 2]$ .

c.  $f(x) = -x - 1, g(x) = x - 1$

$$f(-1) = 1 - 1 = 0$$

$$f(1) = -1 - 1 = -2$$

Also,  $g(-1) = -1 - 1 = -2$

$$g(1) = 1 - 1 = 0$$

These functions do not map  $[-1, 1]$  onto  $[0, 2]$ .

**42. Question**

Mark the correct alternative in each of the following:

Let  $f : [2, \infty) \rightarrow X$  be defined by  $f(x) = 4x - x^2$ . Then,  $f$  is invertible, if  $X =$

A.  $[2, \infty)$

B.  $(-\infty, 2]$

C.  $(-\infty, 4]$

D.  $[4, \infty)$

**Answer**

Given that  $f : [2, \infty) \rightarrow X$  be defined by

$$f(x) = 4x - x^2$$

Let  $y = f(x)$

$$\Rightarrow y = 4x - x^2$$

$$\Rightarrow -y + 4 = 4 - 4x + x^2$$

$$\Rightarrow 4 - y = (x - 2)^2$$

$$\Rightarrow x - 2 = \sqrt{4 - y}$$

$$\Rightarrow x = 2 + \sqrt{4 - y}$$

$$\text{So, } f^{-1}(x) = 2 + \sqrt{4 - x}$$

where  $x < 4$

So,  $x \in (-\infty, 4]$

### 43. Question

Mark the correct alternative in each of the following:

If  $f : \mathbb{R} \rightarrow (-1, 1)$  is defined by  $f(x) = \frac{-x|x|}{1+x^2}$ , then  $f^{-1}(x)$  equals

A.  $\sqrt{\frac{|x|}{1-|x|}}$

B.  $-\text{Sgn}(x) \sqrt{\frac{|x|}{1-|x|}}$

C.  $-\sqrt{\frac{x}{1-x}}$

D. none of these

### Answer

Given that  $f : \mathbb{R} \rightarrow (-1, 1)$  is defined by

$$f(x) = \frac{-x|x|}{1+x^2}$$

Here for mod function we will consider three cases,  $x = 0$ ,  $x < 0$  and  $x > 0$ .

For  $x < 0$

$$f(x) = \frac{-x(-x)}{1+x^2}$$

$$y = \frac{x^2}{1+x^2}$$

$$\Rightarrow y(1+x^2) = x^2$$

$$\Rightarrow x^2(1-y) = y$$

$$\Rightarrow x = -\sqrt{\frac{y}{1-y}}$$

$$\Rightarrow x = -\sqrt{\frac{|y|}{1-|y|}}, x < 0$$

Also, checking on  $x > 0$  and  $x = 0$  we find that



$$f^{-1}(x) = -\operatorname{sgn}(x) \sqrt{\frac{|y|}{1-|y|}}$$

#### 44. Question

Mark the correct alternative in each of the following:

Let  $[x]$  denote the greatest integer less than or equal to  $x$ . If  $f(x) = \sin^{-1} x$ ,  $g(x) = [x^2]$  and

$$h(x) = 2x, \frac{1}{2} \leq x \leq \frac{1}{\sqrt{2}}, \text{ then}$$

- A.  $f \circ g \circ h(x) = \pi/2$
- B.  $f \circ g \circ h(x) = \pi$
- C.  $h \circ f \circ g = h \circ g \circ f$
- D.  $h \circ f \circ g \neq h \circ g \circ f$

#### Answer

Given that  $f(x) = \sin^{-1} x$ ,  $g(x) = [x^2]$  and  $h(x) = 2x, \frac{1}{2} \leq x \leq \frac{1}{\sqrt{2}}$

a.  $g \circ h(x) = g(2x)$

$$\Rightarrow g \circ h(x) = [4x^2]$$

$$f \circ g \circ h(x) = f([4x^2])$$

$$\Rightarrow f \circ g \circ h(x) = \sin^{-1} [4x^2]$$

Hence, given option is incorrect.

b. Similarly, this option is also incorrect.

c.  $f \circ g(x) = f([x^2])$

$$\Rightarrow f \circ g(x) = \sin^{-1} [x^2]$$

$$h \circ f \circ g(x) = h(\sin^{-1} [x^2])$$

$$\Rightarrow h \circ f \circ g(x) = 2(\sin^{-1} [x^2])$$

$$g \circ f(x) = g(\sin^{-1} x)$$

$$\Rightarrow g \circ f(x) = [(\sin^{-1} x)^2]$$

$$h \circ g \circ f(x) = h([(\sin^{-1} x)^2])$$

$$\Rightarrow h \circ g \circ f(x) = 2[(\sin^{-1} x)^2]$$

Hence,  $h \circ g \circ f(x) \neq h \circ f \circ g(x)$

#### 45. Question

Mark the correct alternative in each of the following:

If  $g(x) = x^2 + x - 2$  and  $\frac{1}{2}g \circ f(x) = 2x^2 - 5x + 2$ , then  $f(x)$  is equal to

- A.  $2x - 3$
- B.  $2x + 3$
- C.  $2x^2 + 3x + 1$
- D.  $2x^2 - 3x - 1$

**Answer**

Given that  $g(x) = x^2 + x - 2$  and

$$\frac{1}{2}gof(x) = 2x^2 - 5x + 2$$

a. Let  $f(x) = 2x - 3$

$$gof(x) = g(2x - 3)$$

$$\Rightarrow gof(x) = (2x - 3)^2 + 2x - 3 - 2$$

$$\Rightarrow gof(x) = 4x^2 - 12x + 9 + 2x - 5$$

$$\Rightarrow gof(x) = 4x^2 - 10x + 4$$

$$\frac{1}{2}gof(x) = \frac{4x^2 - 10x + 4}{2}$$

$$\Rightarrow \frac{1}{2}gof(x) = 2x^2 - 5x + 2$$

Hence, this option is the required value of  $f(x)$ .

b. Let  $f(x) = 2x + 3$

$$gof(x) = g(2x + 3)$$

$$\Rightarrow gof(x) = (2x + 3)^2 + 2x + 3 - 2$$

$$\Rightarrow gof(x) = 4x^2 + 12x + 9 + 2x + 1$$

$$\Rightarrow gof(x) = 4x^2 + 14x + 10$$

$$\frac{1}{2}gof(x) = \frac{4x^2 + 14x + 10}{2}$$

$$\Rightarrow \frac{1}{2}gof(x) = 2x^2 + 7x + 5$$

Hence, this option is not the required value of  $f(x)$ .

c and d option are incorrect because their degree is more than 1. So, the degree of  $gof$  will be more than 2.

**46. Question**

Mark the correct alternative in each of the following:

If  $f(x) = \sin^2 x$  and the composite function  $g(f(x)) = |\sin x|$ , then  $g(x)$  is equal to

A.  $\sqrt{x-1}$

B.  $\sqrt{x}$

C.  $\sqrt{x+1}$

D.  $-\sqrt{x}$

**Answer**

Given that  $f(x) = \sin^2 x$  and the composite function  $g(f(x)) = |\sin x|$ .

$$g(f(x)) = g(\sin^2 x)$$

a. If  $g(x) = \sqrt{x-1}$

$$g(f(x)) = \sqrt{\sin^2 x - 1}$$

Hence, given option is incorrect.

b. If  $g(x) = \sqrt{x}$

$$g(f(x)) = \sqrt{\sin^2 x}$$

$$\Rightarrow g(f(x)) = |\sin x|$$

Hence, given option is correct.

c. If  $g(x) = \sqrt{x + 1}$

$$g(f(x)) = \sqrt{\sin^2 x + 1}$$

Hence, given option is incorrect.

d. If  $g(x) = -\sqrt{x}$

$$g(f(x)) = -\sqrt{\sin^2 x}$$

$$\Rightarrow g(f(x)) = -\sin x$$

Hence, given option is incorrect.

#### 47. Question

Mark the correct alternative in each of the following:

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = x^3 + 3$ , then  $f^{-1}(x)$  is equal to

A.  $x^{1/3} - 3$

B.  $x^{1/3} + 3$

C.  $(x - 3)^{1/3}$

D.  $x + 3^{1/3}$

#### Answer

Given that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $f(x) = x^3 + 3$

Then  $f^{-1}(x)$ :

$$y = f(x)$$

$$\Rightarrow y = x^3 + 3$$

$$\Rightarrow y - 3 = x^3$$

$$\Rightarrow x = \sqrt[3]{y - 3}$$

$$\text{So, } f^{-1}(x) = \sqrt[3]{x - 3}$$

#### 48. Question

Mark the correct alternative in each of the following:

Let  $f(x) = x^3$  be a function with domain  $\{0, 1, 2, 3\}$ . Then domain of  $f^{-1}$  is

A.  $\{3, 2, 1, 0\}$

B.  $\{0, -1, -2, -3\}$

C.  $\{0, 1, 8, 27\}$

D.  $\{0, -1, -8, -27\}$

#### Answer

Given that  $f(x) = x^3$  be a function with domain  $\{0, 1, 2, 3\}$ .

Then range =  $\{0, 1, 8, 27\}$

$f$  can be written as  $\{(0, 0), (1, 1), (2, 8), (3, 27)\}$

$f^{-1}$  can be written as  $\{(0, 0), (1, 1), (8, 2), (27, 3)\}$

So, the domain of  $f^{-1}$  is  $\{0, 1, 8, 27\}$

#### 49. Question

Mark the correct alternative in each of the following:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = x^2 - 3$ . Then,  $f^{-1}$  is given by

A.  $\sqrt{x+3}$

B.  $\sqrt{x} + 3$

C.  $x + \sqrt{3}$

D. none of these

#### Answer

Given that  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2 - 3$

For  $f^{-1}$ :

$$y = f(x)$$

$$\Rightarrow y = x^2 - 3$$

$$\Rightarrow x = \pm\sqrt{y+3}$$

$$f^{-1}(x) = \pm\sqrt{x+3}$$

#### 50. Question

Mark the correct alternative in each of the following:

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = \tan x$ . Then,  $f^{-1}(1)$  is

A.  $\frac{\pi}{4}$

B.  $\left\{ n\pi + \frac{\pi}{4} : n \in \mathbb{Z} \right\}$

C. does not exist

D. none of these

#### Answer

Given that  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = \tan x$

For  $f^{-1}$ :

$$y = f(x)$$

$$\Rightarrow y = \tan x$$

$$\Rightarrow x = \tan^{-1} y$$

$$f^{-1} = \tan^{-1} x$$

$$\Rightarrow f^{-1}(x) = n\pi + \frac{\pi}{4}; n \in \mathbb{Z}$$

### 51. Question

Mark the correct alternative in each of the following:

$$\text{Let } f: \mathbb{R} \rightarrow \mathbb{R} \text{ be defined as } f(x) = \begin{cases} 2x, & \text{if } x > 3 \\ x^2, & \text{if } 1 < x \leq 3. \\ 3x, & \text{if } x \leq 1 \end{cases}$$

Then, find  $f(-1) + f(2) + f(4)$

- A. 9
- B. 14
- C. 5
- D. none of these

### Answer

Given that  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined as

$$f(x) = \begin{cases} 2x, & \text{if } x > 3 \\ x^2, & \text{if } 1 < x \leq 3 \\ 3x, & \text{if } x \leq 1 \end{cases}$$

For  $f(-1)$ :

$$f(x) = 3x$$

$$\Rightarrow f(-1) = -3$$

For  $f(2)$ :

$$f(x) = x^2$$

$$\Rightarrow f(2) = 4$$

For  $f(4)$ :

$$f(x) = 2x$$

$$\Rightarrow f(4) = 8$$

$$f(-1) + f(2) + f(4) = -3 + 4 + 8$$

$$\Rightarrow f(-1) + f(2) + f(4) = 9$$

### 52. Question

Mark the correct alternative in each of the following:

Let  $A = \{1, 2, \dots, n\}$  and  $B = \{a, b\}$ . Then the number of subjections from A into B is

- A.  ${}^n P_2$
- B.  $2^n - 2$
- C. 0
- D. none of these

### Answer

Given that  $A = \{1, 2, \dots, n\}$  and  $B = \{a, b\}$

The number of functions from a set with n number of elements into a set of 2 number of elements =  $2^n$

But two functions can be many-one into functions.

Hence, answer is  $2^n - 2$ .

### 53. Question

Mark the correct alternative in each of the following:

If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is

- A. 720
- B. 120
- C. 0
- D. none of these

### Answer

Given that set A contains 5 elements and set B contains 6 elements.

Number of one-one and onto mappings from A to B means bijections from A to B.

Number of bijections are possible only when  $n(B) < n(A)$ .

But here,  $n(A) < n(B)$

So, the number of one-one and onto mappings from A to B is 0.

### 54. Question

Mark the correct alternative in each of the following:

If the set A contains 7 elements and the set B contains 10 elements, then the number one-one functions from A to B is

- A.  ${}^{10}C_7$
- B.  ${}^{10}C_7 \times 7!$
- C.  $7^{10}$
- D.  $10^7$

### Answer

Given that set A contains 7 elements and set B contains 10 elements.

The number one-one functions from A to B is  ${}^{10}C_7 \times 7!$ .

### 55. Question

Mark the correct alternative in each of the following:

Let  $f : \mathbb{R} - \left\{ \frac{3}{5} \right\} \rightarrow \mathbb{R}$  be defined by  $f(x) = \frac{3x + 2}{5x - 3}$ . Then,

- A.  $f^{-1}(x) = x$
- B.  $f^{-1}(x) = -f(x)$
- C.  $f \circ f(x) = x$
- D.  $f^{-1}(x) = \frac{1}{19} f(x)$

### Answer

Given that  $f: \mathbb{R} - \left\{\frac{3}{5}\right\} \rightarrow \mathbb{R}$  defined as  $f(x) = \frac{3x + 2}{5x - 3}$

For  $f^{-1}$ :

$$y = \frac{3x + 2}{5x - 3}$$

$$\Rightarrow y(5x - 3) = 3x + 2$$

$$\Rightarrow x(5y - 3) = 2 + 3y$$

$$\Rightarrow x = \frac{2 + 3y}{5y - 3}$$

$$\text{So, } f^{-1}(x) = \frac{2 + 3x}{5x - 3}$$

$$f \circ f(x) = f\left(\frac{3x + 2}{5x - 3}\right)$$

$$\Rightarrow f \circ f(x) = \frac{3 \frac{3x+2}{5x-3} + 2}{5 \frac{3x+2}{5x-3} - 3}$$

$$\Rightarrow f \circ f(x) = \frac{3(3x + 2) + 2(5x - 3)}{5(3x + 2) - 3(5x - 3)}$$

$$\Rightarrow f \circ f(x) = \frac{19x}{19}$$

$$\Rightarrow f \circ f(x) = x$$

Hence, option C is correct.