RELATION & FUNCTION-I

1. INTRODUCTION

In this chapter, we will learn how to create a relation between two sets by linking pairs of objects from two sets. We will learn how a relation qualifies for being a function. Finally, we will see kinds of function, some standard functions etc.

2. RELATIONS

2.1 Cartesian product of sets

Definition : Given two non-empty sets P & Q. The cartesian product $P \times Q$ is the set of all ordered pairs of elements from P & Q i.e.

$$P \times Q = \{(p, q); p \in P; q \in Q\}$$

2.2 Relations

2.2.1 Definition : Let A & B be two non-empty sets. Then any subset 'R' of $A \times B$ is a relation from A to B.

If $(a, b) \in R$, then we write it as a R b which is read as a is related to b' by the relation R', 'b' is also called image of 'a' under R.

2.2.2 Domain and range of a relation : If R is a relation from A to B, then the set of first elements in R is called domain & the set of second elements in R is called range of R. symbolically.

$$\text{Domain of } R = \{ x : (x, y) \in R\}$$

$$\text{Range of } R = \{ y : (x, y) \in R\}$$

The set B is called co-domain of relation R.

Note that range $\subset$ co-domain.

Note..

Total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and total number of relations is 2^{pq}.

2.2.3 Inverse of a relation : Let A, B be two sets and let R be a relation from a set A to set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

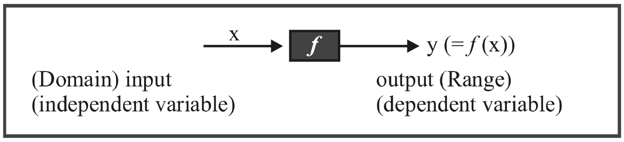
Also, Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1}).

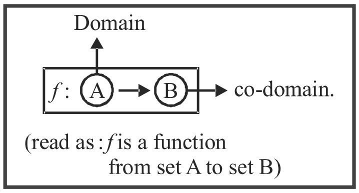
3. FUNCTIONS

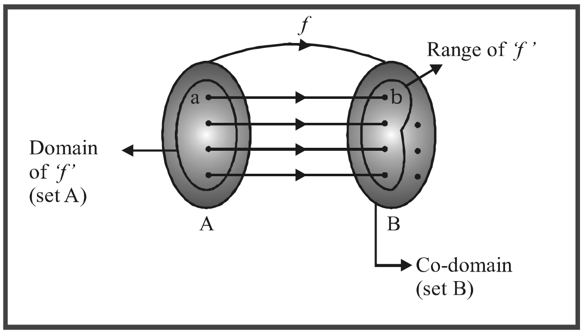
3.1 Definition

A relation 'f' from a set A to set B is said to be a function if every element of set A has one and only one image in set B.

Notations







3.2 Domain, Co-domain and Range of a function

Domain : When we define $y = f(x)$ with a formula and the domain is not stated explicitly, the domain is assumed to be the largest set of x–values for which the formula gives real y–values.

The domain of $y = f(x)$ is the set of all real x for which $f(x)$ is defined (real).

Algo Check : Rules for finding Domain :

(i) Expression under even root (i.e. square root, fourth root etc.) should be non–negative.

(ii) Denominator $\neq 0$.

(iii) $\log_a x$ is defined when $x > 0$, $a > 0$ and $a \neq 1$.

(iv) If domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$. While domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

Range : The set of all f-images of elements of A is known as the range of f & denoted by $f(A)$.

Range $= f(A) = \{f(x) : x \in A\}$;

$f(A) \subseteq B$ {Range $\subseteq$ Co-domain}.

Algo Check : Rule for finding range :

First of all find the domain of $y = f(x)$

(i) If domain $\in$ finite number of points

$\Rightarrow$ range $\in$ set of corresponding $f(x)$ values.

(ii) If domain $\in$ R or R – {some finite points}

Put $y = f(x)$

Then express x in terms of y. From this find y for x to be defined. (i.e., find the values of y for which x exists).

(iii) If domain $\in$ a finite interval, find the least and greater value for range using monotonocity.

Note...

1. Question of format :

$$\left(y = \frac{Q}{Q};\ y = \frac{L}{Q};\ y = \frac{Q}{L}\right) \quad \begin{array}{l} Q \to \text{quadratic} \\ L \to \text{Linear} \end{array}$$

Range is found out by cross-multiplying & creating a quadratic in 'x' & making $D \geq 0$ (as $x \in R$)

2. Questions to find range in which-the given expression $y = f(x)$ can be converted into x (or some function of x) = expression in 'y'.

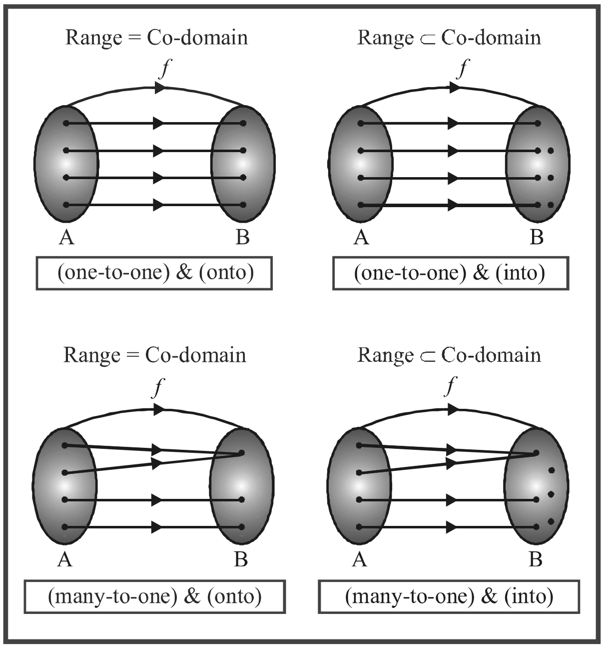
Do this & apply method (ii).

Note...

Two functions f & g are said to be equal iff

1. Domain of f = Domain of g
2. Co-domain of f = Co-domain of g
3. $f(x) = g(x)\ \forall\ x \in$ Domain.

3.3 Kinds of Functions



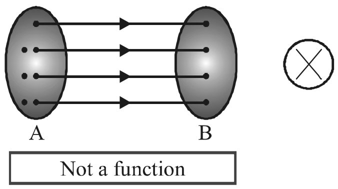
Note...

(a) One-to-One functions are also called Injective functions.

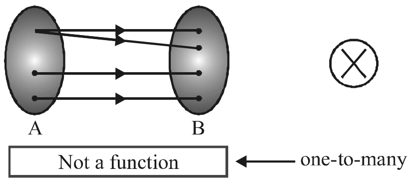
(b) Onto functions are also called Surjective

(c) (one-to-one) & (onto) functions are also called Bijective Functions.

Relations which can not be catagorized as a function



As not all elements of set A are associated with some elements of set B. (violation of– point (i)– definition 2.1)



An element of set A is not associated with a unique element of set B, (violation of point (ii) definition 2.1)

Methods to check one-one mapping

1. **Theoretically :** If $f(x_1) = f(x_2)$

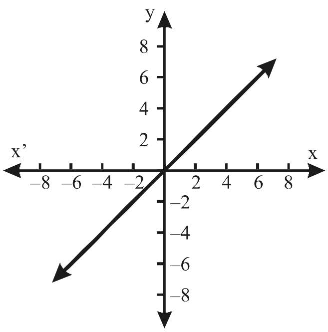
   $\Rightarrow$ $x_1 = x_2$, then $f(x)$ is one-one.

2. **Graphically :** A function is one-one, iff no line parallel to x-axis meets the graph of function at more than one point.

3. **By Calculus :** For checking whether $f(x)$ is One-One, find whether function is only increasing or only decreasing in their domain. If yes, then function is one-one, i.e. if $f'(x) \geq 0, \ \forall \ x \in$ domain or i.e., if $f'(x) \leq 0, \ \forall \ x \in$ domain, then function is one-one.

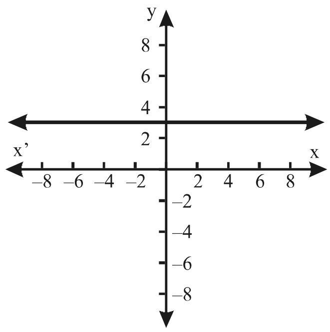
3.4 Some standard real functions & their graphs

3.4.1 Identity Function : The function $f : R \to R$ defined by $y = f(x) = x \ \forall \ x \in R$ is called identity function.



$f(x) = x$

3.4.2 Constant Function : The function $f : R \to R$ defined by $y = f(x) = c, \ \forall \ x \in R$ where c is a constant is called constant function

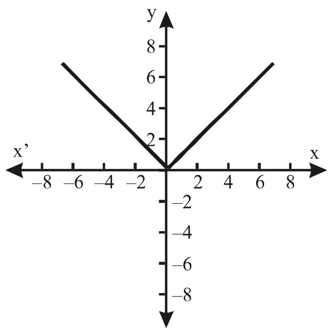


$f(x) = 3$

3.4.3 Modulus Function : The function $f : R \to R$ defined by

$$f(x) = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

is called modulus function. It is denoted by $y = f(x) = |x|$.



$f(x) = |x|$

Its also known as "Absolute value function'.

Properties of Modulus Function :

The modulus function has the following properties :

1. For any real number x, we have $\sqrt{x^2} = |x|$

2. $|x\,y| = |x|\,|y|$

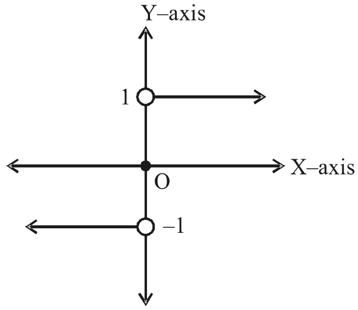
3. $|x + y| \leq |x| + |y|$
4. $|x - y| \geq \big||x| - |y|\big|$

$\left.\begin{matrix} 3. \\ 4. \end{matrix}\right\}$ triangle inequality

3.4.4 Signum Function : The function $f : R \to R$ defined by

$$f(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases}$$

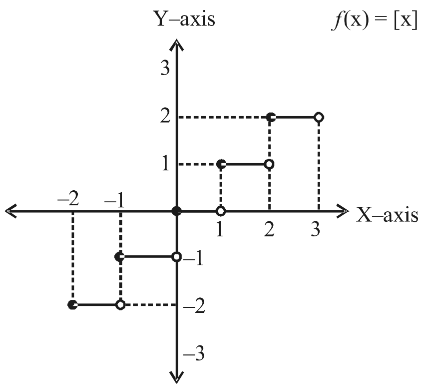
is called signum function. It is usually denoted by $y = f(x) = \text{sgn}(x)$.



Note...

$$\text{Sgn}(x) = \begin{cases} \dfrac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

3.4.5 Greatest Integer Function : The function $f : R \to R$ defined as the greatest integer less than or equal to x. It is usually denoted as $y = f(x) = [x]$



Properties of Greatest Integer Function :

If n is an integer and x is any real number between n and n + 1, then the greatest integer function has the following properties :

(1) $[-n] = -[n]$

(2) $[x+n] = [x] + n$

(3) $[-x] = -[x] - 1$

(4) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$

Note...

Fractional part of x, denoted by $\{x\}$ is given by $x - [x]$. So,

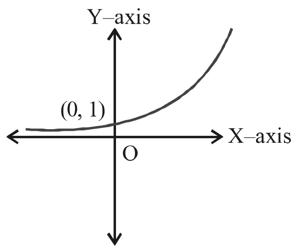
$$\{x\} = x - [x] = \begin{cases} x-1; & 1 \le x < 2 \\ x\ ; & 0 \le x < 1 \\ x+1; & -1 \le x < 0 \end{cases}$$

3.4.6 Exponential Function :

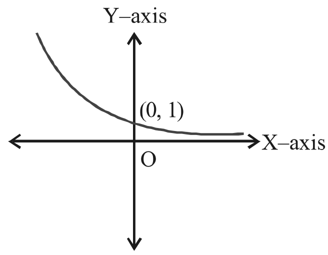
$f(x) = a^x, \quad a > 0, \quad a \neq 1$

Domain : $x \in R$

Range : $f(x) \in (0, \infty)$



$f(x) = a^x$, when $a > 1$



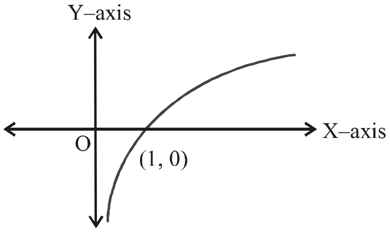
$f(x) = a^x$, when $0 < a < 1$

3.4.7 Logarithm Function :

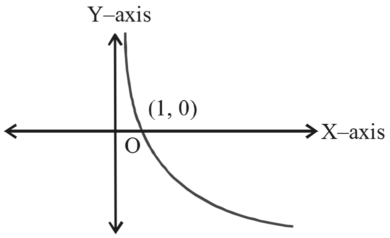
$f(x) = \log_a x, \quad a > 0, \; a \neq 1$

Domain : $x \in (0, \infty)$

Range : $y \in R$



$f(x) = \log_a x$, when $a > 1$



$f(x) = \log_a x$, when $0 < a < 1$

(a) The Principal Properties of Logarithms

Let M & N are arbitrary positive numbers, $a > 0, \; a \neq 1, \; b > 0, b \neq 1$.

(i) $\log_b a = c \quad \Rightarrow \quad a = b^c$

(ii) $\log_a (M \,.\, N) = \log_a M + \log_a N$

(iii) $\log_a (M/N) = \log_a M - \log_a N$

(iv) $\log_a M^N = N \; \log_a M$

(v) $\log_b a = \dfrac{\log_c a}{\log_c b}, \; c > 0, \; c \neq 1.$

(vi) $a^{\log_c b} = b^{\log_c a}, \; a, b, c > 0, \; c \neq 1.$

Note...

(a) $\log_a a = 1$

(b) $\log_b a \,.\, \log_c b \,.\, \log_a c = 1$

(c) $\log_a 1 = 0$

(d) $e^{x \ln a} = e^{\ln a^x} = a^x$

(b) Properties of Monotonocity of Logarithm

(i) If $a > 1, \; \log_a x < \log_a y \quad \Rightarrow \quad 0 < x < y$

(ii) If $0 < a < 1, \; \log_a x < \log_a y \quad \Rightarrow \quad x > y > 0$

(iii) If $a > 1$ then $\log_a x < p \quad \Rightarrow \quad 0 < x < a^p$

(iv) If $a > 1$ then $\log_a x > p \quad \Rightarrow \quad x > a^p$

(v) If $0 < a < 1$ then $\log_a x < p \quad \Rightarrow \quad x > a^p$

(vi) If $0 < a < 1$ then $\log_a x > p \quad \Rightarrow \quad 0 < x < a^p$

Note...

If the exponent and the base are on same side of the unity, then the logarithm is positive.

If the exponent and the base are on different sides of unity, then the logarithm is negative.

4. ALGEBRA OF REAL FUNCTION

In this section, we shall learn how to add two real functins, subtract a real function from another, multiply a real function by a scalar (here by a scalar we mean a real number), multiply two real functions and divide one real function by another.

4.1 Addition of two real functions

Let $f : X \to R$ and $g : X \to R$ by any two real functions, whre $X \subset R$. Then, we define $(f + g) : X \to R$ by

$(f + g)(x) = f(x) + g(x)$, for all $x \in X$.

4.2 Subtraction of a real function from another

Let $f : X \to R$ be any two any two real functions, whre $X \subset R$. Then, we define $(f - g) : X \to R$ by

$(f - g)(x) = f(x) - g(x)$, for all $x \in X$.

4.3 Multiplication by a scalar

Let $f : X \to R$ be a real valued function and α be a scalar. Here by scalar, we mean a real number. Then the product αf is a function from X to R defined by $(\alpha f)(x) = \alpha f(x), x \in X$.

4.4 Multiplication of two real functions

The product (or multiplication) of two real functions $f : X \to R$ and $g : X \to R$ is a function $fg : X \to R$ defined by $(fg)(x) = f(x)\, g(x)$, for all $x \in X$.

This is also called *pointwise multiplication*.

4.5 Quotient of two real functions

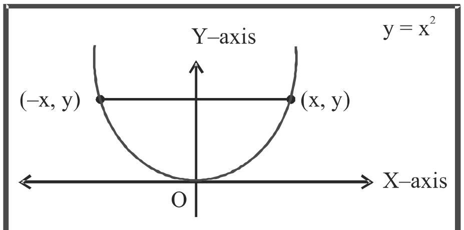
Let f and g be two real functions defined from $X \to R$ where $X \subset R$. The quotient of f by g denoted by f/g is a function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0, x \in X.$$

4.6 Even and Odd Functions

Even Function : $f(-x) = f(x), \ \forall \ x \in$ Domain

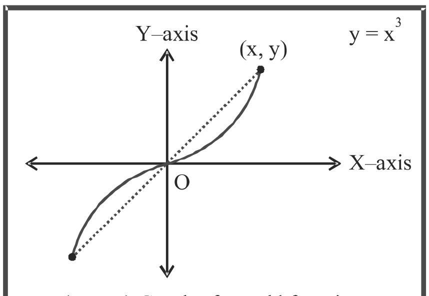
The graph of an even function $y = f(x)$ is symmetric about the y–axis. i.e., (x, y) lies on the graph $\Leftrightarrow$ (–x, y) lies on the graph.



Graph of an even function

Odd Function : $f(-x) = -f(x), \ \forall \ x \in$ Domain

The graph of an odd function $y = f(x)$ is symmetric about origin i.e. if point (x, y) is on the graph of an odd function, then (–x, –y) will also lie on the graph.



Graph of an odd function

5. PERIODIC FUNCTION

Definition : A function $f(x)$ is said to be periodic function, if there exists a positive real number T, such that

$$f(x+T) = f(x), \ \forall x \in R.$$

Then, $f(x)$ is a periodic function where least positive value of T is called fundamental period.

Graphically : If the graph repeats at fixed interval, then function is said to be periodic and its period is the width of that interval.

Some standard results on periodic functions :

	Functions	**Periods**
(i)	$\sin^n x, \cos^n x, \sec^n x, \text{cosec}^n x$	π ; if n is even. 2π ; (if n is odd or fraction)
(ii)	$\tan^n x, \cot^n x$	π ; n is even or odd.
(iii)	$\|\sin x\|, \|\cos x\|, \|\tan x\|$ $\|\cot x\|, \|\sec x\|, \|\text{cosec } x\|$	π
(iv)	$x - [x]$, [.] represents greatest integer function	1
(v)	Algebraic functions e.g., $\sqrt{x}, x^2, x^3 + 5$, ...etc.	period does not exist

Properties of Periodic Function

(i) If $f(x)$ is periodic with period T, then

(a) $c \cdot f(x)$ is periodic with period T.

(b) $f(x \pm c)$ is periodic with period T.

(c) $f(x) \pm c$ is periodic with period T.

where c is any constant.

(ii) If $f(x)$ is periodic with period T, then $kf(cx + d)$ has period $T/|c|$,

i.e. Period is only affected by coefficient of x where k, c, d $\in$ constant.

(iii) If $f_1(x), f_2(x)$ are periodic functions with periods T_1, T_2 respectively, then we have, $h(x) = af_1(x) \pm bf_2(x)$ has period as, LCM of $\{T_1, T_2\}$

Note...

(a) $$\text{LCM of } \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{\text{LCM of } (a, c, e)}{\text{HCF of } (b, d, f)}$$

(b) LCM of rational and rational always exists.

LCM of irrational and irrational sometime exists.

But LCM of rational and irrational never exists.

e.g., LCM of $(2\pi, 1, 6\pi)$ is not possible as

$2\pi, 6\pi \in$ irrational and $1 \in$ rational.