22. Differential Equations

Exercise 22.1

1. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 = e^t$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this question, the order of the differential equation is 3, and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is x and the term $\frac{dx}{dt}$ is multiplied by itself so the given equation is non-linear.

2. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^2y}{dx^2} + 4y = 0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this question, the order of the differential equation is 2, and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

Here dependent variable y and its derivatives are multiplied with a constant or independent variable only so this equation is linear differential equation.

3. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\left(\frac{dy}{dx}\right)^2 + \frac{1}{dy/dx} = 2$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this question, we first need to remove the term $\frac{1}{dy/dx}$ because this can be written as $(\frac{dy}{dx})^{-1}$ which means a negative power.

So, the above equation becomes as

$$(\frac{dy}{dx})^3 + 1 = 2\frac{dy}{dx}$$

So, in this, the order of the differential equation is 3, and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself so the given equation is non-linear.

4. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(c\frac{d^2y}{dx^2}\right)^{1/3}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

In this question we will be raising both the sides to power 6 so as to remove the fractional powers of derivatives of the dependent variable y

So, the equation becomes as

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = \left(c\frac{d^2y}{dx^2}\right)^2$$

$$1+\left(\frac{dy}{dx}\right)^6+3\left(\frac{dy}{dx}\right)^2+3\left(\frac{dy}{dx}\right)^4=(c)^2\left(\frac{d^2y}{dx^2}\right)^2$$

So, in this the order of the differential equation is 2 and the degree of the differential equation is 2.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself and many other are also, so the given equation is non-linear.

5. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+xy=0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this question the order of the differential equation is 2 and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself so the given equation is non-linear.

6. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Squaring on both sides, we get

$$\left(\sqrt[3]{\left(\frac{d^2y}{dx^2}\right)}\right)^2 = \frac{dy}{dx}$$

Cubing on both sides

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3$$

So, in this question, the <u>order of the differential equation is 2,</u> and the <u>degree of the differential equation is 2.</u>

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is <u>**linear**</u>.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself so the given equation is <u>non-linear</u>.

7. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^4y}{dx^4} = \left\{c + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Since this question has fractional powers, we need to remove them.

So, squaring on both sides, we get

$$\left(\frac{d^4y}{dx^4}\right)^2 = \left\{c + \left(\frac{dy}{dx}\right)^2\right\}^3$$

$$\left(\frac{d^4y}{dx^4}\right)^2 = c^3 + \left(\frac{dy}{dx}\right)^6 + 3c^2\left(\frac{dy}{dx}\right)^2 + 3c\left(\frac{dy}{dx}\right)^4$$

So, in this equation, the <u>order of the differential equation is 4,</u> and the <u>degree of the differential equation is 2.</u>

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is <u>**linear**</u>.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, also the degree of the equation is 2 which must be one for the equation to be <u>linear</u> so the given equation is <u>non-linear</u>.

8. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$x + \left(\frac{dy}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Since this question has fractional powers, we need to remove them.

So, squaring on both sides, we get

$$\left(x + \frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$x^2 + \left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$$

$$x^2 + 2x\frac{dy}{dx} - 1 = 0$$

So, in this equation, the order of the differential equation is 1 and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

Here dependent variable y and its derivatives are multiplied with a constant or independent variable only so this equation is linear differential equation.

9. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$y\frac{d^2x}{dy^2} = y^2 + 1$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Here in this question the dependent variable is x, and thus the order of the equation is 2, and the degree of the equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

Here dependent variable x and its derivatives are multiplied with a constant or independent variable only so this equation is linear differential equation.

10. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$s^2\frac{d^2t}{ds^2} + st\frac{dt}{ds} = s$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Here in this question the dependent variable is t, and thus the order of the equation is 2, and the degree of the equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

Here dependent variable t and its derivative is multiplied together $st\frac{dt}{ds}$ so this equation is non-linear differential equation.

11. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^4 = 0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

$$x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^4 = 0$$

So, in this equation the order of the differential equation is 2 and the degree of the differential equation is 3.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

Here dependent variable y and its derivative is multiplied together $y\frac{dy}{dx}$, also y is multiplied by itself so this equation is non-linear differential equation.

12. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} + 4y = \sin x$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this equation the order of the differential equation is 3 and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

Here dependent variable y's derivative is multiplied with itself $\left(\frac{d^2y}{dx^2}\right)^3$, so this equation is non-linear differential equation.

13. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$(xy^2 + x)dx + (y - x^2y)dy = 0$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

The above equation can be written as

$x(y^2+1)dx = y(x^2-1)dy$

$\frac{dy}{dx}(x^2-1)y = x(y^2+1)$

$\frac{dy}{dx}x^2y - \frac{dy}{dx}y = xy^2 + x$

So, from this equation it is clear that order of the differential equation is 1 and the degree of the differential equation is also 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by y and also y is multiplied by itself, so the given equation is non-linear.

14. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\sqrt{1-y^2}\,dx + \sqrt{1-x^2}\,dy = 0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

The above equation can be written as

$$\sqrt{1-x^2}\frac{dy}{dx} = -\sqrt{1-y^2}$$

Since the power of y can't be rational so squaring on both sides

$$\left(\sqrt{1-x^2}\frac{dy}{dx}\right)^2 = \left(-\sqrt{1-y^2}\right)^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 1-y^2$$

So, the order of the above differential equation 1 and the degree of the differential equation is 2

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, so the given equation is non-linear.

15. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^{2/3}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Since the power of $\frac{dy}{dx}$ is not rational we need to make it rational therefore cubing on both sides, we get

$$\left(\frac{d^2y}{dx^2}\right)^3 = \left(\frac{dy}{dx}\right)^2$$

So, the order of the above differential equation 2 and the degree of the differential equation is 3.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, so the given equation is non-linear.

16. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$2\frac{d^2y}{dx^2} + 3\sqrt{1-\left(\frac{dy}{dx}\right)^2 - y} = 0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

The above equation can be written as

$$2\frac{d^2y}{dx^2} = -3\sqrt{1-\left(\frac{dy}{dx}\right)^2 - y}$$

Since the equation has rational powers, we need to remove them so squaring both sides we get

$$4\left(\frac{d^2}{dx^2}y\right)^2 = 9\left(1 - y - \left(\frac{dy}{dx}\right)^2\right)$$

So, the order of the above differential equation 2 and the degree of the differential equation is 2.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, so the given equation is non-linear.

17. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$5\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Since the above equation has rational powers, we need to remove them so squaring on both sides.

$$25\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3$$

$$25\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^6 + 3\left(\frac{dy}{dx}\right)^2 + 3\left(\frac{dy}{dx}\right)^4$$

So, the order of the above differential equation 2 and the degree of the differential equation is 2.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, so the given equation is non-linear.

18. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

First of all, we will rearrange the above equation as follows

$$y - x\frac{dy}{dx} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Since the above equation has rational powers we need to remove them so squaring on both sides.

$$y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = a^2\left\{1+\left(\frac{dy}{dx}\right)^2\right\}$$

So, the order of the above differential equation 1 and the degree of the differential equation is 2.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, so the given equation is non-linear.

19. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$y = px + \sqrt{a^2p^2 + b^2}$, where $p = \frac{dy}{dx}$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

First of all, we will rearrange the above equation as follows

$y-x\frac{dy}{dx}= a\sqrt{\frac{b^2}{a^2}+\left(\frac{dy}{dx}\right)^2}$ here we have substituted the value of p and taken a^2 out from the root

Since the above equation has rational powers we need to remove them so squaring on both sides.

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left\{\frac{b^2}{a^2}+\left(\frac{dy}{dx}\right)^2\right\}$$

$$y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = a^2\left\{\frac{b^2}{a^2}+\left(\frac{dy}{dx}\right)^2\right\}$$

So, the order of the above differential equation 1 and the degree of the differential equation is 2.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself, so the given equation is non-linear.

20. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{dy}{dx} + e^{y} = 0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Concept of the question

$$e^{y} = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots + \frac{y^r}{r!}$$

So, the equation becomes as follows

$$\frac{dy}{dx} + 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots = 0$$

So, the order of the above differential equation 1 and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term y is multiplied by itself, so the given equation is non-linear.

21. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Concept of the question

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^r x^{2r+1}}{(2r+1)!}$$

So, in this question, the x of sin(x) is replaced by $\frac{d^2y}{dx^2}$ which means that the power of $\frac{d^2y}{dx^2}$ is not defined as it approaches to infinity by the above formula.

So, the order of the above differential equation 2 and the degree of the differential equation is **not defined**.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{d^2y}{dx^2}$ is multiplied by itself, so the given equation is non-linear.

22. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$(y'')^2 + (y')^2 + \sin y = 0$

Answer

Here in question $y'' = \frac{d^2y}{dx^2}$ and $y' = \frac{dy}{dx}$

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Concept of the question

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^r x^{2r+1}}{(2r+1)!}$$

So, in this question, the x of sin(x) is replaced by y which means that the power of y is not defined as it approaches infinity by the above formula

So, the order of the above differential equation 2 and the degree of the differential equation is 2.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y, and the term y is multiplied by itself, so the given equation is non-linear.

23. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right) - 6y = \log x$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this question the order of the differential equation is 2 and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself so the given equation is non-linear.

24. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y \sin y = 0$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Concept of the question

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^r x^{2r+1}}{(2r+1)!}$$

So, in this question, the x of sin(x) is replaced by y which means that the power of y is not defined as it approaches infinity by the above formula

So, the order of the above differential equation 3 and the degree of the differential equation is 1.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question the dependent variable is y, and the term y is multiplied by itself, so the given equation is non-linear.

25. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

Concept of the question

For the degree to be defined of any differential equation the euqtion must be expressible in the form of a polynomial.

But, in this question the degree of the differential equation is not defined because the term on the right hand side is not expressible in the form of a polynomial.

Thus, the order of the above equation is 2 whereas the degree is not defined.

Since the degree of the equation is not defined the equation is non-linear.

26. Question

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear.

$$\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x$$

Answer

The order is the highest numbered derivative in the equation with no negative or fractional power of the dependent variable and its derivatives, while the degree is the highest power to which a derivative is raised.

So, in this question, the order of the differential equation is 1, and the degree of the differential equation is 3.

In a **differential equation**, when the dependent variable and their derivatives are only multiplied by constants or independent variable, then the **equation** is **linear**.

So, in this question, the dependent variable is y and the term $\frac{dy}{dx}$ is multiplied by itself so the given equation is non-linear.

Exercise 22.2

1. Question

Form the differential equation of the family of curves represented by $y^2 = (x - c)^3$.

Answer

$y^2 = (x - c)^3$

On differentiating the above equation with respect to x we get

$$2y\frac{dy}{dx} = 3(x-c)^2$$

$$\Rightarrow (x-c)^2 = \frac{2y}{3}\frac{dy}{dx}$$

$$\Rightarrow (x-c) = \left(\frac{2y}{3}\frac{dy}{dx}\right)^{\frac{1}{2}}$$

Putting the value of (x – c) in the given equation, we get,

$$y^2 = \left(\left(\frac{2y}{3}\frac{dy}{dx}\right)^{\frac{1}{2}}\right)^3$$

$$\Rightarrow y^2 = \left(\frac{2y}{3}\frac{dy}{dx}\right)^{\frac{3}{2}}$$

On squaring, both sides we get,

$$y^4 = \left(\frac{2y}{3}\frac{dy}{dx}\right)^3$$

$$\Rightarrow y^4 = \frac{8y^3}{27}\left(\frac{dy}{dx}\right)^3$$

$$\Rightarrow 27y = 8\left(\frac{dy}{dx}\right)^3$$

Hence, $27y = 8\left(\frac{dy}{dx}\right)^3$ is the differential equation which represents the family of curves $y^2 = (x - c)^3$.

2. Question

Form the differential equation corresponding to $y = e^{mx}$ by eliminating m.

Answer

Given equation, $y = e^{mx}$

On differentiating the above equation with respect to x we get

$$\frac{dy}{dx} = me^{mx}$$

But $y = e^{mx}$

$$\therefore \frac{dy}{dx} = my$$

Now we have, $y = e^{mx}$

Applying log on both sides, we get,

$\log y = mx$

which gives $m = \frac{\log y}{x}$

So, putting this value of m in $\frac{dy}{dx} = my$ we get

$$\frac{dy}{dx} = y\frac{\log y}{x}$$

$$\Rightarrow x\frac{dy}{dx} = y\log y$$

Hence, $x\frac{dy}{dx} = y\log y$ is the differential equation corresponding to $y = e^{mx}$.

3 A. Question

Form the differential equation from the following primitives where constants are arbitrary:

$y^2 = 4ax$

Answer

$$\Rightarrow a = \frac{y^2}{4x}$$

On differentiating with respect to x, we get $2y\left(\frac{dy}{dx}\right) = 4a$

On substituting the value of a we get,

$$2y\left(\frac{dy}{dx}\right) = 4\frac{y^2}{4x}$$

$$\Rightarrow 2y\left(\frac{dy}{dx}\right) = \frac{y^2}{x}$$

$$\Rightarrow 2x\left(\frac{dy}{dx}\right) = y$$

Hence, $2x\left(\frac{dy}{dx}\right) = y$ is the differential equation corresponding to

$y^2 = 4ax$.

3 B. Question

Form the differential equation from the following primitives where constants are arbitrary:

$y = cx + 2c^2 + c^3$

Answer

On differentiating with respect to x, we get,

$$\frac{dy}{dx} = c$$

Putting this value of c in the given equation we get

$$y = x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3$$

Hence, $y = x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (\frac{dy}{dx})^3$ is the differential equation corresponding to $y = cx + 2c^2 + c^3$.

3 C. Question

Form the differential equation from the following primitives where constants are arbitrary:

$xy = a^2$

Answer

Again, differentiating with respect to x we get,

$$x\left(\frac{dy}{dx}\right) + y = 0$$

Hence, $x\left(\frac{dy}{dx}\right) + y = 0$ is the differential equation corresponding to $xy = a^2$.

3 D. Question

Form the differential equation from the following primitives where constants are arbitrary:

$y = ax^2 + bx + c$

Answer

As the given equation has 3 different arbitrary constants so we can differentiate it thrice with respect to x

So, differentiating once with respect to x,

$$\frac{dy}{dx} = 2ax + b$$

Differentiating twice with respect to x,

$$\left(\frac{d^2y}{dx^2}\right) = 2a$$

Now, differentiating thrice with respect to x we get,

$$\frac{d^3y}{dx^3} = 0$$

Hence, $\frac{d^3y}{dx^3} = 0$ is the differential equation corresponding to

$y = ax^2 + bx + c$.

4. Question

Form the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.

Answer

$y = Ae^{2x} + Be^{-2x}$

As the equating has two different arbitrary constants so, we can differentiate it twice with respect to x. So, on differentiating once with respect to x we get,

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$$

Again, differentiating it with respect to x, we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4(Ae^{2x} + Be^{-2x})$$

But, $Ae^{2x} + Be^{-2x} = y$ (Given)

$$\therefore \frac{d^2y}{dx^2} = 4y$$

Hence the differential equation corresponding to the curves

$y = Ae^{2x} + Be^{-2x}$ is $\frac{d^2y}{dx^2} = 4y$

5. Question

Form the differential equation of the family of curves,

x = A cos nt + B sin nt, where A and B are arbitrary constant.

Answer

As the given equation has two different arbitrary constants so we can differentiate it twice with respect to x.

x = A cos nt + B sin nt

On differentiating with respect to t we get,

$$\frac{dx}{dt} = -An\sin nt + Bn\cos nt$$

Again, differentiating with respect to x,

$$\frac{d^2x}{dt^2} = -An^2\cos nt - Bn^2\sin nt$$

$$\Rightarrow \frac{d^2x}{dt^2} = -n^2(A\cos nt + B\sin nt)$$

As x = A cos nt + B sin nt

$$\therefore \frac{d^2x}{dt^2} = -n^2x$$

$$\Rightarrow \frac{d^2x}{dt^2} + n^2x = 0$$

Hence, $\frac{d^2x}{dt^2} + n^2x = 0$ is the required differential equation.

6. Question

Form the differential equation corresponding to $y^2 = a(b - x^2)$ by eliminating a and b.

Answer

Given equation $y^2 = a(b - x^2)$

On differentiating with respect to x, we get,

$$2y\frac{dy}{dx} = -2ax \ldots\ldots(1)$$

Again, differentiating with respect to x we get,

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = -2a$$

$$\Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \ldots\ldots(2)$$

From (1) we have $a = -\frac{y}{x}\frac{dy}{dx}$

On putting, this value in (2) we get,

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\left(-\frac{2y}{2x}\frac{dy}{dx}\right)$$

$$\Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \left(\frac{y}{x}\frac{dy}{dx}\right)$$

$$\Rightarrow x\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = \left(y\frac{dy}{dx}\right)$$

So, the required differential equation is $x\left(y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right) = \left(y\frac{dy}{dx}\right)$.

7. Question

Form the differential equation corresponding to $y^2 - 2\,ay + x^2 = a^2$ by eliminating a.

Answer

$y^2 - 2\,a\,y + x^2 = a^2$

On differentiating, with respect to x we get,

$$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$$

$$\Rightarrow y\frac{dy}{dx} + x = a\frac{dy}{dx}$$

$$\Rightarrow a = \frac{\left(y\frac{dy}{dx} + x\right)}{\frac{dy}{dx}}$$

Putting this value of a in the given equation, we get,

$$y^2 - 2a\left[\frac{\left(y\frac{dy}{dx} + x\right)}{\frac{dy}{dx}}\right]y + x^2 = \left[\frac{\left(y\frac{dy}{dx} + x\right)}{\frac{dy}{dx}}\right]^2$$

$$\text{Put } \frac{dy}{dx} = y'$$

$$y^2 - 2a\left(\frac{yy' + x}{y'}\right)y + x^2 = \left(\frac{yy' + x}{y'}\right)^2$$

$$\Rightarrow \frac{y^2y' - 2(y^2y' + xy) + x^2y'}{y'} = \frac{y^2y'^2 + 2xyy' + x^2}{y'^2}$$

$\Rightarrow y^2y'^2 - 2y^2y'^2 - 2xyy' + x^2y'^2 = y^2\,y'^2 + 2xyy' + x^2$

$\Rightarrow y^2y'^2 - 2y^2y'^2 - 2xyy' + x^2y'^2 - y^2\,y'^2 - 2xyy' - x^2 = 0$

$\Rightarrow -\,4xyy' + y'^2x^2 - x^2 - 2y'^2y^2 = 0$

$\Rightarrow y'^2(x^2 - 2y^2) - 4xyy' - x^2 = 0$

So, $y'^2(x^2 - 2y^2) - 4xyy' - x^2 = 0$

8. Question

Form the differential equation corresponding to $(x - a)^2 + (y - b)^2 = r^2$ by eliminating a and b.

Answer

$(x - a)^2 + (y - b)^2 = r^2$ …… (i)

On differentiating with respect to x, we get,

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0$$

$$\Rightarrow (x - a) + (y - b)\frac{dy}{dx} = 0 \ \ldots\ldots (ii)$$

Again, differentiating with respect to x we get,

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow y - b = -\frac{\left(\frac{dy}{dx}\right)^2 + 1}{\frac{d^2y}{dx^2}} \dots\dots (iii)$$

Put the value of (y – b) obtained in (ii) we get,

$$\Rightarrow x - a - \left(\frac{\left(\frac{dy}{dx}\right)^2 + 1}{\frac{d^2y}{dx^2}}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow x - a = \frac{\left(\frac{dy}{dx}\right)^2 + 1}{\frac{d^2y}{dx^2}}\frac{dy}{dx}$$

$$\Rightarrow x - a = \frac{\left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx}}{\frac{d^2y}{dx^2}} \dots\dots (iv)$$

Put the value of (x – a) and (y – b) in (i) we get,

$$\left(\frac{\left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx}}{\frac{d^2y}{dx^2}}\right)^2 + \left(\frac{\left(\frac{dy}{dx}\right)^2 + 1}{\frac{d^2y}{dx^2}}\right)^2 = r^2$$

Put $\frac{dy}{dx}$ as y' and $\frac{d^2y}{dx^2}$ as y'' we get,

$$\left(\frac{y'^3 + y'}{y''}\right)^2 + \left(\frac{y'^2 + 1}{y''}\right)^2 = r^2$$

$\Rightarrow (y'^3 + y')^2 + (y'^2 + 1)^2 = r^2y''^2$

So, the required differential equation is $(y'^3 + y')^2 + (y'^2 + 1)^2 = r^2y''^2$.

9. Question

Form the differential equation of all the circles which pass through the origin and whose centers lie on the y – axis.

Answer

Any circle with centre at (h, k) and radius r is given by,

$(x - h)^2 + (y - k)^2 = r^2$

Here centre is on y – axis, so h = 0

So, we have the equation of circle as, $x^2 + (y - k)^2 = r^2$

Further, it is given that circle passes through the origin (0,0) therefore origin must satisfy the equation of circle. So, we get,

$0 + k^2 = r^2$

So, the equation of circle is $x^2 + (y - k)^2 = k^2$

$\Rightarrow x^2 + y^2 - 2ky = 0$

$\Rightarrow x^2 + y^2 = 2ky$

$$\Rightarrow\ k = \frac{x^2 + y^2}{2y}$$

Now, differentiating it with respect to x we get,

$$0 = \frac{2y\left(2x + 2y\frac{dy}{dx}\right) - (x^2 + y^2)2\frac{dy}{dx}}{(2y)^2}$$

$$\Rightarrow\ 0 = 4xy + 4y^2\frac{dy}{dx} - 2x^2\frac{dy}{dx} - 2y^2\frac{dy}{dx}$$

$$\Rightarrow\ 0 = 2y^2\frac{dy}{dx} - 2x^2\frac{dy}{dx} + 4xy$$

$$\Rightarrow\ -y^2\frac{dy}{dx} + x^2\frac{dy}{dx} = 2xy$$

$$\Rightarrow\ (x^2 - y^2)\frac{dy}{dx} = 2xy$$

Hence, the required differential equation is $(x^2 - y^2)\frac{dy}{dx} = 2xy$

10. Question

Find the differential equation of all the circles which pass through the origin and whose centers lie on the x - axis.

Answer

Any circle with centre at (h, k) and radius r is given by,

$(x – h)^2 + (y – k)^2 = r^2$

Here centre is on x - axis, so k = 0

So, we have the equation of circle as, $(x – h)^2 + y^2 = r^2$

Further it is given that circle passes through origin (0,0) therefore origin must satisfy equation of circle. So, we get,

$0 + h^2 = r^2$

So, the equation of circle is $(x – h)^2 + y^2 = h^2$

$\Rightarrow x^2 – 2hx + y^2 = 0$

$\Rightarrow x^2 + y^2 = 2hx$

$$\Rightarrow\ h = \frac{x^2 + y^2}{2x}$$

Now, differentiating it with respect to x we get,

$$0 = \frac{2x\left(2x + 2y\frac{dy}{dx}\right) - (x^2 + y^2)2}{(2x)^2}$$

$$\Rightarrow\ 2x\left(x + y\frac{dy}{dx}\right) - (x^2 + y^2) = 0$$

$$\Rightarrow\ 2x^2 + 2xy\frac{dy}{dx} - x^2 - y^2 = 0$$

$$\Rightarrow\ (x^2 - y^2) + 2xy\frac{dy}{dx} = 0$$

Hence, the required differential equation is $(x^2 - y^2) + 2xy\frac{dy}{dx} = 0$

11. Question

Assume that a raindrop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the raindrop.

Answer

Let r be the radius of the raindrop, V be its volume and A be its surface area

Given

$$\frac{dV}{dt} \propto A \Rightarrow \frac{dV}{dt} = -kA$$

Negative because V decreases with an increase in t

k is a proportionality constant

Now, we know that

$$V = \frac{4}{3}\pi r^3 \text{ and } A = 4\pi r^2$$

So, we have,

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k(4\pi r^2)$$

$$\Rightarrow 4\pi r^2\frac{dr}{dt} = -k(4\pi r^2)$$

$$\Rightarrow \frac{dr}{dt} = -k$$

Hence, the required differential equation is $\frac{dr}{dt} = -k$

12. Question

Find the differential equation of all the parabolas with latus rectum ‘4a' and whose axes are parallel to the x - axis.

Answer

Equation of parabola with latus rectum ‘4a’ and axes parallel to x - axes and vertex at (h,k) is given by

$(y – k)^2 = 4a(x – h)$

On differentiating with respect to x we get,

$$2(y-k)\frac{dy}{dx} = 4a$$

$$\Rightarrow (y-k)\frac{dy}{dx} = 2a \;\ldots\ldots (i)$$

Again differentiating (i) with respect to x we get,

$$(y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

From (i) we have $(y-k) = \frac{2a}{\frac{dy}{dx}}$, on substituting it in the above equation we get,

$$\frac{2a}{\frac{dy}{dx}}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow 2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$$

Hence, the required differential equation is $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

13. Question

Show that the differential equation of which $y = 2\left(x^2 - 1\right) + ce^{-x^2}$ is a solution, is $\frac{dy}{dx} + 2xy = 4x^3$.

Answer

$$y = 2(x^2 - 1) + c\,e^{-x^2}$$

On differentiating with respect to x we have,

$$\frac{dy}{dx} = 4x - 2cxe^{-x^2}$$

Now,

$$\frac{dy}{dx} + 2xy = 4x - 2cxe^{-x^2} + 2x\{2(x^2 - 1) + ce^{-x^2}\}$$

$$= 4x - 2cxe^{-x^2} + 4x^3 - 4x + 2ce^{-x^2}$$

$= 4x^3$

Which is the given equation.

Hence, $y = 2\left(x^2 - 1\right) + ce^{-x^2}$ is solution to the given differential equation.

14. Question

Form the differential equation having $y = (\sin^{-1} x)^2 + A\cos^{-1} x + B$, where A and B are arbitrary constants, as its general solution.

Answer

$$y = (\sin^{-1}x)^2 + A\cos^{-1}x + B$$

On differentiating with respect to x we get,

$$\frac{dy}{dx} = 2\sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right) + A\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x - A$$

Again, differentiating with respect to x we have,

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right)(-2x) = 2 \times \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2 = 0$$

Hence the required differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2 = 0$$

15. Question

Form the differential equation of the family of curves represented by the equation (a being the parameter):

i. $(2x + a)^2 + y^2 = a^2$

ii. $(2x - a)^2 - y^2 = a^2$

iii. $(x - a)^2 + 2y^2 = a^2$

Answer

(i)

$(2x + a)^2 + y^2 = a^2$

On differentiating, with respect to x we have,

$$2(2x + a) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (2x + a) + y\frac{dy}{dx} = 0$$

$$\Rightarrow a = -2x - y\frac{dy}{dx}$$

Putting this value of a in the given equation we get,

$$\left(2x - 2x - y\frac{dy}{dx}\right)^2 + y^2 = \left(-2x - y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(y\frac{dy}{dx}\right)^2 + y^2 = \left(4x^2 + \left(y\left(\frac{dy}{dx}\right)\right)^2 + 4xy\frac{dy}{dx}\right)$$

$$\Rightarrow y^2 - 4x^2 - 4xy\frac{dy}{dx} = 0$$

ii. $(2x - a)^2 - y^2 = a^2$

$\Rightarrow 4x^2 + a^2 - 4ax - y^2 = a^2$

$\Rightarrow 4x^2 - 4ax - y^2 = 0$

$\Rightarrow 4ax = 4x^2 - y^2$

$$\Rightarrow a = \frac{4x^2 - y^2}{4x}$$

On differentiating with respect to x we get,

$$0 = \left[\frac{\left\{\left(8x - 2\frac{dy}{dx}\right)4x - 4(4x^2 - y^2)\right\}}{4}\right]$$

$$\Rightarrow 8x^2 - 2x\frac{dy}{dx} - 4x^2 + y^2 = 0$$

$$\Rightarrow 4x^2 + y^2 = 2x\frac{dy}{dx}$$

iii. $(x - a)^2 + 2y^2 = a^2$

On differentiating, with respect to x we have,

$$2(x - a) + 4y\frac{dy}{dx} = 0$$

$$\Rightarrow\ (x-a) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ a = x + 2y\frac{dy}{dx}$$

Putting this value of a in the given equation we get,

$$\left(x-\left(x + 2y\frac{dy}{dx}\right)\right)^2 + 2y^2 = \left(x + 2y\frac{dy}{dx}\right)^2$$

$$\Rightarrow\ \left(4y\frac{dy}{dx}\right)^2 + 2y^2 = \left(x^2 + \left(4y\left(\frac{dy}{dx}\right)\right)^2 + 4xy\frac{dy}{dx}\right)$$

$$\Rightarrow\ 2y^2 - x^2 - 4xy\frac{dy}{dx} = 0$$

16 A. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$x^2 + y^2 = a^2$

Answer

On differentiating we get,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ x + y\frac{dy}{dx} = 0$$

16 B. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$x^2 - y^2 = a^2$

Answer

On differentiating we get,

$$2x - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow\ x - y\frac{dy}{dx} = 0$$

16 C. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$y^2 = 4ax$

Answer

$$\frac{y^2}{x} = 4a$$

On differentiating we get,

$$\frac{2xy\frac{dy}{dx} - y^2}{x^2} = 0$$

$$\Rightarrow 2xy\frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow 2x\frac{dy}{dx} - y = 0$$

16 D. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$x^2 + (y - b)^2 = 1$

Answer

On differentiating we get,

$$2x + 2(y - b)\frac{dy}{dx} = 0$$

$$\Rightarrow (y - b)\frac{dy}{dx} = -x$$

$$\Rightarrow (y - b) = \frac{-x}{\frac{dy}{dx}} \;\ldots\ldots (ii)$$

Put (ii) in (i),

$$x^2 + \left(\frac{-x}{\frac{dy}{dx}}\right)^2 = 1$$

$$\Rightarrow x^2\left(\frac{dy}{dx}\right)^2 + x^2 = \left(\frac{dy}{dx}\right)^2$$

16 E. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$(x - a)^2 - y^2 = 1$

Answer

Differentiating with respect to x we get,

$$2(x - a) - 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (x - a) = y\frac{dy}{dx}$$

Now putting the value of (x – a) in the initial equation, we get

$$y^2\left(\frac{dy}{dx}\right)^2 - y^2 = 1$$

16 F. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Answer

$$\frac{b^2x^2-a^2y^2}{a^2b^2} = 1$$

$b^2x^2 - a^2y^2 = a^2b^2$

On differentiating with respect to x we get,

$$2b^2x-2a^2y\frac{dy}{dx} = 0$$

$$\Rightarrow b^2x-a^2y\frac{dy}{dx} = 0 \; \ldots\ldots (i)$$

Again, differentiating with respect to x we get,

$$b^2-a^2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right) = 0$$

$$\Rightarrow b^2 = a^2\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)$$

Putting this value of b^2 in (i) we get,

$$\Rightarrow a^2x\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)-a^2y\frac{dy}{dx} = 0$$

$$\Rightarrow a^2\left(x\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)-y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(x\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)-y\frac{dy}{dx}\right) = 0$$

16 G. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$y^2 = 4a(x - b)$

Answer

On differentiating with respect to x

$$2y\frac{dy}{dx} = 4a$$

Again, differentiating with respect to x we get,

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0$$

16 H. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$y = ax^3$

Answer

$y = ax^3$

On differentiating with respect to x we get,

$$\frac{dy}{dx} = 3ax^2$$

From the given equation $a = \frac{y}{x^3}$

So, we have

$$\frac{dy}{dx} = 3\frac{y}{x^3}x^2$$

$$\Rightarrow \frac{dy}{dx} = 3\frac{y}{x}$$

$$\Rightarrow x\frac{dy}{dx} = 3y$$

16 I. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$x^2 + y^2 = ax^3$

Answer

$x^2 + y^2 = ax^3$

$$a = \frac{x^2 + y^2}{x^3}$$

Differentiating with respect to x,

$$0 = \frac{\left(\left(2x + 2y\frac{dy}{dx}\right)x^3 - 3x^2(x^2 + y^2)\right)}{x^6}$$

$$\Rightarrow 2x^4 + 2x^3y\frac{dy}{dx} - 3x^4 - 3x^2y^2 = 0$$

$$\Rightarrow 2x^3y\frac{dy}{dx} - x^4 - 3x^2y^2 = 0$$

$$\Rightarrow 2x^3y\frac{dy}{dx} = x^4 + 3x^2y^2$$

$$\Rightarrow 2x^3y\frac{dy}{dx} = x^2(x^2 + 3y^2)$$

$$\Rightarrow 2xy\frac{dy}{dx} = (x^2 + 3y^2)$$

16 J. Question

Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):

$y = e^{ax}$

Answer

Differentiating with respect to x

$$\frac{dy}{dx} = a\,e^{ax}$$

$\Rightarrow \frac{dy}{dx} = ay$

From the given equation we have,

$y = e^{ax}$

= >log y = ax

$\Rightarrow a = \frac{\log y}{x}$

Now,

$\Rightarrow \frac{dy}{dx} = ay = y\frac{\log y}{x}$

$\Rightarrow x\frac{dy}{dx} = y\log y$

17. Question

Form the differential equation representing the family of ellipses having the center at the origin and foci on the x - axis.

Answer

Equation of required ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ...... (i)}$

Where a,b are arbitrary constants

Differentiating (i) with respect to x we get,

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\Rightarrow \frac{x}{a^2} = -\frac{y}{b^2}\frac{dy}{dx}$

$\Rightarrow -\frac{b^2}{a^2} = \frac{y}{x}\frac{dy}{dx} \text{ ...... (ii)}$

Now, differentiating (ii) with respect to x we get,

$0 = \frac{y}{x}\frac{d^2y}{dx^2} + \frac{\left(x\frac{dy}{dx} - y\right)}{x^2}\frac{dy}{dx}$

$\Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

The required differential equation is $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

18. Question

Form the differential equation of the family of hyperbolas having foci on x - axis and center at the origin.

Answer

Equation of required hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ ...... (i)}$

Where a,b are arbitrary constants

Differentiating with respect to x we get,

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} - \frac{y}{b^2}\frac{dy}{dx} = 0 \text{ ...... (ii)}$$

Again, differentiating with respect to x we get,

$$\frac{1}{a^2} - \frac{1}{b^2}\left(\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right) = 0$$

$$\Rightarrow \frac{1}{a^2} = \frac{1}{b^2}\left(\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right)$$

Substituting this value of $\frac{1}{a^2}$ in (ii) we get,

$$\Rightarrow x\left(\frac{1}{b^2}\left(\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right)\right) - \frac{y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow x\left(\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right) - y\frac{dy}{dx} = 0$$

$$\Rightarrow x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$$

$$\Rightarrow x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$$

The required differential equation is $x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$

19. Question

Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Answer

Let C denote the family of circles in the second quadrant and touching the coordinate axes and let (– a, a) be co - ordinate of the centre of any member of this circle

Now, the equation representing this family of circle is $(x + a)^2 + (y – a)^2 = a^2$ ...... (i)

$\Rightarrow x^2 + y^2 + 2ax – 2ay + a^2 = 0$ ...... (ii)

Differentiating (ii) with respect to x we get,

$$\Rightarrow 2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} + a - a\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} = -a + a\frac{dy}{dx}$$

$$\Rightarrow a = \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx} - 1}$$

Substituting this value of a in (i) we get,

$$\left[x + \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx} - 1}\right]^2 + \left[y - \frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx} - 1}\right]^2 = \left[\frac{\left(x + y\frac{dy}{dx}\right)}{\frac{dy}{dx} - 1}\right]^2$$

$$\Rightarrow \left[x\left(\frac{dy}{dx} - 1\right) + \left(x + y\frac{dy}{dx}\right)\right]^2 + \left[y\left(\frac{dy}{dx} - 1\right) + \left(x + y\frac{dy}{dx}\right)\right]^2$$
$$= \left[\left(x + y\frac{dy}{dx}\right)\right]^2$$

$$\Rightarrow (x + y)^2\left(\frac{dy}{dx}\right)^2 + (x + y)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

$$\Rightarrow ((x + y)^2 + 1)\left(\frac{dy}{dx}\right)^2 = \left(x + y\frac{dy}{dx}\right)^2$$

The required differential equation is $((x + y)^2 + 1)\left(\frac{dy}{dx}\right)^2 = \left(x + y\frac{dy}{dx}\right)^2$

Exercise 22.3

1. Question

Show that $y = be^x + ce^{2x}$ is a solution of the differential equation, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$

Answer

The differential equation is $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ and the function that is to be proven as solution is

$y = be^x + ce^{2x}$, now we need to find the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$.

$\frac{dy}{dx} = be^x + 2ce^{2x}$

$\frac{d^2y}{dx^2} = be^x + 4ce^{2x}$

Putting the values of these variables in the differential equation, we get,

$be^x + 4ce^{2x} - 3(be^x + 2ce^{2x}) + 2(be^x + ce^{2x}) = 0$,

0 = 0

As, L.H.S = R.H.S. the equation is satisfied. Hence, this function is the solution of the differential equation.

2. Question

Verify that y = 4 sin 3x is a solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 0.$

Answer

The differential equation is $\frac{d^2y}{dx^2} + 9y = 0$ and the function that is to be proven as the solution is

y = 4 sin 3x, now we need to find $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = 12 \cos 3x$

$\frac{d^2y}{dx^2} = -36 \sin 3x$

Putting the values in the equation, we get,

$-36 \sin 3x + 9(4 \sin 3x) = 0$,

$0 = 0$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

3. Question

Show that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

Answer

The differential equation is $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$ and the function that is to be proven as solution is

$y = ae^{2x} + be^{-x}$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = 2ae^{2x} - be^{-x}$

$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$

Putting these values in the equation, we get,

$4ae^{2x} + be^{-x} - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x}) = 0$,

$0 = 0$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

4. Question

Show that the function $y = A \cos x + B \sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0.$

Answer

The differential equation is $\frac{d^2y}{dx^2} + y = 0$ and the function that is to be proven as solution is

$y = A \cos x + B \sin x$, now we need to find the value of $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = -A \sin x + B \cos x$

$\frac{d^2y}{dx^2} = -A \cos x - B \sin x$

Putting the values in equation, we get,

$-A \cos x - B \sin x + A \cos x + B \sin x = 0$,

$0 = 0$

As, L.H.S = R.H.S. the equation is satisfied, hence this function is the solution of the differential equation.

5. Question

Show that the function $y = A \cos 2x - B \sin 2x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 0.$

Answer

The differential equation is $\frac{d^2y}{dx^2} + 4y = 0$ and the function that is to be proven as solution is y = A cos2x – B sin 2x, now we find the value of $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx}$ = –2A sin 2x – 2B cos 2x

$\frac{d^2y}{dx^2}$ = –4A cos 2x + 4B sin 2x

Putting the values in the equation, we get,

–4A cos 2x + 4B sin 2x + 4(A cos 2x – B sin 2x) = 0,

0 = 0

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

6. Question

Show that y = Ae^{Bx} is a solution of the differential equation $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$.

Answer

The differential equation is $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2$ and the function to be proven as the solution is y = Ae^{Bx}, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx}$ = ABe^{Bx}

$\frac{d^2y}{dx^2}$ = AB^2e^{Bx}

Putting values in the equation,

$AB^2e^{Bx} = \frac{1}{Ae^{Bx}}(ABe^{Bx})^2$

$AB^2e^{Bx} = AB^2e^{Bx}$

L.H.S. = R.H.S.

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

7. Question

Verify that $y = \frac{a}{x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$.

Answer

The differential equation is $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$ and the function to be proven as the solution is $y = \frac{a}{x} + b$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = -\frac{a}{x^2}$

$\frac{d^2y}{dx^2} = \frac{2a}{x^3}$

Putting values in equation,

$\frac{2a}{x^3} - \frac{2}{x}\left(\frac{-a}{x^2}\right) = 0,$

0 = 0

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

8. Question

Verify that $y^2 = 4ax$ is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}$.

Answer

The differential equation is $y = x\frac{dy}{dx} + a\frac{dx}{dy}$ and the function to be proven as the solution is $y^2 = 4ax$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{dx}{dy}$.

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\frac{dx}{dy} = \frac{y}{2a}$$

Putting the values,

$$2\sqrt{ax} = x\frac{2a}{2\sqrt{ax}} + a\frac{2\sqrt{ax}}{2a}$$

$$2\sqrt{ax} = \sqrt{ax} + \sqrt{ax}$$

$$2\sqrt{ax} = 2\sqrt{ax}$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

9. Question

Show that $Ax^2 + By^2 = 1$ is a solution of the differential equation $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$.

Answer

The differential equation is $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$ and the function to be proven as the solution is $Ax^2 + By^2 = 1$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$0 = 2Ax + 2By\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-Ax}{By}$$

$$\frac{d^2y}{dx^2} = \frac{-AB^2y^2 - BA^2x^2}{(By)^3}$$

Putting the values in the equation,

$$x\left\{y\left(\frac{-AB^2y^2 - BA^2x^2}{(By)^3}\right) + \frac{BA^2x^2}{B^3y^2}\right\} = y\frac{-Ax}{By}$$

$$x\left[-\left(\frac{A(By^2 + Ax^{2)}}{B^2y^2}\right)\right] + \frac{A^2x^2}{B^2y^2} = y\frac{-Ax}{By}$$

$$-\frac{A^2Bxy^2}{B^2y^2} = y\frac{-Ax}{By}$$

$$y\frac{-Ax}{By} = y\frac{-Ax}{By}$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

10. Question

Show that y = $ax^3 + bx^2 + c$ is a solution of the differential equation $\frac{d^3y}{dx^3} = 6a$.

Answer

The differential equation is $\frac{d^3y}{dx^3} = 6a$ and the function to be proven as the solution is

y = $ax^3 + bx^2 + c$; now we need to find the value of $\frac{d^3y}{dx^3}$.

$$\frac{dy}{dx} = 3ax^2 + 2bx$$

$$\frac{d^2y}{dx^2} = 6ax + 2$$

$$\frac{d^3y}{dx^3} = 6a$$

Putting the value of variables in the equation,

6a = 6a

As, L.H.S = R.H.S. the equation is satisfied, hence this function is the solution of the differential equation.

11. Question

Show that $y = \frac{c-x}{1+cx}$ is a solution of the differential equation $(1+x^2)\frac{dy}{dx} + (1+y^2) = 0$.

Answer

The differential equation is $(1+x^2)\frac{dy}{dx} + (1+y^2)$ and the function to be proven is the solution of equation is $y = \frac{c-x}{1+cx}$, now we need to find the value of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{(1+cx)(0-1)-(c-x)(0+c)}{(1+cx)^2} = \frac{-(c^2+1)}{(1+cx)^2}$$

Putting the value of the variables in the equation,

$$\frac{(1+x^2)(-(c^2+1))}{(1+cx)^2} + [1+\frac{(c-x)^2}{(1+cx)^2}] = 0$$

$$(1+cx)^2 + (1+x^2)(-1-c^2) + (c-x)^2 = 0$$

$1 + c^2x^2 + 2cx - 1 - c^2 - x^2 - c^2x^2 + c^2 + x^2 - 2cx = 0$

0 = 0

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

12. Question

Show that y = e^x (A cosx + B sinx) is a solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

Answer

The differential equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ and the function to be proven as the solution is

y = e^x (A cosx + B sinx), we need to find the value of $\frac{dy}{dx}$.

$\frac{dy}{dx}$ = e^x(A cos x + B sin x) + e^x(–A sin x + B cos x)

$\frac{d^2y}{dx^2}$ = e^x(A cos x + B sin x) + e^x(–A sin x + B cos x) + e^x(–A sin x + B cos x) + e^x(–A cos x – B sin x)

= 2e^x(–A sin x + B cos x)

Putting the values in equation,

2e^x(–A sin x + B cos x) – 2e^x(A cos x + B sin x) – 2e^x(–A sin x + B cos x) + 2 e^x(A cos x + B sin x) = 0

0 = 0

As, L.H.S = R.H.S. the equation is satisfied, hence this function is the solution of the differential equation.

13. Question

Verify that y = cx + 2c^2 is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0.$

Answer

The differential equation is $2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$ and the function to be proven as the solution is

y = cx + 2c^2, now we need to find the value of $\frac{dy}{dx}$.

$\frac{dy}{dx}$ = c + 0

Putting the values,

2c^2 + xc – cx – 2c^2 = 0

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

14. Question

Verify that y = –x – 1 is a solution of the differential equation (y – x)dy – (y^2 – x^2)dx = 0.

Answer

The differential equation is $\frac{dy}{dx} = \frac{y^2-x^2}{y-x} = y + x$ and the function to be proven as the solution is

y = – x – 1, now we need to find the value of $\frac{dy}{dx}$.

$\frac{dy}{dx}$ = –1

Putting the values in equation,

–1 = –x –1 +x

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

15. Question

Verify that $y^2 = 4a(x + a)$ is a solution of the differential equation $y\left\{1-\left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}$.

Answer

The differential equation is $y\left\{1-\left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}$ and the function to be verified as the solution is

$y^2 = 4a(x+a)$, now we need to find the value of $\frac{dy}{dx}$.

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y} = \frac{a}{\sqrt{a(x+a)}}$$

$$y\left\{1-\left(\frac{2a}{y}\right)^2\right\} = 2x\left(\frac{2a}{y}\right)$$

$$\frac{y^2-4a^2}{y} = \frac{4ax}{y}$$

$$y - \frac{4a^2}{y} = \frac{4ax}{y}$$

Putting the value in equation,

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

16. Question

Verify that $y = ce^{\tan^{-1}x}$ is a solution of the differential equation $\left(1+x^2\right)\frac{d^2y}{dx^2} + (2x+1)\frac{dy}{dx} = 0$.

Answer

The differential equation is $(1+x^2)\frac{d^2y}{dx^2} + (2x+1)\frac{dy}{dx} = 0$ and the function to be verified as the solution is $y = ce^{\tan^{-1}x}$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = ce^{\tan^{-1}x}\frac{1}{1+x^2}$$

$$\frac{d^2y}{dx^2} = c[e^{\tan^{-1}x}\frac{1}{(1+x^2)^2} - e^{\tan^{-1}x}\frac{2x}{(1+x^2)^2}]$$

Putting the values,

$$ce^{\tan^{-1}x}\frac{(1-2x)}{(1+x^2)^2}(1+x^2) + (2x-1)\,ce^{\tan^{-1}x}\frac{1}{1+x^2} = 0$$

$$\frac{ce^{\tan^{-1}x}}{1+x^2}[1-2x+2x-1] = 0$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

17. Question

Verify that $y = e^{m\cos^{-1}x}$ is a solution of the differential equation $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$.

Answer

The differential equation is $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - my^2 = 0$ and the function to be verified as the solution is $y = e^{m\cos^{-1}x}$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = e^{m\cos^{-1}x}\, m\, \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = e^{m\cos^{-1}x}\, m^2\, \frac{1}{1-x^2} - m\, e^{m\cos^{-1}x}\, \frac{x}{(1-x^2)^{3/2}}$$

Putting the values in the equation,

$$(1-x^2)e^{m\cos^{-1}x}\, m\, \frac{1}{1-x^2} - m\, e^{m\cos^{-1}x}\, \frac{x}{(1-x^2)^{3/2}} - x\, e^{m\cos^{-1}x}\, m\, \frac{-1}{\sqrt{1-x^2}} - m^2y = 0$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

18. Question

Verify that $y = \log\left(x + \sqrt{x^2 + a^2}\right)^2$ is a solution of the differential equation $\left(a^2 + x^2\right)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$

Answer

The differential equation is $(a^2 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$ and the function to be proven as the solution is $y = \log\left(x + \sqrt{x^2 + a^2}\right)^2$, now we need to find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{2}{\sqrt{x^2 + a^2}}$$

$$\frac{d^2y}{dx^2} = \frac{-2x}{(x^2 + a^2)^{3/2}}$$

Putting the values in the equation,

$$(a^2 + x^2)\frac{-2x}{(x^2 + a^2)^{3/2}} + x\frac{2}{\sqrt{x^2 + a^2}} = 0$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

19. Question

Show that the differential equation of which $y = 2\left(x^2 - 1\right) + ce^{-x^2}$ is a solution is $\frac{dy}{dx} + 2xy = 4x^3$

Answer

The differential equation is $\frac{dy}{dx} + 2xy = 4x^3$ and the function to be proven as the solution is $y = 2(x^2 - 1) + ce^{-x^2}$, now we need to find the value of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 4x - 2cxe^{-x^2}$$

Putting the value,

$$4x - 2cxe^{-x^2} + 2\,x\,y - 4\,x^3 = 0$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

20. Question

Show that $y = e^{-x} + ax + b$ is solution of the differential equation $e^x \frac{d^2y}{dx^2} = 1$.

Answer

The differential equation is $e^x \frac{d^2y}{dx^2} = 1$ and the function to be proven as the solution is

$y = e^{-x} + ax + b$, now we need to find the value of $\frac{d^2y}{dx^2}$.

$\frac{dy}{dx} = -e^{-x} + a$

$\frac{d^2y}{dx^2} = e^{-x}$

Putting the values in equation,

$(e^x)(e^{-x}) = 1$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

21. Question

For each of the following differential equations verify that the accompanying function is a solution.

Differential equation

i. $x\frac{dy}{dx} = y$

ii. $x + y\frac{dy}{dx} = 0$

iii. $x\frac{dy}{dx} + y = y^2$

iv. $x^3\frac{d^2y}{dx^2} = 1$

v. $y = \left(\frac{dy}{dx}\right)^2$

Function

$y = ax$

$y = \pm\sqrt{a^2 - x^2}$

$y = \frac{a}{x + a}$

$y = ax + b + \frac{1}{2x}$

$y = \frac{1}{4}(x \pm a)^2$

Answer

(i). The differential equation is $x\frac{dy}{dx} = y$ and the function to be proven as solution is y = ax, now we need to find the value of $\frac{dy}{dx}$.

$\frac{dy}{dx}$ = a

Putting the value,

ax = y = ax,

As, L.H.S = R.H.S. the equation is satisfied, hence this function is the solution of the differential equation.

(ii). The differential equation is $x + y\frac{dy}{dx} = 0$ and the function to be proven as the solution of this equation is $y = \pm\sqrt{a^2 - x^2}$, now we need to find the value of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$$

Putting the values,

$$x + \sqrt{a^2 - x^2}\frac{-x}{\sqrt{a^2 - x^2}} = 0$$

x – x = 0

As, L.H.S = R.H.S. the equation is satisfied, hence this function is the solution of the differential equation.

(iii). The differential equation is $x\frac{dy}{dx} + y = y^2$ and the function to be proven as solution is

$$y = \frac{x}{x + a}$$

now we need to find the value of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-a}{(x + a)^2}$$

$x\frac{-a}{(x+a)^2} + y = y^2$,

As L.H.S. ≠ R.H.S. the equation is not satisfied, hence this function is not the solution of this differential equation.

(iv). The differential equation is $x^3\frac{d^2y}{dx^2} = 1$ and the function to be proven as solution is $y = ax + b + \frac{1}{2x}$, now we need to find the value of $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = a - \frac{1}{2x^2}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^3}$$

Putting the values,

$x^3\frac{1}{x^3} = 1$,

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

(v). The differential equation is $y = \left(\frac{dy}{dx}\right)^2$ and the function to be proven as solution is $y = \frac{1}{4}(x \pm a)^2$, now we need to find the value of $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{2}(x \pm a)$$

Putting the value, we get,

$$y = [\frac{1}{2}(x \pm a)]^2 = \frac{1}{4}(x \pm a)^2$$

As, L.H.S = R.H.S. the equation is satisfied, so hence this function is the solution of the differential equation.

Exercise 22.4

1. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$x\frac{dy}{dx} = 1, y(1) = 0$$

Function: y = log x

Answer

Verification:

y = log x

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(\log x)}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

Multiplying x on both the sides

$$\Rightarrow x\frac{dy}{dx} = \frac{1}{x}.x \Rightarrow x\frac{dy}{dx} = 1 \text{ (We got the required diffrential equation)}$$

Also, at x=1, y should be equal to 0. Let's check it out.

At x=1, y=log (1)=0. (Hence the initial value condition is also satisfied)

2. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{dy}{dx} = y, y(0) = 1$$

Function: $y = e^x$

Answer

Verification:

$y = e^x$

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(e^x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x$$

Since $y=e^x$, we can replace e^x in the above differential equation

$\therefore \frac{dy}{dx} = y$

Hence $y = e^x$ is the solution of the differential equation.

Also, at x=0, we get $y=e^0$ which is equal to 1.

3. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} + y = 0, y(0) = 0$$

Function: y=sin x

Answer

Verification:

y=sin x

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\sin x$$

$$\therefore \frac{d^2y}{dx^2} + y = -\sin x + \sin x = 0$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Therefore, sin x is the solution to the differential equation.

Also at x=0, we get y=sin 0 which is equal to 0.

4. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0, y(0) = 2$$

Function: $y=e^x+1$

Answer

Verification:

$Y = e^x + 1$

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(e^x + 1)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x$$

$$\therefore \frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x - e^x = 0$$

$$\therefore \frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

Therefore, e^x+1 is the solution of the differential equation.

Also at x=0, we get $y=e^0+1$ which is equal to 2.

5. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{dy}{dx} + y = 2, y(0) = 3$$

Function: $y = e^{-x}+2$

Answer

Verification:

$y = e^{-x} + 2$

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(e^{-x}+2)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -e^{-x}$$

$$\therefore \frac{dy}{dx} + y = -e^{-x} + e^{-x} = 0$$

Therefore, $e^{-x} + 1$ is the solution of the differential equation.

Also at x = 0, we get $y = e^0 + 2$ which is equal to 3.

6. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} + y = 0, y(0) = 1$$

Function: y = sin x + cos x

Answer

y = sin x + cos x

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(\sin x + \cos x)}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \cos x - \sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\sin x - \cos x$$

$$\therefore \frac{d^2y}{dx^2} + y = -\sin x - \cos x + \sin x + \cos x = 0$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Therefore, sin x + cos x is the solution of the differential equation.

Also at x=0, we get y=sin 0+ cos 0 which is equal to 0+1=1.

7. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - y = 0, y(0) = 2$$

Function: y = e^x + e^{-x}

Answer

y = e^x + e^{-x}

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(e^x + e^{-x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x - e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x + e^{-x}$$

$$\therefore \frac{d^2y}{dx^2} - y = e^x + e^{-x} - e^x - e^{-x} = 0$$

$$\therefore \frac{d^2y}{dx^2} - y = 0$$

Therefore, e^x + e^{-x} is the solution of the differential equation.

Also at x=0, we get y=e^0 + e^0 which is equal to 1+1=2.

8. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0, y(0) = 2, y^1(0) = 3$$

Function: y = e^x+e^{2x}

Answer

Verification:

y = e^x + e^{2x}

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(e^x + e^{2x})}{dx}$$

$$\Rightarrow \frac{dy}{dx} = e^x + 2e^{2x} \left[\frac{d(e^{2x})}{dx} = e^{2x}.\frac{d(2x)}{dx}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x + 4e^{2x}$$

$$\therefore \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = (e^x + 4e^{2x}) - 3(e^x + 2e^{2x}) + 2(e^x + e^{2x})$$

$$\therefore \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x(1 - 3 + 2) + e^{2x}(4 - 6 + 2) = e^x(0) + e^{2x}(0) = 0$$

Therefore, $e^x + e^{2x}$ is the solution of the differential equation.

Also at x=0, we get $y=e^0 + e^0$ which is equal to 1+1=2.

Also at x=0, we get $y^1=e^0+2e^0=3$.

9. Question

For each of the following initial value problems verify that the accompanying function is a solution:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0, y(0) = 1, y^1(0) = 2$$

Function: $y=xe^x + e^x$

Answer

Verification:

$Y = x e^x + e^x$

Differentiating both sides we get,

$$\frac{dy}{dx} = \frac{d(e^x(x+1))}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (x+1)e^x + e^x\left[\because \frac{d(uv)}{dx} = u.\frac{d(v)}{dx} + v\frac{d(u)}{dx}\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = (x+1)e^x + e^x + e^x$$

$$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = (xe^x + 3e^x) - 2(xe^x + 2e^x) + (xe^x + e^x)$$

$$\therefore \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x(3 - 4 + 1) + xe^x(1 - 2 + 1) = e^x(0) + xe^x(0) = 0$$

Therefore, $e^x + xe^x$ is the solution of the differential equation.

Exercise 22.5

1. Question

Solve the following differential equations

$$\frac{dy}{dx} = x^2 + x - \frac{1}{x}, x \neq 0$$

Answer

By Separate the variables

$$dy = (x^2 + x - \frac{1}{x})dx$$

Integrate both side

$$\int dy = \int \left(x^2 + x - \frac{1}{x}\right)dx + c$$

$$y = \int x^2 dx + \int x dx - \int \frac{1}{x} dx + c$$

as we known $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$y = \frac{x^3}{2} + \frac{x^2}{2} - \log x + c$$

2. Question

Solve the following differential equations

$$\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}, x \neq 0$$

Answer

By Separate the variables

$$dy = (x^5 + x^2 - \frac{2}{x})dx$$

Integrate both side

$$\int dy = \int x^5 + x^2 - \frac{2}{x} dx + c$$

$$y = \int x^5 dx + \int x^2 dx - \int \frac{2}{x} dx + c$$

as we known $\int x^n dx = \frac{x^{n+1}}{n+1}$

$$y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log x + c$$

3. Question

Solve the following differential equations

$$\frac{dy}{dx} = 2x = e^{3x}$$

Answer

$$\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = e^{3x}$$

$$dy = 2xdx$$

$$dy = e^{3x}dx$$

Integrate both sides we get,

$$\int dy = \int 2xdx + c$$

$$\int dy = \int e^{3x}dx + c$$

as we known $\int x^n dx = \frac{x^{n+1}}{n+1}$ & $\int e^{ax} dx = \frac{e^{ax}}{a}$

$y = x^2 + c$

$y = \frac{e^{3x}}{3} + c$

4. Question

Solve the following differential equations

$$\left(x^2 + 1\right)\frac{dy}{dx} = 1$$

Answer

$$\frac{dy}{dx} = \frac{1}{x^2 + 1}$$

$$dy = \frac{dx}{x^2 + 1}$$

Integrate both sides we get,

$$\int dy = \int \frac{dx}{x^2 + 1} + c$$

we known that $\int \frac{dx}{x^2+1} = \tan^{-1} x + c$

$y = \tan^{-1} x + c$

5. Question

Solve the following differential equations

$$\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$$

Answer

$$dy = \frac{1 - \cos x}{1 + \cos x} dx$$

We known that $1 - \cos x = 2\sin^2 \frac{x}{2}$

$$1 - \cos x = 2\cos^2 \frac{x}{2}$$

$$dy = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} dx$$

$$dy = \tan^2 \frac{x}{2} dx$$

by identity $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Integrate both sides we get,

$$\int dy = \int \tan^2 \frac{x}{2} dx + c$$

we know that $\sec^2 x - 1 = \tan^2 x$

$$y = \int \left(\sec^2\frac{x}{2} - 1\right) dx + c$$

$$y = \int \sec^2\frac{x}{2} dx - \int dx + c$$

$$y = \frac{\tan\frac{x}{2}}{1/2} - x + c$$

$$y = 2\tan\frac{x}{2} - x + c$$

6. Question

Solve the following differential equations

$$(x+2)\frac{dy}{dx} = x^2 + 3x + 7$$

Answer

$$\frac{dy}{dx} = \frac{(x^2 + 3x + 7)}{x + 2}$$

On dividing $\frac{(x^2+3x+7)}{x+2}$

$$\frac{dy}{dx} = (x+3) + (\frac{1}{x+2})$$

$$dy = \left[(x+3) + (\frac{1}{x+2})\right] dx$$

Integrate both sides we get,

$$y = \int (x+3) dx + \int \left(\frac{1}{x+2}\right) dx + c$$

$$y = \frac{x^2}{2} + 3x + \int (\frac{1}{x+2}) dx + c$$

we know $\frac{dx}{x+a} = \log(x+a) + c$

$$y = \frac{x^2}{2} + 3x + \log(x+2) + c$$

7. Question

Solve the following differential equations

$$\frac{dy}{dx} = \tan^{-1} x$$

Answer

$$dy = \tan^{-1} x\, dx$$

Integrate both side

$$y = \int \tan^{-1} x\, dx + c$$

Now using integration by parts we get,

$$y = \tan^{-1} x \int dx - \int ((\frac{d}{dx}\tan^{-1} x) \int dx)dx$$

$$y = x\tan^{-1} x - \int \frac{1}{1 + x^2}.xdx + c$$

Let $1 + x^2 = t$

Differentiate with respect to x.

$$xdx = \frac{dt}{2}$$

$$y = x\tan^{-1} x - \frac{1}{2}\int \frac{dt}{t}$$

$$y = x\tan^{-1} x - \frac{1}{2}\log t + c$$

Put value of $t=1+x^2$

$$y = x\tan^{-1} x - \frac{1}{2}\log(1 + x^2) + c$$

8. Question

Solve the following differential equations

$$\frac{dy}{dx} = \log x$$

Answer

$$dy = \log xdx$$

Integrate both sides we get,

$$\int dy = \int \log xdx + c$$

$$y = \int \log xdx + c$$

Now integrating by parts we get,

$$y = x\log x - \int \frac{1}{x}.x\, dx + c$$

$$y = x\log x - x + c$$

$$y = x(\log x - 1) + c$$

9. Question

Solve the following differential equations

$$\frac{1}{x}\frac{dy}{dx} = \tan^{-1} x,\ x \neq 0$$

Answer

$$\frac{dy}{dx} = x\tan^{-1} x$$

$$dy = x\tan^{-1} xdx$$

Integrate both sides we get,

$$y=\int x\tan^{-1}x dx+c$$

Since, $\int x\tan^{-1}x\,dx=\tan^{-1}x\int xdx-\int\left(\frac{d}{dx}\tan^{-1}x\int xdx\right)dx$

[Using Integration by parts]

$$\int x\tan^{-1}x\,dx=\frac{x^2}{2}\tan^{-1}x-\int\frac{1}{1+x^2}.\frac{x^2}{2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}[\int\frac{x^2+1-1}{1+x^2}]dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}[\int\frac{x^2+1}{1+x^2}dx-\int\frac{1}{1+x^2}dx]$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}[\int dx-\int\frac{1}{1+x^2}dx]$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}(x-\tan^{-1}x)$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{\tan^{-1}x}{2}$$

$$y==\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{\tan^{-1}x}{2}+c$$

10. Question

Solve the following differential equations

$$\frac{dy}{dx}=\cos^3 x\ \sin^2 x+x\sqrt{2x+1}$$

Answer

$$dy\ =(\cos^3 x+\sin^2 x+x\sqrt{2x-1})dx$$

Integrate both sides we get,

$$y=\int\cos^3 x\,dx+\int\sin^2 x\,dx+\int x\sqrt{2x-1}dx$$

$$\int\cos^3\ x\,dx=\int\cos^2 x\cos x\,dx$$

We know that $\cos^2 x=1-\sin^2 x$

$$\int(1-\sin^2 x)\cos x\,dx$$

Let $\sin x=t$

Differentiate with respect to x.

$\cos dx=dt$

$$\int(1-t^2)dt=t-\frac{t^3}{3}$$

put the value of t

$$\int \cos^3 x\,dx = \sin x - \frac{\sin^3 x}{3}$$

$$\int \sin^2 x\,dx$$

we know that $1 - \cos^1 2x = 2\sin^2 x$

therefore $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

$$\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos 2x dx\right]$$

$$= \left[x - \frac{(\sin 2x)}{2}\right]$$

$$= \frac{x}{2} - \frac{\sin 2x}{4}$$

11. Question

Solve the following differential equations

(sinx + cosx)dy + (cosx – sinx) dx = 0

Answer

By separating variables

$$dy = \frac{(\sin x - \cos x)}{(\cos x + \sin x)}dx$$

Integrate both sides we get,

$$y = \int \frac{(\sin x - \cos x)}{(\cos x + \sin x)}dx + c$$

Let $(\cos x + \sin x) = t$

Differentiate with respect to x.

$(-\sin x + \cos x)dx = dt$

$$y = -\int \frac{dt}{t} + c$$

$y = -\log t + c$

Put t value in above eq.

$y = -\log(\cos x + \sin x) + c$

12. Question

Solve the following differential equations

$$\frac{dy}{dx} - x\sin^2 x = \frac{1}{x\log x}$$

Answer

$$\frac{dy}{dx} = \frac{1}{xlogx} + xsin^2x$$

Separate variables

$$dy = (\frac{1}{xlogx} + xsin^2x)dx$$

Integrate both sides we get,

$$\int dy = \int (\frac{1}{xlogx} + xsin^2x)dx \ldots 1$$

$$\int \frac{1}{xlogx}dx$$

logx=t

Differentiate with respect to x.

$$\frac{1}{x}dx = dt$$

$$\int \frac{1}{xlogx}dx = \int \frac{dt}{t}$$

$$\int \frac{1}{xlogx}dx = logt$$

Put value of t

$$\int \frac{1}{xlogx}dx = log(log(x)) \quad \ldots 2$$

$$\int xsin^2x\,dx = \int \frac{x(1 - cos2x)}{2}dx$$

we know that: $1 - cos2x = 2\sin^2 x$

$$= \int \frac{x}{2}dx - \int \frac{xcos2x}{2}dx$$

$$= \frac{x^2}{4} - \int \frac{xcos2x}{2}dx$$

Using Integration by parts we get,

$$= \frac{x^2}{4} - \frac{1}{2}(x\int cos2xdx - \int (\frac{d}{dx}x)\int cos2xdx\,dx$$

$$= \frac{x^2}{4} - \frac{1}{2}(x.\frac{sin2x}{2} - \int \frac{sin2xdx}{2})$$

$$= \frac{x^2}{4} - \frac{1}{2}(x.\frac{sin2x}{2} + \frac{cos2x}{8}) \ldots 3$$

Put values of eq 2 and 3 in eq 1st

$$y = log(log(x) + \frac{x^2}{4} - \frac{1}{2}\left(x.\frac{sin2x}{2} + \frac{cos2x}{8}\right)$$

13. Question

Solve the following differential equations

$$\frac{dy}{dx} = x^5 \tan^{-1}\left(x^3\right)$$

Answer

By separating variables

$dy = x^5 \tan^{-1} x^3 \, dx$

Integrate both sides we get,

$$y = \int x^5 \tan^{-1} x^3 \, dx + c$$

Let $x^3 = z$

Differentiate with respect to x.

$3x^2 = dz$

$$x^2 dx = \frac{dz}{3}$$

$$y = \int x^3 x^2 \tan^{-1} x^3 \, dx + c$$

$$y = \int z \tan^{-1} z \frac{dz}{3} + c$$

$$y = \frac{1}{3}\int z \tan^{-1} z \, dz + c$$

Using integration by parts we get,

$$y = \frac{1}{3}\left[\tan^{-1} z \int zdz - \int \left(\frac{d}{dx}\tan^{-1} z\right)\int z\,dz \;\; dz\right]$$

$$= \frac{1}{3}\left[\tan^{-1} z \frac{z^2}{2} - \int \left(\frac{1}{1+z^2}\frac{z^2}{2}\right) dz\right]$$

$$= \frac{1}{3}\left[\tan^{-1} z \frac{z^2}{2} - \frac{1}{2}\int \left(\frac{z^2+1-1}{z^2+1}\right) dz\right]$$

$$= \frac{1}{3}\left[\tan^{-1} z \frac{z^2}{2} - \frac{1}{2}\int \left(\frac{z^2+1}{z^2+1}\right) dz + \frac{1}{2}\int \frac{1}{z^2+1} dz\right]$$

$$= \frac{1}{3}\left[\tan^{-1} z \frac{z^2}{2} - \frac{1}{2} z + \frac{1}{2}\tan^{-1} z\right]$$

$$= \frac{1}{6}[\tan^{-1} z \,.\, z^2 - z + \tan^{-1} z]$$

Put $z = x^3$

$$= \frac{1}{6}[\tan^{-1} x^3 \,.x^6 - x^3 + \tan^{-1} x^3]$$

14. Question

Solve the following differential equations

$$\sin^4 x \frac{dy}{dx} = \cos x$$

Answer

$$\frac{dy}{dx} = \frac{\cos x}{\sin^4 x}$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x . \sin^3 x}$$

We know the trigonometric identity $\frac{\cos x}{\sin x} = \cot x$ and $\frac{1}{\sin x} = \operatorname{cosec} x$

$$\frac{dy}{dx} = \cot x . \operatorname{cosec}^3 x dx$$

Separate variables and Integrate both sides,

$$y = \int \cot x . \operatorname{cosec} x . \operatorname{cosec}^2 x dx + c$$

Let $\operatorname{cosec} x = z$

Differentiate with respect to x.

$$-\cot x . \operatorname{cosec} x \, dx = dz$$

$$\cot x . \operatorname{cosec} x \, dx = -dz$$

$$y = -\int z^2 dz + c$$

$$y = -z^3 + c$$

put z = cosec x

$$y = -\frac{\operatorname{cosec}^3 x}{3} + c$$

15. Question

Solve the following differential equations

$$\cos x \frac{dy}{dx} - \cos 2x = \cos 3x$$

Answer

$$\frac{\cos x dy}{dx} = \cos 3x + \cos 2x$$

By identity: $\cos 3x = 4\cos^3 x - 3\cos x, \cos 2x = \cos^2 x - \sin^2 x$

$$\frac{dy}{dx} = \frac{4\cos^3 x - 3\cos x + \cos^2 x - \sin^2 x}{\cos x}$$

Separate variables

$$dy = \frac{4\cos^3 x - 3\cos x + \cos^2 x - \sin^2 x}{\cos x} dx$$

Integrate both sides we get,

$$y = 4\int \frac{1 + \cos 2x}{2} dx - 3x + \sin x - \int \sin^2 x / \cos x \, dx + c$$

$$y = 2\int dx + 2\int \cos 2x dx - 3x + \sin x - \int 1 - \cos^2 x / \cos x \, dx + c$$

$y = 2x - 3x + \sin 2x + \sin - \int \sec x dx + \int \cos x dx + c$

$y = \sin 2x + \sin x - x - \ln(\sec x + \tan x) + \sin x + c$

$y = \sin 2x + 2\sin x - x - \ln(\sec x + \tan x) + c$

16. Question

Solve the following differential equations

$\sqrt{1 - x^4}\, dy = x\, dx$

Answer

$dy = \frac{x}{\sqrt{1 - x^4}} dx$

Integrate both sides we get,

$y = \int \frac{x}{\sqrt{1 - (x^2)^2}} dx + c$

Let $x^2 = t$

$2xdx = dt$

$xdx = \frac{dt}{2}$

$y = \frac{1}{2} \int \frac{dt}{\sqrt{1 - t^2}} + c$

we know that: $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$

$y = \frac{1}{2} \sin^{-1} t + c$

Put the value of t,

$y = \frac{1}{2} \sin^{-1} x^2 + c$

17. Question

Solve the following differential equations

$\sqrt{a + x}\, dy + x\, dx = 0$

Answer

$\sqrt{a + x} dy = -xdx$

$dy = -\frac{x}{\sqrt{a + x}} dx$

Integrate both sides we get,

$y = \int -\frac{x}{\sqrt{a + x}} dx + c$

$y = \int -\frac{x + a - a}{\sqrt{a + x}} dx + c$

$$y = \int -\frac{x+a}{\sqrt{a+x}}dx + \int \frac{a}{\sqrt{a+x}}dx + c$$

$$y = \int -\sqrt{a+x}dx + \int \frac{a}{\sqrt{a+x}}dx + c$$

$$y = -\frac{(a+x)^{\frac{3}{2}}}{\frac{3}{2}} + 2a\sqrt{a+x} + c$$

$$y = -\frac{2}{3}(a+x)^{\frac{3}{2}} + 2a\sqrt{a+x} + c$$

18. Question

Solve the following differential equations

$$\left(1+x^2\right)\frac{dy}{dx} - x = 2\ \tan^{-1} x$$

Answer

$$\frac{(1+x^2)dy}{dx} = 2\tan^{-1} x + x$$

$$\frac{dy}{dx} = \frac{2\tan^{-1} x}{1+x^2} + \frac{x}{1+x^2}$$

Separate variables

$$dy = \frac{2\tan^{-1} x}{1+x^2}dx + \frac{x}{1+x^2}\ dx$$

Integrate both sides we get,

$$y = 2\int \frac{\tan^{-1} x}{1+x^2}dx + \int \frac{x}{1+x^2}dx + c$$

Let $\tan^{-1} x = t$

$x^2 = z$

Differentiate with respect to x,

$$\frac{dx}{1+x^2} = dt$$

$2xdx = dz$

$$y = 2\int tdt + \frac{1}{2}\int \frac{dz}{1+z} + c$$

$$y = \frac{2t^2}{2} + \frac{1}{2}\log z + c$$

Put value of t and z

$$y = (\tan^{-1} x)^2 + \frac{1}{2}\log x^2 + c$$

19. Question

Solve the following differential equations

$$\frac{dy}{dx} = x \log x$$

Answer

$dy = x\log x\, dx$

Integrate both sides we get,

$$y = \int x\log x dx + c$$

Integrating by parts we get,

$$y = \log x \cdot \frac{x^2}{2} - \int \left(\frac{d}{dx}\log x. \int x dx\right) dx + c$$

$$y = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2x} dx + c$$

$$y = \log x \cdot \frac{x^2}{2} - \frac{x^2}{4} + c$$

20. Question

Solve the following differential equations

$$\frac{dy}{dx} = x\ e^x - \frac{5}{2} + \cos^2 x$$

Answer

Separate variables

$$dy = (xe^x - \frac{5}{2} + \cos^2 x)dx$$

Integrate both sides we get,

$$y = \int xe^x dx - \frac{5}{2}\int dx + \int \cos^2 x\, dx + c$$

as $\cos^2 x = \frac{1+\cos 2x}{2}$

$$y = \int xe^x dx - \frac{5}{2}\int dx + \int \frac{1+\cos 2x}{2} dx + c$$

$$y = x. \int e^x dx - \int \left(\frac{d}{dx}x \int e^x dx\right) dx - \frac{5}{2}x + \int \frac{dx}{2} + \int \frac{\cos 2x}{2} dx + c$$

$$y = x.e^x - \int e^x dx - \frac{5}{2}x + \frac{x}{2} + \frac{\sin 2x}{4} + c$$

$$y = x.e^x - e^x - 2x + \frac{\sin 2x}{4} + c$$

21. Question

Solve the following differential equations

$$\left(x^3 + x^2 + x + 1\right)\frac{dy}{dx} = 2x^2 + x$$

Answer

$$\frac{dy}{dx} = \frac{2x^2 + x}{x^3 + x^2 + x + 1}$$

$$dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1}dx$$

Integrate both sides we get,

$$y = \int \frac{2x^2+x}{x^3+x^2+x+1}dx + c \ldots 1$$

Solv in partial fraction we get,

$$\frac{2x^2 + x}{(x^2+1)(x+1)}dx = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$

By solving we get

a = 1/2

B = 1

C = -1/2

$$\frac{2x^2 + x}{(x^2+1)(x+1)}dx = \frac{\frac{1}{2}}{x+1} + \frac{x-\frac{1}{2}}{x^2+1}$$

$$y = \int \frac{\frac{1}{2}}{x+1}dx + \int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx + c$$

$$y = \frac{1}{2}\log(x+1) + \int \frac{x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x + c$$

Let $x^2 = t$

X dx = dt/2

$$y = \frac{1}{2}\log(x+1) + \frac{1}{2}\int \frac{dt}{1+t} - \frac{1}{2}\tan^{-1}x + c$$

$$y = \frac{1}{2}\log(x+1) + \frac{1}{2}\log(t+1) - \frac{1}{2}\tan^{-1}x + c$$

Put the value of t

$$y = \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$$

22. Question

Solve the following initial value problem:

$$\sin\left(\frac{dy}{dx}\right) = k; y(0) = 1$$

Answer

$$\sin\left(\frac{dy}{dx}\right) = k$$

$$\frac{dy}{dx} = \sin^{-1}k$$

$$dy = \sin^{-1}k\ dx$$

Integrate both sides we get,

$$y = \int \sin^{-1} k \, dx + c$$

$$y = x \sin^{-1} k + c$$

At x=0 , y=1

Therefore,

$$1 = 0. \sin^{-1} k + c$$

So, c = 1

Putting c = 1 in above eq. we get,

$$y = x \sin^{-1} k + 1$$

23. Question

Solve the following initial value problem:

$$e^{dy/dx} = x + 1; \ y(0) = 3$$

Answer

Taking log

$$\log e^{\frac{dy}{dx}} = \log(x+1)$$

$$\frac{dy}{dx} = \log(x+1)$$

$$dy = \log(x+1)dx$$

Integrate both sides we get,

$$y = \int \log(x+1)dx + c$$

Using Integration by parts we get,

$$y = \log(x+1) \int dx - \int \left(\frac{d}{dx}\log(x+1) \int dx\right) dx + c$$

$$y = x.\log(x+1) - \int \frac{1}{x+1}.xdx + c$$

$$y = x.\log(x+1) - \int \frac{x+1-1}{x+1} dx + c$$

$$y = x.\log(x+1) - \int \frac{x+1}{x+1} dx + \int \frac{1}{x+1} dx + c$$

$$y = x.\log(x+1) - x + \log(x+1) + c$$

At x = 0, y = 3

$$3 = 0.\log(0+1) - 0 + \log(0+1) + c$$

$$3 = 0 - 0 + 0 + c$$

$$c = 3$$

Put on above eq.

$y = x.\log(x+1) - x + \log(x+1) + 3$

24. Question

Solve the following initial value problem:

$$C'(x) = 2 + 0.15\ x; C(0) = 100$$

Answer

$$\frac{dc}{dx} = 2 + 0.15x$$

Separation

$$dc = 2 + 0.15x\ dx$$

Integrate both side

$$c = \int 2dx + \int 0.15xdx + c1$$

$$c = 2x + \frac{0.15x^2}{2} + c1$$

At x=0 c=100

$$100 = 0 + 0 + c1$$

$$c1 = 100$$

Put in above eq.

$$c = 2x + \frac{0.15x^2}{2} + 100$$

25. Question

Solve the following initial value problem:

$$x\frac{dy}{dx} + 1 = 0\ ;\ y(-1) = 0$$

Answer

$$\frac{xdy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{x}$$

Separate variabless

$$dy = -\frac{1}{x}dx$$

Integrate both sides we get,

$$y = -\log x + c$$

At x = -1, y = 0

$$0 = -\log(-1) + c$$

$$c = 0$$

Put in above eq.

$y = -\log x$

26. Question

Solve the following initial value problem:

$$x\left(x^2-1\right)\frac{dy}{dx}=1\,;\ y(2)=0$$

Answer

$$\frac{dy}{dx} = \frac{1}{x(x^2-1)}$$

Separate variables,

$$dy = \frac{1}{x(x^2-1)}dx$$

Integrate both sides we get,

$$y = \int \frac{1}{x(x^2-1)}dx + c$$

$$y = \int \frac{dx}{x^3(1-\frac{1}{x^2})} + c$$

Let $\left(1-\frac{1}{x^2}\right) = t$

$$\frac{2}{x^3}dx = dt$$

$$\frac{dx}{x^3} = \frac{dt}{2}$$

$$y = \frac{1}{2}\int \frac{dt}{t} + c$$

$$y = \frac{1}{2}\log t + c$$

Put value of t,

$$y = \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) + c$$

At x=2 y=0

$$0 = \frac{1}{2}\log\left(1-\frac{1}{4}\right) + c$$

$$c = -\frac{1}{2}\log\left(\frac{3}{4}\right)$$

Put in above eq.

$$y = \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) - \frac{1}{2}\log\left(\frac{3}{4}\right)$$

Exercise 22.6

1. Question

Solve the following differential equations:

$$\frac{dy}{dx}+\frac{1+y^{2}}{y}=0$$

Answer

$$\frac{dy}{dx}=-(\frac{1+y^{2}}{y})$$

Separate variables

$$\frac{y}{1+y^{2}}dy=-dx$$

Integrate both sides we get,

$$\int\frac{y}{1+y^{2}}dy=\int -dx+c$$

Let y^2=t

Differentiate with respect to x

$$ydy=\frac{dt}{2}$$

$$\frac{1}{2}\int\frac{dt}{1+t^{2}}=-x+c$$

$$\frac{1}{2}\tan^{-1}t=-x+c$$

Put value of t,

$$\frac{1}{2}\tan^{-1}y^{2}=-x+c$$

$$\frac{1}{2}\tan^{-1}y^{2}+x=c$$

2. Question

Solve the following differential equations:

$$\frac{dy}{dx}=\frac{1+y^{2}}{y^{3}}$$

Answer

Separate variables

$$\frac{y^{3}}{1+y^{2}}dy=dx$$

Integrate both sides we get,

$$\int\frac{y^{3}}{1+y^{2}}dy=\int dx+c$$

Let y^2=t

Differentiate with respect to x

$$2ydy=dt$$

$\mathrm{ydy} = \frac{\mathrm{dt}}{2}$

$\frac{1}{2}\int \frac{t}{1+t}dt = x + c$

$\frac{1}{2}\int \frac{t+1-1}{1+t}dt = x + c$

$\frac{1}{2}\int dt - \frac{1}{2}\int \frac{1}{1+t} = x + c$

$\frac{1}{2}t - \frac{1}{2}\log(1+t) = x + c$

Put the value of t,

$\frac{1}{2}y^2 - \frac{1}{2}\log(1+y^2) = x + c$

3. Question

Solve the following differential equations:

$\frac{dy}{dx} = \sin^2 y$

Answer

Separate variables

$\frac{dy}{\sin^2 y} = dx$

We know that $\frac{1}{\sin x} = \mathrm{cosecx}$

$\mathrm{cosec}^2 y\, dy = dx$

Integrate both sides we get,

$\int \mathrm{cosec}^2 y\, dy = \int dx + c$

$-\cot y = x + c$

4. Question

Solve the following differential equations:

$\frac{dy}{dx} = \frac{1-\cos 2y}{1+\cos 2y}$

Answer

$1 - \cos 2y = 2\sin^2 y,\ 1 + \cos 2y = 2\cos^2 y$

$\frac{dy}{dx} = (2\sin^2 y)/(2\cos^2 y)$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

$\frac{dy}{dx} = \tan^2 y$

By separate variables,

$$\frac{dy}{\tan^2 y} = dx$$

$$\frac{1}{\tan y} = \cot y$$

$$\cot^2 y dy = dx$$

Integrate both sides we get,

$$\int \cot^2 y \, dy = \int dx + c$$

Using trigonometry identity: $\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\int (\operatorname{cosec}^2 x - 1) dy = x + c$$

$$\int \operatorname{cosec}^2 x dx - \int dy = x + c$$

$$-\cot y - y = x + c$$

$$\cot y + y + x + c = 0$$

Exercise 22.7

1. Question

Solve the following differential equations:

$$(x-1)\frac{dy}{dx} = 2xy$$

Answer

Now separating variable x on one side and variable y on other side, we have

$$\frac{dy}{y} = \frac{2x}{x-1} dx$$

Integrating LHS with respect to y and RHS with respect to x

$$\int \frac{dy}{y} = \int \frac{2x}{x-1} dx$$

Adding 2 and subtracting 2, to the numerator of RHS

$$\int \frac{dy}{y} = \int \frac{2x - 2 + 2}{x - 1} dx$$

Re - writing RHS as

$$\int \frac{dy}{y} = \int (2 + \frac{2}{x-1}) dx$$

Using identities:

$$\int \frac{dy}{y} = \log(y)$$

and

$$\int k dx = kx$$

Integrating both sides, we have

log(y) = 2x + 2log(x – 1) + c

2. Question

Solve the following differential equations:

$(1 + x^2)dy = (xy)dx$

Answer

Now separating variable x on one side and variable y on other side, we have

$$\frac{dy}{y} = \frac{x}{1+x^2}dx$$

Integrating both sides

$$\int \frac{dy}{y} = \int \frac{x}{1+x^2}dx$$

Using identities:

$\int \frac{dy}{y} = \log(y)$ and for RHS assuming $x^2 = t$ (substitution property) and differentiating both sides

Now, 2xdx = dt

$$xdx = \frac{dt}{2}$$

Substituting the above value in the integral and replacing x^2 with t and integrating both sides

$$\int \frac{dy}{y} = \frac{1}{2}\int \frac{dt}{1+t}$$

$$\log(y) = \frac{1}{2}[\log(1+t)]$$

Now replacing t by x^2

$$\log(y) = \frac{1}{2}[\log(1+x^2)]$$

Taking anti - log both sides

$y^2 = 1+x^2$

$$y = \sqrt{1+x^2} + c$$

3. Question

Solve the following differential equations:

$$\frac{dy}{dx} = \left(e^x + 1\right)y$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{y} = (e^x + 1)dx$$

Using identities:

$$\int \frac{dy}{y} = \log(y)$$

and

$$\int e^x dx = e^x$$

and

$$\int k dx = kx$$

Integrating both sides we get,

$$\int \frac{dy}{y} = \int (e^x + 1)dx$$

log(y) = e^x + x + c

4. Question

Solve the following differential equations:

$$(x-1)\frac{dy}{dx} = 2x^3 y$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{y} = \frac{2x^3}{x-1}dx$$

Adding 1 and subtracting 1 to the numerator of RHS

$$\frac{dy}{y} = \frac{2x^3 - 1 + 1}{x-1}dx$$

$$\frac{dy}{y} = 2\left[\frac{x^3-1}{x-1} + \frac{1}{x-1}\right]dx$$

Using formula: $x^3 - 1 = (x - 1)(x^2 + 1 + x)$

$$\frac{dy}{y} = 2\left[x^2 + x + 1 + \frac{1}{x-1}\right]dx$$

Using identities:

$$\int \frac{dy}{y} = \log(y),$$

$$\int k dx = kx,$$

and

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

And integrating both sides we get,

$$\int \frac{dy}{y} = \int 2\left[x^2 + x + 1 + \frac{1}{x-1}\right]dx$$

$$\log(y) = 2\left(\frac{x^3}{3} + \frac{x^2}{2} + x + \log(x-1)\right) + c$$

5. Question

Solve the following differential equations:

$$xy(y+1)dy = (x^2+1)dx$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$y(y+1)dy = \left(\frac{x^2+1}{x}\right)dx$$

$$y(y+1)dy = \left(x+\frac{1}{x}\right)dx$$

Now integrating both sides we get,

$$\int (y+y^2)dy = \int \left(x+\frac{1}{x}\right)dx$$

Using identities:

$$\int kdx = kx$$

and

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\frac{y^2}{2}+\frac{y^3}{3} = x^2+\log x+c$$

6. Question

Solve the following differential equations:

$$5\frac{dy}{dx} = e^x y^4$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$5\frac{dy}{y^4} = e^x dx$$

Integrating both sides using identities:

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

and

$$\int e^x dx = e^x$$

$$\int 5\frac{dy}{y^4} = \int e^x dx$$

$$5\left(\frac{y^{-3}}{-3}\right) = e^x+c$$

7. Question

Solve the following differential equations:

$$x \cos y\, dy = (xe^x \log x + e^x)dx$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\cos(y)\,dy = e^{x}(\log(x) + \frac{1}{x})dx$$

Integrating both sides using identities:

$$\int \cos y\,dy = \sin y$$

and for RHS using property

$$\int e^{x}[f(x) + f'(x)] = e^{x}f(x)$$

$$\int \cos(y)\,dy = \int e^{x}(\log(x) + \frac{1}{x})dx$$

Sin(y) = e^{x}(log(x)) + c

8. Question

Solve the following differential equations:

$$\frac{dy}{dx} = e^{x+y} + x^{2}e^{y}$$

Answer

Re - writing the question as

$$\frac{dy}{dx} = e^{y}(e^{x} + x^{2})$$

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{e^{y}} = (e^{x} + x^{2})dx$$

Integrating both sides using identities

$$\int x^{n}dx = \frac{x^{n+1}}{n+1}$$

and

$$\int e^{kx}dx = \frac{e^{x}}{k}$$

$$\int \frac{dy}{e^{y}} = \int (e^{x} + x^{2})dx$$

$$-e^{-y} = e^{x} + \frac{x^{3}}{3} + c$$

9. Question

Solve the following differential equations:

$$x\frac{dy}{dx} + y = y^{2}$$

Answer

Re - writing the question as:

$$\frac{xdy}{dx} = y^{2} - y$$

$$\frac{dy}{y^2 - y} = \frac{dx}{x}$$

$$\frac{dy}{y(y-1)} = \frac{dx}{x}$$

$$\frac{[y-(y-1)]}{y(y-1)} dy = \frac{dx}{x}$$

$$\left[\frac{1}{y-1} - \frac{1}{y}\right] dy = \frac{dx}{x}$$

Integrating both sides using identities:

$$\int \frac{dy}{y} = \log(y)$$

$$\int \left[\frac{1}{y-1} - \frac{1}{y}\right] dy = \int \frac{dx}{x}$$

log(y – 1) – log(y) = log(x) + c

using:

$$\log(a) - \log(b) = \log\left(\frac{a}{b}\right)$$

$$\log\left[\frac{y-1}{y}\right] = \log(x) + c$$

Taking anti - log both sides we have,

$$1 - \frac{1}{y} = x + c$$

$$\frac{1}{y} = 1 - x - c$$

$$y = \frac{1}{1 - x - c}$$

10. Question

Solve the following differential equations:

$$\left(e^y + 1\right)\cos x \; dx + e^y \sin x \; dy = 0$$

Answer

Now separating variable x on one side and variable y on other side, we have

$$\frac{-\cos x}{\sin x} dx = \frac{e^y}{e^y + 1} dy$$

$$-\cot x \; dx = \frac{e^y}{e^y + 1} dy$$

Using identities:

$$\int \cot x \; dx = \log|\sin x|$$

and on RHS side assuming $e^y = t$, so $e^y dy = dt$ by differentiating both sides.

Now integrating both sides

$$\int -\cot x\ dx = \int \frac{dt}{t+1}$$

– log |sin(x)| = log(t+1) + c

Replacing t by e^y

– log |sin(x)| = log(e^y+1) + c

[sin(x)] (e^y+1) = c

11. Question

Solve the following differential equations:

$$x \cos^2 y\ dx = y \cos^2 x\ dy$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{x}{\cos^2 x} dx = \frac{y}{\cos^2 y} dy$$

$$\frac{1}{\cos^2 x} = \sec^2 x$$

$$x\sec^2 x\, dx = y\sec^2 y\, dy$$

Integrating both sides using integration by parts method.

According to integration by parts method,

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int [\frac{d(f(x))}{dx}\int g(x)dx]\ dx$$

$$\int x\sec^2 x\, dx = \int y\sec^2 y\, dy$$

$$x\int \sec^2 x\, dx = y\int \sec^2 y\, dy$$

$$x\tan(x) - \int \tan x\ dx = y\tan y - \int \tan y\, dy$$

Using identity:

$$\int \tan x\, dx = \log|\sec x\ |$$

$$x\tan(x) - \log|\sec x\ | = y\tan y - \log|\sec y| + c$$

12. Question

Solve the following differential equations:

$$xy\ dy = (y-1)(x+1)dx$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{y}{y-1}dy = \frac{x+1}{x}dx$$

Re – writing LHS as $\frac{y-1+1}{y-1}$

$$\left[1 - \frac{1}{y-1}\right]dy = \left(1 + \frac{1}{x}\right)dx$$

Integrating both sides using identities:

$$\int \frac{dy}{y} = \log(y)$$

and

$$\int k dx = kx$$

$$\int \left[1 - \frac{1}{y-1}\right] dy = \int \left(1 + \frac{1}{x}\right) dx$$

y – log(y – 1) = x + log(x) + c

y – x = x(y – 1) + c

13. Question

Solve the following differential equations:

$$x\frac{dy}{dx} + \cot\ y = 0$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{\cot y} = -\frac{dx}{x}$$

$$\int \tan y\ dy = \int -\frac{dx}{x}$$

Integrating both sides using identities:

$$\int \tan x\, dx = \log|\sec x|$$

and

$$\int \frac{dy}{y} = \log(y)$$

log(|sec(y)|) = – log(x) + c

using log(a)+log(b) = log(ab) formula, we have,

x sec(y) = c

14. Question

Solve the following differential equations:

$$\frac{dy}{dx} = \frac{x\ e^x \log\ x + e^x}{x\ \cos\ y}$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\cos y\, dy = \frac{e^x(x\log(x) + 1)}{x} dx$$

$$\cos y\, dy = e^x\left(\log x + \frac{1}{x}\right) dx$$

Integrating both sides using identities:

$$\int \cos y\, dy = \sin y$$

and property

$\int e^x[f(x) + f'(x)] = e^x f(x)$

$\int \cos y\, dy = \int e^x\left(\log x + \frac{1}{x}\right) dx$

$\mathbf{\sin y = e^x(\log x) + c}$

15. Question

Solve the following differential equations:

$$\frac{dy}{dx} = e^{x+y} + e^y x^3$$

Answer

Re - writing the question as

$$\frac{dy}{dx} = e^y(e^x + x^3)$$

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{e^y} = (e^x + x^3)dx$$

Integrating both sides using identities:

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

and

$$\int e^{kx} dx = \frac{e^x}{k}$$

$$\int \frac{dy}{e^y} = \int (e^x + x^3)dx$$

$$\mathbf{-e^{-y} = e^x + \frac{x^4}{4} + c}$$

16. Question

Solve the following differential equations:

$$y\sqrt{1+x^2} + x\sqrt{1+y^2}\frac{dy}{dx} = 0$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{\sqrt{1+x^2}}{x}dx = -\frac{\sqrt{1+y^2}}{y}dy$$

Re - writing the equation as

$$\frac{x\sqrt{1+x^2}}{x^2}dx = -\frac{y\sqrt{1+y^2}}{y^2}dy$$

Now assuming $1+y^2 = t^2$

Differentiating both sides, we get

ydy = tdt

Similarly, for LHS assuming $1+x^2 = v^2$

differentiating both sides

xdx = vdv

substituting these values in the differential equation

$$\frac{(v^2 dv)}{v^2-1} = -\frac{(t^2 dt)}{t^2-1}$$

Integrating both sides

$$\int \frac{(v^2 dv)}{v^2-1} = -\int \frac{(t^2 dt)}{t^2-1}$$

Re - writing as

$$\int 1\,dv + \int \frac{dv}{v^2-1} = -\int 1\,dt - \int \frac{dt)}{t^2-1}$$

Using identity:

$$\int k\,dx = kx$$

and

$$\int \frac{dv}{v^2-1} = \frac{1}{2}\log(\frac{v-1}{v+1})$$

Integrating both sides, we get

$$v + \frac{1}{2}\log\left(\frac{v-1}{v+1}\right) = -t - \frac{1}{2}\log\left(\frac{t-1}{t+1}\right) + c$$

Substituting the value of v and t in the above equation

$$\sqrt{1+x^2} + \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) = -\sqrt{1+y^2} - \frac{1}{2}\log\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right) + c$$

17. Question

Solve the following differential equations:

$$\sqrt{1+x^2}\ dy + \sqrt{1+y^2}\ dx = 0$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{\sqrt{1+y^2}} = -\frac{dx}{\sqrt{1+x^2}}$$

Integrating both sides using identity:

$$\frac{dy}{\sqrt{1+y^2}} = \log|y + \sqrt{1+y^2}\ |$$

$$\log|y + \sqrt{1+y^2} = -\log\left|x + \sqrt{1+x^2}\ \right| + c$$

$$\log|y + \sqrt{1+y^2} + \log\left|x + \sqrt{1+x^2}\ \right| = \log c$$

$$\mathbf{\left(y + \sqrt{1+y^2}\right)\left(x + \sqrt{1+x^2}\right) = c}$$

18. Question

Solve the following differential equations:

$$\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dx}=0$$

Answer

Re - writing the equation as

$$\sqrt{1+x^2+y^2+x^2y^2} = -xy\frac{dy}{dx}$$

$$\sqrt{(1+x^2)+y^2(1+x^2)} = -xy\frac{dy}{dx}$$

$$\sqrt{(1+y^2)(1+x^2)} = -xy\frac{dy}{dx}$$

Now separating variable x on one side and variable y on another side, we have

$$\frac{\sqrt{1+x^2}dx}{x} = -\frac{ydy}{\sqrt{1+y^2}}$$

Multiplying and dividing the numerator of LHS by x

$$\frac{x\sqrt{1+x^2}dx}{x^2} = -\frac{ydy}{\sqrt{1+y^2}}$$

Assuming $1+y^2 = t^2$ and $1+x^2 = v^2$

Differentiating we get,

ydy = tdt

xdx = vdv

substituting these values in above differential equation

$$\frac{v^2dv}{v^2-1} = -\frac{tdt}{t}$$

Integrating both sides

$$\int\frac{v^2dv}{v^2-1} = -\int\frac{tdt}{t}$$

Adding 1 and subtracting 1 to the numerator of RHS

$$\int dv+\int\frac{1}{v^2-1}dv = -\int dt$$

Using identities:

$$\int k\,dx = kx$$

and

$$\int\frac{dv}{v^2-1} = \frac{1}{2}\log(\frac{v-1}{v+1})$$

$$v+\frac{1}{2}\log\left[\frac{v-1}{v+1}\right] = -t+c$$

Substituting t and v in above equation

$$\sqrt{1+x^2}+\frac{1}{2}\log\left[\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right] = -\sqrt{1+y^2}+c$$

19. Question

Solve the following differential equations:

$$\frac{dy}{dx}=\frac{e^x\left(\sin^2 x+\sin 2x\right)}{y\left(2\log y+1\right)}$$

Answer

Now separating variable x on one side and variable y on another side, we have,

$y(2\log(y) + 1)\ dy = e^x(\sin^2 x + \sin 2x)\ dx$

Integrating both sides we get,

$$\int y(2\log(y)+1)\ dy = \int e^x(\sin^2 x+\sin 2x)\ dx$$

Using integration by parts for LHS and identity:

$$\int e^x[f(x)+f'(x)] = e^x f(x)$$

for RHS.

$$\int y\ dy = \frac{y^2}{2}$$

$$\int 2y\log(y)dy+\int y\ dy = e^x(\sin^2 x)+c$$

$$y^2\log y-\int y dy+\int y dy = e^x(\sin^2 x)+c$$

$$y^2\log y = e^x(\sin^2 x)+c$$

20. Question

Solve the following differential equations:

$$\frac{dy}{dx}=\frac{x\left(2\log x+1\right)}{\sin y+y\cos y}$$

Answer

Now separating variable x on one side and variable y on another side, we have,

$(\sin y + y\cos y)\ dy = (2x\log x + x)\ dx$

Integrating both sides we get,

$$\int (\sin y + y\cos y)\ dy = \int (2x\log x + x)\ dx$$

$$\int \sin y\ dy+\int y\cos y\ dy = \int 2x\log x\, dx+\int x dx$$

Using identity:

$$\int \sin t\, dt = -\cos t$$

$$\int \cos t dt = \sin t$$

and integration by parts we get,

$$-\cos y+y\sin y-\int \sin y\ dy = x^2\log x-\int x\ dx+\int x dx$$

$y\sin y = x^2\log(x) + c$

21. Question

Solve the following differential equations:

$$\left(1-x^2\right)dy + xy\ dx = xy^2 dx$$

Answer

Re - writing the equation as $(1 - x^2)\ dy = xdx(y^2 - y)$

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{y^2 - y} = \frac{xdx}{1 - x^2}$$

$$\frac{dy}{y(y-1)} = \frac{xdx}{1 - x^2}$$

Integrating both sides

$$\int \frac{dy}{y(y-1)} = \int \frac{xdx}{1 - x^2}$$

$$\int \frac{[y - (y-1)]dy}{y(y-1)} = \int \frac{xdx}{1 - x^2}$$

$$\int \frac{dy}{y-1} - \int \frac{dy}{y} = \int \frac{xdx}{1 - x^2}$$

Using identity:

$$\int \frac{dx}{x} = \log x$$

and assuming $x^2 = t$

Differentiating both sides we get,

2 x dx = dt

$$xdx = \frac{dt}{2}$$

Substituting this value in above equation

$$\log(y-1) - \log(y) = -\frac{1}{2}\log(1-t)$$

Replacing t by x^2

$$\log(y-1) - \log(y) = -\frac{1}{2}\log(1-x^2) + c$$

22. Question

Solve the following differential equations:

$$\tan\ y\ dx + \sec^2 y\ \tan\ x\ dy = 0$$

Answer

Now separating variable x on one side and variable y on another side, we have,

$$-\frac{dx}{\tan x} = \frac{\sec^2 y}{\tan y}dy$$

$-\cot(x)\ dx = \sec^2 y\ \cot(y)\ dy$

Integrating both sides we get,

$$-\int \cot(x)\,dx = \int \sec^2 y \cot(y)\,dy$$

$$-\int \cot(x)\,dx = \int \frac{1}{\sin y \cos y}\,dy$$

Sin2x = 2 sinx cosx

$$-\int \cot(x)\,dx = \int \frac{2}{\sin 2y}\,dy$$

$$-\int \cot(x)dx = \int 2\mathrm{cosec}2y\,dy$$

Using identities:

$$\cot x\,dx = \log|\sin x|$$

and

$$\int \mathrm{cosec}\,y\,dy = \log|\tan(\frac{y}{2})|$$

– log| sinx | = log| tany |+ c

Using: log(a)+log(b) = log(ab)

log(sin x tan y) = log c

sin(x) tan(y) = c

23. Question

Solve the following differential equations:

$$(1+x)(1+y^2)dx+(1+y)(1+x^2)dy=0$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{1+x}{1+x^2}dx = -\frac{1+y}{1+y^2}dy$$

Integrating both sides

$$\int \left[\frac{1}{1+x^2}+\frac{x}{1+x^2}\right]dx = -\int \left[\frac{1}{1+y^2}+\frac{y}{1+y^2}\right]dy$$

Using identity:

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x$$

and substituting x^2 = t and y^2 = v

x^2 = t

2xdx = dt

$$xdx = \frac{dt}{2}$$

Similarly, for y

y^2 = v

2ydy = dv

$$\text{ydy} = \frac{dv}{2}$$

Substituting these values in above integral equation

$$\tan^{-1}x + \frac{1}{2}\int \frac{dt}{1+t} = -\tan^{-1}y - \frac{1}{2}\int \frac{dv}{1+v}$$

$$\tan^{-1}x + \frac{1}{2}\log(1+t) = -\tan^{-1}y - \frac{1}{2}\log(1+v) + c$$

Substituting the values of t and v in above equation

$$\tan^{-1}x + \frac{1}{2}\log(1+x^2) = -\tan^{-1}y - \frac{1}{2}\log(1+y^2) + c$$

$$\mathbf{\tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = c}$$

24. Question

Solve the following differential equations:

$$\tan\ y\frac{dy}{dx} = \sin(x+y) + \sin(x-y)$$

Answer

Using the formula:

$$\sin c + \sin d = 2\sin\left[\frac{c+d}{2}\right]\cos\left[\frac{(c-d)}{2}\right]$$

Re - writing the equation as

$$\tan y\frac{dy}{dx} = 2\sin x \cos y$$

Now separating variable x on one side and variable y on another side, we have

$$\frac{\tan y}{\cos y}dy = 2\sin x\ dx$$

Sec y ta ny dy = 2 sin x dx

Integrating both sides using identities :

$$\int \sec x \tan x\, dx = \sec x$$

And

$$\int \sin x\ dx = -\cos x$$

$$\int \sec y \tan y\ dy = \int 2\sin x\ dx$$

sec(y) = – 2cos(x) + c

25. Question

Solve the following differential equations:

$$\cos x\ \cos\ y\frac{dy}{dx} = -\sin\ x\ \sin\ y$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{\cos y}{\sin y}dy = -\frac{\sin x}{\cos x}dx$$

coty dy = – tanx dx

Integrating the above equation using identities:

$$\int \cot y \ dy = \log|\sin y|$$

and

$$\int \tan y \ dy = \log|\cos y|$$

log|sin(y)| = – log|cos(y)|+c

Using: log(a)+log(b) = log(ab)

sin(y) cos(y) = c

26. Question

Solve the following differential equations:

$$\frac{dy}{dx} + \frac{\cos x \sin y}{\cos y} = 0$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$-\frac{\cos y}{\sin y}dy = \cos x \, dx$$

– cot(y)dy = cos(x)dx

Integrating both sides using identities :

$$\int \cot y \ dy = \log|\sin y|$$

and

$$\int \cos x \ dx = \sin x$$

$$-\int \cot(y)dy = \int \cos(x)dx$$

– log|sin(y)| = sin(x) + c

27. Question

Solve the following differential equations:

$$x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\ dy = 0$$

Answer

Now separating variable x on one side and variable y on another side, we have,

$$-\frac{x}{\sqrt{1-x^2}}dx = \frac{y}{\sqrt{1-y^2}}dy$$

Let $1 – x^2 = t^2$ and $1 – y^2 = t^2$

Differentiating we get

xdx = – tdt

ydy = – vdv

substituting these values in above differential equation

$$\frac{tdt}{t} = -\frac{vdv}{v}$$

dt = – dv

integrating both sides, we get

$$\int dt = -\int dv$$

t = – v + c

substituting the value of v and c in above equation

$$\sqrt{1-x^2} = -\sqrt{1-y^2} + c$$

17. Question

Solve the following differential equations:

$$y\left(1+e^x\right)dy = \left(y+1\right)e^x dx$$

Answer

Now separating variable x on one side and variable y on another side, we have

$$\frac{ydy}{y+1} = \frac{e^x dx}{1+e^x}$$

Adding 1 and subtracting 1 to the numerator of LHS, we get

$$dy - \frac{dy}{y+1} = \frac{e^x dx}{1+e^x}$$

Integrating both sides using identities:

$$\int k\,dy = ky$$

And

$$\int \frac{dy}{(y+1)} = \log(y+1)$$

$$\int dy - \int \frac{dy}{y+1} = \int \frac{e^x dx}{1+e^x}$$

$$y - \log(y+1) = \int \frac{e^x dx}{1+e^x}$$

Assuming e^x = t and differentiating both sides we get,

$e^x dx$ = dt

substituting this value in above equation

$$y - \log(y+1) = \int \frac{dt}{1+t}$$

y – log(y+1) = log(1+t)

substituting t as e^x

y – log(y+1) = log(1+e^x) + c

29. Question

Solve the following differential equations:

$$\left(y+xy\right)dx+\left(x-xy^2\right)dy=0$$

Answer

Re – writing the equation as

y(1+x) dx+x(1 – y^2) dy = 0

Now separating variable x on one side and variable y on another side, we have

$$\frac{1+x}{x}dx = -\left(\frac{1-y^2}{y}\right)dy$$

$$\left(\frac{1}{x}+1\right)dx = -\left(\frac{1}{y}-y\right)dy$$

Integrating both sides using identity:

$$\int k\,dx = kx\,,\int x^n dx = \frac{x^{n+1}}{n+1}$$

And

$$\int \frac{1}{x}dx = \log(x)$$

$$\int\left(\frac{1}{x}+1\right)dx = -\int\left(\frac{1}{y}-y\right)dy$$

$$\log(x)+x = -\log(y)+\frac{y^2}{2}+c$$

Using log(a)+log(b) = log(ab)

$$xy+x = \frac{y^2}{2}+c$$

30. Question

Solve the following differential equations:

$$\frac{dy}{dx}=1-x+y-xy$$

Answer

Re - writing the above equation as

$$\frac{dy}{dx} = (1-x)+y(1-x)$$

$$\frac{dy}{dx} = (1+y)(1-x)$$

Now separating variable x on one side and variable y on another side, we have

$$\frac{dy}{1+y} = (1-x)dx$$

Integrating both sides using identities:

$$\int k\,dx = kx\,,\int x^n dx = \frac{x^{n+1}}{n+1}$$

And

$$\int \frac{1}{x}dx = \log(x)$$

$$\int \frac{dy}{1+y} = \int (1-x)dx$$

$$\log(1+y) = x - \frac{x^2}{2} + c$$

31. Question

Solve the following differential equation:

$$\left(y^2+1\right)dx - \left(x^2+1\right)dy = 0$$

Answer

Given. $(y^2 + 1)dx - (x^2 + 1)dy = 0$

Find: Find the general solution of this differential equation.

$= (y^2 + 1)dx = (x^2 + 1)dy$

$= \int \frac{dy}{y^2+1} = \int \frac{dx}{x^2+1}$

$= \tan^{-1}y = \tan^{-1}x + C$

Hence, The solution of the given Differential Equation is $\tan^{-1}y = \tan^{-1}x + C$

32. Question

Solve the following differential equation:

dy + (x + 1)(y + 1) dx = 0

Answer

Given. dy + (x + 1)(y + 1)dx = 0

Find: Find the general solution of this differential equation.

= dy = -(x + 1)(y + 1)dx

$= \int \frac{dy}{y+1} = -\int (x + 1)dx$

$= \log|y + 1| = -\frac{x^2}{2} - x$

$= \log|y + 1| + \frac{x^2}{2} + x + C$

Hence, The solution of the given differential equation is $\log|y + 1| + \frac{x^2}{2} + x + C$.

33. Question

Solve the following differential equation:

$$\frac{dy}{dx} = \left(1+x^2\right)\left(1+y^2\right)$$

Answer

Given $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

Find: Find the general solution of this differential equation.

$= \int \frac{dy}{(1+y^2)} = (1 + x^2)dx$

$= \tan^{-1}y = x + \frac{x^3}{3} + C$

$= \tan^{-1}y - x - \frac{x^3}{3} + C$

Hence, The solution of the given differential equation is $\tan^{-1}y - x - \frac{x^3}{3} + C$

34. Question

Solve the following differential equation:

$$(x-1)\frac{dy}{dx} = 2x^3y$$

Answer

Given $(x-1)\frac{dy}{dx} = 2x^3y$

Find: Find the general solution of this differential equation.

$= \frac{dy}{dx} = \frac{2x^3y}{x-1}$

$= \frac{dy}{y} = \frac{2x^3dx}{x-1}$

Integrate Both side

$= \int \frac{dy}{y} = 2\int x^2 + x + 1 + \frac{1}{x-1}$

35. Question

Solve the following differential equation:

$$\frac{dy}{dx} = e^{x+y} + e^{-x+y}$$

Answer

Given differential equation $\frac{dy}{dx} = e^{x+y} + e^{-x+y}$

Find: Find the general solution of this differential equation.

$= \frac{dy}{dx} = e^{x+y} + e^{-x+y}$

$= \frac{dy}{dx} = e^x e^y + e^{-x}e^y$

$= \frac{dy}{dx} = e^y(e^x + e^{-x})$

$= \frac{dy}{e^y} = (e^x + e^{-x})dx$

Integrate Both Side,

$= \int e^{-y}\,dy = \int (e^x + e^{-x})dx$

$= -e^{-y} = e^x - e^{-x}$

$= e^x = e^{-x} - e^{-y} + C$

Hence, The solution of the given differential equation is $e^x = e^{-x} - e^{-y} + C$

36. Question

Solve the following differential equation:

$$\frac{dy}{dx} = \left(\cos^2 x - \sin^2 x\right)\cos^2 y$$

Answer

Given differential Equation $\frac{dy}{dx} = (\cos^2 x - \sin^2 x)\cos^2 y$

Find: Find the general solution of this differential equation.

$= \frac{dy}{dx} = (\cos^2 x - \sin^2 x)\cos^2 y$

$= \frac{dy}{\cos^2 y} = (\cos^2 x - \sin^2 x)\,dx$

Integrate Both side,

$= \int \frac{dy}{\cos^2 y} = \int (\cos^2 x - \sin^2 x)\,dx$

$= \int \sec^2 y\,dy = \int \cos 2x\,dx$

$= \tan y = \frac{\sin 2x}{2} + C$

Hence,the solution is $\tan y = \frac{\sin 2x}{2} + C$

37 A. Question

Solve the following differential equation:

$$\left(xy^2 + 2x\right)dx + \left(x^2 y + 2y\right)dy = 0$$

Answer

Given $(xy^2 + 2x)dx + (x^2y + 2y)dy = 0$

Find: Find the general solution of this differential equation.

$= (xy^2 + 2x)dx + (x^2y + 2y)dy = 0$

$= (xy^2 + 2x)dx = -(x^2y + 2y)dy$

$= y(x^2 + 2)dy = -x(y^2 + 2)dx$

$= \frac{y}{y^2+2}dy = -\frac{x}{x^2+2}dx$

Multiply by 2 Both side

$= \frac{2y}{y^2+2}dy = -\frac{2x}{x^2+2}dx$

Now, Integrate both sides,

$= \int \frac{2y}{y^2+2}dy = -\int \frac{2x}{x^2+2}dx$

= Let assume $y^{2+}2$ = t Let assume $x^2 + 2$ = v

Then 2y dy =dt 2x dx = dv

$= \int \frac{dt}{t} = -\int \frac{dv}{v}$

= log|t| = -log|v|

Put the value of t and v

$= \log|y^2 + 2| = -\log|x^2 + 2| + \log|c|$

$= |y^2 + 2| = |\frac{c}{x^2+2}|$

Hence, $|y^2 + 2| = |\frac{c}{x^2+2}|$

37 B. Question

Solve the following differential equation:

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0$$

Answer

Given differential equation $\operatorname{cosec} x.\log y \frac{dy}{dx} + x^2y^2 = 0$

Find: Find the general solution of this differential equation.

$= \operatorname{cosec} x.\log y \frac{dy}{dx} + x^2y^2 = 0$

$= \frac{\log y\ dy}{y^2} = -\frac{x^2\,dx}{\operatorname{cosec} x}$

$= \frac{\log y\ dy}{y^2} = -x^2.\sin x\,dx$

Integrating Both side

$= \int \frac{\log y\ dy}{y^2} = -\int x^2.\sin x\,dx$

Using integration by parts both side

$= \frac{\log y + 1}{y} = -x^2 \cos x + 2(x \sin x + \cos x)$

Hence, The Solution is $\frac{\log y + 1}{y} + x^2 \cos x - 2(x \sin x + \cos x) + C$

38 A. Question

Solve the following differential equation:

$$xy\frac{dy}{dx} = 1 + x + y + xy$$

Answer

Given differential equation $xy\frac{dy}{dx} = 1 + x + y + xy$

Find: Find the general solution of this differential equation.

$= xy\frac{dy}{dx} = 1 + x + y + xy$

$= xy\frac{dy}{dx} = (1 + x) + y(1 + x)$

$= xy\frac{dy}{dx} = (1 + y)(1 + x)$

$= \frac{ydy}{(1+y)} = \frac{1+x}{x}$

$= \int \frac{y}{(1+y)} dy = \int \frac{1+x}{x} dx$

$= \int 1 - \frac{1}{y+1} = \int \frac{1}{x} + 1dx$

= y – log|y + 1| = log|x| + x + log|C|

= y = log|x| + x + log|y + 1| + log|C|

Hence, y = log|cx(y + 1)| + x

38 B. Question

Solve the following differential equation:

$$y\left(1-x^2\right)\frac{dy}{dx} = x\left(1+y^2\right)$$

Answer

Given differential equation $y(1-x^2)\frac{dy}{dx} = x(1 + y^2)$

Find: Find the general solution of this differential equation.

$= y(1-x^2)\frac{dy}{dx} = x(1 + y^2)$

$= \frac{ydy}{(1+y^2)} = \frac{xdx}{(1-x^2)}$

$= -\int \frac{2ydy}{(1+y^2)} = \int -\frac{2xdx}{(1-x^2)}$

$= -\log|1 + y^2| = \log|1-x^2| + \log|C1|$

$= -\log|C| = \log|1-x^2| + \log|1 + y^2|$

= C = (1 – x^2)(1 + y^2)

38 C. Question

Solve the following differential equation: $ye^{x/y}dx = \left(xe^{x/y} + y^2\right)dy, y \neq 0$

Answer

Given $ye^{x/y}dx = \left(x e^{\frac{x}{y}} + y^2\right)dy$

Find: Find the general solution of this differential equation.

$= ye^{x/y}dx = \left(x e^{\frac{x}{y}} + y^2\right)dy$

$= ye^{x/y} - x e^{\frac{x}{y}} = y^2 dy$

$= \left(\frac{ydx-xdy}{y^2}\right)e^{\frac{x}{y}} = dy$

$= e^{\frac{x}{y}} d\left(\frac{x}{y}\right) = dy$

= Integrating on the both side we get,

Hence, $e^{\frac{x}{y}} = y + C$

38 D. Question

Solve the following differential equation: $\left(1+y^2\right)\tan^{-1}x\,dx+2y\left(1+x^2\right)dy=0$

Answer

Given differential equation $(1\ +\ y^2)\tan^{-1}x\,dx\ +\ 2y(1\ +\ x^2)dy=0$

Find: Find the general solution of this differential equation.

$=(1\ +\ y^2)\tan^{-1}x\,dx\ +\ 2y(1\ +\ x^2)dy=0$

$=(1\ +\ y^2)\tan^{-1}x\,dx=-2y(1\ +\ x^2)dy$

$=\int-\frac{\tan^{-1}x}{2(1+x^2)}\,dx=\int\frac{y}{(1+y^2)}dy$

$=-\left(\tan^{-1}\left(\frac{1}{2}\tan^{-1}x\right)^2-\int\frac{1}{(1+x^2)}\left(\frac{1}{2}\tan^{-1}x\right)dx\right)=\frac{1}{2}\text{In}(y^2\ +\ 1)\ +C$

$=-\frac{1}{4}(\tan^{-1}x)^2-\frac{1}{2}\text{In}(y^2\ +\ 1)\ +\ C1$

$=\frac{1}{2}(\tan^{-1}x)^2\ +\ \text{In}(y^2\ +\ 1)=C$

39. Question

Solve the following initial value problem:

$$\frac{dy}{dx}=y\ \tan\ 2x, y(0)=2$$

Answer

Given differential equation $\frac{dy}{dx}=y\tan 2x$

Find: Find the general solution of this differential equation.

$=\frac{dy}{dx}=y\tan 2x$

$=\frac{dy}{y}=\tan\ 2x\,dx$

Integrating both sides,

$=\int\frac{dy}{y}=\int\tan\ 2x\,dx$

$=\log|y|\ =\frac{1}{2}\log|\sec 2x|\ +\ \log|C|$

$=y\ =\sqrt{\sec2x}.C$ .....(i)

Put x =0, y = 2

2 $\Longrightarrow\sqrt{\sec2.0}.C$

2 = C

Put value of C in (i)

Hence, y = $\frac{2}{\sqrt{\sec2x}}$

40. Question

Solve the following initial value problem:

$$2x\frac{dy}{dx} = 3y, y(1) = 2$$

Answer

Given differential equation $2x.\frac{dy}{dx} = 3y$

Find: Find the general solution of this differential equation.

$= 2x.\frac{dy}{dx} = 3y$

$= \frac{2dy}{y} = \frac{3dx}{x}$

$= 2\int\frac{dy}{y} = 3\int\frac{dx}{x}$

$= 2\log|y| = 3\log|x| + \log c$

$= y^2 = x^3C$ .....(i)

Put x =1, y = 2

$(2)^2 = (1)^3C$

= C = 4

Put C =4 in equation (i)

$y^2 = x^3 4$

Hence, $y^2 = 4x^3$

41. Question

Solve the following initial value problem:

$$xy\frac{dy}{dx} = y + 2, y(2) = 0$$

Answer

Given differential equation $xy.\frac{dy}{dx} = y + 2$

Find: Find the general solution of this differential equation.

$= xy.\frac{dy}{dx} = y + 2$

$= \frac{ydy}{y+2} = \frac{dx}{x}$

$= \int\left(1 - \frac{2}{y+2}\right)dy = \int\frac{dx}{x}$

On Integrating we get,

= y – 2log|y + 2|=log |x| + log |C| ....(i)

Put y = 0, x = 2

= 0 – 2log2 = log 2 + log c

= -2 log 2 –log 2 = log C

= -3 log 2 = log c

$= \log\left(\frac{1}{8}\right)$=log c

$c = \frac{1}{8}$

Put the value of C in equation (i)

Hence, y – 2log|y + 2|=$\log\left|\frac{x}{8}\right|$

42. Question

Solve the following initial value problem:

$$\frac{dy}{dx} = 2e^x y^3, y(0) = \frac{1}{2}$$

Answer

Given $\frac{dy}{dx} = 2e^x y^3$

$= \frac{dy}{dx} = 2e^x y^3$

$= \frac{dy}{y^3} = 2e^x dx$

Integrating both side

$= \int \frac{dy}{y^3} = \int 2e^x dx$

$= \int y^{-3} = \int 2e^x dx$

$= \left[\frac{y^{-3+1}}{-3+1}\right] = 2.e^x$

$= \frac{y^{-2}}{-2} = 2.e^x$

$= \frac{1}{-2y^2} = 2.e^x + C$ .....(i)

Put x = 0, y = 1/2

$= -\frac{4}{2} = 2e^0$

= -2 = 2 + c

C = -4

Put the value of C = -4 in equation (i)

$= \frac{1}{-2y^2} = 2.e^x$- 4

$= -1 = 4\,e^x y^2 - 8y^2$

$= -1 = -y^2(8 - 4e^x)$

Hence, $y^2(8 - 4e^x) = 1$

43. Question

Solve the following initial value problem:

$$\frac{dr}{dt} = -rt, r(0) = r_0$$

Answer

Given differential equation $\frac{dr}{dt} = -rt$

Find: Find the general solution of this differential equation.

$= \frac{dr}{dt} = -rt$

Integrating both side

$= \int \frac{dr}{r} = \int -t\,dt$

$= \log|r| = -\frac{t^2}{2} + C$ .....(i)

Put t =0, r = r_0 in equation (i).

Now, $\log|r0| = -0 + C$

$\log|r0|$= C

Put the value of C in eq(i)

$\log|r| = -\frac{t^2}{2} + \log|r0|$

$= \frac{r}{r_0} = e^{-\frac{t^2}{2}}$

Hence, $r = r_0 e^{-\frac{t^2}{2}}$

44. Question

Solve the following initial value problem:

$$\frac{dy}{dx} = y \sin 2x, y(0) = 1$$

Answer

Given differential equation $\frac{dy}{dx} = y.\sin 2x$

Find: Find the particular solution of this differential equation.

$= \frac{dy}{dx} = y.\sin 2x$

$= \frac{dy}{y} = \sin 2x\,dx$

Integrating both side

$= \int \frac{dy}{y} = \int \sin 2x\,dx$

$= \log|y| = -\frac{\cos 2x}{2} + C$

Put y =1 and x=1

$= \log|1| = -\frac{\cos 0}{2} + C$

$= 0 = -\frac{1}{2} + C$

$= C = \frac{1}{2}$

So, $\log|y| = -\frac{\cos 2x}{2} + C$

$= \log|y| = -\frac{\cos 2x}{2} + \frac{1}{2}$

$= \log|y| = \frac{1-\cos 2x}{2}$

$= \log|y| = \sin^2 x$

Hence, y = $e^{\sin^2 x}$

45 A. Question

Solve the following initial value problem:

$$\frac{dy}{dx} = y\ \tan\ x, y(0) = 1$$

Answer

The given differential equation is $\frac{dy}{dx} = y\tan x$

Find: Find the particular solution of this differential equation.

$= \frac{dy}{dx} = y\tan x$

$= \frac{dy}{y} = \tan x\, dx$

Integrating both sides,

$= \int \frac{dy}{y} = \int \tan x\, dx$

= log|y|=log|sec x| + log|C| .....(I)

Put y =1, x=0

0 = log(1) + C

C = 0

Put the value of C in equation in equation(I)

= log y = log |sec x|

Hence, y = sec x

45 B. Question

Solve the following initial value problem:

$$2x\frac{dy}{dx} = 5y, y(1) = 1$$

Answer

The given differential Equation is $2x\frac{dy}{dx} = 5y$.

Find: Find the particular solution of this differential equation.

$= 2x\frac{dy}{dx} = 5y$

$= \frac{dy}{5y} = \frac{dx}{2x}$

Now Integrating Both sides

$= \int \frac{dy}{5y} = \int \frac{dx}{2x}$

$= \int \frac{dy}{y} = 5 \int \frac{dx}{x}$

$= 2 \log|y| = 5 \log|x| + \log|C|$ .....(1)

Put x = 1, y = 1

= 2 log(1)=5log(1) + C

= 0 = C

Put the value of C in equation (1)

= 2log |y| = 5log|x|

$= y^5 = |x|^2$

Hence, $y = |x|^{5/2}$

45 C. Question

Solve the following initial value problem:

$$\frac{dy}{dx} = 2e^{2x}y^2, y(0) = -1$$

Answer

The Given equation is $\frac{dy}{dx} = 2e^{2x}y^2$

Find: Find the particular solution of this differential equation.

$= \frac{dy}{dx} = 2e^{2x}y^2$

$= \frac{dy}{y^2} = 2e^{2x}dx$

Integrating both side,

$= \int \frac{dy}{y^2} = \int 2e^{2x}dx$

$= \left[\frac{y^{-2+1}}{-2+1}\right] = .2.\frac{e^{2x}}{2}$

$= -\frac{1}{y} = e^{2x} + C$ .....(i)

Put y = -1, x = 0

$1 = e^{0+} C$

1 = 1 + C

C = 0

Put the value of C in equation (i)

$= -\frac{1}{y} = e^{2x} + 0$

Hence, $y = -e^{-2x}$

45 D. Question

Solve the following initial value problem:

$$\cos\ y\frac{dy}{dx}=e^{x},y(0)=\frac{\pi}{2}$$

Answer

$\cos y\frac{dy}{dx}=e^{x}$

Find: Find the particular solution of this differential equation.

On equating we get,

= cos y dy = e^x dx

Integrating both side

$=\int \cos y\,dy=\int e^{x}dx$

= sin y = e^x + C .....(i)

Put x = 0, y = π/2

$=\sin\left(\frac{\pi}{2}\right)=e^{0}\ +\ C$

= 1 = 1 + C

= C = 0

Put the value of C in equation (i)

Sin y = e^x

Hence, y = $\sin^{-1}(e^x)$

45 E. Question

Solve the following initial value problem:

$$\frac{dy}{dx}=2xy,y(0)=1$$

Answer

$\frac{dy}{dx}=2xy$

Find: Find the particular solution of this differential equation.

$=\frac{dy}{y}=2x\,dx$

Integrating both sides

$=\int\frac{dy}{y}=\int 2x\,dx$

$=\log y=\frac{x^{2}}{2}.2$

$=\log y=\frac{x^{2}}{2}$ + C .....(i)

Put x = 0, y = 1

log(1) = 0 + C

0 = 0 + C

C = 0

Put the value of C in equation (i)

$\log y = \frac{x^2}{2}$

Hence, $y = e^{x^2}$

45 F. Question

Solve the following initial value problem:

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2, y(0) = 1$$

Answer

Given : $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$

Find: Find the particular solution of this differential equation.

$= \frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$= \frac{dy}{1+y^2} = (1 + x^2)dx$

Integrating both side

$= \int \frac{dy}{1+y^2} = \int (1 + x^2)\, dx$

$= \tan^{-1} y = x + \frac{x^3}{3} + C$ ....(i)

Put x = 0, y =1

$= \tan^{-1} 1 = 0 + \frac{0^3}{3} + C$

$= \frac{\pi}{4} = C$

Put the value of C in equation (i)

$= \tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$

Hence, $\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$

45 G. Question

Solve the following initial value problem:

$$xy\frac{dy}{dx} = (x+2)(y+2), y(1) = -1$$

Answer

Given: $xy\frac{dy}{dx} = (x + 2)(y + 2)$

Find: Find the particular solution of this differential equation.

$= \frac{ydy}{y+2} = \frac{x+2}{x}\, dx$

$= \int 1 - \frac{2}{y+2} dy = \int 1 + \frac{2}{x}\, dx$

= y – 2 log|y + 2| = x + 2 log|x| + C

= y – 2 log|y + 2| - x - 2 log|x| = C

Put x =1, y = -1

= -1 -1-2log(-1 + 2) – 2log 1 = C

= -2 = C

Thus, we have

Hence, Y – x – 2log(y + 2) - 2log x = -2

45 H. Question

Solve the following initial value problem:

$\frac{dy}{dx} = 1 + x + y^2 + xy^2$ when $y = 0, x = 0$

Answer

Given: $\frac{dy}{dx} = 1 + x + y^2 + xy^2$

= $\frac{dy}{dx} = (1 + x)(1 + y^2)$

= $\frac{1}{(1+y^2)}dy = (1 + x)dx$

= $\int \frac{1}{(1+y^2)}dy = \int(1 + x)dx$

= $\tan^{-1}y = x + \frac{x^2}{2} + C$ .....(i)

Put y = 0 and x = 0, then

= $\tan^{-1}0 = 0 + \frac{0^2}{2} + C$

= C = 0

Putting the value of C in equation (i) we get.

= $\tan^{-1}y = x + \frac{x^2}{2}$

Hence, $y = \tan\left(x + \frac{x^2}{2}\right)$

45 I. Question

Solve the following initial value problem:

$2(y + 3) - xy\frac{dy}{dx} = 0, y(1) = -2$

Answer

Given: $2(y + 3) - xy\frac{dy}{dx} = 0$

Find: Find the particular solution of this differential equation.

= $2(y + 3) = xy\frac{dy}{dx}$

= $\frac{2}{x}dx = \frac{y}{y+3}dy$

Integrating both sides,

$= \int \frac{2}{x} dx = \int \frac{y}{y+3} dy$

$= 2\log|x| = y + 3 - 3\log|y + 3| + C$ .....(i)

Put x = 1 and y= -2 in equation (i), we get

= 2 log(1) = y + 3 – 3log(-2 + 3) + C

= 0 = 1- 0 + C

= C = -1

Put the value of C in equation (i), we get

$= 2\log|x| = y + 3 - 3\log|y + 3| - 1$

$= \log(x)^2 = y + 2 - \log(y + 3)^3$

$= \log(x)^2 - \log(y + 3)^3 = y + 2$

Hence, $x^2(y + 3)^3 = e^{y+2}$

46. Question

Solve the differential equation $x\frac{dy}{dx} + \cot y = 0$, given that $y = \frac{\pi}{4}$, when $x = \sqrt{2}$.

Answer

Given differential equation. $x\frac{dy}{dx} + \cot y = 0$

Find: Find the particular solution of this differential equation.

$= x\frac{dy}{dx} + \cot y = 0$

$= x\frac{dy}{dx} = -\cot y$

$= \frac{dy}{\cot y} = -\frac{dx}{x}$

Integrating both sides, we get

$= \int \frac{dy}{\cot y} = -\int \frac{dx}{x}$

= log|sec y| = - log|x| + C .....(i)

Put x = $\sqrt{2}$, y = $\frac{\pi}{4}$

= log|sec $\frac{\pi}{4}$| = - log|$\sqrt{2}$| + C

$= \log|\sqrt{2}| = -\frac{1}{2}\log 2$ + C

$= \frac{1}{2}\log 2 = -\frac{1}{2}\log 2$ + C

= C = log2

Put C in equation (i)

= log|sec y| = - log|x| + log2

$= \sec y = \frac{2}{x}$

Hence, $x = \frac{2}{\sec y} = 2\cos y$

47. Question

Solve the differential equation $\left(1+x^2\right)\frac{dy}{dx}+\left(1+y^2\right)=0$, given that $y=1$, when $x=0$.

Answer

Given: $(1 + x^2)\frac{dy}{dx} + (1 + y^2) = 0$

Find: Find the particular solution of this differential equation.

$= (1 + x^2)\frac{dy}{dx} + (1 + y^2) = 0$

$= (1 + x^2)\frac{dy}{dx} = -(1 + y^2)$

$= (1 + x^2)dy = -(1 + y^2)dx$

Integrating both side,

$= \int \frac{dy}{(1+y^2)} = -\frac{\int dx}{(1+x^2)}$

$= \tan^{-1} y = -\tan^{-1} x + C$ ....(i)

Put x = 0, y = 1

$= \tan^{-1} 1 = -\tan^{-1} 0 + C$

$= \frac{\pi}{4} = 0 + C$

Put C in eq (i), we get

$= \tan^{-1} y = -\tan^{-1} x + \frac{\pi}{4}$

$= y = \tan\left(\frac{\pi}{4} - \tan^{-1} x\right)$

$= y = \frac{\tan\left(\frac{\pi}{4}-\tan^{-1} x\right)}{1+\tan\frac{\pi}{4}.\tan(\tan^{-1} x)}$

$= y = \frac{1-x}{1+x}$

= y + xy =1- x

Hence, x + y = 1 – xy

48. Question

Solve the differential equation $\frac{dy}{dx}=\frac{2x\left(\log x+1\right)}{\sin y+y\cos y}$, given that $y=0$, when $x=1$.

Answer

Given: $\frac{dy}{dx} = \frac{2x(\log x + 1)}{\sin y + y\cos y'}$

Find: Find the particular solution of this differential equation.

$= \frac{dy}{dx} = \frac{2x(\log x + 1)}{\sin y + y\cos y}$

$= \sin y + y\cos y)dy = 2x(\log x + 1)\ dx$

Integrating both side

$= \int \sin y + y\cos y\,dy = \int 2x(\log x + 1)dx$

$= \int \sin y\,dy + \int y\cos y\,dy = \int 2x.\log x\,dx + \int 2x\,dx$

$= -\cos y + [y.\int \cos y\,dy - \int 1.\int \cos y\,dy.dy] = 2[\log x \int xdx - \int [\frac{1}{x}\int xdx]dx] + x^2 + C$

$= -\cos y + y\sin y - \int \sin y\,dy = 2.\frac{x^2}{2}.\log x - 2\int \frac{x}{2}dx + x^2 + C$

$= -\cos y + y\sin y + \cos y = x^2\log x - \frac{x^2}{2} + x^2 + C$

$= y\sin y = x^2\log x + \frac{x^2}{2} + C$

Put y = 0, x = 1

= 0 = 0 + 1/2 + C

$= C = -\frac{1}{2}$

Put $C = -\frac{1}{2}$ in equation (i)

$= y\sin y = x^2\log x + \frac{x^2}{2} - \frac{1}{2}$

Hence, The Solution is $2y\sin y = 2x^2\log x + x^2 - 1.$

49. Question

Find the particular solution of $e^{dy/dx} = x + 1,$ given that $y = 3,$ when $x = 0.$

Answer

Given $e^{\frac{dy}{dx}} = x + 1$

Find: Find the particular solution of this differential equation.

$= e^{\frac{dy}{dx}} = x + 1$

$= \frac{dy}{dx} = \log(x + 1)$

$= dy = \log(x + 1)dx$

Integrating both sides

$= \int dy = \int \log(x + 1)dx$

$= y = \log|x + 1|.\int 1.dx - \int \left(\frac{1}{x+1}.\int 1, dx\right)dx$

Using Integration by parts

$= y = x.\log|x + 1| - \int \frac{x}{x+1}dx$

$= y = x.\log|x + 1| - \left(\int \left(1 - \frac{1}{x+1}\right)dx\right)$

$= y = x.\log|x + 1| - (x - \log|x + 1|)$

$= y = x.\log|x + 1| - (x + \log|x + 1|)$

$= y = (x + 1)\log|x + 1| - x + C$ ....(i)

Put y =3 and x = 0

= 3 = 0-0 + C

= C = 3

Put C = 3 in equation (i)

Hence, The Solution is $y = (x + 1)\log|x + 1| - x + 3$.

50. Question

Find the solution of the differential equation $\cos y\, dy + \cos x \sin y dx = 0$ given that $y = \pi/2$, when $x = \pi/2$.

Answer

The given differential Equation is $\cos y\, dy + \cos x \sin y\, dx = 0$.

Find: Find the solution of this differential equation.

$= \cos y\, dy + \cos x \sin y\, dx = 0$.

$= \cos y\, dy = -\cos x \sin y\, dx$

$= \frac{\cos y}{\sin y}\, dy = -\cos x\, dx$

Integrating both side,

$= \int \cot y\, dy = -\int \cos x\, dx$

= log |sin y| = - sin x + C .....(i)

Put y = $\frac{\pi}{2}$ and x = $\frac{\pi}{2}$

= log |sin $\frac{\pi}{2}$ | = - sin $\frac{\pi}{2}$ + C

= 0 = -1 + C

= C =1

Put the value of C in eq(i)

Hence, The solution is log |sin y| + sin x =1

51. Question

Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

Answer

The given differential equation is $\frac{dy}{dx} = -4xy^2$

Find: Find the solution of this differential equation.

$= \frac{dy}{dx} = -4xy^2$

$= \frac{dy}{y^2} = -4x\, dx$

Integrating both sides

$= \int \frac{dy}{y^2} = \int 4x\, dx$

$= \left[\frac{y^{-2+1}}{-2+1}\right] = -4.\frac{x^2}{2}$

$= -\frac{1}{y} = -2x^2 + C$

Put y =1 and x =0

= -1 = 0 + C

= C = -1

Put the value of C in eq (i)

$= -\frac{1}{y} = -2x^2 - 1$

$= \frac{1}{y} = 2x^2 + 1$

$= y = \frac{1}{2x^2+1}$

52. Question

Find the equation of a curve passing through the point (0, 0) and whose differential equation is $\frac{dy}{dx} = e^x \sin x.$

Answer

The given differential equation $\frac{dy}{dx} = e^x . \sin x$

Find: Find the solution of this differential equation.

$= \frac{dy}{dx} = e^x . \sin x$

$= dy = e^x . \sin x\, dx$

Integrating both sides

$= \int dy = \int e^x . \sin x\, dx$

$= I = \sin x \int e^x dx - \int (\frac{d}{dx}(\sin x). \int e^x\, dx) dx$

$= I = \sin x . e^x - \int \cos x . e^x dx$

$= I = \sin x . e^x - [\cos x . \int e^x dx - \int \frac{d}{dx}(\cos x). \int e^x dx$

$= I = \sin x . e^x - [\cos x . e^x - \int (-\sin x). e^x\, dx]$

$= I = \sin x . e^x - \cos x . e^x - I$

$= 2I = e^x(\sin x - \cos x)$

$= I = \left(\frac{e^x(\sin x - \cos x)}{2}\right)$

53. Question

For the differential equation $xy\frac{dy}{dx} = (x+2)\ (y+2)$. Find the solution curve passing through the point (1, -1).

Answer

The given differential equation is $xy\frac{dy}{dx} = (x + 2)(y + 2)$

Find: Find the solution of this differential equation.

$= xy\frac{dy}{dx} = (x + 2)(y + 2)$

$= \frac{ydy}{y+2} = \frac{x+2}{x}dx$

$= \left(1 - \frac{2}{y+2}\right)dy = \left(1 + \frac{2}{x}\right)dx$

Integrating both side

$= \int\left(1 - \frac{2}{y+2}\right)dy = \int\left(1 + \frac{2}{x}\right)dx$

$= \int dy - 2\int\frac{1}{y+2}dy = \int dx + 2\int\frac{1}{x}dx$

$= y - 2\log(y + 2) = x + 2\log x + C$

$= y - x - C = \log x^2 + \log(y + 2)^2 + C$

$= y - x - C = \log[x^2(y + 2)^2]$ .....(i)

Put x =1 and y = -1

$= -1 - 1 - C = \log[1^2(-1 + 2)^2]$

= - 2 - C = 0

= C = -2

Put the value of C in equation (i)

Hence, The solution of the curve is $y - x + 2 = \log[x^2(y + 2)^2]$.

54. Question

The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.

Answer

Let the rate of change of the volume of the balloon be k (where k is constant)

Find: Find the radius of the balloon after t second.

$= \frac{dv}{dt} = k$

$= \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k$

$= \left(\frac{4}{3}\pi.3.r^2.\frac{dr}{dt}\right) = k$

$= 4\pi r^2\ dr = k\,dt$

Integrating both sides, we get:

$= 4\pi\int r^2\ dr = k\int dt$

$= 4\pi.\frac{r^3}{3} = kt$

$= 4\pi r^3\ dr = 3(kt + C)$ .....(i)

Now, at t = 0, r = 3.

$= 4\pi 3^3\ dr = 3(k.0 + C)$

$= 108\pi = 3C$

$= C = 36\pi$

At t = 3, r = 6:

$= 4\pi \times 6^3 = 3(k \times 3 + C)$

$= 864\pi = 3(3k + 36\pi)$

$= 3k = -288\pi - 36\pi = 252\pi$

$= k = 84\pi$

Substitute the value of K and C in equation (1), we get

$= 4\pi r^3 = 3(84\pi + 36\pi)$

$= 4\pi r^3 = 4\pi(63\,t + 27)$

$= r^3 = 63t + 27$

$= r = (63t + 27)^{1/3}$

Hence, the radius of the balloon after t seconds is $(63\ t + 27)^{1/3}$.

55. Question

In a bank principal increases at the rate of r% per year. Find the value of r if ` 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Answer

Let p, t, and r represent the principal, time and rate of interest respectively.

It is the given that the principal increases continuously at the rate r% per year.

Find: Find the value of r?

$= \frac{dp}{dt} = \left(\frac{r}{100}\right)p$

$= \frac{dp}{p} = \left(\frac{r}{100}\right)dt$

Integrating both sides, we get:

$= \int \frac{dp}{p} = \frac{r}{100}\int dt$

$= \log p = \frac{rt}{100} + k$

$= p = e^{\frac{rt}{100}} + k$ .....-(i)

It is given that when t = 0, p = 100

$= 100 = e^k$ .....--(2)

Now, if t =10,then p = 2x100 =200

$= 200 = e^{\frac{r}{10}}.e^k$

$= 200 = e^{r/10}.100$ from (2)

$= e^{\frac{r}{10}} = 2$

$= \frac{r}{10} = \log 2$

$= \frac{r}{10} = 0.6931$

= r = 6.931

Hence, the value of r is 6.93 %.

56. Question

In a bank principal increases at the rate of 5% per year. An amount of ` 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5}$ = 1.648).

Answer

Let p, and t represent the principal, time respectively.

It is the given that the principal increases continuously at the rate of 5% per year.

$= \frac{dp}{dt} = \left(\frac{5}{100}\right)p$

$= \frac{dp}{p} = \left(\frac{1}{20}\right)dt$

Integrating both sides, we get:

$= \int \frac{dp}{p} = \frac{1}{20}\int dt$

$= \log p = \frac{t}{20} + C$

= p = $e^{\frac{t}{20}+C}$ .....-(i)

It is given that when t = 0, p = 1000

= 1000 = e^C .....--(2)

Now, $\log p = \frac{t}{20} + \log 1000$

Putting t = 10,we get

$= \log \frac{p}{1000} = 0.5$

$= \frac{p}{1000} = e^{0.5}$

= p = 1000 x 1.648

= p = 1648

57. Question

In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present.

Answer

Let y be the number of bacteria at any instant t

It is given that the rate of growth of the bacteria is proportional to the number present.

$= \frac{dy}{dt} \propto y$

$= \frac{dy}{dt} = ky$ (where k is a constant)

$= \frac{dy}{y} = kdt$

Integrating both sides, we get

$= \int \frac{dy}{y} = \int kdt$

= log y = kt + C

Let y_0 be the number of bacteria at t = 0.

= log y_0 = C

Substitute the value of C in, we get

$\Longrightarrow$ log y = kt + log y_0

= log y - log y_0 = kt

$= \log\left(\frac{y}{y_0}\right) = kt$

Also, it is given that the number of bacteria increased by 10% in 2 hours.

$= y = \frac{110}{100} y_0$

$= \frac{y}{y_0} = \frac{11}{10}$

Substituting the value,

$= k.2 = \log\left(\frac{11}{10}\right)$

$= k = \frac{1}{2}\log\left(\frac{11}{10}\right)$

Therefore,

$= \frac{1}{2}\log\left(\frac{11}{10}\right).t = \log\left(\frac{y}{y_0}\right)$

$= t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)}$

Now, the time when the number of bacteria increases from 100000 to 200000 be t1.

$= y = 2y_0$ at t = t_1

Now, $t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$

Hence, in $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ hours the number if bacteria increases from 100000 to 200000.

58. Question

If y(x) is a solution of the differential equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$ and y(0) = 1, then find the value of y(π/2).

Answer

Consider the given equation

$= \left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$

$= \frac{dy}{1+y} = -\frac{\cos x \, dx}{2+\sin x}$

Integrating both sides,

$= \int \frac{dy}{1+y} = -\int \frac{\cos x \, dx}{2+\sin x}$

= log(1 + y) ⟹-log(2 + sin x) + log C

= log(1 + y) + log(2 + sin x) = log C

= log(1 + y)(2 + sin x) = log C

= (1 + y)(2 + sin x) = c ....(1)

Given that y(0) = 1

= (1 + 1)(2 + sin 0) = c

= C = 4

Substituting the value of C in eq (1), we get

= (1 + y)(2 + sin x) = 4

= (1 + y) = $\frac{4}{(2+\sin x)}$

= y = $\frac{4}{(2+\sin x)} - 1$ ......(2)

Now, find the value of y(π/2)

Substituting the value of x = $\frac{\pi}{2}$ in equation (2)

= y = $\frac{4}{(2+\sin\frac{\pi}{2})} - 1$

= y = $\frac{4}{(2+1)} - 1$

= y = $\frac{4}{3} - 1$

= y = $\frac{1}{3}$

59. Question

Find the particular solution of the differential equation $(1-y^2)(1+\log x)\ dx + 2xy\ dy = 0$ given that y = 0 whenx = 1.

Answer

Consider the differential equation $(1-y^2)(1 + \log x)dx + 2xy\,dy = 0$

$= (1-y^2)(1 + \log x)dx = -2xy\,dy$

$= \frac{dy}{dx} = -\frac{(1-y^2)(1+\log x)}{2xy}$

$= \frac{ydy}{1-y^2} = -\frac{1+\log x}{2x}dx$

Let 1-y^2 = t then -2y dy = dt Let log x = v

$ydy = -\frac{1}{2}\ dt\ \frac{1}{x}dx = dv$

$= \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\int vdv$

$= \frac{1}{2}\log t = \frac{1}{4}v^2 + C$ .....(1)

Substitute the value of v and t in eq (2)

$= \frac{\log(1-y^2)}{2} = \frac{(1+\log x^2)}{4} + C$ .....(2)

Put x =1 and y =3 in eq (2)

$= \frac{\log(1-3^2)}{2} = \frac{(1+\log 1)}{4} + C$

$= C = \frac{1}{4}$

Put the value of C in eq (2)

$= \frac{\log(1-y^2)}{2} = \frac{(1+\log x^2)}{4} + \frac{1}{4}$

Hence, The particular solution is $(1 + \log x^2) = 2\log(1-y^2) + 1$

Exercise 22.8

1. Question

Solve the following differential equations:

$$\frac{dy}{dx} = (x + y + 1)^2$$

Answer

Given Differential equation is:

$\Rightarrow \frac{dy}{dx} = (x + y + 1)^2$ ......(1)

Let us assume z = x + y + 1

Differentiating w.r.t x on both the sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{d(x+z+1)}{dx}$

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx} + \frac{d(1)}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} + 0$

$\Rightarrow \frac{dz}{dx} - 1 = \frac{dy}{dx}$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow \frac{dz}{dx} - 1 = z^2$

$\Rightarrow \frac{dz}{dx} = 1 + z^2$

Bringing like variables on same (i.e, variable seperable technique) we get,

$\Rightarrow \frac{dz}{1+z^2} = dx$

Integrating on both sides we get,

$\Rightarrow \int \frac{dz}{1+z^2} = \int dx$

We know that $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ and

Also ∫adx = ax + C

⇒ $\frac{1}{1}\tan^{-1}\left(\frac{z}{1}\right) = x + C$

⇒ $\tan^{-1}z = x + C$

We know that z = x + y + 1 , substituting this we get,

⇒ $\tan^{-1}(x + y + 1) = x + C$

∴ The solution for the given Differential equation is $\mathbf{\tan^{-1}(x + y + 1) = x + C}$

2. Question

Solve the following differential equations:

$$\frac{dy}{dx}\cos(x - y) = 1$$

Answer

Given Differential equation is:

⇒ $\frac{dy}{dx}\cos(x - y) = 1$

⇒ $\frac{dy}{dx} = \frac{1}{\cos(x-y)}$

⇒ $\frac{dy}{dx} = \sec(x - y)$ ……(1)

Let us assume z = x – y

Differentiating w.r.t x on both sides we get,

⇒ $\frac{dz}{dx} = \frac{dx}{dx} - \frac{dy}{dx}$

⇒ $\frac{dz}{dx} = 1 - \frac{dy}{dx}$

⇒ $\frac{dy}{dx} = 1 - \frac{dz}{dx}$ ……(2)

Substituting (2) in (1) we get,

⇒ $1 - \frac{dz}{dx} = \sec z$

⇒ $1 - \sec z = \frac{dz}{dx}$

Bringing like variables on same side (i.e., variable seperable technique) we get,

⇒ $\frac{dz}{1-\sec z} = dx$

⇒ $\frac{dz}{1-\frac{1}{\cos z}} = dx$

⇒ $\frac{dz}{\frac{\cos z-1}{\cos z}} = dx$

⇒ $\frac{\cos z dz}{\cos z-1} = dx$

⇒ $\frac{-\cos z dz}{1-\cos z} = dx$

We know that cos2z = $\cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$.

$\Rightarrow \frac{-\left(\cos^2\left(\frac{z}{2}\right)-\sin^2\left(\frac{x}{2}\right)\right)dz}{2\sin^2\left(\frac{z}{2}\right)} = dx$

$\Rightarrow \frac{\sin^2\left(\frac{z}{2}\right)dz}{\sin^2\left(\frac{z}{2}\right)} - \frac{\cos^2\left(\frac{x}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} = 2dx$

$\Rightarrow dz - \cot^2\left(\frac{z}{2}\right)dz = 2dx$

We know $1 + \cot^2 x = \mathrm{cosec}^2 x$

$\Rightarrow \left(1 - \left(\mathrm{cosec}^2\left(\frac{z}{2}\right) - 1\right)\right)dz = 2dx$

$\Rightarrow -\mathrm{cosec}^2\left(\frac{z}{2}\right)dz = 2dx$

Integrating on both sides we get,

$\Rightarrow \int -\mathrm{cosec}^2\left(\frac{z}{2}\right)dz = 2\int dx$

We know that:

(1) $\int \mathrm{cosec}^2 x = -\cot x + C$

(2) $\int f'(ax)dx = \frac{f(ax)}{a} + C$

(3) $\int a dx = ax + C$

$\Rightarrow -\frac{\left(-\cot\left(\frac{z}{2}\right)\right)}{\frac{1}{2}} = 2x + 2C$

$\Rightarrow 2\cot\left(\frac{z}{2}\right) = 2x + 2C$

Since z = x – y substituting this we get,

$\Rightarrow \cot\left(\frac{x-y}{2}\right) = x + C$

∴ The solution for the given Differential equation is $\mathbf{\cot\left(\frac{x-y}{2}\right) = x + C}$.

3. Question

Solve the following differential equations:

$$\frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$$

Answer

Given Differential equation is:

$\Rightarrow \frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$ ......(1)

Let us assume z = x – y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} - \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow 1 - \frac{dz}{dx} = \frac{z+3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = 1 - \frac{z+3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = \frac{2z+5-z-3}{2z+5}$

$\Rightarrow \frac{dz}{dx} = \frac{z+2}{2z+5}$

Bringing like variables on same side(i.e., variable seperable technique) we get,

$\Rightarrow \frac{dz}{\frac{z+2}{2z+5}} = dx$

$\Rightarrow \frac{(2z+5)dz}{z+3} = dx$

$\Rightarrow \frac{(2z+6-1)dz}{z+3} = dx$

$\Rightarrow \frac{(2z+6)dz}{z+3} - \frac{dz}{z+3} = dx$

$\Rightarrow 2dz - \frac{dz}{z+3} = dx$

$\Rightarrow 2dz - \frac{d(z+3)}{z+3} = dx$

Integrating on both sides we get,

$\Rightarrow \int 2dz - \int \frac{d(z+3)}{z+3} = \int dx$

We know that:

(1) $\int adx = ax + C$

(2) $\frac{\int dx}{x} = \log x + C$

$\Rightarrow 2z - \log(z + 3) = x + C$

Since $z = x - y$, we substitute this,

$\Rightarrow 2(x - y) - \log(x-y + 3) = x + C$

$\Rightarrow 2x - 2y - \log(x-y + 3) = x + C$

$\Rightarrow x - 2y - \log(x-y + 3) = C$

∴ The solution for the given Differential equation is: **x – 2y –log(x–y + 3) = C**.

4. Question

Solve the following differential equations:

$$\frac{dy}{dx} = (x + y)^2$$

Answer

Given Differential equation is:

$\Rightarrow \frac{dy}{dx} = (x + y)^2$ ......(1)

Let us assume z = x + y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -1 + \frac{dz}{dx}$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow -1 + \frac{dz}{dx} = z^2$

$\Rightarrow 1 + z^2 = \frac{dz}{dx}$

Bringing the like variables to same side (i.e., Variable seperable technique) we get,

$\Rightarrow \frac{dz}{1 + z^2} = dx$

Integrating on both sides we get,

$\Rightarrow \int \frac{dz}{1 + z^2} = \int dx$

$\Rightarrow \int \frac{dz}{1^2 + z^2} = \int dx$

We know that:

(1) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(2) $\int a dx = ax + C$

$\Rightarrow \frac{1}{1}\tan^{-1}\left(\frac{z}{1}\right) = x + C$

$\Rightarrow \tan^{-1}z = x + C$

Since z = x + y we substitute this,

$\Rightarrow \tan^{-1}(x + y) = x + C$

$\Rightarrow x + y = \tan(x + C)$

∴ The solution for the given Differential equation is **x + y = tan(x + C)**.

5. Question

Solve the following differential equations:

$$(x + y)^2 \frac{dy}{dx} = 1$$

Answer

Given Differential equation is:

$\Rightarrow (x + y)^2 \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{(x + y)^2}$ ......(1)

Let us assume z = x + y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow \frac{dz}{dx} - 1 = \frac{1}{z^2}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{1}{z^2}$

$\Rightarrow \frac{dz}{dx} = \frac{z^2+1}{z^2}$

Bringing like variables on same side (i.e., Variable seperable technique) we get,

$\Rightarrow \frac{dz}{\frac{z^2+1}{z^2}} = dx$

$\Rightarrow \frac{z^2 dz}{z^2+1} = dx$

$\Rightarrow \frac{(z^2+1-1)dz}{z^2+1} = dx$

$\Rightarrow \frac{(z^2+1)dz}{z^2+1} - \frac{dz}{1+z^2} = dx$

$\Rightarrow dz - \frac{dz}{1+z^2} = dx$

Integrating on both sides we get,

$\Rightarrow \int dz - \int \frac{dz}{1+z^2} = \int dx$

$\Rightarrow \int dz - \int \frac{dz}{1^2+z^2} = \int dx$

We know that:

(1) $\int a dx = ax + C$

(2) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$

$\Rightarrow z - \frac{1}{1}\tan^{-1}\left(\frac{z}{1}\right) = x + C$

$\Rightarrow z - \tan^{-1}z = x + C$

Since z = x + y, we substitute this,

$\Rightarrow x + y - \tan^{-1}(x + y) = x + C$

$\Rightarrow y - \tan^{-1}(x + y) = C$

∴ The solution for the given Differential equation is $\mathbf{y - \tan^{-1}(x + y) = C}$.

6. Question

Solve the following differential equations:

$$\cos^2(x - 2y) = 1 - 2\frac{dy}{dx}$$

Answer

Given Differential equation is:

$\Rightarrow \cos^2(x-2y) = 1 - 2\frac{dy}{dx}$

$\Rightarrow 2\frac{dy}{dx} = 1 - \cos^2(x-2y)$

We know that $1 - \cos^2 x = \sin^2 x$

$\Rightarrow \frac{2dy}{dx} = \sin^2(x-2y)$ ……(1)

Let us assume z = x– 2y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} - \frac{2dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 - \frac{2dy}{dx}$

$\Rightarrow \frac{2dy}{dx} = 1 - \frac{dz}{dx}$ ……(2)

Substitute (2) in (1) we get,

$\Rightarrow 1 - \frac{dz}{dx} = \sin^2 z$

$\Rightarrow 1 - \sin^2 z = \frac{dz}{dx}$

$\Rightarrow \cos^2 z = \frac{dz}{dx}$

Bringing like variables on same side (i.e., variable seperable technique) we get,

$\Rightarrow \frac{dz}{\cos^2 z} = dx$

We know that $\frac{1}{\cos^2 x} = \sec^2 x$

$\Rightarrow \sec^2 z dz = dx$

Integrating on both sides we get,

$\Rightarrow \int \sec^2 z dz = \int dx$

We know that:

(1) $\int \sec^2 x dx = \tan x + C$

(2) $\int a dx = ax + C$

$\Rightarrow \tan z = x + C$

Since z = x – 2y we substitute this,

$\Rightarrow \tan(x-2y) = x + C$

∴ The solution for the given Differential Equation is **tan(x–2y) = x + C**.

7. Question

Solve the following differential equations:

$$\frac{dy}{dx} = \sec(x+y)$$

Answer

Given Differential Equation is:

$\Rightarrow \frac{dy}{dx} = \sec(x + y)$ ......(1)

Let us assume z = x + y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow \frac{dz}{dx} - 1 = \sec z$

$\Rightarrow \frac{dz}{dx} = 1 + \sec z$

Bringing like variables on same side(i.e, variable seperable technique) we get,

$\Rightarrow \frac{dz}{1 + \sec z} = dx$

We know that $\sec x = \frac{1}{\cos x}$

$\Rightarrow \frac{dz}{1 + \frac{1}{\cos z}} = dx$

$\Rightarrow \frac{\cos z dz}{\cos z + 1} = dx$

We know that cos2z = $\cos^2$z – $\sin^2$z = 2$\cos^2$z – 1

$\Rightarrow \frac{\cos^2\left(\frac{z}{2}\right) - \sin^2\left(\frac{z}{2}\right)}{2\cos^2\left(\frac{z}{2}\right)} dz = dx$

$\Rightarrow \frac{\cos^2\left(\frac{z}{2}\right) dz}{\cos^2\left(\frac{z}{2}\right)} - \frac{\sin^2\left(\frac{z}{2}\right) dz}{\cos^2\left(\frac{z}{2}\right)} = 2dx$

$\Rightarrow dz - \tan^2\left(\frac{z}{2}\right) dz = 2dx$

We know that 1 + $\tan^2$x = $\sec^2$x

$\Rightarrow dz - \left(\sec^2\left(\frac{z}{2}\right) - 1\right) dz = 2dx$

$\Rightarrow (2 - \sec^2\left(\frac{z}{2}\right)) dz = 2dx$

Integrating on both sides we get,

$\Rightarrow \int 2dz - \int \sec^2\left(\frac{z}{2}\right) dz = 2\int dx$

We know that:

(1) $\int \sec^2 x dx = \tan x + C$

(2) $\int a dx = ax + C$

$\Rightarrow 2z - \tan\left(\frac{z}{2}\right) = 2x + C$

Since z = x + y, we substitute this,

$\Rightarrow 2(x + y) - \tan\left(\frac{z}{2}\right) = 2x + C$

$\Rightarrow 2x + 2y - 2x = \tan\left(\frac{x+y}{2}\right) + C$

$\Rightarrow 2y = \tan\left(\frac{x+y}{2}\right) + C$

∴ the solution for the given differential equation is $\mathbf{2y = \tan\left(\frac{x+y}{2}\right) + C}$.

8. Question

Solve the following differential equations:

$$\frac{dy}{dx} = \tan(x+y)$$

Answer

Given Differential Equation is:

$\Rightarrow \frac{dy}{dx} = \tan(x + y)$ ......(1)

Let us assume z = x + y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ......(2)

Substituting(2) in (1) we get,

$\Rightarrow \frac{dz}{dx} - 1 = \tan z$

$\Rightarrow \frac{dz}{dx} = 1 + \tan z$

Bringing like variables on same side(i.e., variable seperable technique) we get,

$\Rightarrow \frac{dz}{1 + \tan z} = dx$

We know that $\tan x = \frac{\sin x}{\cos x}$

$\Rightarrow \frac{dz}{1 + \frac{\sin z}{\cos z}} = dx$

$\Rightarrow \frac{\cos z dz}{\cos z + \sin z} = dx$

$\Rightarrow \frac{2\cos z dz}{\text{cod}z + \sin z} = 2dx$

$\Rightarrow \frac{(2\cos z + \sin z - \sin z)dz}{\cos z + \sin z} = 2dx$

$\Rightarrow \frac{((\cos z + \sin z) + (\cos z - \sin z))dz}{\cos z + \sin z} = 2dx$

$\Rightarrow \frac{\cos z + \sin z}{\cos z + \sin z}dz + \frac{\cos z - \sin z}{\cos z + \sin z}dz = 2dx$

$\Rightarrow dz + \frac{d(\cos z + \sin z)}{\cos z + \sin z} = 2dx$

Integrating on both sides we get,

$\Rightarrow \int dz + \int \frac{d(\cos z + \sin z)}{\cos z + \sin z} = 2\int dx$

We know that:

(1) $\int \frac{dx}{x} = \log x + C$

(2) $\int a dx = ax + C$

$\Rightarrow z + \log(\cos z + \sin z) = 2x + C$

Since z = x + y, we substitute this,

$\Rightarrow x + y + \log(\cos(x + y) + \sin(x + y)) = 2x + C$

$\Rightarrow y + \log(\cos(x + y) + \sin(x + y)) = x + C$

∴ The solution for the given Differential Equation is **y + log(cos(x + y) + sin(x + y)) = x + C**.

9. Question

Solve the following differential equations:

$(x + y)(dx - dy) = dx + dy$

Answer

Given Differential equation is:

$\Rightarrow (x + y)(dx - dy) = dx + dy$

$\Rightarrow (x + y)dx - (x + y)dy = dx + dy$

$\Rightarrow (x + y - 1)dx = (x + y + 1)dy$

$\Rightarrow \frac{dy}{dx} = \frac{x + y - 1}{x + y + 1}$ ......(1)

Let us assume z = x + y

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow \frac{dz}{dx} - 1 = \frac{z - 1}{z + 1}$

$\Rightarrow \frac{dz}{dx} = \frac{z - 1}{z + 1} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{z - 1 + z + 1}{z + 1}$

$\Rightarrow \frac{dz}{dx} = \frac{2z}{z + 1}$

Bringing like variables on same side(i.e., variable seperable technique) we get,

$\Rightarrow \frac{dz}{\frac{z}{z+1}} = 2dx$

$\Rightarrow \frac{(z + 1)dz}{z} = 2dx$

$\Rightarrow \frac{z}{z}dz + \frac{dz}{z} = 2dx$

$\Rightarrow dz + \frac{dz}{z} = 2dx$

Integrating on both sides we get,

$\Rightarrow \int dz + \int \frac{dz}{z} = 2\int dx$

We know that:

(1) $\int adx = ax + C$

(2) $\int \frac{dx}{x} = \log x + C$

$\Rightarrow z + \log z = 2x + C$

Since z = x + y we substitute this,

$\Rightarrow x + y + \log(x + y) = 2x + C$

$\Rightarrow y + \log(x + y) = x + C$

∴ The solution for the given Differential equation is **y + log(x + y) = x + C**.

10. Question

Solve the following differential equations:

$$(x + y + 1)\frac{dy}{dx} = 1$$

Answer

Given Differential Equation is :

$\Rightarrow (x + y + 1)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y+1}$ ......(1)

Let us assume z = x + y + 1

Differentiating w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx} + \frac{d(1)}{dx}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} + 0$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ......(2)

Substituting (2) in (1) we get,

$\Rightarrow \frac{dz}{dx} - 1 = \frac{1}{z}$

$\Rightarrow \frac{dz}{dx} = 1 + \frac{1}{z}$

$\Rightarrow \frac{dz}{dx} = \frac{z+1}{z}$

Bringing like variables on same side (i.e., variable seperable technique) we get,

$\Rightarrow \frac{dz}{\frac{z+1}{z}} = dx$

$\Rightarrow \frac{zdz}{z+1} = dx$

$\Rightarrow \frac{(z+1-1)dz}{z+1} = dx$

$\Rightarrow \frac{z+1}{z+1}dz - \frac{dz}{z+1} = dx$

$\Rightarrow dz - \frac{dz}{z+1} = dx$

Integrating on both sides we get,

$\Rightarrow \int dz - \int \frac{dz}{z+1} = \int dx$

$\Rightarrow \int dz - \int \frac{d(z+1)}{z+1} = \int dx$

We know that:

(1) $\int adx = ax + C$

(2) $\int \frac{dx}{x} = \log x + C$

$\Rightarrow z - \log(z + 1) = x + C$

Since z = x + y we substitute this,

$\Rightarrow x + y - \log(x + y + 1) = x + C$

$\Rightarrow y - \log(x + y + 1) = C$

$\Rightarrow y = \log(x + y + 1) + C$

∴ The solution for the given Differential Equation is **y = log(x + y + 1) + C**.

11. Question

Solve the following differential equations:

$$\frac{dy}{dx} + 1 = e^{x+y}$$

Answer

Given Differential equation is:

$\Rightarrow \frac{dy}{dx} + 1 = e^{x+y}$ ......(1)

Let us assume z = x + y

Differentiate w.r.t x on both sides we get,

$\Rightarrow \frac{dz}{dx} = \frac{dx}{dx} + \frac{dy}{dx}$

$\Rightarrow \frac{dz}{dx} = \frac{dy}{dx} + 1$ ......(2)

Substitute(2) in (1) we get,

$\Rightarrow \frac{dz}{dx} = e^z$

Bringing like variables on same side (i.e., variable seperable technique) we get,

$\Rightarrow \frac{dz}{e^z} = dx$

$\Rightarrow e^{-z}dz = dx$

Integrating on both sides we get,

$\Rightarrow \int e^{-z}dz = \int dx$

We know that:

(1) $\int a dx = ax + C$

(2) $\int e^{ax} dx = \frac{e^{ax}}{a} + C$

$\Rightarrow \frac{e^{-z}}{-1} = x + C$

$\Rightarrow -e^{-z} = x + C$

$\Rightarrow x + e^{-z} + C = 0$

Since z = x + y we substitute this,

$\Rightarrow x + e^{-(x + y)} + C = 0$

∴ The solution for the given Differential Equation is $\mathbf{x + e^{-(x + y)} + C = 0}$.

Exercise 22.9

1. Question

Solve the following equations:

$x^2dy + y(x + y)dx = 0$

Answer

Let us write the given differential equation in the standard form:

$\Rightarrow \frac{dy}{dx} = \frac{-y(x+y)}{x^2}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^n f(x,y)$ (where n is the order of the homogeneous equation).

Let us assume

$$f(x,y) = \frac{-y(x + y)}{x^2}$$

$\Rightarrow f(zx,zy) = \frac{-zy(zx+zy)}{(zx)^2}$

$\Rightarrow f(zx,zy) = \frac{-z^2 \times (y(x+y))}{z^2x^2}$

$\Rightarrow f(zx,zy) = z^0 \times \frac{-y(x+y)}{x^2}$

$\Rightarrow f(zx,zy) = z^0 f(x,y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation, and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{-vx(x+vx)}{x^2}$

We know that $\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{-x^2(v+v^2)}{x^2}$

$\Rightarrow \frac{xdv}{dx} + v = -v - v^2$

$\Rightarrow \frac{xdv}{dx} = -2v - v^2$

Bringing the like variables on one side

$\Rightarrow \frac{dv}{v^2 + 2v} = -\frac{dx}{x}$

$\Rightarrow \frac{dv}{v^2 + 2v + 1 - 1} = -\frac{dx}{x}$

$\Rightarrow \frac{dv}{(v + 1)^2 - 1^2} = -\frac{dx}{x}$

We know that:

$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$ and

$\int \frac{dx}{x} = \log x + C$

Integrating on both sides we get

$\Rightarrow \int \frac{dv}{(v + 1)^2 - 1^2} = \int \frac{dx}{x}$

$\Rightarrow \frac{1}{2(1)} \log\left|\frac{v + 1 - 1}{v + 1 + 1}\right| = \log x + \log C$

($\because$ logC is also an arbitrary constant)

$\Rightarrow \log\left|\frac{v}{v + 2}\right|^{\frac{1}{2}} = \log\left|\frac{C}{x}\right|$

($\because \log a - \log b = \log\left(\frac{a}{b}\right)$)

($\because$ xloga = $\log a^x$)

Applying exponential on both sides, we get,

$\Rightarrow \left(\frac{v}{v + 2}\right)^{\frac{1}{2}} = \frac{c}{x}$

Squaring on both sides we get,

$\Rightarrow \frac{v}{v + 2} = \left(\frac{c}{x}\right)^2$

Since y = vx

we get $v = \frac{y}{x}$

$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x} + 2} = \frac{c^2}{x^2}$

$\Rightarrow \frac{\frac{y}{x}}{\frac{y + 2x}{x}} = \frac{c^2}{x^2}$

$\Rightarrow \frac{y}{y + 2x} = \frac{c^2}{x^2}$

Cross multiplying on both sides we get,

$\Rightarrow yx^2 = c^2(y + 2x)$

$\therefore$ The solution to the given differential equation is $\mathbf{yx^2 = c^2(y + 2x)}$

2. Question

Solve the following equations:

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

Answer

Given Differential equation is :

$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if f(zx,zy) = z^nf(x,y) (where n is the order of the homogeneous equation).

Let us assume:

$$f(x,y) = \frac{y - x}{y + x}$$

$$\Rightarrow f(zx, zy) = \frac{zy-zx}{zy+zx}$$

$$\Rightarrow f(zx, zy) = \frac{z\times(y-x)}{z\times(y+x)}$$

$$\Rightarrow f(zx, zy) = z^0 \times \frac{y-x}{y+x}$$

$\Rightarrow$ f(zx,zy) = z^0f(x,y)

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation, and the substitution is y = vx.

Let us substitute this in (1)

$$\Rightarrow \frac{d(vx)}{dx} = \frac{vx-x}{vx+x}$$

We know that:

$$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

$$\Rightarrow \frac{vdx}{x} + \frac{xdv}{dx} = \frac{x\times(v-1)}{x\times(v+1)}$$

$$\Rightarrow v + \frac{xdv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow \frac{xdv}{dx} = \frac{v-1}{v+1} - v$$

$$\Rightarrow \frac{xdv}{dx} = \frac{v-1-v^2-v}{v+1}$$

$$\Rightarrow \frac{xdv}{dx} = \frac{-(v^2+1)}{v+1}$$

Bringing like variables on one side we get,

$$\Rightarrow \frac{(v+1)dv}{v^2+1} = -\frac{dx}{x}$$

$$\Rightarrow \frac{vdv}{v^2+1} + \frac{dv}{v^2+1} = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\frac{2vdv}{v^2+1} + \frac{dv}{v^2+1} = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\frac{d(v^2+1)}{v^2+1} + \frac{dv}{v^2+1} = -\frac{dx}{x}$$

We know that:

$\int \frac{dx}{x} = \log x + C$

and Also,

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Integrating on both sides, we get,

$\Rightarrow \frac{1}{2}\int \frac{d(v^2+1)}{v^2+1} + \int \frac{dv}{v^2+1} = -\int \frac{dx}{x}$

$\Rightarrow \frac{1}{2}\log(v^2 + 1) + \frac{1}{1}\tan^{-1}\left(\frac{v}{1}\right) = -\log x + \log C$

($\because$ LogC is an arbitrary constant)

$\Rightarrow \frac{1}{2}\log(v^2 + 1) + \tan^{-1}\left(\frac{v}{1}\right) = \log\left(\frac{c}{x}\right)$

($\because \log a - \log b = \log\left(\frac{a}{b}\right)$)

Since y = vx,

we get $v = \frac{y}{x}$

$\Rightarrow \log\left(\left(\frac{y}{x}\right)^2 + 1\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = \log\left(\frac{c}{x^2}\right)$

($\because$ xloga = $\log a^x$)

$\Rightarrow \log\left(\frac{y^2+x^2}{x^2}\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = \log\left(\frac{c^2}{x^2}\right)$

$\Rightarrow \log(x^2 + y^2) - \log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \log(c^2) - \log(x^2)$

$\Rightarrow \log(x^2 + y^2) - \log(c^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = 0$

$\Rightarrow \log(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = K$

(Assuming $\log(c^2)$ = K a constant)

$\therefore$ The solution to the given differential equation is **$\log(y^2 + x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = K$**

3. Question

Solve the following equations:

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Answer

Given differential equation can be written as:

$\Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^n f(x,y)$ (where n is the order of the homogeneous equation).

Let us assume:

$f(x,y) = \frac{y^2 - x^2}{2xy}$

$\Rightarrow f(zx, zy) = \frac{(zy)^2-(zx)^2}{2(zx)(zy)}$

$\Rightarrow f(zx, zy) = \frac{z^2\times(y^2-x^2)}{z^2\times(2xy)}$

$\Rightarrow f(zx, zy) = z^0 \times \frac{y^2-x^2}{2xy}$

$\Rightarrow f(zx,zy) = z^0f(x,y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{(vx)^2-x^2}{2(vx)x}$

We know that:

$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

$\Rightarrow \frac{vdx}{dx} + \frac{xdv}{dx} = \frac{x^2(v^2-1)}{x^2(2v)}$

$\Rightarrow v + \frac{xdv}{dx} = \frac{v^2-1}{2v}$

$\Rightarrow v + \frac{xdv}{dx} = \frac{v}{2} - \frac{1}{2v}$

$\Rightarrow \frac{xdv}{dx} = -\frac{v}{2} - \frac{1}{2v}$

$\Rightarrow \frac{xdv}{dx} = -\frac{v^2-1}{2v}$

Bringing like variables on one side we get,

$\Rightarrow \frac{2vdv}{v^2-1} = -\frac{dx}{x}$

$\Rightarrow \frac{d(v^2+1)}{v^2+1} = -\frac{dx}{x}$

We know that:

$\int \frac{dx}{x} = \log x + C$

Integrating on both sides, we get,

$\Rightarrow \int \frac{d(v^2+1)}{v^2+1} = -\int \frac{dx}{x}$

$\Rightarrow \log(v^2 + 1) = -\log x + \log C$ (∵ LogC is an arbitrary constant)

Since y = vx,

we get $v = \frac{y}{x}$

$\Rightarrow \log\left(\left(\frac{y}{x}\right)^2 + 1\right) = \log\left(\frac{C}{x}\right)$

(∵ $\log a - \log b = \log\left(\frac{a}{b}\right)$)

Applying exponential on both sides, we get,

$\Rightarrow \frac{y^2+x^2}{x^2} = \frac{C}{x}$

$\Rightarrow \frac{y^2+x^2}{x} = C$

Cross multiplying on both sides we get,

$\Rightarrow y^2 + x^2 = Cx$

∴ The solution for the given differential equation is $\mathbf{y^2 + x^2 = Cx}$.

4. Question

Solve the following equations:

$$x\frac{dy}{dx} = x + y$$

Answer

Give Differential equation is:

$\Rightarrow x\frac{dy}{dx} = x + y$

$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^n f(x,y)$ (where n is the order of the homogeneous equation).

Let us assume:

$f(x,y) = \frac{x+y}{x}$

$\Rightarrow f(zx,zy) = \frac{zx+zy}{zx}$

$\Rightarrow f(zx,zy) = \frac{z(x+y)}{zx}$

$\Rightarrow f(zx,zy) = z^0 \times \frac{x+y}{x}$

$\Rightarrow f(zx,zy) = z^0 f(x,y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{x+vx}{x}$

We know that $\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{x(1+v)}{x}$

$\Rightarrow \frac{xdv}{dx} + v = 1 + v$

$\Rightarrow \frac{xdv}{dx} = 1$

Bringing like coefficients on same sides we get,

$\Rightarrow dv = \frac{dx}{x}$

We know that $\int a dx = ax + C$ and

Also,

$\int \frac{dx}{x} = \log x + C$

Integrating on both sides, we get,

$\Rightarrow \int dv = \int \frac{dx}{x}$

⇒ v = logx + C

Since y = vx,

we get,

$v = \frac{y}{x}$

$\Rightarrow \frac{y}{x} = \log x + C$

Cross multiplying on both sides we get,

⇒ y = xlogx + Cx

∴ The solution for the given differential equation is **y = xlogx + Cx**

5. Question

Solve the following equations:

$(x^2 - y^2)dx - 2xydy = 0$

Answer

Given differential equation is:

$\Rightarrow (x^2 - y^2)dx - 2xydy = 0$

$\Rightarrow (x^2 - y^2)dx = 2xydy$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^n f(x,y)$ (where n is the order of the homogeneous equation).

Let us assume $f(x,y) = \frac{x^2 - y^2}{2xy}$

$\Rightarrow f(zx, zy) = \frac{(zx)^2 - (zy)^2}{2(zx)(zy)}$

$\Rightarrow f(zx, zy) = \frac{z^2x^2 - z^2y^2}{2z^2xy}$

$\Rightarrow f(zx, zy) = \frac{z^2(x^2 - y^2)}{z^2(2xy)}$

$\Rightarrow f(zx, zy) = z^0 \times \frac{x^2 - y^2}{2xy}$

$\Rightarrow f(zx,zy) = z^0 f(x,y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{x^2 - (vx)^2}{2x(vx)}$

We know that:

$$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{x^2 - v^2x^2}{2vx^2}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{x^2(1-v^2)}{x^2(2v)}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{1-v^2}{2v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1-v^2}{2v} - v$

$\Rightarrow \frac{xdv}{dx} = \frac{1-v^2-2v^2}{2v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1-3v^2}{2v}$

Bringing Like variables on same sides we get,

$\Rightarrow \frac{2vdv}{1-3v^2} = \frac{dx}{x}$

$\Rightarrow -\frac{1}{3}\frac{-6vdv}{1-3v^2} = \frac{dx}{x}$

$\Rightarrow -\frac{1}{3}\frac{d(1-3v^2)}{1-3v^2} = \frac{dx}{x}$

We know that:

$$\int \frac{dx}{x} = \log x + C$$

Integrating on both sides, we get,

$\Rightarrow -\frac{1}{3}\int\frac{d(1-3v^2)}{1-3v^2} = \int\frac{dx}{x}$

$\Rightarrow -\frac{1}{3}\log|1-3v^2| = \log x - \log C$

($\because$ logC is an arbitrary constant)

Multiplying with -3 on both sides we get,

$\Rightarrow \log|1-3v^2| = -3\log x + 3\log C$

$\Rightarrow \log|1-3v^2| = 3\log\left(\frac{C}{x}\right)$

($\because \log a - \log b = \log\left(\frac{a}{b}\right)$)

$\Rightarrow \log|1-3v^2| = \log\left(\frac{c}{x}\right)^3$

($\because a\log x = \log x^a$)

$\Rightarrow \log|1-3v^2| = \log\left(\frac{c^3}{x^3}\right)$

Applying exponential on both sides we get,

$\Rightarrow 1-3v^2 = \frac{c^3}{x^3}$

Since y = vx, we get,

$v = \frac{y}{x}$

$\Rightarrow 1 - 3\left(\frac{y}{x}\right)^2 = \frac{c^3}{x^3}$

$\Rightarrow 1 - \frac{3y^2}{x^2} = \frac{c^3}{x^3}$

$\Rightarrow \frac{x^2 - 3y^2}{x^2} = \frac{c^3}{x^3}$

$\Rightarrow x^2 - 3y^2 = \frac{c^3}{x}$

Cross multiplying on both sides we get,

$\Rightarrow x(x^2 - 3y^2) = c^3$

$\Rightarrow x^3 - 3xy^2 = K$ (say any arbitrary constant)

∴ The solution for the differential equation is $\mathbf{x^3 - 3xy^2 = K}$

6. Question

Solve the following equations:

$$\frac{dy}{dx} = \frac{x + y}{x - y}$$

Answer

Given differential equation is:

$\Rightarrow \frac{dy}{dx} = \frac{x + y}{x - y}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^n f(x,y)$ (where n is the order of the homogeneous equation).

Let us assume:

$$f(x,y) = \frac{x + y}{x - y}$$

$\Rightarrow f(zx, zy) = \frac{zx + zy}{zx - zy}$

$\Rightarrow f(zx, zy) = \frac{z(x + y)}{z(x - y)}$

$\Rightarrow f(zx, zy) = z^0 \times \frac{x + y}{x - y}$

$\Rightarrow f(zx,zy) = z^0 f(x,y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{x + vx}{x - vx}$

We know that:

$$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{x(1 + v)}{x(1 - v)}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{1+v}{1-v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1+v}{1-v} - v$

$\Rightarrow \frac{xdv}{dx} = \frac{1+v-v+v^2}{1-v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1+v^2}{1-v}$

Bringing like variables on same side we get,

$\Rightarrow \frac{(1-v)dv}{1+v^2} = \frac{dx}{x}$

$\Rightarrow \frac{1}{1+v^2}dv - \frac{vdv}{1+v^2} = \frac{dx}{x}$

$\Rightarrow \frac{1}{1+v^2}dv - \frac{1}{2}\frac{2vdv}{1+v^2} = \frac{dx}{x}$

$\Rightarrow \frac{1}{1^2+v^2}dv - \frac{1}{2}\frac{d(1+v^2)}{1+v^2} = \frac{dx}{x}$

We know that:

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ and

Also,

$\int \frac{dx}{x} = \log x + C$

Integrating on both sides, we get,

$\Rightarrow \int \frac{1}{1+v^2}dv - \frac{1}{2}\int \frac{d(1+v^2)}{1+v^2} = \int \frac{dx}{x}$

$\Rightarrow \frac{1}{1}\tan^{-1}\left(\frac{v}{1}\right) - \frac{1}{2}\log(1 + v^2) = \log x + C$

$\Rightarrow \tan^{-1} v - \frac{1}{2}\log(1 + v^2) = \log x + C$

Since y = vx, we get,

$v = \frac{y}{x}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(1 + \left(\frac{y}{x}\right)^2\right) = \log x + C$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + C$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2} \times (\log(x^2 + y^2) + \log(x^2)) = \log x + C$

$(\because \log a - \log b = \log\left(\frac{a}{b}\right))$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log(x^2 + y^2) + \frac{1}{2}\log(x^2) = \log x + C$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log(x^2 + y^2) + \log(x^2)^{\frac{1}{2}} = \log x + C$

($\because a\log x = \log x^a$)

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log(x^2 + y^2) + \log x = \log x + C$

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + C$

∴ The solution for the given Differential equation is $\mathbf{\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + C}$

7. Question

Solve the following equations:

$$2xy\frac{dy}{dx} = x^2 + y^2$$

Answer

Given Differential equation is:

$\Rightarrow 2xy\frac{dy}{dx} = x^2 + y^2$

$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if f(zx,zy) = z^nf(x,y) (where n is the order of the homogeneous equation).

Let us assume:

$f(x,y) = \frac{x^2 + y^2}{2xy}$

$\Rightarrow f(zx,zy) = \frac{(zx)^2+(zy)^2}{2(zx)(zy)}$

$\Rightarrow f(zx,zy) = \frac{z^2x^2+z^2y^2}{2z^2xy}$

$\Rightarrow f(zx,zy) = \frac{z^2\times(x^2+y^2)}{z^2\times(2xy)}$

$\Rightarrow f(zx,zy) = z^0 \times \frac{x^2+y^2}{2xy}$

⇒ f(zx,zy) = z^0f(x,y)

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation, and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{x^2 + (vx)^2}{2x(vx)}$

We know that $\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{x^2+v^2x^2}{2vx^2}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{x^2(1+v^2)}{x^2(2v)}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{1+v^2}{2v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1+v^2}{2v} - v$

$\Rightarrow \frac{xdv}{dx} = \frac{1+v^2-2v^2}{2v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1-v^2}{2v}$

Bringing like variables on same side we get,

$\Rightarrow \frac{2vdv}{1-v^2} = \frac{dx}{x}$

$\Rightarrow -\frac{-2vdv}{1-v^2} = \frac{dx}{x}$

We know that:

$\int \frac{dx}{x} = \log x + C$

$\Rightarrow -\int \frac{-2vdv}{1-v^2} = \int \frac{dx}{x}$

$\Rightarrow -\log(1-v^2) = \log x + \log C$

$\Rightarrow \log(1-v^2)^{-1} = \log(Cx)$

($\because a\log x = \log x^a$)

($\because \log a + \log b = \log ab$)

$\Rightarrow \log\left(\frac{1}{1-v^2}\right) = \log(Cx)$

Applying exponential on both sides, we get,

$\Rightarrow \frac{1}{1-v^2} = Cx$

Since y = vx, we get,

$v = \frac{y}{x}$

$\Rightarrow \frac{1}{1-\left(\frac{y}{x}\right)^2} = Cx$

$\Rightarrow \frac{1}{\frac{x^2-y^2}{x^2}} = Cx$

$\Rightarrow \frac{x^2}{x^2-y^2} = Cx$

$\Rightarrow \frac{x}{x^2-y^2} = C$

Cross multiplying on both sides we get,

$\Rightarrow x = C(x^2 - y^2)$

∴ The solution for the given Differential equation is **$x = C(X^2-y^2)$**

8. Question

Solve the following equations:

$$x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$$

Answer

Given Differential equation is:

$\Rightarrow \frac{x^2 dy}{dx} = x^2 - 2y^2 + xy$

$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 2y^2 + xy}{x^2}$ ……(1)

Homogeneous equation: A equation is said to be homogeneous if f(zx,zy) = z^nf(x,y) (where n is the order of the homogeneous equation).

Let us assume $f(x,y) = \frac{x^2-2y^2+xy}{x^2}$

$\Rightarrow f(zx,zy) = \frac{(zx)^2 - 2(zy)^2 + (zx)(zy)}{(zx)^2}$

$\Rightarrow f(zx,zy) = \frac{z^2x^2 - 2z^2y^2 + z^2xy}{z^2x^2}$

$\Rightarrow f(zx,zy) = \frac{z^2(x^2-2y^2+xy)}{z^2(x^2)}$

$\Rightarrow f(zx,zy) = z^0 \times \frac{x^2-2y^2+xy}{x^2}$

$\Rightarrow$ f(zx,zy) = z^0f(x,y)

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{x^2 - 2(vx)^2 + x(vx)}{x^2}$

We know that $\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{x^2 - 2v^2x^2 + vx^2}{x^2}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{x^2(1 = 2v^2 + v)}{x^2}$

$\Rightarrow \frac{xdv}{dx} + v = 1 - 2v^2 + v$

$\Rightarrow \frac{xdv}{dx} = 1 - 2v^2$

Bringing like variables on same side we get,

$\Rightarrow \frac{dv}{1-2v^2} = \frac{dx}{x}$

$\Rightarrow \frac{1}{2}\frac{dv}{\frac{1}{2} - v^2} = \frac{dx}{x}$

$\Rightarrow \frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2} = \frac{2dx}{x}$

We know that:

$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$ and

Also,

$\int \frac{dx}{x} = \log x + C$

Integrating on both sides, we get,

$\Rightarrow \int \frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2} = 2\int \frac{dx}{x}$

$\Rightarrow \frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right| = 2\log x + \log c$

(∵ log C is an arbitrary constant)

$\Rightarrow \frac{1}{\sqrt{2}}\log\left|\frac{\frac{1+\sqrt{2}v}{\sqrt{2}}}{\frac{1-\sqrt{2}v}{\sqrt{2}}}\right| = \log x^2 + \log C$

(∵ $a\log x = \log x^a$)

$\Rightarrow \frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right| = \log(Cx^2)$

(∵ loga + logb = logab)

Since y = vx,

we get,

$v = \frac{y}{x}$

$\Rightarrow \frac{1}{\sqrt{2}}\log\left|\frac{1+\sqrt{2}\left(\frac{y}{x}\right)}{1-\sqrt{2}\left(\frac{y}{x}\right)}\right| = \log(Cx^2)$

$\Rightarrow \log\left|\frac{\frac{x+\sqrt{2}y}{y}}{\frac{x-\sqrt{2}y}{x}}\right|^{\frac{1}{\sqrt{2}}} = \log(Cx^2)$

$\Rightarrow \log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|^{\frac{1}{\sqrt{2}}} = \log(Cx^2)$

Applying exponential on both sides we get,

$\Rightarrow \left(\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right)^{\frac{1}{\sqrt{2}}} = Cx^2$

$\Rightarrow \frac{x+\sqrt{2}y}{x-\sqrt{2}y} = (cx^2)^{\sqrt{2}}$

∴ The solution of the Differential equation is

$$\mathbf{\frac{x + \sqrt{2}y}{x - \sqrt{2}y} = (cx^2)^{\sqrt{2}}}$$

9. Question

Solve the following equations:

$$xy\frac{dy}{dx} = x^2 - y^2$$

Answer

Given Differential equation is:

$\Rightarrow xy\frac{dy}{dx} = x^2 - y^2$

$\Rightarrow \frac{dy}{dx} = \frac{x^2-y^2}{xy}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^n f(x,y)$ (where n is the order of the homogeneous equation).

Let us assume $f(x,y) = \frac{x^2-y^2}{xy}$

$\Rightarrow f(zx, zy) = \frac{(zx)^2-(zy)^2}{(zx)(zy)}$

$\Rightarrow f(zx, zy) = \frac{z^2x^2-z^2y^2}{z^2xy}$

$\Rightarrow f(zx, zy) = \frac{z^2(x^2-y^2)}{z^2 \times xy}$

$\Rightarrow f(zx, zy) = z^0 \times \frac{x^2-y^2}{xy}$

$\Rightarrow f(zx,zy) = z^0 f(x,y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is y = vx.

Let us substitute this in (1)

$\Rightarrow \frac{d(vx)}{dx} = \frac{x^2-(vx)^2}{x(vx)}$

We know that $\frac{d(uv)}{dx} = \frac{vdu}{dx} + \frac{udv}{dx}$

$\Rightarrow \frac{xdv}{dx} + \frac{vdx}{dx} = \frac{x^2-v^2x^2}{vx^2}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{x^2(1-v^2)}{x^2(v)}$

$\Rightarrow \frac{xdv}{dx} + v = \frac{1-v^2}{v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1-v^2}{v} - v$

$\Rightarrow \frac{xdv}{dx} = \frac{1-v^2-v^2}{v}$

$\Rightarrow \frac{xdv}{dx} = \frac{1-2v^2}{v}$

Bringing like on the same side we get,

$\Rightarrow \frac{vdv}{1-2v^2} = \frac{dx}{x}$

$\Rightarrow -\frac{1}{4}\frac{-4vdv}{1-2v^2} = \frac{dx}{x}$

$\Rightarrow -\frac{1}{4}\frac{d(1-2v^2)}{1-2v^2} = \frac{dx}{x}$

We know that $\int \frac{dx}{x} = \log x + C$

Integrating on both sides we get,

$\Rightarrow -\frac{1}{4}\int \frac{d(1-2v^2)}{1-2v^2} = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{4}\log(1-2v^2) = \log x + \log C$

($\because$ logC is an arbitrary constant)

$\Rightarrow \log(1-2v^2) = -4\log x + 4\log C$

$\Rightarrow \log(1–2v^2) = -\log x^4 + \log C^4$

($\because x\log a = \log a^x$)

$\Rightarrow \log(1-2v^2) = \log\left(\frac{c^4}{x^4}\right)$

$(\because \log a - \log b = \log\left(\frac{a}{b}\right))$

Applying exponential on both sides we get,

$\Rightarrow 1-2v^2 = \frac{c4}{x^4}$

Since y = vx, we get,

$v = \frac{y}{x}$

$\Rightarrow 1-2\left(\frac{y}{x}\right)^2 = \frac{c^4}{x^4}$

$\Rightarrow 1-\frac{2y^2}{x^2} = \frac{c^4}{x^4}$

$\Rightarrow \frac{x^2-2y^2}{x^2} = \frac{c^4}{x^4}$

$\Rightarrow x^2-2y^2 = \frac{c^4}{x^2}$

Cross multiplying on both sides we get,

$\Rightarrow x^2(x^2–2y^2) = c^4$

$\Rightarrow x^4–2x^2y^2 = c^4$

∴ The solution for the given differential equation is $\mathbf{x^4–2x^2y^2 = C^4}$.

10. Question

Solve the following equations:

$$y\,e^{x/y}dx = \left(xe^{x/y} + y\right)dy$$

Answer

Given Differential equation is:

$\Rightarrow ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y\right)dy$

$\Rightarrow \frac{dx}{dy} = \frac{xe^{\frac{x}{y}}+y}{ye^{\frac{x}{y}}}$ ......(1)

Homogeneous equation: A equation is said to be homogeneous if $f(zx,zy) = z^nf(x,y)$ (where n is the order of the homogeneous equation).

Let us assume:

$f(x,y) = \frac{xe^{\frac{x}{y}} + y}{ye^{\frac{x}{y}}}$

$\Rightarrow f(zx,zy) = \frac{zxe^{\frac{zx}{zy}}+zy}{zye^{\frac{zx}{zy}}}$

$\Rightarrow f(zx,zy) = \frac{z\left(xe^{\frac{x}{y}}+y\right)}{z\left(ye^{\frac{x}{y}}\right)}$

$\Rightarrow f(zx, zy) = z^0 \times \frac{xe^{\frac{x}{y}} + y}{ye^{\frac{x}{y}}}$

$\Rightarrow f(zx, zy) = z^0 f(x, y)$

So, given differential equation is a homogeneous differential equation.

We need a substitution to solve this type of linear equation and the substitution is x = vy.

Let us substitute this in (1)

$$\Rightarrow \frac{d(vy)}{dy} = \frac{vye^{\frac{vy}{y}} + y}{ye^{\frac{vy}{y}}}$$

We know that:

$$\frac{d(uv)}{dx} = \frac{udv}{dx} + \frac{vdu}{dx}$$

$$\Rightarrow \frac{ydv}{dy} + \frac{vdy}{dy} = \frac{y}{ye^v} + v$$

$$\Rightarrow \frac{ydv}{dy} + v = e^{-v} + v$$

$$\Rightarrow \frac{ydv}{dy} = e^{-v} + v - v$$

$$\Rightarrow \frac{ydv}{dy} = e^{-v}$$

Bringing like variables on the same side we get,

$$\Rightarrow e^v dv = \frac{dy}{y}$$

We know that $\int e^x dx = e^x + C$ and

$$\int \frac{dx}{x} = \log x + C$$

Integrating on both sides, we get,

$$\Rightarrow \int e^v dv = \int \frac{dy}{y}$$

$\Rightarrow e^v = \log y + C$

Since x = vy, we get $v = \frac{x}{y}$

$$\Rightarrow e^{\frac{x}{y}} = \log y + C$$

∴ The solution for the given Differential equation is $\mathbf{e^{\frac{x}{y}} = \log y + C}$.

11. Question

Solve the following differential equations :

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

Answer

Here,

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{(x^2 + xvx + v^2x^2)}{x^2}$$

$$x\frac{dv}{dx} = 1 + v + v^2 - v$$

$$x\frac{dv}{dx} = 1 + v^2$$

Integrating Both Sides we get,

$$\int \frac{dv}{1 + v^2} = \int \frac{dx}{x}$$

$$\tan^{-1} v = \log|x| + c$$

$$\tan^{-1}\frac{y}{x} = \log|x| + c$$

12. Question

Solve the following differential equations :

$$\left(y^2 - 2xy\right)dx = \left(x^2 - 2xy\right)dy$$

Answer

Here, $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - 2xvx}{x^2 - 2xvx}$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v}$$

$$x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - 2v - v + 2v^2}{1 - 2v}$$

$$x\frac{dv}{dx} = \frac{3v^2 - 3v}{1 - 2v}$$

$$\frac{1-2v}{3(v^2-v)}\,dv = \frac{dx}{x}$$

$$\frac{-(2v-1)}{3(v^2-v)}\,dv = \frac{dx}{x}$$

$$\frac{(2v-1)}{(v^2-v)}\,dv = -3\frac{dx}{x}$$

Integrating Both Sides we get,

$$\int \frac{(2v-1)}{(v^2-v)}dv = -3\int \frac{dx}{x}$$

$$\log|v^2-v| = -3\log|x| + \log c$$

$$v^2 - v = \frac{c}{x^3}$$

$$\frac{y^2}{x^2} - \frac{y}{x} = \frac{c}{x^3}$$

$$y^2 - xy = \frac{c}{x}$$

$x(y^2 - xy) = c$

13. Question

Solve the following differential equations :

$$2xy\ dx + \left(x^2 + 2y^2\right)dy = 0$$

Answer

Here, $2xydx + (x^2 + 2y^2) dy = 0$

$$\frac{dy}{dx} = \frac{2xy}{x^2 + 2y^2}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{2xvx}{x^2 + 2v^2x^2}$$

$$v + x\frac{dv}{dx} = \frac{2v}{1 + 2v^2}$$

$$x\frac{dv}{dx} = \frac{2v}{1 + 2v^2} - v$$

$$x\frac{dv}{dx} = \frac{2v - v - 2v^3}{1 + 2v^2}$$

$$x\frac{dv}{dx} = \frac{v - 2v^3}{1 + 2v^2}$$

Integrating Both Sides we get,

$$\int \frac{1 + 2v^2}{v - 2v^3}dv = \int \frac{dx}{x} \quad ...... (1)$$

$$\frac{1 + 2v^2}{v - 2v^3} = \frac{1 + 2v^2}{v(1 - 2v^2)}$$

$$\frac{1 + 2v^2}{v(1 - 2v^2)} = \frac{A}{v} + \frac{Bv + c}{1 - 2v^2}$$

$$\frac{1 + 2v^2}{v(1 - 2v^2)} = \frac{A(1 - 2v^2) + (Bv + c)(v)}{v(1 - 2v^2)}$$

$1 + 2v^2 = A – 2Av^2 + Bv^2 + cv$

$1 + 2v^2 = v^2(– 2A + B) + cv + A$

Comparing the coefficients of like power of v,

A = 1

C = 0

– 2A + B = 2

– 2 + B = 2

B = 4

$$\frac{1 + 2v^2}{v(1 - 2v^2)} = \frac{1}{v} + \frac{4v}{1 - 2v^2}$$

$$\frac{1 + 2\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)(1 - 2\left(\frac{y}{x}\right)^2)} = \frac{1}{\left(\frac{y}{x}\right)} + \frac{4\left(\frac{y}{x}\right)}{1 - 2\left(\frac{y}{x}\right)^2}$$

14. Question

Solve the following differential equations :

$$3x^2 dy = \left(3xy + y^2\right)dx$$

Answer

Here, $3x^2dy = (3xy + y^2)dx$

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{3xvx + v^2x^2}{3x^2}$$

$$v + x\frac{dv}{dx} = \frac{3v + v^2}{3}$$

$$x\frac{dv}{dx} = \frac{3 + v^2}{3} - v$$

$$x\frac{dv}{dx} = \frac{3v + v^2 - 3v}{3}$$

$$x\frac{dv}{dx} = \frac{v^2}{3}$$

Integrating both sides we get,

$$3\int\frac{1}{v^2}dv = \int\frac{dx}{x}$$

$$3\left(-\frac{1}{v}\right) = \log|x| + c$$

$$-\frac{3x}{y} = \log|x| + c$$

15. Question

Solve the following differential equations :

$$\frac{dy}{dx} = \frac{x}{2y + x}$$

Answer

Here, $\frac{dy}{dx} = \frac{x}{2y + x}$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x}{2vx + x}$$

$$v + x\frac{dv}{dx} = \frac{1}{2v + 1}$$

$$x\frac{dv}{dx} = \frac{1}{2v + 1} - v$$

$$x\frac{dv}{dx} = \frac{1 - 2v^2 - v}{2v + 1}$$

Integrating both sides we get,

$$\int\frac{2v + 1}{1 - 2v^2 - v}dv = \int\frac{dx}{x}$$

$$-\int\frac{2v + 1}{2v^2 + v - 1}dv = \int\frac{dx}{x}$$

$$-\frac{1}{2}\int\frac{4v + 2}{2v^2 + v - 1}dv = \int\frac{dx}{x}$$

$$\int\frac{4v + 1 + 1}{2v^2 + v - 1}dv = -2\int\frac{dx}{x}$$

$$\int\frac{4v + 1}{2v^2 + v - 1}dv + \int\frac{1}{2v^2 + v - 1}dv = -2\int\frac{dx}{x}$$

$$\int\frac{4v + 1}{2v^2 + v - 1}dv + \frac{1}{2}\int\frac{1}{v^2 + \frac{v}{2} - \frac{1}{2}}dv = -2\int\frac{dx}{x}$$

$$\int \frac{4v + 1}{2v^2 + v - 1} dv + \frac{1}{2}\int \frac{1}{v^2 + 2v\left(\frac{1}{4}\right) - \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 - \frac{1}{2}} dv = -2\int \frac{dx}{x}$$

$$\int \frac{4v + 1}{2v^2 + v - 1} dv + \frac{1}{2}\int \frac{1}{\left(v + \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2} dv = -2\int \frac{dx}{x}$$

$$\log|2v^2 + v - 1| + \frac{1}{2} \times \frac{1}{2\left(\frac{3}{4}\right)} \log\left|\frac{v + \frac{1}{4} - \frac{3}{4}}{v + \frac{1}{4} + \frac{3}{4}}\right| = -2\log|x| + \log c$$

16. Question

Solve the following differential equations :

$$(x + 2y)dx - (2x - y)dy = 0$$

Answer

Here, (x + 2y)dx – (2x – y)dy = 0

$$\frac{dy}{dx} = \frac{x + 2y}{2x - y}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{x + 2vx}{2x - vx}$$

$$v + x\frac{dv}{dx} = \frac{1 + 2v}{2 - v}$$

$$x\frac{dv}{dx} = \frac{1 + 2v}{2 - v} - v$$

$$x\frac{dv}{dx} = \frac{1 + 2v - 2v + v^2}{2 - v}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2 - v}$$

Integrating both sides we get,

$$\int \frac{2 - v}{1 + v^2} dv = \int \frac{dx}{x}$$

$$\int \frac{2}{1 + v^2} dv - \int \frac{v}{1 + v^2} dv = \int \frac{dx}{x}$$

$$2\tan^{-1} v - \frac{1}{2}\log|1 + v^2| = \log|x| + \log c$$

$$2\tan^{-1} v = \log|xc| + \log|1 + v^2|^{\frac{1}{2}}$$

$$e^{\tan^{-1} v} = \{1 + v^2\}^{\frac{1}{2}} xc$$

$$e^{\tan^{-1}\frac{y}{x}} = \left\{\frac{(y^2 + x^2)}{x^2}\right\}^{\frac{1}{2}} xc$$

$$e^{\tan^{-1}\frac{y}{x}} = \{(y^2 + x^2)\}^{\frac{1}{2}} c$$

17. Question

Solve the following differential equations :

$$\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$$

Answer

Here, $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx}{x} - \sqrt{\frac{v^2x^2}{x^2} - 1}$$

$$v + x\frac{dv}{dx} = v - \sqrt{v^2 - 1}$$

$$x\frac{dv}{dx} = -\sqrt{v^2 - 1}$$

Integrating both sides we get,

$$\int \frac{dv}{\sqrt{v^2 - 1}} = -\int \frac{dx}{x}$$

$$\log\left|v + \sqrt{v^2 - 1}\right| = -\log|x| + \log c$$

$$\left(\frac{y}{x} + \sqrt{v^2 - 1}\right) = \frac{c}{x}$$

$y + \sqrt{(y^2 - x^2)} = c$

18. Question

Solve the following differential equations :

$$\frac{dy}{dx} = \frac{y}{x}\{\log y - \log x + 1\}$$

Answer

$$\frac{dy}{dx} = \frac{y}{x}\left\{\log\left(\frac{y}{x}\right) + 1\right\}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{vx}{x}\left\{\log\left(\frac{vx}{x}\right) + 1\right\}$$

$$v + x\frac{dv}{dx} = v\log v + v$$

$$x\frac{dv}{dx} = v\log v$$

Integrating both sides we get,

$$\int \frac{1}{v\log v} dv = \int \frac{dx}{x}$$

log log v = log|x| + logc

log v = xc

log y/x = xc

$$\frac{y}{x} = e^{xc}$$

y = xe^{xc}

19. Question

Solve the following differential equations :

$$\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$$

Answer

$$\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = v + \sin(v)$$

$$x\frac{dv}{dx} = \sin(v)$$

$$\operatorname{cosec} v \, dv = \frac{dx}{x}$$

Integrating both sides we get,

$$\int \operatorname{cosec} v \, dv = \int \frac{dx}{x}$$

$$\log\tan\frac{v}{2} = \log x + \log c$$

$$\tan\frac{v}{2} = cx$$

$$\tan\frac{y}{2x} = cx$$

20. Question

Solve the following differential equations :

$$y^2dx + \left(x^2 - xy + y^2\right)dy = 0$$

Answer

$y^2 + (x^2 - xy + y^2)dy = 0$

$$\frac{dy}{dx} = -\frac{y^2}{x^2 - xy + y^2}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = -\frac{-x^2v^2}{x^2 - xvx + v^2x^2}$$

$$v + x\frac{dv}{dx} = -\frac{-v^2}{1 - v + v^2}$$

$$x\frac{dv}{dx} = -\frac{-v^2}{1 - v + v^2} - v$$

$$= \frac{-v^2 - v + v^2 - v^3}{1 - v + v^2}$$

$$x\frac{dv}{dx} = \frac{-v - v^3}{1 - v + v^2}$$

$$\frac{1 - v + v^2}{-v(-v - v^3)}dv = \frac{dx}{x}$$

Integrating both sides we get,

$$\int (\frac{1}{1 + v^2} - \frac{1}{v})\, dv = \frac{dx}{x}$$

$$-\int \frac{1}{v}dv + \int \frac{1}{1 + v^2}dv = \int \frac{dx}{x}$$

$$-\log|v| + \tan^{-1} v = \log|x| + \log c$$

$$\log\left|\frac{x}{y}\right| + \tan^{-1}\left(\frac{y}{x}\right) = \log c$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log xc - \log\frac{x}{y}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log\frac{xcy}{x}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \log cy$$

$$\mathbf{e^{\tan^{-1}\frac{y}{x}} = cy}$$

21. Question

Solve the following differential equations :

$$\left[x\sqrt{x^2 + y^2} - y^2\right]dx + xy\ dy = 0$$

Answer

Here, $\left[x\sqrt{x^2 + y^2} - y^2\right]dx + xydy = 0$

$$\frac{dy}{dx} = -\frac{\left[x\sqrt{x^2 + y^2} - y^2\right]}{xy}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = -\frac{\left[x\sqrt{x^2 + v^2x^2} - x^2v^2\right]}{xvx}$$

$$v + x\frac{dv}{dx} = -\frac{\left[\sqrt{1 + v^2} - v^2\right]}{v}$$

$$x\frac{dv}{dx} = -\frac{\left[\sqrt{1 + v^2} - v^2\right]}{v} - v$$

$$x\frac{dv}{dx} = -\frac{\left[\sqrt{1 + v^2} - v^2\right] - v^2}{v}$$

$$x\frac{dv}{dx} = -\frac{\left[\sqrt{1 + v^2}\right]}{v}$$

Integrating both sides we get,

$$\int \frac{v}{\sqrt{1 + v^2}}dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{2v}{\sqrt{1 + v^2}}dv = -\int \frac{dx}{x}$$

Let $1 + v^2 = t$

Differentiating both sides we get,

2vdv = dt

$$\frac{1}{2}\int \frac{1}{\sqrt{t}}dt = \int \frac{dx}{x}$$

$$\frac{1}{2} \times 2\sqrt{t} = -\log|x| + \log c$$

$$\sqrt{1 + v^2} = \log\left|\frac{c}{x}\right|$$

$$\frac{\sqrt{x^2 + y^2}}{x} = \log\left|\frac{c}{x}\right|$$

22. Question

Solve the following differential equations :

$$x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$$

Answer

Here, $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{\left(y - x\cos^2\left(\frac{y}{x}\right)\right)}{x}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{\left(vx - x\cos^2\left(\frac{vx}{x}\right)\right)}{x}$$

$$v + x\frac{dv}{dx} = v - \cos^2 v$$

$$x\frac{dv}{dx} = v - \cos^2 v - v$$

$$x\frac{dv}{dx} = -\cos^2 v$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

Integrating Both sides we get,

$$\int \sec^2 v\, dv = -\int \frac{dx}{x}$$

tanv = – log|x| + logc

$$\tan\frac{y}{x} = -\log\left|\frac{c}{x}\right|$$

23. Question

Solve the following differential equations :

$$\frac{y}{x}\cos\left(\frac{y}{x}\right)dx - \left\{\frac{x}{y}\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\}dy = 0$$

Answer

Here, $\frac{y}{x}\cos\left(\frac{y}{x}\right)dx - \left\{\frac{x}{y}\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\}dy = 0$

$$\frac{dy}{dx} = \frac{\left(\frac{y}{x}\cos\left(\frac{y}{x}\right)\right)}{\frac{x}{y}\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)}$$

It is a homogeneous equation

Put y = vx

And $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So,

$$v + x\frac{dv}{dx} = \frac{\left(\frac{vx}{x}\cos\left(\frac{vx}{x}\right)\right)}{\frac{x}{vx}\sin\left(\frac{vx}{x}\right) + \cos\left(\frac{vx}{x}\right)}$$

$$= \frac{v\cos v}{\frac{1}{v}\sin v + \cos v}$$

$$v + x\frac{dv}{dx} = \frac{v^2\cos v}{\sin v + v\cos v}$$

$$x\frac{dv}{dx} = \frac{v^2\cos v}{\sin v + v\cos v} - v$$

$$x\frac{dv}{dx} = \frac{v^2\cos v - v\sin v - v^2\cos v}{\sin v + v\cos v}$$

$$x\frac{dv}{dx} = \frac{-v\sin v}{\sin v + v\cos v}$$

$$\frac{\sin v + v\cos v}{v\sin v}dv = -\frac{dx}{x}$$

Integrating both sides we get,

$$\int\left(\frac{1}{v} + \cot v\right)dv = -\log|x| + \log c$$

$$\log|v| + \log|\sin v| = \log\left|\frac{c}{x}\right|$$

$$\log|v\sin v| = \log\left|\frac{c}{x}\right|$$

$$|v\sin v| = \left|\frac{c}{x}\right|$$

$$\left|\frac{y}{x}\sin\frac{y}{x}\right| = \left|\frac{c}{x}\right|$$

$$\left|y\sin\frac{y}{x}\right| = c$$

24. Question

Solve the following differential equations :

$$xy\ \log\left(\frac{x}{y}\right)dx + \left\{y^2 - x^2\log\left(\frac{x}{y}\right)\right\}dy = 0$$

Answer

Here, $xy\log\left(\frac{x}{y}\right)dx + \left\{y^2 - x^2\log\left(\frac{x}{y}\right)\right\}dy = 0$

$$\frac{dy}{dx} = \frac{x^2\log\left(\frac{x}{y}\right) - y^2}{xy\log\left(\frac{x}{y}\right)}$$

It is a homogeneous equation

Put x = vy

And $\frac{dx}{dy} = v + y\frac{dv}{dy}$

So,

$$v + y\frac{dv}{dy} = \frac{v^2y^2\log\left(\frac{vy}{y}\right) - y^2}{yvy\log\left(\frac{vy}{y}\right)}$$

$$v + y\frac{dv}{dy} = \frac{v^2\log(v) - 1}{v\log\left(\frac{vy}{y}\right)}$$

$$y\frac{dv}{dy} = \frac{v^2\log(v) - 1}{v\log\left(\frac{vy}{y}\right)} - v$$

$$y\frac{dv}{dy} = \frac{v^2\log(v) - 1 - v^2\log v}{v\log\left(\frac{vy}{y}\right)}$$

$$y\frac{dv}{dy} = -\frac{1}{v\log v}$$

Integrating both sides we get,

$$\int v\log v\, dv = -\int \frac{dy}{y}$$

$$\log v \times \int vdv - \int \frac{1}{v} \times \int vdv\, dv = -\log|y| + \log c$$

Itegration it by parts

$$\frac{v^2}{2}\log v - \int \frac{1}{v} \times \frac{v^2}{2} dv = \log\left|\frac{c}{y}\right|$$

$$\frac{v^2}{2}\log v - \frac{1}{2}\int vdv = \log\left|\frac{c}{y}\right|$$

$$\frac{v^2}{2}\log v - \frac{v^2}{4} = \log\left|\frac{c}{y}\right|$$

$$\frac{v^2}{2}\left[\log v - \frac{1}{2}\right] = \log\left|\frac{c}{y}\right|$$

$$\frac{\frac{x^2}{y}}{2}\left[\log\frac{x}{y} - \frac{1}{2}\right] = \log\left|\frac{c}{y}\right|$$

25. Question

Solve the following differential equations :

$$\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0.$$

Answer

Here, $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

It is a homogeneous equation

Put x = vy

And $y\frac{dv}{dy} + v = \frac{dx}{dy}$

So,

$$y\frac{dv}{dy} + v = -\frac{e^{\frac{vy}{y}}\left(1-\frac{vy}{y}\right)}{\left(1 + e^{\frac{vy}{y}}\right)}$$

$$y\frac{dv}{dy} = -\frac{e^{v}(1-v)}{(1 + e^{v})} - v$$

$$y\frac{dv}{dy} = \frac{-e^{v}(1-v) - v(1 + e^{v})}{1 + e^{v}}$$

Integrating both sides wee get,

$$\int \frac{1 + e^{v}}{-e^{v}(1-v) - v(1 + e^{v})} dv = -\int \frac{1}{y} dy$$

$$x + ye^{\frac{x}{y}} = c$$

26. Question

Solve the following differential equations :

$$(x^2 + y^2)\frac{dy}{dx} = 8x^2 - 3xy + 2y^2$$

Answer

Here, $(x^2 + y^2)\frac{dy}{dx} = 8x^2 - 3xy + 2y^2$

$$\frac{dy}{dx} = \frac{8x^2 - 3xy + 2y^2}{(x^2 + y^2)}$$

It is homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{8x^2 - 3xvx + 2v^2x^2}{(x^2 + v^2x^2)}$$

$$x\frac{dv}{dx} = \frac{8 - 3v + 2v^2}{1 + v^2} - v$$

$$x\frac{dv}{dx} = \frac{8 - 3v + 2v^2 - v^3}{1 + v^2}$$

$$x\frac{dv}{dx} = \frac{8 - 3v + 2v^2 - v^3}{1 + v^2}$$

Integrating both sides we get,

$$\int \frac{1 + v^2}{8 - 4v + 2v^2 - v^3} dv = \int \frac{1}{x} dx$$

$$\int \frac{1 + v^2}{4(2-v) + v^2(2-v)} dv = \int \frac{1}{x} dx$$

$\int \frac{1 + v^2}{(2-v)(v^2 + 4)} dv = \int \frac{1}{x} dx$ ...... (A)

$$\frac{1 + v^2}{(2-v)(v^2 + 4)} = \frac{Ax + B}{4 + v^2} + \frac{C}{2-v}$$

$$1 + v^2 = v^2(-A + C) + v(2A + B) + 2B + 4C$$

Comparing the coefficient of like power of v

– A + C = 1 ......(i)

2A – B = 0

B = 2A ......(ii)

2B + 4C = 1 ...... (iii)

Solving eq. (i),(ii) and (iii),

A = – 3/8,B = – 3/4, C = 5/8

Using eq.(A)

$$\int \frac{-\frac{3}{8}x - \frac{3}{4}}{4 + v^2} dv + \frac{5}{8}\int \frac{C}{2-v} dv = \int \frac{dx}{x}$$

$$-\frac{3}{8}\int \frac{v + 2}{4 + v^2} dv + \frac{5}{8}\int \frac{1}{2-v} dv = \int \frac{dx}{x}$$

$$-\frac{3}{8}\int \frac{v}{4 + v^2} dv - \frac{3}{8}\int \frac{1}{4 + v^2} dv + \frac{5}{8}\int \frac{1}{2-v} dv = \int \frac{dx}{x}$$

$$-\frac{3}{16}\log|4 + v^2| - \frac{3}{8}\tan^{-1}\frac{v}{2} - \frac{5}{8}\log|2-v| = \log|x| + \log c$$

$$\left[\log|4 + v^2|^{\frac{3}{16}} + \log e^{\frac{3}{8}\tan^{-1}\left(\frac{v}{2}\right)} + \log|2-v|^{\frac{5}{8}}\right] = \log xc$$

$$\frac{(4 + v^2)^{\frac{3}{16}}}{x^{\frac{3}{8}}} \times e^{\frac{3}{8}\tan^{-1}\frac{y}{2x}} \frac{(2x-y)^{\frac{5}{8}}}{x^{\frac{5}{8}}} = \frac{c}{x}$$

$$\mathbf{(4 + v^2)^{\frac{3}{16}} \times (2x-y)^{\frac{5}{8}} = ce^{\frac{3}{8}\tan^{-1}\frac{y}{2x}}}$$

27. Question

Solve the following differential equations :

$$\left(x^2 - 2xy\right)dy + \left(x^2 - 3xy + 2y^2\right)dx = 0$$

Answer

$$(x^2 - 2xy)dy + (x^2 - 3xy + 2y^2)dx = 0$$

$$\frac{dy}{dx} = \frac{x^2 - 3xy + 2y^2}{2xy - x^2}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{x^2 - 3xvx + 2v^2x^2}{2xvx - x^2}$$

$$x\frac{dv}{dx} = \frac{1-3v+2v^2}{2v-1} - v$$

$$x\frac{dv}{dx} = \frac{1-2v}{2v-1}$$

$$x\frac{dv}{dx} = -1$$

Integrating both sides we get,

$$\int dv = -\int \frac{1}{x}\,dx$$

$$v = -\log x + c$$

$$\frac{y}{x} = -\log x + c$$

28. Question

Solve the following differential equations :

$$x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$$

Answer

Here, $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{y - x\cos^2\left(\frac{y}{x}\right)}{x}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{vx - x\cos^2\left(\frac{vx}{x}\right)}{x}$$

$$x\frac{dv}{dx} = v - \cos^2 v - v$$

$$x\frac{dv}{dx} = -\cos^2 v$$

Integrating Both Sides we get,

$$\int \frac{1dv}{\cos^2 v} = -\int \frac{1}{x}\,dx$$

$$\int \sec^2 v = -\int \frac{1}{x}\,dx$$

tan v = – log x + log c

$\tan\frac{y}{x} = \log\frac{c}{x}$

29. Question

Solve the following differential equations :

$$x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$$

Answer

Here, $x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

$$\frac{dy}{dx} = \frac{2\sqrt{y^2 - x^2} + y}{x}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{2\sqrt{v^2x^2 - x^2} + vx}{x}$$

$$x\frac{dv}{dx} = 2\sqrt{v^2 - 1}$$

Integrating Both Sides we get,

$$\int \frac{1}{\sqrt{v^2 - 1}} dv = 2\int \frac{1}{x} dx$$

log (v + $\sqrt{v^2 - 1})$ $=$ 2 log x + log c

log (v + $\sqrt{v^2 - 1})$ $=$ log cx^2

(v + $\sqrt{v^2 - 1})$ $=$ cx^2

$$\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 - 1} = cx^2$$

$$y + \sqrt{y^2 - x^2} = cx^3$$

30. Question

Solve the following differential equations :

$$x\cos\left(\frac{y}{x}\right)\{ydx + xdy\} = y\sin\left(\frac{y}{x}\right)\{xdy - ydx\}$$

Answer

$$x\cos\left(\frac{y}{x}\right)\{ydx + xdy\} = y\sin\left(\frac{y}{x}\right)\{xdy - ydx\}$$

$$yx\cos\left(\frac{y}{x}\right) + x^2\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = xy\sin\left(\frac{y}{x}\right) - y^2\sin\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{-xy\cos\left(\frac{y}{x}\right) - y^2\sin\left(\frac{y}{x}\right)}{-yx\sin\left(\frac{y}{x}\right) + x^2\cos\left(\frac{y}{x}\right)}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{-xvx\cos\left(\frac{vx}{x}\right) - v^2x^2\sin\left(\frac{vx}{x}\right)}{-vxx\sin\left(\frac{vx}{x}\right) + x^2\cos\left(\frac{vx}{x}\right)}$$

$$x\frac{dv}{dx} = -\frac{-v\cos v - v^2\sin v}{\cos v - v\sin v} - v$$

$$x\frac{dv}{dx} = -\frac{-2v\cos v}{\cos v - v\sin v}$$

Integrating Both sides we get,

$$\int \frac{\cos v - v\sin v}{v\cos v}dv = -2\int \frac{1}{x}dx$$

$$\int \left(\frac{1}{v} - \tan v\right)dv = -2\int \frac{1}{x}dx$$

log v – log sec v = – 2 log x + c

$\log\left(\frac{v}{\sec v}\right) = \log\frac{c}{x^2}$

$$y\cos\left(\frac{y}{x}\right) = \frac{c}{x}$$

31. Question

Solve the following differential equations :

$(x^2 + 3xy + y^2)dx - x^2dy=0$

Answer

Here, $(x^2 + 3xy + y^2)dx - x^2dy = 0$

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{x^2 + 3xvx + v^2x^2}{x^2}$$

$$x\frac{dv}{dx} = 1 + 3v + v^2 - v$$

$$x\frac{dv}{dx} = (v + 1)^2$$

Integrating Both Sides we get,

$$\int \frac{1}{(v + 1)^2}dv = \int \frac{1}{x}dx$$

$$-\frac{1}{v - 1} = \log x - c$$

$$\frac{x}{x+y} + \log x = c$$

32. Question

Solve the following differential equations :

$$(x-y)\frac{dy}{dx} = x + 2y$$

Answer

Here, $(x-y)\frac{dy}{dx} = x + 2y$

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{x + 2vx}{x - vx}$$

$$x\frac{dv}{dx} = \frac{1 + 2v}{1 - v} - v$$

$$x\frac{dv}{dx} = \frac{1 + v + v^2}{1 - v}$$

Integrating Both Sides we get,

$$\int \frac{1 - v}{1 + v + v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\int \frac{2v - 2}{1 + v + v^2}\,dv = -\int \frac{1}{x}\,dx$$

$$\int \frac{(2v + 1) - 3}{1 + v + v^2}\,dv = -2\int \frac{1}{x}\,dx$$

$$\int \frac{(2v + 1)}{1 + v + v^2}\,dv - \int \frac{3}{v^2 + 2v\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}\,dv = -2\int \frac{1}{x}\,dx$$

$$\log(1 + v + v^2) - 3\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{v + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = -2\log x + c$$

$$\log(y^2 + xy + x^2) = 2\sqrt{3}\tan^{-1}\left(\frac{2y + x}{x\sqrt{3}}\right) + c$$

33. Question

Solve the following differential equations :

$$\left(2x^2y + y^3\right)dx + \left(xy^2 - 3x^3\right)dy = 0$$

Answer

$$(2x^2y + y^3)dx + (xy^2 - 3x^3)dy = 0$$

$$\frac{dy}{dx} = -\frac{(2x^2y + y^3)}{(xy^2 - 3x^3)}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{(2x^2vx + v^3x^3)}{(-xv^2x^2 + 3x^3)}$$

$$x\frac{dv}{dx} = \frac{(2v + v^3)}{(-v^2 + 3)} - v$$

$$x\frac{dv}{dx} = \frac{(2v^3 - v)}{(-v^2 + 3)}$$

Integrating Both Sides We get,

$$\int \frac{(-v^2 + 3)}{(2v^3 - v)} dv = \int \frac{1}{x} dx$$

$$\frac{(-v^2 + 3)}{v(2v^2 - 1)} = \frac{A}{v} + \frac{Bv + C}{(2v^2 - 1)}$$

$$3 - v^2 = A(2v^2 - 1) + (Bv + C)v$$

$$3 - v^2 = (2A + B)v^2 + Cv - A$$

Comparing the coefficient of like power of v

A = – 3

C = 0

And 2A + B = – 1

⇒B = 5

So,

$$\int \frac{-3}{v} dv + \int \frac{5v}{2v^2 - 1} dv = \int \frac{1}{x} dx$$

$$-3\int \frac{1}{v} dv + \frac{5}{4}\int \frac{4v}{2v^2 - 1} dv = \int \frac{1}{x} dx$$

– 3 log v + $\frac{5}{4}\log(2v^2 - 1) = \log x + \log c$

– 12 log v + $5\log(2v^2 - 1) = 4\log x + 4\log c$

$$\frac{(2v^2 - 1)^5}{v^{12}} = x^4c^4$$

$$x^2c^4y^{12} = 2y^2 - x^2$$

34. Question

Solve the following differential equations :

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

Answer

$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$

$\frac{dy}{dx} = \frac{y - x\sin(\frac{y}{x})}{x}$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$x\frac{dv}{dx} + v = \frac{vx - x\sin(\frac{vx}{x})}{x}$

$x\frac{dv}{dx} = v - \sin v - v$

Integrating Both Sides we get,

$\int \operatorname{cosec} v\, dv = -\int \frac{1}{x}\, dx$

log (cosec v + cot v) = – log $\frac{c}{x}$

log (cosec v + cot v) = log $\frac{x}{c}$

$\frac{(1 + \cos(\frac{y}{x}))}{\sin\left(\frac{y}{x}\right)} = \frac{x}{c}$

$x\sin\left(\frac{y}{x}\right) = c(1 + \cos(\frac{y}{x}))$

35. Question

Solve the following differential equations :

$$ydx + \left\{x\log\left(\frac{y}{x}\right)\right\}dy - 2xdy = 0$$

Answer

$ydx + \left\{x\log\left(\frac{y}{x}\right)\right\}dy - 2xdy = 0$

$y + x\log\left(\frac{y}{x}\right)\frac{dy}{dx} - 2x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$

It is a homogeneous equation,

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{vx}{2x - x\log\left(\frac{vx}{x}\right)}$$

$$x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

Integrating both sides we get,

$$\int \frac{2 - \log v}{-v + v\log v} = -\int \frac{1}{x}\,dx$$

Let log v – 1 = t

$$\frac{1}{v}dv = dt$$

$$\int (\frac{t-1}{t})dt = -\int \frac{1}{x}\,dx$$

t – log t = log$\frac{c}{x}$

log v – log(log v – 1) = log$\frac{c}{x}$

$$e^{\log(\frac{v}{e})} = \frac{c}{x}(\log v - 1)$$

$$y = C\{\log\left(\frac{y}{x}\right) - 1\}$$

36 A. Question

Solve each of the following initial value problems

$(x^2 + y^2)$d x = 2xy dy, y(1) = 0

Answer

$(x^2 + y^2)$dx = 2xy dy, y(1) = 0

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

It is a homogenous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{x^2 + v^2x^2}{2xvx}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1 - v^2}{2v}$$

Integrating both sides we get,

$$\int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$$

log (1 – v^2) = – log(x) + log c

log (1 – v^2) = log $\frac{c}{x}$

put v = $\frac{y}{x}$

$$\left(\frac{x^2-y^2}{x^2}\right) = \frac{c}{x}$$

$$x^2-y^2 = cx$$

Put y = 0,x = 1 in eq. (1),

1 – 0 = c

c = 1

put value of c in eq,(1),

$$(x^2-y^2) = x$$

36 B. Question

Solve each of the following initial value problems

$xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$,y(e) = 0

Answer

$xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$,y(e) = 0

$$\frac{dy}{dx} = \frac{y - xe^{\frac{y}{x}}}{x}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{vx - xe^{\frac{vx}{x}}}{x}$$

$$x\frac{dv}{dx} = vx - e^v - v$$

$$x\frac{dv}{dx} = -e^v$$

On integrating both sides,

$$\int -e^v\,dv = \int \frac{1}{x}\,dx$$

e^v = log xc

v = log(log xc)

put value of v,

$\frac{y}{x}$ = log(log x) + k ……(1)

Put y = 0, x = e

0 = e log(log e) + k

k = 0

put in eq. (1),

y = x log(log(x))

36 C. Question

Solve each of the following initial value problems

$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\frac{y}{x} = 0$, y(1) = 0

Answer

$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\frac{y}{x} = 0$, y(1) = 0

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$x\frac{dv}{dx} + v = \frac{vx}{x} + \operatorname{cosec}\frac{vx}{x}$

$x\frac{dv}{dx} = v - \operatorname{cosec} v - v$

$x\frac{dv}{dx} = -\operatorname{cosec} v$

Integrating both sides we get,

$\int \frac{1}{\operatorname{cosec} v}\,dv = -\int \frac{1}{x}\,dx$

$\int \sin v\,dv = -\int \frac{1}{x}\,dx$

– cos v = – log x + c

put value of v,

– cos $\frac{y}{x}$ = – log x + c …… (1)

Put y = 0, x = 1,We have

c = – 1

Now,

– cos $\frac{y}{x}$ = – log x – 1

log x = cos $\frac{y}{x}$ – 1

36 D. Question

Solve each of the following initial value problems

$(xy-y^2)dx+x^2dy=0, y(1) = 1$

Answer

$(xy - y^2)dx - x^2dy = 0, y(1) = 1$

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{xvx - v^2x^2}{x^2}$$

$$x\frac{dv}{dx} = v - v^2 - v$$

$$x\frac{dv}{dx} = -v^2$$

$$-\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$-\left(-\frac{1}{v}\right) = \log x + c$$

$$\frac{x}{y} = \log x + c$$

Put y = 1, x = 1

c = 1

Using equation(1),

x = y(log x + 1)

$$y = \frac{x}{\log x + 1}$$

36 E. Question

Solve each of the following initial value problems

$\frac{dy}{dx} = \frac{y(x+2y)}{x(2x+y)}$, y(1) = 2

Answer

$\frac{dy}{dx} = \frac{y(x+2y)}{x(2x+y)}$, y(1) = 2$\frac{dy}{dx} = \frac{y(x + 2y)}{x(2x + y)}$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{vx(x + 2vx)}{x(2x + vx)}$$

$$x\frac{dv}{dx} = \frac{v(1 + 2v)}{(2 + v)} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - v}{2 + v}$$

$$\frac{2 + v}{v^2 - v}dv = \frac{dx}{x}$$

$\int \frac{2+v}{v^2-v}dv = \int \frac{1}{x}\, dx$ ...... (1)

$$\frac{2 + v}{v(v - 1)} = \frac{A}{v} + \frac{B}{v - 1}$$

2 + v = (A + B)v –A

Comparing the coefficient of like power of v,

A = – 2

A + B = 1

– 2 + B = 1

B = 3

Using in eq. (1)

$$\int \frac{-2}{v}\, dv + 3\int \frac{1}{v - 1}\, dv = \int \frac{1}{x}\, dx$$

– 2 log v + 3log (v – 1) = log x + c

$$(v - 1)^3 = v^2\, cx$$

Put the value of v,

$$\frac{(y - x)^3}{x^3} = \frac{y^2}{x^2}\, cx$$

$$\frac{(y - x)^3}{x^3} = \frac{y^2}{x}\, c$$

36 F. Question

Solve each of the following initial value problems

$$\left(y^4 - 2x^3y\right)dx + \left(x^4 - 2xy^3\right)dy = 0$$

Answer

$$(y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0$$

$$\frac{dy}{dx} = \frac{2x^3y - y^4}{x^4 - 2xy^3}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{2x^3vx - v^4x^3}{x^4 - 2xv^3x^3}$$

$$x\frac{dv}{dx} = \frac{2v - v^4}{1 - 2v^3} - v$$

$$x\frac{dv}{dx} = \frac{v^3 + v}{1 - 2v^3}$$

$$\frac{1 - 2v^3}{v^3 + v}dv = \frac{dx}{x}$$

$\int \frac{1-2v^3}{v^3+v}dv = \int \frac{1}{x} dx$ ...... (a)

$$\frac{1 - 2v^3}{v(v + 1)(v^2 - v + 1)} = \frac{A}{v} + \frac{B}{v + 1} + \frac{Cx + D}{v^2 - v + 1}$$

$$1 - 2v^2 = A(v^3 + 1) + Bv(v^2 - v + 1) + (Cx + D)(v^2 + v)$$

$$1 - 2v^2 = v^3(A + B + C) + v^2(-B + C + D) + v(B + D) + A$$

Comparing coefficients of like power of v,

A = 1 ...... (1)

B + D = 0 ...... (2)

– B + C + D = 0 ...... (3)

A + B + C = – 2 ...... (4)

Solving eq.(1),(2),(3) and (40,We get,

A = 1,B = – 1 C = – 2,D = 1

Using eq. (a)

$$\int \frac{1}{v} dv - \int \frac{1}{v + 1} dv = \int \frac{2v - 1}{v^2 - v + 1} dv = \int \frac{1}{x} dx$$

log v – log (v + 1) – log($v^2 - v + 1$) = log xc

log ($\frac{v}{v^3 + 1}$) = log xc

36 G. Question

Solve each of the following initial value problems

$x\left(x^2 + 3y^2\right)dx + y\left(y^2 + 3x^2\right)dy = 0$,y(1) = 1

Answer

Here, $x(x^2 + 3y^2)dx + y(y^2 + 3x^2)dy = 0$,y(1) = 1

$$\frac{dy}{dx} = -\frac{x(x^2 + 3y^2)}{y(y^2 + 3x^2)}$$

It is homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = -\frac{x(x^2 + 3v^2x^2)}{vx(v^2x^2 + 3x^2)}$$

$$x\frac{dv}{dx} = \frac{1 + 3v^2}{v(v^2 + 3)} - v$$

$$x\frac{dv}{dx} = \frac{-v^4 - 6v - 1}{v(v^2 + 3)}$$

$$\frac{v(v^2 + 3)}{-v^4 - 6v - 1}dv = \frac{dx}{x}$$

$$\int \frac{v(v^2 + 3)}{-v^4 - 6v - 1}dv = \int \frac{1}{x}\,dx$$

$$\int \frac{4v^3 + 12v}{v^4 + 6v + 1}dv = -4\int \frac{1}{x}\,dx$$

$\log(v^4 + 6v + 1) = \log\frac{c}{x^4}$

$(v^4 + 6v + 1) = \frac{c}{x^4}$ ...... (1)

Put value of v,

$(y^4 + 6y^2x^2 + x^4) = c$

Put y = 1,x = 1

c = 8

put in eq. (1),

$(y^4 + 6y^2x^2 + x^4) = 8$

36 H. Question

Solve each of the following initial value problems

$$\left\{x\sin^2\left(\frac{y}{x}\right) - y\right\}dx + xdy = 0$$

Answer

$$\left\{x\sin^2\left(\frac{y}{x}\right) - y\right\}dx + xdy = 0$$

$$\left\{x\sin^2\left(\frac{y}{x}\right) - y\right\}dx = -xdy$$

$\sin^2\left(\frac{y}{x}\right) + \frac{y}{x} = \frac{dy}{dx}$ ...... (1)

Put $v = \frac{y}{x}$

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$\sin^2 v + v = v + x\frac{dv}{dx}$$

$$\frac{1}{\sin^2 v}dv = \frac{1}{x}dx$$

Integrating both sides,

$\int \frac{1}{\sin^2 v} dv = \int \frac{1}{x} dx$

– cot v = log x + c

– cot $(\frac{y}{x})$ = log x + c ...... (2)

Put x = 1, y = $\frac{\pi}{4}$ in eq. (2)

c = – 1

put in eq.(2),

– cot $(\frac{y}{x})$ = log x – 1

36 I. Question

Solve each of the following initial value problems

$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$,y(2) = x

Answer

$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$,y(2) = x

It is homogeneous equation,

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$x\frac{dv}{dx} + v = -\sin\left(\frac{vx}{x}\right) + \frac{vx}{x}$

$x\frac{dv}{dx} = -\sin v$

$\frac{dv}{\sin v} = -\frac{dx}{x}$

cosec v dv = $-\frac{dx}{x}$

– log (cosce v + cot v) = – log x + c

Put y = π,x = 2,We have,

c = 0.301

now,

– log (cosec$\left(\frac{y}{x}\right) + \cot(\frac{y}{x})$) = – log x + 0.301

37. Question

Find the particular solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ given that when x = 1, y = π/4

Answer

Consider the given equation

$$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

this is homogeneous equation,

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{vx\cos\left(\frac{vx}{x}\right) + x}{x\cos\left(\frac{vx}{x}\right)}$$

$$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

$$\frac{dv}{\sin\ v} = \frac{1}{\cos v}$$

cos v dv = $\frac{dx}{x}$

$$\int \cos v\, dv = \int \frac{1}{x}\, dx$$

sin v = log x + c

$\sin\left(\frac{y}{x}\right) = \log x + c$ ...... (1)

Put x = 1, y = $\frac{\pi}{4}$ in eq.(1),

c = $\frac{1}{\sqrt{2}}$

now,

$$\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{\sqrt{2}}$$

38. Question

Find the particular solution of the differential equation $(x-y)\frac{dy}{dx}=x+2y$ given that when x = 1, y = 0

Answer

Consider the given equation,

$$(x-y)\frac{dy}{dx} = x + 2y$$

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}$$

It is a homogeneous equation

Put y = vx

And $x\frac{dv}{dx} + v = \frac{dy}{dx}$

So,

$$x\frac{dv}{dx} + v = \frac{x + 2vx}{x - vx}$$

$$x\frac{dv}{dx} = \frac{1 + 2v}{1 - v} - v$$

$$x\frac{dv}{dx} = \frac{1 + v + v^2}{1 - v}$$

$$\int \frac{1 - v}{1 + v + v^2} dv = \int \frac{1}{x} dx$$

$$\frac{1}{2}\int \frac{2v - 2}{1 + v + v^2} dv = -\int \frac{1}{x} dx$$

$$\int \frac{(2v + 1) - 3}{1 + v + v^2} dv = -2\int \frac{1}{x} dx$$

$$\int \frac{(2v + 1)}{1 + v + v^2} dv - \int \frac{3}{v^2 + 2v\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1} dv = -2\int \frac{1}{x} dx$$

$$\log(1 + v + v^2) - 3\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{v + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = -2\log x + c$$

$\log(y^2 + xy + x^2) = 2\sqrt{3}\tan^{-1}\left(\frac{2y + x}{x\sqrt{3}}\right) + c$ ...... (1)

Put x = 1, y = 0 in eq.(1)

$c = \frac{\pi}{\sqrt{3}}$

Thus,

$$\log(y^2 + xy + x^2) = 2\sqrt{3}\tan^{-1}\left(\frac{2y + x}{x\sqrt{3}}\right) + \frac{\pi}{\sqrt{3}}$$

39. Question

Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ given that when y = 1, x = 0

Answer

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

$\frac{dy}{dx} = \frac{1}{\frac{x}{y} + \frac{y}{x}}$ ...... (1)

Let v = $\frac{y}{x}$

$$x\frac{dv}{dx} + v = \frac{dy}{dx}$$

From (1) we have,

$$x\frac{dv}{dx} + v = \frac{1}{\frac{1}{v} + v}$$

Integrating on both sides we have

$\frac{1}{2v^2} - \log v = \log x + c$

$\frac{x^2}{2y^2} = \log\left(\frac{y}{x} \times x\right) + c$ ...... (2)

Put x = 0,y = 1

0 = log(1) + c

c = 0

From equation (2) we have

$\frac{x^2}{2y^2} = \log(y)$

40. Question

Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ is given by $x^2 - y^2 = Cx$

Answer

Here, $x^2 - y^2 = Cx$

C = $\frac{x^2 - y^2}{x}$ ...... (i)

$x^2 - Cx = y^2$

Differentiate both side,

2x dx – C dx = 2y dy

$\frac{dy}{dx} = \frac{2x - c}{2y}$ ...... (ii)

Put equation (i) in equation(ii),We get,

$\frac{dy}{dx} = \frac{2x - (\frac{x^2 - y^2}{x})}{2y}$

$\frac{dy}{dx} = \frac{2x^2 - x^2 + y^2}{2xy}$

$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$

Hence prove.

Exercise 22.10

1. Question

Solve the following differential equations :

$\frac{dy}{dx} + 2y = e^{3x}$

Answer

Formula:-

(i) if a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int P dx}$

(ii) $\int dx = x + c$

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

Here, $\frac{dy}{dx} + 2y = e^{3x}$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = 2, Q = e^{3x}$

$I.F = e^{\int Pdx}$

$= e^{\int 2dx}$

$= e^{2x}$

multiplying both the sides by I.F

$$e^{2x}\frac{dy}{dx} + e^{2x}2y = e^{2x}.e^{3x}$$

$$\Rightarrow e^{2x}\frac{dy}{dx} + e^{2x}2y = e^{5x}$$

Integrating it with respect to x,

$ye^{2x} = \int e^{5x}\, dx + c$

$$\Rightarrow ye^{2x} = \frac{e^{5x}}{5} + c$$

$$\Rightarrow y = \frac{e^{5x}}{5} + ce^{-2x}$$

2. Question

Solve the following differential equations :

$$4\frac{dy}{dx} + 8y = 5e^{-3x}$$

Answer

Formula:-

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then $y(I.F) = \int Q.(I.F)dx + c$, where $I.F = e^{\int Pdx}$

(ii) $\int dx = x + c$

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

Given:-

$$\frac{dy}{dx} + 2y = \frac{5e^{-3x}}{4}$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

P = 2,

$$Q = \frac{5e^{-3x}}{4}$$

I.F = $e^{\int Pdx}$

= $e^{\int 2Pdx}$

= e^{2x}

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$$\Rightarrow ye^{2x} = \int \frac{5e^{-3x}}{4}.e^{2x}dx + c$$

$$\Rightarrow ye^{2x} = \int \frac{5}{4}e^{-x}dx + c$$

$$\Rightarrow ye^{2x} = \frac{-5}{4}e^{-x}dx + c$$

$$\Rightarrow y = \frac{-5}{4}e^{-3x} + ce^{-2x}$$

3. Question

Solve the following differential equations :

$$\frac{dy}{dx} + 2y = 6e^{x}$$

Answer

Formula:-

(i) if a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫dx = x + c

$$\text{(iii)} \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

Here, $\frac{dy}{dx} + 2y = e^{3x}$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

P = 2, Q = $6e^{x}$

I.F = $e^{\int Pdx}$

= $e^{\int 2Pdx}$

= e^{2x}

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c$

$\Rightarrow ye^{2x} = \int 6e^{x} e^{2x} dx + c$

$\Rightarrow ye^{2x} = \int 6e^{3x} dx + c$

$$\Rightarrow y(e^{2x}) = \frac{6e^{3x}}{3} + c$$

$\Rightarrow ye^{2x} = 2e^{3x} + c$

$\Rightarrow y = 2e^{3x} + ce^{-2x}$

4. Question

Solve the following differential equations :

$$\frac{dy}{dx} + y = e^{-2x}$$

Answer

Formula:-

(i) if a differential equation is $\frac{dy}{dx} + Py = Q$,

then $y(I.F) = \int Q.(I.F)dx + c$, where $I.F = e^{\int Pdx}$

(ii) $\int dx = x + c$

$$\text{(iii)} \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = 1, Q = e^{-2x}$

$I.F = e^{\int Pdx}$

$= e^{\int Pdx}$

$= e^{x}$

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c$

$\Rightarrow y(e^{x}) = \int e^{-2x} e^{x} dx + c$

$\Rightarrow y(e^{x}) = \int e^{-x} dx + c$

$\Rightarrow y(e^{x}) = -e^{-2x} + c$

$\Rightarrow y = -e^{-2x} + c\, e^{-x}$

5. Question

Solve the following differential equations :

$$x\frac{dy}{dx} = x + y$$

Answer

Formula:-

(i) if a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫dx = x + c

(iii) $\int \frac{1}{x} dx = \log x + c$

Given:-

$\frac{dy}{dx} = 1 + \frac{y}{x}$

$$\frac{dy}{dx} - \frac{y}{x} = 1$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{-1}{x}$$

Q = 1

I.F = $e^{\int Pdx}$

$= e^{\int \frac{-1}{x} dx}$

$= e^{-\log x}$

$= e^{\log x^{-1}}$

$= x^{-1}$

multiplying both the sides by I.F

$$\frac{1}{x} \cdot \frac{dy}{dx} - \frac{y}{x} = \frac{1}{x} \cdot 1$$

$$\Rightarrow \frac{1}{x} \cdot \frac{dy}{dx} - \frac{y}{x} = \frac{1}{x}$$

integrating it with respect to x,

$$y\frac{1}{x} = \int \left(\frac{1}{x}\right) dx + c$$

$$\Rightarrow y\frac{1}{x} = \log x + c$$

6. Question

Solve the following differential equations :

$$\frac{dy}{dx} + 2y = 4x$$

Answer

Formula:-

(i) if a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫dx = x + c

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

This is a linear differential equation, comparing it with

$\frac{dy}{dx} + Py = Q$

P = 2, Q = 4x

I.F = $e^{\int Pdx}$

= $e^{\int 2dx}$

= e^{2x}

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

⇒ $y(e^{2x}) = \int 4x.e^{2x}\,dx + c$

⇒ $y(e^{2x}) = 4(x\int e^{2x}\,dx - \int(\int e^{2x}\,dx)dx) + c$

using integration by part

$\Rightarrow\ y(e^{2x}) = \frac{4(xe^{2x})}{2} - \int \frac{e^{2x}}{2}dx + c$

$\Rightarrow\ y(e^{2x}) = 2xe^{2x} - 2.\frac{e^{2x}}{2}dx + c$

$\Rightarrow\ y(e^{2x}) = 2xe^{2x} - e^{2x}dx + c$

$y(e^{2x}) = (2x - 1)e^{2x} + c$

$\Rightarrow\ y = (2x - 1) + c\,e^{-2x}$

7. Question

Solve the following differential equations :

$x\frac{dy}{dx} + y = x\,e^{x}$

Answer

Formula:-

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫dx = x + c

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

(iv) $\int \frac{1}{x}dx = \log x + c$

Given:-

$$\frac{dy}{dx} + \frac{y}{x} = e^x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = \frac{1}{x}$, Q = e^x

I.F = $e^{\int Pdx}$

$= e^{\int \frac{1}{x}dx}$

$= e^{\log x}$

= x

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

⇒ yx = $\int e^x\, x dx + c$

⇒ yx = $x\int e^x\, dx - \int (\int e^x\, dx)dx) + c$

using integration by part

yx = $xe^x - \int e^x\, dx + c$

⇒ yx = $xe^x - e^x + c$

⇒ yx = $(x-1)e^x + c$

$$\Rightarrow y = \left(\frac{x-1}{x}\right)e^x + \frac{c}{x}$$

8. Question

Solve the following differential equations :

$$\frac{dy}{dx} + \frac{4x}{x^2+1}y + \frac{1}{\left(x^2+1\right)^2} = 0$$

Answer

(i) iIf a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫dx = x + c

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

(iv) $\int \frac{1}{ax+b}dx = \frac{1}{a}\log|ax + b|$

Given:-

$$\frac{dy}{dx} + \frac{4x}{x^2 + 1}y = -\frac{1}{(x^2 + 1)^2}$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{4x}{x^2 + 1}, Q = -\frac{1}{(x^2 + 1)^2}$$

I.F = $e^{\int Pdx}$

$$= e^{\int \frac{4x}{x^2+1}dx}$$

$$= e^{2\int \frac{2x}{x^2+1}dx}$$

$= e^{2\log|x^2+1|}$

$= (x^2 + 1)^2$

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$$\Rightarrow y(x^2 + 1)^2 = \int -\frac{1}{(x^2 + 1)^2}(x^2 + 1)^2 x dx + c$$

$\Rightarrow y(x^2 + 1)^2 = -\int dx + c$

$\Rightarrow y(x^2 + 1)^2 = -x + c$

$$\Rightarrow y = -\frac{x}{(x^2 + 1)^2} + \frac{c}{(x^2 + 1)^2}$$

9. Question

Solve the following differential equations :

$$x\frac{dy}{dx} + y = x \log x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) $\int x^n dx = \frac{x^{n+1}}{(n+1)} + c, n \neq -1$

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a}$

(iv) $\int \frac{1}{ax + b}dx = \frac{1}{a}\log|ax + b|$

Given:-

$$\frac{dy}{dx} + \frac{y}{x} = \log x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{x},$$

Q = log x

I.F = $e^{\int P dx}$

$= e^{\int \frac{1}{x} dx}$

= $e^{\log|x|}$

= x, x>0

The solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

⇒ yx = ∫logx.x.dx + c

$\Rightarrow\ yx\ =\ \log x \int xdx - \int (\frac{1}{x} \int xdx)dx\ +\ c$

$\Rightarrow\ yx\ =\ \frac{x^2}{2} \log x - \int \frac{x^2}{2x} dx\ +\ c$

$\Rightarrow\ yx\ =\ \frac{x^2}{2} \log x - \int \frac{x}{2} dx\ +\ c$

$\Rightarrow\ yx\ =\ \frac{x^2}{2} \log x - \frac{x^2}{4} dx\ +\ c$

$\Rightarrow\ y\ =\ \frac{x}{2} \log x - \frac{x}{4}\ +\ \frac{c}{x}$

10. Question

Solve the following differential equations :

$x \frac{dy}{dx} - y = (x-1)e^x$

Answer

Formula:-

(i)∫[f(x) + f'(x)]e^xdx = f(x)e^x + c

(ii) If a differential equation is $\frac{dy}{dx}\ +\ Py\ =\ Q,$

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int P dx}$

(iii) ∫dx = x + c

$(iv)\ \int e^{ax} dx\ =\ \frac{e^{ax}}{a}\ +\ c$

$(v) \int \frac{1}{ax\ +\ b} dx\ =\ \frac{1}{a} \log|ax\ +\ b|\ +\ c$

$(v)\ \int x^n dx\ =\ \frac{x^{n\ +\ 1}}{(n\ +\ 1)}\ +\ c, n \neq -1$

Given:

$\frac{dy}{dx} - \frac{y}{x}\ =\ \frac{x-1}{x} e^x$

This is a linear differential equation, comparing it with

$\frac{dy}{dx} + Py = Q$

$P = -\frac{1}{x},$

$Q = \frac{x-1}{x}$

I.F = $e^{\int Pdx}$

$= e^{-\int \frac{1}{x}dx}$

= $e^{-\log|x|}$

$= \frac{1}{x}, x > 0$

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$\Rightarrow y\left(\frac{1}{x}\right) = \int \frac{x-1}{x} e^x . \frac{1}{x} dx + c$

$\Rightarrow y\left(\frac{1}{x}\right) = \int (\frac{1}{x} - \frac{1}{x^2}). e^x . dx + c$

using formula(v)

$\frac{y}{x} = \frac{1}{x} e^x . dx + c$

⇒ y = e^x + cx

11. Question

Solve the following differential equations :

$\frac{dy}{dx} + \frac{y}{x} = x^3$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q,$

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫dx = x + c

(iii) $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

(iv) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + c$

(v) $\int x^n dx = \frac{x^{n+1}}{(n+1)} + c, n \neq -1$

Given:-

$\frac{dy}{dx} + \frac{y}{x} = x^3$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{x}$$

$Q = x^3$

$I.F = e^{\int Pdx}$

$$= e^{\int \frac{1}{x}dx}$$

$= e^{\log x}$

$= x$

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c$

$\Rightarrow yx = \int x^3 x dx + c$

$$\Rightarrow yx = \frac{x^5}{5} + c$$

$$\Rightarrow y = \frac{x^4}{5} + \frac{c}{x}$$

12. Question

Solve the following differential equations :

$$\frac{dy}{dx} + y = \sin x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then $y(I.F) = \int Q.(I.F)dx + c$, where $I.F = e^{\int Pdx}$

(ii) $\int dx = x + c$

$$\text{(iii) } \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

$$\text{(iv) } \int e^{ax}.\sin(bx + c)dx$$
$$= \frac{e^{ax}}{a^2 + b^2}[a\sin(bx + c) - b\cos(bx + c)] + \text{constant}$$

$$\text{(v) } \int x^n dx = \frac{x^{n+1}}{(n + 1)} + c, n \neq -1$$

given:

$$\frac{dy}{dx} + y = \sin x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = 1, Q = \sin x$

$I.F = e^{\int Pdx}$

$= e^{\int dx}$

$= e^x$

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c$

$\Rightarrow y\,e^x = \int \sin x.\,e^x\,dx + c$

$$\Rightarrow ye^x = \frac{e^x}{2}(\sin x - \cos x) + c$$

$$\Rightarrow y = \frac{1}{2}(\sin x - \cos x) + ce^{-x}$$

13. Question

Solve the following differential equations :

$$\frac{dy}{dx} + y = \cos x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then $y(I.F) = \int Q.(I.F)dx + c$, where $I.F = e^{\int Pdx}$

(ii) $\int dx = x + c$

$$\text{(iii)} \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

$$\text{(iv)} \int e^{ax}.\cos(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}[a\sin(bx + c) + b\cos(bx + c)] + \text{constant}$$

$$\text{(v)} \int x^n dx = \frac{x^{n+1}}{(n + 1)} + c, n \neq -1$$

Given:-

$$\frac{dy}{dx} + y = \cos x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = 1, Q = \cos x$

$I.F = e^{\int Pdx}$

$= e^{\int dx}$

$= e^x$

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c_1$

$\Rightarrow y\,e^x = \int \cos x.\,e^x\,dx + c_1$

let $I = \int e^x \cos x dx$

$= \cos x \int e^x dx \int(\sin x \int e^x dx)dx + c_2$

using integrating by part

$I = e^x \cos x + \int \sin x e^x dx + c$

$= e^x \cos x [\sin x \int e^x dx \int(\cos x \int e^x dx)dx] + c_2$

$\Rightarrow I = e^x \cos x + \sin x e^x – I + C_2$

$\Rightarrow 2I = (\cos x + \sin x)e^x + C_2$

$$\Rightarrow I = \frac{(\cos x + \sin x)e^x}{2} + \frac{C_2}{2}$$

$$\Rightarrow I = \frac{e^x}{2}(\cos x + \sin x) + C_3$$

putting I

$$\Rightarrow ye^x = \frac{e^x}{2}(\cos x + \sin x) + C_1 + C_3$$

$$\Rightarrow ye^x = \frac{e^x}{2}(\cos x + \sin x) + C$$

$$\Rightarrow y = \frac{1}{2}(\cos x + \sin x) + Ce^{-x}$$

14. Question

Solve the following differential equations :

$$\frac{dy}{dx} + 2y = \sin x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then $y(I.F) = \int Q.(I.F)dx + c$, where $I.F = e^{\int Pdx}$

(ii) $\int dx = x + c$

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

$$\text{(iv)} \int e^{ax}.\sin(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}[a\sin(bx + c) - b\cos(bx + c)] + \text{constant}$$

(v) $\int x^n dx = \frac{x^{n+1}}{(n + 1)} + c, n \neq -1$

Given:-

$$\frac{dy}{dx} + 2y = \sin x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

P = 2, Q = sinx

I.F = $e^{\int Pdx}$

= $e^{\int 2dx}$

= e^{2x}

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

⇒ y e^{2x} = ∫sinx. e^{2x} dx + c

$$\Rightarrow ye^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + c$$

$$\Rightarrow y = \frac{1}{5}(2\sin x - \cos x) + ce^{-2x}$$

15. Question

Solve the following differential equations :

$$\frac{dy}{dx} = y\tan x - 2\sin x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫tanxdx = log|secx| + c

$$\text{(iii)} \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

$$\text{(iv)} \int \sin(x)dx = -\cos x + c$$

$$\text{(v)} \int x^n dx = \frac{x^{n+1}}{(n+1)} + c, n \neq -1$$

Given:-

$$\frac{dy}{dx} = y\tan x - 2\sin x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

P = – tanx, Q = – 2 sinx

I.F = $e^{\int Pdx}$

= $e^{\int -\tan x dx}$

= $e^{-\log|\sec x|}$

$$= \frac{1}{\sec x}$$

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$$\Rightarrow \frac{y}{\sec x} = -\int \frac{2\sin x}{\sec x}dx + c_1$$

$\Rightarrow y\cos x = -\int 2\sin x\cos x dx + c_1$

$\Rightarrow y\cos x = -\int \sin 2x dx + c_1$

$$\Rightarrow y\cos x = \frac{\cos 2x}{2} + c_1$$

$$\Rightarrow y = \frac{\cos 2x}{2\cos x} + C$$

16. Question

Solve the following differential equations :

$$(1+x^2)\frac{dy}{dx} + y = \tan^{-1} x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

$$\text{(ii) } \int \frac{1.}{x^2 + 1}dx = \tan - 1\, xdx + c$$

$$\text{(iii) } \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

$$\text{(iv) } \int x^n dx = \frac{x^{n+1}}{(n+1)} + c, n \neq -1$$

Given:-

$$(1 + x^2)\frac{dy}{dx} + y = \tan^{(-1)} x$$

$$\frac{dy}{dx} + \frac{y}{1 + x^2} = \frac{\tan^{(-1)} x}{1 + x^2}$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{y}{1 + x^2},$$

$$Q = \frac{\tan^{(-1)} x}{1 + x^2}$$

I.F = $e^{\int Pdx}$

$$= e^{\int \frac{1.}{x^2 + 1}dx}$$

$$= e^{\tan^{(-1)} x}$$

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$\Rightarrow\ ye^{\tan^{(-1)}x} = \int \frac{\tan^{(-1)}x}{x^2 + 1} e^{\tan^{-1}x} dx + c$

let $\tan^{-1}x = t$

$\frac{1}{x^2 + 1} dx = dt$

so, $ye^t = -\int te^t\, dt + c$

$= t\int e^t\, dt - \int(e^t\, dt)dt + c$

using integration by parts

$y\ e^t = te^t - e^t + c$

$\Rightarrow y = (t-1)ce^{-t}$

$\Rightarrow\ y = \tan^{(-1)}x - 1 + ce\tan^{(-1)}x$

17. Question

Solve the following differential equations :

$\frac{dy}{dx} + y\ \tan\ x = \cos\ x$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q,$

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) $\int \tan x dx = \log|\sec x| + c$

(iii) $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

(iv) $\int x^n dx = \frac{x^{n+1}}{(n + 1)} + c, n \neq -1$

Given:-

$\frac{dy}{dx} + y\tan x = \cos x$

This is a linear differential equation, comparing it with

$\frac{dy}{dx} + Py = Q$

P = tanx, Q = cosx

I.F = $e^{\int Pdx}$

$= e^{\int \tan x dx}$

$= e^{\log|\sec x|}$

= secx

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c$

$\Rightarrow y\sec x = -\int \cos x.\sec x dx + c$

$\Rightarrow \frac{y}{\cos x} = \int dx + c$

$\Rightarrow \frac{y}{\cos x} = x + c$

$\Rightarrow y = x\cos x + C\cos x$

18. Question

Solve the following differential equations :

$$\frac{dy}{dx} + y \cot x = x^2 \cot x + 2x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then $y(I.F) = \int Q.(I.F)dx + c$, where $I.F = e^{\int Pdx}$

(ii) $\int \cot x dx = \log|\sin x| + c$

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

(iv) $\int \cos x dx = \sin x + c$

(v) if u and v are two function then $\int uv dx = u\int v dx - \int (\frac{du}{dv}\int v dx)dx$

given:-

$$\frac{dy}{dx} + y\cot x = x^2\cot x + 2x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = \cot x$, $Q = x^2 \cot x + 2x$

$I.F = e^{\int Pdx}$

$= e^{\int \cot x dx}$

$= e^{\log|\sin x|}$

$= \sin x$

Solution of the equation is given by

$y(I.F) = \int Q.(I.F)dx + c$

$\Rightarrow y\sin x = \int (x^2 \cos x + 2x\sin x)dx + c$

$\Rightarrow y\sin x = \int (x^2 \cos x dx + \int 2x\sin x dx + c$

$\Rightarrow y\sin x = x^2\sin x + c$

19. Question

Solve the following differential equations :

$$\frac{dy}{dx} + y\ \tan x = x^2 \cos^2 x$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) ∫tanxdx = log|secx| + c

(iii) $\int e^{ax}dx = \frac{e^{ax}}{a} + c$

(iv) ∫cosxdx = sinx + c

(v) if u and v are two function then $\int uvdx = u\int vdx - \int (\frac{du}{dv}\int vdx)dx$

given:-

$$\frac{dy}{dx} + y\tan x = x^2\cos^2 x$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

P = tanx, Q = $x^2\cos^2 x$

I.F = $e^{\int Pdx}$

= $e^{\int \tan x dx}$

= $e^{\log|\sec x|}$

= secx

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

⇒ ysecx = $\int (x^2 \cos^2 x(\sec x)dx + c$

⇒ ysinx = $\int (x^2 \cos x dx + c$

⇒ ysecx = $x^2 \int \cos x dx - \int (2x \cos x dx)dx + c$

using integrating by parts

y(secx) = $x^2 \sin x - 2\int x^2 \sin x dx + c$

⇒ y(secx) = $x^2 \sin x - 2(x\int \sin x dx - \int \sin x dx)dx + c$

⇒ y(secx) = $x^2 \sin x + 2x\cos x - 2\sin x + c$

⇒ y = $x^2 \sin x \cos x - 2x\cos^2 x - 2\sin x \cos^2 x - 2\sin x \cos x + c\cos x$

20. Question

Solve the following differential equations :

$$\left(1 + x^2\right)\frac{dy}{dx} + y = e^{\tan^{-1} x}$$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) $\int x^n dx = \frac{x^{n+1}}{(n+1)}, n \neq -1$

(iii) $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

(iv) $\int \frac{1}{x^2 + 1} dx = \tan^{(-1)} x + c$

Given:-

$$\frac{dy}{dx} + \frac{y}{(1 + x^2)} = \frac{e^{\tan^{(-1)} x}}{(1 + x^2)}$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{1 + x^2}, Q = \frac{e^{\tan^{(-1)} x}}{1 + x^2}$$

I.F = $e^{\int Pdx}$

$$= e^{\int \frac{1}{x^2 + 1} dx}$$

$$= e^{\tan^{(-1)} x}$$

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$$\Rightarrow y\, e^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x}}{x^2 + 1} e^{\tan^{-1} x} dx + c$$

$$\text{let } e^{\tan^{-1} x} = t$$

$$\Rightarrow e^{\tan^{-1} x} . \frac{1}{x^2 + 1} dx = dt$$

so,

yt = ∫t.dt + c

$$\Rightarrow yt = \frac{t^2}{2} + c$$

$$\Rightarrow yt = \frac{t^2}{2} + \frac{c}{t}$$

$$\Rightarrow y = \frac{1}{2} e^{\tan^{-1} x} + c e^{\tan^{-1} x}$$

21. Question

Solve the following differential equations :

$xdy = (2y + 2x^4 + x^2)dx$

Answer

(i) If a differential equation is $\frac{dy}{dx} + Py = Q$,

then y(I.F) = ∫Q.(I.F)dx + c, where I.F = $e^{\int Pdx}$

(ii) $\int x^n dx = \frac{x^{n+1}}{(n+1)} + c, n \neq -1$

(iii) $\int e^{ax} dx = \frac{e^{ax}}{a} + c$

(iv) $\int \frac{1}{x^2+1} dx = \tan^{(-1)} x + c$

(v) $\int \frac{1}{x} dx = \log x + c$

Given:-

$$x\frac{dy}{dx} = (2y + 2x^4 + x^2)$$

This is a linear differential equation, comparing it with

$$\frac{dy}{dx} + Py = Q$$

$P = -\frac{2}{x}$,

Q = $2x^3 + x$

I.F = $e^{\int Pdx}$

$$= e^{2\int \frac{-1}{x} dx}.dx$$

= $e^{-2\log x}$

$$= e^{\log\left(\frac{1}{x^2}\right)}$$

$= \frac{1}{x^2}$

Solution of the equation is given by

y(I.F) = ∫Q.(I.F)dx + c

$$\Rightarrow y\left(\frac{1}{x^2}\right) = \int (2x^3 + x)\frac{1}{x^2} dx + c$$

$$\Rightarrow \frac{y}{x^2} = 2.\frac{x^2}{2} + \log|x| + c$$

$$\Rightarrow y = x^4 + x^2\log|x| + cx^2$$

22. Question

Solve the following differential equations:

$$\left(1+y^2\right)+\left(x-e^{\tan^{-1} y}\right)\frac{dy}{dx}=0$$

Answer

Given $(1+y^2)+(x-e^{\tan^{-1} y})\frac{dy}{dx}=0$

$$\Rightarrow \left(x - e^{\tan^{-1}y}\right)\frac{dy}{dx} = -(1+y^2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(1+y^2)}{\left(x - e^{\tan^{-1}y}\right)}$$

$$\Rightarrow \frac{dx}{dy} = -\frac{\left(x - e^{\tan^{-1}y}\right)}{(1+y^2)}$$

$$\Rightarrow \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{e^{\tan^{-1}y}}{1+y^2}$$

This is a first order linear differential equation of the form

$$\frac{dx}{dy} + Px = Q$$

Here, $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dy}$$

$$\Rightarrow I.F = e^{\int \frac{1}{1+y^2} dy}$$

We have $\int \frac{1}{1+y^2} dy = \tan^{-1} y + c$

$$\therefore I.F = e^{\tan^{-1}y}$$

Hence, the solution of the differential equation is,

$$x(I.F) = \int (Q \times I.F) dy + c$$

$$\Rightarrow x\left(e^{\tan^{-1}y}\right) = \int \left(\frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\right) dy + c$$

$$\Rightarrow x\left(e^{\tan^{-1}y}\right) = \int e^{\tan^{-1}y}\left(\frac{e^{\tan^{-1}y}}{1+y^2}\right) dy + c$$

Let $e^{\tan^{-1}y} = t$

$\Rightarrow e^{\tan^{-1}y}\frac{d}{dy}(\tan^{-1}y)dy = dt$ [Differentiating both sides]

$$\Rightarrow e^{\tan^{-1}y}\left(\frac{1}{1+y^2}\right)dy = dt$$

$$\Rightarrow \frac{e^{\tan^{-1}y}}{1+y^2} dy = dt$$

By substituting this in the above integral, we get

$$xt = \int t dt + c$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow xt = \frac{t^{1+1}}{1+1} + c$$

$$\Rightarrow xt = \frac{t^2}{2} + c$$

$$\Rightarrow xt \times \frac{1}{t} = \frac{1}{t}\left(\frac{t^2}{2} + c\right)$$

$$\Rightarrow x = \frac{t}{2} + ct^{-1}$$

$$\Rightarrow x = \frac{e^{\tan^{-1}y}}{2} + c\left(e^{\tan^{-1}y}\right)^{-1} \left[\because t = e^{\tan^{-1}y}\right]$$

$$\therefore x = \frac{1}{2}e^{\tan^{-1}y} + ce^{-\tan^{-1}y}$$

Thus, the solution of the given differential equation is $x = \frac{1}{2}e^{\tan^{-1}y} + ce^{-\tan^{-1}y}$

23. Question

Solve the following differential equations:

$$y^2\frac{dx}{dy} + x - \frac{1}{y} = 0$$

Answer

Given $y^2\frac{dx}{dy} + x - \frac{1}{y} = 0$

$$\Rightarrow y^2\frac{dx}{dy} + x = \frac{1}{y}$$

$$\Rightarrow \left(y^2\frac{dx}{dy} + x\right)\frac{1}{y^2} = \frac{1}{y} \times \frac{1}{y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$$

$$\Rightarrow \frac{dx}{dy} + (y^{-2})x = y^{-3}$$

This is a first order linear differential equation of the form

$$\frac{dx}{dy} + Px = Q$$

Here, $P = y^{-2}$ and $Q = y^{-3}$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dy}$$

$$\Rightarrow I.F = e^{\int y^{-2} dy}$$

We have $\int y^n dy = \frac{y^{n+1}}{n+1} + c$

$$\Rightarrow I.F = e^{\frac{y^{-2+1}}{-2+1}}$$

$$\Rightarrow I.F = e^{\frac{y^{-1}}{-1}}$$

$\therefore \text{I.F} = e^{-\frac{1}{y}}$

Hence, the solution of the differential equation is,

$x(\text{I.F}) = \int (Q \times \text{I.F}) dy + c$

$\Rightarrow x\left(e^{-\frac{1}{y}}\right) = \int \left(y^{-3} \times e^{-\frac{1}{y}}\right) dy + c$

$\Rightarrow x\left(e^{-\frac{1}{y}}\right) = \int e^{-\frac{1}{y}} \frac{1}{y^3} dy + c$

Let $e^{-\frac{1}{y}} = t$

$\Rightarrow e^{-\frac{1}{y}} \frac{d}{dy}\left(-\frac{1}{y}\right) dy = dt$ [Differentiating both sides]

$\Rightarrow e^{-\frac{1}{y}} \frac{1}{y^2} dy = dt$

$\Rightarrow e^{-\frac{1}{y}} \frac{1}{y^2} dy \times \frac{1}{y} = dt \times \frac{1}{y}$

$\Rightarrow e^{-\frac{1}{y}} \frac{1}{y^3} dy = dt \times (-\log t)$

$\Rightarrow e^{-\frac{1}{y}} \frac{1}{y^3} dy = -\log t \, dt$

By substituting this in the above integral, we get

$xt = \int -\log t \, dt + c$

$\Rightarrow xt = -\int (\log t) \times (1) dt + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow xt = -\left\{\log t \left[\int 1dt\right] - \int \left[\frac{d}{dt}(\log t)\left(\int 1dt\right)\right] dt\right\} + c$

$\Rightarrow xt = -\left\{\log t \times t - \int \left[\frac{1}{t} \times t\right] dt\right\} + c$

$\Rightarrow xt = -\left\{t \log t - \int 1dt\right\} + c$

$\Rightarrow xt = -\{t \log t - t\} + c$

$\Rightarrow xt = -t \log t + t + c$

$\Rightarrow x\left(e^{-\frac{1}{y}}\right) = -\left(e^{-\frac{1}{y}}\right) \log\left(e^{-\frac{1}{y}}\right) + e^{-\frac{1}{y}} + c \left[\because t = e^{-\frac{1}{y}}\right]$

$\Rightarrow e^{-\frac{1}{y}} x = -e^{-\frac{1}{y}}\left(-\frac{1}{y}\right) + e^{-\frac{1}{y}} + c$

$\Rightarrow e^{-\frac{1}{y}} x = \frac{1}{y} e^{-\frac{1}{y}} + e^{-\frac{1}{y}} + c$

$\Rightarrow e^{-\frac{1}{y}} x \times \frac{1}{e^{-\frac{1}{y}}} = \left(\frac{1}{y} e^{-\frac{1}{y}} + e^{-\frac{1}{y}} + c\right) \times \frac{1}{e^{-\frac{1}{y}}}$

$\Rightarrow x=\frac{1}{y}+1+\frac{c}{e^{-\frac{1}{y}}}$

$\therefore x=1+\frac{1}{y}+ce^{\frac{1}{y}}$

Thus, the solution of the given differential equation is $x=1+\frac{1}{y}+ce^{\frac{1}{y}}$

24. Question

Solve the following differential equations:

$$\left(2x-10y^{3}\right)\frac{dy}{dx}+y=0$$

Answer

Given $(2x-10y^3)\frac{dy}{dx}+y=0$

$\Rightarrow (2x-10y^3)\frac{dy}{dx}=-y$

$\Rightarrow \frac{dy}{dx}=-\frac{y}{2x-10y^3}$

$\Rightarrow \frac{dx}{dy}=-\frac{2x-10y^3}{y}$

$\Rightarrow \frac{dx}{dy}=-\frac{2x}{y}+\frac{10y^3}{y}$

$\Rightarrow \frac{dx}{dy}+\frac{2x}{y}=10y^2$

$\Rightarrow \frac{dx}{dy}+\left(\frac{2}{y}\right)x=10y^2$

This is a first order linear differential equation of the form

$\frac{dx}{dy}+Px=Q$

Here, $P=\frac{2}{y}$ and $Q = 10y^2$

The integrating factor (I.F) of this differential equation is,

$I.F=e^{\int Pdy}$

$\Rightarrow I.F=e^{\int\frac{2}{y}dy}$

$\Rightarrow I.F=e^{2\int\frac{1}{y}dy}$

We have $\int\frac{1}{y}dy=\log y+c$

$\Rightarrow I.F=e^{2\log y}$

$\Rightarrow I.F=e^{\log y^2}$ [$\because$ m log a = log a^m]

$\therefore$ I.F = y^2 [$\because e^{\log x}=x$]

Hence, the solution of the differential equation is,

$$x(I.F) = \int (Q \times I.F)dy + c$$

$$\Rightarrow x(y^2) = \int (10y^2 \times y^2)dy + c$$

$$\Rightarrow xy^2 = \int 10y^4 dy + c$$

$$\Rightarrow xy^2 = 10\int y^4 dy + c$$

Recall $\int y^n dy = \frac{y^{n+1}}{n+1} + c$

$$\Rightarrow xy^2 = 10\left(\frac{y^{4+1}}{4+1}\right) + c$$

$$\Rightarrow xy^2 = 10\left(\frac{y^5}{5}\right) + c$$

$\Rightarrow xy^2 = 2y^5 + c$

$$\Rightarrow xy^2 \times \frac{1}{y^2} = (2y^5 + c) \times \frac{1}{y^2}$$

$$\Rightarrow x = 2y^3 + \frac{c}{y^2}$$

$\therefore x = 2y^3 + cy^{-2}$

Thus, the solution of the given differential equation is $x = 2y^3 + cy^{-2}$

25. Question

Solve the following differential equations:

$(x + \tan y)dy = \sin 2y\, dx$

Answer

Given $(x + \tan y)dy = \sin 2y\, dx$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin 2y}{x + \tan y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x + \tan y}{\sin 2y}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{\sin 2y} + \frac{\tan y}{\sin 2y}$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{\sin 2y} = \frac{\left(\frac{\sin y}{\cos y}\right)}{2\sin y \cos y}$$

$$\Rightarrow \frac{dx}{dy} - x\,\text{cosec}\,2y = \frac{\sin y}{2\sin y \cos^2 y}$$

$$\Rightarrow \frac{dx}{dy} - x\,\text{cosec}\,2y = \frac{1}{2\cos^2 y}$$

$$\Rightarrow \frac{dx}{dy} + (-\text{cosec}\,2y)x = \frac{1}{2}\sec^2 y$$

This is a first order linear differential equation of the form

$$\frac{dx}{dy} + Px = Q$$

Here, P = –cosec 2y and $Q = \frac{1}{2}\sec^2 y$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int Pdy}$$

$$\Rightarrow I.F = e^{\int -cosec2ydy}$$

$$\Rightarrow I.F = e^{-\int cosec2ydy}$$

We have $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + c$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log|\operatorname{cosec} 2y - \cot 2y|}$$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log\left|\frac{1}{\sin 2y} - \frac{\cos 2y}{\sin 2y}\right|}$$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log\left|\frac{1-\cos 2y}{\sin 2y}\right|}$$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log\left|\frac{2\sin^2 y}{2\sin y \cos y}\right|}$$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log\left|\frac{\sin y}{\cos y}\right|}$$

$$\Rightarrow I.F = e^{-\frac{1}{2}\log|\tan y|}$$

$\Rightarrow I.F = e^{\log|\tan y|^{-\frac{1}{2}}}$ [∵ m log a = log a^m]

$$\Rightarrow I.F = e^{\log\left(\frac{1}{\sqrt{\tan y}}\right)}$$

$\therefore I.F = \frac{1}{\sqrt{\tan y}}$ [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$x(I.F) = \int (Q \times I.F)dy + c$$

$$\Rightarrow x\left(\frac{1}{\sqrt{\tan y}}\right) = \int \left(\frac{1}{2}\sec^2 y \times \frac{1}{\sqrt{\tan y}}\right) dy + c$$

$$\Rightarrow \frac{1}{\sqrt{\tan y}} x = \frac{1}{2}\int (\tan y)^{-\frac{1}{2}} \sec^2 y\, dy + c$$

Let tan y = t

⇒ $\sec^2 y$ dy = dt [Differentiating both sides]

By substituting this in the above integral, we get

$$\frac{x}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}} dt + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow \frac{x}{\sqrt{t}} = \frac{1}{2}\left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + c$$

$\Rightarrow \frac{x}{\sqrt{t}} = \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + c$

$\Rightarrow \frac{x}{\sqrt{t}} = t^{\frac{1}{2}} + c$

$\Rightarrow \frac{x}{\sqrt{t}} = \sqrt{t} + c$

$\Rightarrow x = (\sqrt{t} + c)\sqrt{t}$

$\Rightarrow x = t + c\sqrt{t}$

$\therefore x = \tan y + c\sqrt{\tan y}$ [$\because$ t = tan y]

Thus, the solution of the given differential equation is $x = \tan y + c\sqrt{\tan y}$

26. Question

Solve the following differential equations:

dx + xdy = $e^{-y}\sec^2 y$dy

Answer

Given dx + xdy = $e^{-y}\sec^2 y$dy

$\Rightarrow (dx + xdy) \times \frac{1}{dy} = e^{-y} \sec^2 y\, dy \times \frac{1}{dy}$

$\Rightarrow \frac{dx}{dy} + x = e^{-y} \sec^2 y$

This is a first order linear differential equation of the form

$\frac{dx}{dy} + Px = Q$

Here, P = 1 and $e^{-y}\sec^2 y$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dy}$

$\Rightarrow I.F = e^{\int 1 dy}$

$\Rightarrow I.F = e^{\int dy}$

We have $\int dy = y + c$

$\therefore$ I.F = e^y [$\because e^{\log x} = x$]

Hence, the solution of the differential equation is,

$x(I.F) = \int (Q \times I.F) dy + c$

$\Rightarrow x(e^y) = \int (e^{-y} \sec^2 y \times e^y) dy + c$

$\Rightarrow xe^y = \int \sec^2 y\, dy + c$

Recall $\int \sec^2 x\, dx = \tan x + c$

$\Rightarrow xe^y = \tan y + c$

$\Rightarrow xe^{y} \times \frac{1}{e^{y}} = (\tan y + c) \times \frac{1}{e^{y}}$

∴ x = (tan y + c)e^{-y}

Thus, the solution of the given differential equation is x = (tan y + c)e^{-y}

27. Question

Solve the following differential equations:

$$\frac{dy}{dx} = y \tan x - 2 \sin x$$

Answer

Given $\frac{dy}{dx} = y \tan x - 2 \sin x$

$\Rightarrow \frac{dy}{dx} - y \tan x = -2 \sin x$

$\Rightarrow \frac{dy}{dx} + (-\tan x)y = -2 \sin x$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, P = –tan x and Q = –2 sin x

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int -\tan x dx}$

$\Rightarrow I.F = e^{-\int \tan x dx}$

We have $\int \tan x \, dx = \log(\sec x) + c$

$\Rightarrow I.F = e^{-\log(\sec x)}$

$\Rightarrow I.F = e^{\log\left(\frac{1}{\sec x}\right)}$ [∵ m log a = log a^{m}]

$\Rightarrow I.F = \frac{1}{\sec x}$ [∵ $e^{\log x}$ = x]

∴ I.F = cos x

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F) dx + c$

$\Rightarrow y(\cos x) = \int (-2 \sin x \times \cos x) dx + c$

$\Rightarrow y \cos x = 2 \int \cos x (-\sin x) dx + c$

Let cos x = t

⇒ –sinxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$yt = 2 \int t dt + c$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow yt = 2\left(\frac{t^{1+1}}{1+1}\right) + c$$

$$\Rightarrow yt = 2\left(\frac{t^2}{2}\right) + c$$

⇒ $yt = t^2 + c$

$$\Rightarrow yt \times \frac{1}{t} = (t^2 + c) \times \frac{1}{t}$$

$$\Rightarrow y = t + \frac{c}{t}$$

⇒ $y = \cos x + \frac{c}{\cos x}$ [∵ t = cos x]

∴ y = cos x + c sec x

Thus, the solution of the given differential equation is y = cos x + c sec x

28. Question

Solve the following differential equations:

$$\frac{dy}{dx} + y\cos x = \sin x \cos x$$

Answer

Given $\frac{dy}{dx} + y\cos x = \sin x \cos x$

$$\Rightarrow \frac{dy}{dx} + (\cos x)y = \sin x \cos x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = cos x and Q = sin x cos x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int \cos x dx}$$

We have $\int \cos x\, dx = \sin x + c$

∴ I.F = $e^{\sin x}$

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(e^{\sin x}) = \int (\sin x \cos x \times e^{\sin x})dx + c$$

$$\Rightarrow ye^{\sin x} = \int \sin x\, e^{\sin x} \cos x\, dx + c$$

Let sin x = t

⇒ cosxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$ye^t = \int te^t dt + c$

$\Rightarrow ye^t = \int (t) \times (e^t) dt + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow ye^t = t\left[\int e^t dt\right] - \int \left[\frac{d}{dt}(t)\left(\int e^t dt\right)\right] dt + c$

$\Rightarrow ye^t = t \times e^t - \int 1(e^t) dt + c$

$\Rightarrow ye^t = te^t - \int e^t dt + c$

$\Rightarrow ye^t = te^t - e^t + c$

$\Rightarrow ye^t \times e^{-t} = (te^t - e^t + c)e^{-t}$

$\Rightarrow y = t - 1 + ce^{-t}$

$\therefore y = \sin x - 1 + ce^{-\sin x}$ [$\because t = \sin x$]

Thus, the solution of the given differential equation is $y = \sin x - 1 + ce^{-\sin x}$

29. Question

Solve the following differential equations:

$$\left(1 + x^2\right)\frac{dy}{dx} - 2xy = \left(x^2 + 2\right)\left(x^2 + 1\right)$$

Answer

Given $(1 + x^2)\frac{dy}{dx} - 2xy = (x^2 + 2)(x^2 + 1)$

$\Rightarrow \left[(1 + x^2)\frac{dy}{dx} - 2xy\right] \times \frac{1}{x^2 + 1} = (x^2 + 2)(x^2 + 1) \times \frac{1}{x^2 + 1}$

$\Rightarrow \frac{dy}{dx} - \frac{2xy}{x^2 + 1} = x^2 + 2$

$\Rightarrow \frac{dy}{dx} + \left(\frac{-2x}{x^2 + 1}\right) y = x^2 + 2$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, $P = \frac{-2x}{x^2+1}$ and $Q = x^2 + 2$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int \frac{-2x}{x^2+1} dx}$

$\Rightarrow I.F = e^{-\int \frac{2x}{x^2+1} dx}$

We have $\int \frac{2x}{x^2+1} dx = \log(x^2 + 1) + c$

$\Rightarrow I.F = e^{-\log(x^2+1)}$

$\Rightarrow \text{I.F} = e^{\log\left(\frac{1}{x^2+1}\right)}$ [∵ m log a = log a^m]

$\therefore \text{I.F} = \frac{1}{x^2+1}$ [∵ $e^{\log x}$ = x]

Hence, the solution of the differential equation is,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + c$$

$$\Rightarrow y\left(\frac{1}{x^2+1}\right) = \int\left((x^2+2) \times \frac{1}{x^2+1}\right)dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int\left(\frac{x^2+2}{x^2+1}\right)dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int\left(\frac{x^2+1+1}{x^2+1}\right)dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int\left(1 + \frac{1}{x^2+1}\right)dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int dx + \int \frac{1}{x^2+1}dx + c$$

Recall $\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$ and $\int dx = x + c$

$$\Rightarrow \frac{y}{x^2+1} = x + \tan^{-1}x + c$$

$\therefore y = (x^2 + 1)(x + \tan^{-1}x + c)$

Thus, the solution of the given differential equation is $y = (x^2 + 1)(x + \tan^{-1}x + c)$

30. Question

Solve the following differential equations:

$$(\sin x)\frac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$$

Answer

Given $(\sin x)\frac{dy}{dx} + y\cos x = 2\sin^2 x\cos x$

$$\Rightarrow \frac{1}{\sin x}\left[(\sin x)\frac{dy}{dx} + y\cos x\right] = 2\sin^2 x\cos x \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{\cos x}{\sin x}\right)y = 2\sin x\cos x$$

$$\Rightarrow \frac{dy}{dx} + (\cot x)y = 2\sin x\cos x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = cot x and Q = 2 sin x cos x

The integrating factor (I.F) of this differential equation is,

$$\text{I.F} = e^{\int P dx}$$

$$\Rightarrow \text{I.F} = e^{\int \cot x dx}$$

We have $\int \cot x\, dx = \log(\sin x) + c$

$\Rightarrow \mathrm{I.F} = e^{\log(\sin x)}$

∴ I.F = sin x [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(\mathrm{I.F}) = \int (Q \times \mathrm{I.F})dx + c$$

$$\Rightarrow y(\sin x) = \int (2\sin x \cos x \times \sin x)dx + c$$

$$\Rightarrow y\sin x = 2\int \sin^2 x \cos x\, dx + c$$

Let sin x = t

⇒ cosxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$$yt = 2\int t^2 dt + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow yt = 2\left(\frac{t^{2+1}}{2+1}\right) + c$$

$$\Rightarrow yt = 2\left(\frac{t^3}{3}\right) + c$$

$$\Rightarrow yt = \frac{2t^3}{3} + c$$

$$\Rightarrow yt \times \frac{1}{t} = \left(\frac{2t^3}{3} + c\right) \times \frac{1}{t}$$

$$\Rightarrow y = \frac{2t^2}{3} + \frac{c}{t}$$

$\Rightarrow y = \frac{2(\sin x)^2}{3} + \frac{c}{\sin x}$ [∵ t = sin x]

$$\therefore y = \frac{2}{3}\sin^2 x + c\,\mathrm{cosec}\,x$$

Thus, the solution of the given differential equation is $y = \frac{2}{3}\sin^2 x + c\,\mathrm{cosec}\,x$

31. Question

Solve the following differential equations:

$$\left(x^2 - 1\right)\frac{dy}{dx} + 2\left(x + 2\right)y = 2\left(x + 1\right)$$

Answer

Given $(x^2 - 1)\frac{dy}{dx} + 2(x + 2)y = 2(x + 1)$

$$\Rightarrow \left[(x^2 - 1)\frac{dy}{dx} + 2(x + 2)y\right] \times \frac{1}{x^2 - 1} = 2(x + 1) \times \frac{1}{x^2 - 1}$$

$$\Rightarrow \frac{dy}{dx} + \frac{2(x+2)y}{x^2 - 1} = \frac{2(x+1)}{x^2 - 1}$$

$\Rightarrow \frac{dy}{dx} + \left[\frac{2x+4}{x^2-1}\right]y = \frac{2}{x-1}$ [∵ $x^2 - 1 = (x + 1)(x - 1)$]

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2x+4}{x^2-1}$ and $Q = \frac{2}{x-1}$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \frac{2x+4}{x^2-1} dx}$$

$$\Rightarrow I.F = e^{\int \left(\frac{2x}{x^2-1} + \frac{4}{x^2-1}\right) dx}$$

$$\Rightarrow I.F = e^{\int \left(\frac{2x}{x^2-1}\right) dx + \int \left(\frac{4}{x^2-1}\right) dx}$$

$$\Rightarrow I.F = e^{\int \left(\frac{2x}{x^2-1}\right) dx + 4\int \left(\frac{1}{x^2-1}\right) dx}$$

We have $\int \frac{2x}{x^2-1} dx = \log|x^2 - 1| + c$ and $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

$$\Rightarrow I.F = e^{\log|x^2-1| + 4\left(\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|\right)}$$

$$\Rightarrow I.F = e^{\log|x^2-1| + 2\log\left|\frac{x-1}{x+1}\right|}$$

$\Rightarrow I.F = e^{\log|x^2-1| + \log\left|\frac{x-1}{x+1}\right|^2}$ [∵ m log a = log a^m]

$\Rightarrow I.F = e^{\log\left(|x^2-1| \times \left|\frac{x-1}{x+1}\right|^2\right)}$ [∵ log a + log b = log ab]

$$\Rightarrow I.F = e^{\log\left(|(x-1)(x+1)|\left|\frac{x-1}{x+1}\right|^2\right)}$$

$$\Rightarrow I.F = e^{\log\left[\frac{(x-1)^3}{x+1}\right]}$$

$\therefore I.F = \frac{(x-1)^3}{x+1}$ [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y\left(\frac{(x-1)^3}{x+1}\right) = \int \left(\left(\frac{2}{x-1}\right) \times \frac{(x-1)^3}{x+1}\right) dx + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1} y = \int \frac{2(x-1)^2}{x+1} dx + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1} y = 2\int \frac{(x-1)^2}{x+1} dx + c$$

We can write $(x - 1)^2 = (x + 1)^2 - 4x$

$$\Rightarrow \frac{(x-1)^3}{x+1} y = 2\int \left[\frac{(x+1)^2 - 4x}{x+1}\right] dx + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\int\left[x+1-\frac{4x}{x+1}\right]dx + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\int xdx + \int dx - \int\frac{4x}{x+1}dx\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\int xdx + \int dx - 4\int\frac{x}{x+1}dx\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\int xdx + \int dx - 4\int\left(1-\frac{1}{x+1}\right)dx\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\int xdx + \int dx - 4\left\{\int dx - \int\frac{1}{x+1}dx\right\}\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\int xdx + \int dx - 4\int dx + 4\int\frac{1}{x+1}dx\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\int xdx - 3\int dx + 4\int\frac{1}{x+1}dx\right] + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ and $\int\frac{1}{x+1}dx = \log|x+1| + c$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\frac{x^{1+1}}{1+1} - 3x + 4\log|x+1|\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = 2\left[\frac{x^2}{2} - 3x + 4\log|x+1|\right] + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y = x^2 - 6x + 8\log|x+1| + c$$

$$\Rightarrow \frac{(x-1)^3}{x+1}y \times \frac{x+1}{(x-1)^3} = (x^2 - 6x + 8\log|x+1| + c) \times \frac{x+1}{(x-1)^3}$$

$$\therefore y = \frac{x+1}{(x-1)^3}(x^2 - 6x + 8\log|x+1| + c)$$

Thus, the solution of the given differential equation is $y = \frac{x+1}{(x-1)^3}(x^2 - 6x + 8\log|x+1| + c)$

32. Question

Solve the following differential equations:

$$x\frac{dy}{dx} + 2y = x\cos x$$

Answer

Given $x\frac{dy}{dx} + 2y = x\cos x$

$$\Rightarrow \left(x\frac{dy}{dx} + 2y\right) \times \frac{1}{x} = x\cos x \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \cos x$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right)y = \cos x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2}{x}$ and Q = cos x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \frac{2}{x} dx}$$

$$\Rightarrow I.F = e^{2\int \frac{1}{x} dx}$$

We have $\int \frac{1}{x} dx = \log x + c$

$$\Rightarrow I.F = e^{2\log x}$$

$\Rightarrow I.F = e^{\log x^2}$ [∵ m log a = log a^m]

∴ I.F = x^2 [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(x^2) = \int (\cos x \times x^2) dx + c$$

$$\Rightarrow yx^2 = \int x^2 \cos x \, dx + c$$

$$\Rightarrow yx^2 = \int (x^2) \times (\cos x) dx + c$$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$$\Rightarrow yx^2 = x^2\left[\int \cos x \, dx\right] - \int \left[\frac{d}{dx}(x^2)\left(\int \cos x \, dx\right)\right] dx + c$$

$$\Rightarrow yx^2 = x^2(\sin x) - \int [2x(\sin x)] dx + c$$

$$\Rightarrow yx^2 = x^2 \sin x - 2\int x \sin x \, dx + c$$

$$\Rightarrow yx^2 = x^2 \sin x - 2\left\{x\left[\int \sin x \, dx\right] - \int \left[\frac{d}{dx}(x)\left(\int \sin x \, dx\right)\right] dx\right\} + c$$

$$\Rightarrow yx^2 = x^2 \sin x - 2\left\{x[-\cos x] - \int [1(-\cos x)] dx\right\} + c$$

$$\Rightarrow yx^2 = x^2 \sin x - 2\left\{-x\cos x + \int \cos x \, dx\right\} + c$$

$\Rightarrow yx^2 = x^2 \sin x - 2\{-x \cos x + \sin x\} + c$

$\Rightarrow yx^2 = x^2 \sin x + 2x \cos x - 2 \sin x + c$

$$\Rightarrow yx^2 \times \frac{1}{x^2} = (x^2 \sin x + 2x\cos x - 2\sin x + c) \times \frac{1}{x^2}$$

$$\therefore y = \sin x + \frac{2}{x}\cos x - \frac{2}{x^2}\sin x + \frac{c}{x^2}$$

Thus, the solution of the given differential equation is $y = \sin x + \frac{2}{x}\cos x - \frac{2}{x^2}\sin x + \frac{c}{x^2}$

33. Question

Solve the following differential equations:

$$\frac{dy}{dx} - y = xe^x$$

Answer

Given $\frac{dy}{dx} - y = xe^x$

$$\Rightarrow \frac{dy}{dx} + (-1)y = xe^x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = –1 and Q = xe^x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int Pdx}$$

$$\Rightarrow I.F = e^{\int -1dx}$$

$$\Rightarrow I.F = e^{-\int dx}$$

We have $\int dx = x + c$

$\therefore$ I.F = e^{-x}

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(e^{-x}) = \int (xe^x \times e^{-x})dx + c$$

$$\Rightarrow ye^{-x} = \int xdx + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow ye^{-x} = \frac{x^{1+1}}{1+1} + c$$

$$\Rightarrow ye^{-x} = \frac{x^2}{2} + c$$

$$\Rightarrow ye^{-x} \times e^x = \left(\frac{x^2}{2} + c\right)e^x$$

$$\therefore y = \left(\frac{x^2}{2} + c\right)e^x$$

Thus, the solution of the given differential equation is $y = \left(\frac{x^2}{2} + c\right)e^x$

34. Question

Solve the following differential equations:

$$\frac{dy}{dx} + 2y = xe^{4x}$$

Answer

Given $\frac{dy}{dx} + 2y = xe^{4x}$

$\Rightarrow \frac{dy}{dx} + (2)y = xe^{4x}$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, P = 2 and Q = xe^{4x}

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int 2dx}$

$\Rightarrow I.F = e^{2\int dx}$

We have $\int dx = x + c$

$\therefore$ I.F = e^{2x}

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F)dx + c$

$\Rightarrow y(e^{2x}) = \int (xe^{4x} \times e^{2x})dx + c$

$\Rightarrow ye^{2x} = \int xe^{6x}dx + c$

$\Rightarrow ye^{2x} = \int (x) \times (e^{6x})dx + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow ye^{2x} = x\left[\int e^{6x}dx\right] - \int \left[\frac{d}{dx}(x)\left(\int e^{6x}dx\right)\right]dx + c$

$\Rightarrow ye^{2x} = x\left(\frac{e^{6x}}{6}\right) - \int \left[1\left(\frac{e^{6x}}{6}\right)\right]dx + c$

$\Rightarrow ye^{2x} = \frac{x}{6}e^{6x} - \frac{1}{6}\int e^{6x}dx + c$

$\Rightarrow ye^{2x} = \frac{x}{6}e^{6x} - \frac{1}{6}\left(\frac{e^{6x}}{6}\right) + c$

$\Rightarrow ye^{2x} = \frac{x}{6}e^{6x} - \frac{1}{36}e^{6x} + c$

$\Rightarrow ye^{2x} \times e^{-2x} = \left(\frac{x}{6}e^{6x} - \frac{1}{36}e^{6x} + c\right) \times e^{-2x}$

$\therefore y = \frac{x}{6}e^{4x} - \frac{1}{36}e^{4x} + ce^{-2x}$

Thus, the solution of the given differential equation is $y = \frac{x}{6}e^{4x} - \frac{1}{36}e^{4x} + ce^{-2x}$

35. Question

Solve the differential equation $(x+2y^2)\frac{dy}{dx}=y,$ given that when x = 2, y = 1.

Answer

Given $(x+2y^2)\frac{dy}{dx}=y$ and when x = 2, y = 1

$\Rightarrow \frac{dy}{dx}=\frac{y}{x+2y^2}$

$\Rightarrow \frac{dx}{dy}=\frac{x+2y^2}{y}$

$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+\frac{2y^2}{y}$

$\Rightarrow \frac{dx}{dy}-\frac{x}{y}=2y$

$\Rightarrow \frac{dx}{dy}+\left(-\frac{1}{y}\right)x=2y$

This is a first order linear differential equation of the form

$\frac{dx}{dy}+Px=Q$

Here, $P=-\frac{1}{y}$ and Q = 2y

The integrating factor (I.F) of this differential equation is,

$I.F=e^{\int Pdy}$

$\Rightarrow I.F=e^{\int -\frac{1}{y}dy}$

$\Rightarrow I.F=e^{-\int \frac{1}{y}dy}$

We have $\int \frac{1}{y}dy=\log y+c$

$\Rightarrow I.F=e^{-\log y}$

$\Rightarrow I.F=e^{\log y^{-1}}$ [∵ m log a = log a^m]

∴ I.F = y^{-1} [∵ $e^{\log x}=x$]

Hence, the solution of the differential equation is,

$x(I.F)=\int (Q\times I.F)dy+c$

$\Rightarrow x(y^{-1})=\int (2y\times y^{-1})dy+c$

$\Rightarrow xy^{-1}=\int 2dy+c$

$\Rightarrow xy^{-1}=2\int dy+c$

We know $\int dy=y+c$

⇒ xy^{-1} = 2y + c

⇒ $xy^{-1}\times y$ = (2y + c)y

$\therefore x = (2y + c)y$

However, when x = 2, we have y = 1.

$\Rightarrow 2 = (2 \times 1 + c) \times 1$

$\Rightarrow 2 = 2 + c$

$\therefore c = 2 - 2 = 0$

By substituting the value of c in the equation for x, we get

$x = (2y + 0)y$

$\Rightarrow x = (2y)y$

$\therefore x = 2y^2$

Thus, the solution of the given differential equation is $x = 2y^2$

36 A. Question

Find one-parameter families of solution curves of the following differential equations:

$\frac{dy}{dx} + 3y = e^{mx}$, m is a given real number

Answer

$\frac{dy}{dx} + 3y = e^{mx}$, m is a given real number

Given $\frac{dy}{dx} + 3y = e^{mx}$

$\Rightarrow \frac{dy}{dx} + (3)y = e^{mx}$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, P = 3 and $Q = e^{mx}$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int 3 dx}$

$\Rightarrow I.F = e^{3\int dx}$

We have $\int dx = x + c$

$\therefore I.F = e^{3x}$

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F) dx + c$

$\Rightarrow y(e^{3x}) = \int (e^{mx} \times e^{3x}) dx + c$

$\Rightarrow ye^{3x} = \int e^{mx+3x} dx + c$

$\Rightarrow ye^{3x} = \int e^{(m+3)x} dx + c$

Case (1): m + 3 = 0 or m = –3

When m + 3 = 0, we have $e^{(m+3)x} = e^0 = 1$

$$\Rightarrow ye^{3x} = \int dx + c$$

$\Rightarrow ye^{3x} = x + c$

$\Rightarrow ye^{3x} \times e^{-3x} = (x + c)e^{-3x}$

$\therefore y = (x + c)e^{-3x}$

Case (2): m + 3 ≠ 0 or m ≠ –3

When m + 3 ≠ 0, we have

$$ye^{3x} = \int e^{(m+3)x}dx + c$$

Recall $\int e^x dx = e^x + c$

$$\Rightarrow ye^{3x} = \frac{e^{(m+3)x}}{m+3} + c$$

$$\Rightarrow ye^{3x} \times e^{-3x} = \left(\frac{e^{(m+3)x}}{m+3} + c\right)e^{-3x}$$

$$\Rightarrow y = \left(\frac{e^{mx} \times e^{3x}}{m+3} + c\right)e^{-3x}$$

$$\therefore y = \frac{e^{mx}}{m+3} + ce^{-3x}$$

Thus, the solution of the given differential equation is $y = \begin{cases} (x+c)e^{-3x}, m = -3 \\ \frac{e^{mx}}{m+3} + ce^{-3x}, \text{otherwise} \end{cases}$

36 B. Question

Find one-parameter families of solution curves of the following differential equations:

$$\frac{dy}{dx} - y = \cos 2x$$

Answer

$\frac{dy}{dx} - y = \cos 2x$

Given $\frac{dy}{dx} - y = \cos 2x$

$$\Rightarrow \frac{dy}{dx} + (-1)y = \cos 2x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = –1 and Q = cos 2x

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int -1 dx}$

$\Rightarrow I.F = e^{-\int dx}$

We have $\int dx = x + c$

$\therefore$ I.F = e^{-x}

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(e^{-x}) = \int (\cos 2x \times e^{-x})dx + c$$

$$\Rightarrow ye^{-x} = \int e^{-x} \cos 2x\, dx + c$$

$$\Rightarrow ye^{-x} = \int (e^{-x}) \times (\cos 2x)dx + c$$

Let $I = \int (e^{-x}) \times (\cos 2x)dx$

$$\Rightarrow I = e^{-x}\left[\int \cos 2x\, dx\right] - \int \left[\frac{d}{dx}(e^{-x})\left(\int \cos 2x\, dx\right)\right]dx$$

$$\Rightarrow I = e^{-x}\left(\frac{\sin 2x}{2}\right) - \int \left[-e^{-x}\left(\frac{\sin 2x}{2}\right)\right]dx$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\int e^{-x}\sin 2x\, dx$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\left\{e^{-x}\left[\int \sin 2x\, dx\right] - \int \left[\frac{d}{dx}(e^{-x})\left(\int \sin 2x\, dx\right)\right]dx\right\}$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\left\{e^{-x}\left[-\frac{\cos 2x}{2}\right] - \int \left[-e^{-x}\left(-\frac{\cos 2x}{2}\right)\right]dx\right\}$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\left\{-\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\, dx\right\}$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x - \frac{1}{4}\int e^{-x}\cos 2x\, dx$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x$$

$$\Rightarrow \frac{5}{4}I = \frac{1}{4}e^{-x}(2\sin 2x - \cos 2x)$$

$\Rightarrow 5I = e^{-x}(2 \sin 2x - \cos 2x)$

$$\therefore I = \frac{e^{-x}}{5}(2\sin 2x - \cos 2x)$$

By substituting the value of I in the original integral, we get

$$\Rightarrow ye^{-x} = \frac{e^{-x}}{5}(2\sin 2x - \cos 2x) + c$$

$$\Rightarrow ye^{-x} \times e^{x} = \left[\frac{e^{-x}}{5}(2\sin 2x - \cos 2x) + c\right]e^{x}$$

$$\therefore y = \frac{1}{5}(2\sin 2x - \cos 2x) + ce^{x}$$

Thus, the solution of the given differential equation is $y = \frac{1}{5}(2\sin 2x - \cos 2x) + ce^{x}$

36 C. Question

Find one-parameter families of solution curves of the following differential equations:

$$x\frac{dy}{dx} - y = (x+1)e^{-x}$$

Answer

$x\frac{dy}{dx} - y = (x+1)e^{-x}$

Given $x\frac{dy}{dx} - y = (x+1)e^{-x}$

$\Rightarrow \left(x\frac{dy}{dx} - y\right) \times \frac{1}{x} = (x+1)e^{-x} \times \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \left(\frac{x+1}{x}\right)e^{-x}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{-1}{x}\right)y = \left(\frac{x+1}{x}\right)e^{-x}$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, $P = -\frac{1}{x}$ and $Q = \left(\frac{x+1}{x}\right)e^{-x}$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int -\frac{1}{x}dx}$

$\Rightarrow I.F = e^{-\int \frac{1}{x}dx}$

We have $\int \frac{1}{x}dx = \log x + c$

$\Rightarrow I.F = e^{-\log x}$

$\Rightarrow I.F = e^{\log x^{-1}}$ [∵ m log a = log a^m]

∴ I.F = x^{-1} [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F)dx + c$

$\Rightarrow y(x^{-1}) = \int \left(\left(\frac{x+1}{x}\right)e^{-x} \times x^{-1}\right)dx + c$

$\Rightarrow \frac{y}{x} = \int \left(\frac{x+1}{x^2}\right)e^{-x}dx + c$

$\Rightarrow \frac{y}{x} = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x}dx + c$

Let $\frac{e^{-x}}{x} = t$

$$\Rightarrow \left[(-e^{-x})\left(\frac{1}{x}\right) + e^{-x}\left(-\frac{1}{x^2}\right)\right]dx = dt$$

$$\Rightarrow -e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right)dx = dt$$

$$\Rightarrow \left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x}dx = -dt$$

By substituting this in the above integral, we get

$$\frac{y}{x} = \int -dt + c$$

$$\Rightarrow \frac{y}{x} = -\int dt + c$$

We know $\int dx = x + c$

$$\Rightarrow \frac{y}{x} = -\frac{e^{-x}}{x} + c \left[\because t = \frac{e^{-x}}{x}\right]$$

$$\Rightarrow y = \left(-\frac{e^{-x}}{x} + c\right)x$$

$\therefore$ y = $-e^{-x} + cx$

Thus, the solution of the given differential equation is y = $-e^{-x} + cx$

36 D. Question

Find one-parameter families of solution curves of the following differential equations:

$$x\frac{dy}{dx} + y = x^4$$

Answer

$$x\frac{dy}{dx} + y = x^4$$

Given $x\frac{dy}{dx} + y = x^4$

$$\Rightarrow \left(x\frac{dy}{dx} + y\right) \times \frac{1}{x} = x^4 \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^3$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right)y = x^3$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{1}{x}$ and Q = x^3

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \frac{1}{x} dx}$$

We have $\int \frac{1}{x} dx = \log x + c$

$\Rightarrow \mathrm{I.F} = e^{\log x}$

$\therefore$ I.F = x [$\because e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(\mathrm{I.F}) = \int (Q \times \mathrm{I.F})dx + c$$

$$\Rightarrow y(x) = \int (x^3 \times x)dx + c$$

$$\Rightarrow xy = \int x^4 dx + c$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow xy = \frac{x^{4+1}}{4+1} + c$$

$$\Rightarrow xy = \frac{x^5}{5} + c$$

$$\Rightarrow xy \times \frac{1}{x} = \left(\frac{x^5}{5} + c\right) \times \frac{1}{x}$$

$$\therefore y = \frac{x^4}{5} + \frac{c}{x}$$

Thus, the solution of the given differential equation is $y = \frac{x^4}{5} + \frac{c}{x}$

36 E. Question

Find one-parameter families of solution curves of the following differential equations:

$$(x \log x)\frac{dy}{dx} + y = \log x$$

Answer

$$(x \log x)\frac{dy}{dx} + y = \log x$$

Given $(x \log x)\frac{dy}{dx} + y = \log x$

$$\Rightarrow \left[(x \log x)\frac{dy}{dx} + y\right] \times \frac{1}{x \log x} = \log x \times \frac{1}{x \log x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x \log x}\right) y = \frac{1}{x}$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{1}{x \log x}$ and $Q = \frac{1}{x}$

The integrating factor (I.F) of this differential equation is,

$$\mathrm{I.F} = e^{\int P dx}$$

$\Rightarrow \text{I.F} = e^{\int \frac{1}{x \log x} dx}$

Let t = log x

$\Rightarrow dt = \frac{1}{x} dx$ [Differentiating both sides]

By substituting this in the above integral, we get

$\text{I.F} = e^{\int \frac{1}{t} dt} + c$

We have $\int \frac{1}{x} dx = \log x + c$

$\Rightarrow$ I.F = $e^{\log t}$

$\Rightarrow$ I.F = t [$\because e^{\log x} = x$]

$\therefore$ I.F = log x [$\because$ t = log x]

Hence, the solution of the differential equation is,

$$y(\text{I.F}) = \int (Q \times \text{I.F}) dx + c$$

$$\Rightarrow y(\log x) = \int \left(\frac{1}{x} \times \log x\right) dx + c$$

$$\Rightarrow y \log x = \int \log x \left(\frac{1}{x} dx\right) + c$$

Let t = log x

$\Rightarrow dt = \frac{1}{x} dx$ [Differentiating both sides]

By substituting this in the above integral, we get

$$yt = \int t dt + c$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow yt = \frac{t^{1+1}}{1+1} + c$$

$$\Rightarrow yt = \frac{t^2}{2} + c$$

$$\Rightarrow yt \times \frac{1}{t} = \left(\frac{t^2}{2} + c\right) \times \frac{1}{t}$$

$$\Rightarrow y = \frac{1}{2} t + \frac{c}{t}$$

$\therefore y = \frac{1}{2} \log x + \frac{c}{\log x}$ [$\because$ t = log x]

Thus, the solution of the given differential equation is $y = \frac{1}{2} \log x + \frac{c}{\log x}$

36 F. Question

Find one-parameter families of solution curves of the following differential equations:

$$\frac{dy}{dx} - \frac{2xy}{1+y^2} = x^2 + 2$$

Answer

$\frac{dy}{dx} - \frac{2xy}{1+y^2} = x^2 + 2$

Given $\frac{dy}{dx} - \frac{2xy}{1+y^2} = x^2 + 2$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{-2x}{x^2+1}\right)y = x^2 + 2$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{-2x}{x^2+1}$ and Q = x^2 + 2

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int \frac{-2x}{x^2+1} dx}$

$\Rightarrow I.F = e^{-\int \frac{2x}{x^2+1} dx}$

We have $\int \frac{2x}{x^2+1} dx = \log(x^2+1) + c$

$\Rightarrow I.F = e^{-\log(x^2+1)}$

$\Rightarrow I.F = e^{\log\left(\frac{1}{x^2+1}\right)}$ [∵ m log a = log a^m]

$\therefore I.F = \frac{1}{x^2+1}$ [∵ $e^{\log x}$ = x]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y\left(\frac{1}{x^2+1}\right) = \int \left((x^2+2) \times \frac{1}{x^2+1}\right) dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int \left(\frac{x^2+2}{x^2+1}\right) dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int \left(\frac{x^2+1+1}{x^2+1}\right) dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int \left(1 + \frac{1}{x^2+1}\right) dx + c$$

$$\Rightarrow \frac{y}{x^2+1} = \int dx + \int \frac{1}{x^2+1} dx + c$$

Recall $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$ and $\int dx = x + c$

$$\Rightarrow \frac{y}{x^2+1} = x + \tan^{-1} x + c$$

$\therefore y = (x^2 + 1)(x + \tan^{-1}x + c)$

Thus, the solution of the given differential equation is $y = (x^2 + 1)(x + \tan^{-1}x + c)$

36 G. Question

Find one-parameter families of solution curves of the following differential equations:

$$\frac{dy}{dx} + y\cos x = e^{\sin x}\cos x$$

Answer

$\frac{dy}{dx} + y\cos x = e^{\sin x}\cos x$

Given $\frac{dy}{dx} + y\cos x = e^{\sin x}\cos x$

$$\Rightarrow \frac{dy}{dx} + (\cos x)y = e^{\sin x}\cos x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = cos x and Q = $e^{\sin x}$ cos x

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int \cos x dx}$

We have $\int \cos x\, dx = \sin x + c$

∴ I.F = $e^{\sin x}$

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y\left(e^{\sin x}\right) = \int \left(e^{\sin x}\cos x \times e^{\sin x}\right)dx + c$$

$$\Rightarrow ye^{\sin x} = \int e^{\sin x + \sin x}\cos x\, dx + c$$

$$\Rightarrow ye^{\sin x} = \int e^{2\sin x}\cos x\, dx + c$$

Let sin x = t

⇒ cosxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$$ye^{t} = \int e^{2t}dt + c$$

Recall $\int e^{x}dx = e^{x} + c$

$$\Rightarrow ye^{t} = \frac{e^{2t}}{2} + c$$

$$\Rightarrow ye^{t} \times e^{-t} = \left(\frac{e^{2t}}{2} + c\right) \times e^{-t}$$

$$\Rightarrow y = \frac{1}{2}e^{t} + ce^{-t}$$

∴ $y = \frac{1}{2}e^{\sin x} + ce^{-\sin x}$ [∵ t = sin x]

Thus, the solution of the given differential equation is $y = \frac{1}{2}e^{\sin x} + ce^{-\sin x}$

36 H. Question

Find one-parameter families of solution curves of the following differential equations:

$$(x+y)\frac{dy}{dx}=1$$

Answer

$(x+y)\frac{dy}{dx}=1$

Given $(x+y)\frac{dy}{dx}=1$

$\Rightarrow x+y=\frac{dx}{dy}$

$\Rightarrow \frac{dx}{dy}=x+y$

$\Rightarrow \frac{dx}{dy}-x=y$

$\Rightarrow \frac{dx}{dy}+(-1)x=y$

This is a first order linear differential equation of the form

$\frac{dx}{dy}+Px=Q$

Here, P = –1 and Q = y

The integrating factor (I.F) of this differential equation is,

$I.F=e^{\int Pdy}$

$\Rightarrow I.F=e^{\int -dy}$

$\Rightarrow I.F=e^{-\int dy}$

We have $\int dy=y+c$

$\therefore$ I.F = e^{-y}

Hence, the solution of the differential equation is,

$x(I.F)=\int (Q\times I.F)dy+c$

$\Rightarrow x(e^{-y})=\int (y\times e^{-y})dy+c$

$\Rightarrow xe^{-y}=\int (y)\times(e^{-y})dy+c$

Recall $\int f(x)g(x)=f(x)[\int g(x)dx]-\int[f'(x)(\int g(x)dx)]dx+c$

$\Rightarrow xe^{-y}=y\left[\int e^{-y}dy\right]-\int\left[\frac{d}{dy}(y)\left(\int e^{-y}dy\right)\right]dy+c$

$\Rightarrow xe^{-y}=y[-e^{-y}]-\int[1(-e^{-y})]dy+c$

$\Rightarrow xe^{-y}=-ye^{-y}+\int e^{-y}dy+c$

$\Rightarrow xe^{-y}=-ye^{-y}-e^{-y}+c$

$\Rightarrow xe^{-y} = -e^{-y}(y + 1) + c$

$\Rightarrow xe^{-y} \times e^{y} = [-e^{-y}(y + 1) + c] \times e^{y}$

$\therefore x = -(y + 1) + ce^{y}$

Thus, the solution of the given differential equation is $x = -(y + 1) + ce^{y}$

36 I. Question

Find one-parameter families of solution curves of the following differential equations:

$$\frac{dy}{dx}\cos^2 x = \tan x - y$$

Answer

$\frac{dy}{dx}\cos^2 x = \tan x - y$

Given $\frac{dy}{dx}\cos^2 x = \tan x - y$

$$\Rightarrow \frac{dy}{dx}\cos^2 x \times \frac{1}{\cos^2 x} = (\tan x - y) \times \frac{1}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} = (\tan x - y)\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = \tan x \sec^2 x - y\sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + (\sec^2 x)y = \tan x \sec^2 x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \sec^2 x$ and $Q = \tan x \sec^2 x$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int \sec^2 x dx}$$

We have $\int \sec^2 x\, dx = \tan x + c$

$\therefore I.F = e^{\tan x}$

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(e^{\tan x}) = \int (\tan x \sec^2 x \times e^{\tan x})dx + c$$

$$\Rightarrow ye^{\tan x} = \int \tan x\, e^{\tan x} \sec^2 x\, dx + c$$

Let $\tan x = t$

$\Rightarrow \sec^2 x dx = dt$ [Differentiating both sides]

By substituting this in the above integral, we get

$ye^t = \int te^t dt + c$

$\Rightarrow ye^t = \int (t) \times (e^t) dt + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow ye^t = t\left[\int e^t dt\right] - \int \left[\frac{d}{dt}(t)\left(\int e^t dt\right)\right] dt + c$

$\Rightarrow ye^t = t \times e^t - \int 1(e^t) dt + c$

$\Rightarrow ye^t = te^t - \int e^t dt + c$

$\Rightarrow ye^t = te^t - e^t + c$

$\Rightarrow ye^t \times e^{-t} = (te^t - e^t + c)e^{-t}$

$\Rightarrow y = t - 1 + ce^{-t}$

$\therefore y = \tan x - 1 + ce^{-\tan x}$ [$\because t = \tan x$]

Thus, the solution of the given differential equation is $y = \tan x - 1 + ce^{-\tan x}$

36 J. Question

Find one-parameter families of solution curves of the following differential equations:

$e^{-y}\sec^2 y dy = dx + xdy$

Answer

$e^{-y}\sec^2 y dy = dx + xdy$

Given $e^{-y}\sec^2 y dy = dx + xdy$

$\Rightarrow e^{-y} \sec^2 y\, dy \times \frac{1}{dy} = (dx + xdy) \times \frac{1}{dy}$

$\Rightarrow e^{-y} \sec^2 y = \frac{dx}{dy} + x$

$\Rightarrow \frac{dx}{dy} + x = e^{-y} \sec^2 y$

This is a first order linear differential equation of the form

$\frac{dx}{dy} + Px = Q$

Here, P = 1 and $e^{-y}\sec^2 y$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dy}$

$\Rightarrow I.F = e^{\int 1 dy}$

$\Rightarrow I.F = e^{\int dy}$

We have $\int dy = y + c$

$\therefore$ I.F = e^y [$\because e^{\log x} = x$]

Hence, the solution of the differential equation is,

$x(I.F) = \int (Q \times I.F) dy + c$

$\Rightarrow x(e^y) = \int (e^{-y} \sec^2 y \times e^y) dy + c$

$\Rightarrow xe^y = \int \sec^2 y \, dy + c$

Recall $\int \sec^2 x \, dx = \tan x + c$

$\Rightarrow xe^y = \tan y + c$

$\Rightarrow xe^y \times \frac{1}{e^y} = (\tan y + c) \times \frac{1}{e^y}$

$\therefore x = (\tan y + c)e^{-y}$

Thus, the solution of the given differential equation is $x = (\tan y + c)e^{-y}$

36 K. Question

Find one-parameter families of solution curves of the following differential equations:

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

Answer

$x \log x \frac{dy}{dx} + y = 2 \log x$

Given $x \log x \frac{dy}{dx} + y = 2 \log x$

$\Rightarrow \left(x \log x \frac{dy}{dx} + y\right) \times \frac{1}{x \log x} = 2 \log x \times \frac{1}{x \log x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x \log x}\right) y = \frac{2}{x}$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, $P = \frac{1}{x \log x}$ and $Q = \frac{2}{x}$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int \frac{1}{x \log x} dx}$

Let t = log x

$\Rightarrow dt = \frac{1}{x} dx$ [Differentiating both sides]

By substituting this in the above integral, we get

$I.F = e^{\int \frac{1}{t} dt} + c$

We have $\int \frac{1}{x} dx = \log x + c$

$\Rightarrow$ I.F = $e^{\log t}$

$\Rightarrow$ I.F = t [$\because e^{\log x} = x$]

$\therefore$ I.F = log x [$\because$ t = log x]

Hence, the solution of the differential equation is,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + c$$

$$\Rightarrow y(\log x) = \int \left(\frac{2}{x} \times \log x\right)dx + c$$

$$\Rightarrow y\log x = 2\int \log x\left(\frac{1}{x}dx\right) + c$$

Let t = log x

$\Rightarrow dt = \frac{1}{x}dx$ [Differentiating both sides]

By substituting this in the above integral, we get

$$yt = 2\int tdt + c$$

We know $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow yt = 2\left(\frac{t^{1+1}}{1+1}\right) + c$$

$$\Rightarrow yt = 2\left(\frac{t^2}{2}\right) + c$$

$\Rightarrow yt = t^2 + c$

$$\Rightarrow yt \times \frac{1}{t} = (t^2 + c) \times \frac{1}{t}$$

$$\Rightarrow y = t + \frac{c}{t}$$

$\therefore y = \log x + \frac{c}{\log x}$ [$\because$ t = log x]

Thus, the solution of the given differential equation is $y = \log x + \frac{c}{\log x}$

36 L. Question

Find one-parameter families of solution curves of the following differential equations:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

Answer

$x\frac{dy}{dx} + 2y = x^2 \log x$

Given $x\frac{dy}{dx} + 2y = x^2 \log x$

$$\Rightarrow \left(x\frac{dy}{dx} + 2y\right) \times \frac{1}{x} = x^2 \log x \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = x\log x$$

$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right) y = x \log x$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, $P = \frac{2}{x}$ and Q = x log x

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int \frac{2}{x} dx}$

$\Rightarrow I.F = e^{2\int \frac{1}{x} dx}$

We have $\int \frac{1}{x} dx = \log x + c$

$\Rightarrow I.F = e^{2 \log x}$

$\Rightarrow I.F = e^{\log x^2}$ [∵ m log a = log a^m]

∴ I.F = x^2 [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F) dx + c$

$\Rightarrow y(x^2) = \int (x \log x \times x^2) dx + c$

$\Rightarrow yx^2 = \int x^3 \log x \, dx + c$

$\Rightarrow yx^2 = \int (\log x) \times (x^3) dx + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow yx^2 = \log x \left[\int x^3 dx\right] - \int \left[\frac{d}{dx}(\log x)\left(\int x^3 dx\right)\right] dx + c$

$\Rightarrow yx^2 = \log x \left[\frac{x^{3+1}}{3+1}\right] - \int \left[\frac{1}{x}\left(\frac{x^{3+1}}{3+1}\right)\right] dx + c$

$\Rightarrow yx^2 = \frac{x^4}{4} \log x - \int \left[\frac{1}{x}\left(\frac{x^4}{4}\right)\right] dx + c$

$\Rightarrow yx^2 = \frac{x^4}{4} \log x - \frac{1}{4} \int x^3 dx + c$

$\Rightarrow yx^2 = \frac{x^4}{4} \log x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$

$\Rightarrow yx^2 = \frac{x^4}{4} \log x - \frac{1}{16} x^4 + c$

$\Rightarrow yx^2 = \frac{x^4}{16}(4 \log x - 1) + c$

$\Rightarrow yx^2 \times \frac{1}{x^2} = \left[\frac{x^4}{16}(4\log x - 1) + c\right] \times \frac{1}{x^2}$

$\therefore y = \frac{x^2}{16}(4\log x - 1) + \frac{c}{x^2}$

Thus, the solution of the given differential equation is $y = \frac{x^2}{16}(4\log x - 1) + \frac{c}{x^2}$

37 A. Question

Solve each of the following initial value problems:

y' + y = e^x, $y(0) = \frac{1}{2}$

Answer

y' + y = e^x, $y(0) = \frac{1}{2}$

Given y' + y = e^x and $y(0) = \frac{1}{2}$

$\Rightarrow \frac{dy}{dx} + y = e^x$

$\Rightarrow \frac{dy}{dx} + (1)y = e^x$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, P = 1 and Q = e^x

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int dx}$

We have $\int dx = x + c$

∴ I.F = e^x

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F)dx + c$

$\Rightarrow y(e^x) = \int (e^x \times e^x)dx + c$

$\Rightarrow ye^x = \int e^{x+x}dx + c$

$\Rightarrow ye^x = \int e^{2x}dx + c$

Recall $\int e^x dx = e^x + c$

$\Rightarrow ye^x = \frac{e^{2x}}{2} + c$

$\Rightarrow ye^x \times e^{-x} = \left(\frac{e^{2x}}{2} + c\right)e^{-x}$

$$\Rightarrow y = \frac{e^{2x} \times e^{-x}}{2} + ce^{-x}$$

$$\therefore y = \frac{e^{x}}{2} + ce^{-x}$$

However, when x = 0, we have $y = \frac{1}{2}$

$$\Rightarrow \frac{1}{2} = \frac{e^{0}}{2} + ce^{0}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{2} + c$$

∴ c = 0

By substituting the value of c in the equation for y, we get

$$y = \frac{e^{x}}{2} + 0 \times e^{-x}$$

$$\therefore y = \frac{e^{x}}{2}$$

Thus, the solution of the given initial value problem is $y = \frac{e^{x}}{2}$

37 B. Question

Solve each of the following initial value problems:

$x\frac{dy}{dx} - y = \log x$, y(1) = 0

Answer

$x\frac{dy}{dx} - y = \log x$, y(1) = 0

Given $x\frac{dy}{dx} - y = \log x$ and y(1) = 0

$$\Rightarrow \left(x\frac{dy}{dx} - y\right) \times \frac{1}{x} = \log x \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{-1}{x}\right)y = \frac{\log x}{x}$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = -\frac{1}{x}$ and $Q = \frac{\log x}{x}$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int -\frac{1}{x} dx}$$

$$\Rightarrow I.F = e^{-\int \frac{1}{x} dx}$$

We have $\int \frac{1}{x} dx = \log x + c$

$\Rightarrow \text{I.F} = e^{-\log x}$

$\Rightarrow \text{I.F} = e^{\log x^{-1}}$ [∵ m log a = log a^m]

∴ I.F = x^{-1} [∵ $e^{\log x}$ = x]

Hence, the solution of the differential equation is,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + c$$

$$\Rightarrow y(x^{-1}) = \int \left(\frac{\log x}{x} \times x^{-1}\right) dx + c$$

$$\Rightarrow \frac{y}{x} = \int x^{-2} \log x \, dx + c$$

$$\Rightarrow \frac{y}{x} = \int (\log x) \times (x^{-2}) dx + c$$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$$\Rightarrow \frac{y}{x} = \log x \left[\int x^{-2} dx\right] - \int \left[\frac{d}{dx}(\log x)\left(\int x^{-2} dx\right)\right] dx + c$$

$$\Rightarrow \frac{y}{x} = \log x \left[\frac{x^{-2+1}}{-2+1}\right] - \int \left[\frac{1}{x}\left(\frac{x^{-2+1}}{-2+1}\right)\right] dx + c$$

$$\Rightarrow \frac{y}{x} = \log x \left[\frac{x^{-1}}{-1}\right] - \int \left[\frac{1}{x}\left(\frac{x^{-1}}{-1}\right)\right] dx + c$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{x}\log x + \int x^{-2} dx + c$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{x}\log x + \frac{x^{-2+1}}{-2+1} + c$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{x}\log x + \frac{x^{-1}}{-1} + c$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{x}\log x - \frac{1}{x} + c$$

$$\Rightarrow y = \left(-\frac{1}{x}\log x - \frac{1}{x} + c\right) x$$

∴ y = –log x – 1 + cx

However, when x = 1, we have y = 0

⇒ 0 = –log 1 – 1 + c(1)

⇒ 0 = –0 – 1 + c

⇒ 0 = –1 + c

∴ c = 1

By substituting the value of c in the equation for y, we get

y = –log x – 1 + (1)x

⇒ y = –log x – 1 + x

∴ y = x – 1 – log x

Thus, the solution of the given initial value problem is y = x – 1 – log x

37 C. Question

Solve each of the following initial value problems:

$\frac{dx}{dy} + 2y = e^{-2x} \sin x$, y(0) = 0

Answer

$\frac{dx}{dy} + 2y = e^{-2x} \sin x$, y(0) = 0

Given $\frac{dx}{dy} + 2y = e^{-2x} \sin x$ and y(0) = 0

$\Rightarrow \frac{dy}{dx} + (2)y = e^{-2x} \sin x$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, P = 2 and Q = $e^{-2x}\sin x$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int 2 dx}$

$\Rightarrow I.F = e^{2\int dx}$

We have $\int dx = x + c$

∴ I.F = e^{2x}

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F) dx + c$

$\Rightarrow y(e^{2x}) = \int (e^{-2x} \sin x \times e^{2x}) dx + c$

$\Rightarrow ye^{2x} = \int \sin x \, dx + c$

Recall $\int \sin x \, dx = -\cos x + c$

$\Rightarrow ye^{2x} = -\cos x + c$

$\Rightarrow ye^{2x} \times e^{-2x} = (-\cos x + c) \times e^{-2x}$

$\therefore y = (-\cos x + c)e^{-2x}$

However, when x = 0, we have y = 0

$\Rightarrow 0 = (-\cos 0 + c)e^{0}$

⇒ 0 = (–1 + c) × 1

⇒ 0 = –1 + c

∴ c = 1

By substituting the value of c in the equation for y, we get

$y = (-\cos x + 1)e^{-2x}$

$\therefore y = (1 - \cos x)e^{-2x}$

Thus, the solution of the given initial value problem is y = (1 – cos x)e^{-2x}

37 D. Question

Solve each of the following initial value problems:

$$x\frac{dy}{dx} - y = (x+1)e^{-x}, \ y(1) = 0$$

Answer

$x\frac{dy}{dx} - y = (x+1)e^{-x}$, y(1) = 0

Given $x\frac{dy}{dx} - y = (x+1)e^{-x}$ and y(1) = 0

$$\Rightarrow \left(x\frac{dy}{dx} - y\right) \times \frac{1}{x} = (x+1)e^{-x} \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \left(\frac{x+1}{x}\right)e^{-x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{-1}{x}\right)y = \left(\frac{x+1}{x}\right)e^{-x}$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = -\frac{1}{x}$ and $Q = \left(\frac{x+1}{x}\right)e^{-x}$

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int -\frac{1}{x} dx}$$

$$\Rightarrow I.F = e^{-\int \frac{1}{x} dx}$$

We have $\int \frac{1}{x} dx = \log x + c$

$$\Rightarrow I.F = e^{-\log x}$$

$\Rightarrow I.F = e^{\log x^{-1}}$ [∵ m log a = log a^m]

∴ I.F = x^{-1} [∵ $e^{\log x}$ = x]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(x^{-1}) = \int \left(\left(\frac{x+1}{x}\right)e^{-x} \times x^{-1}\right) dx + c$$

$$\Rightarrow \frac{y}{x} = \int \left(\frac{x+1}{x^2}\right)e^{-x} dx + c$$

$$\Rightarrow \frac{y}{x} = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x} dx + c$$

Let $\frac{e^{-x}}{x} = t$

$$\Rightarrow \left[(-e^{-x})\left(\frac{1}{x}\right) + e^{-x}\left(-\frac{1}{x^2}\right)\right]dx = dt$$

$$\Rightarrow -e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right)dx = dt$$

$$\Rightarrow \left(\frac{1}{x} + \frac{1}{x^2}\right)e^{-x}dx = -dt$$

By substituting this in the above integral, we get

$$\frac{y}{x} = \int -dt + c$$

$$\Rightarrow \frac{y}{x} = -\int dt + c$$

We know $\int dx = x + c$

$$\Rightarrow \frac{y}{x} = -\frac{e^{-x}}{x} + c \left[\because t = \frac{e^{-x}}{x}\right]$$

$$\Rightarrow y = \left(-\frac{e^{-x}}{x} + c\right)x$$

$\therefore$ y = $-e^{-x}$ + cx

However, when x = 1, we have y = 0

$\Rightarrow$ 0 = $-e^{-1}$ + c(1)

$\Rightarrow$ 0 = $-e^{-1}$ + c

$\therefore$ c = e^{-1}

By substituting the value of c in the equation for y, we get

y = $-e^{-x}$ + $(e^{-1})x$

$\therefore$ y = $xe^{-1} - e^{-x}$

Thus, the solution of the given initial value problem is y = $xe^{-1} - e^{-x}$

37 E. Question

Solve each of the following initial value problems:

$(1 + y^2)dx + \left(x - e^{-\tan^{-1} y}\right)dy = 0$, y(0) = 0

Answer

$(1 + y^2)dx + \left(x - e^{-\tan^{-1} y}\right)dy = 0$, y(0) = 0

Given $(1 + y^2)dx + \left(x - e^{-\tan^{-1} y}\right)dy = 0$ and y(0) = 0

$$\Rightarrow (1 + y^2) + \left(x - e^{-\tan^{-1} y}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = -(1 + y^2)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{(1 + y^2)}{\left(x - e^{-\tan^{-1} y}\right)}$$

$$\Rightarrow \frac{dx}{dy} = -\frac{\left(x - e^{-\tan^{-1} y}\right)}{(1 + y^2)}$$

$\Rightarrow \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{e^{-\tan^{-1}y}}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$

$\Rightarrow \frac{dx}{dy} + \left(\frac{1}{1+y^2}\right)x = \frac{e^{-\tan^{-1}y}}{1+y^2}$

This is a first order linear differential equation of the form

$\frac{dx}{dy} + Px = Q$

Here, $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{-\tan^{-1}y}}{1+y^2}$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int Pdy}$

$\Rightarrow I.F = e^{\int \frac{1}{1+y^2}dy}$

We have $\int \frac{1}{1+y^2}dy = \tan^{-1}y + c$

$\therefore I.F = e^{\tan^{-1}y}$

Hence, the solution of the differential equation is,

$x(I.F) = \int (Q \times I.F)dy + c$

$\Rightarrow x(e^{\tan^{-1}y}) = \int \left(\frac{e^{-\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\right)dy + c$

$\Rightarrow xe^{\tan^{-1}y} = \int \left(\frac{1}{1+y^2}\right)dy + c$

We know $\int \frac{1}{1+x^2}dx = \tan^{-1}x + c$

$\Rightarrow xe^{\tan^{-1}y} = \tan^{-1}y + c$

$\Rightarrow xe^{\tan^{-1}y} \times e^{-\tan^{-1}y} = (\tan^{-1}y + c) \times e^{-\tan^{-1}y}$

$\therefore x = (\tan^{-1}y + c)e^{-\tan^{-1}y}$

However, when x = 0, we have y = 0

$\Rightarrow 0 = (\tan^{-1}0 + c)e^{-\tan^{-1}0}$

$\Rightarrow 0 = (0 + c)e^0$

$\Rightarrow 0 = (c) \times 1$

$\therefore c = 0$

By substituting the value of c in the equation for x, we get

$x = (\tan^{-1}y + 0)e^{-\tan^{-1}y}$

$\therefore x = e^{-\tan^{-1}y}\tan^{-1}y$

Thus, the solution of the given initial value problem is $x = e^{-\tan^{-1}y}\tan^{-1}y$

37 F. Question

Solve each of the following initial value problems:

$$\frac{dy}{dx} + y\tan x = 2x + x^2\tan x,\ y(0) = 1$$

Answer

$\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$, y(0) = 1

Given $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$ and y(0) = 1

$$\Rightarrow \frac{dy}{dx} + (\tan x)y = 2x + x^2\tan x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = tan x and Q = 2x + x^2tan x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \tan x dx}$$

We have $\int \tan x\, dx = \log(\sec x) + c$

$$\Rightarrow I.F = e^{\log(\sec x)}$$

∴ I.F = sec x [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(\sec x) = \int ((2x + x^2\tan x) \times \sec x)dx + c$$

$$\Rightarrow y\sec x = \int (2x\sec x + x^2\tan x\sec x)dx + c$$

$$\Rightarrow y\sec x = \int 2x\sec x\, dx + \int x^2\tan x\sec x\, dx + c$$

$$\Rightarrow y\sec x = 2\int x\sec x\, dx + \int x^2\tan x\sec x\, dx + c$$

$$\Rightarrow y\sec x = 2\int (\sec x) \times (x)dx + \int x^2\tan x\sec x\, dx + c$$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int[f'(x)(\int g(x)dx)]dx + c$

$$\Rightarrow y\sec x = 2\left\{\sec x\left[\int x dx\right] - \int\left[\frac{d}{dx}(\sec x)\left(\int x dx\right)\right]dx\right\} + \int x^2\tan x\sec x\, dx + c$$

$$\Rightarrow y\sec x = 2\left\{\sec x\left[\frac{x^{1+1}}{1+1}\right] - \int\left[(\sec x\tan x)\left(\frac{x^{1+1}}{1+1}\right)\right]dx\right\} + \int x^2\tan x\sec x\, dx + c$$

$$\Rightarrow y\sec x = 2\left\{\frac{x^2}{2}\sec x - \int\left[(\sec x\tan x)\left(\frac{x^2}{2}\right)\right]dx\right\} + \int x^2\tan x\sec x\, dx + c$$

$$\Rightarrow y\sec x = 2\left\{\frac{x^2}{2}\sec x - \frac{1}{2}\int x^2 \sec x \tan x\, dx\right\} + \int x^2 \tan x \sec x\, dx + c$$

$$\Rightarrow y\sec x = x^2\sec x - \int x^2 \sec x \tan x\, dx + \int x^2 \tan x \sec x\, dx + c$$

⇒ y sec x = x^2sec x + c

$$\Rightarrow y\sec x \times \frac{1}{\sec x} = (x^2\sec x + c) \times \frac{1}{\sec x}$$

$$\Rightarrow y = x^2 + \frac{c}{\sec x}$$

∴ y = x^2 + c cos x

However, when x = 0, we have y = 1

⇒ 1 = 0^2 + c(cos 0)

⇒ 1 = 0 + c(1)

∴ c = 1

By substituting the value of c in the equation for y, we get

y = x^2 + (1)cos x

∴ y = x^2 + cos x

Thus, the solution of the given initial value problem is y = x^2 + cos x

37 G. Question

Solve each of the following initial value problems:

$\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$, y(0) = 1

Answer

$\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$, y(0) = 1

Given $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$ and y(0) = 1

$$\Rightarrow \frac{dy}{dx} + (\tan x)y = 2x + x^2\tan x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = tan x and Q = 2x + x^2tan x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \tan x dx}$$

We have $\int \tan x\, dx = \log(\sec x) + c$

$$\Rightarrow I.F = e^{\log(\sec x)}$$

∴ I.F = sec x [∵ $e^{\log x}$ = x]

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F)dx + c$

$\Rightarrow y(\sec x) = \int \left((2x + x^2 \tan x) \times \sec x\right)dx + c$

$\Rightarrow y \sec x = \int (2x \sec x + x^2 \tan x \sec x)dx + c$

$\Rightarrow y \sec x = \int 2x \sec x\, dx + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = 2\int x \sec x\, dx + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = 2\int (\sec x) \times (x)dx + \int x^2 \tan x \sec x\, dx + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int[f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow y \sec x = 2\left\{\sec x\left[\int xdx\right] - \int \left[\frac{d}{dx}(\sec x)\left(\int xdx\right)\right]dx\right\} + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = 2\left\{\sec x\left[\frac{x^{1+1}}{1+1}\right] - \int \left[(\sec x \tan x)\left(\frac{x^{1+1}}{1+1}\right)\right]dx\right\} + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = 2\left\{\frac{x^2}{2}\sec x - \int \left[(\sec x \tan x)\left(\frac{x^2}{2}\right)\right]dx\right\} + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = 2\left\{\frac{x^2}{2}\sec x - \frac{1}{2}\int x^2 \sec x \tan x\, dx\right\} + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = x^2 \sec x - \int x^2 \sec x \tan x\, dx + \int x^2 \tan x \sec x\, dx + c$

$\Rightarrow y \sec x = x^2 \sec x + c$

$\Rightarrow y \sec x \times \frac{1}{\sec x} = (x^2 \sec x + c) \times \frac{1}{\sec x}$

$\Rightarrow y = x^2 + \frac{c}{\sec x}$

$\therefore y = x^2 + c \cos x$

However, when x = 0, we have y = 1

$\Rightarrow 1 = 0^2 + c(\cos 0)$

$\Rightarrow 1 = 0 + c(1)$

$\therefore c = 1$

By substituting the value of c in the equation for y, we get

$y = x^2 + (1)\cos x$

$\therefore y = x^2 + \cos x$

Thus, the solution of the given initial value problem is $y = x^2 + \cos x$

37 H. Question

Solve each of the following initial value problems:

$$\frac{dy}{dx}+y\cot x=2\cos x,\ y\left(\frac{\pi}{2}\right)=0$$

Answer

$\frac{dy}{dx}+y\cot x=2\cos x, y\left(\frac{\pi}{2}\right)=0$

Given $\frac{dy}{dx}+y\cot x=2\cos x$ and $y\left(\frac{\pi}{2}\right)=0$

$\Rightarrow \frac{dy}{dx}+(\cot x)y=2\cos x$

This is a first order linear differential equation of the form

$\frac{dy}{dx}+Py=Q$

Here, P = cot x and Q = 2 cos x

The integrating factor (I.F) of this differential equation is,

$I.F=e^{\int Pdx}$

$\Rightarrow I.F=e^{\int \cot x dx}$

We have $\int \cot x\, dx=\log(\sin x)+c$

$\Rightarrow I.F=e^{\log(\sin x)}$

∴ I.F = sin x [∵ $e^{\log x}$ = x]

Hence, the solution of the differential equation is,

$y(I.F)=\int(Q\times I.F)dx+c$

$\Rightarrow y(\sin x)=\int(2\cos x\times\sin x)dx+c$

$\Rightarrow y\sin x=2\int \sin x\cos x\, dx+c$

Let sin x = t

⇒ cosxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$yt=2\int tdt+c$

Recall $\int x^n dx=\frac{x^{n+1}}{n+1}+c$

$\Rightarrow yt=2\left(\frac{t^{1+1}}{1+1}\right)+c$

$\Rightarrow yt=2\left(\frac{t^2}{2}\right)+c$

⇒ yt = t^2 + c

$\Rightarrow yt\times\frac{1}{t}=(t^2+c)\times\frac{1}{t}$

$\Rightarrow y=t+\frac{c}{t}$

$\therefore y = \sin x + \frac{c}{\sin x}$ [∵ t = sin x]

However, when $x = \frac{\pi}{2}$, we have y = 0

$\Rightarrow 0 = \sin\frac{\pi}{2} + \frac{c}{\sin\frac{\pi}{2}}$

$\Rightarrow 0 = 1 + \frac{c}{1}$

⇒ 0 = 1 + c

∴ c = –1

By substituting the value of c in the equation for y, we get

$y = \sin x + \left(\frac{-1}{\sin x}\right)$

$\Rightarrow y = \sin x - \frac{1}{\sin x}$

$\Rightarrow y = \frac{\sin^2 x - 1}{\sin x}$

$\Rightarrow y = \frac{-(1 - \sin^2 x)}{\sin x}$

$\Rightarrow y = \frac{-\cos^2 x}{\sin x}$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

∴ y = –cos x cot x

Thus, the solution of the given initial value problem is y = –cosec x cot x

37 I. Question

Solve each of the following initial value problems:

$$x\frac{dy}{dx} + y = x\cos x + \sin x,\ y\left(\frac{\pi}{2}\right) = 1$$

Answer

$x\frac{dy}{dx} + y = x\cos x + \sin x, y\left(\frac{\pi}{2}\right) = 1$

Given $x\frac{dy}{dx} + y = x\cos x + \sin x$ and $y\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow \left(x\frac{dy}{dx} + y\right) \times \frac{1}{x} = (x\cos x + \sin x) \times \frac{1}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{1}{x}\sin x$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x}\right)y = \cos x + \frac{1}{x}\sin x$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, $P = \frac{1}{x}$ and $Q = \cos x + \frac{1}{x}\sin x$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int P dx}$

$\Rightarrow I.F = e^{\int \frac{1}{x} dx}$

We have $\int \frac{1}{x} dx = \log x + c$

$\Rightarrow I.F = e^{\log x}$

$\therefore$ I.F = x [$\because e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(x) = \int \left(\left(\cos x + \frac{1}{x}\sin x\right) \times x\right) dx + c$$

$$\Rightarrow xy = \int (x\cos x + \sin x) dx + c$$

$$\Rightarrow xy = \int x\cos x \, dx + \int \sin x \, dx + c$$

$$\Rightarrow xy = \int (x) \times (\cos x) dx + \int \sin x \, dx + c$$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int [f'(x)(\int g(x)dx)]dx + c$

$$\Rightarrow xy = x\left[\int \cos x \, dx\right] - \int \left[\frac{d}{dx}(x)\left(\int \cos x \, dx\right)\right] dx + \int \sin x \, dx + c$$

$$\Rightarrow xy = x(\sin x) - \int [1(\sin x)] dx + \int \sin x \, dx + c$$

$$\Rightarrow xy = x\sin x - \int \sin x \, dx + \int \sin x \, dx + c$$

$\Rightarrow xy = x \sin x + c$

$$\Rightarrow xy \times \frac{1}{x} = (x\sin x + c) \times \frac{1}{x}$$

$$\therefore y = \sin x + \frac{c}{x}$$

However, when $x = \frac{\pi}{2}$, we have y = 1

$$\Rightarrow 1 = \sin\frac{\pi}{2} + \frac{c}{\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow 1 = 1 + \frac{2c}{\pi}$$

$$\Rightarrow \frac{2c}{\pi} = 0$$

$\therefore$ c = 0

By substituting the value of c in the equation for y, we get

$$y = \sin x + \left(\frac{0}{\sin x}\right)$$

$\therefore$ y = sin x

Thus, the solution of the given initial value problem is y = sin x

37 J. Question

Solve each of the following initial value problems:

$$\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x,\ y\left(\frac{\pi}{2}\right) = 0$$

Answer

$\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x,\ y\left(\frac{\pi}{2}\right) = 0$

Given $\frac{dy}{dx} + y\cot x = 4x\operatorname{cosec} x$ and $y\left(\frac{\pi}{2}\right) = 0$

$$\Rightarrow \frac{dy}{dx} + (\cot x)y = 4x\operatorname{cosec} x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = cot x and Q = 4x cosec x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \cot x dx}$$

We have $\int \cot x\, dx = \log(\sin x) + c$

$$\Rightarrow I.F = e^{\log(\sin x)}$$

∴ I.F = sin x [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(\sin x) = \int (4x\operatorname{cosec} x \times \sin x)dx + c$$

$$\Rightarrow y\sin x = 4\int \left(x \times \frac{1}{\sin x} \times \sin x\right)dx + c$$

$$\Rightarrow y\sin x = 4\int x dx + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow y\sin x = 4\left(\frac{x^{1+1}}{1+1}\right) + c$$

$$\Rightarrow y\sin x = 4\left(\frac{x^2}{2}\right) + c$$

⇒ y sin x = $2x^2$ + c

$$\Rightarrow y\sin x \times \frac{1}{\sin x} = (2x^2 + c) \times \frac{1}{\sin x}$$

∴ y = ($2x^2$ + c) cosec x

However, when $x = \frac{\pi}{2}$, we have y = 0

$\Rightarrow 0=\left[2\left(\frac{\pi}{2}\right)^2+c\right]\operatorname{cosec}\frac{\pi}{2}$

$\Rightarrow 0=\left[2\times\frac{\pi^2}{4}+c\right]\times 1$

$\Rightarrow 0=\frac{\pi^2}{2}+c$

$\therefore c=-\frac{\pi^2}{2}$

By substituting the value of c in the equation for y, we get

$y=\left(2x^2+\left(-\frac{\pi^2}{2}\right)\right)\operatorname{cosec}x$

$\therefore y=\left(2x^2-\frac{\pi^2}{2}\right)\operatorname{cosec}x$

Thus, the solution of the given initial value problem is $y=\left(2x^2-\frac{\pi^2}{2}\right)\operatorname{cosec}x$

37 K. Question

Solve each of the following initial value problems:

$\frac{dy}{dx}+2y\tan x=\sin x$, y = 0 when $x=\frac{\pi}{3}$

Answer

xi. $\frac{dy}{dx}+2y\tan x=\sin x$, y = 0 when $x=\frac{\pi}{3}$

Given $\frac{dy}{dx}+2y\tan x=\sin x$ and $y\left(\frac{\pi}{3}\right)=0$

$\Rightarrow \frac{dy}{dx}+(2\tan x)y=\sin x$

This is a first order linear differential equation of the form

$\frac{dy}{dx}+Py=Q$

Here, P = 2 tan x and Q = sin x

The integrating factor (I.F) of this differential equation is,

$I.F=e^{\int Pdx}$

$\Rightarrow I.F=e^{\int 2\tan x dx}$

$\Rightarrow I.F=e^{2\int \tan x dx}$

We have $\int \tan x\, dx=\log(\sec x)+c$

$\Rightarrow I.F=e^{2\log(\sec x)}$

$\Rightarrow I.F=e^{\log(\sec x)^2}$ [$\because$ m log a = log a^m]

$\Rightarrow I.F=e^{\log(\sec^2 x)}$

$\therefore$ I.F = $\sec^2 x$ [$\because e^{\log x}=x$]

Hence, the solution of the differential equation is,

$y(\mathrm{I.F}) = \int (Q \times \mathrm{I.F})dx + c$

$\Rightarrow y(\sec^2 x) = \int (\sec^2 x \times \sin x)dx + c$

$\Rightarrow y\sec^2 x = \int \left(\frac{1}{\cos^2 x} \times \sin x\right)dx + c$

$\Rightarrow y\sec^2 x = \int \left(\frac{1}{\cos x} \times \frac{\sin x}{\cos x}\right)dx + c$

$\Rightarrow y\sec^2 x = \int \sec x \tan x \, dx + c$

Recall $\int \sec x \tan x \, dx = \sec x + c$

$\Rightarrow y\sec^2 x = \sec x + c$

$\Rightarrow y\sec^2 x \times \frac{1}{\sec^2 x} = (\sec x + c) \times \frac{1}{\sec^2 x}$

$\Rightarrow y = \frac{1}{\sec x} + \frac{c}{\sec^2 x}$

$\therefore y = \cos x + c\cos^2 x$

However, when $x = \frac{\pi}{3}$, we have y = 0

$\Rightarrow 0 = \cos\frac{\pi}{3} + c\cos^2\frac{\pi}{3}$

$\Rightarrow 0 = \frac{1}{2} + c\left(\frac{1}{2}\right)^2$

$\Rightarrow 0 = \frac{1}{2} + \frac{1}{4}c$

$\Rightarrow \frac{1}{4}c = -\frac{1}{2}$

$\therefore c = -2$

By substituting the value of c in the equation for y, we get

$y = \cos x + (-2)\cos^2 x$

$\therefore y = \cos x - 2\cos^2 x$

Thus, the solution of the given initial value problem is $y = \cos x - 2\cos^2 x$

37 L. Question

Solve each of the following initial value problems:

$\frac{dy}{dx} - 3y\cot x = \sin 2x$, y = 2 when $x = \frac{\pi}{2}$

Answer

$\frac{dy}{dx} - 3y\cot x = \sin 2x$, y = 2 when $x = \frac{\pi}{2}$

Given $\frac{dy}{dx} - 3y\cot x = \sin 2x$ and $y\left(\frac{\pi}{2}\right) = 2$

$\Rightarrow \frac{dy}{dx} + (-3\cot x)y = \sin 2x$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = –3 cot x and Q = sin 2x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int -3 \cot x dx}$$

$$\Rightarrow I.F = e^{-3 \int \cot x dx}$$

We have $\int \cot x \, dx = \log(\sin x) + c$

$$\Rightarrow I.F = e^{-3 \log(\sin x)}$$

$\Rightarrow I.F = e^{\log(\sin x)^{-3}}$ [∵ m log a = log a^m]

$$\Rightarrow I.F = e^{\log\left(\frac{1}{\sin x}\right)^3}$$

$$\Rightarrow I.F = e^{\log(\operatorname{cosec} x)^3}$$

∴ I.F = $\operatorname{cosec}^3 x$ [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(\operatorname{cosec}^3 x) = \int (\sin 2x \times \operatorname{cosec}^3 x) dx + c$$

$$\Rightarrow y \operatorname{cosec}^3 x = \int \left(2 \sin x \cos x \times \frac{1}{\sin^3 x}\right) dx + c$$

$$\Rightarrow y \operatorname{cosec}^3 x = 2 \int \frac{\cos x}{\sin^2 x} dx + c$$

$$\Rightarrow y \operatorname{cosec}^3 x = 2 \int \frac{1}{\sin x} \times \frac{\cos x}{\sin x} dx + c$$

$$\Rightarrow y \operatorname{cosec}^3 x = 2 \int \operatorname{cosec} x \cot x \, dx + c$$

Recall $\int \operatorname{cosec} x \tan x \, dx = -\operatorname{cosec} x + c$

⇒ ycosec^3x = 2(–cosec x) + c

⇒ ycosec^3x = –2cosec x + c

$$\Rightarrow y \operatorname{cosec}^3 x \times \frac{1}{\operatorname{cosec}^3 x} = (-2 \operatorname{cosec} x + c) \times \frac{1}{\operatorname{cosec}^3 x}$$

$$\Rightarrow y = \frac{-2}{\operatorname{cosec}^2 x} + \frac{c}{\operatorname{cosec}^3 x}$$

∴ y = –2$\sin^2$x + c$\sin^3$x

However, when $x = \frac{\pi}{2}$, we have y = 2

$$\Rightarrow 2 = -2 \sin^2 \frac{\pi}{2} + c \sin^3 \frac{\pi}{2}$$

⇒ 2 = –2$(1)^2$ + c$(1)^3$

$\Rightarrow 2 = -2 + c$

$\therefore c = 4$

By substituting the value of c in the equation for y, we get

$y = -2\sin^2 x + (4)\sin^3 x$

$\therefore y = -2\sin^2 x + 4\sin^3 x$

Thus, the solution of the given initial value problem is $y = -2\sin^2 x + 4\sin^3 x$

37 M. Question

Solve each of the following initial value problems:

$$\frac{dy}{dx} + y\cot x = 2\cos x,\ y\left(\frac{\pi}{2}\right) = 0$$

Answer

$$\frac{dy}{dx} + y\cot x = 2\cos x,\ y\left(\frac{\pi}{2}\right) = 0$$

Given $\frac{dy}{dx} + y\cot x = 2\cos x$ and $y\left(\frac{\pi}{2}\right) = 0$

$$\Rightarrow \frac{dy}{dx} + (\cot x)y = 2\cos x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = cot x and Q = 2 cos x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \cot x dx}$$

We have $\int \cot x\, dx = \log(\sin x) + c$

$$\Rightarrow I.F = e^{\log(\sin x)}$$

$\therefore$ I.F = sin x [$\because e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F)dx + c$$

$$\Rightarrow y(\sin x) = \int (2\cos x \times \sin x)dx + c$$

$$\Rightarrow y\sin x = 2\int \sin x \cos x\, dx + c$$

Let sin x = t

$\Rightarrow$ cosxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$$yt = 2\int t dt + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow yt = 2\left(\frac{t^{1+1}}{1+1}\right) + c$

$\Rightarrow yt = 2\left(\frac{t^2}{2}\right) + c$

⇒ yt = t^2 + c

$\Rightarrow yt \times \frac{1}{t} = (t^2 + c) \times \frac{1}{t}$

$\Rightarrow y = t + \frac{c}{t}$

$\therefore y = \sin x + \frac{c}{\sin x}$ [∵ t = sin x]

However, when $x = \frac{\pi}{2}$, we have y = 0

$\Rightarrow 0 = \sin\frac{\pi}{2} + \frac{c}{\sin\frac{\pi}{2}}$

$\Rightarrow 0 = 1 + \frac{c}{1}$

⇒ 0 = 1 + c

∴ c = –1

By substituting the value of c in the equation for y, we get

$y = \sin x + \left(\frac{-1}{\sin x}\right)$

$\Rightarrow y = \sin x - \frac{1}{\sin x}$

$\Rightarrow y = \frac{\sin^2 x - 1}{\sin x}$

$\Rightarrow y = \frac{-(1 - \sin^2 x)}{\sin x}$

$\Rightarrow y = \frac{-\cos^2 x}{\sin x}$ [∵ $\sin^2\theta + \cos^2\theta = 1$]

∴ y = –cos x cot x

Thus, the solution of the given initial value problem is y = –cosec x cot x

37 N. Question

Solve each of the following initial value problems:

dy = cos x (2 – y cosec x)dx

Answer

dy = cos x (2 – y cosec x)dx

Given dy = cos x (2 – y cosec x)dx

$\Rightarrow \frac{dy}{dx} = \cos x\,(2 - y\,\mathrm{cosec}\,x)$

$\Rightarrow \frac{dy}{dx} = 2\cos x - y\cos x\,\mathrm{cosec}\,x$

$\Rightarrow \frac{dy}{dx} = 2\cos x - y\cos x \times \frac{1}{\sin x}$

$\Rightarrow \frac{dy}{dx} = 2\cos x - y\cot x$

$\Rightarrow \frac{dy}{dx} + y\cot x = 2\cos x$

$\Rightarrow \frac{dy}{dx} + (\cot x)y = 2\cos x$

This is a first order linear differential equation of the form

$\frac{dy}{dx} + Py = Q$

Here, P = cot x and Q = 2 cos x

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int Pdx}$

$\Rightarrow I.F = e^{\int \cot x dx}$

We have $\int \cot x\, dx = \log(\sin x) + c$

$\Rightarrow I.F = e^{\log(\sin x)}$

∴ I.F = sin x [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$y(I.F) = \int (Q \times I.F)dx + c$

$\Rightarrow y(\sin x) = \int (2\cos x \times \sin x)dx + c$

$\Rightarrow y\sin x = 2\int \sin x\cos x\, dx + c$

Let sin x = t

⇒ cosxdx = dt [Differentiating both sides]

By substituting this in the above integral, we get

$yt = 2\int tdt + c$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\Rightarrow yt = 2\left(\frac{t^{1+1}}{1+1}\right) + c$

$\Rightarrow yt = 2\left(\frac{t^2}{2}\right) + c$

⇒ yt = t^2 + c

$\Rightarrow yt \times \frac{1}{t} = (t^2 + c) \times \frac{1}{t}$

$\Rightarrow y = t + \frac{c}{t}$

$\therefore y = \sin x + \frac{c}{\sin x}$ [∵ t = sin x]

Thus, the solution of the given differential equation is $y = \sin x + \frac{c}{\sin x}$

37 O. Question

Solve each of the following initial value problems:

$\tan x \frac{dy}{dx} = 2x \tan x + x^2 - y$, tan x ≠ 0 given that y = 0 when $x = \frac{\pi}{2}$

Answer

$\tan x \frac{dy}{dx} = 2x \tan x + x^2 - y$, tan x ≠ 0 given that y = 0 when $x = \frac{\pi}{2}$

Given $\tan x \frac{dy}{dx} = 2x \tan x + x^2 - y$ and $y\left(\frac{\pi}{2}\right) = 0$

$$\Rightarrow \tan x \frac{dy}{dx} \times \frac{1}{\tan x} = (2x \tan x + x^2 - y) \times \frac{1}{\tan x}$$

$$\Rightarrow \frac{dy}{dx} = 2x + \frac{x^2}{\tan x} - \frac{y}{\tan x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{\tan x} = 2x + \frac{x^2}{\tan x}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{\tan x}\right) y = 2x + \frac{x^2}{\tan x}$$

$$\Rightarrow \frac{dy}{dx} + (\cot x) y = 2x + x^2 \cot x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = cot x and Q = 2x + x^2 cot x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \cot x dx}$$

We have $\int \cot x \, dx = \log(\sin x) + c$

$$\Rightarrow I.F = e^{\log(\sin x)}$$

∴ I.F = sin x [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(\sin x) = \int \left((2x + x^2 \cot x) \times \sin x\right) dx + c$$

$$\Rightarrow y \sin x = \int (2x \sin x + x^2 \cot x \sin x) dx + c$$

$$\Rightarrow y \sin x = \int (2x \sin x + x^2 \cos x) dx + c$$

$\Rightarrow y\sin x = \int 2x\sin x\,dx + \int x^2\cos x\,dx + c$

$\Rightarrow y\sin x = \int 2x\sin x\,dx + \int (x^2)\times(\cos x)dx + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int[f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow y\sin x = \int 2x\sin x\,dx + x^2\left[\int \cos x\,dx\right] - \int\left[\frac{d}{dx}(x^2)\left(\int \cos x\,dx\right)\right]dx + c$

$\Rightarrow y\sin x = \int 2x\sin x\,dx + x^2(\sin x) - \int[2x(\sin x)]dx + c$

$\Rightarrow y\sin x = \int 2x\sin x\,dx + x^2\sin x - \int 2x\sin x\,dx + c$

$\Rightarrow$ y sin x = x^2sin x + c

$\Rightarrow y\sin x \times \frac{1}{\sin x} = (x^2\sin x + c)\times\frac{1}{\sin x}$

$\therefore y = x^2 + \frac{c}{\sin x}$

However, when $x = \frac{\pi}{2}$, we have y = 0

$\Rightarrow 0 = \left(\frac{\pi}{2}\right)^2 + \frac{c}{\sin\frac{\pi}{2}}$

$\Rightarrow 0 = \frac{\pi^2}{4} + \frac{c}{1}$

$\Rightarrow 0 = \frac{\pi^2}{4} + c$

$\therefore c = -\frac{\pi^2}{4}$

By substituting the value of c in the equation for y, we get

$y = x^2 + \left(\frac{-\frac{\pi^2}{4}}{\sin x}\right)$

$\therefore y = x^2 - \frac{\pi^2}{4\sin x}$

Thus, the solution of the given initial value problem is $y = x^2 - \frac{\pi^2}{4\sin x}$

38. Question

Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$.

Answer

Given $x\frac{dy}{dx} + 2y = x^2$

$\Rightarrow \left(x\frac{dy}{dx} + 2y\right)\times\frac{1}{x} = x^2\times\frac{1}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = x$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right) y = x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \frac{2}{x}$ and Q = x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int \frac{2}{x} dx}$$

$$\Rightarrow I.F = e^{2 \int \frac{1}{x} dx}$$

We have $\int \frac{1}{x} dx = \log x + c$

$$\Rightarrow I.F = e^{2 \log x}$$

$\Rightarrow I.F = e^{\log x^2}$ [∵ m log a = log a^m]

∴ I.F = x^2 [∵ $e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(x^2) = \int (x \times x^2) dx + c$$

$$\Rightarrow yx^2 = \int x^3 dx + c$$

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\Rightarrow yx^2 = \frac{x^{3+1}}{3+1} + c$$

$$\Rightarrow yx^2 = \frac{x^4}{4} + c$$

$$\Rightarrow yx^2 \times \frac{1}{x^2} = \left(\frac{x^4}{4} + c\right) \times \frac{1}{x^2}$$

$$\therefore y = \frac{x^2}{4} + \frac{c}{x^2}$$

Thus, the solution of the given differential equation is $y = \frac{x^2}{4} + \frac{c}{x^2}$

39. Question

Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$

Answer

Given $\frac{dy}{dx} - y = \cos x$

$$\Rightarrow \frac{dy}{dx} + (-1)y = \cos x$$

This is a first order linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, P = –1 and Q = cos x

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dx}$$

$$\Rightarrow I.F = e^{\int -1 dx}$$

$$\Rightarrow I.F = e^{-\int dx}$$

We have $\int dx = x + c$

∴ I.F = e^{-x}

Hence, the solution of the differential equation is,

$$y(I.F) = \int (Q \times I.F) dx + c$$

$$\Rightarrow y(e^{-x}) = \int (\cos x \times e^{-x}) dx + c$$

$$\Rightarrow ye^{-x} = \int e^{-x} \cos x \, dx + c$$

$$\Rightarrow ye^{-x} = \int (e^{-x}) \times (\cos x) dx + c$$

Let $I = \int (e^{-x}) \times (\cos x) dx$

$$\Rightarrow I = e^{-x}\left[\int \cos x \, dx\right] - \int \left[\frac{d}{dx}(e^{-x})\left(\int \cos x \, dx\right)\right] dx$$

$$\Rightarrow I = e^{-x}(\sin x) - \int [-e^{-x}(\sin x)] dx$$

$$\Rightarrow I = e^{-x} \sin x + \int e^{-x} \sin x \, dx$$

$$\Rightarrow I = e^{-x} \sin x + e^{-x}\left[\int \sin x \, dx\right] - \int \left[\frac{d}{dx}(e^{-x})\left(\int \sin x \, dx\right)\right] dx$$

$$\Rightarrow I = e^{-x} \sin x + e^{-x}[-\cos x] - \int [-e^{-x}(-\cos x)] dx$$

$$\Rightarrow I = e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

⇒ I = e^{-x}(sin x – cos x) – I

⇒ 2I = e^{-x}(sin x – cos x)

$$\therefore I = \frac{e^{-x}}{2}(\sin x - \cos x)$$

By substituting the value of I in the original integral, we get

$$\Rightarrow ye^{-x} = \frac{e^{-x}}{2}(\sin x - \cos x) + c$$

$$\Rightarrow ye^{-x} \times e^{x} = \left[\frac{e^{-x}}{2}(\sin x - \cos x) + c\right] e^{x}$$

$\therefore y=\frac{1}{2}(\sin x-\cos x)+ce^{x}$

Thus, the solution of the given differential equation is $y=\frac{1}{2}(\sin x-\cos x)+ce^{x}$

40. Question

Solve the differential equation $(y+3x^{2})\frac{dx}{dy}=x$

Answer

Given $(y+3x^{2})\frac{dx}{dy}=x$

$\Rightarrow \frac{dx}{dy}=\frac{x}{y+3x^{2}}$

$\Rightarrow \frac{dy}{dx}=\frac{y+3x^{2}}{x}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x}+\frac{3x^{2}}{x}$

$\Rightarrow \frac{dy}{dx}-\frac{y}{x}=3x$

$\Rightarrow \frac{dy}{dx}+\left(-\frac{1}{x}\right)y=3x$

This is a first order linear differential equation of the form

$\frac{dy}{dx}+Py=Q$

Here, $P=-\frac{1}{x}$ and Q = 3x

The integrating factor (I.F) of this differential equation is,

$I.F=e^{\int Pdx}$

$\Rightarrow I.F=e^{\int -\frac{1}{x}dx}$

$\Rightarrow I.F=e^{-\int \frac{1}{x}dx}$

We have $\int \frac{1}{x}dx=\log x+c$

$\Rightarrow I.F=e^{-\log x}$

$\Rightarrow I.F=e^{\log x^{-1}}$ [$\because$ m log a = log a^m]

$\therefore$ I.F = x^{-1} [$\because e^{\log x}=x$]

Hence, the solution of the differential equation is,

$y(I.F)=\int (Q\times I.F)dx+c$

$\Rightarrow y(x^{-1})=\int (3x\times x^{-1})dx+c$

$\Rightarrow yx^{-1}=\int 3dx+c$

$\Rightarrow yx^{-1}=3\int dx+c$

We know $\int dx = x + c$

$\Rightarrow yx^{-1} = 3x + c$

$\Rightarrow yx^{-1} \times x = (3x + c)x$

$\therefore y = (3x + c)x$

Thus, the solution of the given differential equation is y = (3x + c)x

41. Question

Find the particular solution of the differential equation $\frac{dx}{dy} + x\cot y = 2y + y^2\cot y$, $y \neq 0$ given that x = 0 when $y = \frac{\pi}{2}$

Answer

Given $\frac{dx}{dy} + x\cot y = 2y + y^2\cot y$

$$\Rightarrow \frac{dx}{dy} + (\cot y)x = 2y + y^2\cot y$$

This is a first order linear differential equation of the form

$$\frac{dx}{dy} + Px = Q$$

Here, P = cot y and Q = 2y + y^2cot y

The integrating factor (I.F) of this differential equation is,

$$I.F = e^{\int P dy}$$

$$\Rightarrow I.F = e^{\int \cot y dy}$$

We have $\int \cot y\, dy = \log(\sin y) + c$

$$\Rightarrow I.F = e^{\log(\sin y)}$$

$\therefore$ I.F = sin y [$\because e^{\log x} = x$]

Hence, the solution of the differential equation is,

$$x(I.F) = \int (Q \times I.F)dy + c$$

$$\Rightarrow x(\sin y) = \int \left((2y + y^2\cot y) \times \sin y\right)dy + c$$

$$\Rightarrow x\sin y = \int 2y\sin y\, dy + \int y^2\cot y\sin y\, dy + c$$

$$\Rightarrow x\sin y = \int 2y\sin y\, dy + \int y^2\left(\frac{\cos y}{\sin y}\right)\sin y\, dy + c$$

$$\Rightarrow x\sin y = \int 2y\sin y\, dy + \int y^2\cos y\, dy + c$$

$$\Rightarrow x\sin y = \int 2y\sin y\, dy + \int (y^2) \times (\cos y)dy + c$$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int[f'(x)(\int g(x)dx)]dx + c$

$$\Rightarrow x\sin y = \int 2y\sin y\,dy + y^2\left[\int \cos y\,dy\right] - \int \left[\frac{d}{dy}(y^2)\left(\int \cos y\,dy\right)\right]dy + c$$

$$\Rightarrow x\sin y = \int 2y\sin y\,dy + y^2(\sin y) - \int [2y(\sin y)]dy + c$$

$$\Rightarrow x\sin y = \int 2y\sin y\,dy + y^2\sin y - \int 2y\sin y\,dy + c$$

$\Rightarrow$ x sin y = y^2 sin y + c

$$\Rightarrow x\sin y \times \frac{1}{\sin y} = (y^2\sin y + c) \times \frac{1}{\sin y}$$

$$\Rightarrow x = y^2 + \frac{c}{\sin y}$$

$\therefore$ x = y^2 + c cosec y

However, when $y = \frac{\pi}{2}$, we have x = 0.

$$\Rightarrow 0 = \left(\frac{\pi}{2}\right)^2 + c\,\mathrm{cosec}\frac{\pi}{2}$$

$$\Rightarrow 0 = \frac{\pi^2}{4} + c(1)$$

$$\Rightarrow \frac{\pi^2}{4} + c = 0$$

$$\therefore c = -\frac{\pi^2}{4}$$

By substituting the value of c in the equation for x, we get

$$x = y^2 + \left(-\frac{\pi^2}{4}\right)\mathrm{cosec}y$$

$$\therefore x = y^2 - \frac{\pi^2}{4}\mathrm{cosec}y$$

Thus, the solution of the given differential equation is $x = y^2 - \frac{\pi^2}{4}\mathrm{cosec}y$

42. Question

Solve the following differential equation $(\cot^{-1} y + x)dy = (1 + y^2)dx$

Answer

Given $(\cot^{-1}y + x)dy = (1 + y^2)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + y^2}{\cot^{-1}y + x}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cot^{-1}y + x}{1 + y^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cot^{-1}y}{1 + y^2} + \frac{x}{1 + y^2}$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{1 + y^2} = \frac{\cot^{-1}y}{1 + y^2}$$

$\Rightarrow \frac{dx}{dy} + \left(-\frac{1}{1+y^2}\right)x = \frac{\cot^{-1}y}{1+y^2}$

This is a first order linear differential equation of the form

$\frac{dx}{dy} + Px = Q$

Here, $P = -\frac{1}{1+y^2}$ and $Q = \frac{\cot^{-1}y}{1+y^2}$

The integrating factor (I.F) of this differential equation is,

$I.F = e^{\int Pdy}$

$\Rightarrow I.F = e^{\int -\frac{1}{1+y^2}dy}$

We have $\int -\frac{1}{1+y^2}dy = \cot^{-1}y + c$

$\therefore I.F = e^{\cot^{-1}y}$

Hence, the solution of the differential equation is,

$x(I.F) = \int (Q \times I.F)dy + c$

$\Rightarrow x\left(e^{\cot^{-1}y}\right) = \int \left(\frac{\cot^{-1}y}{1+y^2} \times e^{\cot^{-1}y}\right)dy + c$

$\Rightarrow x\left(e^{\cot^{-1}y}\right) = \int \cot^{-1}y\, e^{\cot^{-1}y}\left(\frac{1}{1+y^2}\right)dy + c$

Let $\cot^{-1}y = t$

$\Rightarrow -\frac{1}{1+y^2}dy = dt$ [Differentiating both sides]

$\Rightarrow \frac{1}{1+y^2}dy = -dt$

By substituting this in the above integral, we get

$xe^t = \int -te^t dt + c$

$\Rightarrow xe^t = -\int (t) \times (e^t)dt + c$

Recall $\int f(x)g(x) = f(x)[\int g(x)dx] - \int[f'(x)(\int g(x)dx)]dx + c$

$\Rightarrow xe^t = -\left\{t\left[\int e^t dt\right] - \int\left[\frac{d}{dt}(t)\left(\int e^t dt\right)\right]dt\right\} + c$

$\Rightarrow xe^t = -\left\{t \times e^t - \int 1(e^t)dt\right\} + c$

$\Rightarrow xe^t = -\left\{te^t - \int e^t dt\right\} + c$

$\Rightarrow xe^t = -\{te^t - e^t\} + c$

$\Rightarrow xe^t = -te^t + e^t + c$

$\Rightarrow xe^t \times e^{-t} = (-te^t + e^t + c)e^{-t}$

$\Rightarrow x = -t + 1 + ce^{-t}$

$\therefore x = -\cot^{-1}y + 1 + ce^{-\cot^{-1}y}$ [$\because t = \cot^{-1}y$]

Thus, the solution of the given differential equation is $x = -\cot^{-1}y + 1 + ce^{-\cot^{-1}y}$

Exercise 22.11

1. Question

The surface area of a balloon being inflated, changes at a rate proportional to time t. If initially its radius is 1 unit and after 3 seconds it is 2 units, find the radius after time t.

Answer

Let the surface area of the balloon be S.

$\therefore S = 4\pi r^2$

According to the question,

$\frac{dS}{dt} \propto t$

$\Rightarrow \frac{dS}{dt} = kt$

$\Rightarrow \frac{d(4\pi r^2)}{dt} = kt$

$\Rightarrow 8\pi r\frac{dr}{dt} = kt$

$\Rightarrow 8\pi r dr = kt dt$

Integrating both sides, we have

$\Rightarrow 8\pi\int r dr = k\int t dt$

$\Rightarrow 8\pi \frac{r^2}{2} = \frac{kt^2}{2} + c$

$\Rightarrow 4\pi r^2 = \frac{kt^2}{2} + c$ ......(1)

Given, we have r = 1 unit when the t = 0 sec

Putting the value in equation (1)

$\therefore 4\pi r^2 = \frac{kt^2}{2} + c$

$\Rightarrow 4\pi (1)^2 = k \times 0 + c$

$\Rightarrow c = 4\pi$ ......(2)

Putting the value of c in equation (1) we have,

$4\pi r^2 = \frac{kt^2}{2} + 4\pi$ ......(3)

Given, we have r = 2 units when t = 3 sec

$\therefore 4\pi(2)^2 = \frac{k(3)^2}{2} + 4\pi$

$\Rightarrow 16\pi - 4\pi = \frac{9k}{2}$

$\Rightarrow 12\pi = \frac{9k}{2}$

$\Rightarrow k = \frac{24\pi}{9} = \frac{8\pi}{3}$ ......(4)

Now, putting the value of k in equation (2), We have,

$$4\pi r^2 = \frac{8\pi t^2}{6} + 4\pi$$

$\Rightarrow 4\pi r^2 - 4\pi = \frac{8\pi t^2}{6}$

$\Rightarrow 4\pi(r^2 - 1) = \frac{8\pi t^2}{6}$

$\Rightarrow r^2 - 1 = \frac{t^2}{3}$

$\Rightarrow r^2 = \frac{t^2}{3} + 1$

$\therefore r = \sqrt{\frac{t^2}{3} + 1}$

2. Question

A population grows at the rate of 5% per year. How long does it take for the population to double?

Answer

Let the initial population be P_o.

And the population after time t be P.

According to question,

$$\frac{dP}{dt} = 5\% \times P$$

$\Rightarrow \frac{dP}{dt} = \frac{5}{100}P$

$\Rightarrow \frac{dP}{dt} = \frac{P}{20}$

$\Rightarrow 20\frac{dP}{P} = dt$

Integrating both sides, we have

$\Rightarrow 20\int\frac{dP}{P} = \int dt$

$\Rightarrow 20\log|P| = t + c$ ......(1)

Given, we have $P = P_o$ when $t = 0$ sec

Putting the value in equation (1)

$\therefore 20\log|P| = t + c$

$\Rightarrow 20\log|P_o| = 0 + c$

$\Rightarrow c = 20\log|P_o|$ ......(2)

Putting the value of c in equation (1) we have,

$20\log|P| = t + 20\log|P_o|$

$\Rightarrow 20\log|P| - 20\log|P_o| = t$

$\Rightarrow 20(\log|P| - \log|P_o|) = t$

$[\log a - \log b = \log(\frac{a}{b})]$

$\Rightarrow 20\log(\frac{P}{P_o}) = t$ ......(3)

Now, for the population to be doubled

Let P = $2P_o$ at time t1

∴ t = 20log $\left(\frac{P}{P_o}\right)$

⇒ t1 = 20log $\left(\frac{2P_o}{P_o}\right)$

⇒ t1 = 20 log2

∴ time required for the population to be doubled = 20 log2 years

3. Question

The rate of growth of a population is proportional to the number present. If the population of a city doubled in the past 25 years, and the present population is 100000, when will the city have a population of 500000?

[Given $\log_e 5 = 1.609$, $\log_e 2 = 0.6931$]

Answer

Let the initial population be P_o .

And the population after time t be P.

According to question,

$$\frac{dP}{dt} \propto t$$

⇒ $\frac{dP}{dt} = kP$

⇒ $dP = kPdt$

⇒ $\frac{dP}{P} = kdt$

Integrating both sides, we have

⇒ $\int\frac{dP}{P} = k\int dt$

⇒ log|P| = kt + c ......(1)

Given, we have P = P_o when t = 0 sec

Putting the value in equation (1)

∴ log|P| = kt + c

⇒ log|P_o| = 0 + c

⇒ c = log|P_o| ......(2)

Putting the value of c in equation (1) we have,

log|P| = kt + log|P_o|

⇒ log|P| – log|P_o| = k t

⇒ (log |P| – log|P_o|) = kt [$\log a - \log b = \log\left(\frac{a}{b}\right)$]

⇒ log $\left(\frac{P}{P_o}\right)$ = kt ......(3)

Now, the population doubled in 25 years.

Let P = $2P_o$ at t = 25 years

∴ kt = log $\left(\frac{P}{P_o}\right)$

$\Rightarrow$ k×25 = log $(\frac{2Po}{P^o})$

$\Rightarrow$ k = $\frac{\log 2}{25}$ ……(4)

Now, equation (3) becomes,

$$\log\left(\frac{P}{P^o}\right) = \frac{\log 2}{25}t$$

Now, let t1 be the time for the population to increase from 100000 to 500000

$$\log\left(\frac{500000}{100000}\right) = \frac{\log 2}{25}t1$$

$\Rightarrow \log 5 = \frac{\log 2}{25}t1$

$\Rightarrow t1 = \frac{25\log 5}{\log 2}$

$\Rightarrow t1 = 25 \times \frac{1.609}{0.6931}$ (log5 = 1.609 and log2 = 0.6931)

$\Rightarrow$ t1 = 58

∴ The time require for the population to be 500000 = 58 years.

4. Question

In a culture, the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 20000, if the rate of growth of bacteria is proportional to the number present?

Answer

Let the count of bacteria be C at any time t.

According to question,

$$\frac{dC}{dt} \propto t$$

$\Rightarrow \frac{dC}{dt} = kC$ where k is a constant

$\Rightarrow dC = kCdt$

$\Rightarrow \frac{dC}{C} = kdt$

Integrating both sides, we have

$\Rightarrow \int\frac{dC}{C} = k\int dt$

$\Rightarrow$ log|C| = kt + a ……(1)

Given, we have C = 100000 when t = 0 sec

Putting the value in equation (1)

∴ log|C| = kt + a

$\Rightarrow$ log|100000| = 0 + a

$\Rightarrow$ a = log|100000| ……(2)

Putting the value of a in equation (1) we have,

log|C| = kt + log|100000|

$\Rightarrow$ log|C| – log|100000| = k t $[\log a - \log b = \log(\frac{a}{b})]$

$\Rightarrow \log\left(\frac{C}{100000}\right) = kt$ ......(3)

Also, at t = 2 years, $C = 100000 + 100000 \times \frac{10}{100} = 110000$

From equation(3),we have

$\therefore kt = \log\left(\frac{C}{100000}\right)$

$\Rightarrow k \times 2 = \log\left(\frac{110000}{100000}\right)$

$\Rightarrow k = \frac{1}{2}\frac{\log 11}{\log 10}$ ......(4)

Now, equation (3) becomes,

$$\log\left(\frac{C}{100000}\right) = \frac{1}{2}\frac{\log 11}{\log 10}t$$

Now, let t1 be the time for the population to reach 200000

$$\log\left(\frac{200000}{100000}\right) = \frac{1}{2}\frac{\log 11}{\log 10}t1$$

$\Rightarrow \log 2 = \frac{1}{2}\frac{\log 11}{\log 10}t1$

$\Rightarrow t1 = \frac{2\log 10 \log 2}{\log 11}$

$\therefore$ The time require for the population to be 200000 = $\frac{2\log 10 \log 2}{\log 11}$ hours

5. Question

If the interest is compounded continuously at 6% per annum, how much worth ₹ 1000 will be after ten years? How long will it take to double ₹ 1000?

[Given $e^{0.6} = 1.822$]

Answer

Let the principal, rate and time be Rs P, r and t years.

Also, let the initial principal be P_o.

$$\frac{dP}{dt} = \frac{Pr}{100}$$

$\Rightarrow \frac{dP}{P} = \frac{r}{100}dt$

Integrating both sides, we have

$\Rightarrow \int\frac{dP}{P} = \frac{r}{100}\int dt$

$\Rightarrow \log|P| = \frac{r}{100}t + c$ ......(1)

Now, at t = 0, $P = P_o$

$\log|P_o| = 0 + c$

$\Rightarrow c = \log|P_o|$......(2)

Putting the value of c in equation (1) we have,

$\log|P| = \frac{r}{100}t + \log|P_o|$

$\Rightarrow \log|P| - \log|P_o| = \frac{r}{100}t$

⇒ (log |P| – log|P_0|) = $\frac{r}{100}$t [$\log a - \log b = \log(\frac{a}{b})$]

⇒ log ($\frac{P}{P_o}$) = $\frac{r}{100}$t ……(3)

Now, P_o = 1000, t = 10years, r = 6

∴ log ($\frac{P}{1000}$) = $\frac{6}{100}$×10

⇒ log ($\frac{P}{1000}$) = 0.6

⇒ $\frac{P}{1000} = e^{0.6}$

⇒ P = $e^{0.6}$×1000

⇒ P = 1.822×1000 (Given: $e^{0.6}$ = 1.822)

⇒ P = 1822

Rs 1000 will be Rs 1822 after 10 years at 6% rate.

6. Question

The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given the number triples in 5 hrs, find how many bacteria will be present after 10 hours. Also find the time necessary for the number of bacteria to be 10 times the number of initial present.

[Given $\log_e 3$ = 1.0986, $e^{2.1972}$ = 9]

Answer

Let the count of bacteria be C at any time t.

According to question,

$\frac{dC}{dt} \propto t$

⇒ $\frac{dC}{dt} = kC$ where k is a constant

⇒ $dC = kCdt$

⇒ $\frac{dC}{C} = kdt$

Integrating both sides, we have

⇒ $\int\frac{dC}{C}$ = k∫dt

⇒ log|C| = kt + a……(1)

Given, we have C = C_0 when t = 0 sec

Putting the value in equation (1)

∴ log|C| = kt + a

⇒ log| C_0| = 0 + a

⇒ a = log| C_0| ……(2)

Putting the value of a in equation (1) we have,

log|C| = kt + log|100000|

⇒ log|C| – log| C_0| = k t [$\log a - \log b = \log(\frac{a}{b})$]

⇒ log ($\frac{C}{C0}$) = kt ……(3)

Also, at t = 5 years, C = $3C_0$

From equation(3),we have

∴ $kt = \log\left(\frac{C}{C0}\right)$

⇒ $k\times5 = \log\left(\frac{3C0}{C0}\right)$

⇒ $k = \frac{\log3}{5}$......(4)

Now, equation (3) becomes,

$$\log\left(\frac{C}{C0}\right) = \frac{\log3}{5}t$$

Now, let C1 be the number of bacteria present in 10 hours, as

$$\log\left(\frac{C1}{C0}\right) = \frac{\log3}{5}\times10$$

$$\Rightarrow \log\left(\frac{C1}{C0}\right) = 2\log3$$

$$\Rightarrow \log\left(\frac{C1}{C0}\right) = \log9$$

$$\Rightarrow C1 = 9C0$$

Let the time be t1 for bacteria to be 10 times

$$\log\left(\frac{C}{C0}\right) = \frac{\log3}{5}t1$$

$$\Rightarrow \log\left(\frac{10C0}{C0}\right) = \frac{\log3}{5}t1$$

$$\Rightarrow \log10 = \frac{\log3}{5}t1$$

$$\Rightarrow t1 = \frac{5\log10}{\log3}$$

∴ The time required for = $\frac{5\log10}{\log3}$ hours

7. Question

The population of a city increases at a rate proportional to the number of inhabitants present at any time t. If the population of the city was 200000 in 1990 and 25000 in 2000, what will be the population in 2010?

Answer

Let the initial population be P_o .

And the population after time t be P.

According to question,

$$\frac{dP}{dt}\propto t$$

$$\Rightarrow \frac{dP}{dt} = kP$$

$$\Rightarrow dP = kPdt$$

$$\Rightarrow \frac{dP}{P} = kdt$$

Integrating both sides, we have

⇒ $\int\frac{dP}{P} = k\int dt$

⇒ log|P| = kt + logc……(1)

Given, we have P = 200000 when t = 1990

Putting the value in equation (1)

∴ log|200000| = k× 1990 + logc……(2)

we have P = 250000 when t = 2000

Putting the value in equation (1)

∴ log|250000| = k× 2000 + logc……(3)

On subtracting equation(2) from(3) we have,

log|250000| – log|200000| = k× (2000 – 1990)

⇒ $\log\frac{250000}{200000} = 10k$

⇒ $\log\frac{4}{5} = 10k$

⇒ $k = \frac{1}{10}\log\frac{4}{5}$……(4)

Substituting the value of k from (4) in (2), we have

log|200000| = $\frac{1}{10}\log\frac{4}{5}$× 1990 + logc……(5)

Substituting the value of k, log c and t = 2010 in (2), we have

$$\log P = \frac{1}{10}\log\frac{4}{5}\times 2010 + \log 200000 - 1990\times\frac{4}{5}$$

$$\Rightarrow \log P = \log\left(\frac{4}{5}\right)^{201} + \log(200000\times\left(\frac{5}{4}\right)^{199}$$

$$\Rightarrow P = \left(\frac{4}{5}\right)^{201}\times 200000\times\left(\frac{5}{4}\right)^{199}$$

$$\Rightarrow P = \left(\frac{5}{4}\right)^{2}\times 200000 = \frac{25}{16}\times 200000 = 312500$$

$\therefore P = 312500$

8. Question

If the marginal cost of manufacturing a certain item is given by $C'(x) = \frac{dC}{dx} = 2 + 0.15x$. Find the total cost function C(x), given that C(0) = 100.

Answer

$\frac{dC}{dx} = 2 + 0.15x$

⇒ dC = (2 + 0.15x)dx

Integrating both sides we have

⇒ ∫ dC = ∫ (2 + 0.15x)dx

⇒ ∫dC = 2∫dx + 0.15∫xdx

⇒ $C = 2x + \frac{0.15x^2}{2} + k$……(1)

Now, given C = 100 when x = 0

⇒ 100 = 0 + 0 + k

$\Rightarrow k = 100$……(2)

Putting the value of k in equation (1)

$$C(x) = 2x + \frac{0.15x^2}{2} + 100$$

9. Question

A ban pays interest by continuous compounding, that is, by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accrues at 8% per year, compounded continuously. Calculate the percentage increase in such an account over one year.

[Take $e^{0.08} \approx 1.0833$]

Answer

Let the principal, rate and time be Rs P, r and t years.

Also, let the initial principal be P_o.

$$\frac{dP}{dt} = \frac{Pr}{100}$$

$\Rightarrow \frac{dP}{P} = \frac{r}{100}dt$

Integrating both sides, we have

$\Rightarrow \int\frac{dP}{P} = \frac{r}{100}\int dt$

$\Rightarrow \log|P| = \frac{r}{100}t + c$……(1)

Now, at t = 0, $P = P_o$

$\log|P_o| = 0 + c$

$\Rightarrow c = \log|P_o|$……(2)

Putting the value of c in equation (1) we have,

$\log|P| = \frac{r}{100}t + \log|P_o|$

$\Rightarrow \log|P| - \log|P_o| = \frac{r}{100}t$

$\Rightarrow (\log|P| - \log|P_o|) = \frac{r}{100}t$ $[\log a - \log b = \log(\frac{a}{b})]$

$\Rightarrow \log(\frac{P}{P^\circ}) = \frac{r}{100}t$ ……(3)

Now, t = 1 year, r = 8%

$\therefore \log(\frac{P}{P^\circ}) = \frac{8}{100}\times 1$

$\Rightarrow \log(\frac{P}{P^\circ}) = 0.08$

$\Rightarrow \frac{P}{P^\circ} = e^{0.08}$

$\Rightarrow \frac{P}{P^\circ} = 1.0833$

$\Rightarrow \frac{P}{P^\circ} - 1 = 1.0833 - 1$

(Given: $e^{0.08} = 1.0833$)

$\Rightarrow \frac{P-P^\circ}{P^\circ} = 0.0833$

∴ Percentage increase = 0.0833×100 = 8.33%

10. Question

In a simple circuit of resistance R, self inductance L and voltage E, the current i at any time t is given by $\frac{di}{dt}+R\ i=E.$ If E is constant and initially no current passes through the circuit, prove that $i=\frac{E}{R}\left\{1-e^{-(R/L)t}\right\}.$

Answer

We know that in a circuit of R, L and E we have,

$L\frac{di}{dt} + Ri = E$

$\Rightarrow \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$

We can see that it is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where P = $\frac{R}{L}$ and Q = $\frac{E}{L}$

I.F = $e^{\int Pdt}$

= $e^{\int \frac{R}{L} dt}$

= $e^{\frac{R}{L}t}$

Solution of the given equation is given by

i × I.F = ∫Q × I.F dt + c

$\Rightarrow i \times e^{\frac{R}{L}t} = \int \frac{E}{L} \times e^{\frac{R}{L}t} dt + c$

$\Rightarrow i \times e^{\frac{R}{L}t} = \int \frac{E}{L} \times \frac{L}{R} \times e^{\frac{R}{L}t} dt + c$

$\Rightarrow i = \frac{E}{L} + c\ e^{\frac{R}{L}t}$......(1)

Initially, there was no current

So, at i = 0, t = 0

$0 = \frac{E}{R} + ce^{0}$

$\Rightarrow c = -\frac{E}{R}$

Now, putting the value of c in equation (1)

$i = \frac{E}{L} - \frac{E}{L}\ e^{\frac{R}{L}t}$

$i = \frac{E}{L}\ (1 - e^{\frac{R}{L}t})$

11. Question

The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

Answer

Let the quantity of mass at any time t be A.

According to the question,

$$\frac{dA}{dt} \propto A$$

$\Rightarrow \frac{dA}{dt} = -kA$ where k is a constant

$\Rightarrow dA = -kAdt$

$\Rightarrow \frac{dA}{A} = -kdt$

Integrating both sides, we have

$\Rightarrow \int\frac{dA}{A} = -k\int dt$

$\Rightarrow \log|A| = -kt + c$......(1)

Given, the Initial quantity of masss be A_0 when the t = 0 sec

Putting the value in equation (1)

$\therefore \log|A| = -kt + c$

$\Rightarrow \log|A_0| = 0 + c$

$\Rightarrow c = \log|A_0|$ ......(2)

Putting the value of c in equation (1) we have,

$\log|A| = -kt + \log|A_0|$

$\Rightarrow \log|A| - \log|A_0| = -kt$ $[\log a - \log b = \log(\frac{a}{b})]$

$\Rightarrow \log(\frac{A}{A0}) = -kt$ ......(3)

Let the mass becomes half at time t1, $A = \frac{A0}{2}$

From equation(3),we have

$\therefore -kt = \log(\frac{A}{A0})$

$\Rightarrow -k\times t1 = \log(\frac{A}{2A})$

$\Rightarrow -k\times t1 = \log\frac{1}{2}$

$\Rightarrow -k\times t1 = -\log 2$

$\Rightarrow t1 = \frac{\log 2}{k}$

$\therefore$ Required time $= \frac{\log 2}{k}$ where k is the constant of proportionality.

12. Question

Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its half – life is 1590 years. What percentage will disappear in one year?

$$\left[\text{Use: } e^{-\frac{\log 2}{1590}} = 0.9996\right]$$

Answer

Let the quantity of radium at any time t be A.

According to question,

$\frac{dA}{dt} \propto A$

⇒ $\frac{dA}{dt} = -kA$ where k is a constant

⇒ $dA = -kAdt$

⇒ $\frac{dA}{A} = -kdt$

Integrating both sides, we have

⇒ $\int \frac{dA}{A} = -k\int dt$

⇒ $\log|A| = -kt + c$......(1)

Given, Initial quantity of radium be A_0 when t = 0 sec

Putting the value in equation (1)

∴ $\log|A| = -kt + c$

⇒ $\log|A_0| = 0 + c$

⇒ $c = \log|A_0|$ ......(2)

Putting the value of c in equation (1) we have,

$\log|A| = -kt + \log|A_0|$

⇒ $\log|A| - \log|A_0| = -kt$ [$\log a - \log b = \log(\frac{a}{b})$]

⇒ $\log(\frac{A}{A0}) = -kt$ ......(3)

Given its half life = 1590 years,

From equation(3),we have

∴ $-kt = \log(\frac{A}{A0})$

⇒ $-k\times1590 = \log(\frac{A}{2A})$

⇒ $-k\times1590 = \log\frac{1}{2}$

⇒ $-k\times1590 = -\log 2$

⇒ $k = \frac{\log 2}{1590}$

∴ The equation becomes

$\log(\frac{A}{A0}) = -\frac{\log 2}{1590}t$

$\log(\frac{A}{A0}) = -0.9996\,t$

Percentage Disappeared = (1 – 0.9996) × 100 = 0.04 %

13. Question

The slope of the tangent at a point P(x, y) on a curve is $\frac{-x}{y}$.If the curve passes through the point (3, – 4), find the equation of the curve.

Answer

Given the slope of the tangent $= -\frac{x}{y}$

We know that slope of tangent = $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow ydy = -xdx$

Integrating both sides,

$\Rightarrow \int ydy = -\int xdx$

$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = c$

$\Rightarrow y^2 + x^2 = 2c = c1$……(1)

Now, the curve passes through (3, – 4)

So, it must satisfy the above equation

$\therefore y^2 + x^2 = c1$

$\Rightarrow (-4)^2 + (3)^2 = c1$

⇒ 16 + 9 = c1

⇒ c1 = 25

Putting the value of c1 in equation (1)

$\therefore y^2 + x^2 = 25$

14. Question

Find the equation of the curve which passes through the point (2, 2) and satisfies the differential equation

$y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$.

Answer

Given the differential equation

$y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$

$\Rightarrow y - y^2 = \frac{dy}{dx} + x\frac{dy}{dx}$

$\Rightarrow y(1-y) = \frac{(1+x)dy}{dx}$

$\Rightarrow \frac{dy}{y(1-y)} = \frac{dx}{(1+x)}$

Integrating both sides we have,

$\Rightarrow \int \frac{dy}{y(1-y)} = \int \frac{dx}{(1+x)}$

$\Rightarrow \int \left(\frac{1}{y} + \frac{1}{1-y}\right)dy = \int \frac{dx}{(1+x)}$

⇒ log|y| + log|1 – y| = log|1 + x| + logc

⇒ log|y(1 – y)| = log|c(1 + x)|

⇒ y(1 – y) = c(1 + x) ……(1)

Since, the equation passes through (2,2), So,

$2(1 - 2) = c(1 + 2)$

$\Rightarrow - 2 = c\times3$

$\Rightarrow c = -\frac{2}{3}$

Therefore, equation (1) becomes

$y(1 - y) = -\frac{2}{3}(1 + x)$

15. Question

Find the equation of the curve passing through the point $\left(1, \frac{\pi}{4}\right)$ and tangent at any point of which makes an angle $\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ with the x – axis.

Answer

The equation is $\tan\frac{y}{x} = -\log x + c$

It is passing through $(1, \frac{\pi}{4})$

$\therefore \tan\frac{\pi}{4} = -\log 1 + c$

$\Rightarrow 1 = 0 + c$

$\Rightarrow c = 1$

Putting the value of c in the above equation

$\therefore \tan\frac{y}{x} = -\log x + 1$

16. Question

Find the curve for which the intercept cut – off by a tangent on the x – axis is equal to four times the ordinate of the point of contact.

Answer

Let P(x,y) be the point of contact of tangent and curve y = f(x).

It cuts the axes at A and B so, the equation of the tangent at P(x,y)

$Y - y = \frac{dy}{dx}(X - x)$

Putting X = 0

$Y - y = \frac{dy}{dx}(0 - x)$

$\Rightarrow Y = y - x\frac{dy}{dx}$

So, $A(0, y - x\frac{dy}{dx})$

Now, putting Y = 0

$0 - y = \frac{dy}{dx}(X - x)$

$\Rightarrow X = x - y\frac{dx}{dy}$

So, $B(x - y\frac{dx}{dy}, 0)$

Given, intercept on x – axis = 4× ordinate

$\Rightarrow x - y\frac{dx}{dy} = 4y$

$\Rightarrow y\frac{dx}{dy} + 4y = x$

$\Rightarrow \frac{dx}{dy} + 4 = \frac{x}{y}$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = -4$

We can see that it is a linear differential equation.

Comparing it with $\frac{dx}{dy} + Px = Q$

$P = \frac{1}{y}$, $Q = -4$

I.F $= e^{\int Pdy}$

$= e{-\int \frac{1}{y}dy}$

$= e^{-\log y}$

$= \frac{1}{y}$

Solution of the given equation is given by

$x \times I.F = \int Q \times I.F\, dy + \log c$

$\Rightarrow x \times (\frac{1}{y}) = \int -4 \times \frac{1}{y} dy + \log c$

$\Rightarrow \frac{x}{y} = -4 \log y + \log c$

$\Rightarrow \frac{x}{y} = \log y^{-4} + \log c$

$\Rightarrow \frac{x}{y} = \log c\, y^{-4}$

$\Rightarrow e^{\frac{x}{y}} = c\, y^{-4}$

17. Question

Show that the equation of the curve whose slope at any point is equal to y + 2x and which passes through the origin is $y + 2(x + 1) = 2e^{2x}$.

Answer

Given slope at any point = y + 2x

$\Rightarrow \frac{dy}{dx} = y + 2x$

$\Rightarrow \frac{dy}{dx} - y = 2x$

We can see that it is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$P = -1$, $Q = 2x$

I.F $= e^{\int Pdx}$

$= e^{\int -dx}$

$= e^{-x}$

Solution of the given equation is given by

$y \times I.F = \int Q \times I.F\, dx + c$

$\Rightarrow y \times e^{-x} = \int 2x \times e^{-x}\, dx + c$

$\Rightarrow ye^{-x} = 2\int x \times e^{-x}\, dx + c$

$\Rightarrow ye^{-x} = -2x\, e^{-x} - 2\, e^{-x} + c$

$\Rightarrow y = -2x - 2 + ce^{x}$ ......(1)

As the equation passing through origin,

$0 = 0 - 2 + c \times 1$

$\Rightarrow c = 2$

Putting the value of c in equation (1)

$\therefore y = -2x - 2 + 2e^{x}$

18. Question

The tangent at any point (x, y) of a curve makes an angle $\tan^{-1}(2x + 3y)$ with the x - axis. Find the equation of the curve if it passes through (1, 2).

Answer

19. Question

Find the equation of the curve such that the portion of the x - axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point (1, 2).

Answer

Let P(x,y) be the point of contact of tangent and curve y = f(x).

It cuts the axes at A and B so, equation of tangent at P(x,y)

$Y - y = \frac{dy}{dx}(X - x)$

Putting X = 0

$Y - y = \frac{dy}{dx}(0 - x)$

$\Rightarrow Y = y - x\frac{dy}{dx}$

So, $A(0, y - x\frac{dy}{dx})$

Now, putting Y = 0

$0 - y = \frac{dy}{dx}(X - x)$

$\Rightarrow X = x - y\frac{dx}{dy}$

So, $B(x - y\frac{dx}{dy}, 0)$

Given, intercept on x - axis = 4× ordinate

$\Rightarrow x - y\frac{dx}{dy} = 2x$

$\Rightarrow -y\frac{dx}{dy} = x$

$\Rightarrow -\frac{dx}{x} = \frac{dy}{y}$

$\Rightarrow$ – logx = logy + c ......(1)

As it passes through (1,2)

So, the point must satisfy the equation above

– log1 = log2 + c

$\Rightarrow$ 0 = log2 + c

$\Rightarrow$ c = – log2

Putting the value of c in equation (1)

– logx = logy – log2

$\Rightarrow$ log2 = logx + logy

$\Rightarrow$ log2 = logxy

$\Rightarrow$ xy = 2

20. Question

Find the equation to the curve satisfying $x(x+1)\frac{dy}{dx} - y = x(x+1)$ and passing through (1, 0).

Answer

$\Rightarrow x(x + 1)\frac{dy}{dx} - y = x(x + 1)$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x(x+1)} = 1$

We can see that it is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

P = – $\frac{1}{x(x+1)}$, Q = 1

I.F = $e^{\int P dx}$

= $e^{\int -\frac{1}{x(x+1)} dx}$

= $e^{\int (\frac{1}{x+1} - \frac{1}{x}) dx}$

= $e^{\log|x + 1| - \log|x|}$

= $e^{\log(\frac{x+1}{x})}$

= $\frac{x+1}{x}$

Solution of the given equation is given by

y × I.F = ∫Q × I.F dx + c

$\Rightarrow$ y × $\frac{x+1}{x}$ = ∫ 1 × $\frac{x+1}{x}$ dx + c

$\Rightarrow$ y × $\frac{x+1}{x}$ = ∫ $(1 + \frac{1}{x})$ dx + c

$\Rightarrow$ y × $\frac{x+1}{x}$ = x + logx + c ......(1)

As the equation passing through (1,0),

0 = 1 + log1 + c

$\Rightarrow$ c = – 1

Putting the value of c in equation (1)

$\therefore$ y × $\frac{x+1}{x}$ = x + logx – 1

21. Question

Find the equation of the curve which passes through the point (3, – 4) and has the slope $\frac{2y}{x}$ at any point (x, y) on it.

Answer

Given the slope of the tangent = $\frac{2y}{x}$

We know that slope of tangent = $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2y}{x}$

$\Rightarrow \frac{dy}{y} = \frac{2dx}{x}$

Integrating both sides,

$\Rightarrow \int \frac{dy}{y} = 2\int \frac{dx}{x}$

$\Rightarrow \log y = 2\log x + c$

$\Rightarrow y = x^2c$......(1)

Now, the curve passes through (3, – 4)

So, it must satisfy the above equation

$\therefore y = x^2c$

$\Rightarrow$ – 4 = $(3)^2$× c

$\Rightarrow$ – 4 = 9c

$\Rightarrow$ c = $-\frac{4}{9}$

Putting the value of c in equation (1)

$\therefore y = -\frac{4}{9}x^2$

22. Question

Find the equation of the curve which passes through the origin and has the slope x + 3y – 1 at any point (x, y) on it.

Answer

Given Slope of the equation at any point (x,y) = x + 3y – 1

$\Rightarrow \frac{dy}{dx} = x + 3y - 1$

$\Rightarrow \frac{dy}{dx} - 3y = x - 1$

We can see that it is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

P = – 3, Q = x – 1

I.F = $e^{\int P dx}$

= $e^{\int -3dx}$

= e^{-3x}

Solution of the given equation is given by

y × I.F = ∫Q × I.F dx + c

⇒ $y \times e^{-3x} = \int (x-1) \times e^{-3x} dx + c$

⇒ $y \times e^{-3x} = (x-1) \times -\frac{1}{3}e^{-3x} - \int (1) - \frac{1}{3}e^{-3x}\,dx + c$

⇒ $y \times e^{-3x} = (x-1) \times -\frac{1}{3}e^{-3x} + \left(-\frac{1}{9}e^{-3x}\right) + c$

⇒ $y = -\frac{x}{3} + \frac{1}{3} - \frac{1}{9} + ce^{3x}$

⇒ $y = -\frac{x}{3} + \frac{2}{9} + ce^{3x}$......(1)

As the equation passing through origin(0,0)

$0 = 0 + \frac{2}{9} + c$

⇒ $c = -\frac{2}{9}$

Putting the value of c in equation (1)

$\therefore y = -\frac{x}{3} + \frac{2}{9} - \frac{2}{9}e^{3x}$

23. Question

At every point on a curve, the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through (0, 1). Find the equation of the curve.

Answer

Given the slope at any time = x + xy

We know that slope of tangent = $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = x(1+y)$

⇒ $\frac{dy}{1+y} = xdx$

Integrating both sides,

⇒ $\int \frac{dy}{1+y} = \int xdx$

⇒ $\log|1+y| = \frac{x^2}{2} + c$......(1)

Now, the curve passes through (0,1)

So, it must satisfy the above equation

$\therefore \log|1+y| = \frac{x^2}{2} + c$

⇒ log 2 = 0 + c

⇒ c = log2

Putting the value of c in equation (1)

$\therefore \log|1 + y| = \frac{x^2}{2} + \log 2$

24. Question

A curve is such that the length of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$, and hence find the curve.

Answer

$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2yx}$$

It is a homogenous equation,

Putting y = kx

$$\frac{dy}{dx} = k + x\frac{dk}{dx}$$

So, $k + x\frac{dk}{dx} = \frac{k^2x^2 - x^2}{2kx^2}$

$$\Rightarrow x\frac{dk}{dx} = \frac{k^2x^2 - x^2}{2kx^2} - k$$

$$\Rightarrow x\frac{dk}{dx} = \frac{k^2x^2 - x^2 - 2k^2x^2}{2kx^2}$$

$$\Rightarrow x\frac{dk}{dx} = \frac{-k^2x^2 - x^2}{2kx^2}$$

$$\Rightarrow x\frac{dk}{dx} = \frac{-k^2 - 1}{2k}$$

$$\Rightarrow \frac{2k}{1 + k^2}dk = -\frac{dx}{x}$$

$$\Rightarrow \int \frac{2k}{1 + k^2}dk = -\int \frac{dx}{x}$$

$$\Rightarrow \log|1 + k^2| = -\log x + \log c$$

$$\Rightarrow \log|1 + k^2| = -\log x + \log c$$

$$\Rightarrow \log|1 + k^2| = \log\frac{c}{x}$$

Putting the value of k

$$\log\left|1 + \frac{y^2}{x^2}\right| = \log\frac{c}{x}$$

$$\Rightarrow \frac{x^2 + y^2}{x^2} = \frac{c}{x}$$

$$\therefore x^2 + y^2 - cx = 0$$

Differentiating with respect to x,

$$2x + 2y\frac{dy}{dx} - c = 0$$

$\Rightarrow \frac{dy}{dx} = \frac{c-2x}{2y}$

Let (h,k) be the point where tangent passes through origin and the length is equal to h. So, equation of tangent at (h,k) is

$(y-k) = \frac{dy}{dx}(x-h)$

$\Rightarrow (y-k) = \frac{c-2h}{2k}(x-h)$

$\Rightarrow 2ky - 2k^2 = cx - ch - 2hx + 2h^2$

$\Rightarrow x(c - 2h) - 2ky + 2k^2 - hc + 2h^2 = 0$

$\Rightarrow x(c - 2h) - 2ky + 2(k^2 - 2h) - hc = 0$

$\Rightarrow x(c - 2h) - 2ky + 2(ch) - hc = 0$ ($h^2 + k^2 = ch$ as (h,k) on th curve)

$\Rightarrow x(c - 2h) - 2ky + hc = 0$

Now, Length of perpendicular as tangent from origin is

$L = \frac{ax + by + c}{\sqrt{a^2 + b^2}}$

$\Rightarrow L = \frac{0 \times (c-2h) + 0 \times (-2k) + hc}{\sqrt{(c-2h)^2 + (-2k)^2}}$

$\Rightarrow L = \frac{hc}{\sqrt{c^2 + 4h^2 - 4ch + 4k^2}}$

$\Rightarrow L = \frac{hc}{\sqrt{c^2 + 4(h^2 + k^2 - ch)}}$

$\Rightarrow L = \frac{hc}{\sqrt{c^2 + 4(0)}}$

$\Rightarrow L = \frac{hc}{c} = h$

Hence, $x^2 + y^2 = cx$ is the required curve.

25. Question

Find the equation of the curve which passes through the point (1, 2) and the distance between the foot of the ordinate of the point of contact and the point of intersection of the tangent with the x - axis is twice the abscissa of the point of contact.

Answer

Let P(x,y) be the point of contact of tangent and curve y = f(x).

It cuts the axes at A and B so, the equation of the tangent at P(x,y)

$Y - y = \frac{dy}{dx}(X - x)$

Now, putting Y = 0

$0 - y = \frac{dy}{dx}(X - x)$

$\Rightarrow X = x - y\frac{dx}{dy}$

So, $B(x - y\frac{dx}{dy}, 0)$

Given, the distance between the foot of ordinate of the point of contact and the point of intersection of tangent and x - axis = 2x

BC = 2x

$$\sqrt{\left(x - y\frac{dx}{dy}\right)^2 + (0)^2} = 2x$$

$\Rightarrow y\frac{dx}{dy} = 2x$

$\Rightarrow \frac{dx}{x} = 2\frac{dy}{y}$

Integrating both sides we have

⇒logx = 2logy + c……(1)

As it passes through (1,2)

So, the point must satisfy the equation above

log1 = 2log2 + c

⇒ 0 = 2log2 + c

⇒ c = – 2log2

Putting the value of c in equation (1)

logx = 2logy – 2log2

⇒ logx = 2(logy – log2)

$\Rightarrow \log x = 2\log\frac{y}{2}$

$\Rightarrow \log x = \log\left(\frac{y}{2}\right)^2$

$\Rightarrow x = \frac{y^2}{4}$

26. Question

The normal to a given curve at each point (x, y) on the curve passes through the point (3, 0). If the curve contains the point (3, 4), find its equation.

Answer

Given the equation of normal at point (x,y) on the curve

$\therefore Y - y = -\frac{dx}{dy}(X - x)$

Now, the curve passes through (3,0)

$$0 - y = -\frac{dx}{dy}(3 - x)$$

$$\Rightarrow y = \frac{dx}{dy}(3 - x)$$

$\Rightarrow ydy = (3 - x)dx$

Integrating both sides

$$\Rightarrow \int ydy = \int (3 - x)dx$$

$$\Rightarrow \frac{y^2}{2} = 3x - \frac{x^2}{2} + c$$

$\Rightarrow \frac{y^2}{2} + \frac{x^2}{2} = 3x + c$

It also passes through (3,4),so

$\Rightarrow \frac{(4)^2}{2} + \frac{(3)^2}{2} = 3 \times 3 + c$

$\Rightarrow \frac{16 + 9}{2} - 9 = c$

$\Rightarrow \frac{25}{2} - 9 = c$

$\Rightarrow \frac{25 - 18}{2} = c$

$\Rightarrow c = \frac{7}{2}$

Putting the value of c in equation(1)

$\frac{y^2}{2} + \frac{x^2}{2} = 3x + \frac{7}{2}$

$\Rightarrow y^2 + x^2 = 6x + 7$

27. Question

The rate of increase of bacteria in a culture is proportional to the number of bacteria present, and it is found that the number doubles in 6 hours. Prove that the bacteria becomes eight times at the end of 18 hours.

Answer

Let the count of bacteria be C at any time t.

According to the question,

$\frac{dC}{dt} \propto t$

$\Rightarrow \frac{dC}{dt} = kC$ where k is a constant

$\Rightarrow dC = kCdt$

$\Rightarrow \frac{dC}{C} = kdt$

Integrating both sides, we have

$\Rightarrow \int \frac{dC}{C} = k\int dt$

$\Rightarrow \log|C| = kt + a$......(1)

Given, we have $C = C_0$ when t = 0 sec

Putting the value in equation (1)

$\therefore \log|C| = kt + a$

$\Rightarrow \log|C_0| = 0 + a$

$\Rightarrow a = \log|C_0|$ ......(2)

Putting the value of a in equation (1) we have,

$\log|C| = kt + \log|C_0|$

$\Rightarrow \log|C| - \log|C_0| = k\,t$ $[\log a - \log b = \log(\frac{a}{b})]$

$\Rightarrow \log\left(\frac{C}{C0}\right) = kt$ ......(3)

Also, at t = 6 years, $C = 2C_0$

From equation(3),we have

$\therefore kt = \log\left(\frac{C}{C0}\right)$

$\Rightarrow k\times 6 = \log\left(\frac{2C0}{C0}\right)$

$\Rightarrow k = \frac{\log 2}{6}$......(4)

Now, equation (3) becomes,

$$\log\left(\frac{C}{C0}\right) = \frac{\log 2}{6}t$$

Now, $C = 8C_0$

$$\log\left(\frac{8C0}{C0}\right) = \frac{\log 2}{6}\times t$$

$\Rightarrow \log 8 = \frac{\log 2}{6}\times t$

$\Rightarrow \log 2^3 = \frac{\log 2}{6}\times t$

$\Rightarrow 3\log 2 = \frac{\log 2}{6}\times t$

$\Rightarrow t = 18$

$\therefore$ The time required = 18 hours

28. Question

Radium decomposes at a rate proportional to the quantity of radium present. It is found that in 25 years, approximately 1.1% of a certain quantity of radium has decomposed. Determine approximately how long it will take for one - half of the original amount of radium to decompose?

[Given $\log_e 0.989 = 0.01106$ and $\log_e 2 = 0.6931$]

Answer

Let the quantity of radium at any time t be A.

According to the question,

$$\frac{dA}{dt} \propto A$$

$\Rightarrow \frac{dA}{dt} = -kA$ where k is a constant

$\Rightarrow dA = -kAdt$

$\Rightarrow \frac{dA}{A} = -kdt$

Integrating both sides, we have

$\Rightarrow \int\frac{dA}{A} = -k\int dt$

$\Rightarrow \log|A| = -kt + c$......(1)

Given, Initial quantity of radium be A_0 when t = 0 sec

Putting the value in equation (1)

$\therefore \log|A| = -kt + c$

⇒ log| A_0| = 0 + c

⇒ c = log| A_0| ......(2)

Putting the value of c in equation (1) we have,

log|A| = – kt + log| A_0|

⇒ log|A| – log| A_0| = – k t $[\log a - \log b = \log(\frac{a}{b})]$

⇒ log $(\frac{A}{A0})$ = – kt ......(3)

Given that the radium decomposes 1.1% in 25 years,

A = (100 – 1.1)% = 98.9% = 0.989 A_0 at t = 25 years

From equation(3),we have

∴ – kt = log $(\frac{0.989A0}{A0})$

⇒ – k×25 = log (0.989)

⇒ k = – $\frac{\log 0.989}{25}$

∴ The equation becomes

log $(\frac{A}{A0})$ = – $\frac{\log 0.989}{25}$t

Now, $A = \frac{A0}{2}$

∴ log $(\frac{A}{A0})$ = – $\frac{\log 0.989}{25}$t

⇒ log $(\frac{A}{2A})$ = – $\frac{\log 0.989}{25}$t

⇒ $\log\frac{1}{2}$ = – $\frac{\log 0.989}{25}$t

⇒ $-\log 2 = -\frac{\log 0.989}{25}t$ (log 2 = 0.6931 and log 0.989 = 0.01106)

⇒ $t = \frac{25\times\log 2}{\log 0.989}$

⇒ $t = \frac{25\times 0.6931}{0.01106}$

⇒ t = 1567 years

29. Question

Show that all curves for which the slope at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$ are rectangular hyperbola.

Answer

Given the slope of the tangent = $\frac{x^2+y^2}{2xy}$

∴ $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

It is a homogenous equation,

Putting y = kx

$$\frac{dy}{dx} = k + x\frac{dk}{dx}$$

So, $k + x\frac{dk}{dx} = \frac{x^2 + k^2x^2}{2kx^2}$

$\Rightarrow x\frac{dk}{dx} = \frac{1 + k^2}{2k} - k$

$\Rightarrow x\frac{dk}{dx} = \frac{1 + k^2 - 2k^2}{2k}$

$\Rightarrow x\frac{dk}{dx} = \frac{1 - k^2}{2k}$

$\Rightarrow \frac{2k}{1 - k^2}dk = \frac{dx}{x}$

$\Rightarrow \int \frac{-2k}{1 - k^2}dk = -2\int \frac{dx}{x}$

$\Rightarrow \log|1 - k^2| = -2\log x + \log c$

$\Rightarrow \log|1 - k^2| = -\log x^2 + \log c$

$\Rightarrow \log|1 - k^2| = \log\frac{c}{x^2}$

Putting the value of k

$\log\left|1 - \frac{y^2}{x^2}\right| = \log\frac{c}{x^2}$

$\Rightarrow \frac{x^2 - y^2}{x^2} = \frac{c}{x^2}$

$\therefore x^2 - y^2 = c$

30. Question

The slope of the tangent at each point of a curve is equal to the sum of the coordinates of the point. Find the curve that passes through the origin.

Answer

Given slope at any point = sum of coordinates = x + y

$\Rightarrow \frac{dy}{dx} = y + x$

$\Rightarrow \frac{dy}{dx} - y = x$

We can see that it is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

P = – 1, Q = x

I.F = $e^{\int Pdx}$

= $e^{\int - dx}$

= e^{-x}

Solution of the given equation is given by

y × I.F = ∫Q × I.F dx + c

⇒ y × e^{-x} = ∫ x × e^{-x} dx + c

⇒ ye^{-x} = ∫ x × e^{-x} dx + c (Using integration by parts)

$\Rightarrow ye^{-x} = -xe^{-x} - e^{-x} + c$

$\Rightarrow y = -x - 1 + ce^{x}$......(1)

As the equation passing through origin,

$0 = 0 - 1 + c\times 1$

$\Rightarrow c = 1$

Putting the value of c in equation (1)

$\therefore y = -x - 1 + e^{x}$

$\Rightarrow x + y + 1 = e^{x}$

31. Question

Find the equation of the curve passing through the point (0, 1) if the slope of the tangent to the curve at each of its point is equal to the sum of the abscissa and the product of the abscissa and the ordinate of the point.

Answer

Given slope at any point = sum of the abscissa and the product of the abscissa and the ordinate = x + xy

According to question,

$\Rightarrow \frac{dy}{dx} = x + xy$

$\Rightarrow \frac{dy}{dx} - xy = x$

We can see that it is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

P = – x, Q = x

$I.F = e^{\int Pdx}$

$= e^{\int -xdx}$

$= e^{-\frac{x^2}{2}}$

Solution of the given equation is given by

$y \times I.F = \int Q \times I.F\, dx + c$

$\Rightarrow y \times e^{-\frac{x^2}{2}} = \int x \times e^{-\frac{x^2}{2}}\, dx + c$......(1)

Let $I = \int x \times e^{-\frac{x^2}{2}}\, dx$

Let $t = -\frac{x^2}{2}$

$\Rightarrow dt = -\frac{2xdx}{2} = -xdx$

$\therefore I = \int -e^{t}\, dt = -e^{t} = e^{-\frac{x^2}{2}}$

Now substituting the value of I in equation (1)

$\Rightarrow ye^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}} + c$

$\Rightarrow y = -1 + ce^{\frac{x^2}{2}}$......(2)

As the equation passing through (0,1),

$1 = -1 + c \times 1$

$\Rightarrow c = 2$

Putting the value of c in equation (1)

$\therefore y = -1 + 2e^{\frac{x^2}{2}}$

32. Question

The slope of a curve at each of its points is equal to the square of the abscissa of the point. Find the particular curve through the point (– 1, 1).

Answer

Given the slope of the curve = square of the abscissa = x^2

$\Rightarrow \frac{dy}{dx} = x^2$

$\Rightarrow dy = x^2 dx$

Integrating both sides we have,

$\Rightarrow \int dy = \int x^2 dx$

$\Rightarrow y = \frac{x^3}{3} + c$......(1)

The curve passes through point (– 1,1)

$1 = -\frac{1^3}{3} + c$

$\Rightarrow 1 = -\frac{1}{3} + c$

$\Rightarrow 1 + \frac{1}{3} = c = \frac{4}{3}$

Putting the value of c in equation (1)

$\therefore y = \frac{x^3}{3} + \frac{4}{3}$

33. Question

Find the equation of the curve that passes through the point (0, a) and is such that at any point (x, y) on it, the product of its slope and the ordinate is equal to the ab.

Answer

Given product of slope of the curve and ordinate = x

$\Rightarrow y\frac{dy}{dx} = x$

$\Rightarrow ydy = xdx$

Integrating both sides we have,

$\Rightarrow \int ydy = \int xdx$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$......(1)

The curve passes through point (0,a)

$\frac{a^2}{2} = 0 + c$

$\Rightarrow c = \frac{a^2}{2}$

Putting the value of c in equation (1)

$\therefore \frac{y^2}{2} = \frac{x^2}{2} + \frac{a^2}{2}$

$\Rightarrow y^2 - x^2 = a^2$

34. Question

The x - intercept of the tangent line to a curve is equal to the ordinate of the point of contact. Find the particular curve through the point (1, 1).

Answer

Let P(x,y) be the point on the curve y = f(x) such that tangent at Pcuts the coordinate axes at A and B.

It cuts the axes at A and B so, equation of tangent at P(x,y)

$Y - y = \frac{dy}{dx}(X - x)$

Now, putting Y = 0

$0 - y = \frac{dy}{dx}(X - x)$

$\Rightarrow X = x - y\frac{dx}{dy}$

So, $B(x - y\frac{dx}{dy}, 0)$

Given, intercept on x – axis = y

$\Rightarrow x - y\frac{dx}{dy} = y$

$\Rightarrow - y\frac{dx}{dy} = y - x$

$\Rightarrow -\frac{dx}{dy} = 1 - \frac{x}{y}$

$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = -1$......(1)

We can see that it is a linear differential equation.

Comparing it with $\frac{dy}{dx} + Py = Q$

$P = -\frac{1}{y}$, Q = – 1

I.F = $e^{\int P dy}$

$= e\int -\frac{1}{y}dy$

$= e^{-\log y} = \frac{1}{y}y$

Solution of the given equation is given by

x × I.F = ∫Q × I.F dy + c

$\Rightarrow x \times \frac{1}{y} = \int -1 \times \frac{1}{y} dy + c$

$\Rightarrow \frac{x}{y} = -\log y + c$......(1)

As the equation passing through (1,1)

0 = – 1 + c

⇒ c = 1

Putting the value of c in equation (1)

∴ $\frac{x}{y}$ = – logy + 1

⇒ x = y – ylogy

Very short answer

1. Question

Define a differential equation.

Answer

Differential Equation: An equation containing independent variable, dependent variable, and differential coefficient of dependent variable with respect to independent variable.

$i.e. \frac{dy}{dx}$

Here y is dependent variable and x is independent variable.

Examples: $(1)\ \frac{dy}{dx} = x^2$

$(2) \frac{d^2y}{dx^2} + \frac{2dy}{dx} = \sin x$

$(3)\ \left(\frac{dy}{dx}\right)^2 + y = x$

1. Question

Define a differential equation.

Answer

Differential Equation: An equation containing independent variable, dependent variable, and differential coefficient of dependent variable with respect to independent variable.

$i.e. \frac{dy}{dx}$

Here y is dependent variable and x is independent variable.

Examples: $(1)\ \frac{dy}{dx} = x^2$

$(2) \frac{d^2y}{dx^2} + \frac{2dy}{dx} = \sin x$

$(3)\ \left(\frac{dy}{dx}\right)^2 + y = x$

2. Question

Define order of a differential equation

Answer

ORDER: The order of a differential equation is the order of the highest derivative involved in the equation.

Examples: $(1)\ \frac{dy}{dx} = x^2\ (2) \frac{d^2y}{dx^2} + \frac{2dy}{dx} = \sin x$

⇒ In example 1 order of differential equation is 1.

⇒ In example 2 order of differential equation is 2.

2. Question

Define order of a differential equation

Answer

ORDER: The order of a differential equation is the order of the highest derivative involved in the equation.

Examples: $(1)\ \frac{dy}{dx} = x^2\ (2) \frac{d^2y}{dx^2} + \frac{2dy}{dx} = \sin x$

⇒ In example 1 order of differential equation is 1.

⇒ In example 2 order of differential equation is 2.

3. Question

Define degree of a differential equation

Answer

DEGREE: The degree of differential equation is represented by the power of the highest order derivative in the given differential equation.

The differential equation must be a polynomial equation in derivatives for the degree to be defined and must be free from radicals and fractions.

Examples:

$(1) \frac{d^2y}{dx^2} + \frac{2dy}{dx} = \sin x$

$(2)\ \left(\frac{dy}{dx}\right)^2 + y = x$

$(3) \frac{d^2y}{dx^2} + \cos\left(\frac{d^2y}{dx^2}\right) = 5x$

⇒ In example 1 order of differential equation is 2 and its degree is 1.

⇒ In example 2 order of differential equation is 1 and its degree is 2.

⇒ In example 3, the differential equation is not a polynomial equation in derivatives. Hence, the degree for this equation is not defined.

3. Question

Define degree of a differential equation

Answer

DEGREE: The degree of differential equation is represented by the power of the highest order derivative in the given differential equation.

The differential equation must be a polynomial equation in derivatives for the degree to be defined and must be free from radicals and fractions.

Examples:

$(1) \frac{d^2y}{dx^2} + \frac{2dy}{dx} = \sin x$

$(2)\ \left(\frac{dy}{dx}\right)^2 + y = x$

$(3) \frac{d^2y}{dx^2} + \cos\left(\frac{d^2y}{dx^2}\right) = 5x$

⇒ In example 1 order of differential equation is 2 and its degree is 1.

⇒ In example 2 order of differential equation is 1 and its degree is 2.

⇒ In example 3, the differential equation is not a polynomial equation in derivatives. Hence, the degree for this equation is not defined.

4. Question

Write the differential equation representing the family of straight lines y = Cx + 5, where C is an arbitrary constant.

Answer

We are given

y = Cx + 5 ----(1)

Differentiating w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = C.1 + 0 = C$

Put value of C in (1)

$y = x.\frac{dy}{dx} + 5$ is the required differential equation.

4. Question

Write the differential equation representing the family of straight lines y = Cx + 5, where C is an arbitrary constant.

Answer

We are given

y = Cx + 5 ----(1)

Differentiating w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = C.1 + 0 = C$

Put value of C in (1)

$y = x.\frac{dy}{dx} + 5$ is the required differential equation.

5. Question

Write the differential equation obtained by eliminating the arbitrary constant C in the equation $x^2 - y^2 = C^2$.

Answer

We are given

$x^2 - y^2 = C^2$

Differentiating w.r.t x we get,

$\Rightarrow 2x - 2y.\frac{dy}{dx} = 0$

$\Rightarrow x - y.\frac{dy}{dx} = 0$ is the required differential equation.

5. Question

Write the differential equation obtained by eliminating the arbitrary constant C in the equation $x^2 - y^2 = C^2$.

Answer

We are given

$x^2 - y^2 = C^2$

Differentiating w.r.t x we get,

$$\Rightarrow 2x - 2y.\frac{dy}{dx} = 0$$

$\Rightarrow x - y.\frac{dy}{dx} = 0$ is the required differential equation.

6. Question

Write the differential equation obtained eliminating the arbitrary constant C in the equation xy = C^2.

Answer

We are given

xy = C^2

Differentiating w.r.t x we get,

$$\Rightarrow 1.y + \frac{dy}{dx}.x = 0$$

$\Rightarrow y + \frac{dy}{dx}.x = 0$ is the required differential equation.

6. Question

Write the differential equation obtained eliminating the arbitrary constant C in the equation xy = C^2.

Answer

We are given

xy = C^2

Differentiating w.r.t x we get,

$$\Rightarrow 1.y + \frac{dy}{dx}.x = 0$$

$\Rightarrow y + \frac{dy}{dx}.x = 0$ is the required differential equation.

7. Question

Write the degree of the differential equation $a^2 \frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/4}$.

Answer

We are given

$$a^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{4}}$$

$$\Rightarrow \left(a^2 \frac{d^2y}{dx^2}\right)^4 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^1$$

$$\Rightarrow a^8.\left(\frac{d^2y}{dx^2}\right)^4 = 1 + \left(\frac{dy}{dx}\right)^2$$

Here the order of differential equation is 2 and its degree is 4.

7. Question

Write the degree of the differential equation $a^2 \frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/4}$.

Answer

We are given

$$a^2 \frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{4}}$$

$$\Rightarrow \left(a^2 \frac{d^2y}{dx^2}\right)^4 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^1$$

$$\Rightarrow a^8 . \left(\frac{d^2y}{dx^2}\right)^4 = 1 + \left(\frac{dy}{dx}\right)^2$$

Here the order of differential equation is 2 and its degree is 4.

8. Question

Write the order of the differential equation $1+\left(\frac{dy}{dx}\right)^2 = 7\left(\frac{d^2y}{dx^2}\right)^3$.

Answer

We are given

$$1 + \left(\frac{dy}{dx}\right)^2 = 7.\left(\frac{d^2y}{dx^2}\right)^3$$

Here the Order of differential equation is 2 and its Degree is 3.

8. Question

Write the order of the differential equation $1+\left(\frac{dy}{dx}\right)^2 = 7\left(\frac{d^2y}{dx^2}\right)^3$.

Answer

We are given

$$1 + \left(\frac{dy}{dx}\right)^2 = 7.\left(\frac{d^2y}{dx^2}\right)^3$$

Here the Order of differential equation is 2 and its Degree is 3.

9. Question

Write the order and degree of the differential equation $y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Answer

We are given

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow y - x\frac{dy}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Squaring both sides, we get,

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = a^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2.y.x\frac{dy}{dx} = a^2 + a^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(x^2 - a^2) - 2.y.x\frac{dy}{dx} + y^2 - a^2 = 0$$

Here the order of differential equation is 1 and its degree is 2.

9. Question

Write the order and degree of the differential equation $y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

Answer

We are given

$$y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow y - x\frac{dy}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Squaring both sides, we get,

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)^2 = a^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

$$\Rightarrow y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2.y.x\frac{dy}{dx} = a^2 + a^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(x^2 - a^2) - 2.y.x\frac{dy}{dx} + y^2 - a^2 = 0$$

Here the order of differential equation is 1 and its degree is 2.

10. Question

Write the degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$.

Answer

We are given

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2x^2\log\left(\frac{d^2y}{dx^2}\right)$$

Here the order of differential equation is 2 and its degree is not defined as highest order derivative is a function of logarithmic function and it is not a polynomial equation in derivatives.

10. Question

Write the degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$.

Answer

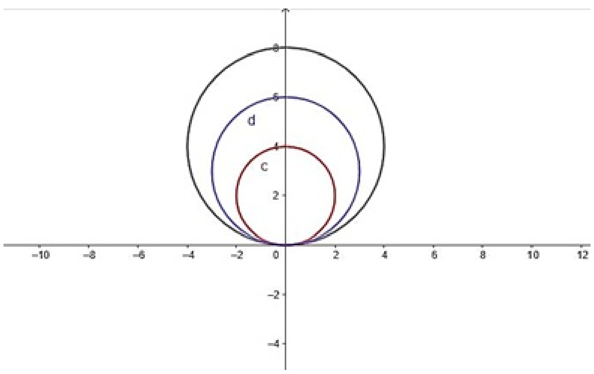
We are given

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$$

Here the order of differential equation is 2 and its degree is not defined as highest order derivative is a function of logarithmic function and it is not a polynomial equation in derivatives.

11. Question

Write the order of the differential equation of the family of circles touching X-axis at the origin.



Answer

Differential Equation of the family of circles touching X-axis at the origin is

$= x^2 + (y - k)^2 = k^2$

Here (0,k) is the center of the circle and radius k.

$= x^2 + y^2 + k^2 - 2 \times k \times y = k^2$

$= x^2 + y^2 - 2 \times k \times y = 0$ ---(1)

Differentiate w.r.t x we get,

$$\Rightarrow 2x + 2y.\frac{dy}{dx} - 2k.\frac{dy}{dx} = 0$$

$$\Rightarrow x + \frac{dy}{dx}(y - k) = 0$$

$$\Rightarrow k = \frac{x + \frac{dy}{dx}.y}{\frac{dy}{dx}}$$

Put value of k in (1) we get,

$$\Rightarrow x^2 + y^2 - 2.y.\left(\frac{x + \frac{dy}{dx}.y}{\frac{dy}{dx}}\right) = 0$$

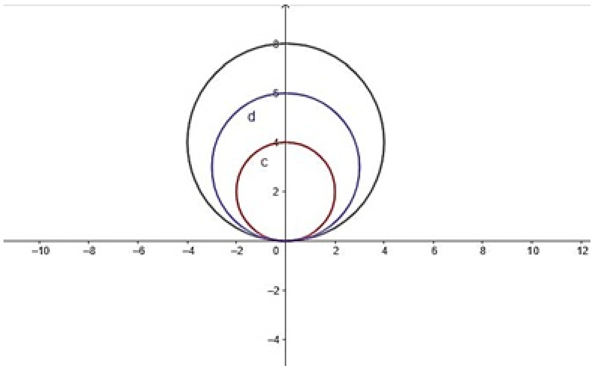
$$\Rightarrow \frac{dy}{dx}(x^2 + y^2) - 2.y.x - 2.y^2\frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx}(x^2 - y^2) - 2.y.x = 0$ is the differential equation of the family of circles touching X-axis at the origin.

Here the order of differential equation of the family of circles touching X-axis at the origin is 1.

11. Question

Write the order of the differential equation of the family of circles touching X-axis at the origin.



Answer

Differential Equation of the family of circles touching X-axis at the origin is

$= x^2 + (y - k)^2 = k^2$

Here (0,k) is the center of the circle and radius k.

$= x^2 + y^2 + k^2 - 2 \times k \times y = k^2$

$= x^2 + y^2 - 2 \times k \times y = 0$ ---(1)

Differentiate w.r.t x we get,

$$\Rightarrow 2x + 2y.\frac{dy}{dx} - 2k.\frac{dy}{dx} = 0$$

$$\Rightarrow x + \frac{dy}{dx}(y - k) = 0$$

$$\Rightarrow k = \frac{x + \frac{dy}{dx}.y}{\frac{dy}{dx}}$$

Put value of k in (1) we get,

$$\Rightarrow x^2 + y^2 - 2.y.\left(\frac{x + \frac{dy}{dx}.y}{\frac{dy}{dx}}\right) = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2 + y^2) - 2.y.x - 2.y^2\frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx}(x^2 - y^2) - 2.y.x = 0$ is the differential equation of the family of circles touching X-axis at the origin.

Here the order of differential equation of the family of circles touching X-axis at the origin is 1.

12. Question

Write the order of the differential equation of all non-horizontal lines in a plane.

Answer

We know equation of a line in a plane is

ax + by = 1

Now equation of non-horizontal lines in a plane is given by, ax + by = 1, a ≠ 0

Now a and b are two constants here we differentiate twice w.r.t y we get,

$\Rightarrow a.\frac{dx}{dy} + b.1 = 0$

$\Rightarrow a.\frac{d^2x}{dy^2} + 0 = 0$

$\Rightarrow a.\frac{d^2x}{dy^2} = 0$

Since a ≠ 0 then,

$\Rightarrow \frac{d^2x}{dy^2} = 0$ is the differential equation of all non-horizontal lines in a plane.

Here the order of differential equation of all non-horizontal lines in a plane is 2.

12. Question

Write the order of the differential equation of all non-horizontal lines in a plane.

Answer

We know equation of a line in a plane is

ax + by = 1

Now equation of non-horizontal lines in a plane is given by,

ax + by = 1, a ≠ 0

Now a and b are two constants here we differentiate twice w.r.t y we get,

$\Rightarrow a.\frac{dx}{dy} + b.1 = 0$

$\Rightarrow a.\frac{d^2x}{dy^2} + 0 = 0$

$\Rightarrow a.\frac{d^2x}{dy^2} = 0$

Since a ≠ 0 then,

$\Rightarrow \frac{d^2x}{dy^2} = 0$ is the differential equation of all non-horizontal lines in a plane.

Here the order of differential equation of all non-horizontal lines in a plane is 2.

13. Question

If sin x is an integrating factor of the differential equation dy/dx + Py = Q, then write the value of P.

Answer

Since $\frac{dy}{dx} + Py = Q$ is a linear differential equation

Integrating factor = $e^{\int p\, dx}$ = sin x (Given)

Taking log both sides we get,

⇒ log ($e^{\int p\, dx}$) = log (sin x)

⇒ ∫ p dx log (e)= log (sin x)

⇒ ∫ p dx = log (sin x) ∵ log (e) = 1

Differentiate w.r.t x we get,

$$\Rightarrow \frac{d}{dx}\left(\int p\,dx\right) = \frac{d}{dx}\log(\sin x)$$

$$\Rightarrow p = \frac{1}{\sin x}.cosx = \cot x$$

$\therefore P = \cot x$

13. Question

If sin x is an integrating factor of the differential equation dy/dx + Py = Q, then write the value of P.

Answer

Since $\frac{dy}{dx} + Py = Q$ is a linear differential equation

Integrating factor = $e^{\int p\,dx}$ = sin x (Given)

Taking log both sides we get,

⇒ log ($e^{\int p\,dx}$) = log (sin x)

⇒ ∫ p dx log (e)= log (sin x)

⇒ ∫ p dx = log (sin x) ∵ log (e) = 1

Differentiate w.r.t x we get,

$$\Rightarrow \frac{d}{dx}\left(\int p\,dx\right) = \frac{d}{dx}\log(\sin x)$$

$$\Rightarrow p = \frac{1}{\sin x}.cosx = \cot x$$

$\therefore P = \cot x$

14. Question

Write the order of the differential equation of the family of circles of radius r.

Answer

To find the Order we first need to find the differential equation of the family of circles of radius r.

In general, the equation of circle with center (a,b) and radius r is given by,

$(x - a)^2 + (y - b)^2 = r^2$ ---(1)

Differentiating above equation w.r.t x

$$\Rightarrow 2(x - a) + 2(y - b).\frac{dy}{dx} = 0$$

$\Rightarrow (x - a) + (y - b).\frac{dy}{dx} = 0 \Rightarrow (x - a) = -(y - b).\frac{dy}{dx}$ ---(2)

Again differentiating above equation w.r.t x

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right).\frac{dy}{dx} + \frac{d^2y}{dx^2}(y - b) = 0$$

$\Rightarrow (y - b) = -\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$ ---(3)

Putting value of (2) and (3) in (1) we get,

$$\Rightarrow \left[-(y-b).\frac{dy}{dx}\right]^2 + \left\{-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\right\}^2 = r^2$$

$$\Rightarrow \left\{-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}.\frac{dy}{dx}\right\}^2 + \left\{-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\right\}^2 = r^2$$

$$\Rightarrow \left\{\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\right\}^2 \left[\left(\frac{dy}{dx}\right)^2 + 1\right] = r^2$$

$$\Rightarrow \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2 .r^2$$

is the required differential equation.

Here the order of differential equation of the family of circles of radius r is 2.

14. Question

Write the order of the differential equation of the family of circles of radius r.

Answer

To find the Order we first need to find the differential equation of the family of circles of radius r.

In general, the equation of circle with center (a,b) and radius r is given by,

$(x - a)^2 + (y - b)^2 = r^2$ ---(1)

Differentiating above equation w.r.t x

$$\Rightarrow 2(x-a) + 2(y-b).\frac{dy}{dx} = 0$$

$\Rightarrow (x-a) + (y-b).\frac{dy}{dx} = 0 \Rightarrow (x-a) = -(y-b).\frac{dy}{dx}$ ---(2)

Again differentiating above equation w.r.t x

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right).\frac{dy}{dx} + \frac{d^2y}{dx^2}(y-b) = 0$$

$\Rightarrow (y-b) = -\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$ ---(3)

Putting value of (2) and (3) in (1) we get,

$$\Rightarrow \left[-(y-b).\frac{dy}{dx}\right]^2 + \left\{-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\right\}^2 = r^2$$

$$\Rightarrow \left\{-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}.\frac{dy}{dx}\right\}^2 + \left\{-\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\right\}^2 = r^2$$

$$\Rightarrow \left\{\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\right\}^2 \left[\left(\frac{dy}{dx}\right)^2+1\right] = r^2$$

$$\Rightarrow \left[\left(\frac{dy}{dx}\right)^2+1\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2 . r^2$$

is the required differential equation.

Here the order of differential equation of the family of circles of radius r is 2.

15. Question

Write the order of the differential equation whose solution is y = a cos x + b sin x + c e^{-x}.

Answer

Solution of differential equation is

y = a cos x + b sin x + c e^{-x} ---(1)

Since it has 3 constants a, b, c we differentiate it by 3 times

Differentiate w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = -a.\sin x + b.\cos x - ce^{-x}$ ---(2)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^2y}{dx^2} = -a.\cos x - b.\sin x + ce^{-x}$ ---(3)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^3y}{dx^3} = a.\sin x - b.\cos x - ce^{-x}$ ---(4)

Equation (3) implies

$\Rightarrow \frac{d^2y}{dx^2} = -(y - ce^{-x}) + ce^{-x}$ From (1)

$$\Rightarrow \frac{d^2y}{dx^2} = 2ce^{-x} - y$$

$\Rightarrow ce^{-x} = \frac{1}{2}\left(\frac{d^2y}{dx^2} + y\right)$ ---(5)

Now, adding equation (2) and (4) we get,

$$\Rightarrow \frac{dy}{dx} + \frac{d^3y}{dx^3} = -2ce^{-x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{d^3y}{dx^3} = -2\left[\frac{1}{2}\left(\frac{d^2y}{dx^2} + y\right)\right]$$

$$\Rightarrow \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

is the required differential equation.

Here the order of differential equation is 3.

15. Question

Write the order of the differential equation whose solution is y = a cos x + b sin x + c e^{-x}.

Answer

Solution of differential equation is

y = a cos x + b sin x + c e^{-x} ---(1)

Since it has 3 constants a, b, c we differentiate it by 3 times

Differentiate w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = -a.\sin x + b.\cos x - ce^{-x}$ ---(2)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^2y}{dx^2} = -a.\cos x - b.\sin x + ce^{-x}$ ---(3)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^3y}{dx^3} = a.\sin x - b.\cos x - ce^{-x}$ ---(4)

Equation (3) implies

$\Rightarrow \frac{d^2y}{dx^2} = -(y - ce^{-x}) + ce^{-x}$ From (1)

$$\Rightarrow \frac{d^2y}{dx^2} = 2ce^{-x} - y$$

$\Rightarrow ce^{-x} = \frac{1}{2}\left(\frac{d^2y}{dx^2} + y\right)$ ---(5)

Now, adding equation (2) and (4) we get,

$$\Rightarrow \frac{dy}{dx} + \frac{d^3y}{dx^3} = -2ce^{-x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{d^3y}{dx^3} = -2\left[\frac{1}{2}\left(\frac{d^2y}{dx^2} + y\right)\right]$$

$$\Rightarrow \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

is the required differential equation.

Here the order of differential equation is 3.

16. Question

Write the order of the differential equation associated with the primitive y = C_1 + C_2 e^x + C_3 e^{-2x} + C_4, where C_1, C_2, C_3, C_4 are arbitrary constants.

Answer

y = C_1 + C_2 e^x + C_3 e^{-2x} + C_4 ---(1)

Since it has 4 arbitrary constants C_1, C_2, C_3, C_4 we differentiate it by 4 times

Differentiate w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = C_2e^x - 2C_3x^{-2x}$ ---(2)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^2y}{dx^2} = C_2e^x + 4C_3x^{-2x}$ ---(3)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^3y}{dx^3} = C_2e^x - 8C_3x^{-2x}$ ---(4)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^4y}{dx^4} = C_2e^x + 16C_3x^{-2x}$ ---(5)

Now, (2) – (3) we get

$\Rightarrow \frac{dy}{dx} - \frac{d^2y}{dx^2} = -6C_3x^{-2x}$ ---(6)

Now, (4) – (5) we get

$$\Rightarrow \frac{d^3y}{dx^3} - \frac{d^4y}{dx^4} = -24C_3x^{-2x} = 4 \times -6C_3x^{-2x}$$

$\Rightarrow \frac{d^3y}{dx^3} - \frac{d^4y}{dx^4} = 4\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right)$ From (6)

$$\Rightarrow \frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} + 4\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right) = 0$$

is the required differential equation

Here the order of differential equation is 4.

16. Question

Write the order of the differential equation associated with the primitive $y = C_1 + C_2\, e^x + C_3\, e^{-2x} + C_4$, where C_1, C_2, C_3, C_4 are arbitrary constants.

Answer

$y = C_1 + C_2\, e^x + C_3\, e^{-2x} + C_4$ ---(1)

Since it has 4 arbitrary constants C_1, C_2, C_3, C_4 we differentiate it by 4 times

Differentiate w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = C_2e^x - 2C_3x^{-2x}$ ---(2)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^2y}{dx^2} = C_2e^x + 4C_3x^{-2x}$ ---(3)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^3y}{dx^3} = C_2e^x - 8C_3x^{-2x}$ ---(4)

Again, differentiate w.r.t x we get,

$\Rightarrow \frac{d^4y}{dx^4} = C_2e^x + 16C_3x^{-2x}$ ---(5)

Now, (2) – (3) we get

$\Rightarrow \frac{dy}{dx} - \frac{d^2y}{dx^2} = -6C_3x^{-2x}$ ---(6)

Now, (4) – (5) we get

$$\Rightarrow \frac{d^3y}{dx^3} - \frac{d^4y}{dx^4} = -24C_3x^{-2x} = 4 \times -6C_3x^{-2x}$$

$\Rightarrow \frac{d^3y}{dx^3} - \frac{d^4y}{dx^4} = 4\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right)$ From (6)

$$\Rightarrow \frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} + 4\left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right) = 0$$

is the required differential equation

Here the order of differential equation is 4.

17. Question

What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x.$$

Answer

$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

17. Question

What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x.$$

Answer

$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

18. Question

Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$

Answer

$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

18. Question

Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$

Answer

$\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

19. Question

Write the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0.$

Answer

$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

19. Question

Write the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0.$

Answer

$5x\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} - 6y = \log x$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

20. Question

Write the differential equation representing family of curves y = mx, where m is arbitrary constant.

Answer

We are given the equation representing family of curves as,

y = mx --(1)

Differentiate w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = m$

Put value of m in equation (1) we get,

$\Rightarrow y = \frac{dy}{dx}.x$

is the required differential equation representing family of curves y = mx.

20. Question

Write the differential equation representing family of curves y = mx, where m is arbitrary constant.

Answer

We are given the equation representing family of curves as,

y = mx --(1)

Differentiate w.r.t x we get,

$$\Rightarrow \frac{dy}{dx} = m$$

Put value of m in equation (1) we get,

$$\Rightarrow y = \frac{dy}{dx}.x$$

is the required differential equation representing family of curves y = mx.

21. Question

Write the degree of the differential equation $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$

Answer

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^2$

∴ Degree = 2

21. Question

Write the degree of the differential equation $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$

Answer

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^2$

∴ Degree = 2

22. Question

Write the degree of the differential equation $\left(1 + \frac{d^2y}{dx^2}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2.$

Answer

$$\left(1 + \frac{d^2y}{dx^2}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow 1^3 + \left(\frac{d^2y}{dx^2}\right)^3 + 3(1)^2\left(\frac{d^2y}{dx^2}\right) + 3(1)\left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow 1 + \left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{d^2y}{dx^2}\right)^2 = 0$$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^3$

∴ Degree = 3

22. Question

Write the degree of the differential equation $\left(1+\frac{d^2y}{dx^2}\right)^3=\left(\frac{d^2y}{dx^2}\right)^2$.

Answer

$\left(1+\frac{d^2y}{dx^2}\right)^3=\left(\frac{d^2y}{dx^2}\right)^2$

$\Rightarrow 1^3+\left(\frac{d^2y}{dx^2}\right)^3+3(1)^2\left(\frac{d^2y}{dx^2}\right)+3(1)\left(\frac{d^2y}{dx^2}\right)^2=\left(\frac{d^2y}{dx^2}\right)^2$

$\Rightarrow 1+\left(\frac{d^2y}{dx^2}\right)^3+3\left(\frac{d^2y}{dx^2}\right)+2\left(\frac{d^2y}{dx^2}\right)^2=0$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^3$

∴ Degree = 3

23. Question

Write degree of the differential equation $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$

Answer

$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$

Here Order of differential equation is 2.

Its degree is not defined as highest order derivative is a function of logarithmic function and it is not a polynomial equation in derivatives.

23. Question

Write degree of the differential equation $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$

Answer

$\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$

Here Order of differential equation is 2.

Its degree is not defined as highest order derivative is a function of logarithmic function and it is not a polynomial equation in derivatives.

24. Question

Write the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=x\sin\left(\frac{dy}{dx}\right)$.

Answer

$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$

Here Order of differential equation is 2.

Its degree is not defined as it is not a polynomial equation in derivatives.

24. Question

Write the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$.

Answer

$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$

Here Order of differential equation is 2.

Its degree is not defined as it is not a polynomial equation in derivatives.

25. Question

Write the order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} + x^{1/5} = 0$.

Answer

$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{4}} + x^{\frac{1}{5}} = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)^{\frac{1}{4}} = -\left(x^{\frac{1}{5}} + \frac{d^2y}{dx^2}\right)$

$\Rightarrow \frac{dy}{dx} = \left(x^{\frac{1}{5}} + \frac{d^2y}{dx^2}\right)^4$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^4$

∴ Degree = 4

25. Question

Write the order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} + x^{1/5} = 0$.

Answer

$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{4}} + x^{\frac{1}{5}} = 0$

$\Rightarrow \left(\frac{dy}{dx}\right)^{\frac{1}{4}} = -\left(x^{\frac{1}{5}} + \frac{d^2y}{dx^2}\right)$

$\Rightarrow \frac{dy}{dx} = \left(x^{\frac{1}{5}} + \frac{d^2y}{dx^2}\right)^4$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^4$

∴ Degree = 4

26. Question

The degree of the differential equation $\frac{d^2y}{dx^2} + e^{dy/dx} = 0.$

Answer

$\frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} = 0$

Here Order of differential equation is 2

Its degree is not defined as one derivative is exponent of exponential function and it is not a polynomial equation in derivatives

26. Question

The degree of the differential equation $\frac{d^2y}{dx^2} + e^{dy/dx} = 0.$

Answer

$\frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} = 0$

Here Order of differential equation is 2

Its degree is not defined as one derivative is exponent of exponential function and it is not a polynomial equation in derivatives

27. Question

How many arbitrary constants are there in the general solution of the differential equation of order 3.

Answer

Let any differential equation of order 3 be

$$\frac{d^3y}{dx^3} = A \quad --(1)$$

Here A is any constant.

Now, to know the number of arbitrary constants in the general solution we integrate both sides of equation (1)

$$\Rightarrow \int \frac{d^3y}{dx^3} dx = \int A\, dx$$

$$\Rightarrow \frac{d^2y}{dx^2} = A\,x + C_1$$

Again integrating

$$\Rightarrow \int \frac{d^2y}{dx^2} dx = \int (A\,x + C_1) dx$$

$$\Rightarrow \frac{dy}{dx} = A\frac{x^2}{2} + C_1x + C_2$$

Again integrating

$$\Rightarrow \int \frac{dy}{dx} dx = \int \left(A\frac{x^2}{2} + C_1 x + C_2 \right) dx$$

$$\Rightarrow y = \frac{Ax^3}{6} + \frac{C_1 x^2}{2} + C_2 x + C_3$$

is the general solution of the differential equation with 3 arbitrary constants C_1, C_2, C_3.

∴ There are 3 arbitrary constants in the general solution of the differential equation of order 3.

27. Question

How many arbitrary constants are there in the general solution of the differential equation of order 3.

Answer

Let any differential equation of order 3 be

$$\frac{d^3y}{dx^3} = A \quad --(1)$$

Here A is any constant.

Now, to know the number of arbitrary constants in the general solution we integrate both sides of equation (1)

$$\Rightarrow \int \frac{d^3y}{dx^3} dx = \int A\, dx$$

$$\Rightarrow \frac{d^2y}{dx^2} = A\,x + C_1$$

Again integrating

$$\Rightarrow \int \frac{d^2y}{dx^2} dx = \int (A\,x + C_1) dx$$

$$\Rightarrow \frac{dy}{dx} = A\frac{x^2}{2} + C_1 x + C_2$$

Again integrating

$$\Rightarrow \int \frac{dy}{dx} dx = \int \left(A\frac{x^2}{2} + C_1 x + C_2 \right) dx$$

$$\Rightarrow y = \frac{Ax^3}{6} + \frac{C_1 x^2}{2} + C_2 x + C_3$$

is the general solution of the differential equation with 3 arbitrary constants C_1, C_2, C_3.

∴ There are 3 arbitrary constants in the general solution of the differential equation of order 3.

28. Question

Write the order of the differential equation representing the family of curves $y = ax + a^3$.

Answer

We are given

$y = ax + a^3$ -- (1)

Differentiating w.r.t x we get

$$\Rightarrow \frac{dy}{dx} = a$$

Put value of a in equation (1) we get,

$$\Rightarrow y = x.\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$$

is the required differential equation.

Since order is the highest order derivative present in the differential equation.

∴ Order = 1 and Degree = 3.

28. Question

Write the order of the differential equation representing the family of curves $y = ax + a^3$.

Answer

We are given

$y = ax + a^3$ -- (1)

Differentiating w.r.t x we get

$$\Rightarrow \frac{dy}{dx} = a$$

Put value of a in equation (1) we get,

$$\Rightarrow y = x.\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3$$

is the required differential equation.

Since order is the highest order derivative present in the differential equation.

∴ Order = 1 and Degree = 3.

29. Question

Find the sum of the order and degree of the differential equation $y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$.

Answer

$$y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$$

Order = Highest order derivative present in the differential equation.

∴ Order = 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

∴ Sum of the order and degree = 2 + 1 = 3

29. Question

Find the sum of the order and degree of the differential equation $y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$.

Answer

$$y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}$$

Order = Highest order derivative present in the differential equation.

∴ Order = 2

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1

∴ Sum of the order and degree = 2 + 1 = 3

30. Question

Find the solution of the differential equation $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0.$

Answer

$x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$

We can write above differential equation as

$\Rightarrow y\sqrt{1+x^2}dy = -x\sqrt{1+y^2}dx$

By the method of variable separable we can write,

$\Rightarrow \frac{y}{\sqrt{1+y^2}}dy = -\frac{x}{\sqrt{1+x^2}}dx$

Integrating both sides,

$\Rightarrow \int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx$

Let 1 + y^2 = t and 1 + x^2 = u

⇒ 2y dy = dt ⇒ 2x dx = du

$\Rightarrow y\,dy = \frac{1}{2}\,dt \;\Rightarrow\; x\,dx = \frac{1}{2}\,du$

Putting values in integral we get,

$\Rightarrow \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$

$\Rightarrow \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) = -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$

$\Rightarrow \sqrt{t} = -\sqrt{u} + C$

Putting values of t and u,

$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + C$

Where C is the arbitrary constant.

$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + C$ is the required solution of the differential equation.

30. Question

Find the solution of the differential equation $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0.$

Answer

$x\sqrt{1+y^2}dx + y\sqrt{1+x^2}dy = 0$

We can write above differential equation as

$\Rightarrow y\sqrt{1+x^2}dy = -x\sqrt{1+y^2}dx$

By the method of variable separable we can write,

$$\Rightarrow \frac{y}{\sqrt{1+y^2}}dy = -\frac{x}{\sqrt{1+x^2}}dx$$

Integrating both sides,

$$\Rightarrow \int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx$$

Let 1 + y^2 = t and 1 + x^2 = u

⇒ 2y dy = dt ⇒ 2x dx = du

$$\Rightarrow y\,dy = \frac{1}{2}\,dt \;\Rightarrow\; x\,dx = \frac{1}{2}\,du$$

Putting values in integral we get,

$$\Rightarrow \frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$\Rightarrow \frac{1}{2}\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) = -\frac{1}{2}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$\Rightarrow \sqrt{t} = -\sqrt{u} + C$$

Putting values of t and u,

$$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + C$$

Where C is the arbitrary constant.

$\Rightarrow \sqrt{1+y^2} + \sqrt{1+x^2} + C$ is the required solution of the differential equation.

MCQ

1. Question

Mark the correct alternative in each of the following:

The integrating factor of the differential equation $(x \log x)\frac{dy}{dx} + y = 2\log x,$ is given by

A. log (log x)

B. e^x

C. log x

D. x

Answer

$(x\log x)\frac{dy}{dx} + y = 2\log x$

Dividing x.log x both sides we get,

$$\Rightarrow \frac{dy}{dx} + y.\frac{1}{x\log x} = \frac{2}{x}$$

The above equation is of the form $\frac{dy}{dx} + Py = Q$ i.e. linear differential equation.

$$Here, P = \frac{1}{x\log x} \; and \; Q = \frac{2}{x}$$

Integrating factor = $e^{\int P\,dx}$

Considering ∫ P

$$\Rightarrow \int P\,dx = \int \frac{1}{x\log x}\,dx$$

Put log x = t

$$\Rightarrow \frac{1}{x}dx = dt$$

Putting values, we get,

$$\Rightarrow \int P\,dx = \int \frac{1}{t}\,dt = \log t$$

$$\Rightarrow \int P\,dx = \log(\log x)$$

$$\therefore e^{\int Pdx} = e^{\log(\log x)} = \log x$$

$\because e^{\log x} = x$

∴ I.F = log x = (C)

2. Question

Mark the correct alternative in each of the following:

The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is:

A. log y = kx

B. y = kx

C. xy = k

D. y = k log x

Answer

Given:

$$\frac{dy}{dx} = \frac{y}{x}$$

By the method of Variable separable we get,

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

Taking integral both sides

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

⇒log y=log x + log k

where log k is an arbitrary constant.

⇒log y=log x . k

∵ log x + log y = log xy

⇒y = kx

=(B)

is the required general solution.

3. Question

Mark the correct alternative in each of the following:

Integrating factor of the differential equation $\cos x \frac{dy}{dx} + y \sin x = 1,$ is

A. sin x

B. sec x

C. tan x

D. cos x

Answer

$\cos x \frac{dy}{dx} + y \sin x = 1$

Dividing cos x both sides we get,

$$\Rightarrow \frac{dy}{dx} + y.\frac{\sin x}{\cos x} = \frac{1}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} + y.\tan x = \sec x$$

The above equation is of the form $\frac{dy}{dx} + Py = Q$ i.e. linear differential equation.

$Here, P = \tan x \ and \ Q = \sec x$

Integrating factor = $e^{\int P\, dx}$

Considering ∫ P

$$\Rightarrow \int P\, dx = \int \tan x \ dx$$

$$\Rightarrow \int P\, dx = \log|\sec x|$$

$\therefore e^{\int Pdx} = e^{\log|\sec x|}$

=sec x

$\because e^{\log x} = x$

∴ I.F = sec x = (B)

4. Question

Mark the correct alternative in each of the following:

The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) = y^3,$ is

A. 1/2

B. 2

C. 3

D. 4

Answer

$\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) = y^3$

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^2$

∴ Degree = 2

= (B)

5. Question

Mark the correct alternative in each of the following:

The degree of the differential equation $\left\{5+\left(\frac{dy}{dx}\right)^2\right\}^{5/3} = x^5\left(\frac{d^2y}{dx^2}\right)$, is

A. 4

B. 2

C. 5

D. 10

Answer

$\left\{5+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{5}{3}} = x^5\left(\frac{d^2y}{dx^2}\right)$

Cubing both sides, we get,

$\Rightarrow \left\{5+\left(\frac{dy}{dx}\right)^2\right\}^5 = x^{15}\left(\frac{d^2y}{dx^2}\right)^3$

Order = Highest order derivative present in the differential equation.

Here Order of differential equation is 2

Degree = Highest power of highest order derivative which is $\left(\frac{d^2y}{dx^2}\right)^3$

∴ Degree = 3

6. Question

Mark the correct alternative in each of the following:

The general solution of the differential equation $\frac{dy}{dx} + y\cot x = \text{cosec } x$, is

A. x + y sinx = C

B. x + y cosx = C

C. y + x (sin x + cos x) = C

D. y sin x = x + C

Answer

$\frac{dy}{dx} + y\cot x = \csc x$

The above equation is of the form $\frac{dy}{dx} + Py = Q$ i.e. linear differential equation.

Here, P = cot x and Q = cosec x

Integrating factor = $e^{\int P\,dx}$

Considering ∫ P

$$\Rightarrow \int P\,dx = \int \cot x\ dx$$

$$\Rightarrow \int P\,dx = \log|\sin x|$$

$\therefore e^{\int Pdx} = e^{\log|\sin x|} = \sin x$ ∵ $e^{\log x} = x$

∴ I.F = sin x

Now, General solution is

⇒ y (I.F) = ∫ Q (I.F) dx + C

⇒ y sin x = ∫ cosec x sin x dx + C

$$\Rightarrow y \sin x = \int \frac{1}{\sin x}.\sin x\,dx + C$$

$$\Rightarrow y \sin x = \int 1\,dx + C$$

⇒y sin x = x + C

=(D)

7. Question

Mark the correct alternative in each of the following:

The differential equation obtained on eliminating A and B from y = A cos ωt + B sin ωt, is

A. $y^n + y' = 0$

B. $y^n - \omega^2 y = 0$

C. $y^n = -\omega^2 y$

D. $y^n + y = 0$

Answer

y = A cos ωt + B sin ωt ---(1)

Since we need to eliminate A and B, so we differentiate (1) twice.

Differentiating (1) w.r.t t we get,

$$\Rightarrow \frac{dy}{dt} = -A\omega \sin \omega t + B\omega \cos \omega t$$

Again differentiating (1) w.r.t t we get,

$$\Rightarrow \frac{d^2y}{dt^2} = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

$$\Rightarrow \frac{d^2y}{dt^2} = -\omega^2(A\cos \omega t + B\sin \omega t) = -\omega^2 y$$

$\Rightarrow y'' = -\omega^2 y$

=(C)

8. Question

Mark the correct alternative in each of the following:

The equation of the curve whose slope is given by $\frac{dy}{dx} = \frac{2y}{x}$; x > 0, y > 0 and which passes through the point (1, 1) is

A. $x^2 = y$

B. $y^2 = x$

C. $x^2 = 2y$

D. $y^2 = 2x$

Answer

Slope of the curve is given by

$$\frac{dy}{dx} = \frac{2y}{x} \; ; x > 0, y > 0$$

By Variable separable

$$\Rightarrow \frac{dy}{2y} = \frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \int \frac{dy}{2y} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log y = \log x + \log C$$

$$\Rightarrow \log y^{\frac{1}{2}} = \log Cx$$

∵ log x + log y = log xy

$\Rightarrow \sqrt{y} = Cx$ -- (1)

Since curve passes through the point (1, 1), we get

$\Rightarrow \sqrt{1} = C.1$

$\Rightarrow C = 1$

Putting value of C in (1) we get,

$\Rightarrow x = \sqrt{y}$

Squaring both sides

$\Rightarrow x^2 = y$

=(A)

9. Question

Mark the correct alternative in each of the following:

The order of the differential equation whose general solution is given by

$y = c_1 \cos(2x + c_2) - (c_3 + c_4)\, a^{x + c_5} + c_6 \sin(x - c_7)$ is

A. 3

B. 4

C. 5

D. 2

Answer

$y = c_1\cos(2x + c_2) - (c_3 + c_4)\, a^{x+c5} + c_6 \sin(x - c_7)$

$\Rightarrow y = c_1 [\cos(2x). \cos c_2 - \sin(2x). \sin c$ �$_2] - (c_3 + c_4)\, a^{c5}. a^x + c_6[\sin(x). \cos c_7 - \cos(x). \sin c_7)$

$\Rightarrow y = c_1.\cos c_2 . \cos(2x) - c_1. \sin c$ �$_2. \sin(2x) - (c_3 + c_4)\, a^{c5}. a^x + c_6. \cos c_7$ � �$. \sin(x) - c_6. \sin c_7.\cos(x)$

Now, $c_1.\cos c_2, c_1. \sin c$ �$_2$, $(c_3 + c_4)\, a^{c5}$, $c_6. \cos c_7$ � �, $c_6. \sin c_7$ are all constants

$\therefore c_1.\cos c_2 = A$

$c_1. \sin c$ �$_2 = B$

$(c_3 + c_4)\, a^{c5} = C$

$c_6. \cos c_7 = D$

$c_6. \sin c_7 = E$

$\Rightarrow y = A. \cos(2x) - B. \sin(2x) - C. a^x + D$ � �$. \sin(x) - E. \cos(x)$

Where A, B, C, D and E are constants

Since there are 5 constants, we have to differentiate y w.r.t x five times.

So, the Order of the differential equation = 5

= (C)

10. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} = \frac{ax + g}{by + f}$ represents a circle when

A. a = b

B. a = –b

C. a = –2b

D. a = 2b

Answer

$\frac{dy}{dx} = \frac{ax+g}{by+f}$

By Variable separable,

$\Rightarrow (by + f)\, dy = (ax + g)\, dx$

Integrating both sides

$\Rightarrow \int (by + f)\, dy = \int (ax + g)\, dx$

$\Rightarrow\ b.\frac{y^2}{2} + fy = a.\frac{x^2}{2} + gx + C$

$\Rightarrow by^2 + 2fy = ax^2 + 2gx + C'$

Where C’ = 2C

We know general solution of circle is

$(x - h)^2 + (y - k)^2 = r^2$

Where (h, k) is center of the circle and r is radius.

By completing square method

$$\Rightarrow b\left[y^2 + \frac{2fy}{b} + \left(\frac{f}{b}\right)^2 - \left(\frac{f}{b}\right)^2\right] = a\left[x^2 + \frac{2gx}{a} + \left(\frac{g}{a}\right)^2 - \left(\frac{g}{a}\right)^2\right] + C'$$

$$\Rightarrow b\left[\left(y + \frac{f}{b}\right)^2 - \left(\frac{f}{b}\right)^2\right] = a\left[\left(x + \frac{g}{a}\right)^2 - \left(\frac{g}{a}\right)^2\right] + C'$$

$$\Rightarrow a.\left(x + \frac{g}{a}\right)^2 - b.\left(y + \frac{f}{b}\right)^2 = \left(\frac{g}{a}\right)^2 - \left(\frac{f}{b}\right)^2 - C'$$

Now to form above equation as equation of circle we need

$$Coefficient\ of\ \left(x + \frac{g}{a}\right)^2 = Coefficient\ of\ \left(y + \frac{f}{b}\right)^2$$

i.e. a = -b

= (B)

11. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} + \frac{2y}{x} = 0$ with y(1) = 1 is given by

A. $y = \frac{1}{x^2}$

B. $x = \frac{1}{y^2}$

C. $x = \frac{1}{y}$

D. $y = \frac{1}{x}$

Answer

$\frac{dy}{dx} + \frac{2y}{x} = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$$

By the method of variable separable

$$\Rightarrow \frac{dy}{2y} = -\frac{dx}{x}$$

Integrating both sides we get,

$$\Rightarrow \int \frac{dy}{2y} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log y = -\log x + \log C$$

$\Rightarrow \log \sqrt{y} = \log C.\ x^{-1}$

$\because \log a + \log b = \log ab$

$\Rightarrow \sqrt{y} = \frac{C}{x}$

Squaring both sides, we get,

$\Rightarrow y = \frac{C}{x^2}$

Now the given condition is y(1) = 1

$\Rightarrow 1 = \frac{C}{1}$

⇒C=1

∴ The solution of the differential equation is

$\Rightarrow y = \frac{1}{x^2}$

=(A)

12. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} - \frac{y(x+1)}{x} = 0$ is given by

A. y = xe^{x+C}

B. x = ye^{x}

C. y = x + C

D. xy e^{x} + C

Answer

$\frac{dy}{dx} - \frac{y(x+1)}{x} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{y(x+1)}{x}$

By the method of variable separable

$\Rightarrow \frac{dy}{y} = \frac{(x+1)}{x} dx$

Integrating both sides we get,

$\Rightarrow \int \frac{dy}{y} = \int \frac{(x+1)}{x} dx$

$\Rightarrow \int \frac{dy}{y} = \int \left(1 + \frac{1}{x}\right) dx$

$\Rightarrow \log y = x + \log x + C$

$\Rightarrow \log y - \log x = x + C$

$\Rightarrow \log \frac{y}{x} = x + c \quad \because \log a - \log b \; = \log \frac{a}{b}$

$\Rightarrow \frac{y}{x} = e^{x+c}$

∴ The solution of the differential equation is

$\Rightarrow y = x.e^{x+c}$

$= (A)$

13. Question

Mark the correct alternative in each of the following:

The order of the differential equation satisfying $\sqrt{1-x^4} + \sqrt{1-y^4} = a\left(x^2 - y^2\right)$ is

A. 1

B. 2

C. 3

D. 4

Answer

The given curve is

$\sqrt{1-x^4} + \sqrt{1-y^4} = a(x^2 - y^2)$

Since the number of constants in the given curve is 1 i.e. a which is an arbitrary constant.

Also, Number of arbitrary constants in the equation of the curve = Order of the differential equation of the curve.

∴ Order = 1

= (A)

14. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $y_1 y_3 = y_2^2$ is

A. $x = C_1 e^{C2\ y} + C_3$

B. $y = C_1 e^{C2\ x} + C_3$

C. $2x = C_1 e^{C2\ y} + C_3$

D. none of these

Answer

$y_1 y_3 = y_2^2$

We can write above differential equation as

$\Rightarrow y' y''' = (y'')^2$

We know that $\int \frac{f'(x)}{f(x)} = \log f(x) + C$

$\Rightarrow \frac{y'''}{y''} = \frac{y''}{y'}$

Integrating both sides we get,

$\Rightarrow \int \frac{y'''}{y''} dx = \int \frac{y''}{y'} dx$

$\Rightarrow \log y'' = \log C_1 y'$

$\Rightarrow y'' = C_1 y'$

$\Rightarrow \frac{y''}{y'} = C_1$

Again, integrating both sides we get,

$\Rightarrow \int \frac{y''}{y'} dx = C_1 \int dx$

$\Rightarrow \log y' = C_1 x + C_2$

$\Rightarrow y' = C_2 e^{C_1 x}$

Again, integrating both sides we get,

$\Rightarrow \int dy = C_2 \int e^{C_1 x} dx$

$\Rightarrow y = \frac{C_2}{C_1} e^{C_1 x} + C_3$

∴ The solution of the differential equation is

$\Rightarrow y = C_1 e^{C_2 x} + C_3$

=(B)

15. Question

Mark the correct alternative in each of the following:

The general solution of the differential equation $\frac{dy}{dx} + yg'(x) = g(x)g'(x)$, where g(x) is a given function of x, is

A. g(x) + log {1 + y + g(x)} = C

B. g(x) + log {1 + y – g(x)} = C

C. g(x) – log {1 + y – g(x)} = C

D. none of these

Answer

$\frac{dy}{dx} + yg'(x) = g(x)g'(x)$

$Here, g'(x) = \frac{d}{dx} g(x)$

Since it is a form of linear differential equation where

P = g'(x) and Q = g(x) g'(x)

Integrating Factor (I.F) = $e^{\int p\, dx}$

I.F = $e^{\int g'(x)\, dx} = e^{g(x)}$

Solution of differential equation is given by

y.(I.F) = ∫ Q.(I.F) dx + C

⇒ y. $e^{g(x)}$ = ∫ g(x). g'(x). $e^{g(x)}$ dx + C

Consider integral ∫ g(x). g'(x). $e^{g(x)}$ dx

Put g(x) = t

⇒ g'(x) dx = dt

⇒ ∫ t. e^t dt

Treating t as first function and e^t as second function, So integrating by Parts we get,

⇒ t. e^t - ∫ 1.e^t dt + C

⇒ e^t (t – 1) + C

Putting value of t we get,

⇒ $e^{g(x)}$ (g(x) – 1) + C

∴ y. $e^{g(x)}$ = $e^{g(x)}$ (g(x) – 1) + C

Dividing $e^{g(x)}$ both sides we get,

⇒ y = (g(x) – 1) + C $e^{-g(x)}$

⇒ y - g(x) + 1 = C $e^{-g(x)}$

Taking log both sides we get,

⇒ log (y – g(x) + 1) = log (C $e^{-g(x)}$)

⇒ log (y – g(x) + 1) = log C – g(x) log e

⇒ log (y – g(x) + 1) = log C – g(x)∵ log e = 1

⇒ g(x) + log {1 + y – g(x)} = C⇒ (B) where log C = C

16. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} = 1 + x + y^2 + xy^2, y(0) = 0$ is

A. $y^2 = \exp\left(x + \frac{x^2}{2}\right) - 1$

B. $y^2 = 1 + C\exp\left(x + \frac{x^2}{2}\right)$

C. y = tan (C + x + x^2)

D. $y = \tan\left(x + \frac{x^2}{2}\right)$

Answer

$\frac{dy}{dx} = 1 + x + y^2 + xy^2$

$\Rightarrow \frac{dy}{dx} = (1 + x) + y^2(1 + x)$

$\Rightarrow \frac{dy}{dx} = (1 + x)(1 + y^2)$

By the method of variable separable

$\Rightarrow \frac{dy}{(1^2 + y^2)} = (1 + x)dx$

Integrating both sides we get,

$$\Rightarrow \int \frac{dy}{(1^2+y^2)} = \int (1+x)dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^2}{2} + C$$

Now the given condition is y(0) = 0

$$\Rightarrow \tan^{-1} 0 = C$$

$$\Rightarrow C = 0$$

$$\therefore y = \tan\left(x + \frac{x^2}{2}\right) = (D)$$

17. Question

Mark the correct alternative in each of the following:

The differential equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = C$ is

A. $\frac{y''}{y'} + \frac{y'}{y} - \frac{1}{x} = 0$

B. $\frac{y''}{y'} + \frac{y'}{y} + \frac{1}{x} = 0$

C. $\frac{y''}{y'} - \frac{y'}{y} - \frac{1}{x} = 0$

D. none of these

Answer

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = C$

Since the equation has 2 constants so we differentiate twice,

Differentiating w.r.t x

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} . \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{y}{b^2} . \frac{dy}{dx} = -\frac{x}{a^2}$$

$\Rightarrow y.y' = -x.\frac{b^2}{a^2}$ --(1)

Again, differentiating w.r.t x

$\Rightarrow y'.y' + y.y'' = -\frac{b^2}{a^2}$ --(2)

Substitute value of (2) in (1) we get,

$$\Rightarrow y.y' = x.[(y')^2 + y.y'']$$

Dividing both sides by y.y’ we get,

$$\Rightarrow 1 = x.\left[\frac{(y')^2}{y.y'} + y.\frac{y''}{y.y'}\right]$$

$\Rightarrow \frac{1}{x} = \left[\frac{y'}{y} + \frac{y''}{y'}\right]$

$\Rightarrow \frac{y''}{y'} + \frac{y'}{y} - \frac{1}{x} = 0 = (A)$

18. Question

Mark the correct alternative in each of the following:

Solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = \sin x$ is

A. x (y + cos x)= sin x + C

B. x (y – cos x)= sin x + C

C. x (y + cos x)= cos x + C

D. none of these

Answer

$\frac{dy}{dx} + \frac{y}{x} = \sin x$

Since it is a form of linear differential equation.

$Where, P = \frac{1}{x}\ and\ \mathrm{Q} = \sin x$

Integrating Factor (I.F) = $e^{\int p\, dx}$

$I.F = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Solution of differential equation is given by

y.(I.F) = ∫ Q.(I.F) dx + C

⇒ y. x = ∫ (sin x).x dx + C

⇒ y. x = ∫ (sin x).x dx + C

Consider integral ∫ (sin x).x dx

Treating x as first function and sin x as second function. So, integrating by Parts we get,

⇒ x. (-cos x) + ∫ 1.cos x dx + C

⇒ – x. cos x + sin x + C

∴ y. x = – x. cos x + sin x + C

⇒ x (y + cos x) = sin x + C = (A) is the required solution.

19. Question

Mark the correct alternative in each of the following:

The equation of the curve satisfying the differential equation

$y(x + y^3)dx = x(y^3 – x)\ dy$ and passing through the point (1, 1) is

A. $y^3 – 2x + 3x^2y = 0$

B. $y^3 + 2x + 3x^2y = 0$

C. $y^3 + 2x – 3x^2y = 0$

D. none of these

Answer

$y(x + y^3)dx = x(y^3 – x)dy$

$\Rightarrow yx\,dx + y^4\,dx = xy^3\,dy – x^2\,dy$

$\Rightarrow xy^3\,dy – x^2\,dy – yx\,dx – y^4\,dx = 0$

$\Rightarrow y^3\,[x\,dy – y\,dx] – x[x\,dy + y\,dx] = 0$

Divide both sides by y^2x^3 we get,

$$\Rightarrow \frac{y^3}{y^2x^3}[x\,dy - y\,dx] - \frac{x}{y^2x^3}[x\,dy + y\,dx] = 0$$

$$\Rightarrow \frac{y}{x^3}[x\,dy - y\,dx] - \frac{1}{y^2x^2}d(x.y) = 0$$

$$\Rightarrow \frac{y}{x}\left[\frac{x\,dy - y\,dx}{x^2}\right] - \frac{d(x.y)}{y^2x^2} = 0$$

$$\Rightarrow \frac{y}{x}\left[d\left(\frac{y}{x}\right)\right] - \frac{d(x.y)}{y^2x^2} = 0$$

Integrating both sides we get,

$$\Rightarrow \int \frac{y}{x}.d\left(\frac{y}{x}\right) - \int \frac{d(x.y)}{y^2x^2} = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{y}{x}\right)^2 - \left(-\frac{1}{xy}\right) = C$$

$$\Rightarrow \frac{1}{2}\left(\frac{y}{x}\right)^2 + \left(\frac{1}{xy}\right) = C \text{ -- (1)}$$

Now the given curve is passing through the point (1, 1)

$$\Rightarrow \frac{1}{2}(1)^2 + (1) = C$$

$$\Rightarrow C = \frac{3}{2}$$

Substituting value of C in (1) we get,

$$\Rightarrow \frac{1}{2}\left(\frac{y^2}{x^2}\right) + \left(\frac{1}{xy}\right) = \frac{3}{2}$$

$$\Rightarrow \frac{y^3 + 2x}{2x^2y} = \frac{3}{2}$$

$\Rightarrow y^3 + 2x = 3x^2y$

$\therefore y^3 + 2x – 3x^2y = 0$ = (C) is the required solution.

20. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represents

A. circles

B. straight lines

C. ellipses

D. parabolas

Answer

$2x.\frac{dy}{dx} - y = 3$

$\Rightarrow 2x.\frac{dy}{dx} = (3 + y)$

By the method of variable separable we get,

$\Rightarrow \frac{dy}{(3+y)} = \frac{dx}{2x}$

Integrating both sides,

$\Rightarrow \int \frac{dy}{(3+y)} = \int \frac{dx}{2x}$

$\Rightarrow \log(3+y) = \frac{1}{2}\log x + \log C$

$\Rightarrow \log(3+y) = \log C\sqrt{x}$ ∵ log a + log b = log ab

$\Rightarrow (y+3) = C\sqrt{x}$

Squaring both sides, we get,

$\Rightarrow (y+3)^2 = Ax$ Where A = C^2

Since it is the form of $(y - k)^2 = 4p(x - h)$ which represents parabolas.

i.e. $(y - (-3))^2 = A(x - 0)$

∴ The solution of the differential equation represents parabolas = (D)

21. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $x\frac{dy}{dx} = y + x\tan\frac{y}{x}$, is

A. $\sin\frac{x}{y} = x + C$

B. $\sin\frac{y}{x} = Cx$

C. $\sin\frac{x}{y} = Cy$

D. $\sin\frac{y}{x} = Cy$

Answer

$x\frac{dy}{dx} = y + x\tan\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ --(1)

The above equation is of the form of Homogeneous differential equation.

$Put\ \frac{y}{x} = v \Rightarrow y = xv$

Differentiate w.r.t x we get,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ --(2)

Putting value of (2) in (1) we get

$\Rightarrow v + x\frac{dv}{dx} = v + \tan v$

$\Rightarrow x\frac{dv}{dx} = v + \tan v - v$

$\Rightarrow \frac{dv}{\tan v} = \frac{dx}{x}$

$\Rightarrow \cot v\, dv = \frac{dx}{x}$

Integrating both sides,

$\Rightarrow \int \cot v\, dv = \int \frac{dx}{x}$

$\Rightarrow \log(\sin v) = \log x + \log C$

$\Rightarrow \log(\sin v) = \log Cx$

$\Rightarrow \sin v = Cx$

Putting value of v we get,

$\Rightarrow \sin\frac{y}{x} = Cx = (B)$

22. Question

Mark the correct alternative in each of the following:

The differential equation satisfied by $ax^2 + by^2 = 1$ is

A. $xyy_2 + y_1^2 + yy_1 = 0$

B. $xyy_2 + xy_1^2 - yy_1 = 0$

C. $xyy_2 - xy_1^2 + yy_1 = 0$

D. none of these

Answer

$ax^2 + by^2 = 1$

Since it has two arbitrary constants, we differentiate it twice w.r.t x

Differentiate w.r.t x

$\Rightarrow 2ax + 2by\frac{dy}{dx} = 0$

$\Rightarrow ax + by\frac{dy}{dx} = 0$ --(1)

Again, differentiate w.r.t x

$\Rightarrow a + b\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}y\right] = 0$ Applying product rule

$$\Rightarrow a = -b\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}y\right]$$

Put value of a in (1) we get,

$$\Rightarrow -xb\left[\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}y\right] + by\frac{dy}{dx} = 0$$

$$-xy\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = 0$$

$$\Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

Which is ⇒ $xyy_2 + xy_1^2 - yy_1 = 0$ =(B)

23. Question

Mark the correct alternative in each of the following:

The differential equation which represents the family of curves $y = e^{Cx}$ is

A. $y_1 = C_2 y$

B. $xy_1 - \ln y = 0$

C. $x \ln y = yy_1$

D. $y \ln y = xy_1$

Answer

$y = e^{Cx}$

Taking log both sides we get,

⇒ $\log y = \log e^{Cx}$

⇒ $\log y = Cx \log e$ ∵ $\log a^x = x \log a$

⇒ $\log y = Cx$ --(1) ∵ $\log e = 1$

Differentiate w.r.t x we get,

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = C$$

Put value of C in (1) we get,

$$\Rightarrow \log y = \frac{x}{y}\frac{dy}{dx}$$

$$\Rightarrow y\log y = x\frac{dy}{dx}$$

Which is ⇒ $y \ln y = xy_1$ = (D)

24. Question

Mark the correct alternative in each of the following:

Which of the following transformations reduce the differential equation $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ into the form $\frac{du}{dx} + P(x)u = Q(x)$.

A. u = log x

B. u = e^x

C. u = $(\log z)^{-1}$

D. u = $(\log z)^2$

Answer

$\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2$

Dividing $z(\log z)^2$ both sides we get,

$$\Rightarrow \frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{z}{x}\log z\times\frac{1}{z(\log z)^2}=\frac{1}{x^2}$$

$\Rightarrow \frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{x(\log z)}=\frac{1}{x^2}$ --(1)

Put $(\log z)^{-1}$ = u

Differentiate w.r.t x

$$\Rightarrow \frac{d}{dx}(\log z)^{-1}=\frac{du}{dx}$$

$$\Rightarrow \frac{-1}{(\log z)^2}.\frac{1}{z}.\frac{dz}{dx}=\frac{du}{dx}$$

$$\Rightarrow \frac{1}{(\log z)^2}.\frac{1}{z}.\frac{dz}{dx}=-\frac{du}{dx}$$

Putting values in (1) we get,

$$\Rightarrow -\frac{du}{dx}+\frac{u}{x}=\frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx}-\frac{u}{x}=\frac{-1}{x^2}$$

$$\Rightarrow \frac{du}{dx}+u\left(-\frac{1}{x}\right)=\frac{-1}{x^2}$$

The above equation is of the form $\frac{du}{dx}+P(x)u=Q(x)$.

$$Where\ P(x)=-\frac{1}{x}\ and\ Q(x)=-\frac{1}{x^2}$$

So the required substitution is u = $(\log z)^{-1}$ = (C)

25. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

A. $\phi\left(\frac{y}{x}\right)=kx$

B. $x\phi\left(\frac{y}{x}\right)=k$

C. $\phi\left(\frac{y}{x}\right)=ky$

D. $y\phi\left(\frac{y}{x}\right)=k$

Answer

$\frac{dy}{dx}=\frac{y}{x}+\frac{\phi\left(\frac{y}{x}\right)}{\phi\prime\left(\frac{y}{x}\right)}$ --(1)

The above differential is homogeneous differential equation with degree 0

$Put\ \frac{y}{x}=v \Rightarrow y=xv$

Differentiate w.r.t x we get,

$\frac{dy}{dx}=v+x\frac{dv}{dx}$ --(2)

Putting value of (2) in (1) we get

$\Rightarrow v+x\frac{dv}{dx}=v+\frac{\phi(v)}{\phi'(v)}$

$\Rightarrow \frac{\phi'(v)}{\phi(v)}dv=\frac{dx}{x}$

Integrate both sides we get,

$\Rightarrow \int\frac{\phi'(v)}{\phi(v)}dv=\int\frac{dx}{x}$

$We\ know\ that\ \int\frac{f'(x)}{f(x)}=\log f(x)+K$

$\Rightarrow \log\phi(v)=\log x+\log K$

$\Rightarrow \log\phi(v)=\log Kx$

$\Rightarrow \phi(v)=Kx$

$\Rightarrow \phi\left(\frac{y}{x}\right)=Kx=(A)$

26. Question

Mark the correct alternative in each of the following:

If m and n are the order and degree of the differential equation $\left(y_2\right)^5+\frac{4\left(y_2\right)^3}{y_3}+y_3=x^2-1,$ then

A. m = 3, n = 3

B. m = 3, n = 2

C. n = 3, n = 5

D. m = 3, n = 1

Answer

$(y_2)^5 + \frac{4(y_2)^3}{y_3} + y_3 = x^2 - 1$

$\Rightarrow y_3(y_2)^5 + 4(y_2)^3 + (y_3)^2 = y_3(x^2 - 1)$

Order = Highest order derivative present in the differential equation.

∴ Order = 3 = m

Degree = Highest power of highest order derivative which is $(y_3)^2$

∴ Degree = 2 = n

∴ m = 3, n = 2 = (B)

27. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$, is

A. (x + y) e^{x+y} = 0

B. (x + C) e^{x+y} = 0

C. (x – C) e^{x+y} = 1

D. (x – C) e^{x+y} + 1 = 0

Answer

$\frac{dy}{dx} + 1 = e^{x+y}$

Put x + y = z

Differentiating w.r.t x we get,

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$

Substituting values, we get

$\Rightarrow \frac{dz}{dx} - 1 \ + 1 = e^z$

$\Rightarrow \frac{dz}{dx} = e^z$

$\Rightarrow \frac{dz}{e^z} = dx$

Integrating both sides

$\Rightarrow \int e^{-z} dz = \int dx$

$\Rightarrow -e^{-z} = x + C$

$\Rightarrow x + e^{-z} = C$

$\Rightarrow e^z x + 1 = e^z C$

$\Rightarrow (x - C)e^z + 1 = 0$

Putting value of z, we get

$\Rightarrow (x - C)e^{x+y} + 1 = 0 = (D)$

28. Question

Mark the correct alternative in each of the following:

The solution of $x^2 + y^2 \frac{dy}{dx} = 4,$ is

A. $x^2 + y^2 = 12x + C$

B. $x^2 + y^2 = 3x + C$

C. $x^3 + y^3 = 3x + C$

D. $x^3 + y^3 = 12x + C$

Answer

$x^2 + y^2 \frac{dy}{dx} = 4$

$\Rightarrow y^2 \frac{dy}{dx} = 4 - x^2$

By variable separable we get,

$\Rightarrow y^2 dy = (4 - x^2)dx$

Integrating both sides

$\Rightarrow \int y^2 dy = \int (4 - x^2)dx$

$\Rightarrow \frac{y^3}{3} = 4x - \frac{x^3}{3} + K$

$\Rightarrow \frac{y^3}{3} = \frac{12x - x^3 + 3K}{3}$

$\Rightarrow y^3 = 12x - x^3 + C$ Where $3k = C$

$\Rightarrow x^3 + y^3 = 12x + C = (D)$

29. Question

Mark the correct alternative in each of the following:

The family of curves in which the subtangent at any point of a curve is double the abscissae, is given by

A. $x = Cy^2$

B. $y = Cx^2$

C. $x^2 = Cy^2$

D. $y = Cx$

Answer

We know the subtangent at any point of curve $= \frac{y}{\frac{dy}{dx}}$

We are given,

The family of curves in which the subtangent at any point of a curve is double the abscissae

$\Rightarrow \dfrac{y}{\frac{dy}{dx}} = 2x$

$\Rightarrow y\dfrac{dx}{dy} = 2x$

By variable separable we get,

$\Rightarrow \dfrac{dx}{2x} = \dfrac{dy}{y}$

Integrating both sides

$\Rightarrow \int \dfrac{dx}{2x} = \int \dfrac{dy}{y}$

$\Rightarrow \dfrac{1}{2}\log x = \log y + \log K$

$\Rightarrow \log\sqrt{x} = \log Ky$ ∵ log a + log b = log ab

$\Rightarrow \sqrt{x} = Ky$

Squaring both sides, we get

$\Rightarrow x = K^2y^2$

⇒ x = Cy^2 = (A) Where K^2 = C

30. Question

Mark the correct alternative in each of the following:

The solution of the differential equation x dx + y dy = x^2y dy – y^2 x dx, is

A. x^2 – 1 = C (1 + y^2)

B. x^2 + 1 = C (1 – y^2)

C. x^3 – 1 = C (1 + y^3)

D. x^2 + 1 = C (1 – y^3)

Answer

x dx + y dy = x^2y dy – y^2 x dx

⇒ x dx + y^2 x dx = x^2y dy – y dy

⇒ x dx(1 + y^2) = y dy(x^2 – 1)

By Variable separable

$\Rightarrow \dfrac{x}{x^2-1}dx = \dfrac{y}{1+y^2}dy$

Integrating both sides we get

$\Rightarrow \int \dfrac{x}{x^2-1}dx = \int \dfrac{y}{1+y^2}dy$

$\Rightarrow \frac{1}{2}\int \frac{2x}{x^2-1}dx = \frac{1}{2}\int \frac{2y}{1+y^2}dy$ -- (1)

Put x^2 – 1 = t and Put 1 + y^2 = u

Diff w.r.t x Diff w.r.t y

2x dx = dt 2y dy = du

Putting values in (1) we get,

$\Rightarrow \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\int \frac{du}{u}$

$\Rightarrow \log t = \log u + \log C$

$\Rightarrow \log t = \log Cu$ ∵ log a + log b = log ab

$\Rightarrow t = Cu$

Putting values of t and u we get,

⇒ $x^2 - 1 = C(1 + y^2)$ = (A)

31. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\left(x^2+1\right)\frac{dy}{dx}+\left(y^2+1\right)=0,$ is

A. $y = 2 + x^2$

B. $y = \frac{1+x}{1-x}$

C. y = x(x –1)

D. $y = \frac{1-x}{1+x}$

Answer

$(x^2+1)\frac{dy}{dx} + (y^2+1) = 0$

$\Rightarrow (x^2+1)\frac{dy}{dx} = -(y^2+1)$

By variable separable we get,

$\Rightarrow \frac{dy}{(y^2+1)} = -\frac{dx}{(x^2+1)}$

Integrating both sides we get,

$\Rightarrow \int \frac{dy}{(1^2+y^2)} = -\int \frac{dx}{(1^2+x^2)}$

$\Rightarrow \tan^{-1} y = -\tan^{-1} x + C$

$\Rightarrow \tan^{-1} y + \tan^{-1} x = C$

$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = C \;\; \because \tan^{-1} y + \tan^{-1} x = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\Rightarrow \frac{x+y}{1-xy} = \tan C$

$\Rightarrow x + y = C(1-xy)$ Where tan C = C

$\Rightarrow x + y = C - Cxy$

$\Rightarrow x + y + Cxy = C$

$\Rightarrow x + y(1+Cx) = C$

$\Rightarrow y = \frac{(C - x)}{(1 + Cx)}$

Now since C is arbitrary constant, we put C = 1 (let) we get,

$\Rightarrow y = \frac{1 - x}{1 + x} = (D)$

32. Question

Mark the correct alternative in each of the following:

The differential equation $x\frac{dy}{dx} - y = x^2$, has the general solution

A. y – x^3 = 2cx

B. 2y – x^3 = cx

C. 2y + x^2 = 2cx

D. y + x^2 = 2cx

Answer

$x\frac{dy}{dx} - y = x^3$

Divide both sides by x we get,

$\Rightarrow \frac{dy}{dx} + y\left(-\frac{1}{x}\right) = x^2$

Since it is a form of linear differential equation where

$P = -\frac{1}{x}$ and Q = x^2

Integrating Factor (I.F) = $e^{\int p\,dx}$

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Solution of differential equation is given by

y.(I.F) = ∫ Q.(I.F) dx + C

$\Rightarrow y\,.\frac{1}{x} = \int x^2\left(\frac{1}{x}\right)dx + C$

$\Rightarrow \frac{y}{x} = \int x dx + C$

$\Rightarrow \frac{y}{x} = \frac{x^2}{2} + C$

$\Rightarrow 2y = x^3 + 2Cx$

⇒ 2y – x^3 = cx = (B)

33. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} - ky = 0,\ y(0) = 1$ approaches to zero when x → ∞, if

A. k = 0

B. k > 0

C. k < 0

D. none of these

Answer

$\frac{dy}{dx} - ky = 0,\ y(0) = 1$

$\Rightarrow \frac{dy}{dx} = ky$

By variable separable and integrating both sides

$\Rightarrow \int \frac{dy}{y} = \int kdx$

$\Rightarrow \log y = kx + C$

$\Rightarrow y = e^{kx+C}$

$\Rightarrow y = e^{kx}\, e^{C}$

$\Rightarrow y = A.e^{kx}$ Where $e^C = A$

We have given condition y(0) = 1

$\Rightarrow 1 = A.e^{k0}$

$\Rightarrow A = 1$

$\therefore Solution\ y = e^{kx}$

We know $e^{\infty} \to \infty$ and $e^{-\infty} \to 0$

So, we are given that y approaches to zero when $x \to \infty$

i.e. $0 = e^{k\infty}$

Which is possible only when k < 0 = (C)

34. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\left(1+x^2\right)\frac{dy}{dx}+1+y^2=0,$ is

A. $\tan^{-1} x - \tan^{-1} y = \tan^{-1} C$

B. $\tan^{-1} y - \tan^{-1} x = \tan^{-1} C$

C. $\tan^{-1} y \pm \tan^{-1} x = \tan C$

D. $\tan^{-1} y + \tan^{-1} x = \tan^{-1} C$

Answer

$(1 + x^2)\frac{dy}{dx} + 1 + y^2 = 0$

$\Rightarrow (x^2 + 1)\frac{dy}{dx} = -(y^2 + 1)$

By variable separable we get,

$\Rightarrow \frac{dy}{(y^2+1)} = -\frac{dx}{(x^2+1)}$

Integrating both sides we get,

$\Rightarrow \int \frac{dy}{(1^2+y^2)} = -\int \frac{dx}{(1^2+x^2)}$

$\Rightarrow \tan^{-1} y = -\tan^{-1} x + \tan^{-1} C$

$\Rightarrow \tan^{-1} y + \tan^{-1} x = \tan^{-1} C$ = (D)

35. Question

Mark the correct alternative in each of the following:

The solution of the differential equation $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$, is

A. $\tan^{-1}\left(\frac{x}{y}\right) = \log y + C$

B. $\tan^{-1}\left(\frac{y}{x}\right) = \log x + C$

C. $\tan^{-1}\left(\frac{x}{y}\right) = \log x + C$

D. $\tan^{-1}\left(\frac{y}{x}\right) = \log y + C$

Answer

$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

$\Rightarrow \frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$ --(1)

The above differential is homogeneous differential equation

$Put\ \frac{y}{x} = v \Rightarrow y = xv$

Differentiate w.r.t x we get,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ --(2)

Putting value of (2) in (1) we get

$\Rightarrow v + x\frac{dv}{dx} = 1 + v + v^2$

$\Rightarrow x\frac{dv}{dx} = 1 + v^2$

By variable separable and integrate both sides we get,

$\Rightarrow \int \frac{dv}{1^2+v^2} = \int \frac{dx}{x}$

$\Rightarrow \tan^{-1} v = \log x + C$

Putting value of v, we get

$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \log x + C = (B)$

36. Question

Mark the correct alternative in each of the following:

The differential equation $\frac{dy}{dx}+Py=Qy^n$, n > 2 can be reduced to linear form by substituting

A. $z = y^{n-1}$

B. $z = y^n$

C. $z = y^{n+1}$

D. $z = y^{1-n}$

Answer

$\frac{dy}{dx}+Py=Qy^n$

Multiply both sides by y^{-n}

$\Rightarrow y^{-n}\frac{dy}{dx}+Py^{-n}y=Qy^{-n}y^n$

$\Rightarrow y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ -- (1)

Put $y^{1-n} = z$

Differentiate w.r.t x we get,

$\Rightarrow (1-n)y^{-n}\frac{dy}{dx}=\frac{dz}{dx}$

$\Rightarrow y^{-n}\frac{dy}{dx}=\frac{1}{(1-n)}\frac{dz}{dx}$ --(2)

Put value of (2) in (1)

$\Rightarrow \frac{1}{(1-n)}\frac{dz}{dx}+Pz=Q$

Multiply both sides by (1 – n)

$\Rightarrow \frac{dz}{dx}+P(1-n).z=Q(1-n)$

Since it is a form of linear differential equation

$i.e.\frac{dy}{dx}+Py=Q$

$Where, P = P(1-n) \; and \; \mathrm{Q} = Q(1-n)$

∴ we put $z = y^{1-n}$ to form linear differential equation = (D)

37. Question

Mark the correct alternative in each of the following:

If p and q are the order and degree of the differential equation $y\frac{dy}{dx}+x^3\frac{d^2y}{dx^2}+xy=\cos x$, then

A. p < q

B. p = q

C. p > q

D. none of theses

Answer

$y\frac{dy}{dx} + x^3\frac{d^2y}{dx^2} + xy = cos\,x,$

Order = Highest order derivative present in the differential equation.

∴ Order = 2 = p

Degree = Highest power of highest order derivative which is $\frac{d^2y}{dx^2}$

∴ Degree = 1 = q

∴ p = 2 > q = 1 = (C)

38. Question

Mark the correct alternative in each of the following:

Which of the following is the integrating factor of $(x\log x)\frac{dy}{dx} + y = 2\log x$?

A. x

B. e^x

C. log x

D. log (log x)

Answer

$(x\,log\,x)\frac{dy}{dx} + y = 2\,log\,x$

Divide both sides by x log x we get,

$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{(x\,log\,x)}\right) = \frac{2}{x}$

Since it is a form of linear differential equation where

$P = \frac{1}{(x\,log\,x)}$ and $Q = \frac{2}{x}$

Integrating Factor (I.F) = $e^{\int p\,dx}$

$I.F = e^{\int\frac{1}{(x\,log x)}dx}$

$Consider\ integral\ \int\frac{1}{(x\,log\,x)}dx$

Put log x = t

Differentiate w.r.t x we get,

$\Rightarrow \frac{1}{x}dx = dt$

Putting value in integral

$\Rightarrow \int\frac{1}{(x\,log\,x)}dx = \int\frac{1}{t}dt$

$\Rightarrow \log t$

Putting value of t, we get,

$\Rightarrow \int\frac{1}{(x\,log\,x)}dx = \log(\log x)$

Now,

$I.F\ =\ e^{\int\frac{1}{(x\,log x)}dx}\ =\ e^{\log(\log x)} = \log x = (C)$

39. Question

Mark the correct alternative in each of the following:

What is integrating factor of $\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y}\sec\mathrm{x} = \tan\mathrm{x}$?

A. sec x + tan x

B. log (sec x + tan x)

C. $e^{\sec x}$

D. sec x

Answer

$\frac{dy}{dx} + y\sec x = \tan x$

Since it is a form of linear differential equation where

$P\ = \sec x$ and $Q\ = \tan x$

Integrating Factor (I.F) = $e^{\int p\,dx}$

$I.F\ =\ e^{\int \sec x\,dx}$

$\Rightarrow\ e^{\int \sec x\,dx}$

$\Rightarrow\ e^{\log(\sec x+\tan x)}$

$\Rightarrow (\sec x + \tan x) = (A)$

40. Question

Mark the correct alternative in each of the following:

Integrating factor of the differential equation $\cos\mathrm{x}\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{y}\sin\mathrm{x} = 1,$ is

A. cos x

B. tan x

C. sec x

D. sin x

Answer

$\cos x\frac{dy}{dx} + y\sin x = 1$

Divide both sides by cos x we get,

$\Rightarrow \frac{dy}{dx} + y\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos x}$

$\Rightarrow \frac{dy}{dx} + y(\tan x) = \sec x$

Since it is a form of linear differential equation where

$P\ = \tan x$ and $Q\ = \sec x$

Integrating Factor (I.F) = $e^{\int p\,dx}$

$I.F = e^{\int \tan x dx}$

$\Rightarrow e^{\int \tan x\, dx}$

$\Rightarrow e^{\log(\sec x)}$

$\Rightarrow \sec x = (C)$

41. Question

Mark the correct alternative in each of the following:

The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0,$ is

A. 3

B. 2

C. 1

D. not defined

Answer

$$\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$$

Here Order of differential equation is 2.

Its degree is not defined as it is not a polynomial equation in derivatives. = (D)

42. Question

Mark the correct alternative in each of the following:

The order of the differential equation $2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0,$ is

A. 2

B. 1

C. 0

D. not defined

Answer

$$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$$

Order = Highest order derivative present in the differential equation which is $\frac{d^2y}{dx^2}$

∴ Order = 2 = (A)

43. Question

Mark the correct alternative in each of the following:

The number of arbitrary constants in the general solution of differential equation of fourth order is

A. 0 B. 0

C. 3 D. 4

Answer

We know,

the number of arbitrary constants of an ordinary differential equation (ODE) is given by the order of the highest derivative.

∵ differential equation is of fourth order then it will have 4 arbitrary constants in the general solution. = (D)

44. Question

Mark the correct alternative in each of the following:

The number of arbitrary constants in the particular solution of a differential equation of third order is

A. 3

B. 2

C. 1

D. 0

Answer

We know,

the number of arbitrary constants of an ordinary differential equation (ODE) is given by the order of the highest derivative and if we give particular values to those arbitrary constants, we get particular solution in which we have 0 arbitrary constants

∴ The number of arbitrary constants in the particular solution of a differential equation of third order is 0 = (D)

45. Question

Mark the correct alternative in each of the following:

Which of the following differential equations has $y = C_1 e^x + C_2 e^{-x}$ as the general solution?

A. $\frac{d^2y}{dx^2} + y = 0$

B. $\frac{d^2y}{dx^2} - y = 0$ $\frac{d^2y}{dx^2} + 1 = 0$

C.

D. $\frac{d^2y}{dx^2} - 1 = 0$

Answer

Solving for (A)

$\Rightarrow \frac{d^2y}{dx^2} + y = 0$

$Put\ \frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y \ldots.\ and\ so\ on$

$\Rightarrow D^2y + y = 0$

$\Rightarrow y(D^2 + 1) = 0$

For general solution put $(D^2 + 1) = 0$

$\Rightarrow (D^2 + 1) = 0$

$\Rightarrow D = \pm i$

General solution, y = C_1 cos x + C_2 sin x

Solving for (B)

$\Rightarrow \frac{d^2y}{dx^2} - y = 0$

$Put\ \frac{dy}{dx} = Dy, \frac{d^2y}{dx^2} = D^2y \ldots.\ and\ so\ on$

$\Rightarrow D^2y - y = 0$

$\Rightarrow y(D^2 - 1^2) = 0$

For general solution put $(D^2 - 1^2) = 0$

$\Rightarrow (D-1)(D+1) = 0$

$\Rightarrow D = 1, -1$

General solution, y = $C_1 e^x + C_2 e^{-x}$ = (B) which is required solution.

46. Question

Mark the correct alternative in each of the following:

Which of the following differential equations has y = x as one of its particular solution?

A. $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x$

B. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = x$

C. $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0$

D. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$

Answer

y = x

Differentiate w.r.t x we get,

$\Rightarrow \frac{dy}{dx} = 1 \ \Rightarrow \frac{d^2y}{dx^2} = 0$

Consider,

$\Rightarrow \frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy$

$\Rightarrow 0 - x^2.1 + x.y$

$\Rightarrow -x^2 + x^2 = 0$ ∵ x = y

$\therefore \frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 0 = (C)$

47. Question

Mark the correct alternative in each of the following:

The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$, is

A. $e^x + e^{-y} = C$

B. $e^x + e^y = C$

C. $e^{-x} + e^y = C$

D. $e^{-x} + e^{-y} = C$

Answer

$\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \frac{dy}{dx} = e^x . e^y$

By variable separable and integrating both sides we get,

$\Rightarrow \int \frac{dy}{e^y} = \int e^x dx$

$\Rightarrow \int e^{-y} dy = \int e^x dx$

$\Rightarrow -e^{-y} = e^x + C$

$\Rightarrow e^x + e^{-y} = C = (A)$

48. Question

Mark the correct alternative in each of the following:

A homogeneous differential equation of the form $\frac{dy}{dx} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution.

A. y = vx

B. v = yx

C. x = vy

D. x = v

Answer

$\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ --(1)

For solving the above homogeneous differential equation we must put

$\Rightarrow \frac{x}{y} = v$

$\Rightarrow x = vy = (C)$

$Such\ that\ \frac{dx}{dy} = v + y.\frac{dv}{dx}$ --(2)

We put value of (2) in (1) for finding the solution.

49. Question

Mark the correct alternative in each of the following:

Which of the following is a homogeneous differential equation?

A. (4x + 6y + 5) dy – (3y + 2x + 4) dx = 0

B. xy dx – (x^3 + y^3) dy = 0

C. (x^3 + $2y^2$) dx + 2xy dy = 0

D. y^2 dx + (x^2 – xy) – y^2) dy = 0

Answer

We know the property of homogeneous differential equation i.e.

$$\frac{dy}{dx} = f(x, y)$$

⇒ f (λ x, λ y) = f (x, y)

In the given set of options, option (D) is correct as addition of power is same throughout the equation.

50. Question

Mark the correct alternative in each of the following:

The integrating factor of the differential equation $x\frac{dy}{dx} - y = 2x^2$.

A. e^{-x}

B. e^{-y}

C. 1/x

D. x

Answer

$x\frac{dy}{dx} - y = 2x^2$

Divide both sides by x we get,

$$\Rightarrow \frac{dy}{dx} + y\left(-\frac{1}{x}\right) = 2x$$

Since it is a form of linear differential equation where

$P = -\frac{1}{x}$ and $Q = 2x$

Integrating Factor (I.F) = $e^{\int p\, dx}$

$$I.F = e^{\int \frac{-1}{x} dx}$$

$$I.F = e^{\int \frac{-1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$$

$$I.F = \frac{1}{x} = (C)$$

51. Question

Mark the correct alternative in each of the following:

The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay\,(-1 < y < 1)$ is

A. $\frac{1}{y^2 - 1}$

B. $\frac{1}{\sqrt{y^2-1}}$

C. $\frac{1}{1-y^2}$

D. $\frac{1}{\sqrt{1-y^2}}$

Answer

$(1-y^2)\frac{dx}{dy}+yx=ay\ (-1<y<1)$

Divide both sides by 1 – y^2 we get,

$$\Rightarrow \frac{dx}{dy}+\left[\frac{y}{(1-y^2)}\right]x=\frac{ay}{(1-y^2)}$$

Since it is a form of linear differential equation where

$P\ =\ \frac{y}{(1-y^2)}$ and $Q\ =\ \frac{ay}{(1-y^2)}$

Integrating Factor (I.F) = $e^{\int p\, dx}$

$$I.F\ =\ e^{\int \frac{y}{(1-y^2)}dy}$$

$$Consider\ integral\ \int \frac{y}{(1-y^2)}dy$$

Put 1 – y^2 = t

Differentiate w.r.t y we get,

$$\Rightarrow y\ dy=-\frac{1}{2}dt$$

Putting value in integral

$$\Rightarrow \int \frac{y}{(1-y^2)}dy=-\frac{1}{2}\int \frac{1}{t}dt$$

$$\Rightarrow -\frac{1}{2}\log t$$

Putting value of t, we get,

$$\Rightarrow \int \frac{y}{(1-y^2)}dy=-\frac{1}{2}\log(1-y^2)=\log(1-y^2)^{-\frac{1}{2}}$$

Now,

$$I.F\ =\ e^{\int \frac{y}{(1-y^2)}dy}\ =\ e^{\log(1-y^2)^{-\frac{1}{2}}}=(1-y^2)^{-\frac{1}{2}}$$

$$I.F=\frac{1}{\sqrt{1-y^2}}=(D)$$

52. Question

Mark the correct alternative in each of the following:

The general solution of the differential equation $\frac{y\,dx-x\,dy}{y}=0$, is

A. xy = C

B. $x = Cy^2$

C. y = Cx

D. $y = Cx^2$

Answer

$\frac{ydx - x\,dy}{y} = 0$

$\Rightarrow dx - \frac{x}{y}dy = 0$

$\Rightarrow dx = \frac{x}{y}dy$

By variable separable and integrating both sides we get,

$\Rightarrow \int \frac{1}{y}dy = \int \frac{dx}{x}$

$\Rightarrow \log y = \log Cx$

⇒ y = Cx = (C)

53. Question

Mark the correct alternative in each of the following:

The general solution of a differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$ is

A. $y e^{\int P_1 dy} = \int \left\{ Q_1 e^{\int P_1 dy} \right\} dy + C$

B. $y e^{\int P_1 dx} = \int \left\{ Q_1 e^{\int P_1 dx} \right\} dx + C$

C. $x e^{\int P_1 dy} = \int \left\{ Q_1 e^{\int P_1 dy} \right\} dy + C$

D. $x e^{\int P_1 dx} = \int \left\{ Q_1 e^{\int P_1 dx} \right\} dx + C$

Answer

$\frac{dx}{dy} + P_1 x = Q_1$

Since it is a form of linear differential equation.

Where, $P = P_1$ and $Q = Q_1$

Integrating Factor (I.F) = $e^{\int p\, dy}$

I.F = $e^{\int P_1\, dy}$

Solution of differential equation is given by

x.(I.F) = ∫ Q.(I.F) dy + C

$$xe^{\int P_1 dy} = \int \left\{Q_1 e^{\int P_1 dy}\right\} dy + C = (C)$$

54. Question

Mark the correct alternative in each of the following:

The general solution of the differential equation $e^x\, dy + (y\, e^x + 2x)dx = 0$ is

A. $x\, e^y + x^2 = C$

B. $x\, e^y + y^2 = C$

C. $y\, e^x + x^2 = C$

D. $y\, e^y + x^2 = C$

Answer

$e^x\, dy + (y\, e^x + 2x)dx = 0$

We can write above equation as

$$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$$

Divide both sides by e^x we get,

$$\Rightarrow \frac{dy}{dx} + y + \frac{2x}{e^x} = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -\frac{2x}{e^x}$$

Since it is a form of linear differential equation.

$$Where, P = 1 \; and \; Q = -\frac{2x}{e^x}$$

Integrating Factor (I.F) = $e^{\int p\, dx}$

I.F = $e^{\int 1\, dx} = e^x$

Solution of differential equation is given by

y.(I.F) = ∫ Q.(I.F) dx + C

$$\Rightarrow y.e^x = \int -\frac{2x}{e^x}.e^x dx + C$$

$$\Rightarrow y.e^x = \int -2x.dx + C$$

$$\Rightarrow y.e^x = -x^2 + C$$

$\Rightarrow y\, e^x + x^2 = C = (C)$