24. Scalar or Dot Products

Exercise 24.1

1 A. Question

Find $\vec{a}.\vec{b}$, when

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\vec{a}.\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$

Given Vectors:

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$

$\vec{a}.\vec{b} = 1 \times 4 + (-2) \times (-4) + 1 \times 7$

$\vec{a}.\vec{b} = 19.$

1 B. Question

Find $\vec{a}.\vec{b}$, when

$\vec{a} = \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\vec{a}.\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$

$\vec{a} = \hat{j} + 2\hat{k}$

$\vec{b} = 2\hat{i} + \hat{k}$

$\vec{a}.\vec{b} = 0 \times 2 + 0 \times 2 + 1 \times 2$

$\vec{a}.\vec{b} = 2$

1 C. Question

Find $\vec{a}.\vec{b}$, when

$\vec{a} = \hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\vec{a}.\vec{b} = x_1x_2 + y_1y_2 + z_1z_2$

$\vec{a} = \hat{j} - \hat{k}$

$\vec{b} = 2\hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{a}.\vec{b} = 0 \times 2 + 1 \times 3 + (-1) \times (-2)$

$\vec{a}.\vec{b} = 5\cdot$

2 A. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 4\hat{i} - 9\hat{j} + 2\hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are ⊥ to each other then $\vec{a}.\vec{b} = 0$

$\vec{a} = \widehat{\lambda\imath} + 2\hat{\jmath} + \hat{k}$

$\vec{b} = 4\hat{\imath} - 9\hat{\jmath} + 2\hat{k}$

Now

$\vec{a}.\vec{b} = 0$

λ × 4 + 2 × (-9) + 1 × 2 = 0

λ × 4 = 16

$\lambda = \frac{16}{4}$

λ = 4

2 B. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$\vec{a} = \lambda\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 5\hat{i} - 9\hat{j} + 2\hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{b} = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are ⊥ to each other then $\vec{a}.\vec{b} = 0$

$\vec{a} = \widehat{\lambda\imath} + 2\hat{\jmath} + \hat{k}$

$\vec{b} = 5\hat{\imath} - 9\hat{\jmath} + 2\hat{k}$

Now

$\vec{a}.\vec{b} = 0$

λ × 5 + 2 × (-9) + 1 × 2 = 0

λ × 5 = 16

$\lambda = \frac{16}{5}$

2 C. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \lambda\hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are ⊥ to each other then $\vec{a}.\vec{b} = 0$

$\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{b} = 3\hat{i} + 2\hat{j} - \lambda\hat{k}$

Now

$\vec{a}.\vec{b} = 0$

2 × 3 + 3 × 2 + 4 × (-λ) = 0

- 4 λ = - 12

$\lambda = \frac{12}{4}$

λ = 3

2 D. Question

For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? Where :

$\vec{a} = \lambda\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$

Answer

For any vector $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

If $\vec{a}$ and $\vec{b}$ are ⊥ to each other then $\vec{a}.\vec{b} = 0$

$\vec{a} = \lambda\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} - \hat{j} + 3\hat{k}$

Now

$\vec{a}.\vec{b} = 0$

λ × 1 + 3 × (-1) + 2 × 3 = 0

λ - 3 + 6 = 0

λ = -3

3. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 4, |\vec{b}| = 3$ and $\vec{a}.\vec{b} = 6.$ Find the angle between $\vec{a}$ and $\vec{b}$.

Answer

Given Data:

$|\vec{a}| = 4, |\vec{b}| = 3$ and $\vec{a}.\vec{b} = 6$

Calculation:

Using formula $\vec{a}.\vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$|\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$

$\cos\theta = \frac{6}{4 \times 3}$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\therefore \theta = \frac{\pi}{3}$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

4. Question

If $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -\hat{j} + 2\hat{k}$, find $\left(\vec{a} - 2\vec{b}\right).\left(\vec{a} + \vec{b}\right)$.

Answer

Given data:

$\vec{a} = \hat{i} - \hat{j}$

$\vec{b} = -\hat{j} + 2\hat{k}$

Now

$\Rightarrow \vec{a} - 2\vec{b} = (\hat{i} - \hat{j}) - 2(-\hat{j} + 2\hat{k})$

$\vec{a} - 2\vec{b} = \hat{i} - \hat{j} + 2\hat{j} - 4\hat{k}$

$\vec{a} - 2\vec{b} = \hat{i} + \hat{j} - 4\hat{k}$

$\Rightarrow \vec{a} + \vec{b} = (\hat{i} - \hat{j}) + (-\hat{j} + 2\hat{k})$

$\vec{a} + \vec{b} = \hat{i} - \hat{j} - \hat{j} + 2\hat{k}$

$\vec{a} + \vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$

Consider

$(\vec{a} - 2\vec{b}).(\vec{a} + \vec{b}) = (\hat{i} + \hat{j} - 4\hat{k})(\hat{i} - 2\hat{j} + 2\hat{k})$

$(\vec{a} - 2\vec{b}).(\vec{a} + \vec{b}) = 1 \times 1 + 1 \times (-2) + (-4) \times 2$

$(\vec{a} - 2\vec{b}).(\vec{a} + \vec{b}) = 1 - 2 - 8$

$(\vec{a} - 2\vec{b}).(\vec{a} + \vec{b}) = -9$

5 A. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{j} + \hat{k}$

Answer

Using formula $\vec{a}.\vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

Given Data:

i. $\vec{a} = \hat{i} - \hat{j}$

$\vec{b} = \hat{j} + \hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$

$\cos\theta = \frac{(\hat{i} - \hat{j})(\hat{j} + \hat{k})}{\sqrt{1^2 + 1^2} \times \sqrt{1^2 + 1^2}}$

$\cos\theta = \frac{1 \times 0 + (-1) \times 1 + 0 \times 1}{\sqrt{2} \times \sqrt{2}}$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\theta = \pi - \frac{\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

5 B. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$\vec{a} = 3\hat{i} - 2\hat{j} - 6\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 8\hat{k}$

Answer

Using formula $\vec{a}.\vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$\vec{a} = 3\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{b} = 4\hat{i} - \hat{j} + 8\hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$

$\cos\theta = \frac{(3\hat{i} - 2\hat{j} - 6\hat{k})(4\hat{i} - \hat{j} + 8\hat{k})}{\sqrt{3^2 + (-2)^2 + (-6)^2} \times \sqrt{4^2 + (-1)^2 + 8^2}}$

$\cos\theta = \frac{3 \times 4 + (-2) \times (-1) + (-6) \times 8}{\sqrt{9 + 4 + 36} \times \sqrt{16 + 1 + 64}}$

$\cos\theta = -\frac{34}{\sqrt{49} \times \sqrt{81}}$

$\cos\theta = -\frac{34}{7 \times 9}$

$\theta = \cos^{-1}\left(-\frac{34}{63}\right)$

$\theta = 122.66°$

Therefore angle between $\vec{a}$ and $\vec{b}$ is 122.66°.

5 C. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} + 4\hat{j} - 2\hat{k}$

Answer

Using formula $\vec{a}.\vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{b} = 4\hat{i} + 4\hat{j} - 2\hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(2\hat{i} - \hat{j} + 2\hat{k})(4\hat{i} + 4\hat{j} - 2\hat{k})}{\sqrt{2^2 + (-1)^2 + (2)^2} \times \sqrt{4^2 + 4^2 + (-2)^2}}$$

$$\cos\theta = \frac{2 \times 4 + (-1) \times 4 + 2 \times (-2)}{\sqrt{4 + 1 + 4} \times \sqrt{16 + 16 + 4}}$$

$$\cos\theta = \frac{0}{\sqrt{9} \times \sqrt{36}}$$

$$\cos\theta = \frac{0}{3 \times 6}$$

$\theta = \cos^{-1}(0)$

$$\theta = \frac{\pi}{2}$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$.

5 D. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$

Answer

Using formula $\vec{a}.\vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(2\hat{i} - 3\hat{j} + \hat{k})(\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{2^2 + (-3)^2 + 1^2} \times \sqrt{1^2 + 1^2 + (-2)^2}}$$

$$\cos\theta = \frac{2 \times 1 + (-3) \times 1 + 1 \times (-2)}{\sqrt{4+9+1} \times \sqrt{1+1+4}}$$

$$\cos\theta = -\frac{3}{\sqrt{14} \times \sqrt{6}}$$

$$\cos\theta = -\frac{3}{\sqrt{84}}$$

$$\theta = \cos^{-1}\left(-\frac{3}{\sqrt{84}}\right)$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(-\frac{3}{\sqrt{84}}\right)$.

5 E. Question

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:

$\vec{a} = \hat{i} + 2\hat{j} - \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}$

Answer

Using formula $\vec{a}.\vec{b} = |\vec{a}| \times |\vec{b}| \times \cos\theta$

$\vec{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

$\vec{b} = \hat{\imath} - \hat{\jmath} + \hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{\imath} + 2\hat{\jmath} - \hat{k})(\hat{\imath} - \hat{\jmath} + \hat{k})}{\sqrt{1^2 + 2^2 + (-1)^2} \times \sqrt{1^2 + (-1)^2 + 1^2}}$$

$$\cos\theta = \frac{1 \times 1 + 2 \times (-1) + (-1) \times 1}{\sqrt{1+4+1} \times \sqrt{1+1+1}}$$

$$\cos\theta = -\frac{2}{\sqrt{2 \times 9}}$$

$$\cos\theta = -\frac{\sqrt{2}}{3}$$

$$\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$$

Therefore angle between $\vec{a}$ and $\vec{b}$ is $\cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$

6. Question

Find the angles which the vector $\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$ makes with the coordinate axes.

Answer

Calculation:

Angle with x-axis

$\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$

unit vector along x axis is $\hat{i}$

So, $\vec{b} = \hat{i}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j} + \sqrt{2}\hat{k})(\hat{i})}{\sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \times \sqrt{1^2}}$$

$$\cos\theta = \frac{1}{\sqrt{4} \times \sqrt{1}}$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = \frac{\pi}{3}$$

Therefore angle between $\vec{a}$ and x axis is $\frac{\pi}{3}$

Angle with y-axis

$\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$

unit vector along y axis is $\hat{j}$

So, $\vec{b} = \hat{j}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j} + \sqrt{2}\hat{k})(\hat{j})}{\sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \times \sqrt{1^2}}$$

$$\cos\theta = -\frac{1}{\sqrt{4} \times \sqrt{1}}$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \cos^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \pi - \frac{\pi}{3}$$

$$\theta = \frac{2\pi}{3}$$

Therefore angle between $\vec{a}$ and y axis is $\frac{2\pi}{3}$

Angle with z-axis

$\vec{a} = \hat{i} - \hat{j} + \sqrt{2}\hat{k}$

unit vector along z axis is $\hat{k}$

So, $\vec{b} = \hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$$

$$\cos\theta = \frac{(\hat{i} - \hat{j} + \sqrt{2}\hat{k})(\hat{k})}{\sqrt{1^2 + (-1)^2 + (\sqrt{2})^2} \times \sqrt{1^2}}$$

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{4} \times \sqrt{1}}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = \frac{\pi}{4}$$

Therefore angle between $\vec{a}$ and z axis is $\frac{\pi}{4}$

7 A. Question

Dot product of a vector with $\hat{i} + \hat{j} - 3\hat{k}$, $\hat{i} + 3\hat{j} - 2\hat{k}$ and $2\hat{i} + \hat{j} + 4\hat{k}$ are 0, 5 and 8 respectively. Find the vector.

Answer

Given Data:

Vectors:

$\vec{a} = \hat{i} + \hat{j} - 3\hat{k}$

$\vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$

$\vec{c} = 2\hat{i} + \hat{j} + 4\hat{k}$

Their Dot products are 0, 5 and 8.

Calculation:

Let the required vector be,

$\vec{h} = x\hat{i} + y\hat{j} + z\hat{k}$

Now,

$\vec{a}.\vec{h} = 0$

$(\hat{i} + \hat{j} - 3\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 0$

x + y -3z =0 ...Eq. 1

Similarly

$\Rightarrow \vec{b}.\vec{h} = 5$

$(\hat{i} + 3\hat{j} - 2\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 5$

x + 3y - 2z = 5 ...Eq. 2

$\Rightarrow \vec{c}.\vec{h} = 8$

$(2\hat{i} + \hat{j} + 4\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 8$

2x + y + 4z = 8 ...Eq. 3

Subtract Eq. 1 from Eq. 2

(x + 3y - 2z) – (x + y -3z) = 5 - 0

⇒ 2y + z = 5 ...Eq. 4

Subtract Eq. 3 from (2 × Eq. 2)

2(x + 3y - 2z) - 2x + y + 4z = (2 × 5) - 8

5y – 8z = 2 ...Eq. 5

Adding Eq. 5 with (8 × Eq. 4)

8(2y + z) + (5y – 8z) = 8 × 5 + 2

⇒ 21y = 42

⇒ y = 2

From Eq. 5,

5×2 - 8z = 2

⇒ z = 1

From Eq. 1

x + y -3z =0

⇒ x + 2 – 3×1 = 0

⇒ x = 1

$\therefore$ required vector is $\vec{h} = \hat{i} + 2\hat{j} + \hat{k}$

7 B. Question

Dot product of a vector with vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively 4, 0 and 2. Find the vector.

Answer

Vectors:

$\vec{a} = \hat{i} - \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\vec{c} = \hat{i} + \hat{j} + \hat{k}$

Their Dot products are 4, 0 and 2.

Calculation:

Let the required vector be,

$\vec{h} = x\hat{i} + y\hat{j} + z\hat{k}$

Now,

$\vec{a}.\vec{h} = 0$

$(\hat{i} - \hat{j} + \hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 4$

x - y + z = 4 ...Eq. 1

Similarly

$\Rightarrow \vec{b}.\vec{h} = 0$

$(2\hat{i} + \hat{j} - 3\hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 5$

2x + y - 3z = 0 ...Eq. 2

$\Rightarrow \vec{c}.\vec{h} = 2$

$(\hat{i} + \hat{j} + \hat{k})(x\hat{i} + y\hat{j} + z\hat{k}) = 2$

x + y + z = 2 ...Eq. 3

Subtract Eq. 1 from Eq. 3

(x + y + z) – (x - y + z) = 2 - 4

⇒ 2y = -2

y = - 1

Now putting the value of y in equation(2) and equation (3) we get,

2 x – 3 z = 1 ...(Eq(4))

x + z = 3 ......(Eq(5))

Eq(4) – 2 × Eq (5)

- 5 z = - 5

z = 1

Now putting value of z in equation (1) we get,

x – y + z = 4

x + 1 + 1 = 4

x = 2

So the vector is,

$\therefore$ required vector is $\vec{h} = 2\hat{i} - \hat{j} + \hat{k}$

8 A. Question

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

$\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$

Answer

Given Data: Two unit vectors inclined at an angle θ

Proof:

Since vectors are unit vectors

$\therefore |\hat{a}| = |\hat{b}| = 1$

Now,

$\Rightarrow |\hat{a}+\hat{b}|^2 = (\hat{a}+\hat{b})^2$

$= (\hat{a})^2 + (\hat{b})^2 + 2\,\hat{a}.\hat{b}$

$= |\hat{a}|^2 + |\hat{b}|^2 + 2 \times |\hat{a}| \times |\hat{b}| \times \cos\theta$

= 1+1+2×1×1×cosθ

= 2 + 2cosθ

= 2(1 + cosθ)

Using the identity, $(1 + \cos\theta) = 2\cos^2\frac{\theta}{2}$

$= 2 \times 2\cos^2\frac{\theta}{2}$

$= 4\cos^2\frac{\theta}{2}$

$\Rightarrow |\hat{a}+\hat{b}|^2 = 4\cos^2\frac{\theta}{2}$

$\Rightarrow |\hat{a}+\hat{b}| = \sqrt{4\cos^2\frac{\theta}{2}}$

$\Rightarrow |\hat{a}+\hat{b}| = 2\cos\frac{\theta}{2}$

$\Rightarrow$ (i) $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a}+\hat{b}|$

8 B. Question

If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

$\tan\frac{\theta}{2} = \frac{|\hat{a}-\hat{b}|}{|\hat{a}+\hat{b}|}$

Answer

$\Rightarrow |\hat{a}-\hat{b}|^2 = (\hat{a}-\hat{b})^2$

$= (\hat{a})^2 + (\hat{b})^2 - 2\,\hat{a}.\hat{b}$

$= |\hat{a}|^2 + |\hat{b}|^2 - 2 \times |\hat{a}| \times |\hat{b}| \times \cos\theta$

= 1+1-2×1×1×cosθ

= 2 - 2cosθ

= 2(1 - cosθ)

Using the identity, $(1 - \cos\theta) = 2\sin^2\frac{\theta}{2}$

$= 2 \times 2\sin^2\frac{\theta}{2}$

$= 4\sin^2\frac{\theta}{2}$

$\Rightarrow |\hat{a} - \hat{b}|^2 = 4\sin^2\frac{\theta}{2}$

$\Rightarrow |\hat{a} - \hat{b}| = \sqrt{4\sin^2\frac{\theta}{2}}$

$\Rightarrow |\hat{a} - \hat{b}| = 2\sin\frac{\theta}{2}$

$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

Dividing above by result (i) we will get,

$$\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\frac{1}{2}|\hat{a} - \hat{b}|}{\frac{1}{2}|\hat{a} + \hat{b}|}$$

$$\text{(ii) } \tan\frac{\theta}{2} = \frac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$$

Proved

9. Question

If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$.

Answer

The sum of two unit vectors is a unit vector

Calculation:

Since $|\hat{a}| = |\hat{b}| = 1$

Also,

$|\hat{a} + \hat{b}| = 1$

Now squaring both sides we get

$\Rightarrow |\hat{a} + \hat{b}|^2 = 1^2$

$|\hat{a}|^2 + |\hat{b}|^2 + 2\,\hat{a}.\hat{b} = 1$

$1^2 + 1^2 + 2\,\hat{a}.\hat{b} = 1$

$\Rightarrow \hat{a}.\hat{b} = -\frac{1}{2}$

Now,

$\Rightarrow |\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b})^2$

$= (\hat{a})^2 + (\hat{b})^2 - 2\,\hat{a}.\hat{b}$

$= |\hat{a}|^2 + |\hat{b}|^2 - 2\,\hat{a}.\hat{b}$

Using the above value,

$= 1^2 + 1^2 - 2\,(-\frac{1}{2})$

= 3

$\therefore \ |\hat{a} - \hat{b}|^2 = 3$

$\Rightarrow \ |\hat{a} - \hat{b}| = \sqrt{3}$

Hence, the magnitude of their difference is √3.

10. Question

If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular unit vectors, then prove that $|\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}.$

Answer

Given Data:

Three mutually perpendicular unit vectors

$\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = 0$

$\text{Since } |\hat{a}| = |\hat{b}| = |\hat{c}| = 1$

Calculation:

$|\hat{a} + \hat{b} + \hat{c}|^2 = (\hat{a} + \hat{b} + \hat{c})^2$

$= (\hat{a})^2 + (\hat{b})^2 + (\hat{c})^2 + 2\,\hat{a}.\hat{b} + 2\,\hat{b}.\hat{c} + 2\,\hat{c}.\hat{a}$

$= |\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 0 + 0 + 0$

= 1+1+1

= 3

$\Rightarrow |\hat{a} + \hat{b} + \hat{c}|^2 = 3$

$\therefore |\hat{a} + \hat{b} + \hat{c}| = \sqrt{3}$

11. Question

If $|\vec{a} + \vec{b}| = 60, |\vec{a} - \vec{b}| = 40$ and $|\vec{b}| = 46,$ find $|\vec{a}|$

Answer

Given Data:

$|\vec{a} + \vec{b}| = 60$

$|\vec{a} - \vec{b}| = 40$

$|\vec{b}| = 46$

Calculation:

$\Rightarrow |\vec{a} + \vec{b}|^2 = 60^2$

$(\vec{a})^2 + (\vec{b})^2 + 2\,\vec{a}.\vec{b} = 3600$

$|\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b} = 3600$ ...Eq. 1

Now,

$\Rightarrow |\vec{a} - \vec{b}|^2 = 40^2$

$(\vec{a} - \vec{b})^2 = 1600$

$(\vec{a})^2 + (\vec{b})^2 - 2\,\vec{a}.\vec{b} = 1600$

$|\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}.\vec{b} = 1600$ ...Eq. 2

Adding Eq. 1 and Eq. 2

$2(|\vec{a}|^2 + |\vec{b}|^2) + 2\,\vec{a}.\vec{b} - 2\,\vec{a}.\vec{b} = 3600 + 1600$

$2(|\vec{a}|^2 + |\vec{b}|^2) = 5200$

$(|\vec{a}|^2 + 46^2) = \frac{5200}{2}$

$(|\vec{a}|^2 + 2116) = 2600$

$|\vec{a}|^2 = 2600 - 2116$

$|\vec{a}|^2 = 484$

$|\vec{a}| = \sqrt{484}$

$|\vec{a}| = 22$

12. Question

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined with the coordinate axes

Answer

Calculation:

Angle with x-axis

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

unit vector along x axis is $\hat{i}$

So, $\vec{b} = \hat{i}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\alpha = \vec{a}.\vec{b}$

$\cos\alpha = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$

$\cos\alpha = \frac{(\hat{i} + \hat{j} + \hat{k})(\hat{i})}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2}}$

$\cos\alpha = \frac{1}{\sqrt{3} \times \sqrt{1}}$

$\cos\alpha = \frac{1}{\sqrt{3}}$

Angle with y-axis

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

unit vector along y axis is $\hat{j}$

So, $\vec{b} = \hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\beta = \vec{a}.\vec{b}$

$\cos\beta = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$

$\cos\beta = \frac{(\hat{i} + \hat{j} + \hat{k})(\hat{j})}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2}}$

$\cos\beta = \frac{1}{\sqrt{3} \times \sqrt{1}}$

$\cos\beta = \frac{1}{\sqrt{3}}$

Angle with z-axis

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

unit vector along z axis is $\hat{k}$

So, $\vec{b} = \hat{k}$

$\Rightarrow |\vec{a}| \times |\vec{b}| \times \cos\gamma = \vec{a}.\vec{b}$

$\cos\gamma = \frac{\vec{a}.\vec{b}}{|\vec{a}| \times |\vec{b}|}$

$\cos\gamma = \frac{(\hat{i} + \hat{j} + \hat{k})(\hat{j})}{\sqrt{1^2 + 1^2 + 1^2} \times \sqrt{1^2}}$

$\cos\gamma = \frac{1}{\sqrt{3} \times \sqrt{1}}$

$\cos\gamma = \frac{1}{\sqrt{3}}$

Hence α = β = γ.

13. Question

Show that the vectors $\vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$, $\vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right)$, $\vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$ are mutually perpendicular unit vectors.

Answer

Given Data:

$\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$

$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$

$\vec{a}.\vec{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}).\frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$

$\vec{a}.\vec{b} = \frac{1}{49}(2 \times 3 + 3 \times (-6) + 6 \times 2)$

$\vec{a}.\vec{b} = \frac{1}{49}(6 + -18 + 12)$

$\vec{a}.\vec{b} = 0$

Similarly,

$\vec{b}.\vec{c} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})\frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$

$\vec{b}.\vec{c} = \frac{1}{49}(3 \times 6 + (-6) \times 2 + 2 \times (-3))$

$\vec{b}.\vec{c} = \frac{1}{49}(18 - 12 - 6)$

$\vec{b}.\vec{c} = 0$

$\vec{c}.\vec{a} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}).\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{c}.\vec{a} = \frac{1}{49}(6 \times 2 + 2 \times 3 + (-3) \times 6)$

$\vec{c}.\vec{a} = \frac{1}{49}(12 + 6 - 18)$

$\vec{c}.\vec{a} = 0$

$\therefore \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = 0$

Hence these vectors are mutually perpendicular.

14. Question

For any two vectors $\vec{a}$ and $\vec{b}$, show that : $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0 \Leftrightarrow |\vec{a}| = |\vec{b}|$.

Answer

Let $(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 0$

$|\vec{a}|^2 - |\vec{b}|^2 = 0$

$|\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{a}| = |\vec{b}|$

Let $\Rightarrow |\vec{a}| = |\vec{b}|$

Squaring both sides

$|\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 0$

$\Rightarrow (\vec{a})^2 - (\vec{b})^2 = 0$

$(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 0$

Hence, $(\vec{a} + \vec{b})(\vec{a} - \vec{b}) = 0 \Leftrightarrow |\vec{a}| = |\vec{b}|$

15. Question

If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$, find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b} + \vec{c}$.

Answer

<u>Given Data</u>:

$\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$

$\vec{b} = (\hat{i} + \hat{j} - 2\hat{k})$

$\vec{c} = (\hat{i} + 3\hat{j} - \hat{k})$

$\vec{d} = \lambda\vec{b} + \vec{c}$

$\vec{d} = \lambda(\hat{i} + \hat{j} - 2\hat{k}) + (\hat{i} + 3\hat{j} - \hat{k})$

$\vec{d} = (\lambda + 1)\hat{i} + (\lambda + 3)\hat{j} - (2\lambda + 1)\hat{k}$

For this vector to be ⊥

$\vec{a}.\vec{d} = 0$

$(2\hat{i} - \hat{j} + \hat{k})\left((\lambda + 1)\hat{i} + (\lambda + 3)\hat{j} - (2\lambda + 1)\hat{k}\right) = 0$

$2(\lambda + 1) - 1(\lambda + 3) - 1.(2\lambda + 1) = 0$

$2(\lambda + 1) - 1(\lambda + 3) - 1.(2\lambda + 1) = 0$

$-\lambda - 2 = 0$

$\therefore \lambda = -2$

16. Question

If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ, so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors.

Answer

Given Data:

$\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$

$\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$

$\vec{p} + \vec{q} = (5\hat{i} + \lambda\hat{j} - 3\hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k})$

$\vec{p} + \vec{q} = (6\hat{i} + (\lambda + 3)\hat{j} - 8\hat{k})$

Also,

$\vec{p} - \vec{q} = (5\hat{i} + \lambda\hat{j} - 3\hat{k}) - (\hat{i} + 3\hat{j} - 5\hat{k})$

$\vec{p} - \vec{q} = (4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k})$

For this vector to be ⊥

$(\vec{p} + \vec{q}).(\vec{p} - \vec{q}) = 0$

$(6\hat{i} + (\lambda + 3)\hat{j} - 8\hat{k}).(4\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}) = 0$

$6 \times 4 + (\lambda+3)(\lambda-3) - 16 = 0$

$24 + \lambda^2 - 9 - 16 = 0$

$\lambda^2 = 1$

$\lambda = \pm 1$

17. Question

If $\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$, then express $\vec{\beta}$ in the form of $\vec{\beta} = \overrightarrow{\beta_1} + \overrightarrow{\beta_2}$, where $\overrightarrow{\beta_1}$ is parallel to

$\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.

Answer

Given Data:

$\vec{\alpha} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{\beta} = 2\hat{i} + \hat{j} - 4\hat{k}$

Now

$\overrightarrow{\beta_1} \parallel \vec{\alpha}$

$\overrightarrow{\beta_1} = \lambda\vec{\alpha}$

$\Rightarrow \overrightarrow{\beta_1} = \lambda(3\hat{i} + 4\hat{j} + 5\hat{k})$

Also,

$\vec{\beta} = \overrightarrow{\beta_1} + \overrightarrow{\beta_2}$

$\Rightarrow \overrightarrow{\beta_2} = \vec{\beta} - \overrightarrow{\beta_1}$

$\overrightarrow{\beta_2} = (2\hat{i} + \hat{j} - 4\hat{k}) - \lambda(3\hat{i} + 4\hat{j} + 5\hat{k})$

$\overrightarrow{\beta_2} = (2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}$

$\overrightarrow{\beta_2} \perp \vec{\alpha}$

$\overrightarrow{\beta_2} . \vec{\alpha} = 0$

$\left((2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k} \right) . \left(3\hat{i} + 4\hat{j} + 5\hat{k}\right) = 0$

3(2-3λ) + 4(1-4λ) - 5(4+5λ) = 0

-50λ = 10

$\therefore \lambda = -\frac{1}{5}$

$\Rightarrow \overrightarrow{\beta_1} = -\frac{1}{5}(3\hat{i} + 4\hat{j} + 5\hat{k})$

Using the above value,

$\Rightarrow \overrightarrow{\beta_2} = \vec{\beta} - \overrightarrow{\beta_1}$

$\overrightarrow{\beta_2} = (2 - 3\lambda)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}$

$\overrightarrow{\beta_2} = (2 - 3)\hat{i} + (1 - 4\lambda)\hat{j} - (4 + 5\lambda)\hat{k}$

$\overrightarrow{\beta_2} = \frac{1}{5}(13\hat{i} + 9\hat{j} - 15\hat{k}$

$\vec{\beta} = \overrightarrow{\beta_1} + \overrightarrow{\beta_2}$

18. Question

If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}.\vec{b} = 0$. But, the converse need not be true. Justify your answer with an example.

Answer

$\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$

$\vec{b} = -\hat{i} + \hat{j} + 3\hat{k}$

$\vec{a}.\vec{b} = \left(2\hat{i} - \hat{j} + \hat{k}\right).\left(-\hat{i} + \hat{j} + 3\hat{k}\right)$

$\vec{a}.\vec{b} = -2 - 1 + 3$

$\vec{a}.\vec{b} = 0$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2}$

$|\vec{a}| = \sqrt{6}$

$|\vec{a}| \neq 0$

Similarly,

$|\vec{b}| = \sqrt{(-1)^2 + 1^2 + 3^2}$

$|\vec{b}| = \sqrt{11}$

$|\vec{b}| \neq 0$

19. Question

Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k},\ \vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}, \vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle.

Answer

Given Vectors:

$\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$

$\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$

First show that the vectors form a triangle, so we use the addition of vector

$\vec{b} + \vec{c} = (\hat{i} - 3\hat{j} + 5\hat{k}) + (2\hat{i} + \hat{j} - 4\hat{k})$

$\therefore \vec{b} + \vec{c} = 3\hat{i} - 2\hat{j} + \hat{k})$

$\therefore \vec{b} + \vec{c} = \vec{a}$

Hence these vectors form a triangle

Now we will use Pythagoras theorem to prove this is a right angle triangle.

$|\vec{a}| = \sqrt{3^2 + (-2)^2 + 1^2}$

$|\vec{a}| = \sqrt{14}$

$|\vec{b}| = \sqrt{1^2 + (-3)^2 + 5^2}$

$|\vec{a}| = \sqrt{35}$

$|\vec{c}| = \sqrt{2^2 + 1^2 + (-4)^2}$

$|\vec{c}| = \sqrt{21}$

$\Rightarrow |\vec{a}|^2 + |\vec{c}|^2 = 14 + 21$

$\Rightarrow |\vec{a}|^2 + |\vec{c}|^2 = 35$

$\Rightarrow |\vec{a}|^2 + |\vec{c}|^2 = |\vec{b}|^2$

Therefore these vectors form a right angled triangle.

20. Question

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Answer

Given Data:

$\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

$\vec{c} = 3\hat{i} + \hat{j}$

$\vec{d} = \vec{a} + \lambda\vec{b}$

$\vec{d} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

$\vec{d} = (-\lambda + 2)\hat{i} + (2\lambda + 2)\hat{j} + (\lambda + 3)\hat{k}$

For this vector to be ⊥

$\vec{c}.\vec{d} = 0$

$(3\hat{i} + \hat{j})\left((-\lambda + 2)\hat{i} + (2\lambda + 2)\hat{j} + (\lambda + 3)\hat{k}\right) = 0$

3(-λ+2)+1(2λ+2)=0

-λ + 8 = 0

λ = 8

The value of λ is 8.

21. Question

Find the angles of a triangle whose vertices are A(0, -1, -2), B(3, 1, 4) and C(5, 7, 1).

Answer

Given Data:

$\vec{A} = -1\hat{j} - 2\hat{k}$

$\vec{B} = 3\hat{i} + \hat{j} + 4\hat{k}$

$\vec{C} = 5\hat{i} + 7\hat{j} + \hat{k}$

$\overrightarrow{AB} = \vec{B} - \vec{A}$

$= (3\hat{i} + \hat{j} + 4\hat{k}) - (-1\hat{j} - 2\hat{k})$

$= 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\overrightarrow{BC} = \vec{C} - \vec{B}$

$= (5\hat{i} + 7\hat{j} + \hat{k}) - (3\hat{i} + \hat{j} + 4\hat{k})$

$= 2\hat{i} + 6\hat{j} - 3\hat{k}$

$\overrightarrow{AC} = \vec{C} - \vec{A}$

$= (5\hat{i} + 7\hat{j} + \hat{k}) - (-1\hat{j} - 2\hat{k})$

$= 5\hat{i} + 8\hat{j} + 3\hat{k}$

Now the angle A

$$\cos A = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}| \times |\overrightarrow{AC}|}$$

$$\cos A = \frac{(3\hat{i} + 2\hat{j} + 6\hat{k})(5\hat{i} + 8\hat{j} + 3\hat{k})}{\sqrt{3^2 + 2^2 + 6^2} \times \sqrt{5^2 + 8^2 + 3^2}}$$

$$\cos A = \frac{15 + 16 + 18}{\sqrt{49} \times \sqrt{98}}$$

$$\cos A = \frac{49}{49\sqrt{2}}$$

$$\cos A = \frac{1}{\sqrt{2}}$$

$$A = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$A = \frac{\pi}{4}$$

Now the angle B

$$\cos B = \frac{\overrightarrow{BC}.\overrightarrow{BA}}{|\overrightarrow{BC}| \times |\overrightarrow{BA}|}$$

$$\cos B = \frac{(2\hat{i} + 6\hat{j} - 3\hat{k})(-3\hat{i} - 2\hat{j} - 6\hat{k})}{\sqrt{2^2 + 6^2 + (-3)^2} \times \sqrt{(-3)^2 + (-2)^2 + (-6)^2}}$$

$$\cos B = \frac{-6 - 12 + 18}{\sqrt{49} \times \sqrt{49}}$$

$$\cos B = \frac{0}{49}$$

$$\cos B = 0$$

$$B = \cos^{-1}(0)$$

$$B = \frac{\pi}{2}$$

Now the sum of angles of a triangle is π

$\therefore A + B + C = \pi$

$$\frac{\pi}{4} + \frac{\pi}{2} + C = \pi$$

$$\therefore C = \pi - \frac{3\pi}{4}$$

$$\therefore C = \frac{\pi}{4}$$

22. Question

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60^o and their scalar product is 1/2.

Answer

Given Data:

$|\vec{a}| = |\vec{b}|$

$\vec{a}.\vec{b} = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$|\vec{a}| \times |\vec{b}| \times \cos\theta = \vec{a}.\vec{b}$

$|\vec{a}| \times |\vec{a}| \times \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$|\vec{a}|^2 \times \frac{1}{2} = \frac{1}{2}$

$|\vec{a}|^2 = \frac{1 \times 2}{2}$

$|\vec{a}|^2 = 1$

$\therefore |\vec{a}| = |\vec{b}| = 1$

Magnitude of vectors is unity.

23. Question

Show that the points whose position vectors are $\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}, \vec{c} = \hat{i} - \hat{j}$ form a right triangle.

Answer

Given Data:

$\vec{a} = 4\hat{i} - 3\hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

$\vec{c} = \hat{i} - \hat{j}$

$\overrightarrow{AB} = \vec{B} - \vec{A}$

$= (2\hat{i} - 4\hat{j} + 5\hat{k}) - (4\hat{i} - 3\hat{j} + \hat{k})$

$= -2\hat{i} - \hat{j} + 4\hat{k}$

$\overrightarrow{BC} = \vec{C} - \vec{B}$

$= (\hat{i} - \hat{j}) - (2\hat{i} - 4\hat{j} + 5\hat{k})$

$= -\hat{i} + 3\hat{j} - 5\hat{k}$

$\overrightarrow{CA} = \vec{A} - \vec{C}$

$= \left(4\hat{i} - 3\hat{j} + \hat{k}\right) - (\hat{i} - \hat{j})$

$= 3\hat{i} - 2\hat{j} + \hat{k}$

$\overrightarrow{AB}.\overrightarrow{CA} = \left(-2\hat{i} - 7\hat{j} + 4\hat{k}\right).(3\hat{i} - 2\hat{j} + \hat{k})$

= -2×3+(-1)×(-2)+1×4

= -6+2+4

$\overrightarrow{AB}.\overrightarrow{CA} = 0$

$\overrightarrow{AB} \perp \overrightarrow{CA}$

Angle A right angle, ABC is right angle triangle.

24. Question

If the vertices A, B, C of ΔABC have position vectors (1, 2, 3),(-1, 0,0),(0, 1, 2) respectively, what is the magnitude of ∠ ABC?

Answer

Given Data:

$\vec{A} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$\vec{B} = -\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$

$\vec{C} = 0\hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\overrightarrow{AB} = \vec{B} - \vec{A}$

$= (-\hat{\imath} + 0\hat{\jmath} + 0\hat{k}) - (\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$= -2\hat{\imath} - 2\hat{\jmath} - 3\hat{k}$

$\overrightarrow{BC} = \vec{C} - \vec{B}$

$= (0\hat{\imath} + \hat{\jmath} + 2\hat{k}) - (-\hat{\imath} + 0\hat{\jmath} + 0\hat{k})$

$= \hat{\imath} + \hat{\jmath} + 2\hat{k}$

$\overrightarrow{AC} = \vec{C} - \vec{A}$

$= \left(0\hat{\imath} + \hat{\jmath} + 2\hat{k}\right) - \left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$

$= -\hat{\imath} - \hat{\jmath} - \hat{k}$

Now the angle B

$$\cos B = \frac{\overrightarrow{BC}.\overrightarrow{BA}}{|\overrightarrow{BC}| \times |\overrightarrow{BA}|}$$

$$\cos B = \frac{(\hat{\imath} + \hat{\jmath} + 2\hat{k})(+2\hat{\imath} + 2\hat{\jmath} + 3\hat{k})}{\sqrt{1^2 + 1^2 + (2)^2} \times \sqrt{2^2 + 2^2 + 3^2}}$$

$$\cos B = \frac{2 + 2 + 6}{\sqrt{6} \times \sqrt{17}}$$

$$\cos B = \frac{10}{\sqrt{102}}$$

$$B = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

25. Question

If A, B, C have position vectors (0, 1, 1), (3, 1, 5), (0, 3, 3) respectively, show that is right angled at C.

Answer

Given Data:

$\vec{A} = \hat{0}\imath + \hat{\jmath} + \hat{k}$

$\vec{B} = 3\hat{\imath} + 1\hat{\jmath} + 5\hat{k}$

$\vec{C} = 0\hat{i} + 3\hat{j} + 3\hat{k}$

$\overrightarrow{AB} = \vec{B} - \vec{A}$

$= (3\hat{i} + 1\hat{j} + 5\hat{k}) - (0\hat{i} + \hat{j} + \hat{k})$

$= 3\hat{i} + 4\hat{k}$

$\overrightarrow{BC} = \vec{C} - \vec{B}$

$= (0\hat{i} + 3\hat{j} + 3\hat{k}) - (3\hat{i} + 1\hat{j} + 5\hat{k})$

$= -3\hat{i} + 2\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = \vec{C} - \vec{A}$

$= \left(0\hat{i} + 3\hat{j} + 3\hat{k}\right) - \left(0\hat{i} + \hat{j} + \hat{k}\right)$

$= 2\hat{j} + 2\hat{k}$

Now the angle C

$$\cos C = \frac{\overrightarrow{BC}.\overrightarrow{AC}}{|\overrightarrow{BC}| \times |\overrightarrow{AC}|}$$

$$\cos C = \frac{(-3\hat{i} + 2\hat{j} - 2\hat{k})(2\hat{j} + 2\hat{k})}{\sqrt{(-3)^2 + 2^2 + (-2)^2} \times \sqrt{2^2 + 2^2}}$$

$$\cos C = \frac{0 + 4 - 4}{\sqrt{6} \times \sqrt{17}}$$

$\cos C = 0$

$$C = \frac{\pi}{2}$$

So angle C is a right angle triangle.

26. Question

Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$.

Answer

we know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos x$ where x is the angle between two vectors, so $\frac{(\vec{a}.\vec{b})}{|\vec{a}|}$ gives the projection of vector b on a

Now applying the formula for projection of $\vec{b} + \vec{c}$ on $\vec{a}$

$\vec{b} + \vec{c} = i + 2j - 2k + 2i - j + 4k$

$\vec{b} + \vec{c} = 3i + j + 2k$

$|\vec{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\vec{a}.\left(\vec{b} + \vec{c}\right) = (2i - 2j + k).(3i + j + 2k)$

$\hat{i}.\hat{i} = 1;\ \hat{j}.\hat{j} = 1;\ \hat{k}.\hat{k} = 1$

$\vec{a}.\left(\vec{b} + \vec{c}\right) = 6 - 2 + 2 = 6$

Substituting these values in above formula, we get

$$\frac{[\vec{a}.(\vec{b}+\vec{c})]}{|\vec{a}|} = \frac{6}{3} = 2$$

27. Question

If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal.

Answer

meaning of orthogonal is that two vectors are perpendicular to each other, so their dot product is zero.

$\vec{a} + \vec{b} = (5\hat{\imath} - \hat{\jmath} - 3\hat{k}) + (\hat{\imath} + 3\hat{\jmath} - 5\hat{k})$

$\vec{a} + \vec{b} = 6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}$

Similarly,

$\vec{a} - \vec{b} = (5\hat{\imath} - \hat{\jmath} - 3\hat{k}) - (\hat{\imath} + 3\hat{\jmath} - 5\hat{k})$

$\vec{a} - \vec{b} = 4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$

So, to satisfy the orthogonal condition $\vec{a}.\vec{b} = 0$

$\hat{\imath}.\hat{\imath} = 1;\ \hat{\jmath}.\hat{\jmath} = 1;\ \hat{k}.\hat{k} = 1$

$(6\hat{\imath} + 2\hat{\jmath} - 8\hat{k}).(4\hat{\imath} - 4\hat{\jmath} + 2\hat{k}) = (24 - 8 - 16) = 0$

Hence proved

28. Question

A unit vector $\vec{a}$ makes angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with $\hat{i}$ and $\hat{j}$ respectively and an acute angle θ with $\hat{k}$. Find the angle θ and components of $\vec{a}$.

Answer

Assume, $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Using formula: $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos x$

$|\vec{a}| = 1$

since it is a unit vector

First taking dot product with $\hat{\imath}$

$\vec{a}.\hat{\imath} = |\vec{a}||\hat{\imath}|\cos x$

$x = \cos\left(\frac{\pi}{4}\right)$

$x = \frac{1}{\sqrt{2}}$

Taking dot product with $\hat{\jmath}$

$\vec{a}.\hat{\jmath} = |\vec{a}||\hat{\jmath}|\cos x$

$y = \cos\left(\frac{\pi}{3}\right)$

$y = \frac{1}{2}$

Now we have $\vec{a}$ as $\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + z\hat{k}$

Since the magnitude of $\vec{a}$ is 1

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + z^2 = 1$$

$$z^2 = 1 - \frac{1}{2} - \frac{1}{4}$$

$$z^2 = \frac{1}{4}$$

$$z = \frac{1}{2} \text{ or } z = -\frac{1}{2}$$

Considering, $z = \frac{1}{2}$

$$\vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

Therefore, angle with $\hat{k}$ is

$$\vec{a}.\hat{k} = |\vec{a}||\hat{k}| \cos x$$

$$\frac{1}{2} = \cos x$$

$$x = \frac{\pi}{3}$$

29. Question

If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a}.\vec{b} = 1,$ then find the value of $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$.

Answer

Expanding the given equation $(3\,\vec{a} - 5\,\vec{b}).\,(2\vec{a} + 7\vec{b})$, we get,

$$6|\vec{a}|^2 + 21(\vec{a}.\vec{b}) - 10(\vec{a}.\vec{b}) - 35|\vec{b}|^2$$

$6(2)^2 + 11(1) - 35(1)^2$

24 + 11 – 35 = 0

Hence, $(3\,\vec{a} - 5\,\vec{b}).\,(2\vec{a} + 7\vec{b}) = 0$.

30. Question

If $\vec{a}$ is a unit vector, then find $|\vec{x}|$ in each of the following

(i) $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 8$

(ii) $(\vec{x} - \vec{a}).(\vec{x} + \vec{a}) = 12$

Answer

(i) Expanding the given equation

$$|\vec{x}|^2 - |\vec{a}|^2 = 8$$

$|\vec{a}| = 1$ as given

$|\vec{x}|^2 = 9$

$|\vec{x}| = 3$ or $|\vec{x}| = -3$

(ii) expanding the given equation

$|\vec{x}|^2 - |\vec{a}|^2 = 12$

$|\vec{a}| = 1$ as given

$|\vec{x}|^2 = 13$

$|\vec{x}| = \sqrt{13}$ or $|\vec{x}| = -\sqrt{13}$

31. Question

Find $\left|\vec{a}\right|$ and $\left|\vec{b}\right|$, if

(i) $\left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right) = 12$ and $\left|\vec{a}\right| = 2\left|\vec{b}\right|$

(ii) $\left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right) = 8$ and $\left|\vec{a}\right| = 8\left|\vec{b}\right|$

(iii) $\left(\vec{a} + \vec{b}\right).\left(\vec{a} - \vec{b}\right) = 3$ and $\left|\vec{a}\right| = 2\left|\vec{b}\right|$

Answer

(i) expanding the given equation

$|\vec{a}|^2 - |\vec{b}|^2 = 12$

Substituting $|\vec{a}| = 2|\vec{b}|$

$4|\vec{b}|^2 - |\vec{b}|^2 = 12$

$3|\vec{b}|^2 = 12$

$|\vec{b}| = 2$ or -2

$|\vec{a}| = 4$ or -4

(ii) expanding the given equation

$|\vec{a}|^2 - |\vec{b}|^2 = 8$

Substituting, $|\vec{a}| = 8|\vec{b}|$

$64|\vec{b}|^2 - |\vec{b}|^2 = 8$

$63|\vec{b}|^2 = 8$

$\left|\vec{b}\right|^2 = \frac{8}{63}$

$|\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$ or $-\frac{2\sqrt{2}}{3\sqrt{7}}$

$|\vec{a}| = \frac{16\sqrt{2}}{3\sqrt{7}}$ or $-\frac{16\sqrt{2}}{3\sqrt{7}}$

(iii) expanding the given equation

$|\vec{a}|^2 - |\vec{b}|^2 = 3$

Substituting, $|\vec{a}| = 2|\vec{b}|$

$4|\vec{b}|^2 - |\vec{b}|^2 = 3$

$3|\vec{b}|^2 = 3$

$|\vec{b}| = 1 \text{ or } -1$

$|\vec{a}| = 2 \text{ or } -2$

32. Question

Find $\left|\vec{a} - \vec{b}\right|$, if

(i) $\left|\vec{a}\right| = 2, \left|\vec{b}\right| = 5$ and $\vec{a} \, . \, \vec{b} = 8$

(ii) $\left|\vec{a}\right| = 3, \left|\vec{b}\right| = 4$ and $\vec{a} \, . \, \vec{b} = 1$

(iii) $\left|\vec{a}\right| = \sqrt{3}, \left|\vec{b}\right| = 2$ and $\vec{a} \, . \, \vec{b} = 4$

Answer

(i) using formula,

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}.\vec{b})}$$

Substituting the given values in above equation we get,

$$|\vec{a} - \vec{b}| = \sqrt{2^2 + 5^2 - 2(8)}$$

$$|\vec{a} - \vec{b}| = \sqrt{13}$$

(ii) using formula,

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}.\vec{b})}$$

Substituting the given values in above equation we get,

$$|\vec{a} - \vec{b}| = \sqrt{3^2 + 4^2 - 2(1)}$$

$$|\vec{a} - \vec{b}| = \sqrt{23}$$

(iii) using formula,

$$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}.\vec{b})}$$

Substituting the given values in above equation we get,

$$|\vec{a} - \vec{b}| = \sqrt{\sqrt{3}^2 + 2^2 - 2(4)}$$

$$|\vec{a} - \vec{b}| = \sqrt{-1}$$

Now this will yield imaginary value.

We know that, $\sqrt{-1} = \text{ i (iota)}$

Therefore, $|\vec{a} - \vec{b}| = \text{i}$

33. Question

Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if

(i) $\left|\vec{a}\right| = \sqrt{3}, \left|\vec{b}\right| = 2$ and $\vec{a}.\vec{b} = \sqrt{6}$

(ii) $\left|\vec{a}\right| = 3, \left|\vec{b}\right| = 3$ and $\vec{a}.\vec{b} = 1$

Answer

(i) we know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos x$ where x is the angle between two vectors

$\cos x = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos x = \frac{(\sqrt{6})}{2\sqrt{3}}$

$\cos x = \frac{1}{\sqrt{2}}$

x = 45°

(ii) we know that,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos x$

Where, x is the angle between two vectors.

$\cos x = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos x = \frac{(1)}{3 \times 3}$

$\cos x = \frac{1}{9}$

$x = \cos^{-1}\left(\frac{1}{9}\right)$

34. Question

Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\vec{b}$.

Answer

let $\vec{a} = \vec{u} + \vec{v}$ where u is vector parallel to b and v is vector perpendicular to b, as given in the question.

$5\hat{i} - 2\hat{j} + 5\hat{k} = \vec{u} + \vec{v}$

So, $\vec{u} = p\vec{b}$; where p is some constant

$\vec{u} = 3p\hat{i} + p\hat{k}$

Substituting this value in above equation

$\vec{v} = (5 - 3p)\hat{i} - 2\hat{j} + (5 - p)\hat{k}$

Now according to conditions since vector v and b are perpendicular to each other $\vec{v}.\vec{b} = 0$

$\hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1$

(5 – 3p) (3) +(5 – p) =0

15 – 9p + 5 – p = 0

20 = 10p

P = 2

So, $\vec{u} = 6\hat{i} + 2\hat{k}$

substituting this value in above equation, we will get $\vec{v}$

$\vec{v} = \left(5\hat{i} - 2\hat{j} + 5\hat{k}\right) - \left(6\hat{i} + 2\hat{k}\right)$

$\vec{v} = -\hat{i} - 2\hat{j} + 3\hat{k}$

35. Question

If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 30^0 such that $\vec{a}.\vec{b} = 3,$ find $\left|\vec{a}\right|, \left|\vec{b}\right|.$

Answer

Let $|\vec{a}| = \left|\vec{b}\right| = x$

The angle between these vectors is 30°

So, applying the formula,

$\vec{a}.\vec{b} = |\vec{a}||\overrightarrow{b}|\cos x$

$3 = x^2 \cos 30$

$x^2 = \dfrac{6}{\sqrt{3}}$

So, the magnitude of $|\vec{a}| = \left|\vec{b}\right| = \frac{6}{\sqrt{3}}$

36. Question

Express $2\hat{i} - \hat{j} + 3\hat{k}$ as the sum of a vector parallel and a vector perpendicular to $2\hat{i} + 4\hat{j} - 2\hat{k}.$

Answer

Let $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

let $\vec{a} = \vec{u} + \vec{v}$ where u is vector parallel to b and v is vector perpendicular to b.

$2\hat{i} - \hat{j} + 3\hat{k} = \vec{u} + \vec{v}$

So, $\vec{u} = p\vec{b}$; where p is some constant

$\vec{u} = p(2\hat{i} + 4\hat{j} - 2\hat{k})$

Substituting this value in above equation

$\vec{v} = (2 - 2p)\hat{i} + (-1 - 4p)\hat{j} + (3 + 2p)\hat{k}$

Now according to conditions since vector v and b are perpendicular to each other $\vec{v}.\vec{b} = 0$

$\hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1$

2(2 – 2p) – 4(1 + 4p) – 2(3 + 2p) = 0

4 – 4p – 4 – 16 p – 6 – 4p = 0

– 24 p = 6

$p = -\frac{1}{4}$

$\vec{u} = -(\frac{1}{2}\hat{i} + 1\hat{j} - \frac{1}{2}\hat{k})$

Substituting this value of u vector in above equation

$2\hat{i} - \hat{j} + 3\hat{k} = \left(-\frac{1}{2}\hat{i} - 1\hat{j} + \frac{1}{2}\hat{k}\right) + \vec{v}$

$\vec{v} = \frac{5}{2}\hat{i} + \frac{5}{2}\hat{k}$

$2\hat{i} - \hat{j} + 3\hat{k} = \left(-\frac{1}{2}\hat{i} - 1\hat{j} + \frac{1}{2}\hat{k}\right) + \left(\frac{5}{2}\hat{i} + \frac{5}{2}\hat{k}\right)$

37. Question

Decompose the vector $6\hat{i} - 3\hat{j} - 6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$.

Answer

let $\vec{a} = 6\hat{i} - 3\hat{j} - 6\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

let $\vec{a} = \vec{u} + \vec{v}$ where u is vector parallel to b and v is vector perpendicular to b

$6\hat{i} - 3\hat{j} - 6\hat{k} = \vec{u} + \vec{v}$

So, $\vec{u} = p\vec{b}$; where p is some constant

$\vec{u} = p(\hat{i} + \hat{j} + \hat{k})$

Substituting this value in above equation

$\vec{v} = (6 - p)\hat{i} + (-3 - p)\hat{j} + (-6 - p)\hat{k}$

Now according to conditions since vector v and b are perpendicular to each other $\vec{v}.\vec{b} = 0$

$\hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1$

6 – p – 3 – p – 6 – p = 0

P = –1

So, $\vec{u} = -(\hat{i} + \hat{j} + \hat{k})$

Substituting this value of $\vec{u}$ in above equation

$6\hat{i} - 3\hat{j} - 6\hat{k} = -(\hat{i} + \hat{j} + \hat{k}) + \vec{v}$

$\vec{v} = 7\hat{i} - 2\hat{j} - 5\hat{k}$

$6\hat{i} - 3\hat{j} - 6\hat{k} = -(\hat{i} + \hat{j} + \hat{k}) + 7\hat{i} - 2\hat{j} - 5\hat{k}$

38. Question

Let $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \lambda\hat{k}$. Find such that $\vec{a} + \vec{b}$ is orthogonal to $\vec{a} - \vec{b}$.

Answer

Meaning of orthogonal is that two vectors are perpendicular to each other, so their dot product is zero.

$\vec{a} + \vec{b} = (5\hat{i} - \hat{j} + 7\hat{k}) + (\hat{i} - \hat{j} + \beta\hat{k})$

$\vec{a}+\vec{b}=6\hat{i}-2\hat{j}+(7+\beta)\hat{k}$

Similarly

$\vec{a}-\vec{b}=\left(5\hat{i}-\hat{j}+7\hat{k}\right)-\left(\hat{i}-\hat{j}+\beta\hat{k}\right)$

$\vec{a}-\vec{b}=4\hat{i}+(7-\beta)\hat{k}$

So, to satisfy the orthogonal condition $\vec{a}.\vec{b}=0$

$\hat{i}.\hat{i}=1;\ \hat{j}.\hat{j}=1;\ \hat{k}.\hat{k}=1$

$\left[6\hat{i}-2\hat{j}+(7+\beta)\hat{k}\right].\left[4\hat{i}+(7-\beta)\hat{k}\right]=24+49-\beta^2=0$

$\beta=\sqrt{73}$

39. Question

If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0,$ what can you conclude about the vector $\vec{b}$?

Answer

it is given that $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

From this, we can say that $|\vec{a}|^2=0$

So $\vec{a}$ is a zero vector

And from the second part $\vec{a}.\vec{b}=0$ we can say that $\vec{b}$ can be any vector perpendicular to zero vector $\vec{a}$.

40. Question

If $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b},$ then prove that it is perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Answer

It is given that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

So, $\vec{c}.\vec{a}=0$ and $\vec{c}.\vec{b}=0$

For $\vec{c}$ to be perpendicular to $\left(\vec{a}+\vec{b}\right)$, $\vec{c}.\left(\vec{a}+\vec{b}\right)=0$

$\vec{c}.\left(\vec{a}+\vec{b}\right)$

$\vec{c}.\vec{a}+\vec{c}.\vec{b}=0$

For the second part.

For $\vec{c}$ to be perpendicular to $\left(\vec{a}-\vec{b}\right)$, $\vec{c}.\left(\vec{a}-\vec{b}\right)=0$

$\vec{c}.\left(\vec{a}-\vec{b}\right)$

$\vec{c}.\vec{a}-\vec{c}.\vec{b}=0$

Hence, proved

41. Question

If $\left|\vec{a}\right|=a$ and $\left|\vec{b}\right|=b,$ prove that $\left(\frac{\vec{a}}{a^2}-\frac{\vec{b}}{b^2}\right)^2=\left(\frac{\vec{a}-\vec{b}}{ab}\right)^2$.

Answer

we know that $|\vec{a}-\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}.\vec{b})}$

Now expanding LHS of given equation we get,

$$= \left[\frac{a^2}{a^4} + \frac{b^2}{b^4} - \frac{2\vec{a}\vec{b}}{a^2b^2}\right]$$

$$= \left[\frac{1}{a^2} + \frac{1}{b^2} - \frac{2\vec{a}\vec{b}}{a^2b^2}\right]$$

Taking LCM we get,

$$= \left[\frac{b^2 + a^2 - 2\vec{a}\vec{b}}{a^2b^2}\right]$$

Using $|\vec{a}-\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2(\vec{a}.\vec{b})}$ re-writing the above equation

$$\left[\frac{\left(\overline{\vec{a}-\vec{b}}\right)^2}{ab}\right]$$

Hence, proved.

42. Question

If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{d}.\vec{a} = \vec{d}.\vec{b} = \vec{d}.\vec{c} = 0,$ then show that $\vec{d}$ is the null vector

Answer

Given that $\vec{a}\,\vec{b}$ and $\vec{c}$ are non-coplanar and $\vec{a}.\vec{d} = 0\ \vec{b}.\vec{d} = 0$ and $\vec{c}.\vec{d} = 0$

From above given conditions we can say that either

(i) $\vec{d} = 0$ or

(ii) $\vec{d}$ is perpendicular to $\vec{a}\,\vec{b}$ and $\vec{c}$

Since $\vec{a}\,\vec{b}$ and $\vec{c}$ are non-coplanar, $\vec{d}$ cannot be simultaneously perpendicular to all three, as only three axes exist that is x, y, z

So $\vec{d}$ must be a null vector which is equal to 0

43. Question

If a vector $\vec{a}$ is perpendicular to two non-collinear vectors $\vec{b}$ and $\vec{c},$ then $\vec{a}$ is perpendicular to every vector in the plane of $\vec{b}$ and $\vec{c}$.

Answer

Given $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$,so $\vec{c}.\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$

Let a random vector $\vec{r} = p\vec{b} + k\vec{c}$ in the plane of $\vec{b}$ and $\vec{c}$ where p and k are some arbitrary constant

Taking dot product of $\vec{r}$ with $\vec{a}$

$$\vec{r}.\vec{a} = (p\vec{b} + k\vec{c}).\vec{a}$$

$$\vec{r}.\vec{a} = (p\vec{b}.\vec{a} + k\vec{c}.\vec{a})$$

Using $\vec{c}.\vec{a} = 0$ and $\vec{a}.\vec{b} = 0$

$\vec{r}.\vec{a} = 0$

Hence, proved.........

44. Question

If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, show that the angle between the vectors $\vec{b}$ and $\vec{c}$ is given by $\cos\theta \frac{|\vec{a}|^2-|\vec{b}|^2-|\vec{c}|^2}{2|\vec{b}||\vec{c}|}$.

Answer

Given $\vec{a}+\vec{b}+\vec{c}=0$

$-\vec{a}=\vec{b}+\vec{c}$

Now squaring both sides, using,

$|\vec{a}+\vec{b}| = \sqrt{|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}.\vec{b})}$ we get,

$|\vec{a}|^2 = |\vec{b}|^2+|\vec{c}|^2+2|\vec{b}||\vec{c}|\cos x$

$\frac{[|\vec{a}|^2-|\vec{b}|^2-|\vec{c}|^2]}{2|\vec{b}||\vec{c}|} = \cos x$

Hence, proved.

45. Question

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vector such $\vec{u}+\vec{v}+\vec{w}=\vec{0}$. If $|\vec{u}|=3, |\vec{v}|=4$ and $|\vec{w}|=5$, then find $\vec{u}.\vec{v}+\vec{v}.\vec{w}+\vec{w}.\vec{u}$.

Answer

Given $\vec{u}+\vec{v}+\vec{w}=0$

Now squaring both sides using:

$(\vec{u}+\vec{v}+\vec{w})^2 = |\vec{u}|^2+|\vec{v}|^2+|\vec{w}|^2+2\vec{u}.\vec{v}+2\vec{w}.\vec{v}+2\vec{w}.\vec{u}$

$0 = 3^2+4^2+5^2+2\vec{u}.\vec{v}+2\vec{w}.\vec{v}+2\vec{w}.\vec{u}$

$2\vec{u}.\vec{v}+2\vec{w}.\vec{v}+2\vec{w}.\vec{u} = -50$

$\vec{u}.\vec{v}+\vec{w}.\vec{v}+\vec{w}.\vec{u} = -25$

46. Question

Let $\vec{a}=x^2\hat{i}+2\hat{j}-2\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{c}=x^2\hat{i}+5\hat{j}-4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{a}$ and $\vec{b}$ is obtuse

Answer

We know that,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos x$

Where, x is the angle between two vectors

Applying for $\vec{a}$ and $\vec{b}$

$(x^2\hat{\imath}+2\hat{\jmath}-2\hat{k}).(\hat{\imath}-\hat{\jmath}+\hat{k}) = \sqrt{x^4+4+4}\sqrt{1+1+1}\cos x$

$$\frac{[x^2 - 2 - 2]}{\sqrt{x^4 + 4 + 4}\sqrt{1 + 1 + 1}} = \cos x$$

$$\frac{x^2 - 4}{\sqrt{x^4 + 8}\sqrt{3}} = \cos x$$

Since angle between $\vec{a}$ and $\vec{b}$ is acute cos x should be greater than 0

$$\frac{x^2 - 4}{\sqrt{x^4 + 8}\sqrt{3}} > 0$$

$x^2 - 4 > 0$

x > 2 and x < -2

applying for $\vec{b}$ and $\vec{c}$

$$(x^2\hat{\imath} + 5\hat{\jmath} - 4\hat{k}).(\hat{\imath} - \hat{\jmath} + \hat{k}) = \sqrt{x^4 + 25 + 16}\sqrt{1 + 1 + 1}\cos x$$

$$\frac{[x^2 - 9]}{\sqrt{x^4 + 25 + 16}\sqrt{1 + 1 + 1}} = \cos x$$

Since angle between $\vec{c}$ and $\vec{b}$ is obtuse cos x should be less than

0

$$\frac{[x^2 - 9]}{\sqrt{x^4 + 41}\sqrt{3}} < 0$$

$x^2 - 9 < 0$

x > -3 and x < 3

47. Question

Find the values of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

Answer

given $\vec{a}$ is perpendicular to $\vec{b}$ so $\vec{b}.\vec{a} = 0$

$\vec{a} = 3\hat{\imath} + x\hat{\jmath} - \hat{k}$

$\vec{b} = 2\hat{\imath} + \hat{\jmath} + y\hat{k}$

Applying, $\vec{b}.\vec{a} = 0$

6 + x - y = 0

X - y = -6...(i)

Since the magnitude of both vectors are equal

$\sqrt{3^2 + x^2 + 1^2} = \sqrt{2^2 + 1^2 + y^2}$

$\sqrt{10 + x^2} = \sqrt{5 + y^2}$

$y^2 - x^2 = 5$

(y-x) (y+x) =5

6x+6y=5...(ii)

Solving equation (i) and (ii) we get

$x = -\frac{31}{12}; y = \frac{41}{12}$

48. Question

If $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors such that $\left|\vec{a}+\vec{b}\right| = \sqrt{3}$, find $\left(2\vec{a}-5\vec{b}\right).\left(3\vec{a}+\vec{b}\right)$.

Answer

Given $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = \sqrt{3}$

$|\vec{a}+\vec{b}| = \sqrt{3}$

Squaring both sides

$|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 3$

$1 + 1 + 2\vec{a}.\vec{b} = 3$

$2\vec{a}.\vec{b} = 1$

$\vec{a}.\vec{b} = \frac{1}{2}$

Now expanding the equation $(2\vec{a} - 5\vec{b})(3\vec{a} + \vec{b})$

$6|\vec{a}|^2 - 5\left|\vec{b}\right|^2 - 13\vec{a}.\vec{b}$

$1 - \frac{13}{2} = -\frac{11}{2}$

49. Question

If $\vec{a}, \vec{b}$ are two vectors such that $\left|\vec{a}+\vec{b}\right| = \left|\vec{b}\right|$, then prove that $\vec{a} = 2\vec{b}$ is perpendicular to $\vec{a}$.

Answer

Given $|\vec{a}+\vec{b}| = |\vec{b}|$

Squaring both sides we get,

$|\vec{a}+\vec{b}|^2 = |\vec{b}|^2$

$|\vec{a}+\vec{b}|.|\vec{a}+\vec{b}| = |\vec{b}|.|\vec{b}|$

$\vec{a}.\vec{a} + \vec{b}.\vec{b} + 2\vec{a}.\vec{b} = |\vec{b}|.|\vec{b}|$

$\vec{a}.\vec{a} + 2\vec{a}.\vec{b} = 0$

$\vec{a}.\left(\vec{a} + 2\vec{b}\right) = 0$

Hence, proved.

Exercise 24.2

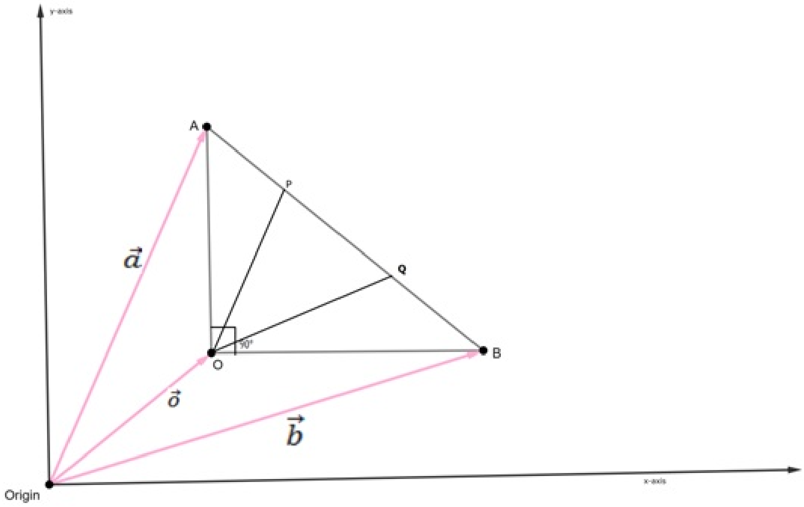
1. Question

In a triangle ΔOAB, $\angle AOB = 90°$. If P and Q are points of trisection of AB, prove that $OP^2 + OQ^2 = \frac{5}{9}AB^2$

Answer

Given:- $\angle AOB = 90°$, P and Q are trisection of AB i.e. AP = PQ = QB or 1:1:1 division of line AB

To Prove:- $OP^2 + OQ^2 = \frac{5}{9}AB^2$



Proof:- Let $\vec{o}, \vec{a}, \text{and } \vec{b}$ be position vector of O, A and B respectively

Now, Find position vector of P, we use section formulae of internal division: Theorem given below

"Let A and B be two points with position vectors $\vec{a} \text{ and } \vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$

By above theorem, here P point divides AB in 1:2, so we get

$\Rightarrow \text{Position vector of P } = \frac{\vec{b} + 2\vec{a}}{1 + 2}$

$\Rightarrow \text{Position vector of P } = \frac{2\vec{a} + \vec{b}}{3}$

Similarly, Position vector of Q is calculated

By above theorem, here Q point divides AB in 2:1, so we get

$\Rightarrow \text{Position vector of Q } = \frac{2\vec{b} + \vec{a}}{2 + 1}$

$\Rightarrow \text{Position vector of Q } = \frac{\vec{a} + 2\vec{b}}{3}$

Length OA and OB in vector form

$\Rightarrow \overrightarrow{OA} = \text{Position vector of A} - \text{Position vector of O}$

$\Rightarrow \overrightarrow{OA} = \vec{a} - \vec{o}$

$\Rightarrow \overrightarrow{OB} = \text{Position vector of B} - \text{Position vector of O}$

$\Rightarrow \overrightarrow{OB} = \vec{b} - \vec{o}$

Now length/distance OP in vector form

$\overrightarrow{OP} = \text{Position vector of P} - \text{Position vector of O}$

$\Rightarrow \overrightarrow{OP} = \frac{2\vec{a} + \vec{b}}{3} - \vec{o}$

$\Rightarrow \overrightarrow{OP} = \frac{2\vec{a} + \vec{b} - 3\vec{o}}{3}$

$\Rightarrow \overrightarrow{OP} = \frac{2\vec{a}+\vec{b}-2\vec{o}-\vec{o}}{3}$

$\Rightarrow \overrightarrow{OP} = \frac{2\vec{a}-2\vec{o}+\vec{b}-\vec{o}}{3}$

Putting $\overrightarrow{OA}$ and $\overrightarrow{OB}$ values

$\Rightarrow \overrightarrow{OP} = \frac{2\overrightarrow{OA}+\overrightarrow{OB}}{3}$

length/distance OQ in vector form

$\overrightarrow{OQ} = \text{Position vector of Q} - \text{Position vector of O}$

$\Rightarrow \overrightarrow{OQ} = \frac{\vec{a}+2\vec{b}}{3} - \vec{o}$

$\Rightarrow \overrightarrow{OQ} = \frac{\vec{a}+2\vec{b}-3\vec{o}}{3}$

$\Rightarrow \overrightarrow{OQ} = \frac{\vec{a}+2\vec{b}-2\vec{o}-\vec{o}}{3}$

$\Rightarrow \overrightarrow{OQ} = \frac{\vec{a}-\vec{o}+2\vec{b}-2\vec{o}}{3}$

Putting $\overrightarrow{OA}$ and $\overrightarrow{OB}$ values

$\Rightarrow \overrightarrow{OQ} = \frac{\overrightarrow{OA}+2\overrightarrow{OB}}{3}$

Taking LHS

$OP^2 + OQ^2$

$= \left(\frac{2\overrightarrow{OA}+\overrightarrow{OB}}{3}\right)^2 + \left(\frac{\overrightarrow{OA}+2\overrightarrow{OB}}{3}\right)^2$

$= \frac{4(\overrightarrow{OA})^2+(\overrightarrow{OB})^2+4(\overrightarrow{OA}).(\overrightarrow{OB})+(\overrightarrow{OA})^2+4(\overrightarrow{OB})^2+4(\overrightarrow{OA}).(\overrightarrow{OB})}{9}$

as we know in case of dot product

$\vec{a}.\vec{a} = |a|^2$

$\vec{a}.\vec{b} = |a||b|\cos\theta$

Angle between OA and OB is 90°,

$\Rightarrow \overrightarrow{OA}.\overrightarrow{OB} = |OA||OB|\cos 90°$

$\Rightarrow \overrightarrow{OA}.\overrightarrow{OB} = 0$

Therefore, $OP^2 + OQ^2$

$= \frac{4(\overrightarrow{OA})^2+(\overrightarrow{OB})^2+0+(\overrightarrow{OA})^2+4(\overrightarrow{OB})^2+0}{9}$

$= \frac{4(\overrightarrow{OA})^2+(\overrightarrow{OB})^2+(\overrightarrow{OA})^2+4(\overrightarrow{OB})^2}{9}$

$= \frac{5(\overrightarrow{OA})^2+5(\overrightarrow{OB})^2}{9}$

$= \frac{5(OA^2+OB^2)}{9}$

As from figure $OA^2 + OB^2 = AB^2$

$= \frac{5(AB)^2}{9}$

= RHS

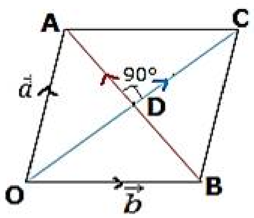
Hence, Proved.

2. Question

Prove that: If the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

Answer

Given:- Quadrilateral OACB with diagonals bisect each other at 90°.



Proof:-It is given diagonal of a quadrilateral bisect each other

Therefore, by property of parallelogram (i.e. diagonal bisect each other) this quadrilateral must be a parallelogram.

Now as Quadrilateral OACB is parallelogram, its opposite sides must be equal and parallel.

⇒ OA = BC and AC = OB

Let, O is at origin.

$\vec{a}$ and $\vec{b}$ are position vector of A and B

Therefore from figure, by parallelogram law of vector addition

$\overrightarrow{OC} = \vec{a} + \vec{b}$

And, by triangular law of vector addition

$\overrightarrow{AB} = \vec{a} - \vec{b}$

As given diagonal bisect each other at 90°

Therefore AB and OC make 90° at their bisecting point D

⇒ $\angle ADC = \angle CDB = \angle BDO = \angle ODA = 90°$

Or, their dot product is zero

⇒ $(\overrightarrow{OC}).(\overrightarrow{AB}) = 0$

⇒ $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$

⇒ $|a|^2 + \vec{a}.\vec{b} - \vec{a}.\vec{b} - |b|^2 = 0$

⇒ $|a|^2 = |b|^2$

⇒ $OA = OB$

Hence we get

OA = AC = CB = OB

i.e. all sides are equal

Therefore by property of rhombus i.e

Diagonal bisect each other at 90°

And all sides are equal

Quadrilateral OACB is a rhombus

Hence, proved.

3. Question

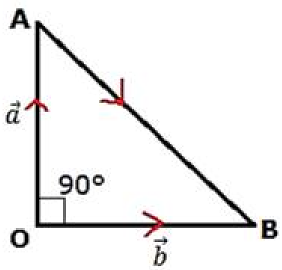
(Pythagoras's Theorem) Prove by vector method that in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Answer

Given:- Right angle Triangle

To Prove:- Square of the hypotenuse is equal to the sum of the squares of the other two sides

Let ΔAOB be right angle triangle with right angle at O



Thus we have to prove

$AB^2 = OA^2 + OB^2$

Proof: - Let, O at Origin, then

$\vec{a}$ and $\vec{b}$ be position vector of A and B respectively

Since OB is perpendicular at OA, their dot product equals to zero

We know that,

(Formula: $\vec{a}.\vec{b} = |a||b|\cos\theta$)

Therefore,

$\Rightarrow (\overrightarrow{OA}).(\overrightarrow{OB}) = 0$

$\Rightarrow \vec{a}.\vec{b} = 0$ ......(i)

Now,We can see that, by triangle law of vector addition, $\overrightarrow{AB} = \vec{b} - \vec{a}$ Therefore,

$(\overrightarrow{AB})^2 = (\vec{b} - \vec{a})^2$

$\Rightarrow (\overrightarrow{AB})^2 = a^2 + b^2 - 2\vec{a}.\vec{b}$

From equation (i)

$\Rightarrow (\overrightarrow{AB})^2 = a^2 + b^2 - 0$

$\Rightarrow AB^2 = OA^2 + OB^2$ (Pythagoras theorem)

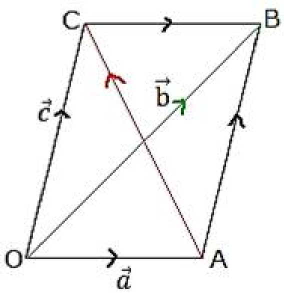
Hence, proved.

4. Question

Prove by vector method that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

Answer

Given:- Parallelogram OABC



To Prove:- $AC^2 + OB^2 = OA^2 + AB^2 + BC^2 + CO^2$

Proof:- Let, O at origin

$\vec{a}, \vec{b}$ and $\vec{c}$ be position vector of A, B and C respectively

Therefore,

$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$

Distance/length of AC

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

By triangular law:-

$\vec{a} + \vec{b} = -\vec{c}$ or $\vec{a} + \vec{b} + \vec{c} = 0$ the the vectors form sides of triangle

$\Rightarrow (\overrightarrow{AC})^2 = (\overrightarrow{AB} + \overrightarrow{BC})^2$

As AB = OC and BC = OA

From figure

$\Rightarrow (\overrightarrow{AC})^2 = (\overrightarrow{OC} - \overrightarrow{OA})^2$

$\Rightarrow (\overrightarrow{AC})^2 = (\vec{c})^2 + (\vec{a})^2 - 2(\vec{c}).(\vec{a})$ ......(i)

Similarly, again from figure

$\Rightarrow (\overrightarrow{OB})^2 = (\overrightarrow{OA} + \overrightarrow{AB})^2$

$\Rightarrow (\overrightarrow{OB})^2 = (\overrightarrow{OA} + \overrightarrow{OC})^2$

$\Rightarrow (\overrightarrow{OB})^2 = (\vec{a} + \vec{c})^2$

$\Rightarrow (\overrightarrow{OB})^2 = (\vec{a})^2 + (\vec{c})^2 + 2(\vec{a}).(\vec{c})$ ......(ii)

Now,

Adding equation (i) and (ii)

$\Rightarrow (\overrightarrow{AC})^2 + (\overrightarrow{OB})^2 = 2|\vec{a}|^2 + 2|\vec{c}|^2$ ......(iii)

Take RHS

$OA^2 + AB^2 + BC^2 + CO^2$

$= (\vec{a})^2 + (\overrightarrow{OC})^2 + (\overrightarrow{OA})^2 + (\vec{c})^2$

$= (\vec{a})^2 + (\vec{c})^2 + (\vec{a})^2 + (\vec{c})^2$

$= 2|\vec{a}|^2 + 2|\vec{c}|^2$ ......(iv)

Thus from equation (iii) and (iv), we get

LHS = RHS

Hence proved

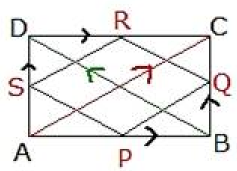
5. Question

Prove using vectors: The quadrilateral obtained by joining mid-points of adjacent sides of a rectangle is a rhombus.

Answer

Given:- ABCD is a rectangle

To prove:- PQRS is rhombus thus finding its properties in PQRS



i.e. All sides equal and parallel

Let, P, Q, R and S are midpoints of sides AB, BC, CD and DA respectively

Therefore

$\overrightarrow{PB} = \frac{\overrightarrow{AB}}{2} = \overrightarrow{AP}$

$\overrightarrow{BQ} = \frac{\overrightarrow{BC}}{2} = \overrightarrow{QC}$

$\overrightarrow{CR} = \frac{\overrightarrow{CD}}{2} = \overrightarrow{RD}$

$\overrightarrow{DS} = \frac{\overrightarrow{DA}}{2} = \overrightarrow{SA}$

also AB = CD, BC = AD (ABCD is rectangle opposite sides are equal)

Therefore

AP = PB = DR = RC and BQ = QC = AS = SD ......(i)

IMP:- Direction/arrow head of vector should be placed correctly

Now, considering in vector notion and applying triangular law of vector addition, we get

$\Rightarrow \overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$

$\Rightarrow \overrightarrow{PQ} = \frac{\overrightarrow{AB}}{2} + \frac{\overrightarrow{BC}}{2}$

$\Rightarrow \overrightarrow{PQ} = \frac{\overrightarrow{AB} + \overrightarrow{BC}}{2}$

$\Rightarrow \overrightarrow{PQ} = \frac{\overrightarrow{AC}}{2}$

Magnitude PQ = AC

and $\overrightarrow{SR} = \overrightarrow{RD} + \overrightarrow{DS}$

$\Rightarrow \overrightarrow{SR} = \frac{\overrightarrow{CD}}{2} + \frac{\overrightarrow{DA}}{2}$

$\Rightarrow \overrightarrow{SR} = \frac{\overrightarrow{CD} + \overrightarrow{DA}}{2}$

$\Rightarrow \overrightarrow{SR} = \frac{\overrightarrow{CA}}{2}$

Magnitude SR = AC

Thus sides PQ and SR are equal and parallel

It shows PQRS is a parallelogram

Now,

$\Rightarrow (\overrightarrow{PQ})^2 = (\overrightarrow{PQ}).(\overrightarrow{PQ})$

$\Rightarrow (\overrightarrow{PQ})^2 = (\overrightarrow{PB} + \overrightarrow{BQ}).(\overrightarrow{PB} + \overrightarrow{BQ})$

$\Rightarrow (\overrightarrow{PQ})^2 = (\overrightarrow{PB}).(\overrightarrow{PB}) + (\overrightarrow{PB}).(\overrightarrow{BQ}) + (\overrightarrow{PB}).(\overrightarrow{BQ}) + (\overrightarrow{BQ}).(\overrightarrow{BQ})$

By Dot product, we know

$\vec{a}.\vec{a} = |a|^2$

$\vec{a}.\vec{b} = 0$; if angle between them is 90°

Here ABCD is rectangle and have 90° at A, B, C, D

$\Rightarrow (\overrightarrow{PQ})^2 = |\overrightarrow{PB}|^2 + |\overrightarrow{BQ}|^2$

And

$\Rightarrow (\overrightarrow{PS})^2 = (\overrightarrow{PS}).(\overrightarrow{PS})$

again by triangular law

$\Rightarrow (\overrightarrow{PS})^2 = (\overrightarrow{PA} + \overrightarrow{AS}).(\overrightarrow{PA} + \overrightarrow{AS})$

$\Rightarrow (\overrightarrow{PS})^2 = \left((\overrightarrow{PA}).(\overrightarrow{PA}) + (\overrightarrow{PA}).(\overrightarrow{AS}) + (\overrightarrow{PA}).(\overrightarrow{AS}) + (\overrightarrow{AS}).(\overrightarrow{AS})\right)$

By Dot product, we know

$\vec{a}.\vec{a} = |a|^2$

$\vec{a}.\vec{b} = 0$; if angle between them is 90°

Here ABCD is rectangle and have 90° at A, B, C, D

$\Rightarrow (\overrightarrow{PS})^2 = |\overrightarrow{PA}|^2 + |\overrightarrow{AS}|^2$

From above similarities of sides of rectangle in eq (i), we have

$\Rightarrow (\overrightarrow{PS})^2 = |\overrightarrow{PB}|^2 + |\overrightarrow{BQ}|^2$

Hence PQ = PS

And from above results we have

All sides of parallelogram are equal

PQ = QR = RS = SP

Hence proved by property of rhombus (all sides are equal and opposite sides are parallel), PQRS is rhombus

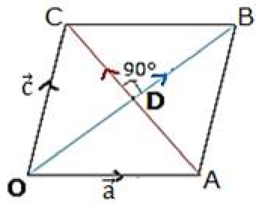
6. Question

Prove that the diagonals of a rhombus are perpendicular bisectors of each other.

Answer

Given:- Rhombus OABC i.e all sides are equal

To Prove:- Diagonals are perpendicular bisector of each other



Proof:- Let, O at the origin

D is the point of intersection of both diagonals

$\vec{a}$ and $\vec{c}$ be position vector of A and C respectively

Then,

$\overrightarrow{OA} = \vec{a}$

$\overrightarrow{OC} = \vec{c}$

Now,

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$\Rightarrow \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$

as AB = OC

$\Rightarrow \overrightarrow{OB} = \vec{a} + \vec{c}$ ……(i)

Similarly

$\Rightarrow \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$\Rightarrow \overrightarrow{AC} = -\vec{a} + \vec{c}$ ……(ii)

Tip:- Directions are important as sign of vector get changed

Magnitude are same AC = OB = $\sqrt{a^2} + c^2$

Hence from two equations, diagonals are equal

Now let’s find position vector of mid-point of OB and AC

$\Rightarrow \overrightarrow{OD} = \overrightarrow{DB} = \frac{\overrightarrow{OA} + \overrightarrow{OC}}{2}$

$\Rightarrow \overrightarrow{OD} = \overrightarrow{DB} = \frac{\vec{a} + \vec{c}}{2}$

and

$\Rightarrow \overrightarrow{AD} = \overrightarrow{DC} = \frac{\overrightarrow{AO} + \overrightarrow{OC}}{2}$

$\Rightarrow \overrightarrow{AD} = \overrightarrow{DC} = \frac{-\vec{a} + \vec{c}}{2}$

Magnitude is same AD = DC = OD = DB = $0.5(\sqrt{a^2} + c^2)$

Thus the position of mid-point is same, and it is the bisecting point D

By Dot Product of OB and AC vectors we get,

$\Rightarrow (\overrightarrow{OB}).(\overrightarrow{AC}) = (\vec{a} + \vec{c}).(\vec{c} - \vec{a})$

$\Rightarrow (\overrightarrow{OB}).(\overrightarrow{AC}) = (\vec{c} + \vec{a}).(\vec{c} - \vec{a})$

$\Rightarrow (\overrightarrow{OB}).(\overrightarrow{AC}) = |\vec{c}|^2 - |\vec{a}|^2$

$\Rightarrow (\overrightarrow{OB}).(\overrightarrow{AC}) = (\overrightarrow{OC})^2 - (\overrightarrow{OA})^2$

As the side of a rhombus are equal OA = OC

$\Rightarrow (\overrightarrow{OB}).(\overrightarrow{AC}) = OC^2 - OC^2$

$\Rightarrow (\overrightarrow{OB}).(\overrightarrow{AC}) = 0$

Hence OB is perpendicular on AC

Thus diagonals of rhombus bisect each other at 90°

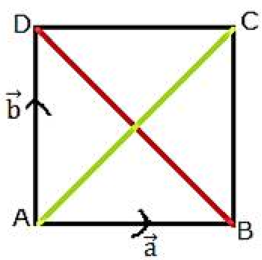
7. Question

Prove that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

Answer

Given:- ABCD is a rectangle i.e AB = CD and AD = BC

To Prove:- ABCD is a square only if its diagonal are perpendicular



Proof:- Let A be at the origin

$\vec{a}$ and $\vec{b}$ be position vector of B and D respectively

Now,

By parallelogram law of vector addition,

$\Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Since in rectangle opposite sides are equal BC = AD

$\Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$

$\Rightarrow \overrightarrow{AC} = \vec{a} + \vec{b}$

and

$\Rightarrow \overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD}$

Negative sign as vector is opposite

$\Rightarrow \overrightarrow{BD} = \vec{a} - \vec{b}$

$\Rightarrow \overrightarrow{BD} = \vec{a} - \vec{b}$

Diagonals are perpendicular to each other only

$\Rightarrow (\overrightarrow{AC}).(\overrightarrow{BD}) = 0$

$\Rightarrow (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 0$

$\Rightarrow |\vec{a}|^2 = |\vec{b}|^2$

$\Rightarrow AB^2 = AD^2$

⇒ AB = AD

Hence all sides are equal if diagonals are perpendicular to each

other

ABCD is a square

Hence proved

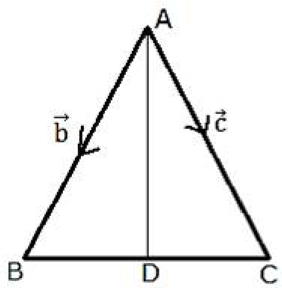
8. Question

If AD is the median of ΔABC, using vectors, prove that $AB^2 + AC^2 = 2(AD^2 + CD^2)$.

Answer

Given:- ΔABC and AD is median

To Prove:- $AB^2 + AC^2 = 2(AD^2 + CD^2)$



Proof:- Let, A at origin

$\vec{b}$ and $\vec{c}$ be position vector of B and C respectively

Therefore,

$\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$

Now position vector of D, mid-point of BC i.e divides BC in 1:1.

Section formula of internal division: Theorem given below

"Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$

Position vector of D is given by

$\Rightarrow \overrightarrow{AD} = \frac{\vec{b} + \vec{c}}{2}$

Now distance/length of CD

$\overrightarrow{CD}$ = position vector of D-position vector of C

$\Rightarrow \overrightarrow{CD} = \frac{\vec{b} + \vec{c}}{2} - \vec{c}$

$\Rightarrow \overrightarrow{CD} = \frac{\vec{b} - \vec{c}}{2}$

Now taking RHS

$= 2(AD^2 + CD^2)$

$= 2\left[\left(\frac{\vec{b}+\vec{c}}{2}\right)^2 + \left(\frac{\vec{b}-\vec{c}}{2}\right)^2\right]$

$= \frac{2}{4}\left[\left(\vec{b} + \vec{c}\right)^2 + \left(\vec{b} - \vec{c}\right)^2\right]$

$= \frac{1}{2}\left[\left(\vec{b}\right)^2 + (\vec{c})^2 + 2\left(\vec{b}\right).(\vec{c}) + \left(\vec{b}\right)^2 + (\vec{c})^2 - 2\left(\vec{b}\right).(\vec{c})\right]$

$= \frac{1}{2}\left[2\left(\vec{b}\right)^2 + 2(\vec{c})^2\right]$

$= \left(\vec{b}\right)^2 + (\vec{c})^2$

$= AB^2 + AC^2$

= LHS

Hence proved

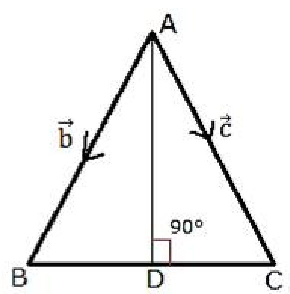
9. Question

If the median to the base of a triangle is perpendicular to the base, then the triangle is isosceles.

Answer

Given:- ΔABC, AD is median

To Prove:- If AD is perpendicular on base BC then ΔABC is isosceles



Proof:- Let, A at Origin

$\vec{b}$ and $\vec{c}$ be position vector of B and C respectively

Therefore,

$\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$

Now position vector of D, mid-point of BC i.e divides BC in 1:1

Section formula of internal division: Theorem given below

"Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$

Position vector of D is given by

$\Rightarrow \overrightarrow{AD} = \frac{\vec{b}+\vec{c}}{2}$

Now distance/length of BC

$\overrightarrow{BC}$ = position vector of C-position vector of B

$\Rightarrow \overrightarrow{BC} = \vec{c} - \vec{b}$

Now, assume median AD is perpendicular at BC

Then by Dot Product

$\Rightarrow (\overrightarrow{AD}).(\overrightarrow{BC}) = 0$

$\Rightarrow \left(\frac{\vec{b} + \vec{c}}{2}\right).(\vec{c} - \vec{b}) = 0$

$\Rightarrow (\vec{c} + \vec{b}).(\vec{c} - \vec{b}) = 0$

$\Rightarrow |\vec{c}|^2 - |\vec{b}|^2 = 0$

$\Rightarrow |\vec{c}|^2 = |\vec{b}|^2$

⇒ AC = AB

Thus two sides of ΔABC are equal

Hence ΔABC is isosceles triangle

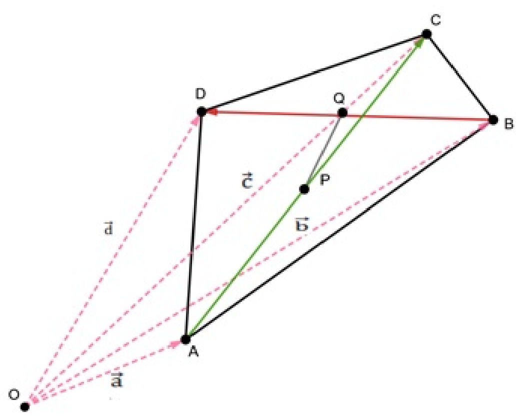
10. Question

In a quadrilateral ABCD, prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$ where P and Q are middle points of diagonals AC and BD.

Answer

Given:- Quadrilateral ABCD with AC and BD are diagonals. P and Q are mid-point of AC and BD respectively

To Prove:- $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2$



Proof:- Let, O at Origin

$\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be position vector of A, B, C and D respectively

As P and Q are mid-point of AC and BD,

Then, position vector of P, mid-point of AC i.e divides AC in 1:1

and position vector of Q, mid-point of BD i.e divides BD in 1:1

Section formula of internal division: Theorem given below

“Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$

respectively, and c be a point dividing AB internally in the ration m:n. Then the position vector of c is given by $\overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}$ “

Hence

Position vector of P is given by

$= \dfrac{\vec{a} + \vec{c}}{2}$

Position vector of Q is given by

$= \dfrac{\vec{b} + \vec{d}}{2}$

Distance/length of PQ

$\Rightarrow \overrightarrow{PQ} = \text{position vector of Q} - \text{position vector of P}$

$\Rightarrow \overrightarrow{PQ} = \frac{\vec{b}+\vec{d}}{2} - \frac{\vec{a}+\vec{c}}{2}$

Distance/length of AC

$\Rightarrow \overrightarrow{AC} = \text{position vector of C} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AC} = \vec{c} - \vec{a}$

Distance/length of BD

$\Rightarrow \overrightarrow{BD} = \text{position vector of D} - \text{position vector of B}$

$\Rightarrow \overrightarrow{BD} = \vec{d} - \vec{b}$

Distance/length of AB

$\Rightarrow \overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$

$\Rightarrow \overrightarrow{AB} = \vec{b} - \vec{a}$

Distance/length of BC

$\Rightarrow \overrightarrow{BC} = \text{position vector of C} - \text{position vector of B}$

$\Rightarrow \overrightarrow{BC} = \vec{c} - \vec{b}$

Distance/length of CD

$\Rightarrow \overrightarrow{CD} = \text{position vector of D} - \text{position vector of C}$

$\Rightarrow \overrightarrow{CD} = \vec{d} - \vec{c}$

Distance/length of DA

$\Rightarrow \overrightarrow{DA} = \text{position vector of A} - \text{position vector of D}$

$\Rightarrow \overrightarrow{DA} = \vec{a} - \vec{d}$

Now, by LHS

$= AB^2 + BC^2 + CD^2 + DA^2$

$= (\vec{b} - \vec{a})^2 + (\vec{c} - \vec{b})^2 + (\vec{d} - \vec{c})^2 + (\vec{a} - \vec{d})^2$

$$= 2\left[|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2 - \vec{a}\vec{b}\cos\theta_1 - \vec{c}\vec{b}\cos\theta_2 - \vec{c}\vec{d}\cos\theta_3 - \vec{a}\vec{d}\cos\theta_4\right]$$

Where $\theta_1, \theta_2, \theta_3, \theta_4$ are angle between vectors

Take RHS

$AC^2 + BD^2 + 4PQ^2$

$$= (\vec{c}-\vec{a})^2 + (\vec{d}-\vec{b})^2 + 4\left(\frac{\vec{b}+\vec{d}}{2} - \frac{\vec{a}+\vec{c}}{2}\right)^2$$

$$= (\vec{c}-\vec{a})^2 + (\vec{d}-\vec{b})^2 + ((\vec{b}+\vec{d}) - (\vec{a}+\vec{c}))^2$$

$$= (\vec{c}-\vec{a})^2 + (\vec{c}+\vec{a})^2 + (\vec{d}-\vec{b})^2 + (\vec{d}+\vec{b})^2 + 2(\vec{b}+\vec{d}).(\vec{a}+\vec{c})$$

$$= 2\left[|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + |\vec{d}|^2 - \vec{a}\vec{b}\cos\theta_1 - \vec{c}\vec{b}\cos\theta_2 - \vec{c}\vec{d}\cos\theta_3 - \vec{a}\vec{d}\cos\theta_4\right]$$

Thus LHS = RHS

Hence proved

Very short answer

1. Question

What $\vec{a}$ and $\vec{b}$ is the angle between vectors and with magnitudes 2 and √3 respectively? Given $\vec{a}\,.\,\vec{b} = \sqrt{3}.$

Answer

We know,

$\vec{a}\cdot\vec{b} = |a||b|cos\theta$ where θ is the angle between a⃗ and b⃗.

Given, |a|=2 |b|=√3

$\vec{a}\cdot\vec{b} = 2.\sqrt{3}\,cos\theta$

So, $cos\theta = \frac{1}{2}$

θ = 60°

2. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}\,.\,\vec{b} = 6, |\vec{a}| = 3$ and $|\vec{b}| = 4.$ Write the projection of on

Answer

$\vec{a}\cdot\vec{b} = |a||b|cos\theta = 6$

Given,

|a|=3, |b|=4

$6 = 3 \times 4\,cos\theta$

$6 = 12cos\theta$

$\cos\theta = \frac{1}{2}$

3. Question

Find the cosine of the angle between the vectors $4\hat{i} - 3\hat{j} + 3\hat{k}$ and $2\hat{i} - \hat{j} - \hat{k}.$

Answer

We know,

If $\vec{A} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$, $\vec{B} = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$

$(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}).(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}) = a_1.a_2 + b_1.b_2 + c_1.c_2$

And $\vec{A} \cdot \vec{B} = |A||B|cos\theta$

So, $\vec{A} \cdot \vec{B} = (a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}).(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k})$

$= a_1.a_2 + b_1.b_2 + c_1.c_2$

$= |A||B|cos\theta$

|A|= √34

|B|= √6

Here, 4×2 + (-3) × (-1) + 3×(-1) = 8

$\vec{A} \cdot \vec{B} = |A||B|cos\theta$

$= \sqrt{34} \times \sqrt{6}\, cos\theta$

$= \sqrt{204}\, cos\theta$

$= 8$

$$cos\theta = \frac{8}{\sqrt{204}} = 0.56$$

4. Question

If the vectors $3\hat{i} + m\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 8\hat{k}$ are orthogonal, find m.

Answer

Orthogonal vectors are perpendicular to each other so their dot product is always 0 as cos90°=0

If $\vec{A} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$, $\vec{B} = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$

$(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}).(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}) = a_1.a_2 + b_1.b_2 + c_1.c_2$

And $\vec{A} \cdot \vec{B} = 3 \times 2 + m \times (-1) + 1 \times (-8) = 0$

6-m-8=0

-m-2=0

m=-2

5. Question

If the vectors $3\hat{i} - 2\hat{j} - 4\hat{k}$ and $18\hat{i} - 12\hat{j} - m\hat{k}$ are parallel, find the value of m.

Answer

If $\vec{A} = a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}$, $\vec{B} = a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}$

And $\vec{A}$ is parallel to $\vec{B}$

Then $\vec{A}$= k$\vec{B}$, where k is some constant

So, $k = \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{3}{18} = \frac{4}{m}$

$k = \frac{4}{m} = \frac{1}{6}$

m=6×4

=24

6. Question

If $\vec{a}$ and $\vec{b}$ are vectors of equal magnitude, write the value of $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)$.

Answer

We know that dot product is distributive.

So

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}|$

$= |\vec{a}|^2 - |\vec{b}|^2$

We know

$|\vec{a}| = |\vec{b}|$

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}|$

$= |\vec{a}|^2 - |\vec{b}|^2$

$= |\vec{a}|^2 - |\vec{a}|^2$

=0

7. Question

If $\vec{a}$ and $\vec{b}$ are two vectors such that $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=0$, find the relation between the magnitudes of $\vec{a}$ and $\vec{b}$.

Answer

We know that dot product is distributive.

So

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}|$

$= |\vec{a}|^2 - |\vec{b}|^2$

Given that,

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}|$

$= |\vec{a}|^2 - |\vec{b}|^2$

$= 0$

$|\vec{a}|^2 - |\vec{b}|^2 = 0$

$|\vec{a}|^2 = |\vec{b}|^2$

Therefore, both the vectors have equal magnitude

8. Question

For any two vectors $\vec{a}$ and $\vec{b}$ write when $|\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$ holds.

Answer

We know,

$$|\vec{a}+\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta}$$

$$|\vec{a}| + |\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta}$$

$$(|\vec{a}| + |\vec{b}|)^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta$$

$$|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}| = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta$$

Comparing LHS and RHS we can conclude that

$$2|\vec{a}||\vec{b}| = 2|\vec{a}||\vec{b}|cos\theta$$

$$cos\theta = 1 \text{ or } \theta = 0°$$

9. Question

For any two vectors $\vec{a}$ and $\vec{b}$ write when $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ holds.

Answer

We know,

$$|\vec{a}+\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta}$$

$$|\vec{a}-\vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|cos\theta}$$

If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

Then, $\sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta} = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|cos\theta}$

$$2|\vec{a}||\vec{b}|cos\theta = -2|\vec{a}||\vec{b}|cos\theta$$

Comparing LHS and RHS we can conclude that

$$cos\theta = 0 \text{ or } \theta = 90°$$

10. Question

If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 60° such that $\vec{a}\,.\,\vec{b} = 8$, write the value of their magnitude.

Answer

Given,

$\theta = 60°$ and $|\vec{a}| = |\vec{b}|$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\theta$$

$$= |\vec{a}|^2 cos60°$$

$$= 8$$

$\vec{a} \cdot \vec{b} = |\vec{a}|^2 \times \frac{1}{2}$

$= 8$

$|\vec{a}|^2 = 16$

$|\vec{a}| = 4$ (as magnitude cannot be negative)

11. Question

If $\vec{a} . \vec{a} = 0 \text{ and } \vec{a} . \vec{b} = 0$, what can you conclude about the vector $\vec{b}$?

Answer

$\vec{a} \cdot \vec{a} = \vec{a} \cdot \vec{b} = 0$

$|\vec{a}||\vec{a}|cos0° = |\vec{a}||\vec{b}|cos\theta$

$= 0$

Possible answers are,

$|\vec{a}| = 0$ i.e. $\vec{a}$ is a null vector

Or

$cos\theta = 0$ or $\theta = 90°$ i.e. $\vec{a}$ and $\vec{b}$ are perpendicular

Or

$|\vec{b}| = 0$ i.e. $\vec{b}$ is a null vector

12. Question

If $\vec{b}$ is a unit vector such that $\left(\vec{a}+\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$, find $|\vec{a}|$.

Answer

$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = |\vec{a}||\vec{a}| - |\vec{a}||\vec{b}| + |\vec{a}||\vec{b}| - |\vec{b}||\vec{b}|$

$= |\vec{a}|^2 - |\vec{b}|^2$

$= 8$

$|\vec{a}|^2 - 1^2 = 8$

$|\vec{a}|^2 = 9$

$|\vec{a}| = 3$

13. Question

If $\hat{a} \text{ and } \hat{b}$ are unit vectors such that $\hat{a} + \hat{b}$ is a unit vector, write the value of $|\hat{a} - \hat{b}|$.

Answer

$|\vec{a} + \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|cos\theta} = 1$ (As given as unit vector)

$\sqrt{|\vec{a}|^2 + |\vec{b}|^2 + |\vec{a}||\vec{b}|2cos\theta} = \sqrt{1^2 + 1^2 + 1 \times 1 \times cos\theta}$

$= 1$

$\sqrt{2+2cos\theta} = 1$

$2 + 2cos\theta = 1$

$cos\theta = {}^{-1}/_{2}$

$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|cos\theta}$

$= \sqrt{1 + 1 - 2 \times 1 \times 1 \times cos\theta}$

$= \sqrt{2 - 2({}^{-1}/_{2})}$

$= \sqrt{3}$

14. Question

If $|\vec{a}| = 2, |\vec{b}|=5$ and $\vec{a}.\vec{b}=2,$ and find $|\vec{a} - \vec{b}|$.

Answer

$|\vec{a}| = 2, |\vec{b}| = 5$

$\vec{a} \cdot \vec{b} = |a||b|cos\theta$

$= 2 \times 5 \times cos\theta$

$= 2$

$cos\theta = \frac{2}{10} = \frac{1}{5}$

$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|cos\theta}$

$= \sqrt{2^2 + 5^2 - 2 \times 2 \times 5 \times cos\theta}$

$= \sqrt{4 + 25 - 20({}^{1}/_{5})}$

$|\vec{a} - \vec{b}| = \sqrt{4 + 25 - 20({}^{1}/_{5})}$

$= \sqrt{29 - 4} = \sqrt{25}$

$= 5$

15. Question

If $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -\vec{j} + \vec{k},$ find the projection of $\vec{a}$ on $\vec{b}$.

Answer

Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$|\vec{b}| = \sqrt{(-1)^2 + 1^2}$

$= \sqrt{2}$

$(a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k}).(a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k}) = a_1.a_2 + b_1.b_2 + c_1.c_2$

$\vec{a}.\vec{b} = 1 \times 0 + (-1) \times (-1) + 0 \times 1$

$= 1$

Therefore projection $= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{2}}$

16. Question

For any two non-zero vectors, write the value of $\frac{|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2}{|\vec{a}|^2 + |\vec{b}|^2}$.

Answer

$\frac{|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2}{|\vec{a}|^2 + |\vec{b}|^2} . = \frac{\left(|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\vec{b}\right) + \left(|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\vec{b}\right)}{|\vec{a}|^2 + |\vec{b}|^2}$

$= \frac{2\left(|\vec{a}|^2 + |\vec{b}|^2\right)}{|\vec{a}|^2 + |\vec{b}|^2}$

= 2

17. Question

Write the projections of $\vec{r} = 3\hat{i} - 4\hat{j} + 12\hat{k}$ on the coordinate axes.

Answer

x-axis=$\hat{\imath}$

y-axis=$\hat{\jmath}$

z-axis=$\hat{k}$

$proj_{\vec{b}}\vec{a} = \frac{\vec{a}.\vec{b}}{|b|^2}\vec{b}$

Projection along x-axis= $\frac{3}{1}\hat{\imath}$

$= 3\hat{\imath}$

Projection along y-axis= $\frac{-4}{1}\hat{\jmath}$

$= -4\hat{\jmath}$

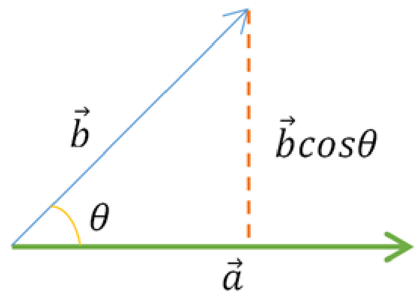
Projection along z-axis= $\frac{12}{1}\hat{k}$

$= 12\hat{k}$

18. Question

Write the component of $\vec{b}$ along $\vec{a}$.

Answer



Component of a given vector $\vec{b}$ along $\vec{a}$ is given by the length of $\vec{b}$ on $\vec{a}$.

Let θ be the angle between both the vectors.

So the length of $\vec{b}$ on $\vec{a}$ is given as: $|b|\cos\theta$

By vector dot product, we know that:

Cosθ = $\frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|}$

Therefore, $|b|\cos\theta = |b|\frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\vec{b}}{|\vec{a}|}$

Hence, $\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\vec{b}}{|\vec{a}|}$

19. Question

Write the value of $(\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k},$ where $\vec{a}$ is any vector.

Answer

Let $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\vec{a}.\hat{\imath} = x$ (1)

$\vec{a}.\hat{\jmath} = y$ (2)

$\vec{a}.\hat{k} = z$ (3)

Put the values obtained in the given equation

We get:

$$(\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k}, = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

i.e.

$$(\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k}, = \vec{a}$$

20. Question

Find the value of θϵ (0, π/2) for which vectors $\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ and $\vec{b} = \hat{\imath} - \sqrt{3}\hat{\jmath} + 2\hat{k}$ are perpendicular.

Answer

Given:

$$\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$$

$\vec{b} = \hat{\imath} - \sqrt{3}\hat{\jmath} + 2\hat{k}$

$\vec{a}.\vec{b} = 0$ (perpendicular)

So,

$\vec{a}.\vec{b} = \{(sin\theta\hat{\imath} + cos\theta\hat{\jmath}).(\hat{\imath} - \sqrt{3}\hat{\jmath} + 2\hat{k})\}$ =0

Therefore;

sin$\theta - \sqrt{3cos\theta} = 0$

Multiply and divide the whole equation by 2:

We get

$\frac{1}{2}sin\theta - \frac{\sqrt{3}}{2}cos\theta = 0$

By the identity:

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

We have:

$sin\left(\theta - \frac{\pi}{3}\right) = 0$

$sin\left(\theta - \frac{\pi}{3}\right) = sin\ n\pi$

So

$\left(\theta - \frac{\pi}{3}\right) = n\pi$

$\theta = n\pi + \frac{\pi}{3}$, n$\in I$

21. Question

Write the projection of $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$.

Answer

Let, $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ & $\vec{b} = \hat{\jmath}$

We know that, projection of $\vec{a}$ along $\vec{b}$ is given by:

$$proj_{\vec{b}}\vec{a} = \frac{\overrightarrow{\vec{a}.b}}{|\vec{b}|^2}\vec{b}$$

Also, $\vec{a}.\vec{b} = 1$

& $|b| = 1$

So, $proj_{\vec{b}}\vec{a} = 1(\hat{\jmath}) = \hat{\jmath}$

22. Question

Write a vector satisfying $\vec{a}.\hat{i} = \vec{a}.\left(\hat{i} + \hat{j}\right) = \vec{a}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 1.$

Answer

Let $\vec{a} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$\vec{a}\hat{\imath} = x$

$\vec{a}(\hat{\imath}+\hat{\jmath}) = x + y$

$\vec{a}(\hat{\imath}+\hat{\jmath}+\hat{k}) = x + y + z$

For all the equations to be equal to 1;

i.e. x= x + y

= x + y + z

=1

So, x=1;

& x + y=1

& x + y + z=1

We get: x=1,y=z=0

Therefore, $\vec{a} = \hat{\imath}$

23. Question

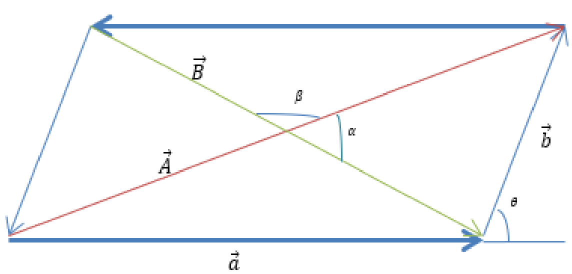
If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

Answer

Since, $|\vec{a}| = |\vec{b}| = 1$

Let $\vec{A} = \vec{a} + \vec{b}$ & $\vec{B} = \vec{a} - \vec{b}$

Angle between $\vec{a}\&\vec{b}$ is θ and angle between $\vec{A}\&\vec{B}$ is $\alpha\&\beta$



By vector addition method;

we have:

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos\theta$

=2(1+cos θ)

$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

=2(1-cos θ)

So,

$|\vec{a}+\vec{b}| = 2cos\frac{\theta}{2}$

$|\vec{a}-\vec{b}| = 2sin\frac{\theta}{2}$

Now in the parallelogram:

Area of parallelogram= (product of two sides and the sine of angle between them) i.e. $area = |\vec{a}| \times |\vec{b}| \times sin\,\theta$ (1)

Also area of parallelogram= sum of area of all four triangle

And area of each triangle =$\frac{1}{2}bh$

So, Area = $2\left\{\frac{1}{2} * \frac{|\vec{A}|}{2} * \frac{|\vec{B}|}{2} \sin\alpha\right\} + 2\left\{\frac{1}{2} * \frac{|\vec{A}|}{2} * \frac{|\vec{B}|}{2} \sin\beta\right\}$

Since α & β are supplementary

A= $4\left\{\frac{1}{2} * \frac{|\vec{A}|}{2} * \frac{|\vec{B}|}{2} \sin\alpha\right\} = \frac{1}{2} * |\vec{A}||\vec{B}| \sin\alpha$ (2)

From (1) &(2) we get:

$$\sin\alpha = \frac{2|\vec{a}||\vec{b}| \sin\theta}{|\vec{A}||\vec{B}|} = \frac{2|\vec{a}||\vec{b}| \sin\theta}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|}$$

$$\sin\alpha = \frac{2 * 1 * 1 * \sin\theta}{2sin\frac{\theta}{2} * 2cos\frac{\theta}{2}} = \frac{2\sin\theta}{2\sin\theta} = 1$$

$$\alpha = \sin^{-1} 1 = \frac{\pi}{2}$$

24. Question

If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors, write the value of $|\vec{a}\,|\,\vec{b}|$.

Answer

Since $\vec{a}$ & $\vec{b}$ are mutually perpendicular;

Then, cos$\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ (1)

And sin$\theta = \frac{(\vec{a}x\vec{b})}{|\vec{a}||\vec{b}|}$ (2)

Squaring and adding both equations, we get;

$$(\sin\theta)^2 + (\cos\theta)^2 = \left(\frac{(\vec{a}X\vec{b})}{|\vec{a}||\vec{b}|}\right)^2 + \left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)^2$$

1=$\frac{(\vec{a}x\vec{b})^2+(\vec{a}.\vec{b})^2}{(|\vec{a}||\vec{b}|)^2}$

So, $(|\vec{a}||\vec{b}|)^2 = (\vec{a}X\vec{b})^2 + (\vec{a}.\vec{b})^2$

Hence, $|\vec{a}||\vec{b}| = \sqrt{}(\vec{a}X\vec{b})^2 + (\vec{a}.\vec{b})^2$

25. Question

If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors, write the value of $|\vec{a}+\vec{b}+\vec{c}|$.

Answer

Since all three vectors are mutually perpendicular, so dot product of each vector with another is zero.

i.e. $\vec{a}.\vec{b} = 0, \vec{b}.\vec{c} = 0, \vec{c}.\vec{a} = 0$

Also, $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

So, $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$

$\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2$

i.e. $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = 3$

So, $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \sqrt{3}$

26. Question

Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

Answer

By vector dot product, we know that:

$\vec{a}.\vec{b} = |a||b|\cos\theta$

So, cosθ=$\frac{\vec{a}.\vec{b}}{|a||b|}$

$\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$.

$\vec{a}.\vec{b} = -1$

$|a|$=$\sqrt{3}$ & $|b| = \sqrt{3}$

Therefore,

cosθ = $\frac{-1}{\sqrt{3}*\sqrt{3}}$

cosθ= $\frac{-1}{3}$

So, $\theta = \cos^{-1}\left(\frac{-1}{3}\right)$

27. Question

For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$. perpendicular to each other?

Answer

Let $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$.

for $\vec{a}$ to be perpendicular to $\vec{b}$ $\cos\theta = 0$

i.e. $\vec{a}.\vec{b} = 0$ [vector dot product]

(2$\hat{\imath} + \lambda\hat{\jmath} + \hat{k}$).($\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$)=0

2-2λ+3=0

5-2λ=0

Hence, λ=$\frac{5}{2}$

28. Question

Find the projection of $\vec{a}$ on $\vec{b}$, if $\vec{a}.\vec{b}$ $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

Answer

We know that;

$proj_{\vec{b}}\vec{a} = \frac{\vec{a}.\vec{b}}{|b|^2}\vec{b}$

So,

$proj_{\vec{b}}\vec{a} = \frac{2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}}{|b|^2}$

29. Question

Write the value of p for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel vectors.

Answer

$\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$

for $\vec{a}$ to be parallel to $\vec{b}$ $\sin\theta = 0$

i.e. $(\vec{a} X \vec{b}) = 0$ [vector cross product]

$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 2 & 9 \\ 1 & p & 3 \end{vmatrix} = 0$

$\hat{\imath}(6 - 9p) - \hat{\jmath}(9 - 9) + \hat{k}(3p - 2) = 0$

$\hat{\imath}(6 - 9p) + \hat{k}(3p - 2) = 0\hat{\imath} + 0\hat{k}$

(6-9p)=0 & (3p-2)=0

Hence, $p = \frac{2}{3}$

30. Question

Find the value of λ if the vectors $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other.

Answer

Let $\vec{a} = 2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - 4\hat{k}$

for $\vec{a}$ to be perpendicular to $\vec{b}$ $\cos\theta = 0$

i.e. $\vec{a}.\vec{b} = 0$ [vector dot product]

$(2\hat{\imath} + \lambda\hat{\jmath} + 3\hat{k}).(3\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) = 0$

6+2λ-12=0

2λ-6=0

Hence, λ=3

31. Question

If $|\vec{a}| = 2, |\vec{b}| = 3, \vec{a}.\vec{b} = 3,$ find the projection of $\vec{b}$ on $\vec{a}$

Answer

Given $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and $|\vec{a}.\vec{b}| = 3$

The projection of $\vec{b}$ vector $\vec{a}$ on a is given by,

$\vec{b}.\hat{a} = \vec{b}.\frac{\vec{a}}{|\vec{a}|}$

$= \frac{\vec{a}.\vec{b}}{|\vec{a}|}$ (since scalar product is commutative)

$= \frac{3}{2}$

32. Question

Write the angle between the two vectors $\vec{a}$ and $\overrightarrow{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a}.\vec{b} = \sqrt{6}$

Answer

We know that the scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined as,

$\vec{a}.\vec{b} = \overrightarrow{|a|}\ \overrightarrow{|b|}\ cos\theta$

$\sqrt{6} = \sqrt{3} \times 2cos\theta$

$cos\theta = \frac{\sqrt{3} \times \sqrt{2}}{\sqrt{3} \times 2}$

$= \frac{1}{\sqrt{2}}$

$\theta = cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4}$

33. Question

Write the projection of vector $\hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ on the vector $2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$.

Answer

Let $\vec{a} = \hat{\imath} + 3\hat{\jmath} + 7\hat{k}$ and $\vec{b} = \hat{2\imath} - 3\hat{\jmath} + 6\hat{k}$

Then the projection of vector $\vec{a}$ on $\overrightarrow{b}$ is given by,

$\vec{a}.\hat{b} = \frac{\vec{a}.\overleftarrow{b}}{|\vec{b}|}$

$= \vec{a}.\vec{b} = (\hat{\imath} + 3\hat{\jmath} + 7\hat{k}).(2\hat{\imath} - 3\hat{\jmath} + 6\hat{k})$

$=1\times 2-3\times 3+7\times 6$

$=2-9+42$

$=35$

Now, $|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2}$

$= \sqrt{4 + 36 + 9}$

$= \sqrt{49}$

$=7$

Therefore projection of $\vec{a}$ on $\vec{b} = \frac{35}{7}$

$=5$

34. Question

Find λ, when the projection of $\vec{a} = \lambda\hat{\imath} + \hat{\jmath} + 4\hat{k}$ on $\vec{b} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$ is 4 units.

Answer

Given $\vec{a}=\widehat{\lambda\imath}+\hat{\jmath}+4\hat{k}$ and $\vec{b}=\widehat{2\imath}+6\hat{\jmath}+3\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is given by $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$\vec{a}.\vec{b}=\left(\lambda\hat{\imath}+\hat{\jmath}+\widehat{4k}\right).\left(2\hat{\imath}+6\hat{\jmath}+\widehat{3k}\right)$

=2λ+6+12

=2λ+18

Now, |b|=$\sqrt{2^2+6^2+3^2}$ =$\sqrt{49}$=7

$\frac{2\lambda+18}{7}=4$

2λ+18=28

2λ=10

λ=5

35. Question

For what value of λ are the vectors $\vec{a}=2\hat{\imath}+\lambda\hat{\jmath}+\hat{k}$ *and* $\vec{b}=\hat{\imath}-2\hat{\jmath}+3\hat{k}$ perpendicular to each other?

Answer

Given $\vec{a}=2\hat{\imath}+\lambda\hat{\jmath}+\hat{k}$ and $\vec{b}=\hat{\imath}-2\hat{\jmath}+3\hat{k}$

For two vectors to be perpendicular, the angle between them must be 90^0 or $\frac{\pi}{2}$

We know that cos 90=0

$\vec{a}.\vec{b}=\left(2\hat{\imath}+\lambda\hat{\jmath}+\hat{k}\right).\left(\hat{\imath}-2\hat{\jmath}+\widehat{3k}\right)$

=2-2λ+3

=5-2λ

By scalar product, $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|cos\theta$

5-2λ=0

$\lambda=\frac{5}{2}$

36. Question

Write the projection of the vector $7\hat{\imath}+\hat{\jmath}-4\hat{k}$ on the vector $2\hat{\imath}+6\hat{\jmath}+3\hat{k}$.

Answer

Let $\vec{a}=\hat{\imath}+\hat{\jmath}+\hat{k}$ and $\vec{b}=\hat{\imath}+\hat{\jmath}+\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is given by,

$\vec{a}.\hat{b}=\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$\vec{a}.\vec{b}=\left(7\hat{\imath}+\hat{\jmath}-4\hat{k}\right).\left(2\hat{\imath}+6\hat{\jmath}+3\hat{k}\right)$

=7×2+1× 6 – 4× 3

=14+6-12

=14-6

=8

$|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2}$

$= \sqrt{4 + 36 + 9}$

$= \sqrt{49}$

$= 7$

Therefore, projection of $\vec{a}\, on\, \vec{b}$ is $\frac{8}{7}$

37. Question

Write the value of λ so that the vectors $\vec{a} = 2\hat{\imath} + \lambda\hat{\jmath} + \hat{k}\ and\ \vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ perpendicular to each other?

Answer

Given $\vec{a} = 2\hat{\imath} + \lambda\hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$

For two vectors to be perpendicular, the angle between them must be 90^0 or $\frac{\pi}{2}$

We know that Cos90=0

$\vec{a}.\vec{b} = \left(2\hat{\imath} + \lambda\hat{\jmath} + \hat{k}\right).\left(\hat{\imath} - 2\hat{\jmath} + \widehat{3k}\right)$

=2-2λ+3

=5-2λ

By scalar product, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|cos\theta$

Therefore, 5-2λ=0

$\lambda = \frac{5}{2}$

38. Question

Write the projection of $\vec{b} + \vec{c}\ on\ \vec{a}$, when $\vec{a} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}, \vec{b} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$,and $\vec{c} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$.

Answer

Given, $\vec{a} = \widehat{2\imath} - 2\hat{\jmath} + \hat{k}$

$\vec{b} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\vec{c} = \widehat{2\imath} - \hat{\jmath} + 4\hat{k}$

So, $\vec{b} + \vec{c} = \left(\hat{\imath} + \widehat{2\jmath} + 2\hat{k}\right) + (2\hat{\imath} - \hat{\jmath} + \widehat{4k})$

$= \widehat{3\imath} + \hat{\jmath} + 6\hat{k}$

$= \vec{d}$

Now, to find projection of $\vec{b} + \vec{c}\ on\ \vec{a}\ i.e.\ \vec{d}\ on\ \vec{a}$

$\vec{d}.\hat{a} = \frac{\vec{d}.\vec{a}}{|\vec{a}|}$

Now,$\vec{d}.\vec{a} = \left(3\hat{\imath} + \hat{\jmath} + \widehat{6k}\right).(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$

=3× 2-1× 2+6× 1

=6-2+6

=10

$|\vec{a}| = \sqrt{2^2 + (-2)^2 + 1^2}$

$= \sqrt{4 + 4 + 1}$

=√9

=3

Therefore, $\frac{\vec{d}.\vec{a}}{|\vec{a}|} = \frac{10}{3}$

39. Question

If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a} + \vec{b}| = 3 \; and \; |\vec{a}| = 5,$find the value of $|\vec{b}|$.

Answer

Given, $|\vec{a} + \vec{b}| = 3$ and $|\vec{a}| = 5$

Also given $\vec{a}$ and $\vec{b}$ are perpendicular

$\vec{a}.\vec{b} = 0$

$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b})^2$

$3^2 = |\vec{a}|^2 + 2|\vec{a}.\vec{b}\,| + |\vec{b}|^2$

$3^2 = 5^2 + |\vec{b}|^2$

$9 = 25 + |\vec{b}|^2$

$-|\vec{b}|^2 \; = 16$

$|\vec{b}| = -4$

40. Question

If $\vec{a} \; and \; \vec{b}$ vectors are such that$|\vec{a}| = 3, |\vec{b}| = \frac{2}{3} \; and \; \vec{a} \, X \, \vec{b}$ is a unit vector, then write the angle between $\vec{a} \; and \; \vec{b}.$

Answer

Given $|\vec{a}| = 3 \; |\vec{b}| = \frac{2}{3}$

Also given, $\vec{a} \times \vec{b}$ is a unit vector

$\Rightarrow |\vec{a} \times \vec{b}\,| = 1$

By vector product,

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|Sin\theta$

Therefore, $1 = 3 \times \frac{2}{3} \; sin\theta$

$\Rightarrow \; sin\theta = \frac{1}{2}$

$\Rightarrow \; \theta = \frac{\pi}{6}$

41. Question

If $\vec{a}\ and\ \vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}\ and\ \vec{b}$.

Answer

Given $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$

Now, $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$\Rightarrow 1 = |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$

$\Rightarrow 1 = 1 + 2|\vec{a}.\vec{b}| + 1$

$\Rightarrow -1 = 2 + 2|\vec{a}.\vec{b}|$

$\Rightarrow -\frac{1}{2} = |\vec{a}.\vec{b}|$

Also, $|\vec{a}.\vec{b}| = |\vec{a}||\vec{b}|cos\theta$

Therefore, $-\frac{1}{2} = 1 \times 1 \times cos\theta$

$\Rightarrow Cos\,\theta = -\frac{1}{2}$

We know that $\cos 60° = \frac{1}{2}$ and cos is negative in 2nd quadrant

Therefore, θ =180-60

=120

$= \frac{2\pi}{3}$

42. Question

If $\vec{a}\ and\ \vec{b}$ are unit vectors, then find the angle between $\vec{a}\ and\ \vec{b}$, given that $\sqrt{3}\vec{a} - \vec{b}$ is a unit vector.

Answer

Given, $|\vec{a}| = |\vec{b}| = 1$ and $|\sqrt{3}\,\vec{a} + \vec{b}| = 1$

By scalar product, $|\vec{a}.\vec{b}| = |\vec{a}||\vec{b}|cos\theta$

By substituting the values, we get

$\vec{a}.\vec{b} = cos\theta$

$|\sqrt{3}a - b|^2 = 1$

$(\sqrt{3}a)^2 - 2\sqrt{3}ab + b^2$=1

$3a^2$ -2√3 cos θ + b^2 =1

⇒3-2√3 cos θ +1=1

⇒4-1=2√3 cos θ

⇒3=2√3 cos θ

$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$

MCQ

1. Question

Mark the correct alternative in each of the following:

The vector $\vec{a}$ and $\vec{b}$ satisfy the equation $2\vec{a}+\vec{b}=\vec{p}$ and $\vec{a}+2\vec{b}=\vec{q}$, where $\vec{p}=\hat{i}+\hat{j}$ and $\vec{q}=\hat{i}-\hat{j}$. If θ is the angle between $\vec{a}$ and $\vec{b}$, then

A. $\cos\theta=\frac{4}{5}$

B. $\sin\theta=\frac{1}{\sqrt{2}}$

C. $\cos\theta=-\frac{4}{5}$

D. $\cos\theta=-\frac{3}{5}$

Answer

Here, $2\vec{a}+\vec{b}=\vec{p}$ and $\vec{a}+2\vec{b}=\vec{q}$

Also, $\vec{p}=\hat{\imath}+\hat{\jmath}$ and $\vec{q}=\hat{\imath}-\hat{\jmath}$

$\therefore 2\vec{a}+\vec{b}=\hat{\imath}+\hat{\jmath}$ and $\vec{a}+2\vec{b}=\hat{\imath}-\hat{\jmath}$

Solving above two equations for $\vec{a}$ and $\vec{b}$ we get,

$\therefore \vec{a}=\frac{2}{6}\hat{\imath}+\hat{\jmath}$ and $\vec{b}=\frac{2}{6}\hat{\imath}-\hat{\jmath}$

$\therefore \vec{a}\cdot\vec{b}=\frac{2}{6}\times\frac{2}{6}+1\times(-1)$

$=\frac{4}{36}-1$

$=-\frac{32}{36}$

Also, $|\vec{a}|=\left\{\left(\frac{2}{6}\right)^2+(1)^2\right\}^{\frac{1}{2}}$

$=\sqrt{\frac{40}{36}}$

$=\frac{\sqrt{40}}{6}$

Ands, $|\vec{b}|=\left\{\left(\frac{2}{6}\right)^2+(1)^2\right\}^{\frac{1}{2}}$

$=\sqrt{\frac{40}{36}}$

$=\frac{\sqrt{40}}{6}$

Now, θ is the angle between $\vec{a}$ and $\vec{b}$

So, $cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$= \frac{\left(-\frac{32}{36}\right)}{\left(\frac{\sqrt{40}}{6}\times\frac{\sqrt{40}}{6}\right)}$

$= -\frac{32}{40}$

$= -\frac{4}{5}$

2. Question

Mark the correct alternative in each of the following:

If $\vec{a}.\hat{i} = \vec{a}.\left(\hat{i}+\hat{j}\right) = \vec{a}.\left(\hat{i}+\hat{j}+\hat{k}\right) = 1,$ then $\vec{a}$ =

A. $\vec{0}$

B. $\hat{i}$

C. $\hat{j}$

D. $\hat{i}+\hat{j}+\hat{k}$

Answer

Here, $\vec{a}\hat{\imath} = 1$ __________(1)

$\vec{a}(\hat{\imath}+\hat{\jmath}) = 1$ _________(2)

and $\vec{a}\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right) = 1$_________(3)

From (2),

$\vec{a}\hat{\imath} + \vec{a}\hat{\jmath} = 1$

$\therefore \vec{a}\hat{\jmath} = 0$ $(\because \vec{a}\hat{\imath} = 1)$___________(4)

From (3) and (4)

$\vec{a}\hat{\imath} + \vec{a}\hat{k} = 1$ $(\because a\vec{\ }\hat{j} = 0)$

$\therefore\therefore \vec{a}\hat{k} = 0$ $(\because \vec{a}\hat{\imath} = 1)$

So, $\vec{a} = \vec{a}\hat{\imath} + \vec{a}\hat{\jmath} + \vec{a}\hat{k}$

$= \hat{\imath}$

3. Question

Mark the correct alternative in each of the following:

If $\vec{a}+\vec{b}+\vec{c} = \vec{0}, |\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7,$ then the angle between $\vec{a}$ and $\vec{b}$ is

A. $\frac{\pi}{6}$

B. $\frac{2\pi}{3}$

C. $\frac{5\pi}{3}$

D. $\frac{\pi}{3}$

Answer

Here, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ ____________(1)

$\Rightarrow \vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=\vec{0}$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=-\vec{a}\cdot\vec{a}=-|\vec{a}|^2$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=-9$ $(\because |\vec{a}|=3)$ __________(2)

From (1)

$\Rightarrow \vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}$

$\Rightarrow \vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}=\vec{0}$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}=-\vec{b}\cdot\vec{b}=-\left|\vec{b}\right|^2$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}=-25$ $(\because \left|\vec{b}\right|=5)$ ________(3)

From (1)

$\Rightarrow \vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}$

$\Rightarrow \vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=\vec{0}$

$\Rightarrow\ \vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=-\vec{c}\cdot\vec{c}=-|\vec{c}|^2$

$\Rightarrow\ \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}=-49$ $(\because |\vec{c}|=7)$ _________(4)

From (2) and (3)

$\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=24$ ___________(5)

From (2) and (5)

$2\left(\vec{a}\cdot\vec{b}\right)=15$

$\Rightarrow \vec{a}\cdot\vec{b}=\frac{15}{2}$

Let θ be the angle between $\vec{a}$ and $\vec{b}$

Then, $cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$=\frac{\frac{15}{2}}{3\times 5}$

$=\frac{1}{2}$

So, $\theta = \frac{\pi}{3}$

i.e. angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

4. Question

Mark the correct alternative in each of the following:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and α be the angle between them, then $\vec{a} + \vec{b}$ is a unit vector, if

A. $\alpha = \frac{\pi}{4}$

B. $\alpha = \frac{\pi}{3}$

C. $\alpha = \frac{2\pi}{3}$

D. $\alpha = \frac{\pi}{2}$

Answer

Here, $\vec{a}$ and $\vec{b}$ are unit vectors.

i.e. $|\vec{a}| = 1$ and $|\vec{b}| = 1$

If $\vec{a} + \vec{b}$ is unit vector then

$|\vec{a} + \vec{b}| = 1$

$\Rightarrow |\vec{a} + \vec{b}|^2 = 1$

$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = 1$ $(\because |\vec{a}|^2 = \vec{a} \cdot \vec{a})$

$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = 1$

$\Rightarrow 2(\vec{a} \cdot \vec{b}) + 2 = 1$ $(\because |\vec{a}|^2 = \vec{a} \cdot \vec{a} = 1;\ |b|^2 = \vec{b} \cdot \vec{b} = 1; \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a})$

$\Rightarrow (\vec{a} \cdot \vec{b}) = -\frac{1}{2}$

Now, $cos\alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$= -\frac{1}{2}$

We know $\cos\frac{\pi}{3} = \frac{1}{2}$ and cosine is negative in second quadrant.

$\therefore \alpha = \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

5. Question

Mark the correct alternative in each of the following:

The vector $(\cos\alpha + \cos\beta)\hat{i} + (\cos\alpha + \sin\beta)\hat{j} + (\sin\alpha)\hat{k}$ is a

A. null vector

B. unit vector

C. constant vector

D. none of these

Answer

Let $\vec{a} = (\cos\alpha\cos\beta)\hat{\imath} + (\cos\alpha\sin\beta)\hat{\jmath} + (\sin\alpha)\hat{k}$

So, $|\vec{a}|^2 = (\cos\alpha\cos\beta)^2 + (\cos\alpha\sin\beta)^2 + (\sin\alpha)^2$

$=\cos^2\alpha(\cos^2\beta+\sin^2\beta)+\sin^2\alpha$

$=\cos^2\alpha(1)+\sin^2\alpha$

$=1$

i.e. $|\vec{a}| = 1$

So, $\vec{a}$ is a unit vector.

6. Question

Mark the correct alternative in each of the following:

If the position vectors of P and Q are $\hat{i} + 3\hat{j} - 7\hat{k}$ and $5\hat{i} - 2\hat{j} + 4\hat{k}$ then the cosine of the angle between $\vec{PQ}$ and y-axis is

A. $\frac{5}{\sqrt{162}}$

B. $\frac{4}{\sqrt{162}}$

C. $-\frac{5}{\sqrt{162}}$

D. $\frac{11}{\sqrt{162}}$

Answer

Let $\vec{r}$ be the direction of $\overrightarrow{PQ}$

Then, $\vec{r} = Q - P = 4\hat{\imath} - 5\hat{\jmath} + 11\hat{k}$

Let θ be the angle between $\vec{r}$ and Y-axis

Then, $\cos\theta = \frac{\vec{r}\cdot\hat{\jmath}}{|\vec{r}|\times|\hat{\jmath}|}$

$= -\frac{5}{(16 + 25 + 121)^{\frac{1}{2}}}$

$= -\frac{5}{\sqrt{162}}$

7. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are unit vectors, then which of the following values of $\vec{a}\,.\,\vec{b}$ is not possible?

A. $\sqrt{3}$

B. $\sqrt{3}/2$

C. $1/\sqrt{2}$

D. -1/2

Answer

Here, $\vec{a}$ and $\vec{b}$ are unit vectors.

i.e. $|\vec{a}| = 1$ and $\overline{|\vec{b}|} = 1$

Now, Let θ be the angle between $\vec{a}$ and $\vec{b}$

So, $cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \vec{a}\cdot\vec{b} = cos\theta$

Now, we know $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

; $cos\frac{2\pi}{3} = -\frac{1}{2}$

; $cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Therefore, $\vec{a}\cdot\vec{b} = cos\theta = \sqrt{3}$ is not possible.

8. Question

Mark the correct alternative in each of the following:

If the vectors $\hat{i} - 2x\hat{j} + 2y\hat{k}$ and $\hat{i} + 2x\hat{j} - 3y\hat{k}$ are perpendicular, then the locus of (x, y) is

A. a circle

B. an ellipse

C. a hyperbola

D. none of these

Answer

Let $\vec{a} = \hat{\imath} - 2x\hat{\jmath} + 2y\hat{k}$ and $\vec{b} = \hat{\imath} + 2x\hat{\jmath} - 3y\hat{k}$

Given that $\vec{a}$ and $\vec{b}$ are perpendicular.

So, $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow 1-4x^2 -6y^2 =0$

$\Rightarrow 4x^2+6y^2=1$

Here, vectors are in 3-Dimensions

∴ above equation represents an ellipse .i.e. locus of (x, y) is an ellipse.

9. Question

Mark the correct alternative in each of the following:

The vector component of $\vec{b}$ perpendicular to $\vec{a}$ is

A. $\left(\vec{b}\,.\,\vec{c}\right)\vec{a}$

B. $\dfrac{\vec{a}\times\left(\vec{b}\times\vec{c}\right)}{|\,\vec{a}\,|^2}$

C. $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$

D. none of these

Answer

Let $\vec{r}$ be the vector projection of $\vec{b}$ onto $\vec{a}$

Then, $\vec{r}=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}$

Now, vector component of $\vec{b}$ perpendicular to $\vec{a}$ is

$\vec{x}=\vec{b}-\vec{r}$

$=\vec{b}-\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}$

$=\dfrac{\vec{b}(\vec{a}\cdot\vec{a})-(\vec{a}\cdot\vec{b})\vec{a}}{|\vec{a}|^2}$

$=\dfrac{\vec{a}\times(\vec{b}\times\vec{c})}{|\vec{a}|^2}$

10. Question

Mark the correct alternative in each of the following:

The length of the longer diagonal of the parallelogram constructed on $5\vec{a}+2\vec{b}$ and $\vec{a}-3\vec{b}$ if its is given that $|\,\vec{a}\,|=2\sqrt{2},|\,\vec{b}\,|=3$ and angle between $\vec{a}$ and $\vec{b}$ is π/4, is

A. 15

B. $\sqrt{113}$

C. $\sqrt{593}$

D. $\sqrt{369}$

Answer

Here, $|\vec{a}|=2\sqrt{2}$ and $|\vec{b}|=3$

The parallelogram is constructed on $5\vec{a}+2\vec{b}$ and $\vec{a}-3\vec{b}$

Then its one diagonal is $5\vec{a}+2\vec{b}+\vec{a}-3\vec{b}=6\vec{a}-\vec{b}$

And other diagonal is $5\vec{a}+2\vec{b}-\vec{a}+3\vec{b}=4\vec{a}+5\vec{b}$

Length of one diagonal is $=|6\vec{a}-\vec{b}|$

$=\left\{\left(6\vec{a}-\vec{b}\right)\cdot\left(6\vec{a}-\vec{b}\right)\right\}^{\frac{1}{2}}$

$=\left(36\vec{a}^2+\vec{b}^2-2\times 6|\vec{a}||\vec{b}|\cos\frac{\pi}{4}\right)^{\frac{1}{2}}$

$= \left(36 \times 8 + 9 - 12 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}$

$= (288 + 9 - 12 \times 6)^{\frac{1}{2}}$

=√225

=15

Length of other diagonal is $= |4\vec{a} + 5\vec{b}|$

$= \left\{\left(4\vec{a} + 5\vec{b}\right) \cdot \left(4\vec{a} + 5\vec{b}\right)\right\}^{\frac{1}{2}}$

$= \left(16\vec{a}^2 + 25\vec{b}^2 + 2 \times 4 \times 5|\vec{a}||\vec{b}|\cos\frac{\pi}{4}\right)^{\frac{1}{2}}$

$= \left(16 \times 8 + 25 \times 9 + 40 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}$

$= (128 + 225 + 40 \times 6)^{\frac{1}{2}}$

=√593

So, Length of the longest diagonal is √593.

11. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ is a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if

A. λ = 1

B. λ = –1

C. a = |λ|

D. $a = \frac{1}{|\lambda|}$

Answer

Here,$|\vec{a}| = a$

Now,$\lambda\vec{a}$ is unit vector if $|\lambda\vec{a}| = 1$

i.e. $|\lambda||\vec{a}| = 1$

i.e. $|\lambda|a = 1$

i.e. $a = \frac{1}{|\lambda|}$

12. Question

Mark the correct alternative in each of the following:

If θ is the angle between two vectors $\vec{a} \text{ and } \vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when

A. $0 < \theta < \frac{\pi}{2}$

B. $0 \leq \theta \leq \frac{\pi}{2}$

C. $0 < \theta < \pi$

D. $0 \le \theta \le \pi$

Answer

Here, θ be the angle between $\vec{a}$ and $\vec{b}$

Then, $cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

Now, $\vec{a} \cdot \vec{b} \ge 0$

$\Rightarrow cos\theta|\vec{a}||\vec{b}| \ge 0$

$\Rightarrow \cos\theta \ge 0$

We know cosine is positive in first quadrant.

$\therefore 0 \le \theta \le \frac{\pi}{2}$

13. Question

Mark the correct alternative in each of the following:

The values of x for which the angle between $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}, \vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\vec{b}$ and the z-axis is acute and less than π/6 are

A. $x > \frac{1}{2} \text{ or } x < 0$

B. $0 < x < \frac{1}{2}$

C. $\frac{1}{2} < x < 15$

D. ϕ

Answer

Here, angle between $\vec{a}$ and $\vec{b}$ is obtuse

So, $\vec{a} \cdot \vec{b} \le 0$

$\Rightarrow 14x^2 - 8x + x \le 0$

$\Rightarrow 14x^2 - 7x \le 0$

$\Rightarrow 2x^2 - x \le 0$

$\Rightarrow x(2x - 1) \le 0$

$\Rightarrow x \le 0 \text{ and } x \ge \frac{1}{2}$

or $x \ge 0 \text{ and } x \le \frac{1}{2}$ __________(1)

Now, angle between $\vec{b}$ and Z-axis is acute

So, $\vec{b} \cdot \hat{k} \ge 0$

$\Rightarrow x \ge 0$ ___________(2)

∴ From (1) and (2) $0 \le x \le \frac{1}{2}$.

14. Question

Mark the correct alternative in each of the following:

If $\vec{a}, \vec{b}, \vec{c}$ are any three mutually perpendicular vectors of equal magnitude a, then $|\vec{a}+\vec{b}+\vec{c}|$ is equal to

A. a

B. $\sqrt{2}a$

C. $\sqrt{3}a$

D. 2a

Answer

We know that,

$\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \overline{|a|}^2 + \overline{|b|}^2 + \overline{|c|}^2 + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a}$ (i)

Since, they are mutually perpendicular vectors

$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ (ii)

And according to question

$|\vec{a}| = |\vec{b}| = |\vec{c}|$

Using (i) and (ii) ,

$\left|\vec{a}+\vec{b}+\vec{c}\right| = \sqrt{\overline{|a|}^2 + \overline{|b|}^2 + \overline{|c|}^2}$

$= \sqrt{3}|\vec{a}|$ Ans.

Smart Approach

In case of such mutually perpendicular vectors, assume vectors to be $\hat{\imath}, \hat{\jmath}, \hat{k}$ and verify your answer from options.

15. Question

Mark the correct alternative in each of the following:

If the vectors $3\hat{i}+\lambda\hat{j}+\hat{k}$ and $2\hat{i}-\hat{j}+8\hat{k}$ are perpendicular, then λ is equal to

A. –14

B. 7

C. 14

D. $\frac{1}{7}$

Answer

We have,

$\vec{a}=3\vec{\imath}+\lambda\vec{\jmath}+\vec{k}$

$\vec{b}=2\vec{\imath}-\vec{\jmath}+8\vec{k}$

Given that $\vec{a}$ and $\vec{b}$ are perpendicular

$\Rightarrow \vec{a}.\vec{b}=0$

$\Rightarrow (3\vec{i}+\lambda\vec{j}+\vec{k}).(2\vec{i}-\vec{j}+8\vec{k})=0$

$\Rightarrow$ 6- λ+8=0

$\therefore \lambda = 14$ Ans.

16. Question

Mark the correct alternative in each of the following:

The projection of the vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$ is

A. 1

B. 0

C. 2

D. –1

Answer

Projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$

Projection of $\hat{i}+\hat{j}+\hat{k}$ on $\hat{j}$ is

$$\frac{(\hat{i}+\hat{j}+\hat{k}).\hat{j}}{|\hat{j}|}$$

$= \frac{1}{1}$

= 1 Ans.

17. Question

Mark the correct alternative in each of the following:

The vectors $2\hat{i}+3\hat{j}-4\hat{k}$ and $a\hat{i}+b\hat{j}+c\hat{k}$ are perpendicular, if

A. a = 2, b = 3, c = –4

B. a = 4, b = 4, c = 5

C. a = 4, b = 4, c = –5

D. a = –4, b = 4, c = –5

Answer

The given two vectors,

$\Rightarrow$ Their dot-product is zero

$\Rightarrow (2\hat{\imath}+3\hat{\jmath}-4\hat{k}).(a\hat{\imath}+b\hat{\jmath}+c\hat{k})=0$

2a+3b-4c=0

From the given options only option B satisfies the above equation

Hence option B is correct answer.

18. Question

Mark the correct alternative in each of the following:

If $|\vec{a}| = |\vec{b}|$, then $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) =$

A. positive

B. negative

C. 0

D. none of these

Answer

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 + (\vec{b}.\vec{a}) - (\vec{a}.\vec{b}) - |\vec{b}|^2$

$= |\vec{a}|^2 - |\vec{b}|^2 \; (|\vec{a}| = |\vec{b}|)$

=0 Ans.

19. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ, then the value of $|\vec{a}-\vec{b}|$ is

A. $2\sin\frac{\theta}{2}$

B. 2 sinθ

C. $2\cos\frac{\theta}{2}$

D. 2 cosθ

Answer

$|\vec{a}-\vec{b}| = \sqrt{|a|^2 + |b|^2 - 2|a||b|cos\theta}$

Given that,

$|\vec{a}| = |\vec{b}| = 1$

$|\vec{a}-\vec{b}| = \sqrt{2-2cos\theta} \left\{(1-cos\theta) = 2sin^2\frac{\theta}{2}\right\}$

$|\vec{a}-\vec{b}| = \sqrt{(2)2sin^2\frac{\theta}{2}}$

$|\vec{a}-\vec{b}| = \left|2sin\frac{\theta}{2}\right|$ Ans.

20. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are unit vectors, then the greatest value of $\sqrt{3}|\vec{a}+\vec{b}| + |\vec{a}-\vec{b}|$ is

A. 2

B. $2\sqrt{2}$

C. 4

D. none of these

Answer

If $\vec{a}\ and\ \vec{b}$ are unit vector then

$|\vec{a}+\vec{b}| = \left|2cos\frac{\theta}{2}\right|$

$|\vec{a}-\vec{b}| = \left|2sin\frac{\theta}{2}\right|$

$\sqrt{3}|\vec{a}\ +\ \vec{b}|\ +\ |\vec{a}-\vec{b}| = 2\sqrt{3}\ cos\frac{\theta}{2}\ +\ 2sin\frac{\theta}{2}$

Maximum value of a sin θ +bcosΘ is $\sqrt{a^2+b^2}$

Maximum value of $2\sqrt{3}cos\frac{\theta}{2}$ + $2sin\frac{\theta}{2}$ is 4 Ans.

21. Question

Mark the correct alternative in each of the following:

If the angle between the vectors $x\hat{i}+3\hat{j}-7\hat{k}$ and $x\hat{i}-x\hat{j}+4\hat{k}$ is acute, then x lies in the interval.

A. (-4, 7)

B. [-4, 7]

C. R – [4, 7]

D. R – (4, 7)

Answer

$cos\Theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

If the angle is acute $cos\Theta < 0$

$\Longrightarrow \vec{a}.\vec{b}$<0

$\Longrightarrow$($x\hat{\imath}$+3$\hat{\jmath}-7\hat{k}$).($x\hat{\imath}-x\hat{\jmath}+4\hat{k}$)<0

$\Longrightarrow$ x^2-3x-28<0

$\Longrightarrow$ (x-7) (x+4)<0

$\Longrightarrow$ x$\in$R-(4,7) Ans.

22. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle θ such that $|\vec{a}+\vec{b}|<1,$ then

A. $\theta<\frac{\pi}{3}$

B. $\theta>\frac{2\pi}{3}$

C. $\frac{\pi}{3}<\theta<\frac{2\pi}{3}$

D. $\frac{2\pi}{3} < \theta < \pi$

Answer

We know that,

If $\vec{a}$ and $\vec{b}$ are two-unit vectors inclined at an angle θ

$|\vec{a} + \vec{b}| = \left|2cos\frac{\theta}{2}\right|$

According to question,

$|\vec{a} + \vec{b}|$<1

$\Rightarrow \left|2cos\frac{\theta}{2}\right| < 1$

$\Rightarrow \frac{-1}{2} < cos\frac{\theta}{2} < \frac{1}{2}$

$\Rightarrow \frac{2\pi}{3} > \frac{\theta}{2} > \frac{\pi}{3}$

$\Rightarrow \frac{2\pi}{3} < \theta < \frac{4\pi}{3}$ Ans.

23. Question

Mark the correct alternative in each of the following:

Let $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $|\vec{a} + \vec{b} + \vec{c}| = 1$ and $\vec{a}$ is perpendicular to $\vec{b}$. If $\vec{c}$ makes angle α and β $\vec{a}$ and $\vec{b}$ respectively, then cos α + cosβ =

A. $-\frac{3}{2}$

B. $\frac{3}{2}$

C. 1

D. –1

Answer

We know that,

$\left|\vec{a} + \vec{b} + \vec{c}\right|^2 = \overrightarrow{|a|}^2 + \overrightarrow{|b|}^2 + \overrightarrow{|c|}^2 + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a}$ (i)

Since,

$\vec{a}$ is perpendicular to $\vec{b}$

$\Rightarrow \vec{a}.\vec{b}$=0

And according to question

$|\vec{a}| = |\vec{b}| = |\vec{c}|$=1

We can rewrite equation (i) as

$\left|\vec{a} + \vec{b} + \vec{c}\right|^2 = \overrightarrow{|a|}^2 + \overrightarrow{|b|}^2 + \overrightarrow{|c|}^2 + 0 + 2cos\,\beta + 2cos\,\alpha$

1=1+1+1+0+2(cos α + cosβ)

$\Longrightarrow$ cos α + cosβ = -1 Ans.

24. Question

Mark the correct alternative in each of the following:

The orthogonal projection of $\vec{a}$ and $\vec{b}$ is

A. $\frac{(\vec{a}.\vec{b})\vec{a}}{|\vec{a}|^2}$

B. $\frac{(\vec{a}.\vec{b})\vec{b}}{|\vec{b}|^2}$

C. $\frac{\vec{a}}{|\vec{a}|^2}$

D. $\frac{\vec{b}}{|\vec{b}|^2}$

Answer

Key Concept/Trick: Magnitude of Projection of any vector $\vec{a}$ on $\vec{b}$

is given by $\vec{a}.\hat{b}$

Now, Since it is the magnitude or length(a$\cos\theta$) we have to give the length a direction in the direction of $\vec{b}$

So, we multiply the projection by unit vector of $\vec{b}$

$\overrightarrow{(a}.\hat{b}).\hat{b}$ which on simplification gives option B Ans.

25. Question

Mark the correct alternative in each of the following:

If θ is an acute angle and the vector $(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ is perpendicular to the vector $\hat{i}-\sqrt{3}\,\hat{j}$, then θ =

A. $\frac{\pi}{6}$

B. $\frac{\pi}{5}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

Answer

Since, the given two vectors are given as perpendicular their dot product must be zero

$\left((\sin\theta)\hat{i}+(\cos\theta)\hat{j}\right)(\hat{\imath}-\sqrt{3}\,\hat{\jmath}) = 0$

$\sin\theta - \sqrt{3}\cos\theta = 0$

$\tan\theta=\sqrt{3}$

Since θ is acute then, $\theta=\frac{\pi}{3}$ Ans

26. Question

Mark the correct alternative in each of the following:

If $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector, if θ =

A. $\frac{\pi}{4}$

B. $\frac{\pi}{3}$

C. $\frac{\pi}{2}$

D. $\frac{2\pi}{3}$

Answer

We know that,

$|\vec{a}+\vec{b}|=\left|2cos\frac{\theta}{2}\right|$

According to Question,

$|\vec{a}+\vec{b}|=1$

$\Rightarrow \left|2cos\frac{\theta}{2}\right|=1$

$\Rightarrow cos\frac{\theta}{2}=\frac{1}{2}$

$\Rightarrow \frac{\theta}{2}=\frac{\pi}{3}$

$\theta=\frac{2\pi}{3}$ Ans.