20. Definite Integrals

Exercise 20.1

1. Question

Evaluate the following definite integrals:

$$\int_4^9 \frac{1}{\sqrt{x}} dx$$

Answer

Using the formula:

$$\int_a^b (x)^n dx = \left[\frac{(x)^{n+1}}{n+1}\right]_a^b$$

$$\Rightarrow \int_4^9 (x)^{-\frac{1}{2}} = \left[\frac{(x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_4^9$$

$$\Rightarrow \left[\frac{(x)^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9$$

$$\Rightarrow \left[\frac{(9)^{\frac{1}{2}}}{\frac{1}{2}}\right] - \left[\frac{(4)^{\frac{1}{2}}}{\frac{1}{2}}\right] = \left[\frac{3}{\frac{1}{2}}\right] - \left[\frac{2}{\frac{1}{2}}\right]$$

⇒ 3 × 2 – 2 × 2

⇒ 6 – 4 = 2

$$\Rightarrow \int_4^9 (x)^{-\frac{1}{2}} = 2$$

2. Question

Evaluate the following definite integrals:

$$\int_{-2}^3 \frac{1}{x+7} dx$$

Answer

Using the formula:

$$\int_a^b \left(\frac{1}{x}\right) dx = [\log|x|]_a^b$$

$$\Rightarrow \int_{-2}^3 \left(\frac{1}{x+7}\right) = [\log|x+7|]_{-2}^3$$

⇒ log 3 + 7 – log –2+7

⇒ log |10|– log |5|

$$\Rightarrow \log\left|\frac{10}{5}\right| = \log|2|$$

$$\int_a^b \left(\frac{1}{x}\right) dx = \log|2|$$

3. Question

Evaluate the following definite integrals:

$$\int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx$$

Answer

Using the formula:

$$\int_a^b \frac{1}{\sqrt{(1-x^2)}} dx = [-\cos^{-1} x]_a^b$$

$$\int_0^{1/2} \frac{1}{\sqrt{(1-x^2)}} = [-\cos^{-1} x]_0^{1/2}$$

$$\Rightarrow -[\cos^{-1} 1/2 - \cos^{-1} 0]$$

$$\Rightarrow -\left[\frac{\pi}{3} - \frac{\pi}{2}\right]$$

$$\Rightarrow \left[\frac{\pi}{6}\right]$$

$$\int_0^{1/2} \frac{1}{\sqrt{(1-x^2)}} = \left[\frac{\pi}{6}\right]$$

4. Question

Evaluate the following definite integrals:

$$\int_0^1 \frac{1}{1+x^2} dx$$

Answer

Using the formula:

$$\int_a^b \frac{1}{(1+x^2)} dx = [\tan^{-1} x]_a^b$$

$$\int_0^1 \frac{1}{(1+x^2)} dx = [\tan^{-1} x]_0^1$$

$$\Rightarrow [\tan^{-1} 1 - \tan^{-1} 0]$$

$$\Rightarrow \left[\frac{\pi}{4} - 0\right]$$

$$\Rightarrow \pi/4$$

$$\int_a^b \frac{1}{(1+x^2)} dx = \pi/4$$

5. Question

Evaluate the following definite integrals:

$$\int_2^3 \frac{x}{x^2+1} dx$$

Answer

Let $x^2 + 1 = t$

⇒ On differentiation, we get

2 x dx = dt

$$\Rightarrow\ xdx = \frac{dt}{2}$$

⇒ Hence the question will become:

$$\int_2^3 \frac{dt/2}{t} = \int_2^3 \frac{dt}{2t}$$

$$\Rightarrow\ \frac{1}{2}\int_2^3 \frac{dt}{t}$$

Using the formula:

$$\int_a^b \left(\frac{1}{x}\right) dx = [\log|x|]_a^b$$

$$1/2\int_2^3 \left(\frac{1}{t}\right) dt = [\log|t|]_2^3$$

$$\Rightarrow [\log|x^2+1|]_2^3$$

$$\Rightarrow \log|3^2+1| - \log|2^2+1|$$

$$\Rightarrow \log|10| - \log|5|$$

$$\Rightarrow \log|10/5|$$

$$\Rightarrow \log|2|$$

6. Question

Evaluate the following definite integrals:

$$\int_0^\infty \frac{1}{a^2+b^2x^2} dx$$

Answer

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)} dx = \int_0^\infty \frac{1}{b^2\left(\left(\frac{a}{b}\right)^2+x^2\right)} dx$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = 1/b^2 \int_0^\infty \frac{1}{\left(\left(\frac{a}{b}\right)^2+x^2\right)}dx$$

Now, Using the formula:

$$\int_a^b \frac{1}{(a^2+x^2)}dx = \frac{1}{a}[\tan^{-1} x/a]_a^b$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = \frac{1}{\frac{a}{b}}[\tan^{-1}\frac{x}{\frac{a}{b}}]_0^\infty$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = \left(\frac{b}{a}\right)[\tan^{-1}\frac{bx}{a}]_0^\infty$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = \left(\frac{b}{a}\right)[\tan^{-1}\frac{b\times\infty}{a} - \tan^{-1} b\times\frac{0}{a}]$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = \left(\frac{b}{a}\right)[\tan^{-1}\infty - \tan^{-1} 0]$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = \left(\frac{b}{a}\right)[\frac{\pi}{2} - 0]$$

$$\Rightarrow \int_0^\infty \frac{1}{(a^2+b^2x^2)}dx = \left(\frac{b}{a}\right)\left(\frac{\pi}{2}\right)$$

7. Question

Evaluate the following definite integrals:

$$\int_{-1}^1 \frac{1}{1+x^2}dx$$

Answer

Using the formula:

$$\int_a^b \frac{1}{(1+x^2)}dx = [\tan^{-1} x]_a^b$$

$\Rightarrow \int_a^b \frac{1}{(1+x^2)} = [\tan^{-1} x]_{-1}^1$

⇒ [$\tan^{-1}$ (1) – $\tan^{-1}$ (– 1)]

$\Rightarrow [\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)]$

$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

8. Question

Evaluate the following definite integrals:

$$\int_0^\infty e^{-x}\ dx$$

Answer

Using the formula:

$$\int_a^b e^x dx = [e^x]_a^b$$

$$\Rightarrow \int_0^\infty e^{-x} dx = -[e^{-x}]_0^\infty$$

$$\Rightarrow \int_0^\infty e^{-x} dx = -[e^{-\infty} - e^0]$$

$$\Rightarrow \int_0^\infty e^{-x} dx = -[0 - 1]$$

$$\Rightarrow \int_0^\infty e^{-x} dx = 1$$

9. Question

Evaluate the following definite integrals:

$$\int_0^1 \frac{x}{x+1} dx$$

Answer

$$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = \int_0^1 \left(\frac{x+1-1}{x+1}\right) dx$$

$$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = \int_0^1 \left(\frac{x+1}{x+1}\right) dx - \int_0^1 \left(\frac{1}{x+1}\right) dx$$

$$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = \int_0^1 1. dx - \int_0^1 \left(\frac{1}{x+1}\right) dx$$

Using the formula:

$$\int_a^b \left(\frac{1}{x}\right) dx = [\log|x|]_a^b$$

$$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = [x]_0^1 - [\log|x+1|]_0^1$$

$$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = [1-0] - [\log|1+1| - \log|0+1|]$$

$$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = [1] - [\log|2|] - 0$$

($\because \log 1 = 0$)

$\Rightarrow \int_0^1 \left(\frac{x}{x+1}\right) dx = [1] - [\log|2|]$

10. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} (\sin x + \cos x) dx$$

Answer

$$\Rightarrow \int_0^{\pi/2} (\sin(x) + \cos(x)) dx = \int_0^{\pi/2} \sin(x) dx + \int_0^{\pi/2} \cos(x) dx$$

Using the formula:

$$\int_a^b \sin(x) dx = -[\cos(x)]_a^b$$

and

$$\int_a^b \cos(x) dx = [\sin(x)]_a^b$$

$$\Rightarrow \int_0^{\pi/2} (\sin(x) + \cos(x)) dx = -[\cos(x)]_0^{\frac{\pi}{2}} + [\sin(x)]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin(x) + \cos(x)) dx = -\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] + \left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin(x) + \cos(x)) dx = -[0 - 1] + [1 - 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin(x) + \cos(x)) dx = -[-1] + [1]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin(x) + \cos(x)) dx = 2$$

11. Question

Evaluate the following definite integrals:

$$\int_{\pi/4}^{\pi/2} \cot x \, dx$$

Answer

Using the formula:

$$\int_a^b \cot(x)\,dx = [\log|\sin(x)|]_a^b$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} \cot(x)\,dx = [\log|\sin(x)|]_{\pi/4}^{\pi/2}$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} \cot(x)\,dx = \left[\log\left|\sin\left(\frac{\pi}{2}\right)\right| - \log\left|\sin\left(\frac{\pi}{4}\right)\right|\right]$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} \cot(x)\,dx = [\log|1| - \log|1/\sqrt{2}|]$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} \cot(x)\,dx = \left[\log\left|1/(\frac{1}{\sqrt{2}})\right|\right]$$

$$\Rightarrow \int_{\pi/4}^{\pi/2} \cot(x)\,dx = [\log\sqrt{2}]$$

12. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/4} \sec x\ dx$$

Answer

Using the formula:

$$\int_a^b \sec(x)\,dx = [\log|\sec(x) + \tan(x)|]_a^b$$

$$\Rightarrow \int_0^{\pi/4} \sec(x)\,dx = [\log|\sec(x) + \tan(x)|]_0^{\pi/4}$$

$$\Rightarrow \int_0^{\pi/4} \sec(x)\,dx = [\log\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \log|\sec(0) + \tan(0)|$$

$$\Rightarrow \int_0^{\pi/4} \sec(x)\,dx = [\log|\sqrt{2} + 1|\,| - \log|1 + 0|$$

$$\Rightarrow \int_0^{\pi/4} \sec(x)\,dx = [\log|\sqrt{2} + 1|\,| - \log|1|$$

$$\Rightarrow \int_0^{\pi/4} \sec(x)\,dx = [\log|\sqrt{2} + 1|\,| - 0$$

($\because \log 1 = 0$)

$$\Rightarrow \int_{0}^{\pi/4} \sec(x)\,dx = [\log|\sqrt{2}+1|\,|]$$

13. Question

Evaluate the following definite integrals:

$$\int_{\pi/6}^{\pi/4} \text{cosec } x\ dx$$

Answer

Using the formula :

$$\int_{a}^{b} \text{cosec}(x)\,dx = [\log|\text{cosec}(x) - \cot(x)|]_{a}^{b}$$

$$\Rightarrow \int_{\pi/6}^{\pi/4} \text{cosec}(x)\,dx = [\log|\text{cosec}(x) - \cot(x)|]_{\pi/6}^{\pi/4}$$

$$\Rightarrow \int_{\pi/6}^{\pi/4} \text{cosec}(x)\,dx = [\log\left|\text{cosec}\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4}\right)\right| - \log\left|\text{cosec}\left(\frac{\pi}{6}\right) - \cot\left(\frac{\pi}{6}\right)\right|$$

$$\Rightarrow \int_{\pi/6}^{\pi/4} \text{cosec}(x)\,dx = [\log|\sqrt{2}-1| - \log|2-\sqrt{3}|]$$

$$(\because\ \text{cosec}\left(\frac{\pi}{4}\right) = \sqrt{2}, \cot\left(\frac{\pi}{4}\right) = 1, \text{cosec}\left(\frac{\pi}{6}\right) = 2, \cot\left(\frac{\pi}{6}\right) = \sqrt{3})$$

$$\Rightarrow \int_{\pi/6}^{\pi/4} \text{cosec}(x)\,dx = \left[\frac{\log|\sqrt{2}-1|}{|2-\sqrt{3}|}\right]$$

14. Question

Evaluate the following definite integrals:

$$\int_{0}^{1} \frac{1-x}{1+x}\,dx$$

Answer

$$\int_{0}^{1} \left(\frac{1-x}{1+x}\right) dx = \int_{0}^{1} \left(\frac{1}{x+1}\right) dx - \int_{0}^{1} \left(\frac{x}{x+1}\right) dx$$

$$\Rightarrow \int_{0}^{1} \left(\frac{1-x}{1+x}\right) dx = \int_{0}^{1} \left(\frac{1}{x+1}\right) dx - \int_{0}^{1} \left(\frac{x+1-1}{x+1}\right) dx$$

$$\Rightarrow \int_{0}^{1} \left(\frac{1-x}{1+x}\right) dx = \int_{0}^{1} \left(\frac{1}{x+1}\right) dx - \int_{0}^{1} \left(\frac{x+1}{x+1}\right) dx + \int_{0}^{1} \left(\frac{1}{x+1}\right) dx$$

$$\Rightarrow \int_{0}^{1} \left(\frac{1-x}{1+x}\right) dx = 2 \times \int_{0}^{1} \left(\frac{1}{x+1}\right) dx - \int_{0}^{1} \left(\frac{x+1}{x+1}\right) dx$$

$$\Rightarrow \int_0^1 \left(\frac{1-x}{1+x}\right) dx = 2 \times \int_0^1 \left(\frac{1}{x+1}\right) dx - \int_0^1 1 \times dx$$

Using the formula:

$$\int_a^b \left(\frac{1}{x}\right) dx = [\log|x|]_a^b$$

$$\Rightarrow \int_0^1 \left(\frac{1-x}{1+x}\right) dx = 2 \times [\log|x+1|]_0^1 - [x]_0^1$$

$$\Rightarrow \int_0^1 \left(\frac{1-x}{1+x}\right) dx = 2 \times [\log|1+1| - \log|0+1|] - [1-0]$$

$$\Rightarrow \int_0^1 \left(\frac{1-x}{1+x}\right) dx = 2 \times [\log|2| - \log|1|] - [1]$$

$$\Rightarrow \int_0^1 \left(\frac{1-x}{1+x}\right) dx = 2 \times [\log|2| - 0] - 1$$

$$\Rightarrow \int_0^1 \left(\frac{1-x}{1+x}\right) dx = 2 \times \log|2| - 1$$

15. Question

Evaluate the following definite integrals:

$$\int_0^\pi \frac{1}{1+\sin x} dx$$

Answer

$$\Rightarrow \int_0^\pi \left(\frac{1}{1+\sin x}\right) dx = \int_0^\pi \left(\frac{dx}{1+\sin x}\right) \times \left[\frac{1-\sin x}{1-\sin x}\right]$$

$$\Rightarrow \int_0^\pi \left(\frac{1}{1+\sin x}\right) dx = \int_0^\pi \frac{1-\sin x}{1-\sin^2 x} dx$$

$$\Rightarrow \int_0^\pi \left(\frac{1}{1+\sin x}\right) dx = \int_0^\pi \frac{1-\sin x}{\cos^2 x} dx$$

($\because 1 - \sin^2 x = \cos^2 x$)

$$\Rightarrow \int_0^\pi \left(\frac{1}{1+\sin x}\right) dx = \int_0^\pi \frac{1}{\cos^2 x} dx - \int_0^\pi \frac{\sin x}{\cos^2 x} dx$$

$$\Rightarrow \int_0^\pi \left(\frac{1}{1+\sin x}\right) dx = \int_0^\pi \sec^2 x \, dx - \int_0^\pi \tan(x)\sec(x) \, dx$$

Now, we know,

$$\int_a^b \sec^2 x\,dx = [\tan(x)]_a^b$$

And,

$$\int_a^b \tan(x)\sec(x)\,dx = [\sec(x)]_a^b$$

$$\therefore \int_0^{\pi}\left(\frac{1}{1+\sin x}\right)dx = [\tan(x)]_0^{\pi} - [\sec(x)]_0^{\pi}$$

$$\Rightarrow \int_0^{\pi}\left(\frac{1}{1+\sin x}\right)dx = [\tan(\pi)-\tan(0)] - [\sec(\pi)-\sec(0)]$$

$$\Rightarrow \int_0^{\pi}\left(\frac{1}{1+\sin x}\right)dx = [0-0] - [-1-1]$$

$$\Rightarrow \int_0^{\pi}\left(\frac{1}{1+\sin x}\right)dx = -[-2]$$

$$\Rightarrow \int_0^{\pi}\left(\frac{1}{1+\sin x}\right)dx = 2$$

16. Question

Evaluate the following definite integrals:

$$\int_{-\pi/4}^{\pi/4} \frac{1}{1+\sin x}dx$$

Answer

$$\Rightarrow \int_{-\pi/4}^{\pi/4}\left(\frac{1}{1+\sin x}\right)dx = \int_{-\pi/4}^{\pi/4}\left(\frac{dx}{1+\sin x}\right)\times\left[\frac{1-\sin x}{1-\sin x}\right]$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4}\left(\frac{1}{1+\sin x}\right)dx = \int_{-\pi/4}^{\pi/4}\frac{1-\sin x}{1-\sin^2 x}dx$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4}\left(\frac{1}{1+\sin x}\right)dx = \int_{-\pi/4}^{\pi/4}\frac{1-\sin x}{\cos^2 x}dx$$

($\because 1 - \sin^2 x = \cos^2 x$)

$$\Rightarrow \int_{-\pi/4}^{\pi/4}\left(\frac{1}{1+\sin x}\right)dx = \int_{-\pi/4}^{\pi/4}\frac{1}{\cos^2 x}dx - \int_{-\pi/4}^{\pi/4}\frac{\sin x}{\cos^2 x}dx$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4}\left(\frac{1}{1+\sin x}\right)dx = \int_{-\pi/4}^{\pi/4}\sec^2 x\,dx - \int_{-\pi/4}^{\pi/4}\tan(x)\sec(x)\,dx$$

Now, we know,

$$\int_a^b \sec^2 x\, dx = [\tan(x)]_a^b$$

And,

$$\int_a^b \tan(x)\sec(x)\, dx = [\sec(x)]_a^b$$

$$\therefore \int_{-\pi/4}^{\pi/4} \left(\frac{1}{1+\sin x}\right) dx = [\tan(x)]_{-\pi/4}^{\pi/4} - [\sec(x)]_{-\pi/4}^{\pi/4}$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4} \left(\frac{1}{1+\sin x}\right) dx = [\tan(\pi/4) - \tan(-\pi/4)] - [\sec(\pi/4) - \sec(-\pi/4)]$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4} \left(\frac{1}{1+\sin x}\right) dx = [1 - (-1)] - [-\sqrt{2} - (\sqrt{2}]$$

(∵ sec (– θ) = sec θ)

$$\Rightarrow \int_{-\pi/4}^{\pi/4} \left(\frac{1}{1+\sin x}\right) dx = [2] - [-2\sqrt{2}]$$

$$\Rightarrow \int_{-\pi/4}^{\pi/4} \left(\frac{1}{1+\sin x}\right) dx = 2 + 2\sqrt{2}$$

17. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \cos^2 x\ dx$$

Answer

Let I = $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2x)}{2} dx$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{2} dx + \int_0^{\frac{\pi}{2}} \frac{\cos(2x)}{2} dx$$

Using the formula:

$$\int_a^b \cos(ax)dx = [\frac{\sin(ax)}{a}]_a^b$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \left(\frac{1}{2}\right)[x]_0^{\pi/2} + \left(\frac{1}{2}\right)\left[\frac{\sin(2x)}{2}\right]_0^{\frac{\pi}{2}}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \left(\frac{1}{2}\right)\left[\frac{\pi}{2} - 0\right] + \left(\frac{1}{4}\right)\left[\left(\sin 2 \times \frac{\pi}{2}\right) - (\sin 2 \times 0)\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4} + \left(\frac{1}{4}\right)[\sin\pi - \sin 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4} + \left(\frac{1}{4}\right)[0 - 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{\pi}{4}$$

18. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \cos^3 x\ dx$$

Answer

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos(x) \times \cos^2(x)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos(x) \times (1 - \sin^2(x))\,dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos(x)\,dx - \int_0^{\frac{\pi}{2}} \cos(x)\sin^2(x)\,dx$$

Let sin x = t. Hence, cos x dx = dt, for the second expression.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [\sin x]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} t^2 dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [\sin x]_0^{\pi/2} - \left[\frac{t^3}{3}\right]_0^{\pi/2}$$

Put back t = sin(x)

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [\sin\frac{\pi}{2} - \sin 0] - (\frac{1}{3})[\sin^3 x]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [1 - 0] - \left(\frac{1}{3}\right)[\sin^3\frac{\pi}{2} - \sin^3 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [1] - \left(\frac{1}{3}\right)[1^3 - 0^3]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [1] - \left(\frac{1}{3}\right)[1]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [1] - \left(\frac{1}{3}\right)$$

$$\Rightarrow \frac{\int_0^{\frac{\pi}{2}} \cos^3 x dx = 2}{3}$$

19. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/6} \cos x \cos 2x \, dx$$

Answer

$$\int_0^{\frac{\pi}{6}} \cos x \times \cos(2x) dx = \int_0^{\frac{\pi}{6}} \cos x \times (2\cos^2 x - 1) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x) dx = \int_0^{\frac{\pi}{6}} (2\cos^3 x - \cos x \, dx)$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x) dx = 2\int_0^{\frac{\pi}{6}} \cos^3 x \, dx - \int_0^{\frac{\pi}{6}} \cos x \, dx$$

We know,

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x \, dx = \int_0^{\frac{\pi}{6}} \cos(x) \times \cos^2(x)$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x \, dx = \int_0^{\pi/6} \cos(x) \times (1 - \sin^2(x)) dx$$

$$\Rightarrow \int_0^{\pi/6} \cos^3 x \, dx = \int_0^{\frac{\pi}{6}} \cos(x) \, dx - \int_0^{\frac{\pi}{6}} \cos(x) \sin^2(x) \, dx$$

Let sin x = t. Hence, cos x dx = dt. For second expression,

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = [\sin x]_0^{\pi/6} - \int_0^{\frac{\pi}{6}} t^2 dt$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = [\sin x]_0^{\pi/6} - \left[\frac{t^3}{3}\right]_0^{\pi/6}$$

Put t = sin(x)

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = [\sin x]_0^{\pi/6} - \left[\frac{\sin^3 x}{3}\right]_0^{\pi/6}$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = \left[\sin\frac{\pi}{6} - \sin 0\right] - \left(\frac{1}{3}\right) [\sin^3 \frac{\pi}{6} - \sin^3 0]$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = \left[\frac{1}{2}\right] - \left(\frac{1}{3}\right) \left[\frac{1}{2}\right]^3$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = \left[\frac{1}{2}\right] - \left[\frac{1}{24}\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = (12-1)/24$$

$\Rightarrow \int_0^{\frac{\pi}{6}} \cos^3 x dx = (11)/24$ (equation 2)

From equation 2 put value of $\int_0^{\frac{\pi}{6}} \cos^3 x dx$ in equation 1.

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x) dx = 2\int_0^{\frac{\pi}{6}} (\cos^3 x) dx - \int_0^{\frac{\pi}{6}} \cos x dx$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x)\, dx = 2 \times \left(\frac{11}{24}\right) - \int_0^{\frac{\pi}{6}} \cos x dx$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x)\, dx = 2 \times \left(\frac{11}{24}\right) - [\sin x]_0^{\pi/6}$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x)\, dx = 2 \times \left(\frac{11}{24}\right) - \left[\sin\frac{\pi}{6} - \sin 0\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x)\, dx = 2 \times \left(\frac{11}{24}\right) - \left[\frac{1}{2} - 0\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x)\, dx = \left(\frac{11}{12}\right) - \left[\frac{1}{2}\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{6}} \cos x \times \cos(2x)\, dx = \left(\frac{5}{12}\right)$$

20. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \sin x \sin 2x\, dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \sin x \times \sin(2x) dx = \int_0^{\frac{\pi}{2}} \sin x \times 2 \times \sin x \cos x\, dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x) dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x)dx = 2\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos x dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x)dx = 2\int_0^{\frac{\pi}{2}} \cos x dx - 2\int_0^{\frac{\pi}{2}} \cos^3 x dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x)dx = 2[\sin x]_0^{\frac{\pi}{2}} - 2 \times \frac{2}{3}$$

First let us find,

$$\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \cos(x) \times \cos^2(x)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \int_0^{\pi/2} \cos(x) \times (1-\sin^2(x))\, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \cos(x)\, dx - \int_0^{\frac{\pi}{2}} \cos(x) \sin^2(x)\, dx$$

Let sin x = t. Hence, cos x dx = dt. For second expression,

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [\sin x]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} t^2 dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [\sin x]_0^{\pi/2} - \left[\frac{t^3}{3}\right]_0^{\pi/2}$$

Put t = sin(x)

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = [\sin x]_0^{\pi/2} - \left[\frac{\sin^3 x}{3}\right]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = \left[\sin\frac{\pi}{2} - \sin 0\right] - \left(\frac{1}{3}\right)\left[\sin^3\frac{\pi}{2} - \sin^3 0\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = 1 - \left(\frac{1}{3}\right)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^3 x dx = \frac{2}{3}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x)dx = 2 \times [1-0] - 2 \times \frac{2}{3}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x)dx = 2 - \left(\frac{4}{3}\right)$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin x \times \sin(2x)dx = \left(\frac{2}{3}\right)$$

21. Question

Evaluate the following definite integrals:

$$\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx$$

Answer

$$\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = \int_{\pi/3}^{\pi/4} (\tan^2 x + \cot^2 x + 2 \times \tan x \times \cot x)dx$$

We know, tan x × cot x = 1

$$\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = \int_{\pi/3}^{\pi/4} (\tan^2 x + \cot^2 x + 2)dx$$

We know, $\tan^2 x = \sec^2 x$ 1 1 and $\cot^2 x = \text{cosec}^2 x - 1$

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = \int_{\pi/3}^{\pi/4} (\sec^2 x - 1 + \text{cosec}^2 x - 1 + 2)dx$$

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = \int_{\pi/3}^{\pi/4} (\sec^2 x + \text{cosec}^2 x)dx$$

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = \int_{\pi/3}^{\pi/4} (\sec^2 x)dx + \int_{\pi/3}^{\pi/4} (\text{cosec}^2 x)dx$$

We know integration of $\sec^2 x$ is tanx and of $\text{cosec}^2 x$ is –cotx. Therefore,

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = [\tan x]_{\pi/3}^{\pi/4} + -[\cot x]_{\pi/3}^{\pi/4}$$

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = [\tan\frac{\pi}{4} - \tan\frac{\pi}{3}] - \left[\cot\frac{\pi}{4} - \cot\frac{\pi}{3}\right]$$

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = [1 - \sqrt{3}] - \left[1 - 1/\sqrt{3}\right]$$

$$\Rightarrow \int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx = \frac{1}{\sqrt{3}} - \sqrt{3}$$

22. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \cos^4 x \, dx$$

Answer

$$\int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x \times \cos^2 x dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right) \times \left(\frac{1+\cos 2x}{2}\right) dx$$

($\because 1 + \cos 2\theta = 2\cos^2\theta$)

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{4}\right) \int_0^{\frac{\pi}{2}} (1+\cos 2x)(1+\cos 2x) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{4}\right) \int_0^{\frac{\pi}{2}} (1+\cos^2 2x + 2\cos 2x) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{4}\right) \int_0^{\frac{\pi}{2}} (1+\frac{(1+\cos 4x)}{2} + 2\cos 2x) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) \int_0^{\frac{\pi}{2}} (2+(1+\cos 4x) + 4\cos 2x) dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) [\int_0^{\frac{\pi}{2}} (2dx + \int_0^{\frac{\pi}{2}} (1+\cos 4x) dx + \int_0^{\frac{\pi}{2}} (4\cos 2x) dx]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) \left[[2x]_0^{\frac{\pi}{2}} + [x]_0^{\frac{\pi}{2}} + [(\sin 4x)/4]_0^{\frac{\pi}{2}} + 4[(\sin 2x)/2]_0^{\frac{\pi}{2}}\right]$$

$$\int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) \left[\left[2 \times \frac{\pi}{2} - 0\right] + \left[\frac{\pi}{2} - 0\right] + \left(\frac{1}{4}\right)[\sin 4 \times \frac{\pi}{2} - \sin 0] + (4/2)[\sin 2 \times \frac{\pi}{2} - \sin 0]\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) \left[[\pi] + \left[\frac{\pi}{2}\right] + \left(\frac{1}{4}\right)[\sin 2 \times \pi] + (2)[\sin \pi]\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) \left[[\pi] + \left[\frac{\pi}{2}\right] + 0 + (2)[0]\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left(\frac{1}{8}\right) \left[[\pi] + \left[\frac{\pi}{2}\right]\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \left[\frac{3\pi}{16}\right]$$

23. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \left(a^2 \cos^2 x + b^2 \sin^2 x\right) dx$$

Answer

$$\int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} [a^2 \cos^2 x + b^2 (1 - \cos^2 x)] dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} [a^2 \cos^2 x + b^2 (1) - b^2 \cos^2 x)] dx$$

$$\int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx = \int_0^{\frac{\pi}{2}} [a^2 \cos^2 x] dx + b^2 \int_0^{\frac{\pi}{2}} 1 \times dx - b^2 \int_0^{\frac{\pi}{2}} [\cos^2 x] dx$$

$$\int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx = \frac{a^2}{2} [1 + \cos 2x]_0^{\pi/2} + b^2 [x]_0^{\pi/2} - \frac{b^2}{2} [1 + \cos 2x]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx$$

$$= \frac{a^2}{2} \left[\left(1 + \cos 2 \times \frac{\pi}{2}\right) - (1 - \cos 0)\right] + b^2 \left[\frac{\pi}{2} - 0\right]$$
$$- \frac{b^2}{2} \left[\left(1 + \cos 2 \times \frac{\pi}{2}\right) - (1 - \cos 0)\right]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx = \frac{a^2}{2} [(1 + -1) - (1 - 1)] + b^2 \left[\frac{\pi}{2}\right] -$$

$$\frac{b^2}{2} [(1 + -1) - (1 - 1)]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (a^2 \cos^2 x + b^2 \sin^2 x)\, dx = b^2 \left[\frac{\pi}{2}\right]$$

24. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \sqrt{1 + \sin x}\, dx$$

Answer

$$\int_0^{\frac{\pi}{2}} (\sqrt{1 + \sin x}) dx = \int_0^{\frac{\pi}{2}} (\sqrt{1 + \sin x}) \times \frac{\sqrt{1 - \sin x}}{\sqrt{1 - \sin x}} dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{\left(\sqrt{1-\sin^2 x}\right)}{\sqrt{1-\sin x}}\right)]dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\right] dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{\cos x}{\sqrt{1-\sin x}}\right] dx$$

Let 1 – sin x = t^2. Hence, – cos x dx = 2 t dt and cos x dx = – 2 t dt.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{-2t}{\sqrt{t^2}}\right] dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{-2t}{t}\right] dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = (-2)\int_0^{\frac{\pi}{2}} [1]dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = (-2)[t]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = (-2)\left[\sqrt{1-\sin x}\right]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = (-2)[\sqrt{1-\sin\left(\frac{\pi}{2}\right)} - \sqrt{1-\sin 0}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = (-2)[\sqrt{1-1} - \sqrt{1-0}\,]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = (-2)[-1\,]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\sin x}\right)dx = 2$$

25. Question

Evaluate the following definite integrals:

$$\int_0^{\pi/2} \sqrt{1+\cos x}\, dx$$

Answer

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right) \times \frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}}dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} [\frac{\left(\sqrt{1-\cos^2 x}\right)}{\sqrt{1-\cos x}})]dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{\sin^2 x}}{\sqrt{1-\cos x}}\right]dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{\sin x}{\sqrt{1-\cos x}}\right]dx$$

Let 1 – cos x = t^2 hence sin x dx = 2 t dt

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{2t}{\sqrt{t^2}}\right]dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} \left[\frac{2t}{t}\right]dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = (2)\int_0^{\frac{\pi}{2}} [1]dt$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = (2)[t]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = (2)\left[\sqrt{1-\cos x}\right]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = (2)[\sqrt{1-\cos\left(\frac{\pi}{2}\right)} - \sqrt{1-\cos 0}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = (2)[\sqrt{1-0} - \sqrt{1-1}\,]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = (2)[1\,]$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left(\sqrt{1+\cos x}\right)dx = 2$$

26. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/2} x^2 \sin x \, dx$$

Answer

We are asked to calculate $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use "ILATE" rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is choosen to make the integration simpler.

Now, In the given question x is an algebraic function and it is chosen as u (A comes first in "ILATE" rule)

So first let us integrate the equation and then let us substitute the limits in it

$$\int x \sin x \, dx = x \int \sin x dx - \int \int \sin x \times \left(\frac{dx}{dx}\right) dx$$

$$= -x\cos x + \int \cos x \, dx$$

Therefore, now substitute the limits given:

Note that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$\int_0^{\pi/2} x \sin x \, dx = [(-x\cos x + \int \cos x \, dx)]_0^{\pi/2}$$

$$= [(-x\cos x + \sin x)]_0^{\pi/2}$$

First we have to substitute the upper limit and then subtract the second limit value from it

$$= -\left(\frac{\pi}{2}\right) \times \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) + -(0 \times \cos 0 + \sin 0)$$

Note that sin0= 0 and cos0=1

=0+1+0–0

=1

27. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/2} x\cos x\, dx$$

Answer

We are asked to calculate $\int_0^{\frac{\pi}{2}} x \cos x \, dx$

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it

$$\int x \cos x \, dx = x\int \cos x dx - \int \int \cos x \times \left(\frac{dx}{dx}\right) dx$$

$$= x\sin x - \int \sin x \, dx$$

Therefore, now substitute the limits given:

Note that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx = [x\sin x - \int \sin x \, dx]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx = [x\sin x + \cos x]_0^{\frac{\pi}{2}}$$

First we have to substitute the upper limit and then subtract the second limit value from it

$$= \left[\frac{\pi}{2}\right] \sin\frac{\pi}{2} + \cos\frac{\pi}{2} - [0 \sin 0 + \cos 0]$$

$$= \left[\frac{\pi}{2}\right] 1 + 0 - [0 + 1]$$

$$= \frac{\pi - 2}{2}$$

28. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/2} x^2 \cos x \, dx$$

Answer

For this, we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference,, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

$$\int x^2 \cos x \, dx = x^2 \int \cos x dx - \int \int \cos x \times 2x) \, dx$$

$$= x^2 \sin x - \int 2x \times \sin x \, dx$$

$$= x^2 \sin x - 2[x \int \sin x \, dx - \int \int \sin x \, dx \, dx]$$

$$= x^2 \sin x - 2[-x \cos x - \int \cos x \, dx]$$

So now we have to substitute the limits in this equation.

And should subtract upper limit value from lower limit value

$$\int_0^{\pi/2} x^2 \cos x \, dx = [x^2 \sin x - 2[-x \cos x - \int \cos x \, dx]]_0^{\pi/2}$$

Sin $\frac{\pi}{2} = 1$, cos $\frac{\pi}{2} = 0$, sin 0 =0, cos0 = 1.

$$= [x^2 \sin x - 2[-x \cos x + \sin x]]_0^{\pi/2}$$

$$= \frac{\pi^2}{4} \sin\frac{\pi}{2} - 2\left[-\frac{\pi}{2} \cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - \{0 \times \sin 0 - 2[-0 \cos 0 + \sin 0]\}$$

$$= [\frac{\pi^2}{4} \times 1 - 2[0 + 1] - \{0 - 2[-0 \times 1 + 0]\}]$$

$$= \frac{\pi^2}{4} - 2 \cdot$$

29. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/4} x^2 \sin x \, dx$$

Answer

For this, we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference,, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

$$\int x^2 \sin x \, dx = x^2 \int \sin x dx - \int 2x(\int \sin x \, dx) \, dx$$

$$= -x^2 \cos x + 2[x \int \cos x \; dx - \int \int \cos x \, dx \; dx]$$

$$= -x^2 \cos x + 2[x \sin x - \int \sin x \, dx]$$

So now we have to substitute the limits in this equation.

And should subtract upper limit value from lower limit value

$$\int_0^{\pi/4} x^2 \sin x \, dx = [-x^2 \cos x + 2[x \sin x - \int \sin x \, dx]]_0^{\pi/4}$$

$\text{Sin}\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$, sin 0 =0, cos0 = 1

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$= [-x^2 \cos x + 2[x \sin x + \cos x]]_0^{\pi/4}$$

$$= [\frac{-\pi^2}{16} \times \cos\frac{\pi}{4} + 2[\frac{\pi}{4} \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \{-0 \times \cos 0 + [0 \times \sin 0 + 2\cos 0]\}]]$$

$$= [\frac{-\pi^2}{16} \times \frac{1}{\sqrt{2}} + 2\left[\frac{\pi}{4} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - \{0 + 2[0 + 1]\}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{-\pi^2}{16} + \frac{\pi}{2} + 2\right] - 2$$

$$= \sqrt{2} + \frac{\pi}{2\sqrt{2}} - \frac{\pi^2}{16\sqrt{2}} - 2$$

30. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/2} x^2 \cos 2x \, dx$$

Answer

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use "ILATE" rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in "ILATE" rule)

So first let us integrate the equation and then let us substitute the limits in it.

Note that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$\int x^2 \cos 2x \, dx = x^2 \int \cos 2x dx - \int \int \cos 2x \times \left(\frac{d2x}{dx}\right) dx$$

$$= \frac{x^2 \sin 2x}{2} - \int 2x \times \frac{\sin 2x}{2} \, dx$$

$$= \frac{x^2 \sin 2x}{2} - [x \int \sin 2x \, dx - \int \int \sin 2x \, dx \, dx]$$

$$= \frac{x^2 \sin 2x}{2} + [\frac{x \cos 2x}{2} - \int \frac{\cos 2x}{2}]$$

$$= \frac{x^2 \sin 2x}{2} + [\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}$$

$$\int_0^{\pi/2} x^2 \cos 2x \, dx = [\frac{x^2 \sin 2x}{2} + [\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}]_0^{\pi/2}$$

$$= \frac{\pi^2}{4} \times \sin\left(2 \times \frac{\pi}{2}\right) \times \frac{1}{2} + \left[\frac{\pi}{2} \times \cos\left(2 \times \frac{\pi}{2}\right) \times \frac{1}{2} - \sin\left(2 \times \frac{\pi}{2}\right) \times \frac{1}{4}\right]$$
$$- \{\frac{\pi^2}{4} \times \sin(2 \times 0) \times \frac{1}{2} + [\frac{\pi}{2} \times \cos(2 \times 0) \times \frac{1}{2} - \sin(2 \times 0) \times \frac{1}{4}]\}$$

$$= \frac{\pi^2}{8} \times 0 - 1 \times \frac{1}{4} \times 0 - \{0 + [\frac{\pi}{4}]\}$$

$$= -\frac{\pi}{4}$$

31. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/2} x^2 \cos^2 x \, dx$$

Answer

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

Let us recall a formula cos2x=2$\cos^2 x$–1

Now substitute it

$$\int x^2 \cos^2 x \, dx = \int x^2 \left(1 + \frac{\cos 2x}{2}\right) dx$$

$$= \frac{1}{2} \int (x^2 + x^2 \cos 2x) \, dx$$

$$= \frac{1}{2} \int (x^2 dx + \int x^2 \cos 2x dx)$$

Now let us recall other formula i.e $\int x^n = \frac{x^{n+1}}{n+1}$

$\int \sin x = -\cos x$ and $\int \cos x = \sin x$

Using them we can write the equation as

$$\int x^2 dx = [\frac{x^3}{3}]_0^{\frac{\pi}{2}} = \frac{\pi^2}{24}$$

$$\int x^2 \cos 2x \, dx = x^2 \int \cos 2x dx - \int \int \cos 2x \times (\frac{d2x}{dx}) \, dx$$

$$= \frac{x^2 \sin 2x}{2} = \int 2x \times \frac{\sin 2x}{2} \, dx$$

$$= \frac{x^2 \sin 2x}{2} - [x \int \sin 2x \, dx - \int \int \sin 2x \, dx \, dx]$$

$$= \frac{x^2 \sin 2x}{2} + [\frac{x \cos 2x}{2} - \int \frac{\cos 2x}{2}]$$

$= \frac{x^2 \sin 2x}{2} + [\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}$

$\int_0^{\pi/2} x^2 \cos 2x \, dx = [\frac{x^2 \sin 2x}{2} + [\frac{x \cos 2x}{2} - \frac{\sin 2x}{4}]_0^{\pi/2}$

$= \frac{\pi^2}{4} \times \sin\left(2 \times \frac{\pi}{2}\right) \times \frac{1}{2} + \left[\frac{\pi}{2} \times \cos\left(2 \times \frac{\pi}{2}\right) \times \frac{1}{2} - \sin\left(2 \times \frac{\pi}{2}\right) \times \frac{1}{4}\right]$

$- \{\frac{\pi^2}{4} \times \sin(2 \times 0) \times \frac{1}{2} + [\frac{\pi}{2} \times \cos(2 \times 0) \times \frac{1}{2} - \sin(2 \times 0) \times \frac{1}{4}]$

$= \frac{\pi^2}{8} \times 0 - 1 \times \frac{1}{4} \times 0 - \{0 + [\frac{\pi}{4}]\}$

$= \frac{\pi}{4}$

On substituting these values we get

$\int_0^{\frac{\pi}{2}} x^2 \cos^2 x \, dx = \frac{1}{2}\left[\frac{\pi^3}{24} - \frac{\pi}{4}\right]$

$= \frac{\pi^3}{48} - \frac{\pi}{8}$

32. Question

Evaluate the following definite Integrals:

$\int_1^2 \log x \, dx$

Answer

For this we have to apply integration by parts

Let u and v be two functions then

$\int udv = uv - \int vdu$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question 1 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

$\int \log x dx = \int 1 \times \log x = \log x \int x dx - \int \frac{1}{x}(\int dx) \, dx$

Let us recall that derivative of logx is 1/x

$= x\log x - \int x \times \frac{1}{x} dx$

$= x\log x - \int dx$

=xlogx–x

Now let us substitute the limits

$\int_1^2 \log x\, dx = [x\log x - x]_1^2$

= 2 log2–2–[1log1–1]

=2log2–1

33. Question

Evaluate the following definite Integrals:

$$\int_1^3 \frac{\log x}{(x+1)^2} dx$$

Answer

For this we have to apply integration by parts

Let u and v be two functions then

$\int udv = uv - \int vdu$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

$\int \frac{\log x dx}{(x+1)^2} = \int \frac{1}{(x+1)^2} \times \log x\, dx$

$= \log x \times \int \frac{1}{(x+1)^2} dx - \int (\int \frac{1}{(x+1)^2} dx) \frac{1}{x} dx$

$= -\frac{\log x}{x+1} + \int \frac{1}{x(x+1)} dx$

$= -\frac{\log x}{x+1} + \int \frac{(x+1-x)}{x(x+1)} dx$

$= -\frac{\log x}{x+1} + \int (\frac{1}{x} - \frac{1}{x+1}) dx$

Now we will substitute the limits

$\int_1^3 \frac{\log x dx}{(x+1)^2} dx = [-\frac{\log x}{x+1} + \int (\frac{1}{x} - \frac{1}{x+1}) dx]_1^3$

$\int_1^3 \frac{\log x dx}{(x+1)^2} dx = [-\frac{\log x}{x+1} + \log x - \log(x+1)]_1^3$

$= -\frac{\log 3}{3+1} + \log 3 - \log(3+1) - \{-\frac{\log 1}{1+1} + \log 1 - \log(1+1)\}$

$\log 1 = 0$ and $\log 4 = 2\log 2$

$= \frac{3\log 3}{4} - \log 2$

34. Question

Evaluate the following definite Integrals:

$$\int_1^e \frac{e^x}{x}(1 + x \log x)dx$$

Answer

let us assume that the given equation is L

$L = \int_1^e \frac{e^x}{x}(1 + x\log x)dx$

$l = \int_1^e \frac{e^x}{x}dx + \int_1^e e^x \log x dx$

For this we have to apply integration by parts

Let u and v be two functions then

$\int udv = uv - \int vdu$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

$= [e^x \log x]_1^e - \int_1^e e^x \log x\, dx + \int_1^e e^x \log x\, dx$

$= [e^x \log x]_1^e$

$= e^e \log e - e\log 1$

We know that loge=1

$= e^e - 0$

$= e^e$

35. Question

Evaluate the following definite Integrals:

$$\int_1^e \frac{\log x}{x}dx$$

Answer

Here in this question by observation we can notice that the derivative of logx is 1/x and the function integral is like

$$\int f(x)d(f(x))\,dx = f(x)2/2$$

Here to solve these kinds of question let us assume logx=t

Now $\frac{1}{x}dx = dt$

Now let us change the limits

x=1 then t=0

x=e then t=1

$$\int_1^e \frac{\log x}{x}dx = \int_0^1 t\,dt$$

$$= [\frac{t^2}{2}]_0^1$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

36. Question

Evaluate the following definite Integrals:

$$\int_e^{e^2}\left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\}dx$$

Answer

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use "ILATE" rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

$$\int \frac{1}{\log x}dx - \frac{1}{\log x}\int dx - \iint dx\,\frac{d}{dx}\,(\frac{1}{\log x})dx$$

$$= \frac{x}{\log x} + \int \frac{1dx}{(\log x)^2}$$

Now let us substitute in the given question equation

$$\int_e^{e^2} \frac{1}{\log x} - \frac{1}{(\log x)^2} dx = [\frac{x}{\log x}]_e^{e^2} + \int_e^{e^2} \frac{1dx}{(\log x)^2} - \int_e^{e^2} \frac{1dx}{(\log x)^2}$$

$$= \left[\frac{x}{\log x}\right]_e^{e^2}$$

$$= \frac{e^2}{\log e^2} - \frac{e}{\log e}$$

$$= \frac{e^2}{2} - e$$

37. Question

Evaluate the following definite Integrals:

$$\int_1^2 \frac{x+3}{x(x+2)} dx$$

Answer

$$= \int_1^2 \frac{x}{x(x+2)} dx + \int_1^2 \frac{3}{x(x+2)} dx$$

$$= \int_1^2 \frac{1}{(x+2)} dx + \int_1^2 \frac{3}{x(x+2)} dx$$

Here we are solving the equation, recall $\frac{1}{x+2}$ is the derivative of log(x+2) and splitting the second one

$$= [\log(x+2)]_1^2 + \frac{3}{2}\int (\frac{1}{x} - \frac{1}{x+2}) dx$$

$$= [\log(x+2)]_1^2 - [\frac{3}{2}\log(x) - \frac{3}{2}\log(x+2)]_1^2$$

$$= [\frac{3}{2}\log(x) - \frac{1}{2}\log(x+2)]_1^2$$

$$= \frac{3}{2}\log(2) - \frac{1}{2}\log(2+2) - \{\frac{3}{2}\log(1) - \frac{1}{2}\log(1+2)\}$$

$$= \frac{1}{2}[3\log 2 - \log 4 + \log 3]$$

Note that log4=2log2 and log1=0

$$= \frac{1}{2}[3\log 2 - 2\log 2 + \log 3]$$

$$= \frac{1}{2}[\log 2 + \log 3]$$

$$= \frac{1}{2}[\log 6]$$

38. Question

Evaluate the following definite Integrals:

$$\int_0^1 \frac{2x+3}{5x^2+1} dx$$

Answer

If the equation is in this form then convert the numerator as sum of derivative of denominator and some constant

Here we know that denominator derivative is 10x

So to get it in the numerator multiply and divide by 5

Now you get the equation as

$$\int_0^1 \frac{2x+3dx}{5x^2+1} = \int_0^1 \frac{5(2x+3)}{5x^2+1} dx \times \frac{1}{5}$$

$$= \int_0^1 \frac{10x+15}{5x^2+1} dx \times \frac{1}{5}$$

$$= \int_0^1 \frac{10x}{5x^2+1} dx \times \frac{1}{5} + \int_0^1 \frac{15}{5x^2+1} dx \times \frac{1}{5}$$

$$= \int_0^1 \frac{10x}{5x^2+1} dx \times \frac{1}{5} + \int_0^1 \frac{3}{5x^2+1} dx$$

$$= \int_0^1 \frac{10x}{5x^2+1} dx \times \frac{1}{5} + \int_0^1 \frac{3}{5(x^2+\frac{1}{5})} dx \times$$

We already know that derivative of logx is 1/x

Using that here derivative of $\log(5x^2+1) = \frac{10x}{5x^2+1}$

And derivative of $\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}$

So $= \frac{1}{5}\log(5x^2+1) + \frac{3}{5} \times \frac{1}{\frac{1}{\sqrt{5}}} \times \tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$

$$= \frac{1}{5}\log(5x^2+1) + \frac{3}{\sqrt{5}} \times \tan^{-1}\sqrt{5x}$$

Now substitute limits 1 and 0

$$= \frac{1}{5}\log(5\times1+1) + \frac{3}{\sqrt{5}} \times \tan^{-1}\sqrt{5\times1} - \{\frac{1}{5}\log(5\times0+1) + \frac{3}{\sqrt{5}} \times \tan^{-1}\sqrt{5\times0}\}$$

$$= \frac{1}{5}\log(6) - \frac{3}{\sqrt{5}} \times \tan^{-1}\sqrt{5} - \{\frac{1}{5}\log(1) + 0\}$$

$$= \frac{1}{5}\log(6) - \frac{3}{\sqrt{5}} \times \tan^{-1}\sqrt{5}$$

39. Question

Evaluate the following definite Integrals:

$$\int_0^2 \frac{1}{4+x-x^2} dx$$

Answer

$$= \int_0^2 \frac{1}{4+x-x^2} dx$$

Since it is a quadratic equation we are trying to make it a complete square

$= \int_0^2 \frac{1}{-(-4-x+x^2)} dx$

$= \int_0^2 \frac{1}{-(-4-x+x^2+\frac{1}{4}-\frac{1}{4})} dx$

$= \int_0^2 \frac{1}{-((x-\frac{1}{2})^2-\frac{17}{4})} dx$

$= \int_0^2 \frac{1}{(\frac{\sqrt{17}}{2})^2-((x-\frac{1}{2})^2)} dx$

=here the equation is in the form of integral of $\frac{1}{x^2+a^2}$ the integral is equal to $\frac{1}{2a} \times \log(\frac{a+x}{a-x})$

Here let us assume that t=x-$\frac{1}{2}$

So that dx=dt

When x=0 t=-1/2

And when x=-1/2, t=3/2

$\int_0^2 \frac{1}{(\frac{\sqrt{17}}{2})^2-((x-\frac{1}{2})^2)} dx = \int_{\frac{-1}{2}}^{\frac{3}{2}} \frac{1}{(\frac{\sqrt{17}}{2})^2-(t)^2)} dt$

$= \frac{1}{2\frac{\sqrt{17}}{2}} \times \log(\frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t})_{\frac{-1}{2}}^{\frac{3}{2}}$

$= \frac{1}{\sqrt{17}} \times [\log(\frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \log(\frac{\frac{\sqrt{17}}{2}-\frac{1}{2}}{\frac{\sqrt{17}}{2}+\frac{1}{2}})\ ]$

$= \frac{1}{\sqrt{17}} \times [\log(\frac{\sqrt{17}+3}{\sqrt{17}-3} - \log(\frac{\sqrt{17}-1}{\sqrt{17}+1})\ ]$

$= \frac{1}{\sqrt{17}} \times [\log(\frac{\sqrt{17}+3}{\sqrt{17}-3}) \times (\frac{\sqrt{17}+1}{\sqrt{17}-1}\ )]$

$= \frac{1}{\sqrt{17}} \times [\log(\frac{4\sqrt{17}+17+3}{17-4\sqrt{17}+3})$

$= \frac{1}{\sqrt{17}} \times [\log(\frac{\sqrt{17}+5}{-\sqrt{17}+5})$

Now rationalize the denominator

We get as

$= \frac{1}{\sqrt{17}} \times [\log(\frac{5\sqrt{17}+21}{4})$

40. Question

Evaluate the following definite Integrals:

$\int_0^1 \frac{1}{2x^2+x+1} dx$

Answer

Since the denominator is a quadratic equation let us make it in form of a perfect square

$$2x^2 + x + 1 = (x^2 + \frac{x}{2} + \frac{1}{2}) \times 2$$

$$= ((x + \frac{1}{4})^2 + \frac{1}{2} - \frac{1}{16}) \times 2$$

$$= ((x + \frac{1}{4})^2 + \frac{7}{16}) \times 2$$

$$= ((x + \frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2) \times 2$$

Now the equation

$$\int_0^1 \frac{1}{2x^2 + x + 1} = \int_0^1 \frac{1}{((x + \frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2) \times 2}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{((x + \frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2)}$$

derivative of $\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}$ using this we can write it as

$$= \frac{4}{2\sqrt{7}}[\tan^{-1}(\frac{x + \frac{1}{4}}{\frac{\sqrt{7}}{4}})]_0^1$$

$$= \{\frac{2}{\sqrt{7}}\{\tan^{-1}\frac{5}{\sqrt{7}} - \tan^{-1}\frac{1}{\sqrt{7}}\}\}$$

41. Question

Evaluate the following definite Integrals:

$$\int_0^1 \sqrt{x(1-x)}\, dx$$

Answer

To solve these kinds of equations we generally take x=$\sin^2\theta, \cos^2\theta$

So now here let $x = \sin^2\theta$

So now $dx = 2\sin\theta\cos\theta d\theta$

Now change the limits

X=0 then θ=0

X=1 then $\theta=\frac{\pi}{2}$

So it is equal to

$$= \int_0^{\frac{\pi}{2}} \sqrt{\sin^2\theta\,(1 - \sin^2\theta)}\; 2\sin\theta\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} 2\sin^2\theta\cos^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta d\theta \; ½$$

now use formula sin2x=2sinxcosx

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2\,\theta\,d\theta$$

Now use formula that $\cos 2x = 1 - 2\sin^2\theta$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4\theta}{2}\,d\theta$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}(1-\cos 4\theta\,d\theta)$$

$$=\frac{1}{4}\int_0^{\frac{\pi}{2}}d\theta-\int_0^{\frac{\pi}{2}}\cos 4\theta d\theta\frac{1}{4}$$

Now let us recall other formula :

$\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$=\frac{1}{4}[\theta]_0^{\frac{\pi}{2}}-\frac{1}{4}[\frac{\sin 4\theta}{4}]_0^{\frac{\pi}{2}}$$

Now recall that sin0=0, cos0=1

$$=\frac{1}{4}[\frac{\pi}{2}-\frac{1}{4}\times\sin 4\times\frac{\pi}{2}-\{0-\frac{1}{4}\times\sin 4\times 0\}]$$

$$=\frac{\pi}{8}-0-\{0-0\}$$

$$=\frac{\pi}{8}$$

42. Question

Evaluate the following definite Integrals:

$$\int_0^2\frac{1}{\sqrt{3+2x-x^2}}dx$$

Answer

Here the equation is of form that a quadratic equation is in the root so now to solve this make the equation in the root in the form of a^2-x^2, a^2+x^2

Here $3+2x-x^2=3+1-(1-2x+x^2)$

$=4-(x-1)^2$

$=2^2-(x-1)^2$

Now just recall a formula that is derivative of $\sin^{-1}\frac{x}{a}=\frac{1}{\sqrt{a^2-x^2}}$

Here a=2 and x =x–1

Now we get

$$\int_0^2\frac{1}{\sqrt{3+2x-x^2}}dx=[\sin^{-1}\frac{x-1}{2}]_0^2$$

$$=\sin^{-1}\frac{2-1}{2}-\sin^{-1}\frac{0-1}{2}$$

$$=\frac{\pi}{6}+\frac{\pi}{6}$$

$= \frac{\pi}{3}$

43. Question

Evaluate the following definite Integrals:

$$\int_0^4 \frac{1}{\sqrt{4x - x^2}} dx$$

Answer

Here first we are converting the quadratic equation in to a perfect square

$4x - x^2 = 4 - 4 + 4x - x^2$

$= 4 - (4 - 4x + x^2)$

$= 4 - (x - 2)^2$

$= (2)^2 - (x - 2)^2$

Now just recall a formula that is derivative of $\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$

Here a=2 and x =x–2

Now we get

$$\int_0^4 \frac{1}{\sqrt{4x - x^2}} dx = [\sin^{-1}\frac{x-2}{2}]_0^4$$

$$= \sin^{-1}\frac{4-2}{2} - \sin^{-1}\frac{0-2}{2}$$

$= \sin^{-1} 1 - \sin^{-1} -1$

$= \frac{\pi}{2} - (-\frac{\pi}{2})$

$= \pi$

44. Question

Evaluate the following definite Integrals:

$$\int_{-1}^{1} \frac{1}{x^2 + 2x + 5} dx$$

Answer

Now denominator is in a quadratic form so let us make it in other form

$x^2 + 2x + 5 = x^2 + 2x + 1 + 4$

$= (x + 1)^2 + 4$

$= (x + 1)^2 + 2^2$

Recall a formula $\tan^{-1}(\frac{x}{a}) = \frac{a}{a^2 + x^2}$

So now $.\int_{-1}^{1} \frac{1}{x^2+2x+5} = [\tan^{-1}(\frac{x+1}{2})]_{-1}^{1} \times \frac{1}{2}$

$$= (\tan^{-1}(\frac{1+1}{2}) - \tan^{-1}\left(\frac{-1+1}{2}\right)) \times \frac{1}{2}$$

$= (\tan^{-1}(1) - \tan^{-1}(0)) \times \frac{1}{2}$

$= \frac{\pi}{4} \times \frac{1}{2} = \frac{\pi}{8}$

45. Question

Evaluate the following definite Integrals:

$$\int_1^4 \frac{x^2 + x}{\sqrt{2x+1}} dx$$

Answer

To solve this let us assume that 2x+1=t^2

2 dx=2t dt

So now x=1, $t = \sqrt{3}$

X=4, t=3

So now after substitution we get

$$\int_1^4 \frac{x^2 + x}{\sqrt{2x+1}} dx = \int_{\sqrt{3}}^3 \frac{\frac{t^2-1}{2})^2 + \frac{t^2-1}{2}}{t} tdt$$

$$= \frac{1}{4}\int_{\sqrt{3}}^3 (t^4 - 2t^2 + 1 + 2t^2 - 2)dt$$

$$= \frac{1}{4}\int_{\sqrt{3}}^3 (t^4 - 1)dt$$

Now let us recall other formula i.e $\int x^n = \frac{x^{n+1}}{n+1}$

$$= \frac{1}{4}[\frac{t^5}{5} - t]_{\sqrt{3}}^3$$

$$= \frac{1}{4}\left[\frac{3^5}{5} - 3\right] - \frac{1}{4}\left[\frac{\sqrt{3}^5}{5} - \sqrt{3}\right]$$

$$= \frac{57 - \sqrt{3}}{5}$$

46. Question

Evaluate the following definite Integrals:

$$\int_0^1 x(1-x)^5 dx$$

Answer

Now to make it simpler problem let us expand $(1-x)^5$using binomial theorem

So $(1-x)^5 = 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$

Let us also recall other formula i.e.,$\int x^n = \frac{x^{n+1}}{n+1}$

So now

$$\int_0^1 x(1-x)^5\,dx = \int_0^1 x(1-5x+10x^2-10x^3+5x^4-x^5)\,dx$$

$$= [\frac{x^2}{2}-\frac{5x^3}{3}+\frac{10x^4}{4}-\frac{10x^5}{5}+\frac{5x^6}{6}-\frac{x^7}{7}]_0^1$$

$$= \frac{1}{2}-\frac{5}{3}+\frac{10}{4}-\frac{10}{5}+\frac{5}{6}-\frac{1}{7}-\{\frac{0}{2}-\frac{0}{3}+\frac{0}{4}-\frac{0}{5}+\frac{0}{6}-\frac{0}{7}\}$$

$$= \frac{1}{42}$$

47. Question

Evaluate the following definite Integrals:

$$\int_1^2 \left(\frac{x-1}{x^2}\right)e^x\,dx$$

Answer

$$\int_1^2 \frac{x-1}{x^2}e^x dx = \int_1^2 \frac{1}{x}e^x dx - \int_1^2 \frac{1}{x^2}e^x dx$$

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

Here we are expanding only first integral first

$$\int_1^2 \frac{1}{x}e^x dx - \int_1^2 \frac{1}{x^2}e^x dx = \frac{1}{x}\int_1^2 e^x dx - \int_1^2 \int e^x dx \times \frac{d\frac{1}{x}}{dx} - \int_1^2 \frac{1}{x^2}e^x dx$$

$$= [\frac{e^x}{x}]_1^2 + \int_1^2 \frac{1}{x^2}e^x dx - \int_1^2 \frac{1}{x^2}e^x dx$$

$$= \left[\frac{e^x}{x}\right]_1^2$$

$$= \frac{e^2}{2} - \frac{e^1}{1}$$

$= \frac{e^2}{2} - e$

48. Question

Evaluate the following definite Integrals:

$$\int_0^1 \left(xe^{2x} + \sin\frac{\pi x}{2} \right) dx$$

Answer

First split the integral $\int_0^1 (xe^{2x})dx + \int_0^1 (\sin\frac{\pi x}{2})dx$

Now integrate by parts the first one

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

Remember $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$= \int_0^1 (xe^{2x})dx + \int_0^1 (\sin\frac{\pi x}{2})dx$$

$$= \frac{xe^x}{2} - \frac{1}{2}\int e^{2x} + \frac{2}{\pi}[1-0]$$

$$= \frac{2e^2}{2} - \frac{e^2}{2} + \frac{1}{4} + \frac{2}{\pi}$$

$$= \frac{2e^2}{2} + \frac{1}{4} + \frac{2}{\pi}$$

49. Question

Evaluate the following definite Integrals:

$$\int_0^1 \left(xe^{x} + \cos\frac{\pi x}{4} \right) dx$$

Answer

First split the integral $\int_0^1 (xe^{x})dx + \int_0^1 (\cos\frac{\pi x}{4})dx$

Now integrate by parts the first one

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

Remember $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$= \int_0^1 (xe^x)dx + \int_0^1 (\cos\frac{\pi x}{4})dx$$

$$= [xe^x]_0^1 - \int e^x + \frac{4}{\pi}[\cos\frac{\pi x}{4}]_0^1$$

$$= [xe^x]_0^1 - [e^x]_0^1 + \frac{4}{\pi}[\cos\frac{\pi x}{4}]_0^1$$

$$= [e - 0] - [e - 0] + \frac{4}{\pi}[\cos\frac{\pi}{4} - \cos\frac{\pi \times 0}{4}]$$

$$= 1 + \frac{2\sqrt{2}}{\pi}$$

50. Question

Evaluate the following definite Integrals:

$$\int_{\pi/2}^{\pi} e^x \left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Answer

Now using the formula

Sin2x=2sinxcosx

$\text{Cos}2x = 1 - 2\sin^2 x$

$$\int_{\frac{\pi}{2}}^{\pi} e^x(\frac{1-\sin x}{1-\cos x})dx = \int_{\frac{\pi}{2}}^{\pi} e^x(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}})dx$$

$$= -\int_{\frac{\pi}{2}}^{\pi} e^x(-\frac{1}{2}\text{cosec}^2\frac{x}{2} + \cot\frac{x}{2})dx$$

Here we know that derivative of $\cot\frac{x}{2}$ is $= -\frac{1}{2}\mathrm{cosec}^2\frac{x}{2}$

And it is in the form of $e^x(\text{function} + \text{derivative of function})$ so the equation integral will be e^xfunction

$= -[e^x \cot\frac{x}{2}]_{\frac{\pi}{2}}^{\pi}$

$= -[e^\pi \cot\frac{\pi}{2} - e^{\frac{\pi}{2}} \cot\frac{\frac{\pi}{2}}{2}]$

$\cot 90 = 0, \cot 45 = 1$

$= e^{\frac{\pi}{2}}$

51. Question

Evaluate the following definite Integrals:

$$\int_0^{2\pi} e^{x/2} \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$$

Answer

We know that $\sin\left(\frac{x}{2} + \frac{\pi}{4}\right) = \sin\left(\frac{x}{2}\right)\cos\left(\frac{\pi}{4}\right) + \sin(\frac{\pi}{4})\cos(\frac{x}{2})$

So the equation will be

$= \int_0^{2\pi} e^{\frac{x}{2}} (\sin\left(\frac{x}{2}\right)\cos\left(\frac{\pi}{4}\right) + \sin(\frac{\pi}{4})\cos(\frac{x}{2}))dx$

We know that cos45=sin45=$\frac{1}{\sqrt{2}}$

Substitute it

$= \int_0^{2\pi} e^{\frac{x}{2}} (\sin\left(\frac{x}{2}\right)\frac{1}{\sqrt{2}} + \cos(\frac{x}{2})\frac{1}{\sqrt{2}})dx$

$= \frac{1}{\sqrt{2}}(\int_0^{2\pi} e^{\frac{x}{2}} (\sin\left(\frac{x}{2}\right) + \cos(\frac{x}{2}))dx)$

$= \frac{1}{\sqrt{2}}(\int_0^{2\pi} e^{\frac{x}{2}} \sin\left(\frac{x}{2}\right)dx) + \frac{1}{\sqrt{2}}(\int_0^{2\pi} e^{\frac{x}{2}} \cos(\frac{x}{2})dx)$

Remember that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

Now integrate by partsFor this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in "ILATE" rule)

So first let us integrate the equation and then let us substitute the limits in it.

$= \frac{1}{\sqrt{2}}\{\sin\left(\frac{x}{2}\right)\int_0^{2\pi} e^{\frac{x}{2}}\,dx - \int_0^{2\pi}\int_0^{2\pi} e^{\frac{x}{2}}\,dx\,d\sin\left(\frac{x}{2}\right) + \frac{1}{\sqrt{2}}\{\cos\left(\frac{x}{2}\right)\int_0^{2\pi} e^{\frac{x}{2}}\,dx -$ $\}$

$\int_0^{2\pi}\int_0^{2\pi} e^{\frac{x}{2}}\,dx\,d\cos\left(\frac{x}{2}\right)$

$= [\frac{1}{\sqrt{2}}\{\sin\left(\frac{x}{2}\right) 2e^{\frac{x}{2}}]_0^{2\pi}$

$= \frac{1}{\sqrt{2}}\{\sin\left(\frac{2\pi}{2}\right) 2e^{\frac{2\pi}{2}} - \frac{1}{\sqrt{2}}\{\sin\left(\frac{0}{2}\right) 2e^{\frac{2\times 0}{2}}$

=0–0=0

52. Question

Evaluate the following definite Integrals:

$$\int_0^{2\pi} e^x \cos\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$$

Answer

Now let us use integration by parts

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use "ILATE" rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in "ILATE" rule)

So first let us integrate the equation and then let us substitute the limits in it.

Remember that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$= [\cos(\frac{x}{2}+\frac{\pi}{4})e^x]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin(\frac{x}{2}+\frac{\pi}{4})dx$

$L = [\cos(\frac{x}{2}+\frac{\pi}{4})e^x]_0^{2\pi} + \frac{1}{2}\{[\sin(\frac{x}{2}+\frac{\pi}{4})e^x]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} e^x \cos\left(\frac{x}{2}+\frac{\pi}{4}\right)dx\}$

$\}$

$L = [\{\cos\left(\frac{5\pi}{4}\right)e^{2\pi} - \cos\left(\frac{\pi}{4}\right)\} + \frac{1}{2}\left[\sin\left(\frac{5\pi}{4}\right)e^{2\pi} - \sin\left(\frac{\pi}{4}\right)\right] - \frac{L}{4}$

$$\frac{5L}{4} = -\frac{1}{\sqrt{2}}(e^{2\pi}+1) - \frac{1}{2\sqrt{2}}(e^{2\pi}+1)$$

$$L = -\frac{3\sqrt{2}}{5}(e^{2\pi}+1)$$

53. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi} e^{2x}\sin\left(\frac{\pi}{4}+x\right)dx$$

Answer

$$\text{let } I = \int_0^{\pi} e^{2x}\sin\left(x+\frac{\pi}{4}\right)dx$$

$$= \sin\left(x+\frac{\pi}{4}\right)\int_0^{\pi} e^{2x}dx - \int_0^{\pi}\frac{e^{2x}}{2}d\left(\sin\left(x+\frac{\pi}{4}\right)\right)dx$$ limit 0 to pi

$$= \sin\left(x+\frac{\pi}{4}\right)\frac{e^{2x}}{2} - \cos\left(x+\frac{\pi}{4}\right)\frac{e^{2x}}{4} - \frac{\int_0^{\pi} e^{2x}\sin\left(x+\frac{\pi}{4}\right)}{4}dx$$

$$\frac{5I}{4} = \sin\left(x+\frac{\pi}{4}\right)\frac{e^{2x}}{2} - \cos\left(x+\frac{\pi}{4}\right)\frac{e^{2x}}{4}$$ with limits 0 to pi

$$I = \frac{4}{5}[\sin\left(x+\frac{\pi}{4}\right)\frac{e^{2x}}{2} - \cos\left(x+\frac{\pi}{4}\right)\frac{e^{2x}}{4}]_0^{\pi}$$

$$= \sin\left(\pi+\frac{\pi}{4}\right)\frac{e^{2\pi}}{2} - \cos\left(\pi+\frac{\pi}{4}\right)\frac{e^{2\pi}}{4} - \{\sin\left(0+\frac{\pi}{4}\right)\frac{e^{0}}{2} - \cos\left(0+\frac{\pi}{4}\right)\frac{e^{0}}{4}\}$$

$$= -\frac{1}{\sqrt{2}}\frac{e^{2\pi}}{2} - \frac{1}{\sqrt{2}}\frac{e^{2\pi}}{4} - \{\frac{1}{2\sqrt{2}} - \frac{1}{4\sqrt{2}}\}$$

$$= -\frac{3}{\sqrt{2}}\frac{e^{2\pi}}{4} - \frac{1}{4\sqrt{2}}$$

54. Question

Evaluate the following definite Integrals:

$$\int_0^{1}\frac{1}{\sqrt{1+x}-\sqrt{x}}dx$$

Answer

Now let it be taken as I

$$I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}dx$$

Now rationalize the denominator

$$I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} \times \frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x}dx$$

$$= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1}dx$$

Let us also recall formula $\int x^n \, dx = \frac{x^{n+1}}{n+1}$

$$= \int_0^1 \sqrt{1+x}dx + \int_0^1 \sqrt{x}dx$$

$$= \frac{2}{3}[(1+x)^{\frac{3}{2}}]_0^1 + \frac{2}{3}[(x)^{\frac{3}{2}}]_0^1$$

$$= \frac{2}{3}(\left(2\right)^{\frac{3}{2}} - 1\right) + \frac{2}{3}$$

$$= \frac{2}{3}((2)^{\frac{3}{2}})$$

$$= \frac{4\sqrt{2}}{3}$$

55. Question

Evaluate the following definite Integrals:

$$\int_1^2 \frac{x}{(x+1)(x+2)} dx$$

Answer

$$= \int_1^2 \frac{2x+2-(x+1)}{(x+1)(x+2)} dx$$

$$= \int_1^2 \frac{2}{(x+2)} dx - \int_1^2 \frac{1}{(x+1)} dx$$

Remember derivative of $\log x = \frac{1}{x}$

So using that

$$= [-\log(x+1) + 2\log(x+2)]_1^2$$

Substitute upper limit and then subtract the lower limit from it

=–(log3–log2)+2(log4–log3)

=–3log3+5log2

$$= \log(\frac{32}{27})$$

56. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi/2} \sin^3 x \, dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x \, dx - \int_0^{\frac{\pi}{2}} (\cos^2 x) \sin x \, dx$$

Remember that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$= [-\cos x]_0^{\frac{\pi}{2}} + [\frac{\cos^3 x}{3}]_0^{\frac{\pi}{2}}$$

$$= -\cos\frac{\pi}{2} - -\cos 0 + \{\frac{\cos^3\frac{\pi}{2}}{3} - \frac{\cos^3 0}{3}\}$$

$\cos 90 = 0, \cos 0 = 1$

$$= -0 + 1 - \frac{1}{3} = \frac{2}{3}$$

57. Question

Evaluate the following definite Integrals:

$$\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) dx$$

Answer

$$I = \int_0^{\pi} \sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} dx$$

$$= -\int_0^{\pi} \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} dx$$

$$= -\int_0^{\pi} \cos x dx$$

Because we have a formula $\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$

Remember that $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$= [-\sin x]_0^{\pi}$$

$$= [-(\sin\pi - \sin 0)]$$

$\sin 180 = 0, \sin 0 = 0$

=0

58. Question

Evaluate the following definite Integrals:

$$\int_1^2 e^{2x} \left(\frac{1}{x} - \frac{1}{2x^2} \right) dx$$

Answer

Let 2x=t then 2dx=dt

When x=1 t=2

And when x=2,t=4

$$\int_1^2 e^{2x} (\frac{1}{x} - \frac{1}{2x^2}) dx = \frac{1}{2} \int_2^4 e^t (\frac{2}{t} - \frac{2}{t^2}) dt$$

$= \int_2^4 e^t(\frac{1}{t} - \frac{1}{t^2})dt$

We can observe here that

Derivative of $\frac{1}{x} = -\frac{1}{x^2}$

Now it is in the form $e^x(\text{function} + \text{derivative of function})$

So the integral will be $e^x(\text{function})$

$\int_2^4 e^t(\frac{1}{t} - \frac{1}{t^2})dt = [\frac{e^t}{t}]_2^4$

$= \frac{e^4}{4} - \frac{e^2}{2}$

$= \frac{(e^4 - 2e^2)}{4}$

59. Question

Evaluate the following definite Integrals:

$$\int_1^2 \frac{1}{(x-1)(2-x)} dx$$

Answer

Let us solve the denominator

$(x-1)(2-x) = 2x - 2 - x^2 + x$

$=3x-x^2-2$

$= -(x - \frac{2}{3})^2 + \frac{1}{4}$

$\int_1^2 \frac{1}{\sqrt{(x-1)(2-x)}} dx = \int_1^2 \frac{1}{\sqrt{-(x-\frac{2}{3})^2 + \frac{1}{4}}} dx$

$= \int_1^2 \frac{1}{\sqrt{-(x-\frac{2}{3})^2 + \frac{1}{2}^2}} dx$

$= \int_1^2 \frac{1}{\sqrt{\frac{1}{2}^2 - (x-\frac{2}{3})^2}} dx$

Now just recall a formula that is derivative of $\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}$

$= [\sin^{-1}(2x-3)]_1^2$

$= \sin^{-1} 1 - \sin^{-1} -1$

$= \pi$

60. Question

If $\int_0^k \frac{1}{2+8x^2} dx = \frac{\pi}{16}$, find value of k

Answer

Given that $\int_0^k \frac{1}{2+8x^2} dx = \frac{\pi}{16}$,k=?

$$\int_0^k \frac{1}{2+8x^2} dx = \frac{\pi}{16}$$

$$\frac{1}{8}\int_0^k \frac{1}{(1/2)^2+x^2} dx = \frac{\pi}{16}$$

derivative of $\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}$

$$\frac{1}{8}[2\tan^{-1} 2x]_0^k = \frac{\pi}{16}$$

$$\frac{1}{8}[\tan^{-1} 2k - \tan^{-1} 0] = \frac{\pi}{16}$$

$$\tan^{-1} 2x = \frac{\pi}{4}$$

2k=1

$$k = \frac{1}{2}$$

61. Question

If $\int_0^a 3x^2\, dx = 8,$ find the value of a.

Answer

$$\int_0^a 3x^2 dx = 8$$

Let us also recall formula $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$[x^3]_0^a = 8$$

$$a^3 = 8$$

a=2

hence a=2.

62. Question

Evaluate the following Integrals:

$$\int_{\pi}^{3\pi/2} \sqrt{1-\cos 2x}\ dx$$

Answer

$$\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\cos 2x} dx$$

We know that $\cos 2x = 1 - 2\sin^2 x$

Now substitute that in the equation

We get

$$\int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-\cos 2x}dx = \int_{\pi}^{\frac{3\pi}{2}} \sqrt{1-(1-2\sin^2 x)}dx$$

$$= \int_{\pi}^{\frac{3\pi}{2}} \sqrt{2\sin^2 x}dx$$

$$= \sqrt{2}\int_{\pi}^{\frac{3\pi}{2}} \sin x dx$$

We already know that integral of sinx is –cosx

$$= \sqrt{2}(-\cos x)_{\pi}^{\frac{3\pi}{2}}$$

$$= -\sqrt{2}(\cos - \cos\pi)$$

$$= -\sqrt{2}(0-(-1))$$

$$= -\sqrt{2}$$

63. Question

Evaluate the following Integrals:

$$\int_{0}^{2\pi} \sqrt{1+\sin\frac{x}{2}}\ dx$$

Answer

Let $I = \int_0^{2\pi} \sqrt{1+\sin(\frac{x}{2})}dx$

Let us recall that $\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) = 1$

And $\sin\left(\frac{x}{2}\right) = 2\sin(\frac{x}{4})\sin(\frac{x}{4})$

$$I = \int_0^{2\pi} \sqrt{\sin^2\left(\frac{X}{2}\right) + \cos^2\left(\frac{X}{2}\right) + 2\sin(\frac{X}{4})\cos(\frac{X}{4})}dx$$

$$I = \int_0^{2\pi} \sqrt{(\sin\left(\frac{X}{4}\right) + \cos\left(\frac{X}{4}\right))^2}dx$$

$$= \int_0^{2\pi} (\sin\left(\frac{X}{4}\right) + \cos\left(\frac{X}{4}\right))dx$$

Recall: $\int \sin x = -\cos x$ and $\int \cos x = \sin x$

$$I = [-\frac{\cos\left(\frac{X}{4}\right)}{\frac{1}{4}} + \frac{\sin\left(\frac{X}{4}\right)}{\frac{1}{4}}]_0^{2\pi}$$

$$= -\frac{\cos\left(\frac{2\pi}{4}\right)}{\frac{1}{4}} + \frac{\sin\left(\frac{2\pi}{4}\right)}{\frac{1}{4}} - \{-\frac{\cos\left(\frac{0}{4}\right)}{\frac{1}{4}} + \frac{\sin\left(\frac{0}{4}\right)}{\frac{1}{4}}\}$$

=4(0+1+1–0)

=8

64. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} (\tan x + \cot x)^{-2} dx$$

Answer

$$I = \int_0^{\frac{\pi}{4}} (\tan x + \cot x)^{-2} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{(\tan x + \cot x)^2} ddx$$

We know that $\tan x = \frac{\sin x}{\cos x}$ and $\cot x = \frac{\cos x}{\sin x}$

Now substitute them in the equation.

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)^2} dx$$

Let us recall that $\sin^2(x) + \cos^2(x) = 1$

$$= \int_0^{\frac{\pi}{4}} (\sin x \cos x)^2 dx$$

Again using $\cos^2(x) = 1 - \sin^2(x)$

$$= \int_0^{\frac{\pi}{4}} (\sin x)^2 (1 - \sin^2(x)) dx$$

$$= \int_0^{\frac{\pi}{4}} ((\sin x)^2) dx - \int_0^{\frac{\pi}{4}} (\sin^4(x)) dx$$

Here we are using reduction formula of sinx

$$\int ((\sin x)^n) dx = \frac{n-1}{n} \times \int ((\sin x)^{n-2}) dx - \frac{\cos x (\sin x)^{n-1}}{n}$$

For n=2

$$\int_0^{\frac{\pi}{4}} ((\sin x)^2) dx = \frac{1}{2} \int_0^{\frac{\pi}{4}} dx - \frac{\cos x \sin x}{2}$$

$$= \frac{1}{2} [\frac{\pi}{4}] - \frac{1/2}{2}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

$\int_0^{\frac{\pi}{4}} (\sin^4(x)) x = \frac{4-1}{4} \int_0^{\frac{\pi}{4}} (\sin^2(x)) x - \frac{\cos x \sin^3 x}{4}$ [limits $0, \frac{\pi}{4}$]

$= \frac{3}{4} \left\{\frac{x}{2} - \frac{\cos x \sin x}{2}\right\} - \frac{\cos x \sin^3 x}{4}$[limits $0, \frac{\pi}{4}$]

Now substitute limits

$$\frac{3}{4} \left\{\frac{\frac{\pi}{4}}{2} - \frac{\cos\frac{\pi}{4} \sin\frac{\pi}{4}}{2}\right\} - \frac{\cos\frac{\pi}{4} \sin^3\frac{\pi}{4}}{4} - \frac{3}{4} \left\{\frac{0}{2} - \frac{\cos 0 \sin 0}{2}\right\} + \frac{\cos 0 \sin^3 0}{4}$$

$\sin 45 = \cos 45 = 0, \cos 0 = 1, \sin 0 = 0$

$= \frac{3}{4}\left\{\frac{\pi}{8} - \frac{1}{4}\right\} - \frac{1}{16}$

Now $\int_0^{\frac{\pi}{4}} (\tan x + \cot x)^{-2} x = \frac{\pi}{8} \frac{-1}{4} - \left\{\frac{3}{4}\left\{\frac{\pi}{8} - \frac{1}{4}\right\} - \frac{1}{16}\right\}$

$= \frac{\pi}{32}$

65. Question

Evaluate the following Integrals:

$$\int_0^1 x \log(1+2x)\, dx$$

Answer

Now let us use integration by parts

For this we have to apply integration by parts

Let u and v be two functions then

$$\int udv = uv - \int vdu$$

To choose the first function u we use “ILATE” rule

That is

I=inverse trigonometric function

L=logarithmic function

A=algebraic function

T=trigonometric functions

E=exponential function

So in this preference, the first function is chosen to make the integration simpler.

Now, In the given question x^2 is an algebraic function and it is chosen as u(A comes first in “ILATE” rule)

So first let us integrate the equation and then let us substitute the limits in it.

$$\int_0^1 x\log(1+2x)dx = \left[\frac{x^2 \log(1+2x)}{2}\right]_0^1 - \int_0^1 \frac{2x^2}{2(2x+1)} dx$$

$$= \left[\frac{\log(1+2)}{2} - 0\right] - \int_0^1 \frac{x}{2} - \frac{1}{4} + \frac{1}{4(2x+1)} dx$$

$$= \frac{\log 3}{2} - \left[\frac{x^2}{4} - \frac{x}{4} + \frac{1}{8}\log(|2x+1|)\right]_0^1$$

$$= \frac{\log 3}{2} - \left\{\left[\frac{1}{4} - \frac{1}{4} + \frac{1}{8}\log(|2+1|\right. - \left[\frac{0}{4} - \frac{0}{4} + \frac{1}{8}\log(|0+1|]\right\}\right)$$

$$= \frac{\log 3}{2} - \frac{\log 3}{8}$$

$$= \frac{3\log 3}{8}$$

66. Question

Evaluate the following Integrals:

$$\int_{\pi/6}^{\pi/3} (\tan x + \cot x)^2 dx$$

Answer

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 x + 2\tan x\cot x + \cot^2 x \, dx$$

recall : $\sec^2 x - \tan^2 x = 1, \text{cosec}^2 x - \cot^2 x = 1$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x - 1 + 2 + \text{cosec}^2 x - 1 \text{ dx}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x + \text{cosec}^2 x \, dx$$

Integral $\sec^2 x$ is $\tan x$ and integral of $\text{cosec}^2 x = -\cot x$

$$= [\tan x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} - [\cot x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\text{Tan}30 = \cot 60 = \frac{1}{\sqrt{3}}$$

$$\text{Tan}60 = \cot 30 = \sqrt{3}$$

$$= \text{Tan}60 - \cot 60 - \{\text{Tan}30 - \cot 30\}$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}} - (\frac{1}{\sqrt{3}} - \sqrt{3})$$

$$= \frac{4}{\sqrt{3}}$$

67. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} (a^2 \cos^2 x + b^2 \sin^2 x) dx$$

Answer

$$I = \int_0^{\frac{\pi}{4}} a^2 \cos^2(x) + b^2 \sin^2(x) dx$$

$$= \int_0^{\frac{\pi}{4}} a^2(1 - \sin^2(x)) + b^2 \sin^2(x) dx$$

$$= \int_0^{\frac{\pi}{4}} a^2 + (b^2 - a^2)\sin^2(x) dx$$

Recall:$\sin^2(x) = \frac{1+\cos 2x}{2}$

$$= \int_0^{\frac{\pi}{4}} a^2 + (b^2 - a^2)\frac{1 + \cos 2x}{2} dx$$

We know integral of cosx is sinx

$$= \left[a^2x + \frac{b^2 - a^2}{2} \times (x + \frac{\sin 2x}{2})\right)]_0^{\frac{\pi}{4}}$$

$$= a^2\frac{\pi}{4} + \frac{b^2 - a^2}{2} \times \left(\frac{\pi}{4} + \frac{\sin 2\frac{\pi}{4}}{2}\right) - \{a^2 \times 0 + \frac{b^2 - a^2}{2} \times (0 + \frac{\sin 0}{2})\}$$

$\sin 0 = 0, \sin 90 = 1$

$$= (b^2 + a^2)\frac{\pi}{8} + \frac{(b^2 - a^2)}{4}$$

68. Question

Evaluate the following Integrals:

$$\int_0^1 \frac{1}{1 + 2x + 2x^2 + 2x^3 + x^4} dx$$

Answer

$$I = \int_0^1 \frac{1}{1 + 2x + 2x^2 + 2x^3 + x^4} dx$$

Now arranging denominator, we get as

$1 + 2x + 2x^2 + 2x^3 + x^4 = (1 + x)^2(x^2 + 1)$

$$\int_0^1 \frac{1}{(1 + x)^2(x^2 + 1)} dx$$

$$= \int_0^1 \frac{-x}{2(x^2 + 1)} dx + \int_0^1 \frac{1}{2(x + 1)} dx + \int_0^1 \frac{1}{2(1 + x)^2} dx$$

Now recall integral $\log x = \frac{1}{x}$

And, $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$= -[\frac{\log((x^2 + 1))}{4}]_0^1 + [\frac{\log((x + 1))}{2}]_0^1 + [\frac{1}{2(x + 1)}]_0^1$$

$$= -[\frac{\log((1 + 1))}{4} - \frac{\log((0 + 1))}{4} + \frac{\log((0 + 1))}{2} - \frac{\log((0 + 1))}{2} + \frac{1}{2(1 + 1)} - \frac{1}{2(0 + 1)}]$$

$$= \frac{\log 2}{4} + \frac{1}{4}$$

Exercise 20.2

1. Question

Evaluate the following Integrals:

$$\int_2^4 \frac{x}{x^2 + 1} dx$$

Answer

Given definite integral is: $\int_2^4 \frac{x}{x^2+1} dx$

Let us assume $I(x) = \int_2^4 \frac{x}{x^2+1} dx$ ......(1)

Assume y = x^2+1

Differentiating w.r.t x on both sides we get,

d(y) = d(x^2 + 1)

dy = 2x dx

$x\,dx = \frac{dy}{2}$ ......(2)

The upper limit for Integral

X = 4 ⇒ y = 4^2 + 1

Upper limit: y = 17......(3)

The lower limit for Integral

X = 2 ⇒ y = 2^2 + 1

Lower limit: y = 5 ......(4)

Substituting (2),(3),(4) in the eq(1), we get,

$$\Rightarrow\ I(x) = \int_5^{17} \frac{1}{y}\frac{dy}{2}$$

$$\Rightarrow\ I(x) = \frac{1}{2}\int_5^{17} \frac{1}{y} dy$$

We know that: $\int \frac{1}{x}\, dx = \log(x) + c$

$$\Rightarrow\ I(x) = \frac{1}{2}\log x|_5^{17}$$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f'(x) is derivative of f(x))

$$\Rightarrow\ I(x) = \frac{1}{2} \times (\log(17) - \log(5))$$

We know that: $\log(\frac{x}{y}) = \log(x) - \log(y)$

$$\Rightarrow\ I(x) = \frac{1}{2}\log\left(\frac{17}{5}\right)$$

$$\therefore\ \int_2^4 \frac{x}{x^2+1}\, dx = \frac{1}{2}\log\left(\frac{17}{5}\right)$$

2. Question

Evaluate the following Integrals:

$$\int_1^2 \frac{1}{x(1+\log x)^2} dx$$

Answer

Given Definite Integral can be assumed as:

$\Rightarrow\ I(x) = \int_1^2 \frac{1}{x(1+\log x)^2}\,dx$ ......(1)

Let us assume y = 1 + log(x)

Differentiating w.r.t x on both sides we get

⇒ d(y) = d(1 + log(x))

$\Rightarrow\ dy = \frac{1}{x}dx$ ......(2)

Lower limit for Definite Integral:

⇒ x = 1 ⇒ y = 1 + log 1

⇒ y = 1 ......(3)

Upper limit for Definite Integral:

⇒ x = 2 ⇒ y = 1 + log2

⇒ y = 1 + log2 ......(4)

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow\ I(x) = \int_1^{1+\log 2} \frac{1}{y^2}\,dy$$

$$\Rightarrow\ I(x) = \int_1^{1+\log 2} y^{-2}\,dy$$

We know that: $\int y^n = \frac{y^{n+1}}{n+1} + c\ (n \neq -1)$

$$\Rightarrow\ I(x) = \left.\frac{y^{-2+1}}{-2+1}\right|_1^{1+\log 2}$$

$$\Rightarrow\ I(x) = \left.\frac{y^{-1}}{-1}\right|_1^{1+\log 2}$$

$$\Rightarrow\ I(x) = \left.-\frac{1}{y}\right|_1^{1+\log 2}$$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f’(x) is derivative of f(x))

$$\Rightarrow\ I(x) = \frac{-1}{1+\log 2} - \frac{-1}{1}$$

$$\Rightarrow\ I(x) = 1 - \frac{1}{1+\log 2}$$

$$\Rightarrow\ I(x) = \frac{1+\log 2 - 1}{1+\log 2}$$

$$\Rightarrow\ I(x) = \frac{\log 2}{1+\log 2}$$

We know that loge=1 and loga+logb=logab

$$\Rightarrow\ I(x) = \frac{\log 2}{\log 2e}$$

$$\therefore \int_1^2 \frac{1}{x(1+\log x)^2} dx = \frac{\log 2}{\log 2e}$$

3. Question

Evaluate the following Integrals:

$$\int_1^2 \frac{3x}{9x^2-1} dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow I(x) = \int_1^2 \frac{3x}{9x^2-1} dx$ ……(1)

Let us assume y = $9x^2$–1

Differentiating w.r.t x on both sides we get

⇒ d(y) = d($9x^2$–1)

⇒ dy = 18 x dx

$\Rightarrow 3x\,dx = \frac{dy}{6}$ ……(2)

Upper limit for Definite Integral:

⇒ x = 1 ⇒ y = (9 × 1^2) – 1

⇒ y = 8……(3)

Lower limit for Definite Integral:

⇒ x = 2 ⇒ y = (9 × 2^2) – 1

⇒ y = 35……(4)

Substituting (2),(3),(4) in the eq(1), we get,

$$\Rightarrow I(x) = \int_8^{35} \frac{1}{y}\frac{dy}{6}$$

$$\Rightarrow I(x) = \frac{1}{6}\int_8^{35} \frac{1}{y} dy$$

We know that: $\int \frac{1}{x} dx = \log x + c$

$$\Rightarrow I(x) = \frac{1}{6} \log x|_8^{35}$$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f’(x) is derivative of f(x))

$$\Rightarrow I(x) = \frac{1}{6}(\log 35 - \log 8)$$

We know that: $\log(\frac{x}{y}) = \log(x) - \log(y)$

$$\Rightarrow I(x) = \frac{1}{6}\log\left(\frac{35}{8}\right)$$

$\therefore \int_1^2 \frac{3x}{9x^2-1}dx = \frac{1}{6}\log\left(\frac{35}{8}\right)$

4. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{1}{5\cos x + 3\sin x}dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{1}{5\cos x+3\sin x}dx$......(1)

We know that: $\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

And $\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$

Let us find the value of 5cosx+3sinx

$$\Rightarrow 5\cos x + 3\sin x = 5 \times \left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right) + 3 \times \left(\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)$$

$$\Rightarrow 5\cos x + 3\sin x = \frac{5-5\tan^2\left(\frac{x}{2}\right)+6\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

We know that: $1+\tan^2 x = \sec^2 x$

$\Rightarrow 5\cos x + 3\sin x = \frac{5-5\tan^2\left(\frac{x}{2}\right)+6\tan\left(\frac{x}{2}\right)}{\sec^2\left(\frac{x}{2}\right)}$......(2)

Substituting (2) in (1) we get,

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{5-5\tan^2\left(\frac{x}{2}\right)+6\tan\left(\frac{x}{2}\right)}dx$$

Let us assume: $\tan\left(\frac{x}{2}\right) = t$

Differentiating on both sides w.r.t x we get,

$$\Rightarrow \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dt$$

$\Rightarrow \sec^2\left(\frac{x}{2}\right)dx = 2dt$......(3)

The upper limit for the Definite Integral:

$$\Rightarrow x = \frac{\pi}{2} \Rightarrow t = \tan\left(\frac{\pi}{4}\right)$$

$\Rightarrow t=1$......(4)

The lower limit for the Definite Integral:

$\Rightarrow\ x = 0\ \Rightarrow\ t\ =\ \tan\left(\frac{0}{2}\right)$

⇒ t=0……(5)

Substituting (3),(4),(5) in the eq(1) we get,

$$\Rightarrow\ I(x)\ =\ \int_0^1 \frac{2dt}{5 - 5t^2 + 6t}$$

$$\Rightarrow\ I(x)\ =\ \frac{2}{5}\int_0^1 \frac{dt}{1 - t^2 + \frac{6t}{5}}$$

We need to convert the denominator into standard forms

$$\Rightarrow\ I(x)\ =\ \frac{2}{5}\int_0^1 \frac{dt}{1 - ((t)^2 - \left(2 \times \frac{3}{5} \times t\right) + \left(\frac{3}{5}\right)^2) + \frac{9}{25}}$$

$$\Rightarrow\ I(x)\ =\ \frac{2}{5}\int_0^1 \frac{dt}{\frac{34}{25} - \left(t - \frac{3}{5}\right)^2}$$

$$\Rightarrow\ I(x)\ =\ \frac{2}{5}\int_0^1 \frac{dt}{\left(\sqrt{\frac{34}{25}}\right)^2 - \left(t - \frac{3}{5}\right)^2}$$

We know that: $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

In this problem the values, $a\ =\ \sqrt{\frac{34}{25}}$ and $x = -\frac{3}{5}$.

Using these values and the standard result, we get,

$$\Rightarrow\ I(x)\ =\ \frac{2}{5}\frac{1}{2 \times \sqrt{\frac{34}{25}}}\log\left(\frac{\sqrt{\frac{34}{25}} + t - \frac{3}{5}}{\sqrt{\frac{34}{25}} - t + \frac{3}{5}}\right)\Bigg|_0^1$$

$$\Rightarrow\ I(x)\ =\ \frac{1}{5}\frac{5}{\sqrt{34}}\ \log\left(\frac{\frac{-3 + \sqrt{34}}{5} + t}{\frac{3 + \sqrt{34}}{5} - t}\right)\Bigg|_0^1$$

$$\Rightarrow\ I(x)\ =\ \frac{1}{\sqrt{34}}\log\left(\frac{\frac{\sqrt{34} - 3 + 5t}{5}}{\frac{3 + \sqrt{34} - 5t}{5}}\right)\Bigg|_0^1$$

$$\Rightarrow\ I(x)\ =\ \frac{1}{\sqrt{34}}\log\left(\frac{\sqrt{34} - 3 + 5t}{3 + \sqrt{34} - 5t}\right)\Bigg|_0^1$$

We know that: $\int_a^b f'(x)dx\ = |f(x)|_b^a = \ f(b) - f(a)$

[here f'(x) is derivative of f(x))

$$\Rightarrow\ I(x)\ =\ \frac{1}{\sqrt{34}} \times \left(\log\left(\frac{\sqrt{34} - 3 + 5}{\sqrt{34} + 3 - 5}\right) - \log\left(\frac{\sqrt{34} - 3 + 0}{\sqrt{34} + 3 - 0}\right)\right)$$

$$\Rightarrow\ I(x) = \frac{1}{\sqrt{34}} \times \left(\log\left(\frac{\sqrt{34}+2}{\sqrt{34}-2}\right) - \log\left(\frac{\sqrt{34}-3}{\sqrt{34}+3}\right)\right)$$

We know that: $\log(\frac{x}{y}) = \log(x) - \log(y)$

$$\Rightarrow\ I(x) = \frac{1}{\sqrt{34}} \times \left(\log\left(\frac{\frac{\sqrt{34}+2}{\sqrt{34}-2}}{\frac{\sqrt{34}-3}{\sqrt{34}+3}}\right)\right)$$

$$\Rightarrow\ I(x) = \frac{1}{\sqrt{34}} \times \log\left(\frac{(\sqrt{34}+2)\times(\sqrt{34}+3)}{(\sqrt{34}-2)\times(\sqrt{34}-3)}\right)$$

$$\Rightarrow\ I(x) = \frac{1}{\sqrt{34}} \times \log\left(\frac{40+5\sqrt{34}}{40-5\sqrt{34}}\right)$$

$$\Rightarrow\ I(x) = \frac{1}{\sqrt{34}}\log\left(\frac{5\times(8+\sqrt{34})}{5\times(8-\sqrt{34})}\right)$$

$$\Rightarrow\ I(x) = \frac{1}{\sqrt{34}}\log\left(\frac{8+\sqrt{34}}{8-\sqrt{34}}\right)$$

$$\therefore\ \int_0^{\frac{\pi}{2}} \frac{1}{5\cos x + 3\sin x}dx = \frac{1}{\sqrt{34}}\log\left(\frac{8+\sqrt{34}}{8-\sqrt{34}}\right)$$

5. Question

Evaluate the following Integrals:

$$\int_0^a \frac{x}{\sqrt{a^2+x^2}}dx$$

Answer

Given Definite integral can be written as:

$\Rightarrow\ I(x) = \int_0^a \frac{x}{\sqrt{a^2+x^2}}dx$(1)

Let us assume y = a^2+x^2

Differentiating w.r.t x on both sides we get,

⇒ d(y) = $d(a^2+x^2)$

⇒ dy = 2xdx

$\Rightarrow\ xdx = \frac{dy}{2}$......(2)

Upper limit for the Definite Integral:

⇒ x=a ⇒ y = a^2+a^2

⇒ y=$2a^2$......(3)

Lower limit for the Definite Integral:

⇒ x=0 ⇒ y = a^2+0^2

⇒ y = a^2......(4)

Substituting (2),(3),(4) in the eq(1), we get,

$$\Rightarrow I(x) = \int_{a^2}^{2a^2} \frac{dy}{2\sqrt{y}}$$

$$\Rightarrow I(x) = \frac{1}{2}\int_{a^2}^{2a^2} y^{-\frac{1}{2}}dy$$

We know that: $\int y^n = \frac{y^{n+1}}{n+1} + c\ (n \neq -1)$

$$\Rightarrow I(x) = \frac{1}{2}\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\Bigg|_{a^2}^{2a^2}$$

$$\Rightarrow I(x) = \frac{1}{2}\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_{a^2}^{2a^2}$$

$$\Rightarrow I(x) = y^{\frac{1}{2}}\Big|_{a^2}^{2a^2}$$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f'(x) is derivative of f(x))

$\Rightarrow I(x) = (2a^2)^{1/2} - (a^2)^{1/2}$

$\Rightarrow I(x) = \sqrt{2}\, a - a$

$\Rightarrow I(x) = a(\sqrt{2}-1)$

$$\therefore \int_0^a \frac{x}{\sqrt{a^2+x^2}}dx = a(\sqrt{2}-1)$$

6. Question

Evaluate the following Integrals:

$$\int_0^1 \frac{e^x}{1+e^{2x}}dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow I(x) = \int_0^1 \frac{e^x}{1+e^{2x}}dx$ ......(1)

Let us assume $y = e^x$

Differentiating w.r.t x on both sides we get,

$\Rightarrow d(y) = d(e^x)$

$\Rightarrow dy = e^x dx$ ......(2)

Upper limit for the Definite Integral:

$\Rightarrow x = 1 \Rightarrow y = e^1$

$\Rightarrow y = e$(3)

Lower limit for the Definite Integral:

$\Rightarrow x = 0 \Rightarrow y = e^0$

$\Rightarrow y = 1(4)$

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow I(x) = \int_1^e \frac{dy}{1+y^2}$$

We know that: $\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$

$$\Rightarrow I(x) = Tan^{-1}(x)|_1^e$$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f'(x) is derivative of f(x))

$$\Rightarrow I(x) = Tan^{-1}(e) - Tan^{-1}(1)$$

$$\Rightarrow I(x) = Tan^{-1}(e) - \frac{\pi}{4}$$

$$\therefore \int_0^1 \frac{e^x}{1+e^{2x}} dx = \tan^{-1} e - \frac{\pi}{4}$$

7. Question

Evaluate the following Integrals:

$$\int_0^1 x e^{x^2} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^1 xe^{x^2} dx (1)$$

Let us assume $y = x^2$

Differentiating w.r.t x on both sides we get,

$\Rightarrow d(y) = d(x^2)$

$\Rightarrow dy = 2xdx$

$\Rightarrow xdx = \frac{dy}{2}$......(2)

Upper limit for the Definite Integral:

$\Rightarrow x = 1 \Rightarrow y = 1^2$

$\Rightarrow y = 1$ ......(3)

Lower limit for the Definite Integral:

$\Rightarrow x = 0 \Rightarrow y = 0^2$

$\Rightarrow y = 0$ ......(4)

Substituting (2),(3),(4) in the eq(1), we get,

$$\Rightarrow I(x) = \int_0^1 \frac{e^y dy}{2}$$

$\Rightarrow\ I(x) = \frac{1}{2}\int_0^1 e^y dy$

We know that: $\int e^x dx = e^x + c$

$\Rightarrow\ I(x) = \frac{1}{2} e^y |_0^1$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f'(x) is derivative of f(x))

$\Rightarrow\ I(x) = \frac{1}{2}(e - e^0)$

$\Rightarrow\ I(x) = \frac{1}{2}(e - 1)$

$\therefore\ \int_0^1 xe^{x^2} dx = \frac{1}{2}(e - 1)$

8. Question

Evaluate the following Integrals:

$$\int_1^3 \frac{\cos(\log x)}{x} dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow\ I(x) = \int_1^3 \frac{\cos(\log x)}{x} dx$ ……(1)

Let us assume y = logx

Differentiating w.r.t x on both sides

⇒ d(y) = d(logx)

$\Rightarrow\ dy = \frac{1}{x}dx$……(2)

Upper limit for the Definite Integral:

⇒ x = 3 ⇒ y = log(3)

⇒ y = log3……(3)

Lower limit for the Definite Integral:

⇒ x = 1 ⇒ y = log(1)

⇒ y = 0……(4)

Substituting (2),(3),(4) in the eq(1) we get,

$\Rightarrow\ I(x) = \int_0^{\log 3} \cos y dy$

We know that $\int \cos x\, dx = \sin x + c$

$\Rightarrow\ I(x) = \sin x|_0^{\log 3}$

We know that: $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

here f'(x) is derivative of f(x))

⇒ I(x) = sin(log3) – sin(0)

⇒ I(x) = sin(log3) – 0

⇒ I(x) = sin(log3)

$\therefore \int_1^3 \frac{\cos(\log x)}{x} dx = \sin(\log 3)$

9. Question

Evaluate the following Integrals:

$$\int_0^1 \frac{2x}{1+x^4} dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow I(x) = \int_0^1 \frac{2x}{1+x^4} dx$ ......(1)

Let us assume $y = x^2$

Differentiating w.r.t x on both sides we get,

⇒ $d(y) = d(x^2)$

⇒ dy = 2xdx......(2)

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ $y = 0^2$

⇒ y = 0......(3)

Upper limit for the Definite Integral:

⇒ x = 1 ⇒ $y = 1^2$

⇒ y = 1......(4)

Substituting (2),(3),(4) in the eq(1), we get,

$$\Rightarrow I(x) = \int_0^1 \frac{dy}{1+y^2}$$

We know that: $\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + c$

$\Rightarrow I(x) = \tan^{-1}(y)|_0^1$

We know that $\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f'(x) is derivative of f(x))

⇒ $I(x) = \tan^{-1}(1) - \tan^{-1}(0)$

$$\Rightarrow I(x) = \frac{\pi}{4} - 0$$

$$\Rightarrow I(x) = \frac{\pi}{4}$$

$\therefore \int_0^1 \frac{2x}{1+x^4} dx = \frac{\pi}{4}$

10. Question

Evaluate the following Integrals:

$$\int_0^a \sqrt{a^2 - x^2}\ dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow\ I(x)\ =\ \int_0^a \sqrt{a^2 - x^2} dx$ ...... (1)

Let us assume x = a sinθ

Differentiating w.r.t x on both sides we get,

⇒ d(x) = d(a sin θ)

⇒ dx = a cos θ dθ ......(2)

Let us find the value of $\sqrt{a^2 - x^2}$

$\Rightarrow\ \sqrt{a^2 - x^2}\ =\ \sqrt{a^2 - (a\sin\theta)^2}$

$\Rightarrow\ \sqrt{a^2 - x^2}\ =\ \sqrt{(a^2 \times (1 - \sin^2\theta))}$

$\Rightarrow\ \sqrt{a^2 - x^2} = a \times \sqrt{\cos^2\theta}$

(∵ 1 – $\sin^2\theta = \cos^2\theta$)

$\Rightarrow\ \sqrt{a^2 - x^2}\ =\ a\cos\theta$ ......(3)

Lower limit for the Definite Integral:

$\Rightarrow\ x\ =\ 0\ \Rightarrow\ \theta\ =\ \sin - 1\left(\frac{0}{a}\right)$

⇒ θ = $\sin^{-1}(0)$

⇒ θ = 0......(4)

Upper limit for the Definite Integral:

$\Rightarrow\ x\ =\ a\ \Rightarrow\ \theta\ =\ \sin - 1\left(\frac{a}{a}\right)$

⇒ θ = $\sin^{-1}(1)$

$\Rightarrow\ \theta\ =\ \frac{\pi}{2}$......(5)

Substituting (2),(3),(4),(5) in eq(1) we get,

$$\Rightarrow\ I(x)\ =\ \int_0^{\frac{\pi}{2}} a\cos\theta \times a\cos\theta d\theta$$

$$\Rightarrow\ I(x)\ =\ a2\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$$

We know that cos2θ = $2\cos^2\theta$ – 1

Then

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Using these result for the integration, we get,

$$\Rightarrow I(x) = a^2 \int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta$$

$$\Rightarrow I(x) = \frac{a^2}{2} \int_0^{\frac{\pi}{2}} 1 + \cos 2\theta d\theta$$

We know that:

∫ adx = ax + c and also

$$\int \cos ax = \frac{-\sin ax}{a} + c.$$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f'(x) is derivative of f(x)).

$$\Rightarrow I(x) = \frac{a^2}{2} \times (\theta - \frac{\sin 2\theta}{2})\Big|_0^{\frac{\pi}{2}}$$

$$\Rightarrow I(x) = \frac{a^2}{2} \times \left(\left(\frac{\pi}{2} - \frac{\sin\left(\frac{2\pi}{2}\right)}{2}\right) - \left(0 - \frac{\sin(2\times 0)}{2}\right)\right)$$

We know that sinnπ = 0 (n∈I)⇒ $I(x) = \frac{a^2}{2} \times \left(\left(\frac{\pi}{2} - 0\right) - (0-0)\right)$

$$\Rightarrow I(x) = \frac{a^2}{2} \times \frac{\pi}{2}$$

$$\Rightarrow I(x) = \frac{a^2\pi}{4}$$

$$\therefore \int_0^a \sqrt{a^2 - x^2} dx = \frac{\pi a^2}{4}$$

11. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \sqrt{\sin\phi} \cos^5\phi \, d\phi$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi} \cos^5\phi d\phi$$

$$\Rightarrow\ I(x)\ =\ \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^4\phi\,\cos\emptyset\,d\phi$$

Let us assume sinϕ = t,

Differentiating w.r.t ϕ on both sides we get,

⇒ d(sinϕ) = d(t)

⇒ dt = cosϕ dϕ……(2)

Upper limit for the Definite Integral:

$$\Rightarrow\ \phi = \frac{\pi}{2} \Rightarrow\ t\ =\ \sin\left(\frac{\pi}{2}\right)$$

⇒ t = 1……(3)

Lower limit for the Definite Integral:

⇒ ϕ=0 ⇒ t = sin(0)

⇒ t = 0……(4)

We know that $\cos^2\phi = 1-\sin^2\phi$

⇒ $\cos^2\phi = 1 - t^2$……(5)

Substituting (2),(3),(4),(5) in the eq(1), we get,

$$\Rightarrow\ I(x)\ =\ \int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$\Rightarrow\ I(x)\ =\ \int_0^1 t^{\frac{1}{2}} \times (1 - 2t^2 + t^4)dt$$

$$\Rightarrow\ I(x)\ =\ \int_0^1 t^{\frac{1}{2}} - 2t^{\frac{5}{2}} + t^{\frac{9}{2}}dt$$

We know that:

$$\int\ x^n dx\ =\ \frac{x^{n+1}}{n+1}\ + c\ (n \neq -1)$$

We know that:

$$\int_a^b f'(x)dx\ = |f(x)|_b^a =\ f(b) - f(a)$$

[here f’(x) is derivative of f(x))

$$\Rightarrow\ I(x)\ =\ \left.\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1 - 2 \times \left.\left(\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right)\right|_0^1 + \left.\frac{t^{\frac{9}{2}+1}}{\frac{9}{2}+1}\right|_0^1$$

$$\Rightarrow\ I(x)\ =\ \left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 - 2 \times \left.\left(\frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right)\right|_0^1 + \left.\frac{t^{\frac{11}{2}}}{\frac{11}{2}}\right|_0^1$$

$$\Rightarrow\ I(x)\ =\ \left(\frac{2}{3} \times \left(1^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)\right) - \left(2 \times \frac{2}{7} \times \left(1^{\frac{7}{2}} - 0^{\frac{7}{2}}\right)\right) + \left(\frac{2}{11} \times \left(1^{\frac{11}{2}} - 0^{\frac{11}{2}}\right)\right)$$

$$\Rightarrow\ I(x)\ =\ \left(\frac{2}{3} \times 1\right) - \left(\frac{4}{7} \times 1\right) + \left(\frac{2}{11} \times 1\right)$$

$\Rightarrow\ I(x) = \frac{2}{3} - \frac{4}{7} + \frac{2}{11}$

$\Rightarrow\ I(x) = \frac{64}{231}$

$\therefore \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi = \frac{64}{231}$

12. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx \ldots\ldots (1)$

Let us assume y = sinx,

Differentiating on both sides w.r.t x we get,

⇒ d(y) = d(sinx)

⇒ dy = cosxdx……(2)

Upper limit for the Definite Integral:

$x = \frac{\pi}{2}, y = \sin\frac{\pi}{2} = 1$

⇒ y = 1……(3)

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ y = sin(0)

⇒ y = 0……(4)

Substituting (2),(3),(4) in the eq(1) we get,

$\Rightarrow\ I(x) = \int_0^1 \frac{dt}{1+t^2}$

We know that:

$\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$

We know that:

$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f’(x) is derivative of f(x))

$\Rightarrow\ I(x) = \tan^{-1} t|_0^1$

$\Rightarrow\ I(x) = (\tan - 1(1) - \tan - 1(0))$

$\Rightarrow\ I(x)\ =\ \frac{\pi}{4} - 0$

$\Rightarrow\ I(x)\ =\ \frac{\pi}{4}$

$\therefore\ \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx = \frac{\pi}{4}$

13. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\sin\,\theta}{\sqrt{1+\cos\theta}} d\theta$$

Answer

Given Definite Integral can be written as:

$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1+\cos\theta}} d\theta \ldots\ldots (1)$

Let us assume 1+cosθ=y

Differentiating w.r.t θ on both sides we get,

⇒ d(1+cosθ) = d(y)

⇒ -sinθdθ = dy

⇒ sinθdθ = -dy……(2)

Upper limit for the Definite Integral

$\Rightarrow\ \theta\ =\ \frac{\pi}{2}\ \Rightarrow\ y\ =\ 1+\cos\left(\frac{\pi}{2}\right)$

⇒ y = 1……(3)

Lower limit for the Definite Integral:

⇒ θ = 0 ⇒ y = 1+cos(0)

⇒ y = 1+1

⇒ y = 2……(4)

Substituting (2),(3),(4) in the eq(1), we get,

$\Rightarrow\ I(x)\ =\ \int_2^1 -\frac{dy}{\sqrt{y}}$

We know that:

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\Rightarrow\ I(x)\ =\ \int_1^2 y^{-\frac{1}{2}} dy$

We know that:

$\int\ x^n dx\ =\ \frac{x^{n+1}}{n+1}\ +\ C$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f'(x) is derivative of f(x))

$$\Rightarrow I(x) = \left.\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right|_1^2$$

$$\Rightarrow I(x) = \left.\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\right|_1^2$$

$$\Rightarrow I(x) = 2 \times \left(2^{\frac{1}{2}} - 1^{\frac{1}{2}}\right)$$

$$\Rightarrow I(x) = 2 \times \left(\sqrt{2} - \sqrt{1}\right)$$

$$\Rightarrow I(x) = 2(\sqrt{2} - 1)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1+\cos\theta}} d\theta = 2(\sqrt{2} - 1)$$

14. Question

Evaluate the following Integrals:

$$\int_0^{\pi/3} \frac{\cos x}{3+4\sin x} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x} dx \ldots\ldots(1)$$

Let us assume 3+4sinx = y

Differentiating w.r.t x on both sides we get,

⇒ d(3+4sinx) = d(y)

⇒ 4cosxdx = dy

$$\Rightarrow \cos x\, dx = \frac{dy}{4} \ldots\ldots(2)$$

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ y = 3+4sin(0)

⇒ y = 3 + 0

⇒ y = 3……(3)

Upper limit for the Definite Integral:

$$\Rightarrow x = \frac{\pi}{3} \Rightarrow y = 3 + 4\sin\left(\frac{\pi}{3}\right)$$

$\Rightarrow\ y\ =\ 3\ +\ (4\times\frac{\sqrt{3}}{2})$

$\Rightarrow\ y\ =\ 3\ +2\sqrt{3}\ ...\ ...\ (4)$

Substituting (2),(3),(4) in the eq(1) we get,

$\Rightarrow\ I(x)\ =\ \int_{3}^{3+2\sqrt{3}}\frac{dy}{4y}$

$\Rightarrow\ I(x)\ =\ \frac{1}{4}\int_{3}^{3+2\sqrt{3}}\frac{dy}{y}$

We know that:

$\int\ \frac{dx}{x}\ =\ \log x\ +\ C$

We know that:

$\int_{a}^{b}f'(x)dx\ =|f(x)|_{b}^{a}=\ f(b)-f(a)$

[here f'(x) is derivative of f(x))

$\Rightarrow\ I(x)\ =\ \frac{1}{4}\times\log y|_{3}^{3+2\sqrt{3}}$

$\Rightarrow\ I(x)\ =\ \frac{1}{4}\times(\log(3+2\sqrt{3})-\log(3))$

We know that $\log\left(\frac{a}{b}\right)\ =\ \log a-\log b$

$\Rightarrow\ I(x)\ =\ \frac{1}{4}\times\log\left(\frac{3+2\sqrt{3}}{3}\right)$

$\therefore\ \int_{0}^{\frac{\pi}{3}}\frac{\cos x}{3+4\sin x}dx=\frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$

15. Question

Evaluate the following Integrals:

$$\int_{0}^{1}\frac{\sqrt{\tan^{-1}x}}{1+x^{2}}dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow\ I(x)=\ \int_{0}^{1}\frac{\sqrt{\tan^{-1}x}}{1+x^{2}}dx\ ...\ ...\ (1)$

Let us assume $\tan^{-1}x = y$

Differentiating w.r.t x on both sides we get,

$\Rightarrow d(\tan^{-1}x) = d(y)$

$\Rightarrow\ \frac{1}{1+x^{2}}\ dx\ =\ dy\ ...\ ...\ (2)$

Upper limit of the Definite Integral:

$\Rightarrow x = 1 \Rightarrow y = \tan^{-1}(1)$

$$\Rightarrow\ y\ =\ \frac{\pi}{4} \ldots\ldots (3)$$

Lower limit of the Definite Integral:

$\Rightarrow x = 0 \Rightarrow y = \tan^{-1}(0)$

$\Rightarrow y = 0\ldots\ldots (4)$

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow\ I(x)\ =\ \int_0^{\frac{\pi}{4}} \sqrt{t}dt$$

$$\Rightarrow\ I(x)\ =\ \int_0^{\frac{\pi}{4}} t^{\frac{1}{2}}dt$$

We know that:

$$\int\ xndx\ =\ \frac{x^{n+1}}{n+1}\ +\ C$$

We know that:

$$\int_a^b f'(x)dx\ = |f(x)|_b^a =\ f(b) - f(a)$$

[here f’(x) is derivative of f(x))

$$\Rightarrow\ I(x)\ =\ \left.\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^{\frac{\pi}{4}}$$

$$\Rightarrow\ I(x)\ =\ \left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^{\frac{\pi}{4}}$$

$$\Rightarrow\ I(x)\ =\ \frac{2}{3}\times\left(\left(\frac{\pi}{4}\right)^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$$

$$\Rightarrow\ I(x)\ =\ \frac{2}{3}\times\left(\frac{\pi^{\frac{3}{2}}}{4^{\frac{3}{2}}}\right)$$

$$\Rightarrow\ I(x)\ =\ \frac{2}{3}\times\left(\frac{\pi^{\frac{3}{2}}}{8}\right)$$

$$\Rightarrow\ I(x)\ =\ \frac{\pi^{\frac{3}{2}}}{12}$$

$$\therefore\ \int_0^1 \frac{\sqrt{\tan^{-1}x}}{1+x^2}dx = \frac{\pi^{\frac{3}{2}}}{12}$$

16. Question

Evaluate the following Integrals:

$$\int_0^2 x\sqrt{x+2}\ dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow\ I(x) = \int_0^2 x\sqrt{x+2}dx \ldots\ldots (1)$$

Let us assume x+2 = y

Then, x = y-2 ……(2)

Differentiating on both side w.r.t x we get,

⇒ d(x+2) = d(y)

⇒ dx = dy ……(3)

Upper limit for the Definite Integral:

⇒ x = 2 ⇒ y = 2+2

⇒ y = 4…… (4)

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ y = 0+2

⇒ y = 2…… (5)

Substituting (2),(3),(4),(5) in the eq(1) we get,

$$\Rightarrow\ I(x)\ =\ \int_2^4 (y-2)y^{\frac{1}{2}}dy$$

$$\Rightarrow\ I(x)\ =\ \int_2^4 (y^{\frac{3}{2}} - 2y^{\frac{1}{2}})dy$$

We know that:

$$\int\ x^n dx\ =\ \frac{x^{n+1}}{n+1}\ +\ C$$

We know that:

$$\int_a^b f'(x)dx\ = |f(x)|_b^a = \ f(b) - f(a)$$

[here f’(x) is derivative of f(x))

$$\Rightarrow\ I(x)\ =\ \left.\frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right|_2^4 - 2\times\left.\left(\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right|_2^4$$

$$\Rightarrow\ I(x)\ =\ \left.\frac{y^{\frac{5}{2}}}{\frac{5}{2}}\right|_2^4 - 2\times\left.\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right)\right|_2^4$$

$$\Rightarrow\ I(x)\ =\ \left.\frac{2}{5}y^{\frac{5}{2}}\right|_2^4 - 2\times\left.\frac{2}{3}y^{\frac{3}{2}}\right|_2^4$$

$$\Rightarrow\ I(x)\ =\ \left(\frac{2}{5}\times\left(4^{\frac{5}{2}}-2^{\frac{5}{2}}\right)\right)-\left(\frac{4}{3}\times\left(4^{\frac{3}{2}}-2^{\frac{3}{2}}\right)\right)$$

$$\Rightarrow\ I(x)\ =\ \left(\frac{2}{5}\times\left(32-4\sqrt{2}\right)\right)-\left(\frac{4}{3}\times\left(8-2\sqrt{2}\right)\right)$$

$$\Rightarrow\ I(x)\ =\ \frac{64}{5}-\frac{8\sqrt{2}}{5}-\frac{32}{3}+\frac{8\sqrt{2}}{3}$$

$$\Rightarrow\ I(x)\ =\ \frac{32}{15}+\frac{16\sqrt{2}}{15}$$

$$\Rightarrow\ I(x)\ =\ \frac{1}{15}\times(16\sqrt{2}+32)$$

$$\Rightarrow\ I(x)\ =\ \frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

$$\therefore\ \int_0^2 x\sqrt{x+2}dx=\frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

17. Question

Evaluate the following Integrals:

$$\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow\ I(x)=\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx\ \ldots\ldots(1)$$

Let us assume x = tany

Differentiating w.r.t x on both sides we get,

⇒ d(x) = d(tany)

⇒ dx = $\sec^2$ydy……-(2)

Then

$$\frac{2x}{1-x^2}=\frac{2\tan y}{1-\tan^2 y}$$

We know that:

$$\tan 2\theta\ =\ \frac{2\tan\theta}{1-\tan^2\theta}$$

Now,

$$\frac{2\,x}{1-\ x^2}\ =\ \tan 2y\ \ldots\ldots(3)$$

Upper limit for the Definite Integral:

⇒ x = 1 ⇒ y = $\tan^{-1}(1)$

$$\Rightarrow\ y\ =\ \frac{\pi}{4}\ldots\ldots(4)$$

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ y = $\tan^{-1}(0)$

⇒ y = 0...... (5)

Substituting (2),(3),(4),(5) in (1) we get,

$$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{4}} \tan^{-1}(\tan 2y)\sec^2 y dy$$

$$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{4}} 2y\sec^2 y\, dy$$

$$\Rightarrow\ I(x) = 2\int_0^{\frac{\pi}{4}} y\sec^2 y dy$$

We know that the By-partss integration is:

$$\Rightarrow \int UVdx = U\int Vdx - \int \left(\frac{d}{dx}(U)\int Vdx\right)dx$$

Now applying by parts Integration:

$$\Rightarrow\ I(x) = 2\times\left(y\int_0^{\frac{\pi}{4}}\sec^2 ydy - \int_0^{\frac{\pi}{4}}\left(\frac{d}{dy}(y)\int \sec^2 ydy\right)dy\right.$$

We know that: $\int \sec^2 xdx = \tan x + C$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x))

$$\Rightarrow\ I(x) = 2\times\left((y\tan y)\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\tan ydy\right)$$

We know that: $\int \tan xdx = -\log(\cos x) + C$

$$\Rightarrow\ I(x) = 2\times\left((y\tan y)\Big|_0^{\frac{\pi}{4}} - (-\log(\cos x))\Big|_0^{\frac{\pi}{4}}\right)$$

$$\Rightarrow\ I(x) = 2\times\left((y\tan y)\Big|_0^{\frac{\pi}{4}} + (\log(\cos x))\Big|_0^{\frac{\pi}{4}}\right)$$

$$\Rightarrow\ I(x) = 2\times\left(\left(\frac{\pi}{4}\tan\left(\frac{\pi}{4}\right) - 0\tan\left(\frac{0}{4}\right)\right) + \left(\log\left(\cos\left(\frac{\pi}{4}\right)\right) - \log\left(\cos\left(\frac{0}{4}\right)\right)\right)\right)$$

$$\Rightarrow\ I(x) = 2\times\left(\frac{\pi}{4} - 0 + \log\left(\frac{1}{\sqrt{2}}\right) - \log(1)\right)$$

$$\Rightarrow\ I(x) = 2\times\left(\frac{\pi}{4} + \log\left(2^{-\frac{1}{2}}\right) - 0\right)$$

We know that: $\log(a^b) = b\log a$

$$\Rightarrow\ I(x) = 2\times\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)$$

$$\Rightarrow\ I(x) = \frac{\pi}{2} - \log 2$$

$\therefore \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{2} - \log 2.$

18. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\sin x \cos x}{1+\sin^4 x} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1+\sin^4 x} dx \ldots\ldots(1)$$

Let us assume, $y = \sin^2 x$

Differentiating w.r.t x on both sides we get,

$\Rightarrow d(y) = d(\sin^2 x)$

$\Rightarrow dy = 2\sin x\cos x dx$

$$\Rightarrow \sin x\cos x dx = \frac{dy}{2} \ldots\ldots(2)$$

Upper limit for the Definite Integral:

$$\Rightarrow x = \frac{\pi}{2} \Rightarrow y = \sin^2\frac{\pi}{2}$$

$\Rightarrow y = 1\ldots\ldots (3)$

Lower limit for the Definite Integral:

$\Rightarrow x = 0 \Rightarrow y = \sin^2 0$

$\Rightarrow y = 0\ldots\ldots(4)$

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow I(x) = \int_0^1 \frac{dy}{2(1+y^2)}$$

$$\Rightarrow I(x) = \frac{1}{2}\int_0^1 \frac{dy}{1+y^2}$$

We know that:

$$\int \frac{1}{1+x^2} dx = \tan - 1x + C$$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x))

$$\Rightarrow I(x) = \frac{1}{2} \times (\tan^{-1} y)|_0^1$$

$$\Rightarrow\ I(x) = \frac{1}{2} \times (\tan - 11 - \tan - 10)$$

$$\Rightarrow\ I(x) = \frac{1}{2} \times \left(\frac{\pi}{4}\right)$$

$$\Rightarrow\ I(x) = \frac{\pi}{8}$$

$$\therefore\ \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1 + \sin^4 x} dx = \frac{\pi}{8}$$

19. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{dx}{a\cos x + b\sin x} a, b > 0$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{2}} \frac{dx}{a\cos x + b\sin x}\ a, b\ >\ 0 \ldots\ldots (1)$$

We know that:

$$\cos x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} \text{ and}$$

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$

Substituting these value in (1) we get,

$$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{2}} \frac{dx}{a\left(\frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right) + b\left(\frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right)}$$

$$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{2}} \frac{1 + \tan^2\left(\frac{x}{2}\right)}{a - a\tan^2\left(\frac{x}{2}\right) + 2b\tan\left(\frac{x}{2}\right)} dx$$

We know that: $1+\tan^2 x = \sec^2 x$

$$\Rightarrow\ I(x) = \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{a\left(1 - \tan^2\left(\frac{x}{2}\right) + \frac{2b}{a}\tan\left(\frac{x}{2}\right)\right)} dx$$

$$\Rightarrow\ I(x) = \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{1\ -\ (\tan^2\left(\frac{x}{2}\right)\ -\ 2 \times \frac{b}{a} \times \tan\left(\frac{x}{2}\right)\ +\ \left(\frac{b}{a}\right)^2) + \left(\frac{b}{a}\right)^2} dx$$

$$\Rightarrow\ I(x) = \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{\frac{a^2 + b^2}{a^2} - \left(\tan\left(\frac{x}{2}\right) - \frac{b}{a}\right)^2} dx$$

$$\Rightarrow\ I(x) = \frac{1}{a}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{\left(\sqrt{\frac{a^2+b^2}{a^2}}\right)^2 - \left(\tan\left(\frac{x}{2}\right) - \frac{b}{a}\right)^2} dx \ldots\ldots(1)$$

Let us assume,

$$y = \tan\left(\frac{x}{2}\right) - \left(\frac{b}{a}\right)$$

Differentiating w.r.t x on both sides we get,

$$\Rightarrow\ d(y) = d(\tan\left(\frac{x}{2}\right) - \left(\frac{b}{a}\right))$$

$$\Rightarrow\ dy = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

$$\Rightarrow\ \sec^2\left(\frac{x}{2}\right) dx = 2dy \ldots\ldots(2)$$

Upper limit for the Definite Integral:

$$\Rightarrow\ x = \frac{\pi}{2} \Rightarrow y = \tan\left(\frac{\pi}{4}\right) - \frac{b}{a}$$

$$\Rightarrow\ y = 1 - \frac{b}{a} \ldots\ldots(3)$$

Lower limit for the Definite Integral:

$$\Rightarrow\ x = 0 \Rightarrow y = \tan(0) - \frac{b}{a}$$

$$\Rightarrow\ y = -\frac{b}{a} \ldots\ldots(4)$$

Substituting (2),(3),(4) in eq(1), we get,

$$\Rightarrow\ I(x) = \frac{1}{a}\int_{-\frac{b}{a}}^{1-\frac{b}{a}} \frac{2dy}{(\left(\sqrt{\frac{a^2+b^2}{a^2}}\right)^2 - y^2)}$$

$$\Rightarrow\ I(x) = \frac{2}{a}\int_{-\frac{b}{a}}^{1-\frac{b}{a}} \frac{dy}{\left(\sqrt{\frac{a^2+b^2}{a^2}}\right)^2 - y^2}$$

We know that:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + C$$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x)).

$$\Rightarrow I(x) = \frac{2}{a} \times \frac{1}{2 \times \sqrt{\frac{a^2+b^2}{a^2}}} \times \log\left(\frac{\sqrt{\frac{a^2+b^2}{a^2}}+y}{\sqrt{\frac{a^2+b^2}{a^2}}-y}\right)\Bigg|_{\frac{b}{a}}^{1-\frac{b}{a}}$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \log\left(\frac{\sqrt{a^2+b^2}+ay}{\sqrt{a^2+b^2}-ay}\right)\Bigg|_{-\frac{b}{a}}^{1-\frac{b}{a}}$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{\sqrt{a^2+b^2}+a\left(1-\frac{b}{a}\right)}{\sqrt{a^2+b^2}-a\left(1-\frac{b}{a}\right)}\right) - \log\left(\frac{\sqrt{a^2+b^2}+a\left(-\frac{b}{a}\right)}{\sqrt{a^2+b^2}-a\left(-\frac{b}{a}\right)}\right)\right)$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{(a^2+b^2)}} \times \left(\log\left(\frac{\sqrt{a^2+b^2}+a-b}{\sqrt{a^2+b^2}-a+b}\right) - \log\left(\frac{\sqrt{a^2+b^2}-b}{\sqrt{a^2+b^2}+b}\right)\right)$$

We know that: $\mathrm{Log}\left(\frac{a}{b}\right) = \log(a) - \log(b)$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{\left(\frac{\sqrt{a^2+b^2}+a-b}{\sqrt{a^2+b^2}-a+b}\right)}{\left(\frac{\sqrt{a^2+b^2}-b}{\sqrt{a^2+b^2}+b}\right)}\right)\right)$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{\left(\sqrt{a^2+b^2}+a-b\right)\times\left(\sqrt{a^2+b^2}+b\right)}{\left(\sqrt{a^2+b^2}-a+b\right)\times\left(\sqrt{a^2+b^2}-b\right)}\right)\right)$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{a^2+b^2+b\sqrt{a^2+b^2}+a\sqrt{a^2+b^2}-b\sqrt{a^2+b^2}+ab-b^2}{a^2+b^2-b\sqrt{a^2+b^2}-a\sqrt{a^2+b^2}+b\sqrt{a^2+b^2}+ab-b^2}\right)\right)$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{a^2+a\sqrt{a^2+b^2}+ab}{a^2+ab-a\sqrt{a^2+b^2}}\right)\right)$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{a\times\left(a+\sqrt{a^2+b^2}+b\right)}{a\times\left(a-\sqrt{a^2+b^2}+b\right)}\right)\right)$$

$$\Rightarrow I(x) = \frac{1}{\sqrt{a^2+b^2}} \times \left(\log\left(\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}\right)\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{dx}{a\cos x + b\sin x}\ a,b > 0 = \frac{1}{\sqrt{a^2+b^2}}\log\left(\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}\right)$$

20. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{1}{5+4\sin x}dx$$

Answer

Given Definite can be written as:

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin x}$$

We know that:

$$\sin x \ = \ \frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\left(\frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right)}$$

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{2}} \frac{1 + \tan^2\left(\frac{x}{2}\right)}{5\left(1 + \tan^2\left(\frac{x}{2}\right)\right) + 8\tan\left(\frac{x}{2}\right)} dx$$

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{2}} \frac{1 + \tan^2\left(\frac{x}{2}\right)}{5 + 5\tan^2\left(\frac{x}{2}\right) + 8\tan\left(\frac{x}{2}\right)} dx$$

We know that: $1+\tan^2 x = \sec^2 x$

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{5\left(1 + \tan^2\left(\frac{x}{2}\right) + \frac{8}{5}\tan\left(\frac{x}{2}\right)\right)} dx$$

$$\Rightarrow \ I(x) \ = \ \frac{1}{5}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{1 + \left(\tan^2\left(\frac{x}{2}\right) + 2 \times \frac{4}{5} \times \tan\left(\frac{x}{2}\right) + \left(\frac{4}{5}\right)^2\right) - \left(\frac{4}{5}\right)^2} dx$$

$$\Rightarrow \ I(x) \ = \ \frac{1}{5}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{\frac{9}{25} + \left(\tan\left(\frac{x}{2}\right) + \frac{4}{5}\right)^2} dx$$

$$\Rightarrow \ I(x) = \frac{1}{5}\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{\left(\frac{3}{5}\right)^2 + \left(\tan\left(\frac{x}{2}\right) + \left(\frac{4}{5}\right)\right)^2} dx \ldots \ldots (1)$$

Let us assume,

$$y \ = \ \tan\left(\frac{x}{2}\right) \ + \ \left(\frac{4}{5}\right)$$

Differentiating w.r.t x on both sides we get,

$$\Rightarrow \ d(y) \ = \ d(\tan\left(\frac{x}{2}\right) \ + \left(\frac{4}{5}\right))$$

$$\Rightarrow \ dy \ = \ \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

$$\Rightarrow \ \sec^2\left(\frac{x}{2}\right) dx \ = \ 2dy \ldots \ldots (2)$$

Upper limit for the Definite Integral:

$$\Rightarrow \ x \ = \ \frac{\pi}{2} \Rightarrow \ y \ = \ \tan\left(\frac{\pi}{4}\right) \ + \ \left(\frac{4}{5}\right)$$

$\Rightarrow\ y = 1 + \frac{4}{5}$

$\Rightarrow\ y = \frac{9}{5} \ldots\ldots (3)$

Lower limit for the Definite Integral:

$\Rightarrow\ x = 0 \Rightarrow y = \tan(0) + \frac{4}{5}$

$\Rightarrow\ y = 0 + \frac{4}{5}$

$\Rightarrow\ y = \frac{4}{5} \ldots\ldots (4)$

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow\ I(x) = \frac{1}{5}\int_{\frac{4}{5}}^{\frac{9}{5}} \frac{2dy}{\left(\frac{3}{5}\right)^2 + y^2}$$

$$\Rightarrow\ I(x) = \frac{2}{5}\int_{\frac{4}{5}}^{\frac{9}{5}} \frac{dy}{\left(\frac{3}{5}\right)^2 + y^2}$$

We know that:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x)).

$$\Rightarrow\ I(x) = \frac{2}{5} \times \frac{1}{\frac{3}{5}} \times \left(\tan^{-1}\left(\frac{5y}{3}\right)\Big|_{\frac{4}{5}}^{\frac{9}{5}}\right)$$

$$\Rightarrow\ I(x) = \frac{2}{3} \times (\tan^{-1}(3) - \tan^{-1}\left(\frac{4}{3}\right))$$

We know that:

$$\tan^{-1}(A) - \tan^{-1}(B) = \tan^{-1}\left(\frac{A - B}{1 + AB}\right) \text{ (if } AB > -1)$$

$$\Rightarrow\ I(x) = \frac{2}{3} \times (\tan^{-1}\left(\frac{3 - \frac{4}{3}}{1 + \left(3 \times \frac{4}{3}\right)}\right))$$

$$\Rightarrow\ I(x) = \frac{2}{3} \times (\tan^{-1}\left(\frac{\frac{5}{3}}{1 + 4}\right))$$

$$\Rightarrow\ I(x) = \frac{2}{3} \times (\tan^{-1}\left(\frac{\frac{5}{3}}{5}\right))$$

$\Rightarrow\ I(x) = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

$\therefore\ \int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x}dx = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

21. Question

Evaluate the following Integrals:

$$\int_0^{\pi} \frac{\sin\ x}{\sin\ x+\cos\ x}dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow\ I(x) = \int_0^{\pi} \frac{\sin x}{\sin x+\cos x}dx \ldots\ldots(1)$

Let us write numerator in terms of the denominator for easy calculation,

$\Rightarrow\ \sin x\ =\ K(\sin x+\cos x)\ +\ L\times\frac{d}{dx}(\sin x+\cos x)$

⇒ sinx = K(sinx+cosx) + L(cosx – sinx)

⇒ sinx = sinx(K-L) +cosx(K+L)

Comparing coefficients of corresponding terms on both sides we get,

⇒ K + L =0

⇒ K – L =1

On solving these two equations we get,

L=$-\frac{1}{2}$ and K = $\frac{1}{2}$.

So numerator can be written as:

$\frac{1}{2}(\sin x+\cos x) - \frac{1}{2}(\cos x-\sin x)$

Substituting these values in(1) we get,

$\Rightarrow\ I(x)\ =\ \int_0^{\pi} \frac{\frac{1}{2}(\sin x+\cos x)-\frac{1}{2}(\cos x-\sin x)}{\sin x+\cos x}dx$

$\Rightarrow\ I(x)\ =\ \int_0^{\pi} \frac{\frac{1}{2}(\sin x+\cos x)}{\sin x+\cos x}dx - \int_0^{\pi} \frac{\frac{1}{2}(\cos x-\sin x)}{\sin x+\cos x}dx$

$\Rightarrow\ I(x)\ =\ \frac{1}{2}\int_0^{\pi} dx - \frac{1}{2}\int_0^{\pi} \frac{d(\sin x+\cos x)}{\sin x+\cos x}$

We know that:

$\int\ dy\ =\ y\ +\ C$ and $\int\ \frac{dy}{y}\ =\ \log|y|\ +\ C$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f'(x) is derivative of f(x)).

$$\Rightarrow I(x) = \frac{1}{2}(x|_0^{\pi}) - \frac{1}{2}(\log(|\sin x + \cos x|)|_0^{\pi})$$

$$\Rightarrow I(x) = \frac{1}{2} \times (\pi - 0) - \frac{1}{2}((\log(|\sin\pi + \cos\pi|)) - (\log(|\sin 0 + \cos 0|))$$

$$\Rightarrow I(x) = \frac{\pi}{2} - \frac{1}{2} \times (\log(|0 - 1|) - \log(|0 + 1|))$$

$$\Rightarrow I(x) = \frac{\pi}{2} - \frac{1}{2} \times (\log 1 - \log 1)$$

$$\Rightarrow I(x) = \frac{\pi}{2} - \frac{1}{2} \times (0 - 0)$$

$$\Rightarrow I(x) = \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{2}$$

22. Question

Evaluate the following Integrals:

$$\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} dx$$

We know that:

$$\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)} \text{ and}$$

$$\cos x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1}{3 + 2\left(\frac{2\tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}\right) + \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}} dx$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1 + \tan^2\left(\frac{x}{2}\right)}{3\left(1 + \tan^2\left(\frac{x}{2}\right)\right) + 4\tan\left(\frac{x}{2}\right) + 1 - \tan^2\left(\frac{x}{2}\right)} dx$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1 + \tan^2\left(\frac{x}{2}\right)}{3 + 3\tan^2\left(\frac{x}{2}\right) + 4\tan\left(\frac{x}{2}\right) + 1 - \tan^2\left(\frac{x}{2}\right)} dx$$

$$\Rightarrow \ I(x) \ = \ \int_0^{\pi} \frac{1+\tan^2\left(\frac{x}{2}\right)}{4+2\tan^2\left(\frac{x}{2}\right)+4\tan\left(\frac{x}{2}\right)} dx$$

We know that: $1+\tan^2x = \sec^2x$

$$\Rightarrow \ I(x) \ = \ \frac{1}{2} \times \int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)}{2+\left(\tan^2\left(\frac{x}{2}\right)+2\tan\left(\frac{x}{2}\right)+1\right)-1} dx$$

$$\Rightarrow \ I(x) = \frac{1}{2} \times \int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)}{1^2+\left(\tan\left(\frac{x}{2}\right)+1\right)^2} dx \ldots\ldots(1)$$

Let us assume, $y \ = \ 1+\tan\left(\frac{x}{2}\right)$

Differentiating w.r.t x on both sides we get,

$$\Rightarrow \ d(y) \ = \ d(1+\tan\left(\frac{x}{2}\right))$$

$$\Rightarrow \ dy \ = \ \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

$$\Rightarrow \ \sec^2\left(\frac{x}{2}\right) \ = \ 2dy \ldots\ldots(2)$$

Upper limit for the Definite Integral:

$$\Rightarrow \ x \ = \ \pi \ \Rightarrow \ y \ = \ 1+\tan\left(\frac{\pi}{2}\right)$$

$\Rightarrow y = 1+\infty$

$\Rightarrow y = \infty$......(3)

Lower limit for the Definite Integral:

$\Rightarrow x = 0 \Rightarrow y = 1+\tan(0)$

$\Rightarrow y = 1+0$

$\Rightarrow y = 1$......(4)

Substituting (2),(3),(4) in the eq(1), we get,

$$\Rightarrow \ I(x) \ = \ \frac{1}{2} \times \int_0^{\infty} \frac{2dy}{1+y^2}$$

$$\Rightarrow \ I(x) \ = \ \int_0^{\infty} \frac{dy}{1+y^2}$$

We know that:

$$\int \frac{1}{1+x^2} dx \ = \ \tan^{-1}x + \ C$$

We know that:

$$\int_a^b f'(x)dx \ = |f(x)|_b^a = \ f(b) - f(a)$$

[here f'(x) is derivative of f(x)).

$$\Rightarrow \ I(x) \ = \ \tan^{-1}y|_0^{\infty}$$

$\Rightarrow$ I(x) = $\tan^{-1}(\infty) - \tan^{-1}(0)$

$$\Rightarrow I(x) = \frac{\pi}{2} - \frac{\pi}{4}$$

$$\Rightarrow I(x) = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} dx = \frac{\pi}{4}$$

23. Question

Evaluate the following Integrals:

$$\int_0^1 \tan^{-1} x \, dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^1 \tan^{-1} x \, dx$$

We will find the value of $\int \tan^{-1}x dx$ using by parts rule

Let us find the value of $\int \tan^{-1}x dx$

$\Rightarrow \int \tan^{-1}x \, dx = \int 1.\tan^{-1}x dx$

$$\Rightarrow \tan^{-1}x \int 1dx - \int \left(\frac{d}{dx}(\tan^{-1}x) \int 1dx\right)dx$$

$$\Rightarrow x\tan^{-1}x - \int \frac{1}{1+x^2}.x \, dx$$

$$\Rightarrow x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$\Rightarrow x\tan^{-1}x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}$$

$$\Rightarrow x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

We substitute this result in the Definite Integral:

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f'(x) is derivative of f(x)).

$$\Rightarrow I(x) = \left(x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right)\Big|_0^1$$

$$\Rightarrow I(x) = (1\tan^{-1}(1) - \frac{1}{2}\log(1+1)) - (0\tan^{-1}(0) - \frac{1}{2}\log(1+0))$$

$$\Rightarrow I(x) = (\frac{\pi}{4} - \frac{1}{2}\log 2) - (0 - 0)$$

$$\Rightarrow \ I(x) \ = \frac{\pi}{4} - \frac{1}{2}\log 2$$

$$\therefore \int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2}\log 2$$

24. Question

Evaluate the following Integrals:

$$\int_0^{1/2} \frac{x \sin^{-1}}{\sqrt{1-x^2}} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow \ I(x) \ = \int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx$$

Let us find the value of $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx$ using by parts integration,

$$\Rightarrow \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx \ = \ \sin^{-1}x \int \frac{x}{\sqrt{1-x^2}} dx - \int \left(\frac{d}{dx}(\sin^{-1}x) \int \frac{x}{\sqrt{1-x^2}} dx\right) dx$$

$$\Rightarrow \ (\sin^{-1}x \times \ -\frac{1}{2} \times \int -\frac{2x}{\sqrt{1-x^2}} dx - \int \left(\frac{1}{\sqrt{1-x^2}} \times -\frac{1}{2} \times \int \frac{-2x}{\sqrt{1-x^2}} dx\right) dx$$

$$\Rightarrow \ (\sin^{-1}x \times \ -\frac{1}{2} \times \int \frac{d(1-x^2)}{\sqrt{1-x^2}} - \int \left(\frac{1}{\sqrt{1-x^2}} \times -\frac{1}{2} \times \int \frac{d(1-x^2)}{\sqrt{1-x^2}}\right) dx$$

$$\Rightarrow \ (\sin^{-1}x \ \times \ -\frac{1}{2} \times \ 2 \times \sqrt{1-x^2}) - \int \left(\frac{1}{\sqrt{1-x^2}} \times -\frac{1}{2} \times 2 \times \sqrt{1-x^2}\right) dx$$

$$\Rightarrow -\sqrt{1-x^2}\sin^{-1}x - \int -1 \, dx$$

$$\Rightarrow \ -\sqrt{1-x^2}\sin^{-1}x + \ x$$

Now we substitute this result in the Definite Integral:

We know that:

$$\int_a^b f'(x) dx \ = |f(x)|_b^a = \ f(b) - f(a)$$

[here f’(x) is derivative of f(x)).

$$\Rightarrow \ I(x) \ = \left(x - \sqrt{1-x^2} \sin^{-1} x\right)\Big|_0^{\frac{1}{2}}$$

$$\Rightarrow \ I(x) \ = \left(\frac{1}{2} - \sqrt{1-\left(\frac{1}{2}\right)^2} \ \sin^{-1}\left(\frac{1}{2}\right)\right) - \left(0 - \sqrt{1-0^2}\sin - 1(0)\right)$$

$$\Rightarrow \ I(x) \ = \left(\frac{1}{2} - \sqrt{1-\frac{1}{4}} \times \frac{\pi}{6}\right) - (0 - 0)$$

$$\Rightarrow \ I(x) \ = \ (\frac{1}{2} - \sqrt{\frac{3\pi}{46}})$$

$$\Rightarrow \ I(x) \ = \ \frac{1}{2} - \frac{\sqrt{3}\pi}{12}$$

$$\therefore \int_0^{\frac{1}{2}} \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx = \frac{1}{2} - \frac{\sqrt{3}\pi}{12}$$

25. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x})dx$$

We know that:

$$\tan x \ = \ \frac{\sin x}{\cos x} \text{ and}$$

$$\cot x \ = \ \frac{\cos x}{\sin x}$$

Substituting in the Definite Integral we get,

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{4}} \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right) dx$$

$$\Rightarrow \ I(x) \ = \ \int_0^{\frac{\pi}{4}} \left(\frac{(\sqrt{\sin x})^2 + (\sqrt{\cos x})^2}{\sqrt{\sin x \cos x}}\right) dx$$

$$\Rightarrow \ I(x) \ = \ \sqrt{2} \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}} dx$$

$$\Rightarrow \ I(x) \ = \ \sqrt{2} \times \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{(1 - (-2\sin x \cos x + 1)}} dx$$

$$\Rightarrow \ I(x) \ = \ \sqrt{2} \times \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2\sin x \cos x)}} dx$$

$$\Rightarrow \ I(x) = \ \sqrt{2} \times \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx \ldots\ldots (1)$$

Let us assume, y = sinx – cosx

Differentiating w.r.t x on both sides we get,

⇒ d(y) = d(sinx – cosx)

⇒ dy = (cosx + sinx)dx……(2)

Upper limit for the Definite Integral:

$\Rightarrow x = \frac{\pi}{4} \Rightarrow y = \sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)$

$\Rightarrow y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$

⇒ y = 0...... (3)

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ y = sin(0) – cos(0)

⇒ y = 0 – 1

⇒ y = -1......(4)

Substituting (2),(3),(4) in the eq(1) we get,

$\Rightarrow I(x) = \sqrt{2} \times \int_{-1}^{0} \frac{dy}{\sqrt{1-y^2}}$

We know that:

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + C$

We know that:

$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$

[here f'(x) is derivative of f(x))

$\Rightarrow I(x) = \sqrt{2} \times (\sin^{-1} y|_{-1}^{0})$

$\Rightarrow I(x) = \sqrt{2} \times (\sin - 1(0) - \sin - 1(-1))$

$\Rightarrow I(x) = \sqrt{2} \times (0 - \left(-\frac{\pi}{2}\right))$

$\Rightarrow I(x) = \sqrt{2} \times \frac{\pi}{2}$

$\Rightarrow I(x) = \frac{\pi}{\sqrt{2}}$

$\therefore \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x})dx = \frac{\pi}{\sqrt{2}}$

26. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x} dx$$

Answer

Given Definite Integral can be written as:

$\Rightarrow I(x) = \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1 + \cos 2x} dx$

We know that: 1+ cos2x = $2\cos^2x$ and

$$\frac{1}{\cos^2 x} = \sec^2 x$$

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{2\cos^2 x} dx$$

$$\Rightarrow I(x) = \frac{1}{2} \times \int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx \ldots\ldots (1)$$

Let us assume, y = tanx

Differentiating w.r.t x on both sides we get,

⇒ d(y) = d(tanx)

⇒ dy = $\sec^2x$dx……(2)

Upper limit for the Definite Integral:

$$\Rightarrow x = \frac{\pi}{4} \Rightarrow y = \tan\left(\frac{\pi}{4}\right)$$

⇒ y = 1……(3)

The lower limit for the Definite Integral:

$$\Rightarrow x = 0 \Rightarrow y = \tan\left(\frac{0}{4}\right)$$

⇒ y = 0……(4)

Substituting (2),(3),(4) in eq(1) we get,

$$\Rightarrow I(x) = \frac{1}{2}\int_0^1 y^3 dy$$

We know that:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x)).

$$\Rightarrow I(x) = \frac{1}{2} \times \left(\frac{y^{3+1}}{3+1} \Big|_0^1 \right)$$

$$\Rightarrow I(x) = \frac{1}{2} \times \left(\frac{y^4}{4} \Big|_0^1 \right)$$

$$\Rightarrow I(x) = \frac{1}{8} \times (14 - 04)$$

$$\Rightarrow I(x) = \frac{1}{8}$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\tan^3 x}{1 + \cos 2x} dx = \frac{1}{8}$$

27. Question

Evaluate the following Integrals:

$$\int_0^{\pi} \frac{1}{5+3\cos x} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1}{5+3\cos x} dx$$

We know that:

$$\cos x = \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1}{5+3\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)} dx$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1+\tan^2\left(\frac{x}{2}\right)}{5\left(1+\tan^2\left(\frac{x}{2}\right)\right)+3\left(1-\tan^2\left(\frac{x}{2}\right)\right)} dx$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1+\tan^2\left(\frac{x}{2}\right)}{5+5\tan^2\left(\frac{x}{2}\right)+3-3\tan^2\left(\frac{x}{2}\right)} dx$$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{1+\tan^2\left(\frac{x}{2}\right)}{8+2\tan^2\left(\frac{x}{2}\right)} dx$$

We know that: $1+\tan^2 x = \sec^2 x$

$$\Rightarrow I(x) = \int_0^{\pi} \frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}{2^2+\left(\tan\left(\frac{x}{2}\right)\right)^2} dx \ldots\ldots (1)$$

Let us assume, $y = \tan\left(\frac{x}{2}\right)$

Differentiating w.r.t x on both the sides we get,

$$\Rightarrow d(y) = d(\tan\left(\frac{x}{2}\right))$$

$$\Rightarrow dy = \frac{1}{2}\sec2\left(\frac{x}{2}\right) dx \ldots\ldots (2)$$

The upper limit for the Definite Integral:

$$\Rightarrow x = \pi \Rightarrow y = \tan\left(\frac{\pi}{2}\right)$$

$\Rightarrow y = \infty$……(3)

Lower limit for the Definite Integral:

$\Rightarrow x = 0 \Rightarrow y = \tan\left(\frac{0}{2}\right)$

⇒ y = 0……(4)

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow I(x) = \int_0^{\infty} \frac{dy}{2^2 + y^2}$$

We know that:

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan - 1\left(\frac{x}{a}\right) + C$$

We know that:

$$\int_a^b f'(x) dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x)).

$$\Rightarrow I(x) = \frac{1}{2} \times (\tan^{-1}\left(\frac{y}{2}\right)\Big|_0^{\infty})$$

$$\Rightarrow I(x) = \frac{1}{2} \times (\tan - 1(\infty) - \tan - 1(0))$$

$$\Rightarrow I(x) = \frac{1}{2} \times \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow I(x) = \frac{\pi}{4}$$

$$\therefore \int_0^{\pi} \frac{dx}{5 + 3\cos x} = \frac{\pi}{4}$$

28. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$$

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x \times \left(\left(\frac{a^2 \sin^2 x}{\cos^2 x}\right) + b^2\right)} dx$$

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{a^2 \tan^2 x + b^2} dx$$

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 \tan^2 x + b^2} dx$$

$$\Rightarrow \ I(x) = \frac{1}{a^2} \times \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(\tan x)^2 + \left(\frac{b}{a}\right)^2} dx \ldots \ldots (1)$$

Let us assume y = tanx

Differentiating w.r.t x on both sides we get,

⇒ d(y) = d(tanx)

⇒ dy = $\sec^2 x dx$......(2)

Upper limit for the Definite Integral:

$$\Rightarrow \ x = \frac{\pi}{2} \Rightarrow y = \tan\left(\frac{\pi}{2}\right)$$

⇒ y = ∞......(3)

Lower limit for the Definite Integral:

⇒ x = 0 ⇒ y = tan(0)

⇒ y = 0......(4)

Substituting (2),(3),(4) in the eq(1) we get,

$$\Rightarrow \ I(x) = \frac{1}{a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$$

We know that:

$$\int \frac{1}{a^2 + x^2} dx = \tan^{-1} x + C$$

We know that:

$$\int_a^b f'(x) dx = |f(x)|_b^a = f(b) - f(a)$$

[here f’(x) is derivative of f(x)).

$$\Rightarrow \ I(x) = \frac{1}{a^2} \times \frac{1}{\frac{b}{a}} \times (\tan^{-1}\left(\frac{x}{\frac{b}{a}}\right)\Bigg|_0^{\infty})$$

$$\Rightarrow \ I(x) = \frac{1}{ab} \times (\tan^{-1}\left(\frac{ax}{b}\right)\Big|_0^{\infty})$$

$$\Rightarrow \ I(x) = \frac{1}{ab} \times (\tan - 1(\infty) - \tan - 1(0))$$

$$\Rightarrow \ I(x) = \frac{1}{ab} \times \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow \ I(x) = \frac{\pi}{2ab}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{a^2 \sin^2 x + b^2 \sin^2 x} dx = \frac{\pi}{2ab}$$

29. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x} dx$$

We know that sin 2x = 2 sinx cosx and 1 + cos2x = 2 $\cos^2$x

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{x + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} dx$$

$$\Rightarrow I(x) = \int_0^{\frac{\pi}{2}} \frac{x\sec^2\left(\frac{x}{2}\right)}{2} dx + \int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx$$

Applying by-parts integration for 1st term only

$$\Rightarrow I(x) = x\int_0^{\frac{\pi}{2}} \frac{\sec^2\left(\frac{x}{2}\right)}{2} dx - \int_0^{\frac{\pi}{2}} \left(\frac{d}{dx}(x) \int \frac{\sec^2\left(\frac{x}{2}\right)}{2} dx\right) dx + \int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx$$

We know that:

$$\int_a^b f'(x)dx = |f(x)|_b^a = f(b) - f(a)$$

[here f'(x) is derivative of f(x)).

$$\Rightarrow I(x) = \left(x\tan\left(\frac{x}{2}\right)\right)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx + \int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx$$

$$\Rightarrow I(x) = \left(\frac{\pi}{2}\tan\left(\frac{\pi}{4}\right) - 0\tan(0)\right)$$

$$\Rightarrow I(x) = \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x} dx = \frac{\pi}{2}$$

30. Question

Evaluate the following Integrals:

$$\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$$

Answer

Given Definite Integral can be written as:

$$\Rightarrow I(x) = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx \ldots\ldots(1)$$

Let us assume y = $\tan^{-1}$x

Differentiating w.r.t x on both sides we get,

$\Rightarrow$ d(y) = d($\tan^{-1}$x)

$\Rightarrow \; dy \; = \; \frac{1}{1+x^2} dx$......(2)

Upper limit for the Definite Integral:

$\Rightarrow$ x = 1 $\Rightarrow$ y = $\tan^{-1}(1)$

$\Rightarrow \; y \; = \; \frac{\pi}{4}$......(3)

Lower limit for the Definite Integral:

$\Rightarrow$ x = 0 $\Rightarrow$ y =$\tan^{-1}(0)$

$\Rightarrow$ y = 0......-(4)

Substitute (2),(3),(4) in the eq(1) we get,

$$\Rightarrow \; I(x) \; = \; \int_0^{\frac{\pi}{4}} y dy$$

We know that:

$$\int x^n dx \; = \; \frac{x^{n+1}}{n+1} \; + \; C$$

We know that:

$$\int_a^b f'(x) dx \; = |f(x)|_b^a = \; f(b) - f(a)$$

[here f'(x) is derivative of f(x)).

$$\Rightarrow \; I(x) \; = \; \frac{y^2}{2}\bigg|_0^{\frac{\pi}{4}}$$

$$\Rightarrow \; I(x) \; = \; \frac{1}{2} \times \left(\left(\frac{\pi}{4}\right)^2 - o^2\right)$$

$$\Rightarrow \; I(x) \; = \; \frac{1}{2} \times \frac{\pi^2}{16}$$

$$\Rightarrow \; I(x) \; = \; \frac{\pi^2}{32}$$

$$\therefore \; \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx = \frac{\pi^2}{32}$$

31. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{3 + \sin 2x} dx$

In the denominator, we have sin 2x = 2 sin x cos x

Note that we can write 2 sin x cos x = 1 – (1 – 2 sin x cos x)

We also have $\sin^2 x + \cos^2 x = 1$

$\Rightarrow 1 - 2\sin x \cos x = \sin^2 x + \cos^2 x - 2\sin x \cos x$

$\Rightarrow \sin 2x = 1 - (\sin x - \cos x)^2$

So, using this, we can write our integral as

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{3 + [1 - (\sin x - \cos x)^2]} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{4 - (\sin x - \cos x)^2} dx$$

Now, put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x)\,dx = dt$ (Differentiating both sides)

When $x = 0$, $t = \sin 0 - \cos 0 = 0 - 1 = -1$

When,

$$x = \frac{\pi}{4}, t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

So, the new limits are -1 and 0.

Substituting this in the original integral,

$$I = \int_{-1}^{0} \frac{1}{4 - t^2} dt$$

$$\Rightarrow I = \int_{-1}^{0} \frac{1}{2^2 - t^2} dt$$

Recall,

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right| + c$$

$$\Rightarrow I = \left[\frac{1}{2(2)} \ln\left|\frac{2 + t}{2 - t}\right|\right]_{-1}^{0}$$

$$\Rightarrow I = \frac{1}{4}\left[\ln\left|\frac{2 + 0}{2 - 0}\right| - \ln\left|\frac{2 + (-1)}{2 - (-1)}\right|\right]$$

$$\Rightarrow I = \frac{1}{4}\left[\ln|1| - \ln\left|\frac{1}{3}\right|\right]$$

$$\Rightarrow I = \frac{1}{4}\ln 3 \ [\because \ln 1 = 0 \ \& \ln\left(\frac{1}{x}\right) = -\ln x]$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{3 + \sin 2x} dx = \frac{1}{4}\ln 3$$

32. Question

Evaluate the following Integrals:

$$\int_0^1 x\ \tan^{-1} x\ dx$$

Answer

Let $I = \int_0^1 x \tan^{-1} x\, dx$

We will use integration by parts.

Recall, $\int f(x)g(x)dx = f(x)[\int g(x)dx] - \int[f'(x)\int g(x)dx]dx + c$

Here, take f(x) = $\tan^{-1}x$ and g(x) = x

$$\Rightarrow \int g(x)dx = \int xdx$$

We have,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

$$\Rightarrow \int g(x)dx = \frac{x^2}{2}$$

Now,

$$f'(x) = \frac{df(x)}{dx} = \frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow f'(x) = \frac{1}{1+x^2}$$

Substituting these values, we evaluate the integral.

$$\Rightarrow\ I = \left[(\tan^{-1} x)\frac{x^2}{2}\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\left(\frac{x^2}{2}\right)dx$$

$$\Rightarrow\ I = \left[\frac{x^2}{2}\tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx$$

We can write,

$$\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

$$\Rightarrow I = \left[\frac{x^2}{2}\tan^{-1} x\right]_0^1 - \frac{1}{2}\int_0^1 \left[1 - \frac{1}{1+x^2}\right]dx$$

$$\Rightarrow I = \left[\frac{1^2}{2}\tan^{-1}(1) - \frac{0^2}{2}\tan^{-1}(0)\right] - \left(\frac{1}{2}\int_0^1 dx - \frac{1}{2}\int_0^1 \frac{1}{1+x^2}dx\right)$$

Recall,

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$$

$$\Rightarrow I = \frac{1}{2}\times\frac{\pi}{4} - \left(\frac{1}{2}[x]_0^1 - \frac{1}{2}[\tan^{-1} x]_0^1\right)$$

$$\Rightarrow I = \frac{\pi}{8} - \left(\frac{1}{2}[1-0] - \frac{1}{2}[\tan^{-1}(1) - \tan^{-1}(0)]\right)$$

$$\Rightarrow I = \frac{\pi}{8} - \left(\frac{1}{2} - \frac{1}{2}\left[\frac{\pi}{4} - 0\right]\right)$$

$$\Rightarrow I = \frac{\pi}{8} - \left(\frac{1}{2} - \frac{\pi}{8}\right)$$

$$\Rightarrow I = \frac{\pi}{4} - \frac{1}{2}$$

$$\therefore \int_0^1 x\tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}$$

33. Question

Evaluate the following Integrals:

$$\int_0^1 \frac{1-x^2}{x^4+x^2+1}dx$$

Answer

Let $I = \int_0^1 \frac{1-x^2}{x^4+x^2+1}dx$

In the denominator, we have $x^4 + x^2 + 1$

Note that we can write $x^4 + x^2 + 1 = (x^4 + 2x^2 + 1) - x^2$

We have $x^4 + 2x^2 + 1 = (1 + x^2)^2$

$\Rightarrow x^4 + x^2 + 1 = (1 + x^2)^2 - x^2$

So, using this, we can write our integral as

$$I = \int_0^1 \frac{1-x^2}{(1+x^2)^2 - x^2}dx$$

Dividing numerator and denominator with x^2, we have

$$I = \int_0^1 \frac{\frac{1-x^2}{x^2}}{\left(\frac{x^2+1}{x}\right)^2 - \frac{x^2}{x^2}}dx$$

$$\Rightarrow I = \int_0^1 \frac{\frac{1}{x^2} - 1}{\left(x + \frac{1}{x}\right)^2 - 1}dx$$

Put, $x + \frac{1}{x} = t$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right)dx = dt$$

(Differentiating both sides)

$$\Rightarrow \left(\frac{1}{x^2} - 1\right)dx = -dt$$

When $x = 0, t = 0 + \frac{1}{0} = \infty$

When $x = 1, t = 1 + \frac{1}{1} = 2$

So, the new limits are ∞ and 2.

Substituting this in the original integral,

$$I = \int_{\infty}^{2} \frac{1}{t^2 - 1}(-dt)$$

$$\Rightarrow I = -\int_{\infty}^{2} \frac{1}{t^2 - 1} dt$$

Recall, $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow I = -\left[\frac{1}{2(1)} \ln\left|\frac{t - 1}{t + 1}\right|\right]_{\infty}^{2}$$

$$\Rightarrow I = -\frac{1}{2}\left[\ln\left|\frac{2 - 1}{2 + 1}\right| - \ln\left|\frac{\infty - 1}{\infty + 1}\right|\right]$$

$$\Rightarrow I = -\frac{1}{2}\left[\ln\frac{1}{3} - 0\right]$$

$$\Rightarrow I = -\frac{1}{2}\ln 3^{-1}$$

$$\Rightarrow I = \ln 3^{\frac{1}{2}} = \ln\sqrt{3}$$

$$\therefore \int_{0}^{1} \frac{1 - x^2}{x^4 + x^2 + 1} dx = \ln\sqrt{3}$$

34. Question

Evaluate the following Integrals:

$$\int_{0}^{1} \frac{24x^3}{\left(1+x^2\right)^4} dx$$

Answer

Let $I = \int_0^1 \frac{24x^3}{(1+x^2)^4} dx$

Put $1 + x^2 = t$

⇒ $2xdx = dt$ (Differentiating both sides)

When $x = 0$, $t = 1 + 0^2 = 1$

When $x = 1$, $t = 1 + 1^2 = 2$

So, the new limits are 1 and 2.

In numerator, we can write $24x^3dx = 12x^2 \times 2xdx$

But, $x^2 = t – 1$ and $2xdx = dt$

$\Rightarrow 24x^3dx = 12(t - 1)dt$

Substituting this in the original integral,

$$I = \int_1^2 \frac{12(t-1)}{t^4}dt$$

$$\Rightarrow I = 12\int_1^2 \left(\frac{t}{t^4} - \frac{1}{t^4}\right)dt$$

$$\Rightarrow I = 12\left(\int_1^2 \frac{1}{t^3}dt - \int_1^2 \frac{1}{t^4}dt\right)$$

$$\Rightarrow I = 12\left(\int_1^2 t^{-3}dt - \int_1^2 t^{-4}dt\right)$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = 12\left[\frac{t^{-3+1}}{(-3+1)} - \frac{t^{-4+1}}{(-4+1)}\right]_1^2$$

$$\Rightarrow I = 12\left[\frac{-1}{2t^2} + \frac{1}{3t^3}\right]_1^2$$

$$\Rightarrow I = 12\left[\left(\frac{-1}{2(2)^2} + \frac{1}{3(2)^3}\right) - \left(\frac{-1}{2(1)^2} + \frac{1}{3(1)^3}\right)\right]$$

$$\Rightarrow I = 12\left[\left(\frac{-1}{8} + \frac{1}{24}\right) - \left(\frac{-1}{2} + \frac{1}{3}\right)\right]$$

$$\Rightarrow I = 12\left[\frac{-1}{12} - \left(\frac{-1}{6}\right)\right]$$

$$\Rightarrow I = 12\left[\frac{1}{6} - \frac{1}{12}\right] = 1$$

$$\therefore \int_0^1 \frac{24x^3}{(1+x^2)^4}dx = 1$$

35. Question

Evaluate the following Integrals:

$$\int_4^{12} x(x-4)^{1/3}dx$$

Answer

Let $I = \int_4^{12} x(x-4)^{\frac{1}{3}}dx$

Put $x - 4 = t^3$

$\Rightarrow dx = 3t^2dt$ (Differentiating both sides)

When $x = 4$, $t^3 = 4 - 4 = 0 \Rightarrow t = 0$

When x = 12, t^3 = 12 – 4 = 8 ⇒ t = 2

So, the new limits are 0 and 2.

We can write x = t^3 + 4

Substituting this in the original integral,

$$I = \int_0^2 (t^3 + 4)t(3t^2)dt$$

$$\Rightarrow I = 3\int_0^2 (t^6 + 4t^3)dt$$

$$\Rightarrow I = 3\left(\int_0^2 t^6 dt + 4\int_0^2 t^3 dt\right)$$

Recall,

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c$$

$$\Rightarrow I = 3\left(\left[\frac{t^{6+1}}{6+1}\right]_0^2 + 4\left[\frac{t^{3+1}}{3+1}\right]_0^2\right)$$

$$\Rightarrow I = 3\left(\left[\frac{t^7}{7}\right]_0^2 + 4\left[\frac{t^4}{4}\right]_0^2\right)$$

$$\Rightarrow I = 3\left[\left(\frac{2^7}{7} - \frac{0^7}{7}\right) + 4\left(\frac{2^4}{4} - \frac{0^4}{4}\right)\right]$$

$$\Rightarrow I = 3\left(\frac{128}{7} + 16\right)$$

$$\Rightarrow I = 3\left(\frac{240}{7}\right) = \frac{720}{7}$$

$$\therefore \int_4^{12} x(x-4)^{\frac{1}{3}}dx = \frac{720}{7}$$

36. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} x^2 \sin x \, dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$

We will use integration by parts.

Recall,

$$\int f(x)g(x)dx = f(x)\left[\int g(x)dx\right] - \int \left[f'(x)\int g(x)dx\right]dx + c$$

Here, take $f(x) = x^2$ and $g(x) = \sin x$

$$\Rightarrow \int g(x)dx = \int \sin x\, dx = -\cos x$$

Now,

$$f'(x) = \frac{df(x)}{dx} = \frac{d}{dx}(x^2)$$

$\Rightarrow f'(x) = 2x$

Substituting these values, we evaluate the integral.

$$\Rightarrow I = [x^2(-\cos x)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2x)(-\cos x)dx$$

$$\Rightarrow I = [-x^2 \cos x]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} x \cos x\, dx$$

Let $I_1 = \int_0^{\frac{\pi}{2}} x \cos x\, dx$

We use integration by parts again.

Here, take $f(x) = x$ and $g(x) = \cos x$

$$\Rightarrow \int g(x)dx = \int \cos x\, dx = \sin x$$

Now, $f'(x) = \frac{df(x)}{dx} = \frac{d}{dx}(x)$

$\Rightarrow f'(x) = 1$

Using these values in equation for I_1

$$\Rightarrow I_1 = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (1)(\sin x)\, dx$$

$$\Rightarrow I_1 = [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$\Rightarrow I_1 = [x \sin x]_0^{\frac{\pi}{2}} + [\cos x]_0^{\frac{\pi}{2}}$$

Substituting I_1 in I, we get

$$I = [-x^2 \cos x]_0^{\frac{\pi}{2}} + 2\left([x \sin x]_0^{\frac{\pi}{2}} + [\cos x]_0^{\frac{\pi}{2}}\right)$$

$$\Rightarrow I = [-x^2 \cos x + 2x \sin x + 2 \cos x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow I = \left[-\left(\frac{\pi}{2}\right)^2 \cos\frac{\pi}{2} + 2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + 2\cos\frac{\pi}{2}\right] - (0 + 0 + 2\cos 0)$$

$\Rightarrow I = (0 + \pi + 0) – (2)$

$\Rightarrow I = \pi – 2$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 \sin x \, dx = \pi - 2$$

37. Question

Evaluate the following Integrals:

$$\int_0^1 \sqrt{\frac{1-x}{1+x}} \, dx$$

Answer

Let $I = \int_0^1 \sqrt{\frac{1-x}{1+x}} dx$

As we have the trigonometric identity $\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$, to evaluate this integral we use x = cos 2θ

⇒ dx = –2sin(2θ)dθ (Differentiating both sides)

When x = 0, cos 2θ = 0 ⇒ 2θ = $\frac{\pi}{2}$ ⇒ θ = $\frac{\pi}{4}$

When x = 1, cos 2θ = 1 ⇒ 2θ = 0 ⇒ θ = 0

So, the new limits are $\frac{\pi}{4}$ and 0.

Substituting this in the original integral,

$$I = \int_{\frac{\pi}{4}}^{0} \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} (-2 \sin 2\theta \, d\theta)$$

$$\Rightarrow I = -2 \int_{\frac{\pi}{4}}^{0} \sqrt{\tan^2 \theta} \sin 2\theta \, d\theta$$

$$\Rightarrow I = -2 \int_{\frac{\pi}{4}}^{0} \tan\theta \sin 2\theta \, d\theta$$

We have $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and sin 2θ = 2 sin θ cos θ

$$\Rightarrow I = -2 \int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} \times 2 \sin\theta \cos\theta \, d\theta$$

$$\Rightarrow I = -4 \int_{\frac{\pi}{4}}^{0} \sin^2\theta \, d\theta$$

But,

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\Rightarrow I = -4 \int_{\frac{\pi}{4}}^{0} \left[\frac{1 - \cos 2\theta}{2}\right] d\theta$$

$$\Rightarrow I = -2\int_{\frac{\pi}{4}}^{0}[1-\cos 2\theta]d\theta$$

$$\Rightarrow I = -2\left(\int_{\frac{\pi}{4}}^{0} d\theta - \int_{\pi/4}^{0}\cos 2\theta\, d\theta\right)$$

$$\Rightarrow I = -2\left([\theta]_{\frac{\pi}{4}}^{0} - \left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{0}\right)$$

$$\Rightarrow I = -2\left[\left(0-\frac{\pi}{4}\right) - \left[\frac{\sin 0}{2} - \frac{\sin(2\times\frac{\pi}{4})}{2}\right]\right]$$

$$\Rightarrow I = -2\left(-\frac{\pi}{4}+\frac{1}{2}\sin\frac{\pi}{2}\right)$$

$$\Rightarrow I = -2\left(-\frac{\pi}{4}+\frac{1}{2}\right) = \frac{\pi}{2}-1$$

$$\therefore \int_{0}^{1}\sqrt{\frac{1-x}{1+x}}dx = \frac{\pi}{2}-1$$

38. Question

Evaluate the following Integrals:

$$\int_{0}^{1}\frac{1-x^2}{\left(1+x^2\right)^2}dx$$

Answer

Let $I = \int_0^1 \frac{1-x^2}{(1+x^2)^2}dx$

As we have the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$, to evaluate this integral we use $x = \tan\theta$

$\Rightarrow dx = \sec^2\theta\, d\theta$ (Differentiating both sides)

When $x = 0$, $\tan\theta = 0 \Rightarrow \theta = 0$

When $x = 1, \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

So, the new limits are 0 and $\frac{\pi}{4}$.

Substituting this in the original integral,

$$I = \int_{0}^{\frac{\pi}{4}}\frac{(1-\tan^2\theta)}{(\sec^2\theta)^2}[\sec^2\theta]d\theta$$

$$\Rightarrow I = \int_{0}^{\frac{\pi}{4}}\frac{(1-\tan^2\theta)}{\sec^2\theta}d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\left(1 - \frac{\sin^2\theta}{\cos^2\theta}\right)}{\left(\frac{1}{\cos^2\theta}\right)} d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} (\cos^2\theta - \sin^2\theta) d\theta$$

We have $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta$$

$$\Rightarrow I = \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow I = \frac{\sin(2 \times \frac{\pi}{4})}{2} - \frac{\sin(2 \times 0)}{2}$$

$$\Rightarrow I = \frac{1}{2}\sin\frac{\pi}{2} - 0 = \frac{1}{2}$$

$$\therefore \int_0^1 \frac{1 - x^2}{(1 + x^2)^2} dx = \frac{1}{2}$$

39. Question

Evaluate the following Integrals:

$$\int_{-1}^{1} 5x^4 \sqrt{x^5 + 1} \, dx$$

Answer

Let $I = \int_{-1}^{1} 5x^4\sqrt{x^5 + 1}dx$

Put $x^5 + 1 = t$

$\Rightarrow 5x^4dx = dt$ (Differentiating both sides)

When $x = -1$, $t = (-1)^5 + 1 = 0$

When $x = 1$, $t = 1^5 + 1 = 2$

So, the new limits are 0 and 2.

Substituting this in the original integral,

$$I = \int_0^2 \sqrt{t}dt$$

$$\Rightarrow I = \int_0^2 t^{\frac{1}{2}}dt$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = \left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^2$$

$$\Rightarrow I = \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2$$

$$\Rightarrow I = \frac{2}{3}\left(2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$$

$$\Rightarrow I = \frac{2}{3} \times \sqrt{8} = \frac{4\sqrt{2}}{3}$$

$$\therefore \int_{-1}^{1} 5x^4\sqrt{x^5+1}dx = \frac{4\sqrt{2}}{3}$$

40. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\cos^2 x}{1+3\sin^2 x}dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1+3\sin^2 x}dx$

Dividing numerator and denominator with $\cos^2$x, we have $I = \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2 x + 3\tan^2 x}dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{1}{1+4\tan^2 x}dx \; [\because \sec^2 x = 1 + \tan^2 x]$$

Put tan x = t

⇒ $\sec^2$x dx = dt (Differentiating both sides)

$$\Rightarrow dx = \frac{dt}{1+\tan^2 x} = \frac{dt}{1+t^2}$$

When x = 0, t = tan 0 = 0

When x = $\frac{\pi}{2}$, t = tan $\frac{\pi}{2}$ = ∞

So, the new limits are 0 and ∞.

Substituting this in the original integral,

$$I = \int_0^{\infty} \frac{1}{1+4t^2}\left(\frac{dt}{1+t^2}\right)$$

$$\Rightarrow I = \int_0^{\infty} \frac{1}{(1+t^2)(1+4t^2)}dt$$

Multiplying numerator and denominator with 3, we have

$$I = \int_0^\infty \frac{3}{3(1 + t^2)(1 + 4t^2)} dt$$

Now, we can write

$$\frac{3}{3(1 + t^2)(1 + 4t^2)} = \frac{[(4 + 4t^2) - (1 + 4t^2)]}{3(1 + t^2)(1 + 4t^2)}$$

$$\Rightarrow \frac{3}{3(1 + t^2)(1 + 4t^2)} = \frac{4(1 + t^2) - (1 + 4t^2)}{3(1 + t^2)(1 + 4t^2)}$$

$$\Rightarrow \frac{3}{3(1 + t^2)(1 + 4t^2)} = \frac{4}{3}\left(\frac{1}{1 + 4t^2}\right) - \frac{1}{3}\left(\frac{1}{1 + t^2}\right)$$

Substituting this in the original integral,

$$I = \int_0^\infty \left[\frac{4}{3}\left(\frac{1}{1 + 4t^2}\right) - \frac{1}{3}\left(\frac{1}{1 + t^2}\right)\right] dt$$

$$\Rightarrow I = \frac{4}{3}\int_0^\infty \left(\frac{1}{1 + 4t^2}\right) dt - \frac{1}{3}\int_0^\infty \left(\frac{1}{1 + t^2}\right) dt$$

$$\Rightarrow I = \frac{1}{3}\int_0^\infty \left(\frac{1}{\frac{1}{4} + t^2}\right) dt - \frac{1}{3}\int_0^\infty \left(\frac{1}{1 + t^2}\right) dt$$

$$\Rightarrow I = \frac{1}{3}\int_0^\infty \left(\frac{1}{\left(\frac{1}{2}\right)^2 + t^2}\right) dt - \frac{1}{3}\int_0^\infty \left(\frac{1}{1^2 + t^2}\right) dt$$

Recall $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow I = \frac{1}{3}\left[\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left(\frac{x}{\left(\frac{1}{2}\right)}\right)\right]_0^\infty - \frac{1}{3}\left[\tan^{-1}\left(\frac{x}{1}\right)\right]_0^\infty$$

$$\Rightarrow I = \frac{2}{3}[\tan^{-1} 2x]_0^\infty - \frac{1}{3}[\tan^{-1} x]_0^\infty$$

$$\Rightarrow I = \frac{2}{3}[\tan^{-1}(\infty) - \tan^{-1}(0)] - \frac{1}{3}[\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$\Rightarrow I = \frac{2}{3}\left(\frac{\pi}{2}\right) - \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{1 + 3\sin^2 x} dx = \frac{\pi}{6}$$

41. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} \sin^3 2t \cos 2t \, dt$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t \, dt$

Put sin 2t = x

⇒ 2cos(2t)dt = dx (Differentiating both sides)

$\Rightarrow \cos(2t)dt = \frac{dx}{2}$

When t = 0, x = sin 0 = 0

When $t = \frac{\pi}{4}, x = \sin 2 \times \frac{\pi}{4} = \sin \frac{\pi}{2} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 x^3 \left(\frac{dx}{2}\right)$$

$$\Rightarrow I = \frac{1}{2} \int_0^1 x^3 dx$$

Recall $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$

$$\Rightarrow I = \frac{1}{2}\left[\frac{x^{3+1}}{3+1}\right]_0^1$$

$$\Rightarrow I = \frac{1}{8}[x^4]_0^1$$

$$\Rightarrow I = \frac{1}{8}(1^4 - 0^4) = \frac{1}{8}$$

$$\therefore \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t \, dt = \frac{1}{8}$$

42. Question

Evaluate the following Integrals:

$$\int_0^{\pi} 5(5 - 4\cos\theta)^{1/4} \sin\theta \, d\theta$$

Answer

Let $I = \int_0^{\pi} 5(5 - 4\cos\theta)^{\frac{1}{4}} \sin\theta \, d\theta$

Put 5 – 4 cos θ = x

⇒ 4sin(θ)dθ = dx (Differentiating both sides)

$\Rightarrow \sin(\theta)d\theta = \frac{dx}{4}$

When θ = 0, x = 5 – 4 cos 0 = 5 – 4 = 1

When θ = π, x = 5 – 4 cos π = 5 – (–4) = 9

So, the new limits are 1 and 9.

Substituting this in the original integral,

$$I = \int_1^9 5x^{\frac{1}{4}}\left(\frac{dx}{4}\right)$$

$$\Rightarrow I = \frac{5}{4}\int_1^9 x^{\frac{1}{4}}dx$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = \frac{5}{4}\left[\frac{x^{\frac{1}{4}+1}}{\frac{1}{4} + 1}\right]_1^9$$

$$\Rightarrow I = \left[x^{\frac{5}{4}}\right]_1^9$$

$$\Rightarrow I = 9^{\frac{5}{4}} - 1$$

$$\Rightarrow I = (3^2)^{\frac{5}{4}} - 1$$

$$\Rightarrow I = \sqrt{3^5} - 1 = 9\sqrt{3} - 1$$

$$\therefore \int_0^{\pi} 5(5 - 4\cos\theta)^{\frac{1}{4}}\sin\theta\, d\theta = 9\sqrt{3} - 1$$

43. Question

Evaluate the following Integrals:

$$\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$$

Answer

Let $I = \int_0^{\frac{\pi}{6}} \cos^{-3} 2\theta \sin 2\theta\, d\theta$

Put cos 2θ = x

⇒ –2sin(2θ)dθ = dx (Differentiating both sides)

$$\Rightarrow \sin(2\theta)d\theta = -\frac{dx}{2}$$

When θ = 0, x = cos 0 = 1

When $\theta = \frac{\pi}{6}, x = \cos 2 \times \frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}$

So, the new limits are 1 and $\frac{1}{2}$.

Substituting this in the original integral,

$$I = \int_{1}^{\frac{1}{2}} x^{-3}\left(-\frac{dx}{2}\right)$$

$$\Rightarrow I = -\frac{1}{2}\int_{1}^{\frac{1}{2}} x^{-3}dx$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = -\frac{1}{2}\left[\frac{x^{-3+1}}{-3 + 1}\right]_1^{\frac{1}{2}}$$

$$\Rightarrow I = \frac{1}{4}[x^{-2}]_1^{\frac{1}{2}}$$

$$\Rightarrow I = \frac{1}{4}\left[\frac{1}{x^2}\right]_1^{\frac{1}{2}}$$

$$\Rightarrow I = \frac{1}{4}\left[\frac{1}{\left(\frac{1}{2}\right)^2} - \frac{1}{1^2}\right]$$

$$\Rightarrow I = \frac{1}{4}(4-1) = \frac{3}{4}$$

$$\therefore \int_0^{\frac{\pi}{6}} \cos^{-3}2\theta \sin 2\theta \, d\theta = \frac{3}{4}$$

44. Question

Evaluate the following Integrals:

$$\int_0^{(\pi)^{2/3}} \sqrt{x}\cos^2 x^{3/2}\,dx$$

Answer

Let $I = \int_0^{\pi^{\frac{2}{3}}} \sqrt{x}\cos^2 x^{\frac{3}{2}} dx$

Put $x^{\frac{3}{2}} = t$

$$\Rightarrow \frac{3}{2}x^{\frac{1}{2}}dx = dt$$

(Differentiating both sides)

$$\Rightarrow \sqrt{x}dx = \frac{2}{3}dt$$

When $x = 0, t = 0^{\frac{3}{2}} = 0$

When $x = \pi^{\frac{2}{3}}, t = \left(\pi^{\frac{2}{3}}\right)^{\frac{3}{2}} = \pi$

So, the new limits are 0 and π.

Substituting this in the original integral,

$$I = \int_0^{\pi} \cos^2 t\left(\frac{2}{3}dt\right)$$

$$\Rightarrow I = \frac{2}{3}\int_0^{\pi} \cos^2 t\,dt$$

But,

$$\cos^2 t = \frac{1\ +\ \cos 2t}{2}$$

$$\Rightarrow I = \frac{2}{3}\int_0^{\pi}\left[\frac{1\ +\ \cos 2t}{2}\right]dt$$

$$\Rightarrow I = \frac{1}{3}\int_0^{\pi}[1 + \cos 2t]dt$$

$$\Rightarrow I = \frac{1}{3}\left(\int_0^{\pi} dt + \int_0^{\pi} \cos 2t\,dt\right)$$

$$\Rightarrow I = \frac{1}{3}\left([t]_0^{\pi} + \left[\frac{\sin 2t}{2}\right]_0^{\pi}\right)$$

$$\Rightarrow I = \frac{1}{3}\left[(\pi - 0) + \left[\frac{\sin 2\pi}{2} - \frac{\sin 0}{2}\right]\right]$$

$$\Rightarrow I = \frac{1}{3}(\pi + 0) = \frac{\pi}{3}$$

$$\therefore \int_0^{\pi^{\frac{2}{3}}} \sqrt{x}\cos^2 x^{\frac{3}{2}}\,dx = \frac{\pi}{3}$$

45. Question

Evaluate the following Integrals:

$$\int_1^2 \frac{1}{x\left(1+\log x\right)^2}dx$$

Answer

Let $I = \int_1^2 \frac{1}{x(1\ +\ \log x)^2}dx$

Put 1 + log x = t

Differentiating both sides, we get,

$$\Rightarrow \frac{1}{x}dx = dt$$

Now, changing the limits,

When x = 1, t = 1 + log 1 = 1

When x = 2, t = 1 + log 2

So, the new limits are 1 and 1 + log 2.

Substituting this in the original integral,

$$I = \int_{1}^{1+\log 2} \frac{1}{t^2} dt$$

$$\Rightarrow I = \int_{1}^{1+\log 2} t^{-2} dt$$

Recall $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$

$$\Rightarrow I = \left[\frac{t^{-2+1}}{-2+1}\right]_{1}^{1+\log 2}$$

$$\Rightarrow I = -[t^{-1}]_{1}^{1+\log 2}$$

$$\Rightarrow I = -\left[\frac{1}{t}\right]_{1}^{1+\log 2}$$

$$\Rightarrow I = -\left[\frac{1}{1+\log 2} - \frac{1}{1}\right]$$

$$\Rightarrow I = 1 - \frac{1}{1+\log 2} = \frac{\log 2}{1+\log 2}$$

$$\therefore \int_{1}^{2} \frac{1}{x(1+\log x)^2} dx = \frac{\log 2}{1+\log 2}$$

46. Question

Evaluate the following Integrals:

$$\int_{0}^{\pi/2} \cos^5 x \, dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \cos^5 x \, dx$

Note that we can write $\cos^5 x = \cos^4 x \times \cos x$

$\Rightarrow \cos^5 x = (\cos^2 x)^2 \times \cos x$

We also have $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \cos^5 x = (1 - \sin^2 x)^2 \cos x$

So, $I = \int_0^{\frac{\pi}{2}} (1 - \sin^2 x)^2 \cos x \, dx$

Put sin x = t

⇒ cos x dx = dt (Differentiating both sides)

When x = 0, t = sin 0 = 0

When $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 (1 - t^2)^2 dt$$

$$\Rightarrow I = \int_0^1 (1 - 2t^2 + t^4) dt$$

$$\Rightarrow I = \int_0^1 dt - 2\int_0^1 t^2 dt + \int_0^1 t^4 dt$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = [t]_0^1 - 2\left[\frac{t^{2+1}}{2+1}\right]_0^1 + \left[\frac{t^{4+1}}{4+1}\right]_0^1$$

$$\Rightarrow I = [t]_0^1 - \frac{2}{3}[t^3]_0^1 + \frac{1}{5}[t^5]_0^1$$

$$\Rightarrow I = (1 - 0) - \frac{2}{3}(1^3 - 0) + \frac{1}{5}(1^5 - 0)$$

$$\Rightarrow I = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$

$$\therefore \int_0^{\frac{\pi}{2}} \cos^5 x \, dx = \frac{8}{15}$$

47. Question

Evaluate the following Integrals:

$$\int_4^9 \frac{\sqrt{x}}{\left(30 - x^{3/2}\right)^2} dx$$

Answer

Let $I = \int_4^9 \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)} dx$

Put $30 - x^{\frac{3}{2}} = t$

$$\Rightarrow -\frac{3}{2}x^{\frac{1}{2}} dx = dt$$

(Differentiating both sides)

$$\Rightarrow \sqrt{x} dx = -\frac{2}{3} dt$$

When $x = 4, t = 30 - (4)^{\frac{3}{2}} = 30 - 8 = 22$

When $x = 9, t = 30 - (9)^{\frac{3}{2}} = 30 - 27 = 3$

So, the new limits are 22 and 3.

Substituting this in the original integral,

$$I = \int_{22}^{3} \frac{1}{t^2}\left(-\frac{2}{3}dt\right)$$

$$\Rightarrow I = -\frac{2}{3}\int_{22}^{3} t^{-2}dt$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = -\frac{2}{3}\left[\frac{t^{-2+1}}{-2+1}\right]_{22}^{3}$$

$$\Rightarrow I = \frac{2}{3}[t^{-1}]_{22}^{3}$$

$$\Rightarrow I = \frac{2}{3}\left[\frac{1}{t}\right]_{22}^{3}$$

$$\Rightarrow I = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right]$$

$$\Rightarrow I = \frac{2}{3} \times \frac{19}{66} = \frac{19}{99}$$

$$\therefore \int_{4}^{9} \frac{\sqrt{x}}{\left(30 - x^{\frac{3}{2}}\right)^2}dx = \frac{19}{99}$$

48. Question

Evaluate the following Integrals:

$$\int_{0}^{\pi} \sin^3 x(1+2\cos x)(1+\cos x)^2 dx$$

Answer

Let $I = \int_0^{\pi} \sin^3 x\,(1 + 2\cos x)(1 + \cos x)^2 dx$

Note that we can write $\sin^3 x = \sin^2 x \times \sin x$

We also have $\sin^2 x + \cos^2 x = 1$

$\Rightarrow \sin^3 x = (1 - \cos^2 x) \sin x$

So, $I = \int_0^{\pi}(1 - \cos^2 x)(1 + 2\cos x)(1 + \cos x)^2 \sin x\, dx$

Put $\cos x = t$

$\Rightarrow -\sin(x)dx = dt$ (Differentiating both sides)

$\Rightarrow \sin(x)dx = -dt$

When $x = 0, t = \cos 0 = 1$

When $x = \pi, t = \cos \pi = -1$

So, the new limits are 1 and -1.

Substituting this in the original integral,

$$I = \int_{1}^{-1} (1-t^2)(1+2t)(1+t)^2(-dt)$$

$$\Rightarrow I = -\int_{1}^{-1} (1-t^2)(1+2t)(1+2t+t^2)dt$$

$$\Rightarrow I = -\int_{1}^{-1} (1+2t-t^2-2t^3)(1+2t+t^2)dt$$

$$\Rightarrow I = -\int_{1}^{-1} \left[1(1+2t+t^2)+2t(1+2t+t^2)-t^2(1+2t+t^2)- 2t^3(1+2t+t^2)\right]dt$$

$$\Rightarrow I = -\int_{1}^{-1} (1+2t+t^2+2t+4t^2+2t^3-t^2-2t^3-t^4-2t^3-4t^4-2t^5)dt$$

$$\Rightarrow I = -\int_{1}^{-1} (1+4t+4t^2-2t^3-5t^4-2t^5)dt$$

$$\Rightarrow I = -\left(\int_{1}^{-1} dt + 4\int_{1}^{-1} tdt + 4\int_{1}^{-1} t^2dt - 2\int_{1}^{-1} t^3dt - 5\int_{1}^{-1} t^4dt - 2\int_{1}^{-1} t^5dt\right)$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = -\left([t]_1^{-1} + 4\left[\frac{t^{1+1}}{1+1}\right]_1^{-1} + 4\left[\frac{t^{2+1}}{2+1}\right]_1^{-1} - 2\left[\frac{t^{3+1}}{3+1}\right]_1^{-1} - 5\left[\frac{t^{4+1}}{4+1}\right]_1^{-1} - 2\left[\frac{t^{5+1}}{5+1}\right]_1^{-1}\right)$$

$$\Rightarrow I = -\left([t]_1^{-1} + 2[t^2]_1^{-1} + \frac{4}{3}[t^3]_1^{-1} - \frac{1}{2}[t^4]_1^{-1} - [t^5]_1^{-1} - \frac{1}{3}[t^6]_1^{-1}\right)$$

$$\Rightarrow I = -\left[(-1-1) + 2((-1)^2-1^2) + \frac{4}{3}((-1)^3-1^3) - \frac{1}{2}((-1)^4-1^4) - ((-1)^5-1^5) - \frac{1}{3}((-1)^6-1^6)\right]$$

$$\Rightarrow I = -\left[-2+2(0)+\frac{4}{3}(-2)-\frac{1}{2}(0)-(-2)-\frac{1}{3}(0)\right]$$

$$\Rightarrow I = 2 + \frac{8}{3} + (-2) = \frac{8}{3}$$

$$\therefore \int_{0}^{\pi} \sin^3 x\,(1+2\cos x)(1+\cos x)^2 dx = \frac{8}{3}$$

49. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

Put sin x = t

⇒ cos x dx = dt (Differentiating both sides)

When x = 0, t = sin 0 = 0

When $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 2t\tan^{-1} t\,dt$$

$$\Rightarrow I = 2\int_0^1 t\tan^{-1} t\,dt$$

We will use integration by parts.

Recall $\int f(x)g(x)dx = f(x)[\int g(x)dx] - \int[f'(x)\int g(x)dx]dx + c$

Here, take f(t) = $\tan^{-1}$t and g(t) = t

$$\Rightarrow \int g(t)\,dt = \int t dt = \frac{t^2}{2}$$

Now,

$$f'(t) = \frac{df(t)}{dt} = \frac{d}{dt}(\tan^{-1} t)$$

$$\Rightarrow f'(t) = \frac{1}{1+t^2}$$

Substituting these values, we evaluate the integral.

$$\Rightarrow\ I = 2\left(\left[\tan^{-1} t\left(\frac{t^2}{2}\right)\right]_0^1 - \int_0^1\left(\frac{1}{1+t^2}\right)\left(\frac{t^2}{2}\right)dt\right)$$

$$\Rightarrow\ I = 2\left[\frac{t^2}{2}\tan^{-1} t\right]_0^1 - \int_0^1\left(\frac{t^2}{1+t^2}\right)dt$$

We can write,

$$\frac{t^2}{1\ +\ t^2} = 1 - \frac{1}{1\ +\ t^2}$$

$$\Rightarrow I = 2\left[\frac{t^2}{2}\tan^{-1} t\right]_0^1 - \int_0^1\left[1 - \frac{1}{1\ +\ t^2}\right]dt$$

$$\Rightarrow I = 2\left[\frac{1^2}{2}\tan^{-1}(1) - \frac{0^2}{2}\tan^{-1}(0)\right] - \left(\int_0^1 dt - \int_0^1 \frac{1}{1+t^2}dt\right)$$

Recall $\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

$$\Rightarrow I = 2 \times \frac{1}{2} \times \frac{\pi}{4} - ([t]_0^1 - [\tan^{-1}t]_0^1)$$

$$\Rightarrow I = \frac{\pi}{4} - ([1-0] - [\tan^{-1}(1) - \tan^{-1}(0)])$$

$$\Rightarrow I = \frac{\pi}{4} - \left(1 - \left[\frac{\pi}{4} - 0\right]\right)$$

$$\Rightarrow I = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

$$\therefore \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\,dx = \frac{\pi}{2} - 1$$

50. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx$

We have sin 2x = 2 sin x cos x

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$$

Put sin x = t

⇒ cos x dx = dt (Differentiating both sides)

When x = 0, t = sin 0 = 0

When $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 2t\tan^{-1}t\,dt$$

$$\Rightarrow I = 2\int_0^1 t\tan^{-1}t\,dt$$

We will use integration by parts.

Recall $\int f(x)g(x)dx = f(x)[\int g(x)dx] - \int[f'(x)\int g(x)dx]dx + c$

Here, take f(t) = $\tan^{-1}t$ and g(t) = t

$$\Rightarrow \int g(t)\,dt = \int t dt = \frac{t^2}{2}$$

Now, $f'(t) = \frac{df(t)}{dt} = \frac{d}{dt}(\tan^{-1}t)$

$$\Rightarrow f'(t) = \frac{1}{1+t^2}$$

Substituting these values, we evaluate the integral.

$$\Rightarrow\ I = 2\left(\left[\tan^{-1}t\left(\frac{t^2}{2}\right)\right]_0^1 - \int_0^1 \left(\frac{1}{1+t^2}\right)\left(\frac{t^2}{2}\right)dt\right)$$

$$\Rightarrow\ I = 2\left[\frac{t^2}{2}\tan^{-1}t\right]_0^1 - \int_0^1 \left(\frac{t^2}{1+t^2}\right)dt$$

We can write $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$

$$\Rightarrow I = 2\left[\frac{t^2}{2}\tan^{-1}t\right]_0^1 - \int_0^1 \left[1 - \frac{1}{1\ +\ t^2}\right]dt$$

$$\Rightarrow I = 2\left[\frac{1^2}{2}\tan^{-1}(1) - 0\right] - \left(\int_0^1 dt - \int_0^1 \frac{1}{1\ +\ t^2}dt\right)$$

Recall $\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

$$\Rightarrow I = \frac{\pi}{4} - ([t]_0^1 - [\tan^{-1}t]_0^1)$$

$$\Rightarrow I = \frac{\pi}{4} - ([1-0] - [\tan^{-1}(1) - \tan^{-1}(0)])$$

$$\Rightarrow I = \frac{\pi}{4} - \left(1 - \left[\frac{\pi}{4} - 0\right]\right)$$

$$\Rightarrow I = \frac{\pi}{4} - 1 + \frac{\pi}{4} = \frac{\pi}{2} - 1$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx = \frac{\pi}{2} - 1$$

51. Question

Evaluate the following Integrals:

$$\int_0^1 \left(\cos^{-1}x^2\right)dx$$

Answer

Let $I = \int_0^1 (\cos^{-1}x)^2 dx$

Put $\cos^{-1}x = t$

⇒ x = cos t

⇒ dx = – sin t dt (Differentiating both sides)

When $x = 0, t = \cos - 1(0) = \frac{\pi}{2}$

When x = 1, t = $\cos^{-1}(1) = 0$

So, the new limits are $\frac{\pi}{2}$ and 0.

Substituting this in the original integral,

$$I = \int_{\frac{\pi}{2}}^{0} t^2(-\sin t\,dt)$$

$$\Rightarrow I = -\int_{\frac{\pi}{2}}^{0} t^2 \sin t\,dt$$

We will use integration by parts.

Recall $\int f(x)g(x)dx = f(x)[\int g(x)dx] - \int [f'(x)\int g(x)dx]dx + c$

Here, take f(t) = t^2 and g(t) = sin t

$$\Rightarrow \int g(t)\,dt = \int \sin t\,dt = -\cos t$$

Now, $f'(t) = \frac{df(t)}{dt} = \frac{d}{dt}(t^2)$

⇒ f’(t) = 2t

Substituting these values, we evaluate the integral.

$$\Rightarrow I = -\left([t^2(-\cos t)]_{\frac{\pi}{2}}^{0} - \int_{\frac{\pi}{2}}^{0} (2t)(-\cos t)dt\right)$$

$$\Rightarrow I = [t^2 \cos t]_{\frac{\pi}{2}}^{0} - 2\int_{\frac{\pi}{2}}^{0} t\cos t\,dt$$

Let $I_1 = \int_{\frac{\pi}{2}}^{0} t\cos t\,dt$

We use integration by parts again.

Here, take f(t) = t and g(t) = cos t

$$\Rightarrow \int g(t)\,dt = \int \cos t\,dt = \sin t$$

Now, $f'(t) = \frac{df(t)}{dt} = \frac{d}{dt}(t)$

⇒ f’(t) = 1

Using these values in equation for I_1

$$\Rightarrow I_1 = [t\sin t]_{\frac{\pi}{2}}^{0} - \int_{\frac{\pi}{2}}^{0} (1)(\sin t)\,dt$$

$$\Rightarrow I_1 = [t\sin t]_{\frac{\pi}{2}}^{0} + [\cos t]_{\frac{\pi}{2}}^{0}$$

Substituting I_1 in I, we get

$$I = [t^2\cos t]_{\frac{\pi}{2}}^{0} - 2\left([t\sin t]_{\frac{\pi}{2}}^{0} + [\cos t]_{\frac{\pi}{2}}^{0}\right)$$

$$\Rightarrow I = [t^2\cos t - 2t\sin t - 2\cos t]_{\frac{\pi}{2}}^{0}$$

$$\Rightarrow I = (0 - 0 - 2\cos 0) - \left[\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - 2\cos\left(\frac{\pi}{2}\right)\right]$$

$\Rightarrow$ I = -2 - (-π) = π - 2

$$\therefore \int_0^1 (\cos^{-1}x)^2\,dx = \pi - 2$$

52. Question

Evaluate the following Integrals:

$$\int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$$

Answer

Let $I = \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

Put x = $a\tan^2\theta$

$\Rightarrow$ x = 2a tan θ $\sec^2\theta$ dθ (Differentiating both sides)

When x = 0, $a\tan^2\theta$ = 0 ⇒ tan θ = 0 ⇒ θ = 0

When $x = a, a\tan 2\theta = a \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

So, the new limits are 0 and $\frac{\pi}{4}$.

Also,

$$\sqrt{\frac{x}{a+x}} = \sqrt{\frac{a\tan^2\theta}{a + a\tan^2\theta}}$$

$$\Rightarrow \sqrt{\frac{x}{a+x}} = \sqrt{\frac{\tan^2\theta}{1 + \tan^2\theta}}$$

We have the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$

$$\Rightarrow \sqrt{\frac{\tan^2\theta}{1 + \tan^2\theta}} = \sqrt{\frac{\tan^2\theta}{\sec^2\theta}} = \sqrt{\frac{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}{\left(\frac{1}{\cos^2\theta}\right)}}$$

$$\Rightarrow \sqrt{\frac{\tan^2\theta}{1 + \tan^2\theta}} = \sqrt{\sin^2\theta} = \sin\theta$$

Substituting this in the original integral,

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin\theta)\,(2a\tan\theta\sec^2\theta\,d\theta)$$

$$\Rightarrow I = 2a\int_0^{\frac{\pi}{4}} \theta\tan\theta\sec^2\theta\,d\theta$$

Now, put $\tan\theta = t$

$\Rightarrow \sec^2\theta\, d\theta = dt$ (Differentiating both sides)

When $\theta = 0$, $t = \tan 0 = 0$

When $\theta = \frac{\pi}{4}, t = \tan\frac{\pi}{4} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = 2a\int_0^1 (\tan^{-1}t)\,(t)\,dt$$

$$\Rightarrow I = 2a\int_0^1 t\tan^{-1}t\,dt$$

We will use integration by parts.

Recall $\int f(x)g(x)dx = f(x)[\int g(x)dx] - \int[f'(x)\int g(x)dx]dx + c$

Here, take $f(t) = \tan^{-1}t$ and $g(t) = t$

$$\Rightarrow \int g(t)\,dt = \int tdt = \frac{t^2}{2}$$

Now,

$$f'(t) = \frac{df(t)}{dt} = \frac{d}{dt}(\tan^{-1}t)$$

$$\Rightarrow f'(t) = \frac{1}{1+t^2}$$

Substituting these values, we evaluate the integral.

$$\Rightarrow\ I = 2a\left(\left[\tan^{-1}t\left(\frac{t^2}{2}\right)\right]_0^1 - \int_0^1\left(\frac{1}{1+t^2}\right)\left(\frac{t^2}{2}\right)dt\right)$$

$$\Rightarrow\ I = 2a\left[\frac{t^2}{2}\tan^{-1}t\right]_0^1 - a\int_0^1\left(\frac{t^2}{1+t^2}\right)dt$$

We can write $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$

$$\Rightarrow I = 2a\left[\frac{t^2}{2}\tan^{-1}t\right]_0^1 - a\int_0^1 \left[1 - \frac{1}{1 + t^2}\right]dt$$

$$\Rightarrow I = 2a\left[\frac{1^2}{2}\tan^{-1}(1) - 0\right] - a\left(\int_0^1 dt - \int_0^1 \frac{1}{1 + t^2}dt\right)$$

Recall $\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

$$\Rightarrow I = 2a \times \frac{1}{2} \times \frac{\pi}{4} - a([t]_0^1 - [\tan^{-1}t]_0^1)$$

$$\Rightarrow I = \frac{\pi}{4}a - a([1-0] - [\tan^{-1}(1) - \tan^{-1}(0)])$$

$$\Rightarrow I = \frac{\pi}{4}a - a\left(1 - \left[\frac{\pi}{4} - 0\right]\right)$$

$$\Rightarrow I = \frac{\pi}{4}a - a + \frac{\pi}{4}a = \left(\frac{\pi}{2} - 1\right)a$$

$$\therefore \int_0^a \sin^{-1}\sqrt{\frac{x}{a+x}}dx = \left(\frac{\pi}{2} - 1\right)a$$

53. Question

Evaluate the following Integrals:

$$\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{3/2}}dx$$

Answer

Let $I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{3/2}}dx$

In the denominator, we can write

$$(1-\cos x)^{\frac{3}{2}} = (1-\cos x)\sqrt{1-\cos x}$$

$$\Rightarrow I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)\sqrt{1 - \cos x}}dx$$

$$\Rightarrow I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{(1 - \cos x)}\sqrt{\frac{1 + \cos x}{1 - \cos x}}dx$$

Recall the trigonometric identity,

$$\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} = \tan^2\theta$$

Here, we have,

$$\sqrt{\frac{1 + \cos x}{1 - \cos x}} = \sqrt{\frac{1}{\tan^2\left(\frac{x}{2}\right)}} = \frac{1}{\tan\left(\frac{x}{2}\right)}$$

We also have,

$1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$

$\Rightarrow I = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2\sin^2\left(\frac{x}{2}\right)}\left(\frac{1}{\tan\left(\frac{x}{2}\right)}\right)dx$

$\Rightarrow I = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \operatorname{cosec}^2\left(\frac{x}{2}\right)\cot\left(\frac{x}{2}\right)dx$

Put $\cot\left(\frac{x}{2}\right) = t$

$\Rightarrow -\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)dx = dt$

(Differentiating both sides)

$\Rightarrow \frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)dx = -dt$

When $x = \frac{\pi}{3}, t = \cot\frac{\left(\frac{\pi}{3}\right)}{2} = \cot\frac{\pi}{6} = \sqrt{3}$

When $x = \frac{\pi}{2}, t = \cot\frac{\left(\frac{\pi}{2}\right)}{2} = \cot\frac{\pi}{4} = 1$

So, the new limits are $\sqrt{3}$ and 1.

Substituting this in the original integral,

$I = \int_{\sqrt{3}}^{1} t(-dt)$

$\Rightarrow I = -\int_{\sqrt{3}}^{1} tdt$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$\Rightarrow I = -\left[\frac{t^{1+1}}{1+1}\right]_{\sqrt{3}}^{1}$

$\Rightarrow I = -\frac{1}{2}[t^2]_{\sqrt{3}}^{1}$

$\Rightarrow I = -\frac{1}{2}\left[1^2 - \left(\sqrt{3}\right)^2\right]$

$\Rightarrow I = -\frac{1}{2}[1 - 3] = 1$

$\therefore \int_{\pi/3}^{\pi/2} \frac{\sqrt{1 + \cos x}}{(1 - \cos x)^{3/2}}dx = 1$

54. Question

Evaluate the following Integrals:

$$\int_0^a x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$$

Answer

Let $I = \int_0^a x\sqrt{\frac{a^2-x^2}{a^2+x^2}}dx$

As we have the trigonometric identity

$$\frac{1-\cos(2\theta)}{1+\cos(2\theta)} = \tan^2\theta$$

to evaluate this integral we use $x^2 = a^2\cos 2\theta$

⇒ $2xdx = -2a^2\sin(2\theta)d\theta$ (Differentiating both sides)

⇒ $xdx = -a^2\sin(2\theta)d\theta$

When $x = 0, a2\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0$

$$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

When $x = a$, $a^2\cos 2\theta = a^2 \Rightarrow \cos 2\theta = 1$

⇒ $2\theta = 0 \Rightarrow \theta = 0$

So, the new limits are $\frac{\pi}{4}$ and 0.

Also,

$$\sqrt{\frac{a^2-x^2}{a^2+x^2}} = \sqrt{\frac{a^2-a^2\cos 2\theta}{a^2+a^2\cos 2\theta}}$$

$$\Rightarrow \sqrt{\frac{a^2-x^2}{a^2+x^2}} = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$$

$$\Rightarrow \sqrt{\frac{a^2-x^2}{a^2+x^2}} = \sqrt{\tan^2\theta} = \tan\theta$$

Substituting this in the original integral,

$$I = \int_{\frac{\pi}{4}}^{0} \tan\theta\,(-a^2\sin 2\theta\,d\theta)$$

$$\Rightarrow I = -a^2\int_{\frac{\pi}{4}}^{0} \tan\theta\sin 2\theta\,d\theta$$

$$\Rightarrow I = -a^2\int_{\frac{\pi}{4}}^{0} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow I = -a^2 \int_{\frac{\pi}{4}}^{0} 2\sin^2\theta \, d\theta$$

But, we have 2 $\sin^2\theta$ = 1 – cos 2θ

$$\Rightarrow I = -a^2 \int_{\frac{\pi}{4}}^{0} (1 - \cos 2\theta) d\theta$$

$$\Rightarrow I = -a^2 \left[\int_{\frac{\pi}{4}}^{0} d\theta - \int_{\frac{\pi}{4}}^{0} \cos 2\theta \, d\theta \right]$$

$$\Rightarrow I = -a^2 \left([\theta]_{\frac{\pi}{4}}^{0} - \left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{0} \right)$$

$$\Rightarrow I = -a^2 \left[\left(0 - \frac{\pi}{4}\right) - \frac{1}{2}\left(\sin 0 - \sin\left(2 \times \frac{\pi}{4}\right)\right) \right]$$

$$\Rightarrow I = -a^2 \left[-\frac{\pi}{4} - \frac{1}{2}\left(-\sin\left(\frac{\pi}{2}\right)\right) \right]$$

$$\Rightarrow I = -a^2 \left[-\frac{\pi}{4} + \frac{1}{2} \right] = a^2 \left(\frac{\pi}{4} - \frac{1}{2} \right)$$

$$\therefore \int_{0}^{a} x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} dx = a^2 \left(\frac{\pi}{4} - \frac{1}{2} \right)$$

55. Question

Evaluate the following Integrals:

$$\int_{-a}^{a} \sqrt{\frac{a - x}{a + x}} \, dx$$

Answer

Let $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} dx$

As we have the trigonometric identity

$$\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} = \tan^2\theta$$

to evaluate this integral we use x = acos 2θ

⇒ dx = –2a sin(2θ) dθ (Differentiating both sides)

When x = –a, acos 2θ = –a ⇒ cos 2θ = –1

$$\Rightarrow 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$$

When x = a, acos 2θ = a ⇒ cos 2θ = 1

⇒ 2θ = 0 ⇒ θ = 0

So, the new limits are $\frac{\pi}{2}$ and 0.

Also,

$$\sqrt{\frac{a-x}{a+x}} = \sqrt{\frac{a - a\cos 2\theta}{a + a\cos 2\theta}}$$

$$\Rightarrow \sqrt{\frac{a-x}{a+x}} = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$$

$$\Rightarrow \sqrt{\frac{a-x}{a+x}} = \sqrt{\tan^2\theta} = \tan\theta$$

Substituting this in the original integral,

$$I = \int_{\frac{\pi}{2}}^{0} \tan\theta\,(-2a\sin 2\theta\, d\theta)$$

$$\Rightarrow I = -2a\int_{\frac{\pi}{2}}^{0} \tan\theta \sin 2\theta\, d\theta$$

$$\Rightarrow I = -2a\int_{\frac{\pi}{2}}^{0} \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow I = -2a\int_{\frac{\pi}{2}}^{0} 2\sin^2\theta\, d\theta$$

But, we have $2\sin^2\theta = 1 - \cos 2\theta$

$$\Rightarrow I = -2a\int_{\frac{\pi}{2}}^{0} (1 - \cos 2\theta)d\theta$$

$$\Rightarrow I = -2a\left[\int_{\frac{\pi}{2}}^{0} d\theta - \int_{\frac{\pi}{2}}^{0} \cos 2\theta\, d\theta\right]$$

$$\Rightarrow I = -2a\left([\theta]_{\frac{\pi}{2}}^{0} - \left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}\right)$$

$$\Rightarrow I = -2a\left[\left(0 - \frac{\pi}{2}\right) - \frac{1}{2}\left(\sin 0 - \sin\left(2 \times \frac{\pi}{2}\right)\right)\right]$$

$$\Rightarrow I = -2a\left[-\frac{\pi}{2} - \frac{1}{2}(0)\right]$$

$$\Rightarrow I = -2a\left[-\frac{\pi}{2}\right] = \pi a$$

$$\therefore \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}dx = \pi a$$

56. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$

In the denominator, we can write

$\cos^2 x + 3 \cos x + 2 = (\cos x + 1)(\cos x + 2)$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{(\cos x + 1)(\cos x + 2)} dx$$

Put cos x = t

⇒ –sin(x)dx = dt (Differentiating both sides)

⇒ sin(x)dx = –dt

When x = 0, t = cos 0 = 1

$\text{When } x = \frac{\pi}{2}, t = \cos\frac{\pi}{2} = 0$

So, the new limits are 1 and 0.

Substituting this in the original integral,

$$I = \int_1^0 \frac{t}{(t + 1)(t + 2)} (-dt)$$

$$\Rightarrow I = -\int_1^0 \frac{t}{(t + 1)(t + 2)} dt$$

We can write,

$$\frac{t}{(t + 1)(t + 2)} = \frac{2t + 2 - (t + 2)}{(t + 1)(t + 2)}$$

$$\Rightarrow \frac{t}{(t + 1)(t + 2)} = \frac{2(t + 1) - (t + 2)}{(t + 1)(t + 2)} = \frac{2}{t + 2} - \frac{1}{t + 1}$$

Using this, we have

$$I = -\int_1^0 \left(\frac{2}{t + 2} - \frac{1}{t + 1}\right) dt$$

$$\Rightarrow I = -\left(2\int_1^0 \frac{1}{t + 2} dt - \int_1^0 \frac{1}{t + 1} dt\right)$$

Recall $\int \frac{1}{x + a} dx = \ln|x + a| + c$

$\Rightarrow I = -(2[\ln|x + 2|]_1^0 - [\ln|x + 1|]_1^0)$

⇒ I = – [2(ln|0+2| – ln|1+2|) – (ln|0+1| – ln|1+1|)]

⇒ I = – [2(ln 2 – ln 3) – (ln 1 – ln 2)]

⇒ I = – (2 ln 2 – 2 ln 3 – 0 + ln 2)

⇒ I = – (3 ln 2 – 2 ln 3)

⇒ I = 2 ln 3 – 3 ln 2

⇒ I = ln 9 – ln 8 = ln $\frac{9}{8}$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx = \ln\frac{9}{8}$$

57. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\tan x}{1+m^2 \tan^2 x} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\tan x}{1 + m^2 \tan^2 x} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{1 + m^2\left(\frac{\sin^2 x}{\cos^2 x}\right)} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx$$

We have $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1 + (m^2 - 1)\sin^2 x} dx$$

Put $\sin^2 x = t$

⇒ 2 sin x cos x dx = dt (Differentiating both sides)

⇒ sin x cos x dx = $\frac{1}{2}$ dt

When x = 0, t = $\sin^2 0$ = 0

When $x = \frac{\pi}{2}, t = \sin^2\frac{\pi}{2} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 \frac{1}{1 + (m^2 - 1)t}\left(\frac{1}{2}dt\right)$$

$$\Rightarrow I = \frac{1}{2}\int_0^1 \frac{1}{1 + (m^2 - 1)t} dt$$

Recall $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$

$$\Rightarrow I = \frac{1}{2}\left(\frac{1}{m^2-1}\left[\ln|1+(m^2-1)t|\right]_0^1\right)$$

$$\Rightarrow I = \frac{1}{2(m^2-1)}(\ln|1+(m^2-1)\times 1| - \ln|1+(m^2-1)\times 0|)$$

$$\Rightarrow I = \frac{1}{2(m^2-1)}(\ln|m^2| - \ln|1|)$$

$$\Rightarrow I = \frac{1}{2(m^2-1)}(2\ln|m|) = \frac{\ln|m|}{m^2-1}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\tan x}{1+m^2\tan^2 x}dx = \frac{\ln|m|}{m^2-1}$$

58. Question

Evaluate the following Integrals:

$$\int_0^{1/2} \frac{1}{(1+x^2)\sqrt{1-x^2}}dx$$

Answer

Let $I = \int_0^{\frac{1}{2}} \frac{1}{(1+x^2)\sqrt{1-x^2}}dx$

Put x = sin θ

⇒ dx = cos θ dθ (Differentiating both sides)

Also, $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

When x = 0, sin θ = 0 ⇒ θ = 0

When $x = \frac{1}{2}, \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

So, the new limits are 0 and $\frac{\pi}{6}$.

Substituting this in the original integral,

$$I = \int_0^{\frac{\pi}{6}} \frac{1}{(1+\sin^2\theta)\cos\theta}(\cos\theta\, d\theta)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{6}} \frac{1}{1+\sin^2\theta}d\theta$$

Dividing numerator and denominator with $\cos^2\theta$, we have

$$I = \int_0^{\frac{\pi}{6}} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{6}} \frac{\sec^2\theta}{1 + 2\tan^2\theta} d\theta$$

[∵ $\sec^2\theta = 1 + \tan^2\theta$]

Put $\tan\theta = t$

$\Rightarrow \sec^2\theta \, d\theta = dt$ (Differentiating both sides)

When $\theta = 0$, $t = \tan 0 = 0$

$$\text{When } \theta = \frac{\pi}{6}, t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

So, the new limits are 0 and $\frac{1}{\sqrt{3}}$.

Substituting this in the original integral,

$$I = \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1 + 2t^2} dt$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\frac{1}{2} + t^2} dt$$

Recall $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

$$\Rightarrow I = \frac{1}{2}\left(\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\left[\tan^{-1}\frac{t}{\left(\frac{1}{\sqrt{2}}\right)}\right]_0^{\frac{1}{\sqrt{3}}}\right)$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\left[\tan^{-1}(\sqrt{2}t)\right]_0^{\frac{1}{\sqrt{3}}}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) - \tan^{-1}(0)\right]$$

$$\Rightarrow I = \frac{1}{\sqrt{2}}\left(\tan^{-1}\left(\sqrt{\frac{2}{3}}\right) - 0\right) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

$$\therefore \int_0^{\frac{1}{2}} \frac{1}{(1 + x^2)\sqrt{1 - x^2}} dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

59. Question

Evaluate the following Integrals:

$$\int_{1/3}^{1} \frac{\left(x - x^3\right)^{1/3}}{x^4} dx$$

Answer

Let $I = \int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4} dx$

$$\Rightarrow I = \int_{\frac{1}{3}}^{1} \frac{\left[x^3\left(\frac{1}{x^2} - 1\right)\right]^{\frac{1}{3}}}{x^4} dx$$

(taking x^3 common)

$$\Rightarrow I = \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^4} dx$$

$$\Rightarrow I = \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2} - 1\right)^{\frac{1}{3}}}{x^3} dx$$

Put $\frac{1}{x^2} - 1 = t$

$\Rightarrow -2x^{-2-1}dx = dt$ (Differentiating both sides)

$$\Rightarrow -\frac{2}{x^3} dx = dt$$

$$\Rightarrow \frac{1}{x^3} dx = -\frac{1}{2} dt$$

When $x = \frac{1}{3}, t = \frac{1}{\left(\frac{1}{3}\right)^2} - 1 = 32 - 1 = 8$

When $x = 1, t = \frac{1}{1^2} - 1 = 0$

So, the new limits are 8 and 0.

Substituting this in the original integral,

$$I = \int_{8}^{0} t^{\frac{1}{3}} \left(-\frac{dt}{2}\right)$$

$$\Rightarrow I = -\frac{1}{2} \int_{8}^{0} t^{\frac{1}{3}} dt$$

Recall $\int x^n dx = \frac{1}{n+1} x^{n+1} + c$

$$\Rightarrow I = -\frac{1}{2} \left[\frac{t^{\frac{1}{3}+1}}{\frac{1}{3} + 1}\right]_8^0$$

$$\Rightarrow I = -\frac{3}{8} \left[t^{\frac{4}{3}}\right]_8^0$$

$$\Rightarrow I = -\frac{3}{8} \left[0 - (8)^{\frac{4}{3}}\right]$$

$$\Rightarrow I = -\frac{3}{8}[-16] = 6$$

$$\therefore \int_{\frac{1}{3}}^{1} \frac{(x - x^3)^{\frac{1}{3}}}{x^4} dx = 6$$

60. Question

Evaluate the following Integrals:

$$\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{\left(\sin^3 x + \cos^3 x\right)^2} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x \cos^2 x}{\left[(\cos^3 x)\left(\frac{\sin^3 x}{\cos^3 x} + 1\right)\right]^2} dx$$

(taking $\cos^3 x$ common)

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\sin^2 x \cos^2 x}{\cos^6 x (\tan^3 x + 1)^2} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\frac{\sin^2 x}{\cos^2 x} \times \frac{1}{\cos^2 x}}{(\tan^3 x + 1)^2} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan^2 x \sec^2 x}{(\tan^3 x + 1)^2} dx$$

Put tan x = t

$\Rightarrow \sec^2 x \, dx = dt$ (Differentiating both sides)

When x = 0, t = tan 0 = 0

When $x = \frac{\pi}{4}, t = \tan\frac{\pi}{4} = 1$

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 \frac{t^2}{(1 + t^3)^2} dt$$

Put $t^3 = u$

$\Rightarrow 3t^2 dt = du$ (Differentiating both sides)

$$\Rightarrow t^2 dt = \frac{1}{3} du$$

When t = 0, u = 0^3 = 0

When t = 1, u = 1^3 = 1

So, the new limits are 0 and 1.

Substituting this in the original integral,

$$I = \int_0^1 \frac{1}{(1 + u)^2}\left(\frac{1}{3}du\right)$$

$$\Rightarrow I = \frac{1}{3}\int_0^1 (u+1)^{-2}du$$

Recall $\int (x+a)^n dx = \frac{1}{n+1}(x+a)^{n+1} + c$

$$\Rightarrow I = \frac{1}{3}\left[\frac{(u + 1)^{-2+1}}{-2 + 1}\right]_0^1$$

$$\Rightarrow I = -\frac{1}{3}\left[\frac{1}{u+1}\right]_0^1$$

$$\Rightarrow I = -\frac{1}{3}\left(\frac{1}{1+1} - \frac{1}{0+1}\right)$$

$$\Rightarrow I = -\frac{1}{3}\left(\frac{1}{2} - 1\right) = \frac{1}{6}$$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx = \frac{1}{6}$$

61. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \sqrt{\cos x - \cos^3 x}\left(\sec^2 x - 1\right)\cos^2 x \, dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}\,(\sec^2 x - 1)\cos^2 x\, dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sqrt{\cos x\,(1 - \cos^2 x)}(\sec^2 x - 1)\cos^2 x\, dx$$

We have $\sin^2 x + \cos^2 x = 1$ and $\sec^2 x - \tan^2 x = 1$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sqrt{\cos x \sin^2 x}\,(\tan^2 x)\cos^2 x\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\sin x\left(\frac{\sin^2 x}{\cos^2 x}\right)\cos^2 x\, dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\sin^3 x\,dx$$

We can write $\sin^3 x = \sin^2 x \times \sin x = (1 - \cos^2 x)\sin x$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}(1 - \cos^2 x)\sin x\,dx$$

Put cos x = t

⇒ –sin(x)dx = dt (Differentiating both sides)

⇒ sin(x)dx = –dt

When x = 0, t = cos 0 = 1

When $x = \frac{\pi}{2}, t = \cos\frac{\pi}{2} = 0$

So, the new limits are 1 and 0.

Substituting this in the original integral,

$$I = \int_1^0 \sqrt{t}(1 - t^2)(-dt)$$

$$\Rightarrow I = -\int_1^0 t^{\frac{1}{2}}(1 - t^2)dt$$

$$\Rightarrow I = -\int_1^0 \left(t^{\frac{1}{2}} - t^{\frac{5}{2}}\right)dt$$

$$\Rightarrow I = -\left(\int_1^0 t^{\frac{1}{2}}dt - \int_1^0 t^{\frac{5}{2}}dt\right)$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = -\left(\left[\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^0 - \left[\frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right]_1^0\right)$$

$$\Rightarrow I = -\left(\frac{2}{3}\left[t^{\frac{3}{2}}\right]_1^0 - \frac{2}{7}\left[t^{\frac{7}{2}}\right]_1^0\right)$$

$$\Rightarrow I = -\left[\frac{2}{3}(0-1) - \frac{2}{7}(0-1)\right]$$

$$\Rightarrow I = -\left(-\frac{2}{3} + \frac{2}{7}\right) = \frac{8}{21}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}(\sec^2 x - 1)\cos^2 x\,dx = \frac{8}{21}$$

62. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^n} dx$$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^n} dx$

We can write,

$$\cos x = \cos\left(2\times\frac{x}{2}\right) = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$\Rightarrow \cos x = \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$$

Putting this value in the integral

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^n} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^{n-1}} dx$$

Put $\cos\frac{x}{2} + \sin\frac{x}{2} = t$

$$\Rightarrow \left(-\frac{1}{2}\sin\frac{x}{2} + \frac{1}{2}\cos\frac{x}{2}\right) dx = dt$$

(Differentiating both sides)

$$\Rightarrow \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) dx = 2dt$$

When x = 0, t = cos 0 + sin 0 = 1

$$\text{When } x = \frac{\pi}{2}, t = \cos\frac{\left(\frac{\pi}{2}\right)}{2} + \sin\frac{\left(\frac{\pi}{2}\right)}{2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

So, the new limits are 1 and $\sqrt{2}$.

Substituting this in the original integral,

$$I = \int_1^{\sqrt{2}} \frac{1}{t^{n-1}}(2dt)$$

$$\Rightarrow I = 2\int_1^{\sqrt{2}} t^{-(n-1)} dt$$

Recall $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$

$$\Rightarrow I = 2\left[\frac{t^{-(n-1)+1}}{-(n-1)+1}\right]_1^{\sqrt{2}}$$

$$\Rightarrow I = \frac{2}{2-n}[t^{2-n}]_1^{\sqrt{2}}$$

$$\Rightarrow I = \frac{2}{2-n}\left[\begin{matrix}(\sqrt{2})^{2-n} \\ -1^{2-n}\end{matrix}\right]$$

$$\Rightarrow I = \frac{2}{2-n}\left[\begin{matrix}\left(2^{\frac{1}{2}}\right)^{2-n} \\ -1\end{matrix}\right] = \frac{2}{2-n}\left(2^{1-\frac{n}{2}} - 1\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^n} dx = \frac{2}{2-n}\left(2^{1-\frac{n}{2}} - 1\right)$$

Exercise 20.3

1 A. Question

Evaluate the following Integrals:

$\int_1^4 f(x)dx$, where $f(x) = \begin{cases} 4x+3, & \text{if } 1 \le x \le 2 \\ 3x+5, & \text{if } 2 \le x \le 4 \end{cases}$

Answer

We have, $\int_1^4 f(x)dx$

$= \int_1^2 (4x + 3)dx + \int_2^4 (3x + 5)dx$

$= \left[\frac{4x^2}{2} + 3x\right]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^5$

$= \left[\left(\frac{16}{2} + 6\right) - \left(\frac{4}{2} + 3\right)\right] + \left[\left(\frac{48}{2} + 20\right) - \left(\frac{12}{2} + 10\right)\right]$

$= [14 - 5] + [44 - 16]$

= 9 + 28

= 37

1 B. Question

Evaluate the following Integrals:

$\int_0^9 f(x)\,dx$, where $f(x) = \begin{cases} \sin x & , \quad 0 \le x \le \pi/2 \\ 1 & , \quad \pi/2 \le x \le 3 \\ e^{x-3} & , \quad 3 \le x \le 9 \end{cases}$

Answer

We have, $\int_0^9 f(x)dx$

$= \int_0^{\frac{\pi}{2}} \sin x\,dx + \int_{\frac{\pi}{2}}^3 1\,dx + \int_3^9 e^{x-3}dx$

$= [-\cos x]_0^{\frac{\pi}{2}} + [x]_{\frac{\pi}{2}}^3 + [e^{x-3}]_3^9$

$= \left[-\cos\frac{\pi}{2} + \cos 0\right] + \left[3 - \frac{\pi}{2}\right] + [e^{9-3} - e^{3-3}]$

$= [0 + 1] + \left[3 - \frac{\pi}{2}\right] + [e^{6} - e^{0}]$

$= 1 + 3 - \frac{\pi}{2} + e^{6} - 1$

Hence, $3 - \frac{\pi}{2} + e^{6}$

1 C. Question

Evaluate the following Integrals:

$$\int_{1}^{4} f(x)\,dx, \text{ where } f(x) = \begin{cases} 7x+3, & \text{if } 1 \le x \le 3 \\ 8x, & \text{if } 3 \le x \le 4 \end{cases}$$

Answer

We have, $\int_{1}^{4} f(x)\,dx$

$= \int_{1}^{3}(7x + 3) + \int_{3}^{4} 8x\,dx$

$= \left[\frac{7x^2}{2} + 3x\right]_{1}^{3} + \left[\frac{8x^2}{2}\right]_{3}^{4}$

$= \left[\frac{7\times9}{2} + 3\times3\right] - \left[\frac{7\times1}{2} + 3\times1\right] + \left[\left(\frac{8\times16}{2}\right) - \left(\frac{8\times9}{2}\right)\right]$

$= \left[\frac{63}{2} + 9 - \frac{7}{2} + 3\right] + [64 - 36]$

= 34 + 28

Hence, 62

2. Question

Evaluate the following Integrals:

$$\int_{-4}^{4} |x + 2|\,dx$$

Answer

We have,

$= \int_{-4}^{4} |x + 2|\,dx$

$= \int_{-4}^{-2} -(x + 2)dx + \int_{-2}^{4}(x + 2)\,dx$

$= -\left[\frac{x^2}{2} + 2x\right]_{-4}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{4}$

$= -\left[\frac{4}{2} - 4\right] - \left[\frac{16}{2} - 8\right] + \left[\left(\frac{16}{2} + 8\right) - \left(\frac{4}{2} - 4\right)\right]$

$= -[2 - 4] - [8 - 8] + [(8 + 8) - (2 - 4)]$

$= -[-2] + 16 + 2$

= 2 + 2 + 16

Hence, 20

3. Question

Evaluate the following Integrals:

$$\int_{-3}^{3} |x+1|\, dx$$

Answer

We have, $\int_{-3}^{3} |x + 1|\, dx$

$= \int_{-3}^{3} |x + 1|\, dx$

$= \int_{-3}^{-1} -(x + 1)dx + \int_{-1}^{3} (x + 1)\, dx$

$= -\left[\frac{x^2}{2} + x\right]_{-3}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{3}$

$= -\left[\frac{1}{2} - 1\right] + \left[\frac{9}{2} - 3\right] + \left[\left(\frac{9}{2} + 3\right) - \left(\frac{1}{2} - 1\right)\right]$

$= -\left[-\frac{1}{2}\right] + \left[\frac{3}{2}\right] + \left[\left(\frac{15}{2}\right) - \left(-\frac{1}{2}\right)\right]$

$= \frac{1}{2} + \frac{3}{2} + \frac{16}{2}$

$= \frac{20}{2} = 10$

Hence, $\int_{-3}^{3} |x + 1|\, dx = 10$

4. Question

Evaluate the following Integrals:

$$\int_{-1}^{1} |2x+1|\, dx$$

Answer

We have, $\int_{-1}^{1} |2x + 1|\, dx$

$= \int_{-1}^{1} |2x + 1|\, dx$

$= \int_{-1}^{-\frac{1}{2}} -(2x + 1)dx + \int_{-\frac{1}{2}}^{1} (2x + 1)\, dx$

$= -\left[\frac{2x^2}{2} + x\right]_{-1}^{-\frac{1}{2}} + \left[\frac{2x^2}{2} + x\right]_{-\frac{1}{2}}^{1}$

$= -\left[\frac{2}{8} - \frac{1}{2}\right] - \left[\frac{2}{2} - 1\right] + \left[\left(\frac{2}{2} + 1\right) - \left(\frac{2}{8} - \frac{1}{2}\right)\right]$

$= -\left[\frac{1}{4} - \frac{1}{2}\right] - [1 - 1] + \left[(1 + 1) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$

$= -\left[-\frac{1}{4}\right] + \left[2 + \left(\frac{1}{4}\right)\right]$

$= \frac{1}{4} + 2 + \frac{1}{4}$

$= \frac{2}{4} + 2$

Hence, $\int_{-1}^{1}|2x + 1|\,dx = \frac{5}{2}$

5. Question

Evaluate the following Integrals:

$$\int_{-2}^{2}|2x+3|dx$$

Answer

We have, $\int_{-2}^{2}|2x + 3|\,dx$

$= \int_{-2}^{2}|2x + 3|\,dx$

$= \int_{-2}^{-\frac{3}{2}}-(2x + 3)dx + \int_{-\frac{3}{2}}^{2}(2x + 3)\,dx$

$= -\left[\frac{2x^2}{2} + 3x\right]_{-2}^{-\frac{3}{2}} + \left[\frac{2x^2}{2} + 3x\right]_{-\frac{3}{2}}^{2}$

$= -\left[\frac{2\times 9}{2\times 4}-\frac{9}{2}\right]-\left[\frac{2\times 4}{2}-6\right] + \left[\left(\frac{2\times 4}{2} + 6\right)-\left(\frac{2\times 9}{2\times 4}-\frac{9}{2}\right)\right]$

$= -\left[\frac{18}{8}-\frac{9}{2}\right]-\left[\frac{8}{2}-6\right] + \left[\left(\frac{8}{2} + 6\right)-\left(\frac{18}{8}-\frac{9}{2}\right)\right]$

$= -\left[\frac{9}{4}-\frac{9}{2}\right]-[-2] + \left[(10)-\left(\frac{9}{4}-\frac{9}{2}\right)\right]$

$= -\left[-\frac{9}{4} + 2\right] + \left[(10) + \frac{9}{4}\right]$

$= \frac{9}{4}-2 + 10 + \frac{9}{4}$

$= \frac{9}{2} + 8$

Hence, $\int_{-2}^{2}|2x + 3|\,dx = \frac{25}{2}$

6. Question

Evaluate the following Integrals:

$$\int_{0}^{2}\left|x^2-3x+2\right|dx$$

Answer

We have, $\int_{0}^{2}|x^2-3x + 2|\,dx$

$= \int_{0}^{2}|x^2-3x + 2|\,dx$

$= \int_{0}^{1}(x^2-3x + 2)dx + \int_{1}^{2}-(x^2-3x + 2)\,dx$

$= \left[\frac{x^3}{3}-\frac{3x^2}{2} + 2x\right]_{0}^{1}-\left[\frac{x^3}{3}-\frac{3x^2}{2} + 2x\right]_{1}^{2}$

$= \left[\frac{1}{3}-\frac{3}{2} + 2-0\right]-\left[\left(\frac{8}{3}-\frac{12}{2} + 4-\frac{1}{3} + \frac{3}{2} + 2\right)\right]$

$= \left[\frac{1}{6}\right]-\left[-\frac{5}{6}\right]$

$= \frac{1}{6} + \frac{5}{6}$

$= \frac{6}{6} = 1$

Hence, $\int_0^2 |x^2 - 3x + 2|\, dx = 1$

7. Question

Evaluate the following Integrals:

$$\int_0^3 |3x - 1| dx$$

Answer

We have, $\int_0^3 |3x - 1|\, dx$

$= \int_0^3 |3x - 1|\, dx$

$= \int_0^{\frac{1}{3}} -(3x - 1)dx + \int_{\frac{1}{3}}^3 (3x - 1)\, dx$

$= -\left[\frac{3x^2}{2} - x\right]_0^{\frac{1}{3}} + \left[\frac{3x^2}{2} - x\right]_{\frac{1}{3}}^3$

$= -\left[\frac{3}{9\times 2} - \frac{1}{3} - 0\right] + \left[\left(\frac{3\times 9}{2} - 3\right) - \left(\frac{3}{9\times 2}\right)\right]$

$= -\left[\frac{1}{6} - \frac{1}{3}\right] + \left[\left(\frac{27}{2} - 3\right) - \left(\frac{1}{6} - \frac{1}{3}\right)\right]$

$= -\left[-\frac{1}{6}\right] + \left[\left(\frac{21}{2}\right) - \left(-\frac{1}{6}\right)\right]$

$= \left[\frac{1}{6}\right] + \left[\left(\frac{21}{2}\right) + \left(\frac{1}{6}\right)\right]$

$= \frac{2}{6} + \frac{21}{2}$

$= \frac{1}{3} + \frac{21}{2} = \frac{2 + 63}{6}$

$= \frac{65}{6}$

Hence, $\int_0^3 |3x - 1|\, dx = \frac{65}{6}$

8. Question

Evaluate the following Integrals:

$$\int_{-6}^6 |x + 2| dx$$

Answer

We have, $\int_{-6}^6 |x + 2| dx$

$= \int_{-6}^2 -(x + 2) + \int_2^6 (x + 2)\, dx$

$= -\left[\frac{x^2}{2} + 2x\right]_{-6}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^6$

$= -\left[\left(\frac{4}{2} + 2 \times 2\right) - \left(\frac{36}{2} - 12\right)\right] + \left[\left(\frac{36}{2} + 12\right) - \left(\frac{4}{2} - 4\right)\right]$

$= -[(2 - 4) - (18 - 12)] + [(18 + 12) - (2 - 4)]$

= –(–8) + (30 + 2)

= 8 + 32

= 40

Hence, $\int_{-6}^{6} |x + 2| dx = 40$

9. Question

Evaluate the following Integrals:

$$\int_{-2}^{2} |x + 1| \, dx$$

Answer

We have, $\int_{-2}^{2} |x + 1| \, dx$

$= \int_{-2}^{-1} -(x + 1)dx + \int_{-1}^{2} (x + 1) \, dx$

$= -\left[\frac{x^2}{2} + x\right]_{-2}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{2}$

$= -\left[\left(\frac{1}{2} - 1\right) - \left(\frac{4}{2} - 2\right)\right] + \left[\left(\frac{4}{2} + 2\right) - \left(\frac{1}{2} - 1\right)\right]$

$= -\left[(-\frac{1}{2}) - 0\right] + \left[4 + \frac{1}{2}\right]$

$= \frac{1}{2} + \frac{9}{2}$

$= \frac{10}{2}$

Hence, $\int_{-2}^{2} |x + 1| \, dx = 5$

10. Question

Evaluate the following Integrals:

$$\int_{1}^{2} |x - 3| \, dx$$

Answer

We have, $\int_{1}^{2} |x - 3| \, dx$

$= \int_{1}^{2} -(x - 3)dx$

= [x – 3 < 0 for 1 > x > 2]

$= -\left[\frac{x^2}{2} - 3x\right]_{1}^{2}$

$= -\left[\left(\frac{4}{2} - 6\right) - \left(\frac{1}{2} - 3\right)\right]$

$= -\left[(-4) - \left(-\frac{5}{2}\right)\right]$

$= -\left[-4 + \frac{5}{2}\right]$

$= -\left[-\frac{3}{2}\right]$

$= \frac{3}{2}$

Hence, $\int_1^2 |x-3|\, dx = \frac{3}{2}$

11. Question

Evaluate the following Integrals:

$$\int_0^{\pi/2} |\cos 2x|\, dx$$

Answer

We have, $\int_0^{\frac{\pi}{2}} |\cos 2x|\, dx$

$= \int_0^{\frac{\pi}{2}} |\cos 2x|\, dx$

$= \int_0^{\frac{\pi}{4}} -(\cos 2x)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos 2x)\, dx$

$= \left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} + \left[\frac{-\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right] + \frac{1}{2}[\sin\pi + \sin\frac{\pi}{2}]$

$= \frac{1}{2}(1) + \frac{1}{2}(1)$

$= \frac{1}{2} + \frac{1}{2}$

$= \frac{2}{2} = 1$

Hence, $\int_0^{\frac{\pi}{2}} |\cos 2x|\, dx = 1$

12. Question

Evaluate the following Integrals:

$$\int_0^{2\pi} |\sin x|\, dx$$

Answer

We have, $\int_0^{\frac{\pi}{2}} |\sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin x\, dx = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} -\sin x\, dx$

$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$

$= [-\cos\pi + \cos 0] + [\cos 2\pi - \cos\pi]$

= [1 + 1] + [1 + 1]

= 2 + 2

= 4

Hence, $\int_0^{\frac{\pi}{2}} |\sin x|\, dx$ = 4

13. Question

Evaluate the following Integrals:

$$\int_{-\pi/4}^{\pi/4} |\sin x|\, dx$$

Answer

We have, $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x|\, dx$

$= \int_{-\frac{\pi}{4}}^{0} -\sin x\, dx + \int_0^{\frac{\pi}{4}} \sin x\, dx$

$= [\cos x]_{-\frac{\pi}{4}}^{0} + [-\cos x]_0^{\frac{\pi}{4}}$

$= [\cos 0 + \cos\left(-\frac{\pi}{4}\right)] + [-\cos\frac{\pi}{4} + \cos 0]$

$= \left[\left(1 - \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} - 1\right)\right]$

$= \left(2 - \sqrt{2}\right)$

Hence, $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sin x|\, dx = \left(2 - \sqrt{2}\right)$

14. Question

Evaluate the following Integrals:

$$\int_2^8 |x - 5|\, dx$$

Answer

We have,

$$\int_2^8 |x - 5| dx$$

We have,

$$|x - 5| = \begin{cases} x - 5 & \text{if } x \in (5, 8) \\ -(x - 5) & \text{if } x \in (2, 5) \end{cases}$$

Hence,

$= \int_2^5 -(x - 5)dx + \int_5^8 (x - 5)dx$

$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$

$= -\left[\left(\frac{25}{2} - 25\right) - \left(\frac{4}{2} - 10\right) + \left(\frac{64}{2} - 40\right) - \left(\frac{25}{2} - 25\right)\right]$

$= -\left[-\frac{25}{2} + 8\right] + \left[(-8) + \left(\frac{25}{2}\right)\right]$

$= \frac{25}{2} + \frac{25}{2} - 8 - 8$

= 25 – 16 = 9

Hence, $\int_2^8 |x - 5|dx$ = 9

15. Question

Evaluate the following Integrals:

$$\int_{-\pi/2}^{\pi/2} (\sin|x| + \cos|x|)dx$$

Answer

We have,

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin|x| + \cos|x|)dx$

Let f(x) = sin|x| + cos|x|

Then, f(x) =f(–x)

Since, (f(x) is an even function.

So, $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin|x| + \cos|x|)dx$

$= 2\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx$

$= 2[-\cos x + \sin x]_0^{\frac{\pi}{2}}$

$= 2[-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \cos 0 - \sin 0]$

= 2[0 + 1 + 1–0]

= 2(2)

Hence, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin|x| + \cos|x|)dx$ = 4

16. Question

Evaluate the following Integrals:

$$\int_0^4 |x - 1|\, dx$$

Answer

We have, $\int_0^4 |x - 1|\, dx$

It can be seen that (x–1)≤0 when 0≤x≤1 and (x–1)≥0 when 1≤x≤4

= $I = \int_1^4 |x - 1|dx$

$= \int_0^1 -(x - 1)dx + \int_1^4 (x - 1)dx$

$= -\left[\frac{x^2}{2} - x\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^4$

$= -\left[\left(\frac{1}{2} - 1 - 0\right) - \left(\frac{16}{2} - 4 - \frac{1}{2} + 1\right)\right]$

$= \left[1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1\right]$

$= \left[\frac{1}{2} + 4 - \frac{1}{2} + 1\right]$

= 5

Hence, $\int_1^4 |x - 1|\, dx = 5$

17. Question

Evaluate the following Integrals:

$$\int_1^4 \{|x - 1| + |x - 2| + |x - 4|\}\, dx$$

Answer

Let I $= \int_1^4 \{|x - 1| + |x - 2| + |x - 4|\} dx$

$= \int_1^2 \{(x - 1 - x + 2 - x + 4)\} dx + \int_2^4 \{(x - 1 + x - 2 - x + 4)\} dx$

$= \int_1^2 (5 - x) dx + \int_2^4 (x + 1) dx$

$= \left[5x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} + x\right]_2^4$

$= \left[10 - \frac{4}{2} - 5 + \frac{1}{2}\right] + \left[\frac{16}{2} + 4 - \frac{4}{2} - 2\right]$

$= \left[5 - \frac{3}{2}\right] + [8 + 4 - 4]$

$= \left[\frac{7}{2}\right] + 8$

Hence, $\int_1^4 \{|x - 1| + |x - 2| + |x - 4|\} dx = \frac{23}{2}$

18. Question

Evaluate the following Integrals:

$$\int_{-5}^{0} f(x) dx, \text{where } f(x) = |x| + |x + 2| + |x + 5|$$

Answer

We have,

I $= \int_{-5}^{0} \{|x| + |x + 2| + |x + 5|\} dx$

$= \int_{-5}^{0} \{|x| dx + {}_{-5}^{0}\int |x + 2| dx + \int_{-5}^{0} |x + 5|\, dx$

$=$ I$= \int_{-5}^{0} -x\, dx + \int_{-5}^{-2} -(x + 2)\, dx + \int_{-2}^{0} (x + 2)\, dx + \int_{-5}^{0} |x + 5|\, dx$

$= \left[-\frac{x}{2}\right]_{-5}^{0} + \left[-\frac{x^2}{2} - 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{0} + \left[\frac{x^2}{2} + 5x\right]_{-5}^{0}$

$= \left[0 + \frac{25}{2}\right] - \left[\frac{4}{2} + 4 - \frac{25}{2} + 10\right] + \left[0 - \frac{4}{2} + 4\right] + \left[0 - \frac{25}{2} + 25\right]$

$= \frac{25}{2} - \left[8 - \frac{25}{2}\right] + (2) + \left[25 - \frac{25}{2}\right]$

$= \frac{25}{2} - 8 + \frac{25}{2} + 2 + 25 - \frac{25}{2}$

$= 19 + \frac{25}{2}$

Hence, $\int_{-5}^{0}\{|x| + |x + 2| + |x + 5|\}dx = \frac{62}{2}$

19. Question

Evaluate the following Integrals:

$$\int_{0}^{4}(|x| + |x - 2| + |x - 4|)dx$$

Answer

We have

$I = \int_{0}^{4}(|x| + |x - 2| + |x - 4|)dx$

$= \int_{0}^{2}|x| + |x - 2| + |x - 4|)dx + \int_{2}^{4}(|x| + |x - 2| + |x - 4|)dx$

$= \int_{0}^{2}(x + 2 - x + 4 - x)dx + \int_{2}^{4}(x + x - 2 + 4 - x)dx$

$= \int_{0}^{2}(6 - x)dx + \int_{2}^{4}(2 + x)dx$

$= \left[6x - \frac{x^2}{2}\right]_{0}^{2} + \left[2x + \frac{x^2}{2}\right]_{2}^{4}$

= [12–2–0–0] + [8 + 8–4–2]

= [10 + 10]

= 20

Hence, $\int_{0}^{4}(|x| + |x - 2| + |x - 4|)dx$ =20

20. Question

Evaluate the following Integrals:

$$\int_{-1}^{2}(|x + 1| + |x| + |x - 1|)dx$$

Answer

We have, $\int_{-1}^{2}|x + 1| + |x| + |x - 1|)dx$

Now, we can write as

$$\int_{-1}^{2}|x + 1|dx \int_{-1}^{2}|x|dx + \int_{-1}^{2}|x - 1|dx$$

$$\int_{-1}^{2}(x + 1)dx - \int_{-1}^{0}xdx + \int_{0}^{2}xdx - \int_{-1}^{1}(x - 1)dx + \int_{1}^{2}(x - 1)dx$$

$$\left\{\frac{x^2}{2}+x\right\}_{-1}^{2}-\left\{\frac{x^2}{2}\right\}_{-1}^{0}+\left\{\frac{x^2}{2}\right\}_{0}^{2}-\left\{\frac{x^2}{2}-x\right\}_{-1}^{1}+\left\{\frac{x^2}{2}-x\right\}_{1}^{2}$$

While putting the limits

$$\left\{\left(\frac{4}{2}+2-\frac{1}{2}-1\right)-\left(0-\frac{1}{2}\right)+\left(\frac{4}{2}-0\right)-\left(\frac{1}{2}-1+\frac{1}{2}+1\right)+\left(\frac{4}{2}-2-\frac{1}{2}+1\right)\right\}$$

$$\left\{\left(4-\left(-\frac{1}{2}\right)\right)-\left(-\frac{1}{2}\right)+(2)-\left(-\frac{1}{2}-\frac{3}{2}\right)+\left(2-2+\frac{1}{2}\right)\right\}$$

$$\left\{\left(4+\frac{1}{2}\right)+\frac{1}{2}+(2)-\left(-\frac{1}{2}-\frac{3}{2}\right)+\left(0+\frac{1}{2}\right)\right\}$$

$$\left\{\left(4+\frac{1}{2}\right)+\frac{1}{2}+(2)-\left(-\frac{1}{2}-\frac{3}{2}\right)+\left(0+\frac{1}{2}\right)\right\}$$

$$\left\{\left(\frac{9}{2}\right)+\frac{1}{2}+(2)+(2)+\left(\frac{1}{2}\right)\right\}$$

$$\left\{\frac{11}{2}+(4)\right\}$$

Hence, $\frac{19}{2}$

21. Question

Evaluate the following Integrals:

$$\int_{-2}^{2} x\, e^{|x|}\, dx$$

Answer

$\int_{-2}^{2} x\, e^{|x|}$

$= \int_{-2}^{0} x\, e^{-x} dx + \int_{0}^{2} x\, e^{x}\, dx$

Let's Say $I = I_1 + I_2$

And, $I_1 = \int_{-2}^{0} x\, e^{-x} dx$

Using Integration By parts

$$\int f'g = fg - \int fg'$$

$f' = e^{-x}$, $g = x$

$f = -e^{-x}$, $g' = 1$

$$\int_{-2}^{0} x\, e^{-x} dx = \{-xe^{-x}\}_{-2}^{0} - \int_{-2}^{0} e^{-x} dx$$

$$\int_{-2}^{0} x\, e^{-x} dx = \{-xe^{-x} - e^{-x}\}_{-2}^{0}$$

$$\int_{-2}^{0} x\, e^{-x} dx = \{(-1) - (2e^2 - e^2)\}$$

$I_1 = \int_{-2}^{0} x\, e^{-x} dx = \{-1 - e^2\}$

For $I_2 = \int_0^2 xe^x\, dx$

Using Integration By parts

$\int f'g = fg - \int fg'$

$f' = e^x$, $g = x$

$f = e^x$, $g' = 1$

$$\int_0^2 x\, e^x dx = \{xe^x\}_0^2 - \int_0^2 e^x dx$$

$$\int_0^2 x\, e^x dx = \{xe^x - e^x\}_0^2$$

$$\int_0^2 x\, e^x dx = \{(2e^2 - e^2 + 1)\}$$

$I_2 = \int_0^2 x\, e^x dx = e^2 + 1$

Now, $I = I_1 + I_2$

$\int_{-2}^{2} x\, e^{|x|} = -1 - e^2 + e^2 + 1 = 0$

Hence, 0

22. Question

Evaluate the following Integrals:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\sin x \left|\sin x\right|\right) dx$$

Answer

$$-\int_{-\frac{\pi}{4}}^{0} \sin^2 x\, dx + \int_0^{\frac{\pi}{2}} \sin^2 x\, dx$$

$$= \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= -\int_{-\frac{\pi}{4}}^{0} \frac{1-\cos 2x}{2} dx + \int_0^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx$$

$$= -\frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}_{-\frac{\pi}{4}}^{0} + \frac{1}{2}\left\{x + \frac{\sin 2x}{2}\right\}_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2}\left\{-0\left(-\frac{\pi}{4} + \frac{1}{2}\right)\right\} + \frac{1}{2}\left\{\frac{\pi}{2} - 0\right\}$$

$$= \left\{-\frac{\pi}{8} + \frac{1}{4}\right\} + \left\{\frac{\pi}{4}\right\}$$

$= \frac{\pi}{8} + \frac{1}{4}$

$= \frac{\pi + 2}{8}$

23. Question

Evaluate the following Integrals:

$$\int_0^{\pi} (\cos x |\cos x|) dx$$

Answer

$\int_0^{\pi/2} \cos^2 x \, dx - \int_{\pi/2}^{\pi} \cos^2 x \, dx$

$= \cos^2 x = \frac{1 + \cos 2x}{2}$

$= \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx - \int_{\frac{\pi}{2}}^{\pi} \frac{1 + \cos 2x}{2} dx$

$= \frac{1}{2}\left\{\frac{1 + \cos 2x}{2}\right\}_0^{\frac{\pi}{2}} - \frac{1}{2}\left\{x + \frac{\sin 2x}{2}\right\}_{\frac{\pi}{2}}^{\pi}$

$= \frac{\pi}{4} - \frac{\pi}{4}$

= 0

24. Question

Evaluate the following Integrals:

$$\int_{-\pi/4}^{\pi/2} (2\sin|x| + \cos|x|) dx$$

Answer

$\int_{-\pi/4}^{\pi/2} (2 \sin|x| + \cos|x|) dx$

$= \int_{-\frac{\pi}{4}}^{0} (-2\sin x + \cos x) dx + \int_0^{\frac{\pi}{2}} (2\sin x + \cos x) dx$

$= [2\cos x + \sin x]_{-\frac{\pi}{4}}^{0} + [-2\cos x + \sin x]_0^{\frac{\pi}{2}}$

$= [2\cos(0) + \sin(0) - 2\cos(-\frac{\pi}{4}) - \sin(-\frac{\pi}{4})] + [-2\cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2}) + 2\cos(0) - \sin(0)]$

= [2 + 0–0 + 1] + [0 + 1 + 2–0]

= 6

25. Question

Evaluate the following Integrals:

$$\int_{-\pi/2}^{\pi} \sin^{-1}(\sin x) dx$$

Answer

$\int_{-\pi/2}^{\pi} \sin^{-1}(\sin x)\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) dx$

$= \left\{\frac{x^2}{2}\right\}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left\{\pi x - \frac{x^2}{2}\right\}_{\frac{\pi}{2}}^{\pi}$

$= \left\{\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right\}$

$= \left\{\left(\frac{2\pi^2 - \pi^2}{2}\right) - \left(\frac{4\pi^2 - \pi^2}{8}\right)\right\}$

$= \left\{\left(\frac{\pi^2}{2} - \frac{3\pi^2}{8}\right)\right\}$

$= \left(\frac{4\pi^2 - 3\pi^2}{8}\right)$

$= \frac{\pi^2}{8}$

26. Question

Evaluate the following Integrals:

$$\int_{-\pi/2}^{\pi/2} \frac{-\pi/2}{\sqrt{\cos x \sin^2 x}} dx$$

Answer

Let f(x) = $\frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}}$

f(– x) = f(x)

And thus f(x) is an even function.

So,

$$\int_{-\pi/2}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}} dx = 2\int_{0}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}} dx$$

$$\int_{-\pi/2}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}} dx = 2\int_{0}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sin x \sqrt{\cos x}} dx$$

Let cos x = t

Differentiating both sides we get,

– Sinx dx = dt

– $\sqrt{(1 - t^2)}$ dx = dt

Limits will also change,

At x = 0, t = 1 and at x = π/2, t = 0

Now the Expression becomes,

$$\int_{-\pi/2}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}} dx = \int_{1}^{0} \frac{\pi}{\sqrt{t}\sqrt{1 - t^2}} dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}} dx = \int_1^0 \frac{\pi}{\sqrt{t - t^3}} dt$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{-\frac{\pi}{2}}{\sqrt{\cos x \sin^2 x}} dx = 1 + \frac{1}{3} = \frac{4}{3}$$

27. Question

Evaluate the following Integrals:

$$\int_0^2 2x[x]\, dx$$

Answer

[x] = 0 for 0

and [x] = 1 for 1

Hence

$= \int_0^1 0 \ + \int_1^2 2x\, dx$

$= 0 + \left[\frac{2x^2}{2}\right]_1^2$

$= [x^2]_1^2$

$=(2^2 - 1^2)$

= 4–1

= 3.

28. Question

Evaluate the following Integrals:

$$\int_0^{2\pi} \cos^{-1}(\cos x) dx$$

Answer

$\int_0^{2\pi} \cos^{-1}(\cos x)\, dx$

$= -\int_0^{\pi} \cos^{-1}(\cos x)\, dx \ + \int_0^{2\pi} \cos^{-1}(\cos x)\, dx$

$= -\int_0^{\pi} x dx \ + \int_0^{2\pi} x dx$

$= -\left[\frac{x^2}{2}\right]_0^{\pi} \ + \left[\frac{x^2}{2}\right]_{\pi}^{2\pi}$

$= -\frac{\pi^2}{2} \ + \frac{4\pi^2}{2} - \frac{\pi^2}{2}$

$= \pi^2$

Exercise 20.4

1. Question

Evaluate of each of the following integral:

$$\int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx$$

Answer

let us assume $I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx$ .... equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_0^{2\pi} \frac{e^{\sin(2\pi-x)}}{e^{\sin(2\pi-x)} + e^{-\sin(2\pi-x)}} dx$ ...... equation 2

Adding equation 1 and 2

$2I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx + \int_0^{2\pi} \frac{e^{\sin(2\pi-x)}}{e^{\sin(2\pi-x)} + e^{-\sin(2\pi-x)}} dx$

We know $\sin(2\pi - x) = -\sin x$

Thus

$2I = \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx + \int_0^{2\pi} \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} dx$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

thus

$$2I = \int_0^{2\pi} \frac{e^{\sin x} + e^{-\sin x}}{e^{\sin x} + e^{-\sin x}} dx$$

$$2I = \int_0^{2\pi} 1\, dx$$

$2I = [x]_0^{2\pi}$

We know

$[f(x)]_a^b = f(b) - f(a)$

2I = [2π – 0]

I = π

2. Question

Evaluate of each of the following integral:

$$\int_0^{2\pi} \log(\sec x + \tan x) dx$$

Answer

Let us assume $I = \int_0^{2\pi} \log(\sec x + \tan x)\, dx$ .... equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_0^{2\pi} \log(\sec(2\pi - x) + \tan(2\pi - x))\, dx$$

We know that sec (2π – x) = sec (x)

tan(2π – x)=–tan (x)

thus $I = \int_0^{2\pi} \log(\sec(x) - \tan(x))\, dx$ .... equation 2

Adding equations 1 and equation 2, we get,

$$2I = \int_0^{2\pi} \log(\sec x + \tan x)\, dx + \int_0^{2\pi} \log(\sec x - \tan x)\, dx$$

We know $\log(a) + \log(b) = \log(ab)$

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{2\pi} \log(\sec x + \tan x)\,(\sec x - \tan x)\, dx$$

$$2I = \int_0^{2\pi} \log(\sec^2 x - \tan^2 x)\, dx$$

We know Trigonometric identity $\sec^2 x - \tan^2 x = 1$

$$2I = \int_0^{2\pi} \log(1)\, dx$$

We know

$[f(x)]_a^b = f(b) - f(a)$ where b is the upper limit and a is lower and f(x) is integral funtion

$2I = [0]_0^{2\pi}$

Thus I = 0.

3. Question

Evaluate of each of the following integral:

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}}\, dx$$

Answer

Let us assume,

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx$ .... equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}{\sqrt{\tan\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\cot\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}\, dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan\left(\frac{\pi}{2} - x\right)}}{\sqrt{\tan\left(\frac{\pi}{2} - x\right)} + \sqrt{\cot\left(\frac{\pi}{2} - x\right)}} dx$$

Trigonometric property

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan x$$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ ..... equation 2

Adding equation 1 and equation 2

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} + \sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1 dx$$

$$2I = [x]_{\pi/6}^{\pi/3}$$

We know $[f(x)]_a^b = f(b) - f(a)$ where b and a are upper and lower limits respectively and f(x) is a function

$$2I = \frac{\pi}{3} - \frac{\pi}{6}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6}$$

$$2I = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

4. Question

Evaluate of each of the following integral:

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

Answer

Let us assume ,

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ .... equation 1

By property, we know that $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Thus

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}} dx$$

Trigonometric property

$\sin\left(\frac{\pi}{2}-x\right) = \cos x$

$\cos\left(\frac{\pi}{2}-x\right) = \sin x$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$ ....equation 2

Adding equations 1 and equation 2, we get,

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

We know

$$\int_{m}^{n} [f(x)+g(x)]dx = \int_{m}^{n} f(x)dx + \int_{m}^{n} g(x)dx$$

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1\, dx$$

$$2I = [x]_{\pi/6}^{\pi/3}$$

Since $[f(x)]_a^b = f(b) - f(a)$ where b and a are upper and lower limits respectively

$$2I = \frac{\pi}{3} - \frac{\pi}{6}$$

$$2I = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$

5. Question

Evaluate of each of the following integral:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} dx$$

Answer

Let us assume $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+e^x} dx$.....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Thus

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2\left(\frac{\pi}{4}+\frac{-\pi}{4}-x\right)}{1+e^{\left(\frac{\pi}{4}+\frac{-\pi}{4}-x\right)}} dx$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2(-x)}{1+e^{(-x)}} dx$$

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2(-x)}{1+\frac{1}{e^x}} dx$$

We know tan(– x) = – tan x

$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x \tan^2(x)}{e^{(x)}+1} dx$ ......equation 2

Adding equations 1 and equation 2, we get,

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2(x)}{1+e^{(x)}} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{e^x \tan^2(x)}{e^{(x)}+1} dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2(x) + e^x \tan^2(x)}{1+e^{(x)}} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+e^{(x)})\tan^2(x)}{1+e^{(x)}} dx$$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2(x)}{1} dx$$

Trigonometric identity $\sec^2 \theta$

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec^2 x - 1)\, dx$$

We know $\int \sec^2\theta\, d\theta = \tan\theta$

Thus

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = [\tan x - x]_{-\pi/4}^{\pi/4}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively.

$$2I = \left[\tan\left(\frac{\pi}{4}\right) - \left(\frac{\pi}{4}\right)\right] - \left[\tan\left(\frac{-\pi}{4}\right) - \left(-\frac{\pi}{4}\right)\right]$$

Since $\tan\frac{\pi}{4} = 1$ and $\tan(-\theta) = -\tan\theta$

Thus

$$2I = \left[1 - \frac{\pi}{4}\right] - \left[-1 + \frac{\pi}{4}\right]$$

$$I = \frac{1}{2}\left[2 - \frac{\pi}{2}\right]$$

6. Question

Evaluate of each of the following integral:

$$\int_{-a}^{a} \frac{1}{1+a^x} dx$$

Answer

Let us assume $I = \int_{-a}^{a} \frac{1}{1+a^x} dx$

By integration property, we know,

$$\int_{-m}^{m} f(x)dx = \int_0^m f(x)dx + \int_0^m f(-x)dx$$

Thus

$$I = \int_0^a \frac{1}{1+a^x} dx + \int_0^a \frac{1}{1+a^{-x}} dx$$

$$I = \int_0^a \frac{1}{1+a^x} dx + \int_0^a \frac{a^x}{1+a^x} dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$I = \int_0^a \frac{1+a^x}{1+a^x} dx$$

$$I = \int_0^a 1dx$$

$$I = [x]_0^a$$

We know $[f(x)]_a^b = f(b) - f(a)$ and a being the upper and lower limits respectively

I = [a – 0]

I = a.

7. Question

Evaluate of each of the following integral:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} dx$$

Answer

Let us assume,

$$I = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} dx$$

By integration property,

$$\int_{-m}^{m} f(x)dx = \int_{0}^{m} f(x)dx + \int_{0}^{m} f(-x)dx$$

Thus

$$I = \int_{0}^{\frac{\pi}{3}} \frac{1}{1+e^{\tan x}} dx + \int_{0}^{\frac{\pi}{3}} \frac{1}{1+e^{-\tan x}} dx$$

We know

$$\int_{m}^{n} [f(x)+g(x)]dx = \int_{m}^{n} f(x)dx + \int_{m}^{n} g(x)dx$$

$$I = \int_{0}^{\frac{\pi}{3}} \frac{1+e^{\tan x}}{1+e^{\tan x}} dx$$

$$I = \int_{0}^{\frac{\pi}{3}} 1dx$$

$$I = [x]_{0}^{\frac{\pi}{3}}$$

we know since $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits

$$I = [\frac{\pi}{3} - 0]$$

$$I = \frac{\pi}{3}$$

8. Question

Evaluate of each of the following integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+e^x} dx.$$

Answer

Let us assume $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 x}{1+e^x} dx$ ....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Thus

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(\frac{\pi}{4} + \frac{-\pi}{4} - x)}{1+e^{(\frac{\pi}{4}+\frac{-\pi}{4}-x)}} dx$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(-x)}{1+e^{(-x)}} dx$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(-x)}{1+\frac{1}{e^x}} dx$$

$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \cos^2(x)}{e^{(x)}+1} dx$ .....equation 2

Adding the equations 1 and 2, we get,

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(x)}{1+e^{(x)}} dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x \cos^2(x)}{e^{(x)}+1} dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(x) + e^x \cos^2(x)}{1+e^{(x)}} dx$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1+e^{(x)})\cos^2(x)}{1+e^{(x)}} dx$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2(x)}{1} dx$$

Trigonometric formula

$$2\cos^2\theta - 1 = \cos 2\theta$$

$$2I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos 2x + 1}{2} dx$$

We know $\int \cos\theta \, d\theta = \sin\theta$

Thus

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2x dx + 1/2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx$$

Thus

$$2I = \frac{\left[\frac{\sin 2x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + [x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}{2}$$

we know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limit

$$4I = \left[\frac{\sin\left(\frac{2\pi}{2}\right)}{2} - \frac{\sin\left(\frac{-2\pi}{2}\right)}{2}\right] + \left[\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right)\right]$$

Since sinπ = 0 and sin (– θ) = – sinθ

Thus

$$4I = [0 - 0] + \left[\frac{\pi}{1}\right]$$

$$I = \left[\frac{\pi}{4}\right]$$

9. Question

Evaluate of each of the following integral:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^{11} - 3x^9 + 5x^7 - x^5 + 1}{\cos^2 x} dx$$

Answer

Let us assume $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^{11}-3x^9+5x^7-x^5+1}{\cos^2 x} dx$

By property, we know,

$$\int_{-m}^{m} f(x)dx = \int_0^m f(x)dx + \int_0^m f(-x)dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{x^{11} - 3x^9 + 5x^7 - x^5 + 1}{\cos^2 x} dx + \int_0^{\frac{\pi}{4}} \frac{-x^{11} + 3x^9 - 5x^7 + x^5 + 1}{\cos^2 x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{x^{11} - 3x^9 + 5x^7 - x^5 + 1 - x^{11} + 3x^9 - 5x^7 + x^5 + 1}{\cos^2 x} dx$$

$$I = \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2 x} dx$$

$$I = \int_0^{\frac{\pi}{4}} 2\sec^2 x \, dx$$

$$I = [2\tan x]_0^{\frac{\pi}{4}}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits

$I = [2(1 - 0)]$

Since $\tan\frac{\pi}{4} = 1$

I = 2.

10. Question

Evaluate of each of the following integral:

$$\int_a^b \frac{x^{1/n}}{x^{1/n} + (a + b - x)^{1/n}} dx, \; n \in N, n \geq 2$$

Answer

Let us assume $I = \int_a^b \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}} dx$ ..... equation 1

By property, we know that,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$I = \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+(x)^{1/n}} dx$ ...... equation 2

Adding equation 1 and equation 2

$$2I = \int_a^b \frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}} dx + \int_a^b \frac{(a+b-x)^{1/n}}{(a+b-x)^{1/n}+(x)^{1/n}} dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_a^b \frac{x^{1/n}+(a+b-x)^{1/n}}{x^{1/n}+(a+b-x)^{1/n}} dx$$

$$I = \int_a^b 1dx$$

$$2I = [x]_a^b$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limit

$$I = \frac{b-a}{2}$$

11. Question

Evaluate of each of the following integral:

$$\int_0^{\frac{\pi}{2}} 2\log\cos x - \log\sin 2x \, dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} 2\log\cos x - \log\sin 2x \, dx$

We know $n\log m = \log m^n$ and $\sin 2\theta = 2\sin\theta\cos\theta$

Thus

$$I = \int_0^{\frac{\pi}{2}} \log\cos^2 x - \log 2\sin x\cos x \, dx$$

We know

$$\log m - \log n = \log\left(\frac{m}{n}\right)$$

Thus

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cos^2 x}{2\sin x\cos x}\right) dx$$

Since $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cos x}{2\sin x}\right) dx$$

$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cot x}{2}\right) dx$ ...... equation 1

By property, we know that,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Thus

$$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\cot\left(\frac{\pi}{2}+0-x\right)}{2}\right) dx$$

Since $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

$I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan x}{2}\right) dx$ .... equation 2

Adding equations 1 and 2

$$2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan x}{2}\right) dx + \int_0^{\frac{\pi}{2}} \log\left(\frac{\cot x}{2}\right) dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log\left(\frac{\tan x}{2}\right) + \log\left(\frac{\cot x}{2}\right) dx$$

Since we know that $\log m + \log n = \log mn$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{\tan x}{2} * \frac{\cot x}{2} dx$$

Since $\tan x = 1/\cot x$

$$2I = \int_0^{\frac{\pi}{2}} \log \frac{1 \; * 1}{2 * 2} dx$$

$$2I = \left[\log\left(\frac{1}{4}\right) x\right]_0^{\frac{\pi}{2}}$$

We know $[f(x)]_a^b = f(b) - f(a)$

$$2I = \log\left(\frac{1}{4}\right)\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4} \log\left(\frac{1}{4}\right)$$

12. Question

Evaluate of each of the following integral:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$$

Answer

Let us assume $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx$.....equation 1

By property, we know that,

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}} dx$ .....equation 2

Adding equation 1 and 2 $2I = \int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} dx + \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}} dx$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}} dx$$

$$2I = \int_0^a 1dx$$

We know $[f(x)]_a^b = f(b) - f(a)$

$$2I = [x]_0^a$$

$$2I = [a-0]$$

2I = a

$$I = \frac{a}{2}$$

13. Question

Evaluate of each of the following integral:

$$\int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}} dx$$

Answer

Let us assume $I = \int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}} dx$ ......equation 1

By property we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_0^5 \frac{\sqrt[4]{9-x}}{\sqrt[4]{9-x}+\sqrt[4]{x+4}} dx$ .....equation 2

Adding equation 1 and 2

$$2I = \int_0^5 \frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}} dx + \int_0^5 \frac{\sqrt[4]{9-x}}{\sqrt[4]{9-x}+\sqrt[4]{x+4}} dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^5 \frac{\sqrt[4]{9-x} + \sqrt[4]{x+4}}{\sqrt[4]{9-x} + \sqrt[4]{x+4}} dx$$

$$2I = \int_0^5 1dx$$

$$2I = [x]_0^5$$

We know $[f(x)]_a^b = f(b) - f(a)$

$$2I = [5 - 0]$$

$$2I = 5$$

$$I = \frac{5}{2}$$

14. Question

Evaluate of each of the following integral:

$$\int_0^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{7-x}} dx$$

Answer

Let us assume $I = \int_0^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{7-x}} dx$...... equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$I = \int_0^7 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[4]{x}} dx$ .....equation 2

Adding equation 1 and 2

$$2I = \int_0^7 \frac{\sqrt[3]{x}}{\sqrt[3]{x} + \sqrt[3]{7-x}} dx + \int_0^7 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[3]{x}} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^7 \frac{\sqrt[3]{7-x} + \sqrt[3]{x}}{\sqrt[3]{7-x} + \sqrt[3]{x}} dx$$

$$2I = \int_0^7 1dx$$

$$2I = [x]_0^7$$

We know $[f(x)]_a^b = f(b) - f(a)$

2I = [7 – 0]

2 I = 7

$$I = \frac{7}{2}$$

15. Question

Evaluate of each of the following integral:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}} dx$$

Answer

Let us assume $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}} dx$

We know

$$\tan x = \frac{\sin x}{\cos x}$$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$ .... equation 1

By property,s we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Thus

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}} dx$$

Trigonometric property

$$\sin\left(\frac{\pi}{2}-x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$

$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$ ....equation 2

Adding equations 1 and 2, we get,

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1\, dx$$

$$2I = [x]_{\pi/6}^{\pi/3}$$

We know $[f(x)]_a^b = f(b) - f(a)$

$$2I = \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$2I = \frac{\pi}{6}$

$I = \frac{\pi}{12}$

16. Question

If f(a + b – x) = f(x), then prove that $\int_a^b x f(x)\, dx = \frac{a+b}{2} \int_a^b f(x)\, dx.$

Answer

LHS

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

thus

$$\int_a^b xf(x).dx = \int_a^b (a+b-x)f(a+b-x).dx$$

Given $f(a+b-x) = f(x)$

$$\int_a^b xf(x).dx = \int_a^b (a+b-x)f(x).dx$$

We know

$$\int_m^n [f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$\int_a^b xf(x).dx = \int_a^b (a+b)f(x).dx - \int_a^b xf(x).dx$$

$$2\int_a^b xf(x).dx = \int_a^b (a+b)f(x).dx$$

$$\int_a^b xf(x).dx = (a+b)/2\int_a^b f(x).dx$$

Hence proved

Exercise 20.5

1. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan x} dx$.....equation 1

We know that $\tan x = \frac{\sin x}{\cos x}$

Substituting the value in equation 1 we have,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sin x}{\cos x}} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$$.....equation 2

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus in equation 2

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2} - x)}{\cos(\frac{\pi}{2} - x) + \sin(\frac{\pi}{2} - x)} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$.....equation 3

Adding equation 2 and 3

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

2. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cot x} dx$.....equation 1

We know that $\tan x = \frac{\cos x}{\sin x}$ $\cot x = \frac{1}{\tan x}$

Substituting the value in equation 1 we have,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\cos x}{\sin x}} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx$.....equation 2

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus in equation 2

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$.....equation 3

Adding equation 2 and 3

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x + \sin x}{\cos x + \sin x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

3. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$ .....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus in equation 2

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot\left(\frac{\pi}{2} - x\right)}}{\sqrt{\tan\left(\frac{\pi}{2} - x\right)} + \sqrt{\cot\left(\frac{\pi}{2} - x\right)}} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$ .....equation 2

Adding equation 1 and 2

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

We know

$$\int_m^n [f(x) + g(x)] dx = \int_m^n f(x) dx + \int_m^n g(x) dx$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x} + \sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

4. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$.....equation 1

By property, we know that $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Thus in equation 2

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}}(\frac{\pi}{2} - x)}{\sin^{\frac{3}{2}}(\frac{\pi}{2} - x) + \cos^{\frac{3}{2}}(\frac{\pi}{2} - x)} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$.....equation 2

Adding equation 1 and 2

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

5. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}$.....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus in equation 2

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^n(\frac{\pi}{2} - x)}{\sin^n(\frac{\pi}{2} - x) + \cos^n(\frac{\pi}{2} - x)} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}$.....equation 2

Adding equation 1 and 2

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos^n x + \sin^n x}{\sin^n x + \cos^n x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

6. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\tan x}} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sqrt{\tan x}} dx$.....equation 1

We know that $\tan x = \frac{\sin x}{\cos x}$

Substituting the value in equation 1 we have,

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \frac{\sqrt{\sin x}}{\sqrt{\cos x}}} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$.....equation 2

By property, we know that $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

Thus in equation 2

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$.....equation 3

Adding equation 2 and 3

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

We know

$$\int_m^n [f(x) + g(x)] dx = \int_m^n f(x) dx + \int_m^n g(x) dx$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\,dx$$

$$2I = [x]_0^{\frac{\pi}{2}}$$

$$2I = \left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{4}$$

7. Question

Evaluate the following integral:

$$\int_0^a \frac{1}{x + \sqrt{a^2 - x^2}}\,dx$$

Answer

Let us assume $I = \int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx$..........equation 1

Let $x = a\cos\theta$

thus

$x = a\cos\theta$

Differentiating both sides, we get,

$dx = -a\sin\theta\, d\theta$

Thus substituting old limits, we get a new upper limit and lower limit

For a = a cos θ

0 = θ

For 0 = acos θ

$\frac{\pi}{2} = \theta$

We know that $\int_a^b -f(x) = \int_b^a f(x)$

thus

Substituting the values in equation 1

$$I = \int_{\frac{\pi}{2}}^0 \frac{1}{a\cos\theta + \sqrt{a^2 - a^2\cos^2\theta}}(-a\sin\theta)\,d\theta$$

We know that $\int_a^b -f(x) = \int_b^a f(x)$

Trigonometric identity 1 – $\cos^2\theta = \sin^2\theta$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos\theta + \sqrt{a^2(1-\cos^2\theta)}}(a\sin\theta)\,d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos\theta + \sqrt{a^2\sin^2\theta}}(a\sin\theta)\,d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos\theta + a\sin\theta}(a\sin\theta)\,d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + \sin\theta}(\sin\theta)\,d\theta$$.......equation 2

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

thus

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{\cos(\frac{\pi}{2} - \theta) + \sin\left(\frac{\pi}{2} - \theta\right)}(\sin(\frac{\pi}{2} - \theta))\,d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta + \cos\theta}(\cos\theta)\,d\theta$$.........equation 3

Adding equation 3 and equation 2

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + \sin\theta}(\sin\theta)\,d\theta + \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + \sin\theta}(\cos\theta)\,d\theta$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + \sin\theta}(\sin\theta + \cos\theta)\,d\theta$$

$$2I = \int_0^{\frac{\pi}{2}} 1\,d\theta$$

$$2I = [\theta]_0^{\frac{\pi}{2}}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively.

$$2I = [\frac{\pi}{2} - 0]$$

$$I = \frac{\pi}{4}$$

8. Question

Evaluate the following integral:

$$\int_0^{\infty} \frac{\log x}{1+x^2}dx$$

Answer

let us assume $I = \int_0^{\infty} \frac{\log x}{1 + x^2}dx$

let x = tany

differentiating both sides

dx = $\sec^2$ y dy

for x = ∞

$\frac{\pi}{2} = y$

For x = 0

0 = y

thus

$$I = \int_0^{\frac{\pi}{2}} \frac{\log(\tan y)}{1 + \tan^2 y} \sec^2 y \, dy$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\log(\tan y)}{\sec^2 y} \sec^2 y \, dy$$

(since $\sec^2 y - \tan^2 y = 1$)

$I = \int_0^{\frac{\pi}{2}} \log(\tan y) \, dy$ .....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log\left(\tan\left(\frac{\pi}{2} - y\right)\right) dy$$

$I = \int_0^{\frac{\pi}{2}} \log(\cot y) \, dy$ ....equation 2

Adding equations 1 and 2, we get,

$$2I = \int_0^{\frac{\pi}{2}} \log(\tan y) \, dy + \int_0^{\frac{\pi}{2}} \log(\cot y) \, dy$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^{\frac{\pi}{2}} [\log(\tan y) + \log(\cot y)] \, dy$$

$2I = \int_0^{\frac{\pi}{2}} [\log(\tan y \times \cot y)] \, dy$ since logm + logn = logmn

$2I = \int_0^{\frac{\pi}{2}} [\log 1] \, dy$ since tany = 1/coty

$2I = \int_0^{\frac{\pi}{2}} 0 \, dy$ since log 1 = 0

Thus

2I = 0

I = 0

9. Question

Evaluate the following integral:

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Answer

Let us assume $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$ ....equation 1

Let x= tan θ thus

Differentiating both sides, we get,

$dx = \sec^2\theta\, d\theta$

Thus substituting old limits, we get a new upper limit and lower limit

For 1 = tan θ

$\frac{\pi}{4} = \theta$

For 0 = tan θ

0 = θ

substitute the values in equation 1

we get $I = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \sec^2\theta\, d\theta$.........................equation 2

trigonometric identity we know

$1 + \tan^2\theta = \sec^2\theta$

Thus substituting in equation 2 we have

$$I = \int_0^{\frac{\pi}{4}} \frac{\log(1 + \tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$$

$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta)\, d\theta$ ....equation 3

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus

$I = \int_0^{\frac{\pi}{4}} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta$.....equation 4

Trigonometric formula:

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Thus $\tan(\frac{\pi}{4} - \theta) = \frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$

We know by trigonometric property:

$\tan\frac{\pi}{4} = 1$

thus

$$\tan(\frac{\pi}{4} - \theta) = \frac{1 - \tan\theta}{1 + \tan\theta}$$

Substituting in equation 4

$$I = \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{1 + \tan\theta + 1 - \tan\theta}{1 + \tan\theta}\right) d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1 + \tan\theta}\right) d\theta$$

We know $\log\left(\frac{m}{n}\right) = \log m - \log n$

Thus

$I = \int_0^{\frac{\pi}{4}} \log(2) - \log(1 + \tan\theta)\, d\theta$ ......equation 6

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Adding equation 3 and equation 6

$$2I = \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta)\, d\theta + \int_0^{\frac{\pi}{4}} \log(2) - \log(1 + \tan\theta)\, d\theta$$

Thus

$$2I = \int_0^{\frac{\pi}{4}} \log(1 + \tan\theta) + \log(2) - \log(1 + \tan\theta)\, d\theta$$

$$2I = \int_0^{\frac{\pi}{4}} \log(2)\, d\theta$$

$$2I = \log(2) \int_0^{\frac{\pi}{4}} 1\, d\theta$$

$$2I = \log 2[\theta]_0^{\frac{\pi}{4}}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively.

$$2I = \log 2\left[\frac{\pi}{4} - 0\right]$$

$$I = \frac{\pi}{8}\log 2$$

10. Question

Evaluate the following integral:

$$\int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx$$

Answer

Let us assume $I = \int_0^{\infty} \frac{x}{(1 + x)(1 + x^2)} dx$

Adding – 1 and + 1

$$I = \int_0^{\infty} \frac{x + 1 - 1}{(1 + x)(1 + x^2)} dx$$

$$I = \int_0^{\infty} \frac{x + 1}{(1 + x)(1 + x^2)} - \frac{1}{(1 + x)(1 + x^2)} dx$$

$$I = \int_0^{\infty} \frac{1}{(1 + x^2)} dx - \int_0^{\infty} \frac{1}{(1 + x)(1 + x^2)} dx$$

Let $I_1 = \int_0^{\infty} \frac{1}{(1 + x^2)} dx$

$$I_2 = \int_0^{\infty} \frac{1}{(1 + x)(1 + x^2)} dx$$

Thus $I = I_1 - I_2$ ....equation 1

Solving for I_1

$$I_1 = \int_0^{\infty} \frac{1}{(1 + x^2)} dx$$

$$I_1 = [\tan^{-1} x]_0^{\infty}$$

since $\int \frac{1}{(1 + x^2)} dx = \tan^{-1} x$

$I_1 = [\tan^{-1}(\infty) - \tan^{-1}(0)]$

$I_1 = \pi/2$ ....equation 2

Solving for I_2

$$I_2 = \int_0^{\infty} \frac{1}{(1 + x)(1 + x^2)} dx$$

Let $\frac{1}{(1 + x)(1 + x^2)} = \frac{a}{1 + x} + \frac{bx + c}{1 + x^2}$ ....equation 3

$$\frac{1}{(1 + x)(1 + x^2)} = \frac{a(1 + x^2) + (bx + c)(1 + x)}{(1 + x)(1 + x^2)}$$

$$\frac{1}{(1 + x)(1 + x^2)} = \frac{ax^2 + a + bx^2 + bx + cx + c}{(1 + x)(1 + x^2)}$$

a + b = 0; a + c = 1; b + c = 0

solving we get

a = c = 1/2

b = – 1/2

substituting the values in equation 3

$$\frac{1}{(1 + x)(1 + x^2)} = \frac{\frac{1}{2}}{1 + x} + \frac{-\frac{1}{2}x + \frac{1}{2}}{1 + x^2}$$

$$\frac{1}{(1 + x)(1 + x^2)} = \frac{\frac{1}{2}}{1 + x} + \frac{-\frac{1}{2}x}{1 + x^2} + \frac{1/2}{1 + x^2}$$

Thus substituting the values in I_2, thus

$$I_2 = \int_0^{\infty} \frac{\frac{1}{2}}{1 + x} + \frac{-\frac{1}{2}x}{1 + x^2} + \frac{1/2}{1 + x^2} dx$$

$$I_2 = \int_0^\infty \frac{\frac{1}{2}}{1 + x} dx + \int_0^\infty \frac{-\frac{1}{2}x}{1 + x^2} dx + \int_0^\infty \frac{1/2}{1 + x^2} dx$$

Solving :

$$\int_0^\infty \frac{-\frac{1}{2}x}{1 + x^2} dx$$

Let $1 + x^2 = y$

2xdx = dy

For x = ∞

y = ∞

For x = 0

y = 0

substituting values

$$-\frac{1}{2}\int_0^\infty \frac{dy}{2y}$$

$$-\frac{1}{4}[\log y]_0^\infty$$

Thus

$$I_2 = \int_0^\infty \frac{\frac{1}{2}}{1 + x} dx + \int_0^\infty \frac{-\frac{1}{2}x}{1 + x^2} dx + \int_0^\infty \frac{1/2}{1 + x^2} dx$$

$$I_2 = \frac{1}{2}[\log(1 + x)]_0^\infty + -\frac{1}{4}[\log x]_0^\infty + \frac{1}{2}[\tan^{-1} x]_0^\infty$$

$I_2 = \frac{1}{2}\left[\frac{\pi}{2}\right]$..........equation 4

Substituting values equation 2 and equation 4 in equation 1

Thus

$I = I_1 - I_2$

I = π/2 – π/4

I = π/4

11. Question

Evaluate the following integral:

$$\int_0^\pi \frac{x\tan x}{\sec x \operatorname{cosec} x} dx$$

Answer

Let us assume $I = \int_0^\pi \frac{x\tan x}{\sec x \operatorname{cosec} x} dx$.........equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

THUS

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x)\ \text{cosec}(\pi - x)} dx$$

We know

$$\tan(\pi - x) = -\tan x$$

$$\sec(\pi - x) = -\sec x$$

$$\text{cosec}(\pi - x) = \text{cosec}x$$

Thus substituting values

$I = \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x\ \text{cosec} x} dx$ ....equation 2

Adding equation 1 and 2

$$2I = \int_0^{\pi} \frac{(x)(\tan x)}{\sec x\ \text{cosec} x} dx + \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x\ \text{cosec} x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{\pi} \frac{(\pi)(\tan x)}{\sec x\ \text{cosec} x} dx$$

We know

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\text{cosec}\theta = \frac{1}{\sin\theta}$$

Substituting the values we have

$$2I = \pi \int_0^{\pi} \sin^2 x\, dx$$

$2I = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx$ by trigonometric formula

$$2I = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively.

$$2I = \frac{\pi}{2}\left[\left[\pi - \frac{\sin 2\pi}{2}\right] - [0]\right]$$

$$I = \frac{\pi^2}{4}$$

12. Question

Evaluate the following integral:

$$\int_0^{\pi} x \sin x \cos^4 x \, dx$$

Answer

Let us assume $I = \int_0^{\pi} x \sin x \cos^4 x \, dx$.....equation 1

By property, we know that $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

$$I = \int_0^{\pi} (\pi - x) \sin(\pi - x) \cos^4(\pi - x) \, dx$$

$I = \int_0^{\pi} (\pi - x) \sin x \cos^4 x \, dx$.......equation 2

Adding equation 1 and equation 2

$$2I = \int_0^{\pi} x \sin x \cos^4 x \, dx + \int_0^{\pi} (\pi - x) \sin x \cos^4 x \, dx$$

We know

$$\int_m^n [f(x) + g(x)] dx = \int_m^n f(x) dx + \int_m^n g(x) dx$$

$2I = \int_0^{\pi} \pi \sin x \cos^4 x \, dx$ ......equation 3

Let cosx = y

Differentiating both sides

– sinxdx = dy

sinxdx = – dy

for x = 0

cos0 = y

1 = y

For x = π

cosπ = y

– 1 = y

Substituting equation 3 becomes

$$2I = \int_1^{-1} -\pi y^4 \, dy$$

$$2I = \int_{-1}^{1} \pi y^4 \, dy$$

$$2I = \pi \left[\frac{y^5}{5}\right]_{-1}^{1}$$

$$2I = \pi \left[\frac{(1)^5}{5} - \frac{(-1)^5}{5}\right]$$

2I = 2π/5

I = π/5

13. Question

Evaluate the following integral:

$$\int_0^{\pi} x\sin^3 x dx$$

Answer

Let us assume $I = \int_0^{\pi} x\sin^3 x dx$

$$I = \int_0^{\pi} x\sin^2 x \sin x dx$$

$I = \int_0^{\pi} x(1 - \cos^2 x)\sin x dx$.........equation 1

By property we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus

$$I = \int_0^{\pi} (\pi - x)(1 - \cos^2(\pi - x)\sin(\pi - x)dx$$

$I = \int_0^{\pi} (\pi - x)(1 - \cos^2 x)\sin x dx$.....equation 2

Adding equation 1 and equation 2

$$2I = \int_0^{\pi} x(1 - \cos^2 x)\sin x dx + \int_0^{\pi} (\pi - x)(1 - \cos^2 x)\sin x dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$2I = \int_0^{\pi} \pi(1 - \cos^2 x)\sin x dx$......equation

Let cosx = y

Differentiating both sides

– sinxdx = dy

sinxdx = – dy

for x = 0

cos0 = y

1 = y

For x = π

cosπ = y

– 1 = y

Substituting equation 3 becomes

$$2I = \int_1^{-1} -\pi(1 - y^2)\, dy$$

$$2I = \int_{-1}^{1} \pi(1 - y^2)\, dy$$

$$2I = \pi\left[y - \frac{y^3}{3}\right]_{-1}^{1}$$

$$2I = \pi\left[\frac{3y - y^3}{3}\right]_{-1}^{1}$$

2I = π[{3(1) – (1)^3} – {3(– 1) – (– 1)^3}]/3

2I = π[2 – { – 3 + 1}]/3

2I = π[2 + 2]/3

I = 2π/3

14. Question

Evaluate the following integral:

$$\int_0^{\pi} x \log \sin x \, dx$$

Answer

Let us assume $I = \int_0^{\pi} x \log \sin x \, dx$........equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$I = \int_0^{\pi} (\pi - x)\log\sin(\pi - x) \, dx$.............equation 2

Adding equation 1 and equation 2

$$2I = \int_0^{\pi} x\log\sin x \, dx + \int_0^{\pi} (\pi - x)\log\sin(\pi - x) \, dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$2I = \int_0^{\pi} \pi\log\sin x \, dx$ ....equation 3

We know $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a - x)dx$

$= 2\int_0^a f(x)dx$ if f(2a – x) = f(x)

= 0 if f(2a – x) = – f(x)

Thus equation 3 becomes

$2I = 2\pi\int_0^{\frac{\pi}{2}} \log\sin x \, dx$ ....equation 4 since logsin(π – x) = logsinx

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$$2I = 2\pi\int_0^{\frac{\pi}{2}} \log\sin(\frac{\pi}{2} - x) \, dx$$

$2I = 2\pi\int_0^{\frac{\pi}{2}} \log\cos x \, dx$ ....equation 5

Adding equation 4 and equation 5

$$4I = 2\pi \int_0^{\frac{\pi}{2}} \log\sin x \, dx + 2\pi \int_0^{\frac{\pi}{2}} \log\cos x \, dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$4I = 2\pi \int_0^{\frac{\pi}{2}} \log\sin x + \log\cos x \, dx$$

We know logm + logn = logmn thus

$$4I = 2\pi \int_0^{\frac{\pi}{2}} \log[(2\sin x\cos x)/2] \, dx$$

$$2I = \pi \int_0^{\frac{\pi}{2}} \log[(\sin 2x)/2] \, dx$$

$2I = \pi \int_0^{\frac{\pi}{2}} \log\sin 2x - \log 2 \, dx$ since log(m/n) = logm – logn

$2I = \pi \int_0^{\frac{\pi}{2}} \log\sin 2x dx - \pi \int_0^{\frac{\pi}{2}} \log 2 dx$ .....equation 6

Let $I_1 = \pi \int_0^{\frac{\pi}{2}} \log\sin 2x dx$

Let 2x = y

2dx = dy

dx = dy/2

For x = 0

y = 0

for $x = \frac{\pi}{2}$

y = π

thus substituting value in I1

$$I1 = \frac{1}{2} \int_0^{\pi} \pi \log\sin y dy$$

From equation 3 we get

$$I_1 = \frac{1}{2}(2I)$$

$$I_1 = I$$

Thus substituting the value of I1 in equation 6

$$2I = I - \pi \int_0^{\frac{\pi}{2}} \log 2 dx$$

$$I = -\pi \log 2 \int_0^{\frac{\pi}{2}} 1 dx$$

$$I = -\pi \log 2 [x]_0^{\frac{\pi}{2}}$$

$I = -\pi\log 2\left[\frac{\pi}{2}\right]$

$I = -\log 2\left[\frac{\pi^2}{2}\right]$

15. Question

Evaluate the following integral:

$$\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$$

Answer

Let us assume $I = \int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$.........equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

thus

$I = \int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}dx$.................equation 2

Since $\sin(\pi-x) = \sin x$

Adding equation 1 and equation 2

$$2I = \int_0^{\pi}\frac{x\sin x}{1+\sin x}dx + \int_0^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}dx$$

We know

$$\int_m^n[f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_0^{\pi}\frac{x\sin x}{1+\sin x} + \frac{(\pi-x)\sin x}{1+\sin x}dx$$

$$2I = \int_0^{\pi}\frac{x\sin x + (\pi-x)\sin x}{1+\sin x}dx$$

$$2I = \int_0^{\pi}\frac{\pi\sin x}{1+\sin x}dx$$

Adding and subtracting 1

$$2I = \pi\int_0^{\pi}\frac{\sin x + 1 - 1}{1+\sin x}dx$$

$$2I = \pi\left(\int_0^{\pi}\frac{\sin x+1}{1+\sin x} + \frac{-1}{1+\sin x}dx\right)$$

We know

$$\int_m^n[f(x)+g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \pi\left(\int_0^{\pi}\frac{\sin x+1}{1+\sin x}dx + \int_0^{\pi}\frac{-1}{1+\sin x}dx\right)$$

Let $I_1 = \int_0^{\pi} \frac{\sin x + 1}{1 + \sin x} dx$

Let $I_2 = \int_0^{\pi} \frac{-1}{1 + \sin x} dx$

$2I = \pi(I_1 + I_2)$.......equation 3

Solving I_1:

$$I_1 = \int_0^{\pi} 1\, dx$$

$$I_1 = [x]_0^{\pi}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively.

$$I_1 = [\pi - 0]$$

$$I_1 = \pi$$

Solving I2:

$$I_2 = \int_0^{\pi} \frac{-1}{1 + \sin x} dx$$

Using trigonometric identity and formula

$$I_2 = \int_0^{\pi} \frac{-1}{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} dx$$

$$I_2 = \int_0^{\pi} \frac{-1}{\left[\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right]^2} dx$$

Taking $\cos(x/2)$ common

$$I_2 = \int_0^{\pi} \frac{-1}{\cos^2\left(\frac{x}{2}\right)\left[\sin\left(\frac{x}{2}\right)/\cos\left(\frac{x}{2}\right) + 1\right]^2} dx$$

$$I_2 = \int_0^{\pi} \frac{-1}{\cos^2\left(\frac{x}{2}\right)\left[\tan\left(\frac{x}{2}\right) + 1\right]^2} dx$$

$$I_2 = \int_0^{\pi} \frac{-\sec^2\left(\frac{x}{2}\right)}{\left[\tan\left(\frac{x}{2}\right) + 1\right]^2} dx$$

Let $\tan\left(\frac{x}{2}\right) + 1 = y$

Differentiating both sides, we get,

$$\frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx = dy$$

$$\sec^2\left(\frac{x}{2}\right) dx = 2dy$$

For x = 0

$$\tan\left(\frac{0}{2}\right) + 1 = y$$

$$1 = y$$

For x = π

$\tan\left(\frac{\pi}{2}\right) + 1 = y$

$\infty + 1 = y$

$\infty = y$

Substituting the values

Thus

$I_2 = \int_1^{\infty} \frac{-2}{y^2} dy$

$I_2 = -2\int_1^{\infty} y^{-2} dy$

$I_2 = -2\left[\frac{y^{-2+1}}{-2+1}\right]_1^{\infty}$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively

$I_2 = 2[\frac{1}{\infty} - \frac{1}{1}]$

$I_2 = -2$

Substituting values in equation 3

$2I = \pi(\pi - 2)$

$I = \pi\left(\frac{\pi}{2} - 1\right)$

16. Question

Evaluate the following integral:

$\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x} dx$, 0 < α < π

Answer

Let I = $\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x} dx$

We know that,

$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$

Therefore,

I = $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos\alpha\sin x} dx + \int_0^{\frac{\pi}{2}} \frac{\pi-x}{1+\cos\alpha\sin x} dx$

2 I = $\int_0^{\frac{\pi}{2}} \frac{\pi}{1+\cos\alpha\alpha\sin x} dx$

2I = 2$\pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos\alpha\alpha\sin x} dx$

I = $\frac{\pi\alpha}{\sin\alpha}$

17. Question

Evaluate the following integral:

$$\int_0^{\pi} x\cos^2 x.dx$$

Answer

Let us assume $I = \int_0^{\pi} x\cos^2 x.dx$.....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus

$$I = \int_0^{\pi} (\pi - x)\cos^2(\pi - x).dx$$

We know $\cos(\pi - x) = -\cos x$

Thus

$I = \int_0^{\pi} (\pi - x)\cos^2(x).dx$........equation 2

Adding equation 1 and equation 2

$$2I = \int_0^{\pi} x\cos^2 x.dx + \int_0^{\pi} (\pi - x)\cos^2 x.dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus

$$2I = \int_0^{\pi} (x\cos^2 x + (\pi - x)\cos^2 x).dx$$

$$2I = \int_0^{\pi} (\pi\cos^2 x).dx$$

We know

$2\cos^2 x = 1 + \cos 2x$

$$2I = \pi\int_0^{\pi} \frac{1 + \cos 2x}{2}.dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus

$$2I = \frac{\pi}{2}\int_0^{\pi} 1.dx + \frac{\pi}{2}\int_0^{\pi} \cos 2x\, dx$$

since $\int \cos y dy = \sin y$

$$2I = \frac{\pi}{2}[x]_0^{\pi} + \frac{\pi}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively

$$2I = \frac{\pi}{2}[\pi - 0] + \frac{\pi}{2}\left[\frac{\sin 2\pi}{2} - \frac{\sin 20}{2}\right]$$

Thus

$$I = \frac{\pi}{4}[\pi]$$

18. Question

Evaluate the following integral:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\cot^{\frac{3}{2}} x} dx$$

Answer

Let us assume $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\cot^{\frac{3}{2}} x} dx$.........equation 1

We know that $\tan x = \frac{\cos x}{\sin x}$ $\cot x = \frac{1}{\tan x}$

Substituting the value in equation 1 we have,

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1 + \left(\frac{\cos x}{\sin x}\right)^{\frac{3}{2}}} dx$$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx$.....equation 2

By property we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus in equation 2 $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^{\frac{3}{2}}(\frac{\pi}{2} - x)}{\sin^{\frac{3}{2}}(\frac{\pi}{2} - x) + \cos^{\frac{3}{2}}(\frac{\pi}{2} - x)} dx$$

$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$.....equation 3

Adding equation 2 and 3

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^{\frac{3}{2}} x}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x} dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}{\cos^{\frac{3}{2}} x + \sin^{\frac{3}{2}} x} dx$$

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\, dx$$

$2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$2I = \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$

$I = \frac{\pi}{12}$

19. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\tan^7 x + \cot^7 x} .dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\tan^7 x + \cot^7 x} . dx$ .....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

We know $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

$\cot\left(\frac{\pi}{2} - x\right) = \tan x$

Thus substituting the values in equation 1

$I = \int_0^{\frac{\pi}{2}} \frac{\cot^7 x}{\cot^7 x + \tan^7 x} . dx$ ....equation 2

Adding equation 1 and equation 2

$$2I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\tan^7 x + \cot^7 x} . dx + \int_0^{\frac{\pi}{2}} \frac{\cot^7 x}{\tan^7 x + \cot^7 x} . dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\tan^7 x + \cot^7 x} + \frac{\cot^7 x}{\tan^7 x + \cot^7 x} . dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x} . dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1. dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1. dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1. dx$$

$2I = [x]_0^{\frac{\pi}{2}}$

$2I = \left[\frac{\pi}{2} - 0\right]$

$I = \frac{\pi}{4}$

20. Question

Evaluate the following integral:

$$\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$$

Answer

Let us assume $I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$ ......equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

thus

$I = \int_2^8 \frac{\sqrt{10-(8+2-x)}}{\sqrt{(8+2-x)}+\sqrt{10-(8+2-x)}}dx$ .....equation 2

Adding equation 1 and 2 $2I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx + \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx$

We know

$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$2I = \int_2^8 \frac{\sqrt{x} + \sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}dx$

$2I = \int_2^8 1dx$

We know $[f(x)]_a^b = f(b) - f(a)$

$2I = [x]_2^8$

$2I = [8 - 2]$

$2I = 6$

$I = 3$

21. Question

Evaluate the following integral:

$$\int_0^{\pi} x\sin x\cos^2 x dx$$

Answer

Let us assume $I = \int_0^{\pi} x\sin x\cos^2 x dx$ ....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus

$I = \int_0^{\pi}(\pi - x)\sin(\pi - x)\cos^2(\pi - x)\,dx$........equation 2

Adding equation 1 and equation 1

$$2I = \int_0^{\pi} x\sin x\cos^2 x\,dx + \int_0^{\pi}(\pi - x)\sin(\pi - x)\cos^2(\pi - x)\,dx$$

We know $\int_m^n[f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$2I = \int_0^{\pi} x\sin x\cos^2 x + (\pi - x)\sin(\pi - x)\cos^2(\pi - x)\,dx$$

We know $\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$$2I = \int_0^{\pi}(\pi - x + x)\sin x\cos^2 x\,dx$$

Let cos x = y

Differentiating both sides

– sin x dx = dy

sinx dx = – dy

For x = 0

Cos x = y

Cos 0 = y

y = 1

for x = π

$\cos\pi = y$

y = - 1

Substituting the given values

$$2I = \int_1^{-1} -(\pi)y^2\,dy$$

We know that $\int_a^b -f(x) = \int_b^a f(x)$

$$2I = \int_{-1}^{1}(\pi)y^2\,dy$$

$$2I = \pi\left[\frac{y^3}{3}\right]_{-1}^{1}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively

$$2I = \pi\left[\frac{1^3}{3} - \frac{(-1)^3}{3}\right]$$

$$2I = \pi\left[\frac{2}{3}\right]$$

$$I = \pi\left[\frac{1}{3}\right]$$

22. Question

Evaluate the following integral:

$$\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx$.....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos x^4\left(\frac{\pi}{2} - x\right)} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\cos x\sin x}{\cos^4 x + \sin^4 x} dx$ ....equation 2

Adding equation 1 and 2

Thus

$$2I = \int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx + \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)\cos x\sin x}{\cos^4 x + \sin^4 x} dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}\sin x\cos x}{\cos^4 x(\sin^4 x/\cos^4 x + 1)} dx$$

Since $\tan x = \frac{\sin x}{\cos x}$ and $\frac{1}{\cos x} = \sec x$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}\sin x}{\cos^3 x(\tan^4 x + 1)} dx$$

$2I = \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}\tan x\sec^2 x}{(\tan^4 x + 1)} dx$......equation 3

Let $\tan^2 x = y$

Differentiating both sides

$2\tan x\sec^2 x dx = dy$

For $x = \frac{\pi}{2}$

$\tan^2\frac{\pi}{2} = y$

$y = \infty$

For $x = 0$

$\tan^2 0 = y$

y = 0

substituting values in equation 3

$$2I = \int_0^{\infty} \frac{\frac{\pi}{2}}{2(y^2+1)} dy$$

$2I = \frac{\pi}{4}[\tan^{-1} y]_0^{\infty}$ since $\int \frac{1}{x^2+1} dx = \tan^{-1} x$

$$I = \frac{\pi}{8}[\tan^{-1} \infty - \tan^{-1} 0]$$

$$I = \frac{\pi}{8}[\frac{\pi}{2} - 0]$$

$$I = \frac{\pi^2}{16}$$

23. Question

Evaluate the following integral:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^3 x dx$$

Answer

Let us assume $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^3 x dx$

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \sin x dx$$

$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x dx$$

By property, we know that $\int_{-a}^{a} f(x)dx = \int_0^a f(x) + f(-x) dx$

Thus

$$I = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\sin x + (1 - \cos^2(-x)\sin(-x) dx$$

$$I = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\sin x - (1 - \cos^2 x)\sin x dx$$

I = 0

24. Question

Evaluate the following integral:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^4 x dx$$

Answer

Let us assume $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^4 x dx$

By property, we know that $\int_{-a}^{a} f(x)dx = \int_0^a f(x) + f(-x)\, dx$

Thus

$$I = \int_0^{\frac{\pi}{2}} \sin^4 x dx + \sin^4(-x) dx$$

$$I = 2\int_0^{\frac{\pi}{2}} \sin^4 x dx$$ .....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus

$$I = 2\int_0^{\frac{\pi}{2}} \sin^4\left(\frac{\pi}{2} - x\right) dx$$

$$I = 2\int_0^{\frac{\pi}{2}} \cos^4 x dx$$ ....equation 2

Adding equation 1 and 2

$$2I = 2\int_0^{\frac{\pi}{2}} \sin^4 x dx + 2\int_0^{\frac{\pi}{2}} \cos^4 x dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$2I = 2\int_0^{\frac{\pi}{2}} \sin^4 x dx + \cos^4 x dx$$

$$2I = 2\int_0^{\frac{\pi}{2}} (\sin^4 x + \cos^4 x) dx$$

$$2I = 2\int_0^{\frac{\pi}{2}} (\sin^2 x)^2 + (\cos^2 x)^2\, dx$$

Since $(a + b)^2 = a^2 + b^2 + 2ab$

$$2I = 2\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x)^\wedge 2 - 2\sin^2 x\cos^2 x\, dx$$

$$I = \int_0^{\frac{\pi}{2}} 1 - 2\sin^2 x\cos^2 x\, dx$$

$$I = \int_0^{\frac{\pi}{2}} 1 - [2(2)\sin^2 x\cos^2 x]\frac{1}{2}\, dx$$

$$I = \int_0^{\frac{\pi}{2}} 1 - (\sin^2 2x)\frac{1}{2}\, dx$$

$$I = \int_0^{\frac{\pi}{2}} 1 - (1 - \cos 4x)\frac{1}{2 \times 2}\, dx$$

$$I = \frac{1}{4}\int_0^{\frac{\pi}{2}} 3 + \cos 4x\, dx$$

$$I = \frac{1}{4}\left[3 + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{2}}$$

$$I = \frac{1}{4}\left[\frac{3\pi}{2} + \frac{\sin 2\pi}{4}\right] - [0]$$

$I = \frac{3}{4}\left[\frac{\pi}{2}\right]$

25. Question

Evaluate the following integral:

$$\int_{-1}^{1} \log \frac{2-x}{2+x} dx$$

Answer

Let us assume $I = \int_{-1}^{1} \log\frac{2-x}{2+x} dx$ ....equation 1

By property, we know that $\int_{a}^{-a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(-x)dx$

thus

$$I = \int_{0}^{1} \log\frac{2-x}{2+x} dx + \int_{0}^{1} \log\frac{2+x}{2-x} dx$$

We know $\int_{m}^{n}[f(x) + g(x)]dx = \int_{m}^{n} f(x)dx + \int_{m}^{n} g(x)dx$

$$I = \int_{0}^{1} [\log\frac{2-x}{2+x} + \log\frac{2+x}{2-x}] dx$$

Since we know $\log(mn) = \log m + \log n$

$$I = \int_{0}^{1} [\log\frac{2-x}{2+x} * \frac{2+x}{2-x}] dx$$

$$I = \int_{0}^{1} \log 1 \ dx$$

$$I = \int_{0}^{1} 0 \ dx$$

Thus

I = 0

26. Question

Evaluate the following integral:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$$

Answer

Let us assume $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

By property, we know that $\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x) + f(-x) dx$

$$I = \int_{0}^{\frac{\pi}{4}} \sin^2 x dx + \sin^2(-x) dx$$

$I = 2\int_0^{\frac{\pi}{4}} \sin^2 x dx$

$I = 2\int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{2} dx$

$I = \int_0^{\frac{\pi}{4}} 1 - \cos 2x \ \ dx$

$I = \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}$

$I = \left[\frac{\pi}{4} - \frac{\sin(2\frac{\pi}{4})}{2}\right] - [0]$

$I = \left[\frac{\pi}{4} - \frac{1}{2}\right]$

27. Question

Evaluate the following integral:

$$\int_0^{\pi} \log(1 - \cos x) dx$$

Answer

Let us assume $I = \int_0^{\pi} \log(1 - \cos x) dx$

$I = \int_0^{\pi} \log(2\sin^2 x) dx$

$I = \int_0^{\pi} \log(2) + \log(\sin^2 x) dx$ since logmn = logm + logn and $\log(m)^n = n\log m$

$I = \int_0^{\pi} \log 2 dx + \int_0^{\pi} \log \sin^2 x dx$

$I = \int_0^{\pi} \log 2 dx + 2\int_0^{\pi} \log \sin x dx$.......equation (a)

Let $I_1 = \int_0^{\pi} \log \sin x dx$

We know that $\int_0^{2a} f(x) dx = \int_0^{a} f(x) dx + \int_0^{a} f(2a - x) dx$

If f(2a – x) = f(x)

than $\int_0^{2a} f(x) dx = 2\int_0^{a} f(x) dx$

thus

$I_1 = 2\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ ....equation 1

since logsin(π – x) = logsinx

By property, we know that $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$

$I_1 = 2\int_0^{\frac{\pi}{2}} \log \sin(\frac{\pi}{2} - x) \, dx$

$I_1 = 2\int_0^{\frac{\pi}{2}} \text{logcosx}\, dx$ ....equation 2

Adding equation 1 and equation 2

$2I_1 = 2\int_0^{\frac{\pi}{2}} \text{logsinx}\, dx + 2\int_0^{\frac{\pi}{2}} \text{logcosx}\, dx$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I_1 = 2\int_0^{\frac{\pi}{2}} \text{logsinx} + \text{logcosx}\, dx$$

We know logm + logn = logmn thus

$$I_1 = \int_0^{\frac{\pi}{2}} \log[(2\text{sinxcosx})/2]\, dx$$

$$I_1 = \int_0^{\frac{\pi}{2}} \log[(\sin 2x)/2]\, dx$$

$I_1 = \int_0^{\frac{\pi}{2}} \text{logsin2x} - \log 2\, dx$ since log(m/n) = logm – logn

$I_1 = \int_0^{\frac{\pi}{2}} \text{logsin2xdx} - \int_0^{\frac{\pi}{2}} \text{log2dx}$ .....equation 3

Let $I_2 = \int_0^{\frac{\pi}{2}} \text{logsin2xdx}$

Let 2x = y

2dx = dy

dx = dy/2

For x = 0

y = 0

for $x = \frac{\pi}{2}$

y = π

thus substituting value in I1

$$I_2 = \frac{1}{2}\int_0^{\pi} \text{logsinydy}$$

From equation 3 we get

$$I_2 = \frac{1}{2}(I1)$$

$$I_2 = \frac{I1}{2}$$

Thus substituting the value of I2 in equation 3

$$I_1 = I2 - \int_0^{\frac{\pi}{2}} \text{log2dx}$$

$$\frac{I_1}{2} = -\log 2 \int_0^{\frac{\pi}{2}} 1 dx$$

$$I_1 = -2\log 2 [x]_0^{\frac{\pi}{2}}$$

$$I_1 = -2\log 2 \left[\frac{\pi}{2}\right]$$

$$I_1 = -2\log 2 \left[\frac{\pi}{2}\right]$$

$$I_1 = -\pi\log 2$$

Substituting in equation (a) i.e

$$I = \int_0^{\pi} \log 2dx + 2\int_0^{\pi} \log\sin xdx$$

$$I = \int_0^{\pi} \log 2dx - 2\pi\log 2$$

$$I = \pi\log 2 - 2\pi\log 2$$

I = $-\pi\log 2$

28. Question

Evaluate the following integral:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\frac{2-\sin x}{2+\sin x} dx$$

Answer

Let us assume $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\frac{2-\sin x}{2+\sin x} dx$ ....equation 1

By property, we know that $\int_a^{-a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx$

thus

$$I = \int_0^{\frac{\pi}{2}} \log\frac{2-\sin x}{2+\sin x} dx + \int_0^{\frac{\pi}{2}} \log\frac{2+\sin x}{2-\sin x} dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$I = \int_0^{\frac{\pi}{2}} [\log\frac{2-\sin x}{2+\sin x} + \log\frac{2+\sin x}{2-\sin x}] dx$$

Since we know $\log(mn) = \log m + \log n$

$$I = \int_0^{\frac{\pi}{2}} [\log\frac{2-\sin x}{2+\sin x} * \frac{2+\sin x}{2-\sin x}] dx$$

$$I = \int_0^{\frac{\pi}{2}} \log 1 \ dx$$

$$I = \int_0^{\frac{\pi}{2}} 0 \, dx$$

Thus

I = 0

29. Question

Evaluate the following integral:

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$$

Answer

Let us assume $I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$.........equation 1

By property, we know that $\int_a^{-a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx$

$$I = \int_0^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{-2x(1 + \sin(-x))}{1 + \cos^2(-x)} dx$$

We know that

Sin(– x) = – sin x

Cos(– x) = cos x

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus substituting the values, we get,

$$I = \int_0^{\pi} [\frac{2x(1 + \sin x)}{1 + \cos^2 x} + \frac{2(-x)(1 - \sin x)}{1 + \cos^2 x}] dx$$

$I = \int_0^{\pi} \frac{4x(\sin x)}{1 + \cos^2 x} dx$.....................equation 2

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

$$I = \int_0^{\pi} \frac{4(\pi - x)(\sin(\pi - x))}{1 + \cos^2(\pi - x)} dx$$

$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

Thus substituting the values

$I = \int_0^{\pi} \frac{4(\pi - x)(\sin(x))}{1 + \cos^2(x)} dx$ ....equation 3

Adding equation 2 and equation 3

$$2I = \int_0^{\pi} \frac{4x(\sin x)}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{4(\pi - x)(\sin(x))}{1 + \cos^2(x)} dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$2I = \int_0^{\pi} [\frac{4x(\sin x)}{1 + \cos^2 x} + \frac{4(\pi - x)(\sin(x))}{1 + \cos^2(x)}] dx$$

$$2I = \int_0^{\pi} \frac{4x(\sin x) + 4(\pi - x)(\sin x)}{1 + \cos^2 x} dx$$

$$2I = \int_0^{\pi} \frac{4\pi(\sin x)}{1 + \cos^2 x} dx$$

Let $\cos x = y$

Differentiating both sides

$-\sin x dx = dy$

$\sin x dx = -dy$

For $x = \pi$

$\cos\pi = y$

y= -1

For x = 0

Cos 0 = y

y = 1

Thus substituting the given values

$2I = \int_{-1}^{1} \frac{4\pi}{1 + y^2} dy$ ....equation 4

Now let $y = \tan\theta$

Differentiating both sides

$dy = \sec^2\theta d\theta$

For $y = -1$

$\tan\theta = -1$

$-\frac{\pi}{4} = \theta$

For $y = 1$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

Substituting the values in equation 4

$$2I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4\pi}{1 + \tan^2\theta} \sec^2\theta \, d\theta$$

$$2I = 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 \, d\theta$$

$$2I = 4\pi[\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$2I = 4\pi\left[\frac{\pi}{4} - \left(\frac{-\pi}{4}\right)\right]$$

$I = \pi^2$

30. Question

Evaluate the following integral:

$$\int_{-a}^{a} \log \frac{a - \sin\theta}{a + \sin\theta} dx$$

Answer

Let us assume $I = \int_{-a}^{a} \log \frac{a - \sin\theta}{a + \sin\theta} dx$ ....equation 1

By property, we know that $\int_{a}^{-a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx$

thus

$$I = \int_0^a \log\frac{a - \sin\theta}{a + \sin\theta} dx + \int_0^a \log\frac{a + \sin\theta}{a - \sin\theta} dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$I = \int_0^a [\log\frac{a - \sin\theta}{a + \sin\theta} + \log\frac{a + \sin\theta}{a - \sin\theta}] dx$$

Since we know $\log(mn) = \log m + \log n$

$$I = \int_0^a [\log\frac{a - \sin\theta}{a + \sin\theta} * \frac{a + \sin\theta}{a - \sin\theta}] dx$$

$$I = \int_0^a \log 1 \ dx$$

$$I = \int_0^a 0 \ dx$$

Thus

$I = 0$

31. Question

Evaluate the following integral:

$$\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx$$

Answer

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\, dx$$

And also,

We know that if f(x) is an odd function,

$$\int_{-a}^{a} f(x)dx = 0$$

As we know the property,

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx$$

Applying this property we get,

$$\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx = \int_{0}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx + \frac{-3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx$$

$$\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx = \int_{0}^{2} \frac{2(2x + 1)}{x^2 + x + 1} dx$$

Let $x^2 + x + 1 = t$

(2 x + 1) dx = dt

And for limits,

At x = 0, t = 1

At x = 2, t = 7

Therefore, we get,

$$\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx = \int_{1}^{7} \frac{2}{t} dt$$

$$\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx = 2[\log(7) - \log(1)]$$

$$\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1} dx = 2\log_e 7$$

32. Question

Evaluate the following integral:

$$\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \{\sin^2(3\pi + x) + (\pi + x)^3\}dx$$

Answer

Let us assume that $I = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \{\sin^2(3\pi + x) + (\pi + x)^3\}dx$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$I = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \{\sin^2(3\pi + x)\}dx + \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} (\pi + x)^3 \ dx$$

$$I = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \{\sin^2(3\pi + x)\}dx + \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} (\pi + x)^3 \ dx$$

$$I = \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \{1 - \cos 2(3\pi + x)\}/2 \ dx + \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} (\pi + x)^3 \ dx$$

$$I = \frac{1}{2}\left[x + \frac{\sin 2(3\pi + x)}{2}\right]_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} + \left[\frac{(\pi + x)^4}{4}\right]_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively

thus

$$I = \frac{1}{2}\left[\left\{-\frac{\pi}{2} + \frac{\sin 2\left(3\pi - \frac{\pi}{2}\right)}{2}\right\} - \left\{-\frac{3\pi}{2} + \frac{\sin 2\left(3\pi - \frac{3\pi}{2}\right)}{2}\right\}\right] + \left[\left\{\frac{\left(\pi - \frac{\pi}{2}\right)^4}{4}\right\} - \left\{\frac{\left(\pi - \frac{3\pi}{2}\right)^4}{4}\right\}\right]$$

Thus solving the above equation, we get

$$I = \frac{\pi}{2} + \frac{\left(\frac{\pi}{2}\right)^4}{4} - \frac{\left(-\frac{\pi}{2}\right)^4}{4}$$

$$I = \frac{\pi}{2}$$

33. Question

Evaluate the following integral:

$$\int_0^2 x\sqrt{2 - x}\,dx$$

Answer

Let $I = \int_0^2 x\sqrt{2 - x}\,dx$ ......equation 1

Put

$2 - x = y^2$

Differentiating both sides

$-dx = 2ydy$

For x = 2

$2 - x = y^2$

$2 - 2 = y^2$

$y = 0$

For x = 0

$2 - x = y^2$

$2 - 0 = y^2$

$y = \sqrt{2}$

Substituting the values in equation 1

$$I = \int_{\sqrt{2}}^{0} -2y(2 - y^2)ydy$$

$$I = \int_0^{\sqrt{2}} 2y(2 - y^2)ydy$$

$$I = 2\int_0^{\sqrt{2}}(2y^2 - y^4)dy$$

$$I = 2\left[\frac{2y^3}{3} - \frac{y^5}{5}\right]_0^{\sqrt{2}}$$

We know $[f(x)]_a^b = f(b) - f(a)$ b and a being the upper and lower limits respectively

thus

$$I = 2[\frac{2(\sqrt{2})^3}{3} - \frac{(\sqrt{2})^5}{5}] - [0]$$

Solving this we get

$$I = 2[\frac{(\sqrt{2})^5}{3} - \frac{(\sqrt{2})^5}{5}]$$

$$I = 2(\sqrt{2})^5[\frac{1}{3} - \frac{1}{5}]$$

$$I = 2(\sqrt{2})^5[\frac{5-3}{15}]$$

Thus

$$I = \sqrt{2}2[\frac{16}{15}]$$

34. Question

Evaluate the following integral:

$$\int_0^1 \log\left(\frac{1}{x} - 1\right)dx$$

Answer

Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right)dx$

$$I = \int_0^1 \log\frac{(1-x)}{x}dx$$

We know $\log\left(\frac{m}{n}\right) = \log m - \log n$

thus

$I = \int_0^1 (\log(1-x) - \log(x))\,dx$ ....equation 2

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

thus

$$I = \int_0^1 (\log 1 - (1-x) - \log(1-x))\,dx$$

$I = \int_0^1 (\log(x) - \log(1-x))\,dx$ .....equation 2

Adding equation 2 and equation 3 we have

$$2I = \int_0^1 (\log(1-x) - \log(x))\,dx + \int_0^1 (\log(x) - \log(1-x))\,dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus on solving we get

$$2I = \int_0^1 0\,dx$$

Thus

$I = 0$

35. Question

Evaluate the following integral:

$$\int_{-1}^{1} |x \cos \pi x| dx$$

Answer

Let f(x) = $|x \cos\pi x|$

Substituting x = – x in f(x)

f(– x) = $|-x\cos(-\pi x)| = |-x\cos(\pi x)| = |x\cos\pi x| = f(x)$

f(x) = f(– x)

∴it is an even function

$\int_{-1}^{1} |x \cos\pi x| dx = 2\int_0^1 |x\cos\pi x| dx$.........(1)

Now,

f(x) = |x cosπx| = x cosπx; for x∈ [0,1\2]

= – x cosπx; for x∈ [1/2,1]

Using interval addition property of integration, we know that

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Equation 1 can be written as

$2[\int_0^{1/2} x\cos\pi x\,dx + \int_{1/2}^{1} -x\cos\pi x dx]$

Putting the limits in above equation

= $2\{[(x/\pi)\sin\pi x + (1/\pi^2)\cos\pi x]_0^{1/2} - [(x/\pi)\sin\pi x + (1/\pi^2)\cos\pi x]^1_{1/2}\}$

= $2\{[(1/2\pi) - (1/\pi^2)] - [(-1/\pi^2) - (1/2\pi)]\}$

= 2/π

36. Question

Evaluate the following integral: $\int_0^{\pi} \frac{x}{1+\sin^2 x} + \cos^7 x\,dx$

Answer

Let us assume $I = \int_0^{\pi} \frac{x}{1+\sin^2 x} \cos^7 x\,dx$.........equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_0^{\pi} \frac{\pi - x}{1+\sin^2(\pi - x)} + \cos^7(\pi - x)\, dx$..........................equation 2

Adding equations 1 and 2, we get,

$2I = \int_0^{\pi} \frac{x}{1+\sin^2 x} + \cos^7 x\, dx + \int_0^{\pi} \frac{\pi - x}{1+\sin^2(x)} - \cos^7(x)\, dx$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus

$$2I = \int_0^{\pi} \frac{x}{1+\sin^2 x} + \cos^7 x + \frac{\pi - x}{1+\sin^2(x)} - \cos^7(x)\, dx$$

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin^2 x} dx$$

We know that $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

If f(2a – x) = f(x)

than $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$

thus

$2I = 2\int_0^{\frac{\pi}{2}} \frac{\pi}{1+\sin^2 x} dx$ since sinx = sinπ – x

Now

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$2I = 2\int_0^{\frac{\pi}{2}} \frac{\pi}{1+\sin^2(\frac{\pi}{2} - x)} dx$$

$$2I = 2\int_0^{\frac{\pi}{2}} \frac{\pi}{1+\cos^2(x)} dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\pi \sec^2 x}{\sec^2(x) + 1} dx$$

since 1/cosx = secx

$$I = \int_0^{\frac{\pi}{2}} \frac{\pi \sec^2 x}{\tan^2(x) + 2} dx$$

since $\tan^2 x + 1 = \sec^2 x$

Let tanx = y

$Sec^2 xdx = dy$

Thus

For $x = \frac{\pi}{2}$

$$\tan\frac{\pi}{2} = y$$

$y = \infty$

For x = 0

tan 0 = y

y = 0

thus substituting in

$$I = \int_0^{\frac{\pi}{2}} \frac{\pi \sec^2 x}{\tan^2(x) + 2} dx$$

$$I = \int_0^{\infty} \frac{\pi}{y^2 + 2} dy$$

$$I = \int_0^{\infty} \frac{\pi}{2[\{y^2/(\sqrt{2})^2\} + 1]} dy$$

$$I = \frac{\pi}{2}[\sqrt{2}\tan^{-1}[y/\sqrt{2}]_0^{\infty}$$

$$I = \frac{\pi}{\sqrt{2}}[\tan^{-1}(\infty) - \tan^{-1} 0]$$

$$I = \frac{\pi}{\sqrt{2}}[\frac{\pi}{2}]$$

37. Question

Evaluate the following integral: $\int_0^{\pi} \frac{x}{1+\sin\alpha\sin x} dx$

Answer

Let I = $\int_0^{\pi} \frac{x}{1+\sin\alpha\sin x} dx$

We know that,

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$$

Therefore,

I = $\int_0^{\frac{\pi}{2}} \frac{x}{1+\sin\alpha\sin x} dx + \int_0^{a} \frac{\pi-x}{1+\sin\alpha\sin x} dx$

2 I = $\int_0^{\frac{\pi}{2}} \frac{\pi}{1+\sin\alpha\sin x} dx$

2I = $2\pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\alpha\sin x} dx$

I = $\frac{\pi(\frac{\pi}{2}-\alpha)}{\cos\alpha}$

38. Question

Evaluate the following integral: $\int_0^{2\pi} \sin^{100} x\cos^{101} x dx$

Answer

Let us assume $I = \int_0^{2\pi} \sin^{100} x\cos^{101} x dx$

We know that $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$

If f(2a – x) = f(x)

than $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$

$I = 2\int_0^{\pi} \sin^{100}x\cos^{101}xdx$ since sin2π – x = – sinx and cos2π – x = cosx and $(-\sin x)^{100} = \sin^{100}x$ ……equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = 2\int_0^{\pi} \sin^{100}(\pi - x)\cos^{101}(\pi - x)dx$$

$I = -2\int_0^{\pi} \sin^{100}x\cos^{101}xdx$……equation 2 since cosπ – x = cosx

Adding equation 1 and equation 2

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

thus

$$2I = 2\int_0^{\pi} \sin^{100}x\cos^{101}xdx - 2\int_0^{\pi} \sin^{100}x\cos^{101}xdx$$

2I = 0

I = 0

39. Question

Evaluate the following integral: $\int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$

Answer

Let us assume $I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$ ………equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Thus

$$I = \int_0^{\frac{\pi}{2}} \frac{a\sin(\frac{\pi}{2} - x) + b\cos(\frac{\pi}{2} - x)}{\sin x + \cos x} dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{\sin x + \cos x} dx$……equation 2

Adding the equation 1 and 2

$$2I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{\sin x + \cos x} dx$$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

$$2I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x} + \frac{a\cos x + b\sin x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x + a\cos x + b\sin x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{a(\sin x + \cos x) + b(\cos x + \sin x)}{\sin x + \cos x} dx$$

$$2I = \int_0^{\frac{\pi}{2}} a + b \; dx$$

$$2I = a + b[x]_0^{\frac{\pi}{2}}$$

$$I = \frac{(a + b)(\pi)}{2}$$

40. Question

Evaluate the following integrals: $\int_0^{\frac{3}{2}} |x\cos\pi x| dx$

Answer

Let us assume $I = \int_0^{\frac{3}{2}} |x\cos\pi x| dx$

We know| cosx| = cosx for 0 < x<π/2 & |cosx| = – cosx for π/2<x<3π/2

We know that $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$ given a<c<b

Thus

$$I = \int_0^{\frac{1}{2}} x\cos\pi x dx - \int_{\frac{1}{2}}^{\frac{3}{2}} x\sin\pi x dx$$

By partial integration $\int (u)(v) = (u) \int (v) - \int du \int v$

Thus

$$I = [x \int \cos\pi x dx - \int \frac{dx}{dx} \int \cos\pi x dx]_0^{\frac{1}{2}} - [x \int \cos\pi x dx - \int \frac{dx}{dx} \int \cos\pi x dx]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$I = \left[\frac{x\{\sin\pi x\}}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_0^{\frac{1}{2}} - \left[\frac{x\{\sin\pi x\}}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

Since $[f(x)]_a^b = f(b) - f(a)$

$$I = \left[\left[\frac{1/2\left\{\sin(\frac{\pi 1}{2})\right\}}{\pi} + \frac{\cos(\frac{\pi 1}{2})}{\pi^2}\right] - \left[\frac{0\{\sin\pi 0\}}{\pi} + \frac{\cos\pi 0}{\pi^2}\right]\right] - \left[\left[\frac{3/2\{\sin\pi 3/2\}}{\pi} + \frac{\cos\pi 3/2}{\pi^2}\right] - \left[\frac{1/2\left\{\sin(\frac{\pi 1}{2})\right\}}{\pi} + \frac{\cos(\frac{\pi 1}{2})}{\pi^2}\right]\right]$$

$$I = \left[\left[\frac{1}{2\pi} + \frac{0}{\pi^2}\right] - \left[\frac{0\{0\}}{\pi} + \frac{1}{\pi^2}\right]\right] - \left[\left[\frac{3/2\{-1\}}{\pi} + \frac{0}{\pi^2}\right] - \left[\frac{1}{2\pi} + \frac{0}{\pi^2}\right]\right]$$

$$I = \left[\left[\frac{1}{2\pi}\right] - \left[\frac{1}{\pi^2}\right]\right] - \left[\left[\frac{-3}{2\pi}\right] - \left[\frac{1}{2\pi}\right]\right]$$

$I = \frac{5}{2\pi} - \frac{1}{\pi^2}$

41. Question

Evaluate the following integrals: $\int_0^1 |x\sin \pi x| dx$

Answer

Let us assume $I = \int_0^1 |x\sin\pi x| dx$

We know| sinx| = sinx for 0 < x<π & |sinx| = – sinx for π<x<2π

We know that $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$ given a<c<b

Thus

$I = \int_0^1 x\sin\pi x dx$

By partial integration $\int (u)(v) = (u)\int (v) - \int du \int v$

Thus

$I = [x\int \sin\pi x dx - \int \frac{dx}{dx}\int \sin\pi x dx]_0^1$

$I = \left[\frac{x\{-\cos\pi x\}}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_0^1$

Since $[f(x)]_a^b = f(b) - f(a)$

$I = \left[\left[\frac{1\{-\cos\pi 1\}}{\pi} + \frac{\sin\pi 1}{\pi^2}\right] - \left[\frac{0\{-\cos\pi 0\}}{\pi} + \frac{\sin\pi 0}{\pi^2}\right]\right]$

$I = \left[\left[\frac{1}{\pi} + \frac{0}{\pi^2}\right] - \left[\frac{0\{-1\}}{\pi} + \frac{0}{\pi^2}\right]\right]$

$I = \left[\left[\frac{1}{\pi}\right]\right]$

$I = \frac{1}{\pi}$

42. Question

Evaluate the following integrals: $\int_0^{\frac{3}{2}} |x\sin \pi x| dx$

Answer

Let us assume, $I = \int_0^{\frac{3}{2}} |x\sin\pi x| dx$

We know|sinx| = sinx for 0 < x<π & |sinx| = – sinx for π<x<2π

We know that $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$ given a < c < b

Thus

$$I = \int_0^1 x\sin\pi x dx - \int_1^{\frac{3}{2}} x\sin\pi x dx$$

By partial integration, $\int (u)(v) = (u)\int(v) - \int du \int v$

Thus,

$$I = [x\int \sin\pi x dx - \int \frac{dx}{dx}\int \sin\pi x dx]_0^1 - [x\int \sin\pi x dx - \int \frac{dx}{dx}\int \sin\pi x dx]_1^{\frac{3}{2}}$$

$$I = \left[\frac{x\{-\cos\pi x\}}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_0^1 - \left[\frac{x\{-\cos\pi x\}}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_1^{\frac{3}{2}}$$

Since, $[f(x)]_a^b = f(b) - f(a)$

$$I = \left[\left[\frac{1\{-\cos\pi 1\}}{\pi} + \frac{\sin\pi 1}{\pi^2}\right] - \left[\frac{0\{-\cos\pi 0\}}{\pi} + \frac{\sin\pi 0}{\pi^2}\right]\right] - \left[\left[\frac{3/2\{-\cos\pi 3/2\}}{\pi} + \frac{\sin\pi 3/2}{\pi^2}\right] - \left[\frac{1\{-\cos\pi 1\}}{\pi} + \frac{\sin\pi 1}{\pi^2}\right]\right]$$

$$I = \left[\left[\frac{1}{\pi} + \frac{0}{\pi^2}\right] - \left[\frac{0\{-1\}}{\pi} + \frac{0}{\pi^2}\right]\right] - \left[\left[\frac{3/2\{0\}}{\pi} + \frac{-1}{\pi^2}\right] - \left[\frac{1}{\pi} + \frac{0}{\pi^2}\right]\right]$$

$$I = \left[\left[\frac{1}{\pi}\right]\right] - \left[\left[\frac{-1}{\pi^2}\right] - \left[\frac{1}{\pi}\right]\right]$$

$$I = \frac{2}{\pi} + \frac{1}{\pi^2}$$

43. Question

If f is an integrable function such that f(2a – x) = f(x), then prove that

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$$

Answer

Using interval addition property of integration, we know that

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

So L.H.S can be written as,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Let us assume x = 2a – t

Differentiating it we get,

dx = – dt

from above assumption

when x = 2a ⇒ t = 0

and when x = a ⇒ t = a

substituting above assumptions in L.H.S

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt$$

Using the property of integration $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\int_a^{2a} f(x)\,dx = \int_0^a f(2a-t)\,dt$$

Using integration property

$$\int_a^{2a} f(x)\,dx = \int_a^{2a} f(2a-x)\,dx$$

Substituting above value in equation 1

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(2a-x)\,dx$$

Now using the property $\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b \{f(x) + g(x)\}dx$

$$\int_0^{2a} f(x)\,dx = \int_0^a \{f(x) + f(2a-x)\}dx$$

Since, f(2a – x) = f(x)

$$\therefore \int_0^{2a} f(x)dx = \int_0^a \{f(x) + f(x)\}$$

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

Hence proved.

44. Question

if $f\left(2a-x\right)=-f\left(x\right)$ prove that $\int_0^{2a} f\left(x\right)dx=0$

Answer

Let us assume $I = \int_0^{2a} f(x)dx$ ....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

Thus

$$I = \int_0^{2a} f(2a-x)\,dx$$

Given: $f(2a-x) = -f(x)$

Equation 1 becomes

$I = \int_0^{2a} -f(x)dx$..........equation 2

Adding equation 2 and 3

$2I = \int_0^{2a} f(2a - x)dx + \int_0^{2a} -f(x)dx$

We know $\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$

Thus

$$2I = \int_0^{2a} [f(x) - f(x)]dx$$

Thus

$2I = 0$

$I = 0$

$$\int_0^{2a} f(x)dx = 0$$

45. Question

If f is an integrable function, show that

(i) $\int_{-a}^{a} f\left(x^2\right)dx = 2\int_0^a f\left(x^2\right)dx$

(ii) $\int_{-a}^{a} xf\left(x^2\right)dx = 0$

Answer

(i) Let us check the given function for being even and odd.

$f((-x)^2) = f(x^2)$

The function does not change sign and therefore the function is even.

We know that if f(x) is an even function,

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx$$

Therefore,

$$\int_{-a}^{a} f(x^2)dx = 2\int_0^a f(x^2)\,dx$$

Hence, Proved.

(ii) Let us check the given function for even and odd.

Let $g(x) = xf(x^2)$

$g(-x) = -x\ f((-x)^2)$

$g(-x) = -\ xf(x^2)$

$g(-x) = -\ g(x)$

Therefore, the function is odd.

We know that if f(x) is an odd function,

$$\int_{-a}^{a} f(x)dx = 0$$

Therefore,

$$\int_{-a}^{a} xf(x^2)dx = 0$$

Hence, Proved.

46. Question

If f(x) is a continuous function defined on [0,2a]. Then Prove that

$$\int_0^{2a} f(x)dx = \int_0^a \{f(x)+f(2a-x)\}dx$$

Answer

Using interval addition property of integration, we know that

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

So L.H.S can be written as,

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \ldots \ldots (1)$$

Let us assume x = 2a – t

Differentiating it we get,

dx = – dt

from above assumption

when x = 2a⇒ t = 0

and when x = a⇒ t = a

substituting above assumptions in L.H.S

$$\int_a^{2a} f(x)dx = -\int_a^0 f(2a-t)dt$$

Using the property of integration $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

$$\int_a^{2a} f(x)dx = \int_0^a f(2a-t)dt$$

Using integration property

$$\int_a^{2a} f(x)dx = \int_a^{2a} f(2a-x)dx$$

Substituting above value in equation 1

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(2a-x)dx$$

Now using the property $\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b \{f(x) + g(x)\}dx$

$$\int_0^{2a} f(x)dx = \int_0^a \{f(x) + f(2a-x)\}dx$$

∴L.H.S = R.H.S

Hence, proved.

47. Question

If f(a + b – x) = f(x) prove that:

$$\int_a^b x f(x).dx = (a+b)/2\int_a^b f(x).dx$$

Answer

LHS

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

thus

$$\int_a^b xf(x).dx = \int_a^b (a + b - x)f(a + b - x).dx$$

Given f(a + b – x) = f(x)

$$\int_a^b xf(x).dx = \int_a^b (a + b - x)f(x).dx$$

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

Thus

$$\int_a^b xf(x).dx = \int_a^b (a + b)f(x).dx - \int_a^b xf(x).dx$$

$$2\int_a^b xf(x).dx = \int_a^b (a + b)f(x).dx$$

$$\int_a^b xf(x).dx = (a + b)/2\int_a^b f(x).dx$$

Hence proved

48. Question

If f(x) is a continuous function defined on [– a,a] ,then prove that

$$\int_{-a}^a f(x)dx = \int_0^a \{f(x)+f(-x)\}dx$$

Answer

Using interval addition property of integration, we know that

$$\int_a^c f(x)\,dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

So L.H.S can be written as,

$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$ .....(1)

Now let us take x = – t

Differentiating it, we get,

dx = – dt

from above assumption

when x = – a ⇒ t = a

and when x = 0 ⇒ t = 0

Substituting the above assumptions in equation 1 $\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-t)(-dt) = -\int_{a}^{0} f(-t)dt$

Using the property of integration $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_{-a}^{0} f(x)\,dx = \int_0^a f(-t)\,dt$ ....(2)

Using integration property

$\int_0^a f(-t)\,dt = \int_0^a f(-x)\,dx$ .....(3)

Using equation 2 and 3, now equation 1 can be rewritten as

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

Now using the property $\int_a^b f(x)dx + \int_a^b g(x)dx = \int_a^b \{f(x) + g(x)\}dx$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} \{f(-x) + f(x)\}dx$$

$\therefore$ L. H. S = R. H. S

Hence proved.

49. Question

Prove that: $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x).dx$

Answer

LHS

Let $I = \int_0^{\pi} xf(\sin x)dx$.....equation 1

By property, we know that $\int_a^b f(x)dx = \int_a^b f(a + b - x)dx$

thus

$I = \int_0^{\pi} xf(\sin x).dx = \int_0^{\pi} (\pi - x)f\sin(\pi - x).dx$...............equation 2

Adding equations 1 and equation 2, we get,

$$2I = \int_0^{\pi} xf(\sin x)dx + \int_0^{\pi} (\pi - x)f(\sin(\pi - x)).dx$$

Since we know, sin(π – x) = sinx

We know

$$\int_m^n [f(x) + g(x)]dx = \int_m^n f(x)dx + \int_m^n g(x)dx$$

$$2I = \int_0^\pi [xf(\sin x) + (\pi - x)f(\sin x)]dx$$

Thus on solving

$$2I = \int_0^\pi [(\pi)f(\sin x)]dx$$

We know that by integration property:

$$\int_a^b [(m)f(x)]dx = m\int_a^b [f(x)]dx$$

Thus we have

$$2I = \pi\int_0^\pi [f(\sin x)]dx$$

$$I = \frac{\pi}{2}\int_0^\pi [f(\sin x)]dx$$

Putting back the value of I we have

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi [f(\sin x)]dx$$

Hence proved

Exercise 20.6

1. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^3 (x+4)dx$$

Answer

To find: $\int_0^3 (x+4)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 3

Therefore,

$$h = \frac{3-0}{n}$$

⇒ nh = 3

Let,

$$I = \int_0^3 (x + 4)\, dx$$

Here, f(x) = x + 4 and a = 0

$$\Rightarrow I = \lim_{h \to 0} h[f(0) + f(0 + h) + f(0 + 2h) + \cdots + f(0 + (n - 1)h)]$$

$$\Rightarrow I = \lim_{h \to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n - 1)h\big)\big]$$

Now, By putting x = 0 in f(x) we get,

f(0) = 0 + 4 = 4

Similarly, f(h) = h + 4

$$\Rightarrow I = \lim_{h \to 0} h[4 + h + 4 + 2h + 4 + \cdots + (n - 1)h + 4]$$

In this series, 4 is getting added n times

$$\Rightarrow I = \lim_{h \to 0} h[4n + h + 2h + \cdots + (n - 1)h]$$

Now take h common in remaining series

$$\Rightarrow I = \lim_{h \to 0} h[4n + h(1 + 2 + \cdots + (n - 1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n - 1) = \frac{n(n - 1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h \to 0} h\left[4n + h\left\{\frac{n(n - 1)}{2}\right\}\right]$$

Put,

$$h = \frac{3}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{3}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n \to \infty} \frac{3}{n}\left[4n + \frac{3}{n}\left\{\frac{n(n - 1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n \to \infty} \frac{3}{n}\left[4n + \frac{3(n - 1)}{2}\right]$$

$$\Rightarrow I = \lim_{n \to \infty} \left\{12 + \frac{9(n - 1)}{2n}\right\}$$

$$\Rightarrow I = \lim_{n \to \infty} \left\{12 + \frac{9}{2}\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = 12 + \frac{9}{2}\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 12 + \frac{9}{2}(1 - 0)$$

$$\Rightarrow I = 12 + \frac{9}{2}$$

$$\Rightarrow I = \frac{24 + 9}{2}$$

$\Rightarrow I = \frac{33}{2}$

Hence, the value of $\int_0^3 (x+4)\,dx = \frac{33}{2}$

2. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 (x+3)dx$$

Answer

To find: $\int_0^2 (x+3)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$$\Rightarrow h = \frac{2}{n}$$

Let,

$$I = \int_0^2 (x+3)\,dx$$

Here, f(x) = x + 3 and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, By putting x = 0 in f(x) we get,

f(0) = 0 + 3 = 3

Similarly, f(h) = h + 3

$$\Rightarrow I = \lim_{h\to 0} h[3 + h + 3 + 2h + 3 + \cdots + (n-1)h + 3]$$

In this series, 3 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[3n + h + 2h + \cdots + (n-1)h]$$

Now take h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[3n + h(1 + 2 + \cdots + (n-1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[3n + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[3n + \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}[3n + (n-1)]$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{6 + \frac{2(n-1)}{n}\right\}$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{6 + 2\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = 6 + 2\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 6 + 2(1-0)$$

$$\Rightarrow I = 6 + 2$$

$$\Rightarrow I = 8$$

$$\text{Hence, the value of } \int_0^2 (x+3)\,dx = 8$$

3. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^3 (3x-2)\,dx$$

Answer

$$\text{To find: } \int_1^3 (3x-2)\,dx$$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 3

Therefore,

$h = \frac{3-1}{n}$

$\Rightarrow \ h \ = \frac{2}{n}$

Let,

$I = \int_{1}^{3} (3x - 2) \, dx$

Here, f(x) = 3x – 2 and a = 1

$\Rightarrow I = \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$

Now, By putting x = 1 in f(x) we get,

f(1) = 3(1) – 2 = 3 – 2 = 1

Similarly, f(1 + h)

= 3(1 + h) – 2

= 3 + 3h – 2

= 3h + 1

$\Rightarrow I = \lim_{h \to 0} h[1 + 3h + 1 + 3(2h) + 1 + \cdots + 3(n-1)h + 1]$

In this series, 1 is getting added n times

$\Rightarrow I = \lim_{h \to 0} h[1 \times n + 3h + 3(2h) + \cdots + 3(n-1)h]$

Now take 3h common in remaining series

$\Rightarrow I = \lim_{h \to 0} h[n + 3h(1 + 2 + \cdots + (n-1)]$

$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$

$\Rightarrow I = \lim_{h \to 0} h\left[n + 3h\left\{\frac{n(n-1)}{2}\right\}\right]$

Put,

$h = \frac{2}{n}$

Since,

$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$

$\Rightarrow I = \lim_{n \to \infty} \frac{2}{n}\left[n + \frac{3(2)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$

$\Rightarrow I = \lim_{n \to \infty} \frac{2}{n}\left[n + \frac{6(n-1)}{2}\right]$

$\Rightarrow I = \lim_{n \to \infty} \frac{2}{n}[n + 3(n-1)]$

$\Rightarrow I = \lim_{n \to \infty} \left\{2 + \frac{6(n-1)}{n}\right\}$

$$\Rightarrow I = \lim_{n\to\infty}\left\{2 + 6\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = 2 + 6\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 2 + 6(1 - 0)$$

$$\Rightarrow I = 2 + 6$$

$$\Rightarrow I = 8$$

Hence, the value of $\int_1^3 (3x - 2)\, dx = 8$

4. Question

Evaluate the following integrals as a limit of sums:

$$\int_{-1}^{1} (x + 3)\, dx$$

Answer

To find: $\int_{-1}^{1} (x + 3)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a + h) + f(a + 2h) + \cdots + f(a + (n - 1)h)],$$

where,

$$h = \frac{b - a}{n}$$

Here, a = -1 and b = 1

Therefore,

$$h = \frac{1 - (-1)}{n}$$

$$\Rightarrow h = \frac{1 + 1}{n}$$

$$\Rightarrow h = \frac{2}{n}$$

Let,

$$I = \int_{-1}^{1} (x + 3)\, dx$$

Here, f(x) = x + 3 and a = -1

$$\Rightarrow I = \lim_{h\to 0} h[f(-1) + f(-1 + h) + f(-1 + 2h) + \cdots + f(-1 + (n - 1)h)]$$

Now, By putting x = -1 in f(x) we get,

f(-1) = -1 + 3 = 2

Similarly, f(-1 + h)

= -1 + h + 3

= h + 2

$\Rightarrow I = \lim_{h\to 0} h[2 + h + 2 + 2h + 2 + \cdots + (n-1)h + 2]$

In this series, 2 is getting added n times

$\Rightarrow I = \lim_{h\to 0} h[2n + h + 2h + \cdots + (n-1)h]$

Now take h common in remaining series

$\Rightarrow I = \lim_{h\to 0} h[2n + h(1 + 2 + \cdots + (n-1)]$

$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$

$\Rightarrow I = \lim_{h\to 0} h\left[2n + h\left\{\frac{n(n-1)}{2}\right\}\right]$

Put,

$h = \frac{2}{n}$

Since,

$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$

$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[2n + \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$

$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}[2n + (n-1)]$

$\Rightarrow I = \lim_{n\to\infty} \left\{4 + \frac{2(n-1)}{n}\right\}$

$\Rightarrow I = \lim_{n\to\infty} \left\{4 + 2\left(1 - \frac{1}{n}\right)\right\}$

$\Rightarrow I = 4 + 2\left(1 - \frac{1}{\infty}\right)$

$\Rightarrow I = 4 + 2(1 - 0)$

$\Rightarrow I = 4 + 2$

$\Rightarrow I = 6$

Hence, the value of $\int_{-1}^{1} (x+3)\, dx = 6$

5. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^5 (x+1)dx$$

Answer

To find: $\int_0^5 (x+1)\, dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 5

Therefore,

$$h = \frac{5-0}{n}$$

$$\Rightarrow\ h\ = \frac{5}{n}$$

Let,

$$I = \int_0^5 (x+1)\,dx$$

Here, f(x) = x + 1 and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, By putting x = 0 in f(x) we get,

f(0) = 0 + 1 = 1

Similarly, f(h) = h + 1

$$\Rightarrow I = \lim_{h\to 0} h[1 + h + 1 + 2h + 1 + \cdots + (n-1)h + 1]$$

In this series, 1 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[1\times n + h + 2h + \cdots + (n-1)h]$$

Now take h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[n + h(1 + 2 + \cdots + (n-1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[n + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{5}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{5}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{5}{n}\left[n + \frac{5}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{5}{n}\left[n + \frac{5(n-1)}{2}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{5 + \frac{25(n-1)}{2n}\right\}$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{5 + \frac{25}{2}\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = 5 + \frac{25}{2}\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 5 + \frac{25}{2}(1-0)$$

$$\Rightarrow I = 5 + \frac{25}{2}$$

$$\Rightarrow I = \frac{10+25}{2}$$

$$\Rightarrow I = \frac{35}{2}$$

Hence, the value of $\int_0^5 (x+1)\,dx = \frac{35}{2}$

6. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^3 (2x+3)\,dx$$

Answer

To find: $\int_1^3 (2x+3)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 3

Therefore,

$$h = \frac{3-1}{n}$$

$$\Rightarrow h = \frac{2}{n}$$

Let,

$$I = \int_1^3 (2x+3)\,dx$$

Here, f(x) = 2x + 3 and a = 1

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

Now, By putting x = 1 in f(x) we get,

f(1) = 2(1) + 3 = 2 + 3 = 5

Similarly, f(1 + h)

= 2(1 + h) + 3

= 2 + 2h + 3

= 2h + 5

$$\Rightarrow I = \lim_{h\to 0} h[5 + 2h + 5 + 2(2h) + 5 + \cdots + 2(n-1)h + 5]$$

In this series, 5 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[5 \times n + 2h + 2(2h) + \cdots + 2(n-1)h]$$

Now take 2h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[5n + 2h(1 + 2 + \cdots + (n-1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[5n + 2h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[5n + \frac{2(2)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[5n + \frac{4(n-1)}{2}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}[5n + 2(n-1)]$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{10 + \frac{4(n-1)}{n}\right\}$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{10 + 4\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = 10 + 4\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 10 + 4(1 - 0)$$

$$\Rightarrow I = 10 + 4$$

$$\Rightarrow I = 14$$

Hence, the value of $\int_1^3 (2x+3)\,dx = 14$

7. Question

Evaluate the following integrals as a limit of sums:

$$\int_{3}^{5}(2-x)\,dx$$

Answer

To find: $\int_{3}^{5}(2-x)\,dx$

Formula used:

$$\int_{a}^{b} f(x)\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 3 and b = 5

Therefore,

$$h = \frac{5-3}{n}$$

$$\Rightarrow\ h\ = \frac{2}{n}$$

Let,

$$I = \int_{3}^{5}(2-x)\,dx$$

Here, f(x) = 2 – x and a = 3

$$\Rightarrow I = \lim_{h\to 0} h[f(3)+f(3+h)+f(3+2h)+\cdots+f(3+(n-1)h)]$$

Now, By putting x = 3 in f(x) we get,

f(3) = 2 – 3 = -1

Similarly, f(3 + h)

= 2 – (3 + h)

= 2 – 3 – h

= -1 – h

$$\Rightarrow I = \lim_{h\to 0} h[-1-1-h-1-2h-\cdots-1-(n-1)h]$$

In this series, -1 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[-1\times n-h-2h-\cdots-(n-1)h]$$

Now take -h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[-n-h(1+2+\cdots+(n-1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1+2+\cdots+(n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[-n - h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[-n - \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}[-n - (n-1)]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}[-n - n + 1]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\{-2n + 1\}$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{-4 + \frac{2}{n}\right\}$$

$$\Rightarrow I = -4 + \frac{2}{\infty}$$

$$\Rightarrow I = -4 + 0$$

$$\Rightarrow I = -4$$

Hence, the value of $\int_3^5 (2 - x)\, dx = -4$

8. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 \left(x^2 + 1\right) dx$$

Answer

To find: $\int_0^2 (x^2 + 1)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$\Rightarrow\ h = \frac{2}{n}$

Let,

$I = \int_0^2 (x^2 + 1)\, dx$

Here, f(x) = x^2 + 1 and a = 0

$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0 + h) + f(0 + 2h) + \cdots + f(0 + (n-1)h)]$

$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$

Now, By putting x = 0 in f(x) we get,

f(0) = 0^2 + 1 = 0 + 1 = 1

Similarly, f(h) = h^2 + 1

$\Rightarrow I = \lim_{h\to 0} h[1 + h^2 + 1 + (2h)^2 + 1 + \cdots + \{(n-1)h\}^2 + 1]$

$\Rightarrow I = \lim_{h\to 0} h[1 + h^2 + 1 + h^2(2)^2 + 1 + \cdots + h^2(n-1)^2 + 1]$

In this series, 1 is getting added n times

$\Rightarrow I = \lim_{h\to 0} h[1 \times n + h^2 + h^2(2)^2 + \cdots + h^2(n-1)^2]$

Now take h^2 common in remaining series

$\Rightarrow I = \lim_{h\to 0} h[n + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\}]$

$\left\{\because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right\}$

$\Rightarrow I = \lim_{h\to 0} h\left[n + h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$

Put,

$h = \frac{2}{n}$

Since,

$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$

$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[n + \left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$

$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[n + \frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$

$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[n + \frac{2(n-1)(2n-1)}{3n}\right]$

$\Rightarrow I = \lim_{n\to\infty} \left[2 + \frac{4(n-1)(2n-1)}{3n \times n}\right]$

$\Rightarrow I = \lim_{n\to\infty} \left[2 + \frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)\right]$

$$\Rightarrow I = \lim_{n\to\infty}\left[2 + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 2 + \frac{4}{3}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 2 + \frac{4}{3}(1 - 0)(2 - 0)$$

$$\Rightarrow I = 2 + \frac{4}{3} \times 1 \times 2$$

$$\Rightarrow I = 2 + \frac{8}{3}$$

$$\Rightarrow I = \frac{6 + 8}{3}$$

$$\Rightarrow I = \frac{14}{3}$$

Hence, the value of $\int_0^2 (x^2 + 1)\, dx = \frac{14}{3}$

9. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^2 x^2\, dx$$

Answer

To find: $\int_1^2 x^2\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a + h) + f(a + 2h) + \cdots + f(a + (n - 1)h)],$$

where,

$$h = \frac{b - a}{n}$$

Here, a = 1 and b = 2

Therefore,

$$h = \frac{2 - 1}{n}$$

$$\Rightarrow h = \frac{1}{n}$$

Let,

$$I = \int_1^2 x^2\, dx$$

Here, $f(x) = x^2$ and a = 1

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1 + h) + f(1 + 2h) + \cdots + f(1 + (n - 1)h)]$$

Now, by putting x = 1 in f(x) we get,

$f(1) = 1^2 = 1$

$f(1 + h)$

$= (1 + h)^2$

$= h^2 + 1^2 + 2(h)(1)$

$= h^2 + 1 + 2(h)$

Similarly, $f(1 + 2h)$

$= (1 + 2h)^2$

$= (2h)^2 + 1^2 + 2(2h)(1)$

$= (2h)^2 + 1 + 2(2h)$

$\{\because (x + y)^2 = x^2 + y^2 + 2xy\}$

$$\Rightarrow I = \lim_{h\to 0} h[1 + h^2 + 1 + 2(h) + (2h)^2 + 1 + 2(2h) + \cdots + \{(n-1)h\}^2 + 1 + 2\{(n-1)h\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[1 + h^2 + 1 + 2h + h^2(2)^2 + 1 + 2(2h) + \cdots + h^2(n-1)^2 + 1 + 2h(n-1)]$$

In this series, 1 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[1 \times n + h^2 + 2h + h^2(2)^2 + 2(2h) + \cdots + h^2(n-1)^2 + 2h(n-1)]$$

Now take h^2 and 2h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[n + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 2h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{\begin{array}{l} \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6} ; \\ \qquad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \dfrac{n(n-1)}{2} \end{array}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[n + h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 2h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{1}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{1}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[n + \left(\frac{1}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2(1)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[n + \frac{(n-1)(2n-1)}{6n} + (n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[1 + \frac{(n-1)(2n-1)}{6n \times n} + \frac{1}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[1 + \frac{1}{6}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + \left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[1 + \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \left(1-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = 1 + \frac{1}{6}\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right) + \left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = 1 + \frac{1}{6}(1-0)(2-0) + (1-0)$$

$$\Rightarrow I = 1 + \frac{1}{6} \times 1 \times 2 + 1$$

$$\Rightarrow I = 2 + \frac{1}{3}$$

$$\Rightarrow I = \frac{6+1}{3}$$

$$\Rightarrow I = \frac{7}{3}$$

Hence, the value of $\int_1^2 x^2\, dx = \frac{7}{3}$

10. Question

Evaluate the following integrals as a limit of sums:

$$\int_2^3 \left(2x^2 + 1\right) dx$$

Answer

To find: $\int_2^3 (2x^2 + 1)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 2 and b = 3

Therefore,

$$h = \frac{3-2}{n}$$

$$\Rightarrow h = \frac{1}{n}$$

Let,

$$I = \int_2^3 (2x^2 + 1)\, dx$$

Here, $f(x) = 2x^2 + 1$ and a = 2

$$\Rightarrow I = \lim_{h\to 0} h[f(2) + f(2+h) + f(2+2h) + \cdots + f(2+(n-1)h)]$$

Now, by putting x = 2 in f(x) we get,

$f(2) = 2(2^2) + 1 = 2(4) + 1 = 8 + 1 = 9$

f(1 + h)

$= 2(2 + h)^2 + 1$

$= 2\{h^2 + 2^2 + 2(h)(2)\} + 1$

$= 2(h)^2 + 8 + 2(4h) + 1$

$= 2(h)^2 + 9 + 8(h)$

Similarly, f(2 + 2h)

$= 2(2 + 2h)^2 + 1$

$= 2\{2(2h)^2 + 2^2 + 2(2h)(2)\} + 1$

$= 2(2h)^2 + 8 + 8(2h) + 1$

$= 2(2h)^2 + 9 + 8(2h)$

$\{\because (x + y)^2 = x^2 + y^2 + 2xy\}$

$$\Rightarrow I = \lim_{h\to 0} h[9 + 2(h)^2 + 9 + 8(h) + 2(2h)^2 + 9 + 8(2h) + \cdots + 2\{(n-1)h\}^2 + 9 + 8\{(n-1)h\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[9 + 2h^2 + 9 + 8h + 2h^2(2)^2 + 9 + 8(2h) + \cdots + 2h^2(n-1)^2 + 9 + 8h(n-1)]$$

In this series, 9 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[9 \times n + 2h^2 + 8h + 2h^2(2)^2 + 8(2h) + \cdots + 2h^2(n-1)^2 + 8h(n-1)]$$

Now take $2h^2$ and 4h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[9n + 2h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 8h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{\begin{array}{l}\because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}\ ; \\ \qquad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \dfrac{n(n-1)}{2}\end{array}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[9n + 2h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 8h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{1}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{1}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[9n + 2\left(\frac{1}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{8(1)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[9n + \frac{2}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{8}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[9n + \frac{(n-1)(2n-1)}{3n} + 4(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[9 + \frac{(n-1)(2n-1)}{3n \times n} + \frac{4}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[9 + \frac{1}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 4\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[9 + \frac{1}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 4\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 9 + \frac{1}{3}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + 4\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 9 + \frac{1}{3}(1-0)(2-0) + 4(1-0)$$

$$\Rightarrow I = 9 + \frac{1}{3} \times 1 \times 2 + 4$$

$$\Rightarrow I = 13 + \frac{2}{3}$$

$$\Rightarrow I = \frac{39+2}{3}$$

$$\Rightarrow I = \frac{41}{3}$$

Hence, the value of $\int_2^3 (2x^2+1)\,dx = \frac{41}{3}$

11. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^2 \left(x^2 - 1\right) dx$$

Answer

To find: $\int_1^2 (x^2 - 1)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 2

Therefore,

$$h = \frac{2-1}{n}$$

$\Rightarrow\ h = \frac{1}{n}$

Let,

$$I = \int_1^2 (x^2 - 1)\, dx$$

Here, f(x) = x^2 – 1 and a = 1

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

Now, by putting x = 1 in f(x) we get,

f(1) = 1^2 – 1 = 1 – 1 = 0

f(1 + h)

= $(1 + h)^2$ – 1

= $h^2 + 1^2$ + 2(h)(1) – 1

= h^2 + 2(h)

Similarly, f(1 + 2h)

= $(1 + 2h)^2$ – 1

= $(2h)^2 + 1^2$ + 2(2h)(1) – 1

= $(2h)^2$ + 2(2h)

{∵ $(x + y)^2 = x^2 + y^2 + 2xy$}

$$\Rightarrow I = \lim_{h\to 0} h[0 + h^2 + 2(h) + (2h)^2 + 2(2h) \ldots\ldots + \{(n-1)h\}^2 + 2\{(n-1)h\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[h^2 + 2h + h^2(2)^2 + 2(2h) \ldots\ldots + h^2(n-1)^2 + 2h(n-1)]$$

Now take h^2 and 2h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 2h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 \ldots\ldots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \quad \sum_{i=1}^{n-1} i = 1 + 2 \ldots\ldots + (n-1) = \frac{n(n-1)}{2} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 2h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{1}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{1}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2(1)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\frac{1}{n}\left[\frac{(n-1)(2n-1)}{6n}+(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[\frac{(n-1)(2n-1)}{6n\times n}+\frac{1}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)+\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = \frac{1}{6}\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right)+\left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = \frac{1}{6}(1-0)(2-0)+(1-0)$$

$$\Rightarrow I = \frac{1}{6}\times 1\times 2+1$$

$$\Rightarrow I = 1+\frac{1}{3}$$

$$\Rightarrow I = \frac{3+1}{3}$$

$$\Rightarrow I = \frac{4}{3}$$

Hence, the value of $\int_1^2 (x^2-1)\,dx = \frac{4}{3}$

12. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 \left(x^2+4\right)\,dx$$

Answer

To find: $\int_0^2 (x^2+4)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$\Rightarrow h = \frac{2}{n}$

Let,

$$I = \int_0^2 (x^2 + 4)\, dx$$

Here, f(x) = x^2 + 4 and a = 0

$$\Rightarrow I = \lim_{h \to 0} h[f(0) + f(0 + h) + f(0 + 2h) + \cdots + f(0 + (n-1)h)]$$

$$\Rightarrow I = \lim_{h \to 0} h\left[f(0) + f(h) + f(2h) + \cdots + f\left((n-1)h\right)\right]$$

Now, by putting x = 0 in f(x) we get,

f(0) = 0^2 + 4 = 0 + 4 = 4

f(h)

= $(h)^2$ + 4

= h^2 + 4

Similarly, f(2h)

= $(2h)^2$ + 4

$$\Rightarrow I = \lim_{h \to 0} h[4 + h^2 + 4 + (2h)^2 + 4 + \cdots + \{(n-1)h\}^2 + 4]$$

$$\Rightarrow I = \lim_{h \to 0} h[4 + h^2 + 4 + h^2(2)^2 + 4 + \cdots + h^2(n-1)^2 + 4]$$

In this series, 4 is getting added n times

$$\Rightarrow I = \lim_{h \to 0} h[4 \times n + h^2 + h^2(2)^2 + \cdots + h^2(n-1)^2]$$

Now take h^2 common in remaining series

$$\Rightarrow I = \lim_{h \to 0} h[4n + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\}]$$

$$\left\{\because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right\}$$

$$\Rightarrow I = \lim_{h \to 0} h\left[4n + h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n \to \infty} \frac{2}{n}\left[4n + \left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$\Rightarrow I = \lim_{n \to \infty} \frac{2}{n}\left[4n + \frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$\Rightarrow I = \lim_{n \to \infty} \frac{2}{n}\left[4n + \frac{2(n-1)(2n-1)}{3n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + \frac{4(n-1)(2n-1)}{3n \times n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + \frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 8 + \frac{4}{3}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 8 + \frac{4}{3}(1-0)(2-0)$$

$$\Rightarrow I = 8 + \frac{4}{3} \times 1 \times 2$$

$$\Rightarrow I = 8 + \frac{8}{3}$$

$$\Rightarrow I = \frac{24+8}{3}$$

$$\Rightarrow I = \frac{32}{3}$$

Hence, the value of $\int_0^2 (x^2+4)\,dx = \frac{32}{3}$

13. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^4 \left(x^2 - x\right)dx$$

Answer

To find: $\int_1^4 (x^2 - x)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 4

Therefore,

$$h = \frac{4-1}{n}$$

$$\Rightarrow\ h\ = \frac{3}{n}$$

Let,

$$I = \int_1^4 (x^2 - x)\,dx$$

Here, $f(x) = x^2 - x$ and $a = 1$

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

Now, by putting $x = 1$ in $f(x)$ we get,

$f(1) = 1^2 - 1 = 1 - 1 = 0$

$f(1 + h)$

$= (1 + h)^2 - (1 + h)$

$= h^2 + 1^2 + 2(h)(1) - 1 - h$

$= h^2 + 2h - h$

$= h^2 + h$

Similarly, $f(1 + 2h)$

$= (1 + 2h)^2 - (1 + 2h)$

$= (2h)^2 + 1^2 + 2(2h)(1) - 1 - 2h$

$= (2h)^2 + 4h - 2h$

$= (2h)^2 + 2h$

$\{\because (x + y)^2 = x^2 + y^2 + 2xy\}$

$$\Rightarrow I = \lim_{h\to 0} h[0 + h^2 + h + (2h)^2 + 2h + \cdots + \{(n-1)h\}^2 + (n-1)h]$$

$$\Rightarrow I = \lim_{h\to 0} h[h^2 + h + h^2(2)^2 + 2h + \cdots + h^2(n-1)^2 + h(n-1)]$$

Now take h^2 and h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{\begin{aligned} \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \\ \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \end{aligned}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{3}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{3}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[\left(\frac{3}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{3}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[\frac{9}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{3}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[\frac{3(n-1)(2n-1)}{2n} + \frac{3}{2}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[\frac{9(n-1)(2n-1)}{2n \times n} + \frac{9}{2n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[\frac{9}{2}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + \frac{9}{2}\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[\frac{9}{2}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{9}{2}\left(1-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = \frac{9}{2}\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right) + \frac{9}{2}\left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = \frac{9}{2}(1-0)(2-0) + \frac{9}{2}(1-0)$$

$$\Rightarrow I = \frac{9}{2} \times 1 \times 2 + \frac{9}{2}$$

$$\Rightarrow I = 9 + \frac{9}{2}$$

$$\Rightarrow I = \frac{18+9}{2}$$

$$\Rightarrow I = \frac{27}{2}$$

Hence, the value of $\int_1^4 (x^2 - x)\, dx = \frac{27}{2}$

14. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^1 \left(3x^2 + 5x\right) dx$$

Answer

To find: $\int_0^1 (3x^2 + 5x)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 1

Therefore,

$$h = \frac{1-0}{n}$$

$$\Rightarrow h = \frac{1}{n}$$

Let,

$$I = \int_0^1 (3x^2 + 5x)\, dx$$

Here, $f(x) = 3x^2 + 5x$ and $a = 0$

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = 3(0)^2 + 5(0) = 0 + 0 = 0$

f(h)

$= 3(h)^2 + 5(h)$

$= 3h^2 + 5h$

Similarly, f(2h)

$= 3(2h)^2 + 5(2h)$

$= 3h^2(2)^2 + 5h(2)$

$$\Rightarrow I = \lim_{h\to 0} h[0 + 3h^2 + 5h + 3h^2(2)^2 + 5h(2) + \cdots + 3h^2(n-1)^2 + 5h(n-1)]$$

Now take $3h^2$ and 5h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[3h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 5h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \begin{aligned} \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \\ \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \end{aligned} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[3h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 5h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{1}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{1}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[3 \times \left(\frac{1}{n}\right)^2 \left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{5(1)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[\frac{3}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{5}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[\frac{(n-1)(2n-1)}{2n} + \frac{5}{2}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{(n-1)(2n-1)}{2n \times n} + \frac{5}{2n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{1}{2}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + \frac{5}{2}\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{1}{2}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + \frac{5}{2}\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = \frac{1}{2}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + \frac{5}{2}\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = \frac{1}{2}(1-0)(2-0) + \frac{5}{2}(1-0)$$

$$\Rightarrow I = \frac{1}{2} \times 1 \times 2 + \frac{5}{2}$$

$$\Rightarrow I = 1 + \frac{5}{2}$$

$$\Rightarrow I = \frac{2+5}{2}$$

$$\Rightarrow I = \frac{7}{2}$$

Hence, the value of $\int_0^1 (3x^2 - 5x)\, dx = \frac{7}{2}$

15. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 e^x\, dx$$

Answer

To find: $\int_0^2 e^x\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$$\Rightarrow h = \frac{2}{n}$$

Let,

$$I = \int_0^2 e^x\, dx$$

Here, $f(x) = e^x$ and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = e^0 = 1$

f(h)

$= (e)^h$

$= e^h$

Similarly, f(2h)

$= e^{2h}$

$$\Rightarrow I = \lim_{h\to 0} h\left[1 + e^h + e^{2h} + \cdots + e^{(n-1)h}\right]$$

This is G.P. (Geometric Progression) of n terms whose first term(a) is 1

and common ratio(r) $= \frac{e^h}{1} = e^h$

Sum of n terms of a G.P. is given by,

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

Therefore,

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{1\{(e^h)^n - 1)}{e^h - 1}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{e^{nh} - 1}{e^h - 1}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{e^2 - 1}{e^h - 1}\right]$$

$$\left\{\because h = \frac{2}{n} \Rightarrow nh = 2\right\}$$

$$\Rightarrow I = (e^2 - 1)\lim_{h\to 0}\left[\frac{1}{\frac{e^h - 1}{h}}\right]$$

$$\Rightarrow I = (e^2 - 1)\frac{1}{\lim_{h\to 0}\frac{e^h - 1}{h}}$$

$$\left\{\because \lim_{x\to 0}\frac{e^x - 1}{x} = 1\right\}$$

$$\Rightarrow I = e^2 - 1$$

Hence, the value of $\int_0^2 e^x\,dx = e^2 - 1$

16. Question

Evaluate the following integrals as a limit of sums:

$$\int_a^b e^x\ dx$$

Answer

To find: $\int_a^b e^x\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

$$\Rightarrow nh = b - a$$

Let,

$$I = \int_a^b e^x\,dx$$

Here, f(x) = e^x

$$\Rightarrow I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)]$$

Now, by putting x = a in f(x) we get,

f(a) = e^a

f(a + h)

= $(e)^{a+h}$

= e^{a+h}

Similarly, f(a + 2h)

= e^{a+2h}

$$\Rightarrow I = \lim_{h\to 0} h\left[e^a + e^{a+h} + e^{a+2h} + \cdots + e^{a+(n-1)h}\right]$$

This is G.P. (Geometric Progression) of n terms whose first term(a) is 1

and common ratio(r) $= \frac{e^{a+h}}{e^a} = \frac{e^a \times e^h}{e^a} = e^h$

Sum of n terms of a G.P. is given by,

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

Therefore,

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{e^a\{(e^h)^n - 1)}{e^h - 1}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[e^a\left\{\frac{e^{nh} - 1}{e^h - 1}\right\}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{e^a(e^{b-a} - 1)}{e^h - 1}\right]$$

$$\left\{\because h = \frac{2}{n} \Rightarrow nh = 2\right\}$$

$$\Rightarrow I = e^{a}(e^{b-a} - 1)\lim_{h\to 0}\left[\frac{1}{\frac{e^{h}-1}{h}}\right]$$

$$\Rightarrow I = e^{a}\left(\frac{e^{b}}{e^{a}} - 1\right)\frac{1}{\lim_{h\to 0}\frac{e^{h}-1}{h}}$$

$$\left\{\because \lim_{x\to 0}\frac{e^{x}-1}{x} = 1\right\}$$

$$\Rightarrow I = e^{b} - e^{a}$$

Hence, the value of $\int_a^b e^{x}\,dx = e^{b} - e^{a}$

17. Question

Evaluate the following integrals as a limit of sums:

$$\int_a^b \cos x\,dx$$

Answer

To find: $\int_a^b \cos x\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

$$\Rightarrow nh = b - a$$

Let,

$$I = \int_a^b \cos x\,dx$$

Here, f(x) = cos x

$$\Rightarrow I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)]$$

Now, by putting x = a in f(x) we get,

f(a) = cos a

f(a + h)

= cos (a + h)

Similarly, f(a + 2h)

= cos (a + 2h)

$$\Rightarrow I = \lim_{h\to 0} h[\cos a + \cos(a+h) + \cos(a+2h) + \cdots + \cos\{a+(n-1)h\}]$$

We know,

$$\cos A + \cos(A+B) + \cos(A+2B) + \cdots + \cos\{A+(n-1)B\} = \frac{\cos\left\{A + \frac{(n-1)B}{2}\right\}\sin\frac{nB}{2}}{\sin\frac{B}{2}}$$

Therefore,

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{a + \frac{(n-1)h}{2}\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{a + \frac{nh}{2} - \frac{h}{2}\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

{∵ nh = b – a}

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{a + \frac{b-a}{2} - \frac{h}{2}\right\}\sin\frac{b-a}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{a + \frac{b}{2} - \frac{a}{2} - \frac{h}{2}\right\}\sin\frac{b-a}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{\frac{a}{2} + \frac{b}{2} - \frac{h}{2}\right\}\sin\frac{b-a}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \sin\frac{b-a}{2} \times \lim_{h\to 0}\left[\frac{2 \times \frac{h}{2} \times \cos\left\{\frac{a}{2} + \frac{b}{2} - \frac{h}{2}\right\}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = 2\sin\frac{b-a}{2} \times \lim_{h\to 0}\left[\frac{\cos\left\{\frac{a+b}{2} - \frac{h}{2}\right\}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}}\right]$$

$$\Rightarrow I = 2\sin\frac{b-a}{2} \times \frac{\lim_{h\to 0}\left[\cos\left\{\frac{a+b}{2} - \frac{h}{2}\right\}\right]}{\lim_{h\to 0}\left[\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]}$$

$$\text{As, } h \to 0 \Rightarrow \frac{h}{2} \to 0$$

$$\Rightarrow I = 2\sin\frac{b-a}{2} \times \frac{\lim_{\frac{h}{2}\to 0}\left[\cos\left\{\frac{a+b}{2} - \frac{h}{2}\right\}\right]}{\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]}$$

$$\left\{\because \lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 1\right\}$$

$$\Rightarrow I = 2\sin\frac{b-a}{2} \times \frac{\cos\left\{\frac{a+b}{2} - 0\right\}}{1}$$

$$\Rightarrow I = 2\cos\frac{a+b}{2}\sin\frac{b-a}{2}$$

{∵ 2 cos A sin B = sin(A + B) – sin(A – B)}

$$\Rightarrow I = \sin\left\{\frac{a+b}{2} + \frac{b-a}{2}\right\} - \sin\left\{\frac{a+b}{2} - \frac{b-a}{2}\right\}$$

$$\Rightarrow I = \sin\left\{\frac{a+b+b-a}{2}\right\} - \sin\left\{\frac{a+b-b+a}{2}\right\}$$

$$\Rightarrow I = \sin\left\{\frac{2b}{2}\right\} - \sin\left\{\frac{2a}{2}\right\}$$

$$\Rightarrow I = \sin b - \sin a$$

Hence, the value of $\int_a^b \cos x\, dx = \sin b - \sin a$

18. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^{\pi/2} \sin x\, dx$$

Answer

To find: $\int_0^{\frac{\pi}{2}} \sin x\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, $a = 0$ and $b = \frac{\pi}{2}$

$$\Rightarrow h = \frac{\frac{\pi}{2} - 0}{n}$$

$$\Rightarrow h = \frac{\pi}{2n}$$

$$\Rightarrow nh = \frac{\pi}{2}$$

Let,

$$I = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

Here, f(x) = sin x and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + \cdots + f\left((n-1)h\right)\right]$$

Now, by putting x = 0 in f(x) we get,

f(0) = sin 0

f(h)

= sin h

Similarly, f(2h)

= sin 2h

$$\Rightarrow I = \lim_{h\to 0} h[\sin 0 + \sin h + \sin 2h + \cdots + \sin(n-1)h]$$

We know,

$$\sin A + \sin(A + B) + \sin(A + 2B) + \cdots + \sin\{A + (n-1)B\} = \frac{\sin\left\{A + \frac{(n-1)B}{2}\right\}\sin\frac{nB}{2}}{\sin\frac{B}{2}}$$

Here A = 0 and B = h

Therefore,

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\sin\left\{0 + \frac{(n-1)h}{2}\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\sin\left\{\frac{nh}{2} - \frac{h}{2}\right\}\sin\frac{nh}{2}}{\sin\frac{h}{2}}\right]$$

$$\left\{\because\ nh = \frac{\pi}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\sin\left\{\frac{\frac{\pi}{2}}{2} - \frac{h}{2}\right\}\sin\frac{\frac{\pi}{2}}{2}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\sin\left\{\frac{\pi}{4} - \frac{h}{2}\right\}\sin\frac{\pi}{4}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\sin\left\{\frac{\pi}{4} - \frac{h}{2}\right\}\sin\frac{\pi}{4}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \sin\frac{\pi}{4} \times \lim_{h\to 0}\left[\frac{2 \times \frac{h}{2} \times \sin\left\{\frac{\pi}{4} - \frac{h}{2}\right\}}{\sin\frac{h}{2}}\right]$$

$$\left\{\because \sin\frac{\pi}{4} = \sin 45° = \frac{1}{\sqrt{2}}\right\}$$

$$\Rightarrow I = 2 \times \frac{1}{\sqrt{2}} \times \lim_{h\to 0}\left[\frac{\sin\left\{\frac{\pi}{4} - \frac{h}{2}\right\}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}}\right]$$

As, $h \to 0 \Rightarrow \frac{h}{2} \to 0$

$$\Rightarrow I = \sqrt{2} \times \frac{\lim_{\frac{h}{2}\to 0}\left[\sin\left\{\frac{\pi}{4} - \frac{h}{2}\right\}\right]}{\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]}$$

$$\left\{\because \lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 1\right\}$$

$$\Rightarrow I = \sqrt{2} \times \frac{\sin\left\{\frac{\pi}{4} - 0\right\}}{1}$$

$$\Rightarrow I = \sqrt{2} \times \sin\frac{\pi}{4}$$

$$\left\{\because \sin\frac{\pi}{4} = \sin 45° = \frac{1}{\sqrt{2}}\right\}$$

$$\Rightarrow I = \sqrt{2} \times \frac{1}{\sqrt{2}}$$

$$\Rightarrow I = 1$$

Hence, the value of $\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$

19. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^{\pi/2} \cos x \, dx$$

Answer

To find: $\int_0^{\frac{\pi}{2}} \cos x \, dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, $a = 0$ and $b = \frac{\pi}{2}$

$$\Rightarrow h = \frac{\frac{\pi}{2} - 0}{n}$$

$$\Rightarrow h = \frac{\pi}{2n}$$

$$\Rightarrow nh = \frac{\pi}{2}$$

Let,

$$I = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

Here, f(x) = cos x and a = 0

$$\Rightarrow I = \lim_{h \to 0} h[f(0) + f(0 + h) + f(0 + 2h) + \cdots + f(0 + (n - 1)h)]$$

$$\Rightarrow I = \lim_{h \to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n - 1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

f(0) = cos 0

f(h)

= cos h

Similarly, f(2h)

= cos 2h

$$\Rightarrow I = \lim_{h \to 0} h[\cos 0 + \cos h + \cos 2h + \cdots + \cos(n - 1)h]$$

We know,

$$\cos A + \cos(A + B) + \cos(A + 2B) + \cdots + \cos\{A + (n - 1)B\} = \frac{\cos\left\{A + \frac{(n-1)B}{2}\right\} \sin \frac{nB}{2}}{\sin \frac{B}{2}}$$

Here A = 0 and B = h

Therefore,

$$\Rightarrow I = \lim_{h \to 0} h \left[\frac{\cos\left\{0 + \frac{(n-1)h}{2}\right\} \sin \frac{nh}{2}}{\sin \frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h \to 0} h \left[\frac{\cos\left\{\frac{nh}{2} - \frac{h}{2}\right\} \sin \frac{nh}{2}}{\sin \frac{h}{2}}\right]$$

$$\left\{\because \; nh = \frac{\pi}{2}\right\}$$

$$\Rightarrow I = \lim_{h \to 0} h \left[\frac{\cos\left\{\frac{\frac{\pi}{2}}{2} - \frac{h}{2}\right\} \sin \frac{\frac{\pi}{2}}{2}}{\sin \frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{\frac{\pi}{4}-\frac{h}{2}\right\}\sin\frac{\pi}{4}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{\cos\left\{\frac{\pi}{4}-\frac{h}{2}\right\}\sin\frac{\pi}{4}}{\sin\frac{h}{2}}\right]$$

$$\Rightarrow I = \sin\frac{\pi}{4}\times\lim_{h\to 0}\left[\frac{2\times\frac{h}{2}\times\cos\left\{\frac{\pi}{4}-\frac{h}{2}\right\}}{\sin\frac{h}{2}}\right]$$

$$\left\{\because \sin\frac{\pi}{4} = \sin 45° = \frac{1}{\sqrt{2}}\right\}$$

$$\Rightarrow I = 2\times\frac{1}{\sqrt{2}}\times\lim_{h\to 0}\left[\frac{\cos\left\{\frac{\pi}{4}-\frac{h}{2}\right\}}{\frac{\sin\frac{h}{2}}{\frac{h}{2}}}\right]$$

$$\text{As, } h \to 0 \Rightarrow \frac{h}{2}\to 0$$

$$\Rightarrow I = \sqrt{2}\times\frac{\lim_{\frac{h}{2}\to 0}\left[\cos\left\{\frac{\pi}{4}-\frac{h}{2}\right\}\right]}{\lim_{\frac{h}{2}\to 0}\left[\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right]}$$

$$\left\{\because \lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 1\right\}$$

$$\Rightarrow I = \sqrt{2}\times\frac{\cos\left\{\frac{\pi}{4}-0\right\}}{1}$$

$$\Rightarrow I = \sqrt{2}\times\cos\frac{\pi}{4}$$

$$\left\{\because \cos\frac{\pi}{4} = \cos 45° = \frac{1}{\sqrt{2}}\right\}$$

$$\Rightarrow I = \sqrt{2}\times\frac{1}{\sqrt{2}}$$

$$\Rightarrow I = 1$$

Hence, the value of $\int_0^{\frac{\pi}{2}}\cos x\,dx = 1$

20. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^4\left(3x^2+2x\right)dx$$

Answer

To find: $\int_1^4 (3x^2 + 2x)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 4

Therefore,

$$h = \frac{4-1}{n}$$

$$\Rightarrow h = \frac{3}{n}$$

Let,

$$I = \int_1^4 (3x^2 + 2x)\,dx$$

Here, $f(x) = 3x^2 + 2x$ and a = 1

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

Now, by putting x = 1 in f(x) we get,

$f(1) = 3(1)^2 + 2(1) = 3 + 2 = 5$

f(1 + h)

$= 3(1 + h)^2 + 2(1 + h)$

$= 3\{h^2 + 1^2 + 2(h)(1)\} + 2 + 2h$

$= 3h^2 + 3 + 6h + 2 + 2h$

$= 3h^2 + 8h + 5$

Similarly, f(1 + 2h)

$= 3(1 + 2h)^2 + 2(1 + 2h)$

$= 3\{(2h)^2 + 1^2 + 2(2h)(1)\} + 2 + 4h$

$= 3(2h)^2 + 3 + 6(2h) + 2 + 2(2h)$

$= 3(2h)^2 + 8(2h) + 5$

$\{\because (x + y)^2 = x^2 + y^2 + 2xy\}$

$$\Rightarrow I = \lim_{h\to 0} h[5 + 3h^2 + 8h + 5 + 3(2h)^2 + 8(2h) + 5 + \cdots + 3\{(n-1)h\}^2 + 8(n-1)h + 5]$$

$$\Rightarrow I = \lim_{h\to 0} h[5 + 3h^2 + 8h + 5 + 3h^2(2)^2 + 8h(2) + 5 + \cdots + 3h^2(n-1)^2 + 8h(n-1) + 5]$$

Since 5 is repeating n times in series

$$\Rightarrow I = \lim_{h\to 0} h[5n + 3h^2 + 8h + 3h^2(2)^2 + 8h(2) + \cdots + 3h^2(n-1)^2 + 8h(n-1)]$$

Now take $3h^2$ and 8h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[5n + 3h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 8h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}; \quad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[5n + 3h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 8h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{3}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{3}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[5n + 3\left(\frac{3}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{3(8)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[5n + \frac{27}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{24}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[5n + \frac{9(n-1)(2n-1)}{2n} + 12(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[15 + \frac{27(n-1)(2n-1)}{2n \times n} + \frac{36}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[15 + \frac{27}{2}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 36\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[15 + \frac{27}{2}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 36\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 15 + \frac{27}{2}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + 36\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 15 + \frac{27}{2}(1-0)(2-0) + 36(1-0)$$

$$\Rightarrow I = 15 + \frac{27}{2} \times 1 \times 2 + 36$$

$$\Rightarrow I = 15 + 27 + 36$$

$$\Rightarrow I = 78$$

Hence, the value of $\int_1^4 (3x^2 + 2x)\, dx = 78$

21. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 (3x^2 - 2)dx$$

Answer

To find: $\int_0^2 (3x^2 - 2)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$$\Rightarrow\ h\ = \frac{2}{n}$$

Let,

$$I = \int_0^2 (3x^2 - 2)\, dx$$

Here, $f(x) = 3x^2 - 2$ and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = 3(0)^2 - 2 = 0 - 2 = -2$

f(h)

$= 3(h)^2 - 2$

Similarly, f(2h)

$= 3(2h)^2 - 2$

$$\Rightarrow I = \lim_{h\to 0} h[-2 + 3h^2 - 2 + 3(2h)^2 - 2 + \cdots + 3\{(n-1)h\}^2 - 2]$$

$$\Rightarrow I = \lim_{h\to 0} h[-2 + 3h^2 - 2 + 3h^2(2)^2 - 2 + \cdots + 3h^2(n-1)^2 - 2]$$

Since -2 is repeating n times in series

$$\Rightarrow I = \lim_{h\to 0} h[-2n + 3h^2 + 3h^2(2)^2 \ldots + 3h^2(n-1)^2]$$

Now take $3h^2$ common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[-2n + 3h^2\{1^2 + 2^2 + \cdots + (n-1)^2\}]$$

$$\left\{\because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[-2n + 3h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[-2n + 3\left(\frac{2}{n}\right)^2 \left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[-2n + \frac{12}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[-2n + \frac{2(n-1)(2n-1)}{n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[-4 + \frac{4(n-1)(2n-1)}{n \times n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[-4 + 4\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[-4 + 4\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = -4 + 4\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = -4 + 4(1-0)(2-0)$$

$$\Rightarrow I = -4 + 4 \times 1 \times 2$$

$$\Rightarrow I = -4 + 8$$

$$\Rightarrow I = 4$$

Hence, the value of $\int_0^2 (3x^2 - 2)\, dx = 4$

22. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 \left(x^2 + 2\right) dx$$

Answer

To find: $\int_0^2 (x^2 + 2)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h=\frac{2-0}{n}$$

$$\Rightarrow\ h\ =\frac{2}{n}$$

Let,

$$I=\int_0^2 (x^2+2)\,dx$$

Here, f(x) = x^2 + 2 and a = 0

$$\Rightarrow I=\lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\cdots+f(0+(n-1)h)]$$

$$\Rightarrow I=\lim_{h\to 0} h\big[f(0)+f(h)+f(2h)+\cdots+f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

f(0) = $(0)^2$ + 2 = 0 + 2 = 2

f(h)

= $(h)^2$ + 2

Similarly, f(2h)

= $(2h)^2$ + 2

$$\Rightarrow I=\lim_{h\to 0} h[2+h^2+2+(2h)^2+2+\cdots+\{(n-1)h\}^2+2]$$

$$\Rightarrow I=\lim_{h\to 0} h[2+h^2+2+h^2(2)^2+2+\cdots+h^2(n-1)^2+2]$$

Since 2 is repeating n times in series

$$\Rightarrow I=\lim_{h\to 0} h[2n+h^2+h^2(2)^2+\cdots+h^2(n-1)^2]$$

Now take h^2 common in remaining series

$$\Rightarrow I=\lim_{h\to 0} h[2n+h^2\{1^2+2^2+\cdots+(n-1)^2\}]$$

$$\left\{\because \sum_{i=1}^{n-1} i^2=1^2+2^2+\cdots+(n-1)^2=\frac{n(n-1)(2n-1)}{6}\right\}$$

$$\Rightarrow I=\lim_{h\to 0} h\left[2n+h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

Put,

$$h=\frac{2}{n}$$

Since,

$$h\to 0 \text{ and } h=\frac{2}{n}\Rightarrow n\to\infty$$

$$\Rightarrow I=\lim_{n\to\infty}\frac{2}{n}\left[2n+\left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$\Rightarrow I=\lim_{n\to\infty}\frac{2}{n}\left[2n+\frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[2n + \frac{2(n-1)(2n-1)}{3n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{4(n-1)(2n-1)}{3n \times n}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 4 + \frac{4}{3}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 4 + \frac{4}{3}(1-0)(2-0)$$

$$\Rightarrow I = 4 + \frac{4}{3} \times 1 \times 2$$

$$\Rightarrow I = 4 + \frac{8}{3}$$

$$\Rightarrow I = \frac{12+8}{3}$$

$$\Rightarrow I = \frac{20}{3}$$

Hence, the value of $\int_0^2 (x^2 + 2)\, dx = \frac{20}{3}$

23. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^4 \left(x + e^{2x}\right) dx$$

Answer

To find: $\int_0^4 (x + e^{2x})\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 4

Therefore,

$$h = \frac{4-0}{n}$$

$$\Rightarrow h = \frac{4}{n}$$

Let,

$$I = \int_0^4 (x + e^{2x})\,dx$$

Here, $f(x) = x + e^{2x}$ and $a = 0$

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = 0 + e^{2(0)} = 0 + e^0 = 0 + 1 = 1$

f(h)

$= h + (e)^{2h}$

$= h + e^{2h}$

Similarly, f(2h)

$= 2h + (e)^{2(2h)}$

$= 2h + e^{4h}$

$$\Rightarrow I = \lim_{h\to 0} h\big[1 + h + e^{2h} + 2h + e^{4h} + \cdots + (n-1)h + e^{2(n-1)h}\big]$$

Take h common in some of the terms of series

$$\Rightarrow I = \lim_{h\to 0} h\big[1 + e^{2h} + e^{4h} + \cdots + e^{2(n-1)h} + h\{1 + 2 + \cdots + (n-1)\}\big]$$

This is G.P. (Geometric Progression) of n terms whose first term(a) is 1

and common ratio$(r) = \frac{e^{2h}}{1} = e^{2h}$

Sum of n terms of a G.P. is given by,

$$S_n = \frac{a(r^n - 1)}{r - 1}, r > 1$$

and

$$\sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

Therefore,

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{1\{(e^{2h})^n - 1)}{e^{2h} - 1} + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{h\to 0} h\left[\frac{e^{2nh} - 1}{e^{2h} - 1} + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{h\to 0}\left[\frac{2h}{2} \times \frac{e^8 - 1}{e^{2h} - 1} + h^2\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\left\{\because h = \frac{4}{n} \Rightarrow nh = 4\right\}$$

$$\Rightarrow I = \frac{e^8 - 1}{2}\lim_{h\to 0}\left[\frac{1}{\frac{e^{2h} - 1}{2h}}\right] + \lim_{h\to 0}\left[\left(\frac{4}{n}\right)^2 \times \left\{\frac{n(n-1)}{2}\right\}\right]$$

As, $h \to 0 \Rightarrow 2h \to 0$ and $\frac{4}{n} \to 0 \Rightarrow n \to \infty$

$$\Rightarrow I = \frac{e^8 - 1}{2} \times \frac{1}{\lim_{2h \to 0} \frac{e^{2h} - 1}{2h}} + \lim_{n \to \infty} \left[\frac{16}{n^2} \times \left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\left\{\because \lim_{x \to 0} \frac{e^x - 1}{x} = 1\right\}$$

$$\Rightarrow I = \frac{e^8 - 1}{2} \times \frac{1}{1} + \lim_{n \to \infty} \left[8 \times \left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = \frac{e^8 - 1}{2} + 8 \times \left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = \frac{e^8 - 1}{2} + 8 \times (1 - 0)$$

$$\Rightarrow I = \frac{e^8 - 1 + 16}{2}$$

$$\Rightarrow I = \frac{e^8 + 15}{2}$$

Hence, the value of $\int_0^4 (x + e^x)\, dx = \frac{e^8 + 15}{2}$

24. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 \left(x^2 + x\right) dx$$

Answer

To find: $\int_0^2 (x^2 + x)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h[f(a) + f(a + h) + f(a + 2h) + \cdots + f(a + (n - 1)h)],$$

where,

$$h = \frac{b - a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2 - 0}{n}$$

$$\Rightarrow h = \frac{2}{n}$$

Let,

$$I = \int_0^2 (x^2 + x)\, dx$$

Here, $f(x) = x^2 + x$ and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = 0^2 + 0 = 0 + 0 = 0$

f(h)

$= (h)^2 + (h)$

$= h^2 + h$

Similarly, f(2h)

$= (2h)^2 + (2h)$

$$\Rightarrow I = \lim_{h\to 0} h[h^2 + h + (2h)^2 + 2h + \cdots + \{(n-1)h\}^2 + (n-1)h]$$

$$\Rightarrow I = \lim_{h\to 0} h[h^2 + h + h^2(2)^2 + h(2) + \cdots + h^2(n-1)^2 + h(n-1)]$$

Now take h^2 and h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \begin{array}{l} \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}\,; \\ \qquad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \dfrac{n(n-1)}{2} \end{array} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[\left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[\frac{2(n-1)(2n-1)}{3n} + (n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{4(n-1)(2n-1)}{3n \times n} + \frac{2}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 2\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 2\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = \frac{4}{3}\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right)+2\left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = \frac{4}{3}(1-0)(2-0)+2(1-0)$$

$$\Rightarrow I = \frac{4}{3}\times 1\times 2+2$$

$$\Rightarrow I = \frac{8}{3}+2$$

$$\Rightarrow I = \frac{8+6}{3}$$

$$\Rightarrow I = \frac{14}{3}$$

Hence, the value of $\int_0^2 (x^2+x)\,dx = \frac{14}{3}$

25. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 \left(x^2+2x+1\right)dx$$

Answer

To find: $\int_0^2 (x^2+2x+1)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+\cdots+f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$$\Rightarrow\ h\ = \frac{2}{n}$$

Let,

$$I = \int_0^2 (x^2+2x+1)\,dx$$

Here, $f(x) = x^2 + 2x + 1$ and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0)+f(0+h)+f(0+2h)+\cdots+f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0)+f(h)+f(2h)+\cdots+f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = 0^2 + 2(0) + 1 = 0 + 0 + 1 = 1$

$f(h)$

$= (h)^2 + 2(h) + 1$

$= h^2 + 2h + 1$

Similarly, $f(2h)$

$= (2h)^2 + 2(2h) + 1$

$$\Rightarrow I = \lim_{h\to 0} h[1 + h^2 + 2h + 1 + (2h)^2 + 2(2h) + 1 + \cdots + \{(n-1)h\}^2 + 2(n-1)h + 1]$$

$$\Rightarrow I = \lim_{h\to 0} h[1 + h^2 + 2h(1) + 1 + h^2(2)^2 + 2h(2) + 1 + \cdots + h^2(n-1)^2 + 2h(n-1) + 1]$$

Since 1 is repeating n times in the series

$$\Rightarrow I = \lim_{h\to 0} h[1 \times n + h^2 + 2h(1) + h^2(2)^2 + 2h(2) + \cdots + h^2(n-1)^2 + 2h(n-1)]$$

Now take h^2 and 2h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[n + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 2h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} ; \quad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[n + h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 2h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[n + \left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2(2)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[n + \frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2(2)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[n + \frac{2(n-1)(2n-1)}{3n} + 2(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[2 + \frac{4(n-1)(2n-1)}{3n \times n} + \frac{4}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[2 + \frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 4\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[2 + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 4\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 2 + \frac{4}{3}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + 4\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 2 + \frac{4}{3}(1-0)(2-0) + 4(1-0)$$

$$\Rightarrow I = 2 + \frac{4}{3} \times 1 \times 2 + 4$$

$$\Rightarrow I = 6 + \frac{8}{3}$$

$$\Rightarrow I = \frac{18+8}{3}$$

$$\Rightarrow I = \frac{26}{3}$$

Hence, the value of $\int_0^2 (x^2 + 2x + 1)\,dx = \frac{26}{3}$

26. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^3 \left(2x^2 + 3x + 5\right)dx$$

Answer

To find: $\int_0^3 (2x^2 + 3x + 5)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a + (n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 3

Therefore,

$$h = \frac{3-0}{n}$$

$$\Rightarrow \ h \ = \frac{3}{n}$$

Let,

$$I = \int_0^3 (2x^2 + 3x + 5)\,dx$$

Here, $f(x) = 2x^2 + 3x + 5$ and a = 0

$$\Rightarrow I = \lim_{h \to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0 + (n-1)h)]$$

$$\Rightarrow I = \lim_{h \to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

$f(0) = 2(0)^2 + 3(0) + 5 = 0 + 0 + 5 = 5$

$f(h)$

$= 2(h)^2 + 3(h) + 5$

$= 2h^2 + 3h + 5$

Similarly, $f(2h)$

$= 2(2h)^2 + 3(2h) + 5$

$$\Rightarrow I = \lim_{h\to 0} h[5 + 2h^2 + 3h + 5 + 2(2h)^2 + 3(2h) + 5 + \cdots + 2\{(n-1)h\}^2 + 3(n-1)h + 5]$$

$$\Rightarrow I = \lim_{h\to 0} h[5 + 2h^2(1)^2 + 3h(1) + 5 + 2h^2(2)^2 + 3h(2) + 5 + \cdots + 2h^2(n-1)^2 + 3h(n-1) + 5]$$

Since 5 is repeating n times in the series

$$\Rightarrow I = \lim_{h\to 0} h[5 \times n + 2h^2(1)^2 + 3h(1) + 2h^2(2)^2 + 3h(2) + \cdots + 2h^2(n-1)^2 + 3h(n-1)]$$

Now take h^2 and 2h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[5n + 2h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 3h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6} ; \quad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[5n + 2h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 3h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{3}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{3}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[5n + 2\left(\frac{3}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{3(3)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[5n + \frac{2(9)}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{9}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{3}{n}\left[5n + \frac{18(n-1)(2n-1)}{6n} + \frac{9}{2}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[15 + \frac{54(n-1)(2n-1)}{6n \times n} + \frac{27}{2n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[15 + 9\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + \frac{27}{2}\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[15 + 9\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + \frac{27}{2}\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 15 + 9\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + \frac{27}{2}\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 15 + 9(1-0)(2-0) + \frac{27}{2}(1-0)$$

$$\Rightarrow I = 15 + 9 \times 1 \times 2 + \frac{27}{2}$$

$$\Rightarrow I = 15 + 18 + \frac{27}{2}$$

$$\Rightarrow I = 33 + \frac{27}{2}$$

$$\Rightarrow I = \frac{66 + 27}{2}$$

$$\Rightarrow I = \frac{93}{2}$$

Hence, the value of $\int_0^3 (2x^2 + 3x + 5)\,dx = \frac{93}{2}$

27. Question

Evaluate the following integrals as a limit of sums:

$$\int_a^b x\ dx$$

Answer

To find: $\int_a^b x\,dx$

We know,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Let,

$$I = \int_a^b x\,dx$$

Here, f(x) = x

$$\Rightarrow I = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)]$$

Now, By putting x = a in f(x) we get,

f(a) = a

Similarly, f(a + h) = a + h

$$\Rightarrow I = \lim_{h\to 0} h[a + a + h + a + 2h + \cdots a + +(n-1)h]$$

In this series, a is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[an + h + 2h + \cdots + (n-1)h]$$

Now take h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[an + h(1 + 2 + \cdots + (n-1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[an + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{b-a}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{b-a}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{b-a}{n}\left[an + \frac{b-a}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{b-a}{n}\left[an + \left(\frac{b-a}{2}\right)(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{a(b-a) + \left(\frac{b-a}{2}\right)\frac{(b-a)(n-1)}{n}\right\}$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{a(b-a) + \left(\frac{(b-a)^2}{2}\right)\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = a(b-a) + \left(\frac{(b-a)^2}{2}\right)\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = a(b-a) + \left(\frac{(b-a)^2}{2}\right)(1-0)$$

$$\Rightarrow I = a(b-a) + \frac{(b-a)^2}{2}$$

$$\Rightarrow I = (b-a)\left(a + \frac{b-a}{2}\right)$$

$$\Rightarrow I = (b-a)\left(\frac{2a+b-a}{2}\right)$$

$$\Rightarrow I = (b-a)\left(\frac{b+a}{2}\right)$$

$$\Rightarrow I = \frac{b^2-a^2}{2}$$

Hence, the value of $\int_a^b x\,dx = \frac{b^2-a^2}{2}$

28. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^5 (x+1)dx$$

Answer

To find: $\int_0^5 (x+1)\,dx$

We know,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 5

Therefore,

$$h = \frac{5-0}{n}$$

$$\Rightarrow\ h\ = \frac{5}{n}$$

Let,

$$I = \int_0^5 (x+1)\,dx$$

Here, f(x) = x + 1 and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0+(n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, By putting x = 0 in f(x) we get,

f(0) = 0 + 1 = 1

Similarly, f(h) = h + 1

$$\Rightarrow I = \lim_{h\to 0} h[1 + h + 1 + 2h + 1 + \cdots + (n-1)h + 1]$$

In this series, 1 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[1 \times n + h + 2h + \cdots + (n-1)h]$$

Now take h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[n + h(1 + 2 + \cdots + (n-1)]$$

$$\left\{\because \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[n + h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{5}{n}$$

Since,

$h \to 0$ and $h = \frac{5}{n} \Rightarrow n \to \infty$

$$\Rightarrow I = \lim_{n\to\infty} \frac{5}{n}\left[n + \frac{5}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{5}{n}\left[n + \frac{5(n-1)}{2}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{5 + \frac{25(n-1)}{2n}\right\}$$

$$\Rightarrow I = \lim_{n\to\infty} \left\{5 + \frac{25}{2}\left(1 - \frac{1}{n}\right)\right\}$$

$$\Rightarrow I = 5 + \frac{25}{2}\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 5 + \frac{25}{2}(1-0)$$

$$\Rightarrow I = 5 + \frac{25}{2}$$

$$\Rightarrow I = \frac{10+25}{2}$$

$$\Rightarrow I = \frac{35}{2}$$

Hence, the value of $\int_0^5 (x+1)\, dx = \frac{35}{2}$

29. Question

Evaluate the following integrals as a limit of sums:

$$\int_2^3 x^2\, dx$$

Answer

To find: $\int_2^3 x^2\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 2 and b = 3

Therefore,

$$h = \frac{3-2}{n}$$

$$\Rightarrow h = \frac{1}{n}$$

Let,

$$I = \int_2^3 x^2\, dx$$

Here, $f(x) = x^2$ and $a = 2$

$$\Rightarrow I = \lim_{h\to 0} h[f(2) + f(2+h) + f(2+2h) + \cdots + f(2+(n-1)h)]$$

Now, by putting x = 2 in f(x) we get,

$f(2) = 2^2 = 4$

$f(2 + h)$

$= (2 + h)^2$

$= h^2 + 2^2 + 2(h)(2)$

$= h^2 + 4 + 4(h)$

Similarly, $f(2 + 2h)$

$= (2 + 2h)^2$

$= (2h)^2 + 2^2 + 2(2h)(2)$

$= (2h)^2 + 4 + 4(2h)$

$\{\because (x + y)^2 = x^2 + y^2 + 2xy\}$

$$\Rightarrow I = \lim_{h\to 0} h[4 + h^2 + 4 + 4(h) + (2h)^2 + 4 + 4(2h) + \cdots + \{(n-1)h\}^2 + 4 + 4\{(n-1)h\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[4 + h^2 + 4 + 4h + h^2(2)^2 + 4 + 4(2h) + \cdots + h^2(n-1)^2 + 4 + 4h(n-1)]$$

In this series, 4 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[4 \times n + h^2 + 4h + h^2(2)^2 + 4(2h) + \cdots + h^2(n-1)^2 + 4h(n-1)]$$

Now take h^2 and 4h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[4n + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 4h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{\because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ;\quad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[4n + h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 4h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{1}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{1}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[4n + \left(\frac{1}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{4(1)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{(n-1)(2n-1)}{6n} + 2(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{(n-1)(2n-1)}{6n \times n} + \frac{2}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{1}{6}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 2\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 2\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 4 + \frac{1}{6}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + 2\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 4 + \frac{1}{6}(1-0)(2-0) + 2(1-0)$$

$$\Rightarrow I = 4 + \frac{1}{6} \times 1 \times 2 + 2$$

$$\Rightarrow I = 6 + \frac{1}{3}$$

$$\Rightarrow I = \frac{18+1}{3}$$

$$\Rightarrow I = \frac{19}{3}$$

Hence, the value of $\int_2^3 x^2\,dx = \frac{19}{3}$

30. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^3 \left(x^2 + x\right)\,dx$$

Answer

To find: $\int_1^3 (x^2 + x)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 3

Therefore,

$$h = \frac{3-1}{n}$$

$\Rightarrow \ h \ = \frac{2}{n}$

Let,

$$I = \int_1^3 (x^2 + x)\, dx$$

Here, f(x) = x^2 + x and a = 1

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

Now, by putting x = 1 in f(x) we get,

f(1) = 1^2 + 1 = 1 + 1 = 2

f(1 + h)

= $(1 + h)^2$ + (1 + h)

= h^2 + 1^2 + 2(h)(1) + 1 + h

= h^2 + 2h + h + 1 + 1

= h^2 + 3h + 2

Similarly, f(1 + 2h)

= $(1 + 2h)^2$ + (1 + 2h)

= $(2h)^2$ + 1^2 + 2(2h)(1) + 1 + 2h

= $(2h)^2$ + 4h + 2h + 1 + 1

= $(2h)^2$ + 6h + 2

{∵ $(x + y)^2 = x^2 + y^2 + 2xy$}

$$\Rightarrow I = \lim_{h\to 0} h[2 + h^2 + 3h + 2 + (2h)^2 + 6h + 2 + \cdots + \{(n-1)h\}^2 + 3(n-1)h + 2]$$

$$\Rightarrow I = \lim_{h\to 0} h[2 + h^2 + 3h(1) + 2 + h^2(2)^2 + 3h(2) + 2 + \cdots + h^2(n-1)^2 + 3h(n-1) + 2]$$

In this series, 2 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[2n + h^2 + 3h(1) + h^2(2)^2 + 3h(2) + \cdots + h^2(n-1)^2 + 3h(n-1)]$$

Now take h^2 and 3h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[2n + h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 3h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \quad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[2n + h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 3h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$h \to 0$ and $h = \frac{2}{n} \Rightarrow n \to \infty$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[2n + \left(\frac{2}{n}\right)^2 \left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2(3)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[2n + \frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{6}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[2n + \frac{2(n-1)(2n-1)}{3n} + 3(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{4(n-1)(2n-1)}{3n \times n} + \frac{6}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 6\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[4 + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) + 6\left(1 - \frac{1}{n}\right)\right]$$

$$\Rightarrow I = 4 + \frac{4}{3}\left(1 - \frac{1}{\infty}\right)\left(2 - \frac{1}{\infty}\right) + 6\left(1 - \frac{1}{\infty}\right)$$

$$\Rightarrow I = 4 + \frac{4}{3}(1-0)(2-0) + 6(1-0)$$

$$\Rightarrow I = 4 + \frac{4}{3} \times 1 \times 2 + 6$$

$$\Rightarrow I = 10 + \frac{8}{3}$$

$$\Rightarrow I = \frac{30+8}{3}$$

$$\Rightarrow I = \frac{38}{3}$$

Hence, the value of $\int_1^3 (x^2 + x)\, dx = \frac{38}{3}$

31. Question

Evaluate the following integrals as a limit of sums:

$$\int_0^2 \left(x^2 - x\right) dx$$

Answer

To find: $\int_0^2 (x^2 - x)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 0 and b = 2

Therefore,

$$h = \frac{2-0}{n}$$

$$\Rightarrow\ h\ = \frac{2}{n}$$

Let,

$$I = \int_0^2 (x^2 - x)\,dx$$

Here, f(x) = x^2 – x and a = 0

$$\Rightarrow I = \lim_{h\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0 + (n-1)h)]$$

$$\Rightarrow I = \lim_{h\to 0} h\big[f(0) + f(h) + f(2h) + \cdots + f\big((n-1)h\big)\big]$$

Now, by putting x = 0 in f(x) we get,

f(0) = 0^2 – 0 = 0 – 0 = 0

f(h)

= $(h)^2$ – (h)

= h^2 – h

Similarly, f(2h)

= $(2h)^2$ – (2h)

= $(2h)^2$ – 2h

$$\Rightarrow I = \lim_{h\to 0} h[0 + h^2 - h + (2h)^2 - 2h + \cdots + \{(n-1)h\}^2 - (n-1)h]$$

$$\Rightarrow I = \lim_{h\to 0} h[h^2 - h + h^2(2)^2 - 2h + \cdots + h^2(n-1)^2 - h(n-1)]$$

Now take h^2 and -h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} - h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \begin{array}{l} \because \displaystyle\sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \\ \qquad\quad \displaystyle\sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \end{array} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} - h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[\left(\frac{2}{n}\right)^2 \left\{\frac{n(n-1)(2n-1)}{6}\right\} - \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} - \frac{2}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[\frac{2(n-1)(2n-1)}{3n} - (n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{4(n-1)(2n-1)}{3n \times n} - \frac{2}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{4}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) - 2\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[\frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) - 2\left(1-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = \frac{4}{3}\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right) - 2\left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = \frac{4}{3}(1-0)(2-0) - 2(1-0)$$

$$\Rightarrow I = \frac{4}{3} \times 1 \times 2 - 2$$

$$\Rightarrow I = \frac{8}{3} - 2$$

$$\Rightarrow I = \frac{8-6}{3}$$

$$\Rightarrow I = \frac{2}{3}$$

Hence, the value of $\int_0^2 (x^2 - x)\, dx = \frac{2}{3}$

32. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^3 \left(2x^2 + 5x\right)dx$$

Answer

To find: $\int_1^3 (2x^2 + 5x)\, dx$

Formula used:

$$\int_a^b f(x)\, dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 3

Therefore,

$h = \frac{3-1}{n}$

$\Rightarrow\ h\ = \frac{2}{n}$

Let,

$I = \int_1^3 (2x^2 + 5x)\, dx$

Here, f(x) = $2x^2$ + 5x and a = 1

$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$

Now, by putting x = 1 in f(x) we get,

f(1) = $2(1)^2$ + 5(1) = 2 + 5 = 7

f(1 + h)

= $2(1 + h)^2$ + 5(1 + h)

= 2{h^2 + 1^2 + 2(h)(1)} + 5 + 5h

= $2h^2$ + 4h + 2 + 5 + 5h

= $2h^2$ + 9h + 7

Similarly, f(1 + 2h)

= $2(1 + 2h)^2$ + 5(1 + 2h)

= 2{$(2h)^2$ + 1^2 + 2(2h)(1)} + 5 + 10h

= $2(2h)^2$ + 2 + 8h + 5 + 10h

= $2(2h)^2$ + 18h + 7

= $2(2h)^2$ + 9(2h) + 7

{∵ $(x + y)^2 = x^2 + y^2 + 2xy$}

$$\Rightarrow I = \lim_{h\to 0} h[7 + 2h^2 + 9h + 7 + 2(2h)^2 + 9(2h) + 7 + \cdots + 2\{(n-1)h\}^2 + 9(n-1)h + 7]$$

$$\Rightarrow I = \lim_{h\to 0} h[7 + 2h^2(1) + 9h(1) + 7 + 2h^2(2)^2 + 9h(2) + 7 + \cdots + 2h^2(n-1)^2 + 9h(n-1) + 7]$$

In this series, 7 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[7n + 2h^2(1) + 9h(1) + 2h^2(2)^2 + 9h(2) + \cdots + 2h^2(n-1)^2 + 9h(n-1)]$$

Now take $2h^2$ and 9h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[7n + 2h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 9h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{ \begin{array}{l} \because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \\ \qquad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2} \end{array} \right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[7n + 2h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 9h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[7n + 2\left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{2(9)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[7n + \frac{8}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{18}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[7n + \frac{4(n-1)(2n-1)}{3n} + 9(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[14 + \frac{8(n-1)(2n-1)}{3n \times n} + \frac{18}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[14 + \frac{8}{3}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 18\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \left[14 + \frac{8}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 18\left(1-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = 14 + \frac{8}{3}\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right) + 18\left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = 14 + \frac{8}{3}(1-0)(2-0) + 18(1-0)$$

$$\Rightarrow I = 14 + \frac{8}{3} \times 1 \times 2 + 18$$

$$\Rightarrow I = 32 + \frac{16}{3}$$

$$\Rightarrow I = \frac{96+16}{3}$$

$$\Rightarrow I = \frac{112}{3}$$

Hence, the value of $\int_1^3 (2x^2 + 5x)\,dx = \frac{112}{3}$

33. Question

Evaluate the following integrals as a limit of sums:

$$\int_1^3 \left(3x^2 + 1\right)dx$$

Answer

To find: $\int_1^3 (3x^2 + 1)\,dx$

Formula used:

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f(a+(n-1)h)],$$

where,

$$h = \frac{b-a}{n}$$

Here, a = 1 and b = 3

Therefore,

$$h = \frac{3-1}{n}$$

$$\Rightarrow\ h\ = \frac{2}{n}$$

Let,

$$I = \int_1^3 (3x^2+1)\,dx$$

Here, $f(x) = 3x^2 + 1$ and a = 1

$$\Rightarrow I = \lim_{h\to 0} h[f(1) + f(1+h) + f(1+2h) + \cdots + f(1+(n-1)h)]$$

Now, by putting x = 1 in f(x) we get,

$f(1) = 3(1^2) + 1 = 3(1) + 1 = 3 + 1 = 4$

f(1 + h)

$= 3(1 + h)^2 + 1$

$= 3\{h^2 + 1^2 + 2(h)(1)\} + 1$

$= 3(h)^2 + 3 + 3(2h) + 1$

$= 3(h)^2 + 4 + 6h$

Similarly, f(1 + 2h)

$= 3(1 + 2h)^2 + 1$

$= 3\{2(2h)^2 + 1^2 + 2(2h)(1)\} + 1$

$= 3(2h)^2 + 3 + 3(4h) + 1$

$= 3(2h)^2 + 4 + 12h$

$\{\because (x + y)^2 = x^2 + y^2 + 2xy\}$

$$\Rightarrow I = \lim_{h\to 0} h[4 + 3(h)^2 + 4 + 6h + 3(2h)^2 + 4 + 12h + \cdots + 3\{(n-1)h\}^2 + 4 + 6\{(n-1)h\}]$$

$$\Rightarrow I = \lim_{h\to 0} h[4 + 3h^2(1^2) + 4 + 6h + 3h^2(2)^2 + 4 + 6(2h) + \cdots + 3h^2(n-1)^2 + 4 + 6h(n-1)]$$

In this series, 4 is getting added n times

$$\Rightarrow I = \lim_{h\to 0} h[4\times n + 3h^2(1^2) + 6h + 3h^2(2)^2 + 6(2h) + \cdots + 3h^2(n-1)^2 + 6h(n-1)]$$

Now take $3h^2$ and 6h common in remaining series

$$\Rightarrow I = \lim_{h\to 0} h[4n + 3h^2\{1^2 + 2^2 + \cdots + (n-1)^2\} + 6h\{1 + 2 + \cdots + (n-1)\}]$$

$$\left\{\begin{array}{l}\because \sum_{i=1}^{n-1} i^2 = 1^2 + 2^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}\ ; \\ \qquad \sum_{i=1}^{n-1} i = 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}\end{array}\right\}$$

$$\Rightarrow I = \lim_{h\to 0} h\left[4n + 3h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + 6h\left\{\frac{n(n-1)}{2}\right\}\right]$$

Put,

$$h = \frac{2}{n}$$

Since,

$$h \to 0 \text{ and } h = \frac{2}{n} \Rightarrow n \to \infty$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[4n + 3\left(\frac{2}{n}\right)^2\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{6(2)}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[4n + \frac{12}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{12}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$\Rightarrow I = \lim_{n\to\infty} \frac{2}{n}\left[4n + \frac{2(n-1)(2n-1)}{n} + 6(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + \frac{4(n-1)(2n-1)}{n \times n} + \frac{12}{n}(n-1)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + 4\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) + 12\left(\frac{n-1}{n}\right)\right]$$

$$\Rightarrow I = \lim_{n\to\infty}\left[8 + 4\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 12\left(1-\frac{1}{n}\right)\right]$$

$$\Rightarrow I = 8 + 4\left(1-\frac{1}{\infty}\right)\left(2-\frac{1}{\infty}\right) + 12\left(1-\frac{1}{\infty}\right)$$

$$\Rightarrow I = 8 + 4(1-0)(2-0) + 12(1-0)$$

$$\Rightarrow I = 8 + 4 \times 1 \times 2 + 12$$

$$\Rightarrow I = 20 + 8$$

$$\Rightarrow I = 28$$

Hence, the value of $\int_1^3 (3x^2 + 1)\, dx = 28$

Very short answer

1. Question

Evaluate $\int_0^{\pi/2} \sin^2 x\, dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} sin^2 x dx$-(1)

Using the property that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_0^{\frac{\pi}{2}} sin^2\left(\frac{\pi}{2} - x\right)dx$$

$= \int_0^{\frac{\pi}{2}} cos^2 xdx$ -(2)

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} sin^2 xdx + \int_0^{\frac{\pi}{2}} cos^2 xdx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

Hence, $I = \frac{\pi}{4}$

2. Question

Evaluate $\int_0^{\pi/2} \cos^2 x\, dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} cos^2 xdx$-(1)

Using the property that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_0^{\frac{\pi}{2}} cos^2(\frac{\pi}{2} - x)dx = \int_0^{\frac{\pi}{2}} sin^2 xdx$-(2)

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} sin^2 xdx + \int_0^{\frac{\pi}{2}} cos^2 xdx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}$$

Hence, $I = \frac{\pi}{4}$

3. Question

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2 xdx$

Since $cos2x = 1 - 2sin^2 x \Longrightarrow sin^2 x = \frac{1-cos2x}{2}$

$$\mathrm{I} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - cos2x)dx$$

$$= \frac{x}{2}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \frac{\sin 2x}{4}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \frac{1}{4}(sin\pi - \sin(-\pi))$$

$$= \frac{\pi}{2}$$

4. Question

Evaluate $\int_{-\pi/2}^{\pi/2} \cos^2 x \, dx$

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^2 x dx$

Since cos2x=2cos^2 x-1

$\Rightarrow cos^2 x = \frac{1 + cos2x}{2}$

$I = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + cos2x) dx$

$= \frac{x}{2} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{sin2x}{4} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - \frac{1}{4}(sin\pi - \sin(-\pi))$

$= \frac{\pi}{2}$

5. Question

Evaluate $\int_{-\pi/2}^{\pi/2} \sin^3 x \, dx$

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^3 x dx$

f(x)=sin^{3}x

f(-x)=sin^3(-x)=-sin^{3}x

Hence, f(x) is an odd function.

Since, $\int_{-a}^{a} f(x) dx = 0$ if f(x) is an odd function.

Therefore, I=0.

6. Question

Evaluate $\int_{-\pi/2}^{\pi/2} x \cos^2 x \, dx$

Answer

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x cos^2 x \, dx$

f(x)=xcos^{2}x

f(-x)=(-x)cos^2(-x)

=-xcos^{2}x

=-f(x)

Hence, f(x) is an odd function.

Since, $\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function.

Therefore, I=0.

7. Question

Evaluate $\int_{0}^{\pi/4} \tan^2 x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} tan^2 x dx = \int_0^{\frac{\pi}{4}} (sec^2 x - 1)dx$

Let tan x=t

⇒ $sec^2xdx=dt$

When x=0, t=0 and when $x = \frac{\pi}{4}, t = 1$

Hence, $I = \int_0^1 dt + \int_0^{\frac{\pi}{4}} -dx = 1 - \frac{\pi}{4}$

8. Question

Evaluate $\int_{0}^{1} \frac{1}{x^2 + 1} dx$

Answer

Let $\mathrm{I} = \int_0^1 \frac{1}{1+x^2} dx$

Substituting x=tanθ ⇒ $dx=sec^2θdθ$ (By differentiating both sides)

Also, when x=0, θ=0 and x=1, θ=$\frac{\pi}{4}$

We get $\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{1}{1+tan^2\theta} sec^2\theta d\theta$

Since $sec^2θ=1+tan^2θ$

We get $\mathrm{I} = \int_0^{\frac{\pi}{4}} d\theta$

$= \frac{\pi}{4}$

9. Question

Evaluate $\int_{-2}^{1} \frac{|x|}{x} dx$

Answer

Let $I = \int_{-2}^{1} \frac{|x|}{x} dx = \int_{-2}^{0} \frac{|x|}{x} dx + \int_{0}^{1} \frac{|x|}{x} dx$

|x|=-x, if x<0

And |x|=x, if x≥0

Hence, $I = \int_{-2}^{0} - dx + \int_{0}^{1} dx$

$I = -2 + 1 = -1$

10. Question

Evaluate $\int_{0}^{\infty} e^{-x}\ dx$

Answer

Let $I = \int_0^\infty e^{-x} dx$

$= -e^{-x}\Big|_0^\infty$

=-(0-1)

=1

11. Question

Evaluate $\int_{0}^{4} \frac{1}{\sqrt{16-x^2}}\ dx$

Answer

Let $I = \int_0^4 \frac{1}{\sqrt{16-x^2}} dx$

Substituting x=4sinθ ⇒ dx=4cosθdθ

Also, When x=0, θ=0 and x=4, θ=$\frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{16-16sin^2\theta}} 4cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}$$

12. Question

Evluate $\int_{0}^{3} \frac{1}{x^2+9}\ dx$

Answer

Substituting x=3tanθ ⇒ dx=3sec^{2}θdθ (By differentiating both sides)

Also, when x=0, θ=0 and x=3, θ=$\frac{\pi}{4}$

We get $I = \int_0^{\frac{\pi}{4}} \frac{1}{9+9tan^2\theta} 3sec^2\theta d\theta$

Since sec^{2}θ=1+tan^{2}θ

We get $I = \int_0^{\frac{\pi}{4}} \frac{d\theta}{3}$

$$= \frac{\pi}{12}$$

13. Question

Evaluate $\int_0^{\pi/2} \sqrt{1-\cos 2x}\ dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1 - cos2x}\, dx$

Since, $\cos 2x = 1 - 2\sin^2 x \Rightarrow 2\sin^2 x = 1 - \cos 2x$

Hence, $I = \int_0^{\frac{\pi}{2}} \sqrt{2} sinx dx = -\sqrt{2} cosx \Big|_0^{\frac{\pi}{2}}$

$= -\sqrt{2}(0-1) = \sqrt{2}$

14. Question

Evaluate $\int_0^{\pi/2} \log\ \tan x\ dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \log tanx\, dx$

Using the property that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log\tan\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log cotx\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{1}{tanx} dx = -\int_0^{\frac{\pi}{2}} \log tanx\, dx$$

=-I

(Since $\log_a \frac{1}{b} = -\log_a b$)

Since I=-I, therefore I=0

15. Question

Evaluate $\int_0^{\pi/2} \log\left(\frac{3+5\cos x}{3+5\sin x}\right) dx$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \log \frac{3+5cosx}{3+5sinx} dx$

Using the property that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log \frac{3 + 5\cos\left(\frac{\pi}{2} - x\right)}{3 + 5sin\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \log \frac{3 + 5sinx}{3 + 5cosx} dx$$

$$= -\int_0^{\frac{\pi}{2}} \log \frac{3 + 5cosx}{3 + 5sinx} dx$$

=-I (Since $\log_a \frac{1}{b} = -\log_a b$)

Since I=-I, therefore I=0

16. Question

Evaluate $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx, n \in N.$

Answer

Let $I = \int_0^{\frac{\pi}{2}} \frac{sin^n x}{sin^n x + cos^n x} dx$ -(1)

Using the property that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

We get $I = \int_0^{\frac{\pi}{2}} \frac{sin^n x}{sin^n x + cos^n x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{sin^n (\frac{\pi}{2} - x)}{sin^n (\frac{\pi}{2} - x) + cos^n (\frac{\pi}{2} - x)} dx$$

Since sin($\frac{\pi}{2}$-x) =cosx and cos($\frac{\pi}{2}$-x) =sinx

We get $I = \int_0^{\frac{\pi}{2}} \frac{cos^n x}{sin^n x + cos^n x} dx$-(2)

Add (1) and (2)

$$2I = \int_0^{\frac{\pi}{2}} \frac{sin^n x + cos^n x}{sin^n x + cos^n x} dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$= \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

17. Question

Evaluate $\int_0^{\pi} \cos^5 x \, dx$

Answer

Let $I = \int_0^{\pi} cos^5 x dx$

Consider $\cos^5$x= $\cos^4$x × cosx

= $(\cos^2 x)^2$ × cosx

= $(1- \sin^2 x)^2$cosx

Let sinx=y ⇒ cosxdx=dy (Differentiating both sides)

Also, when x=0, y=0 and x=π, y=0

Hence, I become $\int_0^0 (1-y^2)^2 dy$

Since $\int_{a}^{a} f(x)dx = 0$, We get $\mathrm{I} = \int_{0}^{\pi} cos^{5} x dx$

$$= \int_{0}^{0} (1-y^2)^2 dy$$

=0

18. Question

Evaluate $\int_{-\pi/2}^{\pi/2} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) d\theta$

Answer

Let $\mathrm{I} = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} log \frac{(a-sin\theta)}{(a+sin\theta)} d\theta$

Let $f(\theta) = \ log \frac{(a-sin\theta)}{(a+sin\theta)}$

Then $f(-\theta) = log \frac{(a-\sin(-\theta))}{(a+\sin(-\theta))}$

$$= log \frac{(a+sin\theta)}{(a-sin\theta)}$$

$$= -log \frac{(a-sin\theta)}{(a+sin\theta)}$$

$$= -f(\theta)$$

(Since sin(-θ) =-sin(θ) and $\log_a b = -\log_a \frac{1}{b}$)

From this, we infer that f(θ) is an odd function.

Using $\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function, we get that I=0

19. Question

Evaluate $\int_{-1}^{1} x\,|\,x\,|\,dx$

Answer

Let $\mathrm{I} \ = \int_{-1}^{1} x|x|dx$

|x|=-x, if x<0

And |x|=x, if x≥0

Therefore f(x)=x|x|=-x^2, if x<0

And f(x)=x|x|=x^2, if x≥0

Consider x≥0 ⇒ f(x)=x^2

Then -x<0 ⇒ f(-x) = -(-x)^2 = -f(-x)

Now Consider x<0 ⇒ f(x)=-x^2

Then -x≥0 ⇒ f(-x) =-(-x)^2=x^2=-f(x)

Hence f(x) is an odd function. An odd function is a function which satisfies the property f(-x) =-f(-x), ∀ x∈ Domain of f(x)

There is a property of integration of odd functions which states that

$\int_{-a}^{a} f(x)dx = 0$ if f(x) is an odd function.

Therefore $\mathrm{I} = \int_{-1}^{1} x|x|dx = 0$

20. Question

Evaluate $\int_{a}^{b} \frac{f(x)}{f(x)+f(a+b-x)} dx$

Answer

Let $\mathrm{I} = \int_{a}^{b} \frac{f(x)}{f(x)+f(a+b-x)} dx$ - (1)

Using the property that $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx$

We get $\mathrm{I} = \int_{a}^{b} \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))} dx$

$\mathrm{I} = \int_{a}^{b} \frac{f(a+b-x)}{f(a+b-x)+f(x)} dx$ - (2)

Adding (1) and (2) we get

$$2\mathrm{I} = \int_{a}^{b} \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)} dx$$

$$= \int_{a}^{b} dx$$

$$= b - a$$

$$\mathrm{I} = \frac{b-a}{2}$$

21. Question

Evaluate $\int_{0}^{1} \frac{1}{1+x^2} dx$

Answer

Let $\mathrm{I} = \int_{0}^{1} \frac{1}{1+x^2} dx$

Substituting x=tanθ ⇒ dx=$\sec^2$θdθ (By differentiating both sides)

Also, when x=0, θ=0 and x=1, θ=$\frac{\pi}{4}$

We get $\mathrm{I} = \int_{0}^{\frac{\pi}{4}} \frac{1}{1+tan^2\theta} sec^2\theta d\theta$

Since $\sec^2\theta = 1+\tan^2\theta$

We get $\mathrm{I} = \int_{0}^{\frac{\pi}{4}} d\theta$

$$= \frac{\pi}{4}$$

22. Question

Evaluate $\int_0^{\pi/4} \tan x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} tanx dx = \int_0^{\frac{\pi}{4}} \frac{sinx}{cosx} dx$

Substituting cosx=y ⇒ -sinxdx=dy (By differentiating both sides)

Also, when x=0, y=0 and x=$\frac{\pi}{4}$, y=$\frac{1}{\sqrt{2}}$

We get $I = -\int_1^{\frac{1}{\sqrt{2}}} \frac{1}{y} dx$

We get $I = -\log_e y \Big|_1^{\frac{1}{\sqrt{2}}}$ (Check Q23. For proof)

$= -(\log_e(\frac{1}{\sqrt{2}}) - \log_e 1)$

Since $\log_a 1 = 0 \; and \; -\log_a b = \log_a \frac{1}{b}$, We get

$I = \log_e \sqrt{2}$

23. Question

Evaluate $\int_2^3 \frac{1}{x} dx$

Answer

Let $I = \int_2^3 \frac{1}{x} dx$

Substitute x=e^y⇒ dx=e^ydy (Differentiating both sides)

Since $x = e^y \Longrightarrow y = \log_e x \; and \; when \; x = 2, y = \log_e 2 \,, \; and \; when \; x = 3, y = \log_e 3$

We get $I = \int_{\log_e 2}^{\log_e 3} \frac{1}{e^y} e^y dy$

$I = \int_{\log_e 2}^{\log_e 3} dy$

$= \log_e 3 - \log_e 2$

$= \log_e \frac{3}{2}$ (Since $\log_a b - \log_a c = \log_a \frac{b}{c}$)

24. Question

Evaluate $\int_0^2 \sqrt{4 - x^2} \; dx$

Answer

Let $I = \int_0^2 \sqrt{4 - x^2} \, dx$

Substituting x=2sinθ ⇒ dx=2sinθdθ

Also, When x=0, θ=0 and x=2, θ=$\frac{\pi}{2}$

We get $I = \int_0^{\frac{\pi}{2}} \sqrt{4-(2sin\theta)^2}\, 2cos\theta d\theta$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{4-4sin^2\theta}\, 2cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1-sin^2\theta}\, 4cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4cos^2\theta d\theta$$

Since $cos2\theta = 2cos^2\theta - 1 \Longrightarrow cos^2\theta = \frac{1+cos2\theta}{2}$

$$I = \int_0^{\frac{\pi}{2}} (2+2cos2\theta) d\theta$$

$$= 2\theta \Big|_0^{\frac{\pi}{2}} + sin2\theta \Big|_0^{\frac{\pi}{2}}$$

$=\pi+\sin\pi-\sin 0$

$=\pi$

25. Question

Evaluate $\int\limits_0^1 \frac{2x}{1+x^2} dx$

Answer

Let $I = \int_0^1 \frac{2x}{1+x^2} dx$

Substituting $1+x^2= t$

$\Rightarrow 2x\, dx=dt$

Also, When x=0, t=1 and x=1, t=2

We get $I = \int_1^2 \frac{1}{t} dt$ (For proof check Q23.)

$= \log_e t \,|_1^2 = \log_e 2 - \log_e 1 = \log_e 2$ (Since $\log_e 1 = 0$)

26. Question

Evaluate $\int\limits_0^1 x\, e^{x^2}\, dx$

Answer

Let $I = \int_0^1 xe^{x^2} dx$

Substitute $x^2=y$

$\Rightarrow 2xdx=dy$

Also, when x=0, y=0 and x=1, y=1

We get $I = \int_0^1 \frac{e^y}{2} dy$

Since $\int e^y dy = e^y$

We get $I = \frac{e^y}{2}\Big|_0^1 = \frac{e^1 - e^0}{2} = \frac{e-1}{2}$

27. Question

Evaluate $\int_0^{\pi/4} \sin 2x \, dx$

Answer

Let $I = \int_0^{\frac{\pi}{4}} sin2x \, dx$

Substitute 2x=y ⇒ 2dx=dy

Also, when x=0, y=0 and x=$\frac{\pi}{4}$, y=$\frac{\pi}{2}$

We get $I = \int_0^{\frac{\pi}{2}} \frac{siny}{2} dy$

$$= -\frac{1}{2} cosy \Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}\left(\cos\left(\frac{\pi}{2}\right) - cos0\right) = \frac{1}{2}$$

28. Question

Evaluate $\int_e^{e^2} \frac{1}{x \log x} dx$

Answer

Let $I = \int_e^{e^2} \frac{1}{x \log_e x} dx$

Substitute $\log_e x = y \Longrightarrow \frac{dx}{x} = dy$

Also, When x=e, y=1 and x=e^2, y=2

We get $I = \int_1^2 \frac{1}{y} dy$ (Check Q23. For proof)

$$= \log_e y \Big|_1^2 = \log_e 2 - \log_e 1 = \log_e 2$$

29. Question

Evaluate $\int_e^{\pi/2} e^x (\sin x - \cos x) dx$

Answer

Let $I = \int_e^{\frac{\pi}{2}} e^x (sinx - cosx) \, dx$

Substitute $-e^x$cosx=t ⇒ e^x(-cosx+sinx)dx=dt

(Differentiating both sides by using multiplication rule)

Also, When x=e, t=$-e^e$cose and x=$\frac{\pi}{2}$, t=0

We get $I = \int_{-e^e cose}^0 dt = t\Big|_{-e^e cose}^0 = 0 - (-e^e cose) = e^e cose$

30. Question

Evaluate $\int_{2}^{4} \frac{x}{x^2+1} dx$

Answer

Let $I = \int_2^4 \frac{x}{1+x^2} dx$

Substitute 1+x2=t ⇒ 2xdx=dt

Also, When x=2, t=5 and x=4, t=17

We get $I = \frac{1}{2} \int_5^{17} \frac{1}{t} dt$

$= \frac{1}{2} \log_e t \Big|_5^{17} = \frac{1}{2}(\log_e 17 - \log_e 5) = \frac{1}{2} \log_e \frac{17}{5}$

(Since $\log_a b - \log_a c = \log_a \frac{b}{c}$)

31. Question

If $\int_{0}^{1} \left(3x^2 + 2x + k\right) dx = 0,$ find the value of k.

Answer

To find the value of K, First we have to integrate above integral for which we have to apply simple formulas of integration $\int x^2 dx$ and $\int x dx$,so

$\int_0^1 (3x^2 + 2x + k)$=[3(x^3/3) + 2(x^2/2) + k.x]$_0^1$ = 0

Put the upper limit and lower limit in above equation-

$\Rightarrow \left[3.\left(\frac{1}{3}\right) + 2\left(\frac{1}{2}\right) + k.1\right] - [3.(0) + 2.(0) + k.(0)] = 0$

=(1+1+k) = 0

K = -2

32. Question

If $\int_{0}^{a} 3x^2 dx = 8,$ write the value of a.

Answer

Doing integration yields-

$\left[3.\frac{x^3}{3}\right]_0^a = 8$

a^3 =8

$a = \sqrt[3]{8}$

a=2

33. Question

If $f(x) = \int_{0}^{x} t \sin t \, dt,$ then write the value of f’(x).

Answer

Doing integration yields-

$$f(x) = t.(-cost)_0^x - \int_0^x \frac{d}{dt}(t).(-cost)dt$$

$$= -(xcosx - 0) + \int_0^x costdt$$

$$= -xcosx + [sint]_0^x$$

then finally f(x)=-x cos x + sin x

To calculate derivative of the above function f(x) we have to apply formula of derivation of products of two functions-

F(x) =-x cos(x)+sin(x)

F'(x) =-[x(-sin x)+cos x.1]+cos x ;{by formula d/dx (f.g)=f.g'+g.f'}

F'(x) =-(-x sin x) - cos x + cos x

F'(x) = x.sin(x)

34. Question

If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$, find the value of a.

Answer

Doing integration yields-

$$[\frac{1}{2}\tan^{-1}(\frac{x}{2})]_0^a = \frac{\pi}{8}$$

$$\frac{1}{2}(\tan^{-1}\frac{a}{2} - \tan^{-1} 0) = \frac{\pi}{8}$$

$$\tan^{-1}\frac{a}{2} - 0 = \frac{\pi}{4}$$

$$a = \tan\frac{\pi}{4} + 1$$

a=2

35. Question

Write the coefficient a, b, c of which the value of the integral $\int_{-3}^{3} \left(ax^2 - bx + c\right) dx$ is independent.

Answer

Doing integration yields-

$$\left(a.\frac{x^3}{3} - b.\frac{x^2}{2} + c.x\right)_{-3}^{3}$$

By substituting upper and lower limit in above equation=$\left(9a - b.\frac{9}{2} + 3c\right) - (9a - b.\frac{9}{2} - 3c)$

=18a+bc

so now we can say this is independent of variable b

36. Question

Evaluate $\int_{2}^{3} 3^{x}\, dx$

Answer

Doing integration yields-

$= \left[\frac{3^x}{log3}\right]_2^3$

$= \frac{1}{log3}(3^3 - 3^2)$

$= \frac{18}{log3}$

37. Question

I = $\int_{0}^{2} [x]\, dx$

Answer

we know that

$[x] = \begin{cases} 0\,, 0 < x < 1 \\ 1\,, 1 < x < 2 \end{cases}$

$I = \int_0^1 [x]dx + \int_1^2 [x]dx = \int_0^1 0dx + \int_1^2 1.dx$

$= 0 + (x)_1^2$

= (2-1)

=1

38. Question

I = $\int_{0}^{15} [x]\, dx$

Answer

we know that

[x]=$\begin{cases} 0\,, 0 < x < 1 \\ 1\,, 1 < x < 2 \end{cases}$

$I = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx \dots\dots\dots\dots\dots + \int_{14}^{15} [x].dx$

=(0)+(1)+(2)………………..+14

=105 ans

39. Question

I = $\int_{0}^{1} \{x\}\, dx$

Answer

we all know that-

{x} = x , 0 < x < 1

$$I = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}\ ans$$

40. Question

$$I = \int_0^1 e^{\{x\}} dx$$

Answer

we all know that-

{x} = x , 0 < x < 1

$$I = \int_0^1 e^x dx = (e^1 - e^0) = (e - 1)$$

41. Question

$$I = \int_0^2 x[x] dx$$

Answer

we know that

$$[x] = \begin{cases} 0, 0 < x < 1 \\ 1, 1 < x < 2 \end{cases}$$

$$I = \int_0^1 x[x]dx + \int_1^2 x[x]dx = \int_0^1 0dx + \int_1^2 x.1dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2}$$

$$= \frac{3}{2}$$

42. Question

$$I = \int_0^1 2^{x-[x]} dx$$

Answer

we know that

$$[x] = \begin{cases} 0, 0 < x < 1 \\ 1, 1 < x < 2 \end{cases}$$

$$I = \int_0^1 2^{(x-0)} dx = \int_0^1 2^x dx = \left[\frac{2^x}{log2}\right]_0^1 = \frac{1}{log2}[2^1 - 2^0]$$

$$= \frac{1}{log2}$$

43. Question

$$I = \int_0^2 \log_e [x]\, dx$$

Answer

we know that

$$[x]=\begin{cases}0, 0 < x < 1\\ 1, 1 < x < 2\end{cases}$$

$$I = \int_0^1 log[x]dx + \int_1^2 log[x]dx = \int_0^1 0dx + \int_1^2 log1.dx$$

=o

44. Question

$$I = \int_0^{\sqrt{2}} \left[x^2\right] dx$$

Answer

we know that

$$[x]=\begin{cases}0, 0 < x < 1\\ 1, 1 < x < 2\end{cases}$$

$$I = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx = \int_0^1 0dx + \int_1^{\sqrt{2}} 1.dx$$

=(√2-1)

45. Question

$$I = \int_0^{\pi/4} \sin\{x\}\, dx$$

Answer

we all know that-

{x} = x , 0 < x < 1

$$I = \int_0^{\pi/4} \sin(x)\, dx = (-cosx)_0^{\pi/4} = -(\cos\frac{\pi}{4} - \cos 0)$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right)$$

MCQ

1. Question

$\int_0^1 \sqrt{x(1-x)}\, dx$ equals

A. π/2

B. π/4

C. π/6

D. π/8

Answer

Let, x = $\sin^2$ t

Differentiating both sides with respect to t

$$\frac{dx}{dt} = 2\sin t \cos t$$

⇒ dx = 2 sin t cos t dt

At x = 0, t = 0

At x = 1, $t = \frac{\pi}{2}$

$$y = \int_0^{\frac{\pi}{2}} \sqrt{sin^2\, t\,(1 - sin^2\, t)} \times 2\sin t \cos t\ dt$$

$$= \int_0^{\frac{\pi}{2}} 2\, sin^2 t\, cos^2 t\ dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^2 2t\, dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2(2t)}{2}\ dt$$

$$= \frac{1}{2}\left(\frac{t}{2} - \frac{\sin 4t}{8}\right)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - 0\right) - (0 - 0)\right]$$

$$y = \frac{\pi}{8}$$

2. Question

$\int_0^{\pi} \frac{1}{1+\sin x} dx$ equals

A. 0

B. 1/2

C. 2

D. 3/2

Answer

$y = \int_0^{\pi} \frac{1}{1+\sin x}\ dx$

Multiply by 1 – sin x in numerator and denominator

$$y = \int_0^{\pi} \frac{1 - \sin x}{(1 + \sin x)(1 - \sin x)}\ dx$$

$$= \int_0^{\pi} \frac{1 - \sin x}{1 - sin^2\, x}\ dx$$

$$= \int_0^{\pi} \frac{1 - \sin x}{cos^2\, x}\ dx$$

$= \int_0^{\pi} \frac{1}{cos^2 x} - \frac{\sin x}{cos^2 x}\ dx$

$= \int_0^{\pi} sec^2 x - \tan x \sec x\ dx$

$= (\tan x - \sec x)_0^{\pi}$

y = [(0 – (-1)) – (0 – 1)]

y = 2

3. Question

$\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x} dx$

A. $\frac{\pi^2}{4}$

B. $\frac{\pi^2}{2}$

C. $\frac{3\pi^2}{2}$

D. $\frac{\pi^2}{3}$

Answer

In this question we can use the king rule

$y = \int_0^{\pi} \frac{x \tan x}{sec\, x + cos x} dx$ ...(1)

$= \int_0^{\pi} \frac{(\pi - x)\ tan(\pi - x)}{sec(\pi - x) + cos(\pi - x)} dx$

$= \int_0^{\pi} \frac{-\pi \tan x}{-(sec\, x + cos x)} - \frac{-x \tan x}{-(\sec x + cos x)} dx$ ...(2)

On adding eq(1) and eq(2)

$2y = \int_0^{\pi} \frac{\pi\ tan\, x}{sec\ x + cos\ x} dx$

$= \int_0^{\pi} \frac{\pi sinx}{1 + cos^2 x}\ dx$

Let, cos x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = -\sin x$ ⇒ -dt = sin x dx

At x = 0, t = 1

At x = π, t = -1

$2y = \int_1^{-1} \frac{-\pi}{1 + t^2}\ dt$

$= -\pi(\tan^{-1} t)_1^{-1}$

$y = \frac{-\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$

$= \frac{\pi^2}{4}$

4. Question

The value of $\int_0^{2\pi} \sqrt{1 + \sin\frac{x}{2}}\, dx$ is

A. 0

B. 2

C. 8

D. 4

Answer

$y = \int_0^{2\pi} \sqrt{sin^2\frac{x}{4} + cos^2\frac{x}{4} + 2\,sin\frac{x}{4}\cos\frac{x}{4}}\ dx$

$= \int_0^{2\pi} sin\frac{x}{4} + \cos\frac{x}{4}\ dx$

$= 4\left(-\cos\frac{x}{4} + \sin\frac{x}{4}\right)_0^{2\pi}$

= 4[(0 + 1) – (-1 – 0)]

y = 8

5. Question

The value of the integral $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{cos\,x} + \sqrt{sin\,x}} dx$ is

A. 0

B. π/2

C. π/4

D. none of these

Answer

Mistake: limit should be 0 to π\2

Right sol. In this question we apply the king rule

$y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{cos\,x} + \sqrt{sin\,x}} dx$ ...(1)

$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{cos\left(\frac{\pi}{2} - x\right)} + \sqrt{sin\left(\frac{\pi}{2} - x\right)}} dx$

$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ ...(2)

On adding eq(1) and eq(2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$y = \frac{1}{2}(x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

6. Question

$$\int_0^{\infty} \frac{1}{1+e^x} dx$$ equals

A. log 2 – 1

B. log 2

C. log 4 – 1

D. –log 2

Answer

Take e^x out from the denominator

$$y = \int_0^{\infty} \frac{1}{e^{x(e^{-x}+1)}} dx$$

$$y = \int_0^{\infty} \frac{e^{-x}}{e^{-x}+1} dx$$

Let, $e^{-x} + 1 = t$

Differentiating both side with respect to t

$\frac{dt}{dx} = -e^{-x} \Rightarrow -dt = e^{-x} dx$

At x = 0, t = 2

At x = ∞, t = 1

$$y = -\int_2^1 \frac{1}{t} dt$$

$$y = -(\log t)_2^1$$

y = -(0 – log 2)

y = log 2

7. Question

$$\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$$ equals

A. 2

B. 1

C. π/4

D. $\pi^2/8$

Answer

Let, $\sqrt{x} = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow 2dt = \frac{1}{\sqrt{x}}\, dx$$

At x = 0, t = 0

At x = $\pi^2\backslash 4$, t = $\pi\backslash 2$

$$y = 2\int_0^{\frac{\pi}{2}} \sin t \; dt$$

$$y = 2(-\cos t)_0^{\frac{\pi}{2}}$$

y = 2[0 – (-1)]

y = 2

8. Question

$$\int_0^{\pi/2} \frac{\cos x}{(2+\sin x)(1+\sin x)} dx$$ equals

A. $\log\left(\frac{2}{3}\right)$

B. $\log\left(\frac{3}{2}\right)$

C. $\log\left(\frac{3}{4}\right)$

D. $\log\left(\frac{4}{3}\right)$

Answer

Let, sin x = t

Differentiating both side with respect to x

$\frac{dt}{dx} = \cos x \Rightarrow$ dt = cos x dx

At x = 0, t = 0

At x = $\pi\backslash 2$, t = 1

$$y = \int_0^1 \frac{1}{(2+t)(1+t)}\; dt$$

By using the concept of partial fraction

$$\frac{1}{(2+t)(1+t)} = \frac{A}{(2+t)} + \frac{B}{(1+t)}$$

1 = A(1 + t) + B(2 + t)

1 = (A + 2B) + t(A + B)

A + 2B = 1, A + B = 0

A = -1, B = 1

$$y = \int_0^1 \frac{-1}{(2+t)} + \frac{1}{(1+t)}\ dt$$

$$y = [-\log(2+t) + \log(1+t)]_0^1$$

y = [(-log 3 + log 2) – (-log 2 + log 1)]

$$y = log\left(\frac{4}{3}\right)$$

9. Question

$\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx$ equals

A. $\frac{1}{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

B. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

C. $\sqrt{3}\tan^{-1}\left(\sqrt{3}\right)$

D. $2\sqrt{3}\tan^{-1}\sqrt{3}$

Answer

$$y = \int_0^{\frac{\pi}{2}} \frac{1}{2+\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}\ dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{1+tan^2\frac{x}{2}}{2\left(1+tan^2\frac{x}{2}\right)+1-tan^2\frac{x}{2}}\ dx$$

$$y = \int_0^{\frac{\pi}{2}} \frac{sec^2\frac{x}{2}}{3+tan^2\frac{x}{2}}\ dx$$

Let, $\tan\frac{x}{2} = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = \frac{1}{2}sec^2\frac{x}{2} \Rightarrow 2\,dt = sec^2\frac{x}{2}\ dx$$

At x = 0, t = 0

At x = π\2, t = 1

$$y = \int_0^1 \frac{2}{(\sqrt{3})^2+t^2}dt$$

$$y = 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right)_0^1$$

$$y = \frac{2}{\sqrt{3}}\left(\tan^{-1}\frac{1}{\sqrt{3}} - 0\right)$$

$$y = \frac{2}{\sqrt{3}} tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

10. Question

$\int_0^1 \sqrt{\frac{1-x}{1+x}}\ dx =$

A. $\frac{\pi}{2}$

B. $\frac{\pi}{2} - 1$

C. $\frac{\pi}{2} + 1$

D. π + 1

Answer

Right sol. Let, x = sin t

Differentiating both side with respect to t

$\frac{dx}{dt} = \cos t \Rightarrow$ dx = cos t dt

At x = 0, t = 0

At x = 1, t = π\2

$$y = \int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin t}{1+\sin t}} \cos t\ \ dt$$

Multiply by 1 – sin t in numerator and denominator

$$y = \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin t)(1-\sin t)}{(1+\sin t)(1-\sin t)}} \cos t\ \ dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{1-\sin t}{\cos t} \cos t\ \ dt$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin t)\ dt$$

$$= (t + \cos t)_0^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{\pi}{2} + 0\right) - (0+1)\right]$$

$$y = \frac{\pi}{2} - 1$$

11. Question

$$\int_0^{\pi} \frac{1}{a + b\cos x}\,dx =$$

A. $\frac{\pi}{\sqrt{a^2 - b^2}}$

B. $\frac{\pi}{ab}$

C. $\frac{\pi}{a^2 + b^2}$

D. (a +b)π

Answer

$$y = \int_0^{\pi} \frac{1}{a + b\frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}}\ dx$$

$$y = \int_0^{\pi} \frac{1 + tan^2\frac{x}{2}}{a\left(1 + tan^2\frac{x}{2}\right) + b(1 - tan^2\frac{x}{2})}\ dx$$

$$y = \int_0^{\pi} \frac{sec^2\frac{x}{2}}{(a + b) + (a - b)tan^2\frac{x}{2}}\ dx$$

Let, $\tan\frac{x}{2} = t$

Differentiating both side with respect to x

$$\frac{dt}{dx} = \frac{1}{2}sec^2\frac{x}{2} \Rightarrow 2\,dt = sec^2\frac{x}{2}\ dx$$

At x = 0, t = 0

At x = π, t = ∞

$$y = \int_0^{\infty} \frac{2}{(\sqrt{a + b})^2 + (t\sqrt{a - b})^2}dt$$

$$y = \frac{2}{\sqrt{a - b}}\left(\frac{1}{\sqrt{a + b}}\tan^{-1}\frac{t\sqrt{a - b}}{\sqrt{a + b}}\right)_0^{\infty}$$

$$y = \frac{2}{\sqrt{a^2 - b^2}}\left(\frac{\pi}{2} - 0\right)$$

$$y = \frac{\pi}{\sqrt{a^2 - b^2}}$$

12. Question

$\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\cot x}}\ dx$ is

A. π/3

B. π/6

C. π/12

D. π/2

Answer

$y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ ...(1)

$$y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}} dx$$

$y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ ...(2)

On adding eq(1) and eq(2)

$$2y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$2y = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx$$

$$y = \frac{1}{2} (x)_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$y = \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$y = \frac{\pi}{12}$$

13. Question

Given that $\int_0^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)} dx = \frac{\pi}{2(a+b)(b+c)(c+a)}$, the value of $\int_0^{\infty} \frac{dx}{(x^2+4)(x^2+9)}$, is

A. $\frac{\pi}{60}$

B. $\frac{\pi}{20}$

C. $\frac{\pi}{40}$

D. $\frac{\pi}{80}$

Answer

In this question we use the method of partial fraction

$$\frac{1}{(x^2+4)(x^2+9)} = \frac{A}{(x^2+4)} + \frac{B}{(x^2+9)}$$

$1 = A(x^2 + 9) + B(x^2 + 4)$

A + B = 0, 9A + 4B = 1

A = 1\5, B = -1\5

$$y = \int_0^{\infty} \frac{\frac{1}{5}}{(x^2+4)} + \frac{\frac{-1}{5}}{(x^2+9)}\ dx$$

$$y = \left(\frac{1}{5\times 2}\tan^{-1}\frac{x}{2} - \frac{1}{5\times 3}\tan^{-1}\frac{x}{3}\right)_0^{\infty}$$

$$y = \left(\frac{\pi}{20} - \frac{\pi}{30}\right)$$

$$y = \frac{\pi}{60}$$

14. Question

$$\int_1^e \log x\ dx =$$

A. 1

B. e – 1

C. e + 1

D. 0

Answer

$$y = \int_1^e 1 \times \log x\, dx$$

By using integration by parts

Let, log x as Ist function and 1 as IInd function

Use formula $\int I \times II\, dx = I \int II\, dx - \int \left(\frac{d}{dx} I\right)\left(\int II\, dx\right) dx$

$$y = \log x \int_1^e dx - \int_1^e \left(\frac{d}{dx}\log x\right)\left(\int_1^e dx\right) dx$$

$$y = (\log x)x - \int_1^e \left(\frac{1}{x}\right)(x)\, dx$$

$$y = x\log x - \int_1^e dx$$

$$y = (x\log x - x)_1^e$$

y = [(e – e) – (0 – 1)]

y = 1

15. Question

$\int_1^{\sqrt{3}} \frac{1}{1+x^2}\ dx$ is equal to

A. $\frac{\pi}{12}$

B. $\frac{\pi}{6}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

Answer

$y = \int_1^{\sqrt{3}} \frac{1}{1+x^2}\ dx$

$y = (\tan^{-1} x)_1^{\sqrt{3}}$

$y = \left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$y = \frac{\pi}{12}$

16. Question

$\int_0^3 \frac{3x+1}{x^2+9} dx =$

A. $\frac{\pi}{12} + \log\left(2\sqrt{2}\right)$

B. $\frac{\pi}{2} + \log\left(2\sqrt{2}\right)$

C. $\frac{\pi}{6} + \log\left(2\sqrt{2}\right)$

D. $\frac{\pi}{3} + \log\left(2\sqrt{2}\right)$

Answer

$y = \int_0^3 \frac{3x}{x^2+9} + \frac{1}{x^2+9}\ dx$

$A = \frac{3}{2}\int_0^3 \frac{2x}{x^2+9}\ dx$

Let, $x^2 + 9 = t$

Differentiating both side with respect to x

$\frac{dt}{dx} = 2x \Rightarrow dt = 2x\ dx$

At x = 0, t = 9

At x = 3, t = 18

$A = \frac{3}{2}\int_9^{18} \frac{1}{t}\ dt$

$= \frac{3}{2}(\log t)_9^{18}$

$= \frac{3}{2}(\log 18 - \log 9)$

$= \frac{3}{2}\log 2 = \log 2\sqrt{2}$

$B = \int_0^3 \frac{1}{x^2+9}\, dx$

$= \left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right)_0^3$

$= \frac{1}{3}(\frac{\pi}{4} - 0)$

$= \frac{\pi}{12}$

So, the complete solution is y = A + B

$y = \frac{\pi}{12} + \log 2\sqrt{2}$

17. Question

The value of the integral $\int_0^\infty \frac{x}{(1+x)(1+x^2)} dx$ is

A. $\frac{\pi}{2}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{3}$

Answer

Let, x = tan t

Differentiating both side with respect to t

$\frac{dx}{dt} = sec^2 t$ ⇒ dx = sec^2t dt

At x = 0, t = 0

At x = ∞, t = π\2

$y = \int_0^{\frac{\pi}{2}} \frac{\tan t}{(1+\tan t)(1+tan^2 t)} sec^2 t\, dt$

$y = \int_0^{\frac{\pi}{2}} \frac{\tan t}{(1+\tan t)} dt$

$y = \int_0^{\frac{\pi}{2}} \frac{\sin t}{(\cos t + \sin t)} dt$ ...(1)

By using the king rule

$y = \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2} - t)}{\left(\cos\left(\frac{\pi}{2} - t\right) + \sin\left(\frac{\pi}{2} - t\right)\right)} dt$

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos t}{(\cos t + \sin t)} dt$$ ...(2)

On adding eq(1) and eq(2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sin t + \cos t}{(\cos t + \sin t)} dt$$

$$2y = \int_0^{\frac{\pi}{2}} dt$$

$$y = \frac{1}{2}(t)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

18. Question

$\int_{-\pi/2}^{\pi/2} \sin|x|$ dx is equal to

A. 1

B. 2

C. –1

D. –2

Answer

In this question, we break the limit in two-part

$$y = \int_{\frac{-\pi}{2}}^{0} \sin(-x)\ dx + \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$y = (\cos x)_{\frac{-\pi}{2}}^{0} + (-\cos x)_0^{\frac{\pi}{2}}$$

y = (1 – 0) + [0 – (-1)]

y = 2

19. Question

$\int_0^{\pi/2} \frac{1}{1+\tan x} dx$ is equal to

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. π

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(\cos x + \sin x)} dx$ ...(1)

By using the king rule

$$y = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2} - x\right)}{\left(\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)\right)} dx$$

$y = \int_0^{\frac{\pi}{2}} \frac{\sin x}{(\cos x + \sin x)} dx$ ...(2)

On adding eq(1) and eq(2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{(\cos x + \sin x)} dx$$

$$2y = \int_0^{\frac{\pi}{2}} dx$$

$$y = \frac{1}{2}(x)_0^{\frac{\pi}{2}}$$

$$y = \frac{\pi}{4}$$

20. Question

The value of $\int_0^{\pi/2} \cos x\, e^{\sin x}\, dx$ is

A. 1

B. e – 1

C. 0

D. –1

Answer

Let, sin x = t

Differentiating both sides with respect to x

$\frac{dt}{dx} = \cos x \Rightarrow dt = \cos x\, dx$

At x = 0, t = 0

At x = π/2, t = 1

$$y = \int_0^1 e^t dt$$

$$y = (e^t)_0^1$$

$y = e^1 - e^0$

y = e – 1

21. Question

If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{4}$ then a equals

A. $\frac{\pi}{2}$

B. $\frac{1}{2}$

C. $\frac{\pi}{4}$

D. 1

Answer

Given, $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{4}$

$= \int_0^a \frac{1}{(1)^2 + (2x)^2} dx = \frac{\pi}{4}$

$= \left[\frac{1}{1}\tan^{-1}\left(\frac{2x}{1}\right)\right]_0^a + c = \frac{\pi}{4}$

$= [\tan^{-1}(2a) - \tan^{-1}(0)] = \frac{\pi}{4}$

$= [\tan^{-1}(2a) - 0] = \frac{\pi}{4}$

$= \tan^{-1}(2a) = \frac{\pi}{4}$

$= 2a = \tan\frac{\pi}{4}$

= 2a=1

= a= �

Option A: it’s not option A, because clearly we got the value of ‘a’ as $\frac{1}{2}$ after solving.

Option C: it’s not option C, because clearly we got the value of ‘a’ as $\frac{1}{2}$ after solving.

Option D: it’s not option D, because clearly we got the value of ‘a’ as $\frac{1}{2}$ after solving.

22. Question

If $\int_0^1 f(x)dx = 1,\ \int_0^1 xf(x)dx = a,\ \int_0^1 x^2f(x)dx = a^2$, then $\int_0^1 (a-x)^2 f(x)dx$ equals

A. $4a^2$

B. 0

C. $2a^2$

D. none of these

Answer

Given, $\int_0^1 f(x)dx = 1,\ \int_0^1 xf(x)dx = a,\ \int_0^1 x^2f(x)dx = a^2,$

Now, $\int_0^1 (a-x)^2 f(x)dx$

$= \int_0^1 (a^2 - 2ax + x^2) f(x) dx$

$= \int_0^1 a^2 f(x)\, dx - \int_0^1 2axf(x)\, dx + \int_0^1 x^2 f(x) dx$

$= a^2 \int_0^1 f(x) dx - 2a \int_0^1 x\, f(x) dx + \int_0^1 x^2 f(x) dx$

$= a^2(1) - 2a(a) + a^2$

$= 2a^2 - 2a^2$

= 0

Option A:- it's not option A , this is clearly justified on solving.

Option C: - it's not option C , this is clearly justified on solving.

Option D: - it's not option D, this is clearly justified on solving.

23. Question

The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx$ is

A. $\frac{\pi^4}{2}$

B. $\frac{\pi^4}{4}$

C. 0

D. none of these

Answer

let, $I_{3,2} = \int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx$

$[\, if, I_{m,n} = \int \sin^m x cos^n x\ dx\ then\ I_{m,n} = -\frac{\sin^{m-1} x cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}]$

$= \frac{-\sin^2 x \cos^3 x}{5} + \frac{2}{5} I_{1,2}$

$= \frac{-\sin^2 x cos^3 x}{5} + \frac{2}{5}\left[\frac{-\sin^0 x cos^3 x}{3} + 0\right]$

$= \left[\frac{-\sin^2 x cos^3 x}{5} - \frac{2}{15}\cos^3 x\right]_{-\pi}^{\pi}$

$= \left[\frac{-\sin^2 \pi \cos^2 \pi}{5} - \frac{2}{15}\cos^3 \pi\right] - \left[\frac{-\sin^2(-\pi)\cos^2(-\pi)}{5} - \frac{2}{15}\cos^3(-\pi)\right]$

$= 0 - \frac{2}{15}(-1) - 0 + \frac{2}{15}(-1)$

= 0

Option A:- it's not option A , this is clearly justified on solving.

Option B: - it's not option B , this is clearly justified on solving.

Option D: - it's not option D, this is clearly justified on solving.

24. Question

$\int_{\pi/6}^{\pi/3} \frac{1}{\sin 2x} dx$ is equal to

A. $\log_e 3$

B. $\log_e \sqrt{3}$

C. $\frac{1}{2}\log(-1)$

D. log (–1)

Answer

Given, $\int_{\pi/6}^{\pi/3} \frac{1}{\sin 2x}\, dx$

$$= \int_{\pi/6}^{\pi/3} cosec\ 2x\, dx$$

$$= \left[\frac{\log|cosec\ 2x - \cot 2x|}{2}\right]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\log|cosec\ \frac{2\pi}{3} - \cot\frac{2\pi}{3}|\right] - \frac{1}{2}\left[\log|cosec\frac{\pi}{3} - \cot\frac{\pi}{3}|\right]$$

$$= \frac{1}{2}\log\left[\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right] - \frac{1}{2}\log\left[\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right] =$$

$$= \frac{1}{2}\log\left[\frac{3}{\sqrt{3}}\right] - \frac{1}{2}\log\left[\frac{1}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\log[\sqrt{3}] - \frac{1}{2}\log[\sqrt{3}]^{-1}$$

$$= \frac{1}{2}\log[\sqrt{3}] + \frac{1}{2}\log[\sqrt{3}]$$

$$= \log[\sqrt{3}]$$

Option A:- it's not option A , this is clearly justified on solving.

Option C: - it's not option C , this is clearly justified on solving.

Option D: - it's not option D, this is clearly justified on solving.

25. Question

$\int_{-1}^{1} |1 - x|\, dx$ is equal to

A. –2

B. 2

C. 0

D. 4

Answer

Given, $\int_{-1}^{1}|1-x|dx$

Now, $|1-x| = \begin{cases}(1-x)\ if\ x>1\\(1-x) if\ x<1\end{cases}$

$$= \int_{-1}^{0}(x-1)dx + \int_{0}^{1}(x-1)dx$$

$$= \left[x-\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{2}\right]_{0}^{1}$$

$$= \left[(0-0)-\left((-1)-\frac{1}{2}\right)\right] + \left[\left(1-\frac{1}{2}\right)-(0-0)\right]$$

$$= 0+1+\frac{1}{2}+1-\frac{1}{2}$$

= 2

Option A:- it's not option A , this is clearly justified on solving.

Option C:- it's not option C , this is clearly justified on solving.

Option D:- it's not option D, this is clearly justified on solving.

26. Question

The derivative of f(x) = $\int_{x^2}^{x^3}\frac{1}{\log_e t}dt, (x>0),$ is

A. $\frac{1}{3\ln x}$

B. $\frac{1}{3\ln x}-\frac{1}{2\ln x}$

C. $(\ln x)^{-1}$ x (x – 1)

D. $\frac{3x^2}{\ln x}$

Answer

$$f'(x) = \frac{1}{\log_e x^3}\frac{d}{dx}x^3 - \frac{1}{\log_e x^2}\frac{d}{dx}x^2$$

$$f'(x) = \frac{3x^2}{3\log_e x} - \frac{2x}{2\log_e x}$$

$$f'(x) = \frac{x^2-x}{\log_e x}$$

$$f'(x) = \frac{x(x-1)}{\log_e x}$$

f'(x) = $(\ln x)^{-1}$x(x-1)

27. Question

If $I_{10} = \int_0^{\pi/2} x^{10}$ sin x dx, then the value of $I_{10} + 90I_8$ is

A. $9\left(\frac{\pi}{2}\right)^9$

B. $10\left(\frac{\pi}{2}\right)^9$

C. $\left(\frac{\pi}{2}\right)^9$

D. $9\left(\frac{\pi}{2}\right)^8$

Answer

Use method of integration by parts

$$I_{10} = x^{10}\int \sin x\,dx - \int_0^{\frac{\pi}{2}} \frac{d}{dx}x^{10}\left(\int \sin x\,dx\right)dx$$

$$I_{10} = (x^{10}(-\cos x))_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 10x^9(-\cos x)dx$$

$$I_{10} = \left(\frac{\pi}{2}\right)^{10} \times (-0) - 0 \times (-1) - \int_0^{\frac{\pi}{2}} 10x^9(-\cos x)dx$$

$$I_{10} = 0 + 10\left(x^9\int \cos x\,dx - \int_0^{\frac{\pi}{2}} \frac{d}{dx}x^9\left(\int \cos x\,dx\right)dx\right)$$

$$I_{10} = 10\left((x^9 \sin x)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 9x^8 \sin x\,dx\right)$$

$$I_{10} = 10\left(\frac{\pi}{2}\right)^9 - 90I_8$$

$$I_{10} + 90I_8 = 10\left(\frac{\pi}{2}\right)^9$$

28. Question

$$\int_0^1 \frac{x}{(1-x)^{54}}dx =$$

A. $\frac{15}{16}$

B. $\frac{3}{16}$

C. $-\frac{3}{16}$

D. $-\frac{16}{3}$

Answer

We know that $y = \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$y = \int_0^1 \frac{x}{(1-x)^{54}}dx = \int_0^1 \frac{1+0-x}{(1-(1+0-x))^{54}}dx$$

$$y = \int_0^1 \frac{1-x}{x^{54}}dx$$

$$y = \int_0^1 \frac{1}{x^{54}} - \frac{1}{x^{53}}dx$$

$$y = \left(\frac{1}{-53x^{53}} - \frac{1}{-52x^{52}}\right)_0^1$$

29. Question

$\lim\limits_{n \to \infty}\left\{\frac{1}{2n+1} + \frac{1}{2n+2} + \ldots + \frac{1}{2n+n}\right\}$ is equal to

A. $\ln\left(\frac{1}{3}\right)$

B. $\ln\left(\frac{2}{3}\right)$

C. $\ln\left(\frac{3}{2}\right)$

D. $\ln\left(\frac{4}{2}\right)$

Answer

Given, $\lim\limits_{n\to\infty}\left\{\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{2n+n}\right\}$

$$= \lim_{n\to\infty}\left\{\frac{1}{n\left(2+\frac{1}{n}\right)} + \frac{1}{n\left(2+\frac{2}{n}\right)} + \cdots + \frac{1}{n(2+1)}\right\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\left\{\frac{1}{2+\frac{1}{n}} + \frac{1}{2+\frac{2}{n}} + \cdots + \frac{1}{2+1}\right\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{2+\frac{r}{n}}$$

Now for easy solvation, replace $\lim\limits_{n\to\infty}\sum_{r=1}^{n}$ $with$ $\int_0^1$, $\frac{r}{n}$ $with\ x\ and$ $\frac{1}{n}$ $with\ dx$

$$= \int_0^1 \frac{1}{2+x}\,dx$$

$$= [\log(2+x)]_0^1$$

$= \log(2+1) - \log(2+0)$

$= \log 3 - \log 2$

$= log\left(\frac{3}{2}\right)$

$= \ln\left(\frac{3}{2}\right)$

Option A:- it’s not option A , this is clearly justified on solving.

Option B:- it’s not option B, this is clearly justified on solving.

Option D:- it’s not option D, this is clearly justified on solving.

30. Question

The value of the integral $\int_{-2}^{2} |1-x^2|\, dx$ is

A. 4

B. 2

C. -2

D. 0

Answer

Given, $\int_{-2}^{2} |1-x^2|\, dx$

Now, $|1-x^2| = \begin{cases} -(1-x^2)\ if\ x < 0 \\ 1-x^2\ if\ x > 0 \end{cases}$

$= \int_{-2}^{0} |1-x^2|\, dx + \int_{0}^{2} |1-x^2| dx$

$= \int_{-2}^{0} -(1-x^2) dx + \int_{0}^{2} (1-x^2)\, dx$

$= \left[-\left(x - \frac{x^3}{3}\right)\right]_{-2}^{0} + \left[x - \frac{x^3}{3}\right]_{0}^{2}$

$= \left[(0-0) - \left(\frac{(-2)^3}{3} - (-2)\right)\right] + \left[\left(2 - \frac{(2)^3}{3}\right) - (0-0)\right]$

$= \frac{8}{3} - 2 + 2 - \frac{8}{3}$

= 0

Option A: - it’s not option A , this is clearly justified on solving.

Option B: - it’s not option B , this is clearly justified on solving.

Option C: - it’s not option C, this is clearly justified on solving.

31. Question

$\int_{0}^{\pi/2} \frac{1}{1+\cot^3 x} dx$ is equal to

A. 0

B. 1

C. π/2

D. π/4

Answer

$y = \int_0^{\frac{\pi}{2}} \frac{sin^3 x}{sin^3 x + cos^3 x} dx$ ...(1)

Use king's property $y = \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$y = \int_0^{\frac{\pi}{2}} \frac{sin^3\left(\frac{\pi}{2} - x\right)}{sin^3\left(\frac{\pi}{2} - x\right) + cos^3\left(\frac{\pi}{2} - x\right)} dx$$

$y = \int_0^{\frac{\pi}{2}} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$ ...(2)

On adding eq.(1) and (2)

$$2y = \int_0^{\frac{\pi}{2}} \frac{sin^3 x}{sin^3 x + cos^3 x} dx + \int_0^{\frac{\pi}{2}} \frac{cos^3 x}{sin^3 x + cos^3 x} dx$$

$$2y = \int_0^{\frac{\pi}{2}} \frac{sin^3 x + cos^3 x}{sin^3 x + cos^3 x} dx$$

$$y = \frac{1}{2} \int_0^{\frac{\pi}{2}} dx$$

$$y = \frac{1}{2}\left(\frac{\pi}{2} - 0\right)$$

$$y = \frac{\pi}{4}$$

32. Question

$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ equals to

A. π

B. π/2

C. π/3

D. π/4

Answer

Given, $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\ dx$

Now, $\sin x = A(\sin x + \cos x) + B\frac{d}{dx}(\sin x + \cos x)$

$\sin x = A(\sin x + \cos x) + B(\cos x - \sin x)$

Equating '$\sin x$' coeff:- Equating '$\cos x' coeff:-$

1=A-B 0=A+B

A-B=1

A+B=0

————2A=1

A=$\frac{1}{2}$

B=-$\frac{1}{2}$

$$\sin x=\frac{1}{2}(\sin x+\cos x)-\frac{1}{2}(\cos x-\sin x)$$

$$=\int_0^{\pi/2}\frac{\frac{1}{2}(\sin x+\cos x)-\frac{1}{2}(\cos x-\sin x)}{\sin x+\cos x}dx$$

$$=\int_0^{\pi/2}\frac{1}{2}\left(\frac{\sin x+\cos x}{\sin x+\cos x}\right)dx-\int_0^{\pi/2}\frac{1}{2}\left(\frac{\cos x-\sin x}{\sin x+\cos x}\right)dx$$

$$=\frac{1}{2}\int_0^{\pi/2}1\,dx-\frac{1}{2}\left[\log(\sin x+\cos x)\right]_0^{\pi/2}$$

$$=\frac{1}{2}[x]_0^{\pi/2}-\frac{1}{2}\left[\log(\sin x+\cos x)\right]_0^{\pi/2}$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-0\right]-\frac{1}{2}\left[\log(\sin\frac{\pi}{2}+\cos\frac{\pi}{2})-\log(\sin 0+\cos 0)\right]$$

$$=\frac{\pi}{4}-\frac{1}{2}[\log(1+0)-\log(0+1)]$$

$$=\frac{\pi}{4}-\frac{1}{2}[0-0]$$

$$=\frac{\pi}{4}-0$$

$$=\frac{\pi}{4}$$

Option A: - it’s not option A , this is clearly justified on solving.

Option B: - it’s not option B , this is clearly justified on solving.

Option C: - it’s not option C, this is clearly justified on solving.

33. Question

$\int_0^1\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\}dx$ is equal to

A. 0

B. π

C. π/2

D. π/4

Answer

Given, $\int_0^1\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\}dx$

$$= \int_0^1 \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \cdot \frac{d}{dx}\left(\frac{2x}{1+x^2}\right) dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{(1+x^2)^2-4x^2}} \cdot \frac{(1+x^2)2-2x(2x)}{(1+x^2)^2} dx$$

$$= \int_0^1 \frac{1}{\sqrt{(1-x^2)^2}} \cdot \frac{2+2x^2-4x^2}{1+x^2} dx$$

$$= \int_0^1 \frac{1}{1-x^2} \cdot \frac{2-2x^2}{1+x^2} dx$$

$$= \int_0^1 \frac{1}{1-x^2} \cdot \frac{2(1-x^2)}{1+x^2} dx$$

$$= \int_0^1 \frac{2}{1+x^2} dx$$

$$= 2[\tan^{-1}(x)]_0^1$$

$$= 2[\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= 2\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{2}$$

Option A: - it’s not option A , this is clearly justified on solving.

Option B: - it’s not option B , this is clearly justified on solving.

Option D: - it’s not option D, this is clearly justified on solving.

34. Question

$\int_0^{\pi/2} x \sin x \, dx$ is equal to

A. π/4

B. π/2

C. π

D. 1

Answer

Given, $\int_0^{\pi/2} x \sin x \, dx$

$$= \left[x \int_0^{\pi/2} \sin x \, dx\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} \left[1 \int_0^{\pi/2} \sin x \, dx\right] dx$$

$= [x(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)dx$

$= \left[\frac{\pi}{2}(0) - 0\right] + [\sin x]_0^{\pi/2}$

= 0 + (1-0)

= 1

Option A: - it’s not option A , this is clearly justified on solving.

Option B: - it’s not option B , this is clearly justified on solving.

Option C: - it’s not option C, this is clearly justified on solving.

35. Question

$\int_0^{\pi/2} \sin 2x \log \tan x \, dx$ is equal to

A. π

B. π/ 2

C. 0

D. 2π

Answer

Given, $\int_0^{\pi/2} \sin 2x \log(\tan x) dx$

$= \left[\log(\tan x) \int \sin 2x \, dx\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{d}{dx}[\log(\tan x) \int \sin 2x \, dx$

$= \left[\log(\tan x).\frac{-\cos 2x}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{1}{\tan x}(\sec^2 x).\frac{-\cos 2x}{2} \, dx$

= (0-0) $-\int_0^{\pi/2} \frac{\cos x}{\sin x}.\frac{1}{\cos^2 x}.\frac{-\cos 2x}{2} \, dx$

$= 0 - \int_0^{\pi/2} \frac{-\cos 2x}{2 \sin x \cos x} \, dx$

$= \int_0^{\pi/2} \frac{\cos 2x}{\sin 2x} \, dx$

$= \int_0^{\pi/2} \tan 2x \, dx)$

$= \left[\frac{\log|\sec 2x|}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}[(\log|\sec \pi| - \log|\sec 0|)]$

$= \frac{1}{2}(\log 1 - \log 1)$

$= \frac{1}{2}(0 - 0)$

= 0

Option A: - it’s not option A , this is clearly justified on solving.

Option B: - it’s not option B , this is clearly justified on solving.

Option D: - it’s not option D , this is clearly justified on solving

36. Question

The value of $\int_0^{\pi} \frac{1}{5 + 3\cos x} dx$ is

A. π/4

B. π/8

C. π/2

D. 0

Answer

Given, $\int_0^{\pi} \frac{1}{5+3\cos x}\, dx$

Now put, $\tan\frac{x}{2} = t$

$dx = \frac{2}{1+t^2} dt \; and \; \cos x = \frac{1-t^2}{1+t^2}$

Limits will also be changed accordingly,

From (0 to π) To (0 to ∞)

$= \int_0^{\infty} \frac{1}{5 + 3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt$

$= \int_0^{\infty} \frac{1+t^2}{5+5t^2+3-3t^2} \cdot \frac{2}{1+t^2} dt$

$= \int_0^{\infty} \frac{2}{2t^2+8}\, dt$

$= \int_0^{\infty} \frac{1}{4+t^2}\, dt$

$= \int_0^{\infty} \frac{1}{(t)^2 + (2)^2}\, dt$

$= \left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^{\infty}$

$= \frac{1}{2}[\tan^{-1}(\infty) - \tan^{-1}(0)]$

$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$

$= \frac{\pi}{4}$

Option B: - it’s not option B , this is clearly justified on solving.

Option C: - it’s not option C , this is clearly justified on solving.

Option D: - it’s not option D , this is clearly justified on solving

37. Question

$$\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{1}{1+x^2}\,dx =$$

A. π ln 2

B. –π ln 2

C. 0

D. $-\frac{\pi}{2}\ln 2$

Answer

Given, $\int_0^{\infty} \log\left(x+\frac{1}{x}\right)\frac{1}{1+x^2}\,dx$

Put $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Limits also will be changed accordingly,

x = 0 $\rightarrow\ \theta = 0$

x = ∞ $\rightarrow\ \theta = \frac{\pi}{2}$

$$= \int_0^{\pi/2} \log(\tan\theta + \cot\theta)\frac{1}{1+\tan^2\theta}\sec^2\theta\ d\theta$$

$$= \int_0^{\pi/2} \log(tan\theta + \cot\theta)\frac{1}{\sec^2\theta}\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \log(tan\theta + \cot\theta)\ d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} \log(\sin\theta\cos\theta)^{-1}\ d\theta$$

$$= -\int_0^{\pi/2} \log(\sin\theta\cos\theta)\, d\theta$$

$$= -\left[\int_0^{\pi/2} \log(\sin\theta) d\theta + \int_0^{\pi/2} \log(\cos\theta) d\theta\right]$$

(Some standard notations which we need to remember)

$$= -\left[-\frac{\pi}{2}\log 2 - \frac{\pi}{2}\log 2\right]$$

$$= -[-\pi \log 2]$$

$$= \pi \log 2$$

Option B: - it’s not option B, this is clearly justified on solving.

Option C: - it’s not option C , this is clearly justified on solving.

Option D: - it’s not option D , this is clearly justified on solving

38. Question

$\int_0^{2a} f(x)\, dx$ is equal to

A. $2\int_0^{a} f(x)\, dx$

B. 0

C. $\int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx$

D. $\int_0^{a} f(x)\, dx + \int_0^{2a} f(2a - x)\, dx$

Answer

We know that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

$$y = \int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_a^{2a} f(x)dx$$

$$A = \int_0^{a} f(x)dx$$

$$B = \int_a^{2a} f(x)dx$$

Let, t = 2a – x ⇒ x = 2a - t

Differentiating both side with respect to x

$\frac{dt}{dx} = -1$ ⇒ dx = -dt

At x = a, t =a

At x = 2a, t = 0

$$B = -\int_a^0 f(2a-t)dt$$

Use $\int_a^b f(x)dx = -\int_b^a f(x)dx$ and $\int_a^b f(x)dx = \int_a^b f(t)dt$

$$B = \int_0^a f(2a-x)dx$$

The final is y = A + B

$$y = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

39. Question

If f(a + b – x) = f(x), then $\int_a^b x f(x)\,dx$ is equal to

A. $\frac{a+b}{2}\int_a^b f(b-x)\,dx$

B. $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

C. $\frac{b-a}{2}\int_a^b f(x)\,dx$

D. $\frac{a+b}{2}\int_a^b f(x)\,dx$

Answer

Given, f (a + b - x) =f(x)

a + b – x = x

a + b = 2x

x = $\frac{a+b}{2}$

Now, $\int_a^b xf(x)dx$

$$= \int_a^b \frac{a+b}{2} f(x)dx$$

$$= \frac{a+b}{2}\int_a^b f(x)\,dx$$

Option A: - it’s not option A , this is clearly justified on solving.

Option B: - it’s not option B , this is clearly justified on solving.

Option C: - it’s not option C , this is clearly justified on solving

40. Question

The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$, is

A. 1

B. 0

C. –1

D. π/4

Answer

$y = \int_0^1 \tan^{-1}\left(\frac{x+x-1}{1-x(x-1)}\right) dx$

Use $\tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1} a + \tan^{-1} b$

$y = \int_0^1 \tan^{-1} x + \tan^{-1}(x-1)\, dx$

$y = \int_0^1 \tan^{-1} x - \tan^{-1}(1-x)\, dx$ ...(1)

Use king's property $y = \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$y = \int_0^1 \tan^{-1}(1-x) - \tan^{-1}(1-(1-x))\, dx$

$y = \int_0^1 \tan^{-1}(1-x) - \tan^{-1} x\, dx$ ...(2)

On adding eq(1) and (2)

$2y = \int_0^1 \tan^{-1} x - \tan^{-1}(1-x)\, dx + \int_0^1 \tan^{-1}(1-x) - \tan^{-1} x\, dx$

2y = 0

y = 0

41. Question

The value of $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

A. 2

B. 3/4

C. 0

D. –2

Answer

$y = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ ...(1)

Use king's property $y = \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$y = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right) dx$

$y = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$ ...(2)

On adding eq.(1) and (2)

$$2y = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx + \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$$

$$2y = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$$

$$2y = \int_0^{\frac{\pi}{2}} \log 1\, dx$$

y = 0

42. Question

The value of $\int\limits_{-\pi/2}^{\pi/2} \left(x^3 + x\cos x + \tan^5 x + 1\right) dx,$ is

A. 0

B. 2

C. π

D. 1

Answer

Given, $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\, dx$

$$= \left[\frac{x^4}{4}\right]_{-\pi/2}^{\pi/2} + \left\{[x(\sin x)]_{-\pi/2}^{\pi/2} - \int\limits_{-\pi/2}^{\pi/2} 1.\sin x\, dx\right\} + \int\limits_{-\pi/2}^{\pi/2} \tan^3 x.\tan^2 x + (x)_{-\pi/2}^{\pi/2}$$

$$= \left[\frac{x^4}{4}\right]_{-\pi/2}^{\pi/2} + \left[\frac{\pi}{2}\left(\sin\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\ (\sin\frac{-\pi}{2})\right)\right] + \int\limits_{-\pi/2}^{\pi/2} tan^3 x(1+\sec^2 x)\, dx + (x)_{-\pi/2}^{\pi/2}$$

$$= \left[\frac{x^4}{4}\right]_{-\pi/2}^{\pi/2} + \frac{\pi}{2}(1) - \frac{\pi}{2}(1) + \int_{-\pi/2}^{\pi/2} \tan x(1+\sec^2 x)dx + \int_{-\pi/2}^{\pi/2} \tan^3 x.\sec^2 x\, dx + (x)_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{2}\right)^4 - \left(\frac{-\pi}{2}\right)^4\right] + 0 + \int_{-\pi/2}^{\pi/2} \tan x + \int_{-\pi/2}^{\pi/2} \tan x.\sec^2 x\, dx + \left[\frac{\tan^4 x}{4}\right]_{\pi/2}^{\pi/2} + \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{4}(0) + 0 + [\log|\sec x|]_{-\pi/2}^{\pi/2} + \left[\frac{\tan^2 x}{2}\right]_{-\pi/2}^{\pi/2} + \left[\frac{\tan^4 x}{4}\right]_{-\pi/2}^{\pi/2} + \pi$$

$$= 0 + 0 + 0 + 0 + \pi$$

$$= \pi$$

Option A: - it’s not option A, this is clearly justified on solving.

Option B: - it’s not option B, this is clearly justified on solving.

Option D: - it’s not option D, this is clearly justified on solving