29. The Plane

Exercise 29.1

1 B. Question

Find the equation of the plane passing through the following points:

(2, 1, 0), (3, – 2, – 2), and (3, 1, 7)

Answer

If the given points are $(x_1,y_1,z_1),(x_2,y_2,z_2),(x_3,y_3,z_3)$ then the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Now substitute the values given

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 3-2 & -2-1 & -2-0 \\ 3-2 & 1-1 & 7-0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf) - b(di-fg) + c(dh-eg)$$

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = (x-2)(-21-0) - (y-1)(7+2) + (z)(0+3)$$

$$(x-2)(-21-0) - (y-1)(7+2) + (z)(0+3) = 0 \ldots.(\text{given})$$

– 21x + 42 – 9y + 9 + 3z = 0

– 21x – 9y + 3z + 51 = 0

– 3(7x + 3y – z – 17 = 0)

7x + 3y – z – 17 = 0

This is the equation of the plane.

1 B. Question

Find the equation of the plane passing through the following points:

(– 5, 0, – 6), (– 3, 10, – 9) and (– 2, 6, – 6)

Answer

If the given points are $(x_1,y_1,z_1),(x_2,y_2,z_2),(x_3,y_3,z_3)$ then the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Now substitute the values given

$$\begin{vmatrix} x+5 & y-0 & z+6 \\ -3+5 & 10-0 & -9+6 \\ -2+5 & 6-0 & -6+6 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+5 & y-0 & z+6 \\ 2 & 10 & -3 \\ 3 & 6 & 0 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf) - b(di-fg) + c(dh-eg)$$

(x + 5)(0 + 18) – y(0 + 9) + (z + 6)(12 – 30) = 0

(x + 5)(18) – y(9) + (z + 6)(– 18) = 0

18x + 90 – 9y – 18z – 108 = 0

Now divide both sides 9 then we get the plane equation as

2x – y – 2z – 2 = 0

1 C. Question

Find the equation of the plane passing through the following points:

(1, 1, 1), (1, – 1, 2) and (– 2, – 2, 2)

Answer

If the given points are $(x_1,y_1,z_1),(x_2,y_2,z_2),(x_3,y_3,z_3)$ then the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Now substitute the values given

$$\begin{vmatrix} x-1 & y-1 & z-1 \\ 1-1 & -1-1 & 2-1 \\ -2-1 & -2-1 & 2-1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+5 & y-0 & z+6 \\ 0 & -2 & 1 \\ -3 & -3 & 1 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf) - b(di-fg) + c(dh-eg)$$

(x – 1)(– 2 + 3) – (y – 1)(0 + 3) + (z – 1)(0 – 6) = 0

(x – 1)1 – (y – 1)3 + (z – 1)(– 6) = 0

x – 3y – 6z + 8 = 0

this is the equation of plane.

1 D. Question

Find the equation of the plane passing through the following points:

(2, 3, 4), (– 3, 5, 1) and (4, – 1, 2)

Answer

If the given points are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ then the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Now substitute the values given

$$\begin{vmatrix} x - 2 & y - 3 & z - 4 \\ -3 - 2 & 5 - 3 & 1 - 4 \\ 4 - 2 & -1 - 3 & 2 - 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} x + 5 & y - 0 & z + 6 \\ -5 & 2 & -3 \\ 2 & -4 & -2 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)$$

(x – 2)(– 4 – 12) – (y – 3)(10 + 6) + (z – 4)(20 – 4) = 0

(x – 2)(– 16) – (y – 3)(16) + (z – 4)(16) = 0

– 16x + 32 – 16y + 48 + 16z – 64 = 0

– 16x – 16y + 16z + 16 = 0

(x + y – z – 1) × – 16 = 0

The equation of plane is $x + y - z - 1 = 0$

1 E. Question

Find the equation of the plane passing through the following points:

(0, – 1, 0), (3, 3, 0) and (1, 1, 1)

Answer

If the given points are $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ then the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Now substitute the values given

$$\begin{vmatrix} x - 0 & y + 1 & z - 0 \\ 3 - 0 & 3 + 1 & 0 - 0 \\ 1 - 0 & 1 + 1 & 1 - 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y + 1 & z \\ 3 & 4 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)$$

(x)(4 – 0) – (y + 1)(3 – 0) + z(6 – 4) = 0

4x – (y + 1)(3) + z(2) = 0

4x – 3y – 3 + 2z = 0

This is the equation of the plane.

2. Question

Show that the four point (0, – 1, – 1), (4, 5, 1), (3, 9, 4) and (– 4, 4, 4) are coplanar and find the equation of the common plane.

Answer

Given that these four points are coplanar so these four points lie on the same plane.

So first let us take three points and find the equation of the plane passing through these four points and then let us substitute the fourth point in it. If it is 0 then the point lies on the plane formed by these three points then they are coplanar.

The equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Now let us take (0, – 1, – 1), (4, 5, 1), (3, 9, 4) and find plane equation.

$$\begin{vmatrix} x - 0 & y + 1 & z + 1 \\ 4 - 0 & 5 + 1 & 1 + 1 \\ 3 - 0 & 9 + 1 & 4 + 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y + 1 & z + 1 & \\ 4 & 6 & 2 & = \\ 3 & 10 & 5 & \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)$$

x(30 – 20) – (y + 1)(20 – 6) + (z + 1)(40 – 18) = 0

10x – (y + 1)(14) + (z + 1)(22) = 0

10x – 14y + 22z + 8 = 0 now divide by 2 on both sides

The equation is 5x – 7y + 11z + 4 = 0

Now let us substitute fourth point (– 4, 4, 4) we get

5(– 4) – 7(4) + 11(4) + 4 = 0

– 20 – 28 + 44 + 4 = 0

– 48 + 48 = 0

0 = 0

L.H.S = R.H.S

So as said above this fourth point satisfies so this point also lies on the same plane.

Hence they are coplanar.

3 A. Question

Show that the following points are coplanar.

(0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0)

Answer

Given that these four points are coplanar so these four points lie on the same plane

So first let us take three points and find the equation of plane passing through these four points and then let us substitute the fourth point in it. If it is 0 then the point lies on the plane formed by these three points then they are coplanar.

the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Now let us take (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and find plane equation.

$$\begin{vmatrix} x - 0 & y + 1 & z + 1 \\ 2 - 0 & 1 + 1 & -1 + 0 \\ 1 - 0 & 1 + 1 & 1 - 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y + 1 & z \\ 2 & 2 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) - b(di - fg) + c(dh - eg)$$

x(2 + 2) – (y + 1)(2 + 1) + z(4 – 2) = 0

4x – 3y – 3 + 2z = 0

4x – 3y + 2z – 3 = 0

Now let us substitute (3, 3, 0) in plane equation

4x – 3y + 2z – 3 = 0

4(3) – 3(3) + 2(0) – 3 = 0

12 – 9 + 0 – 3 = 0

12 – 12 = 0

0 = 0

So this point lies on the plane

Hence they are coplanar.

3 B. Question

Show that the following points are coplanar.

(0, 4, 3), (– 1, – 5, – 3), (– 2, – 2, 1) and (1, 1, – 1)

Answer

Given that these four points are coplanar so these four points lie on the same plane

So first let us take three points and find the equation of plane passing through these four points and then let us substitute the fourth point in it. If it is 0 then the point lies on the plane formed by these three points then they are coplanar.

the equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Now let us take (0, 4, 3), (– 1, – 5, – 3), (– 2, – 2, 1) and find plane equation.

$$\begin{vmatrix} x-0 & y-4 & z-3 \\ -1-0 & -5-4 & -3-3 \\ -2-0 & -2-4 & 1-3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y+1 & z \\ -1 & -9 & -6 \\ -2 & -6 & -2 \end{vmatrix} = 0$$

Now apply the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei-hf) - b(di-fg) + c(dh-eg)$$

x(18 – 36) – (y – 4)(2 – 12) + (z – 3)(6 – 18) = 0

x(– 18) – (y – 4)(– 10) + (z – 3)(– 12) = 0

– 18x + 10y – 40 – 12z + 36 = 0

– 18x + 10y – 12z – 4 = 0

Now let us substitute (1, 1, – 1) in plane equation

– 18x + 10y – 12z – 4 = 0

– 18(1) + 10(1) – 12(– 1) – 4 = 0

– 18 + 10 + 12 – 4 = 0

– 22 + 22 = 0

0 = 0

Lhs = rhs

So this point lies on the plane

Hence they are coplanar.

4. Question

Find the coordinates of the point P where the line through A(3, – 4, – 5) and B(2, – 3, 1) crosses the plane passing through three points L(2, 2, 1), M(3, 0, 1) and N(4, – 1, 0). Also, find the ratio in which P divides the line segment AB.

Answer

We know that the equation passing through two point (a, b, c) and (d, e, f) is given by

$$\frac{x-a}{d-a} = \frac{y-b}{e-b} = \frac{z-c}{f-c}$$

the line through A(3, – 4, – 5) and B(2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{6}$$

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6}$$

Now let us see how a point is going to be on the line

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = k$$

X = – k + 3, y = k – 4, z = 6k – 5

So now let a point P be the point of the intersection of a line and the plane so let the coordinates of P = (– k + 3, k – 4, 6k – 5)

Now let us find the equation of the plane passing through L(2, 2, 1), M(3, 0, 1) and N(4, – 1, 0).

The equation of the plane passing through these three points is given by the following equation.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 3 - 2 & 0 - 2 & 1 - 1 \\ 4 - 2 & -1 - 2 & 0 - 1 \end{vmatrix} = 0$$

(x – 2)2 – (y – 2)(– 1) + (z – 1)(– 3 + 4) = 0

2x – 4 + y – 2 – z + 1 = 0

2x + y – z = 5

Now point P lies on this plane so

2(3 – k) + (k – 4) – (6k – 5) = 5

6 – 2k + k – 4 – 6k + 5 = 5

– 7k = – 2

$$k = \frac{2}{7}$$

So point P is = $\left(\frac{19}{7}, \frac{-26}{7}, \frac{-23}{7}\right)$

Now we have to find the ration in which this P divides AB

Let the ratio b m:1

We know the section formula

That is if a line AB is divided by P in ratio m:1 then

$$P = \left(\frac{mx_2 + x_1}{m + 1}, \frac{my_2 + y_1}{m + 1}, \frac{mz_2 + z_1}{m + 1}\right)$$

$$\left(\frac{19}{7}, \frac{-26}{7}, \frac{-23}{7}\right) = \frac{m2 + 3}{m + 1}, \frac{m(-3) - 4}{m + 1}, \frac{m1 + 5}{m + 1}$$

Solving

$$\frac{19}{7} = \frac{m2 + 3}{m + 1}$$

We get 19m + 19 = 14m + 21

5m = – 2

$$m = \frac{-2}{5}$$

So point P divides line in ration 2:5 externally since ratio is negative.

Exercise 29.2

1. Question

Write the equation of the plane whose intercepts on the coordinate axes are 2, – 3 and 4.

Answer

Given

intercepts on the coordinate axes are 2, – 3 and 4.

The equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here a = 2, b = – 3, c = 4

So now let us substitute in the equation of the plane

$$\frac{x}{2} + \frac{y}{-3} + \frac{z}{4} = 1$$

L.C.M of 2, 3, 4 is 12

$$\frac{6x - 4y + 3z}{12} = 1$$

So the equation is 6x - 4y + 3z = 0

2 A. Question

Reduce the equations of the following planes in the intercept form and find its intercepts on the coordinate axes:

4x + 3y – 6z – 12 = 0

Answer

given equation is 4x + 3y – 6z – 12 = 0

4x + 3y – 6z = 12

Now let us divide both sides by 12

We get,

$$\frac{4x + 3y - 6z}{12} = \frac{12}{12}$$

$$\frac{x}{3} + \frac{y}{4} - \frac{z}{2} = 1$$

$$\frac{x}{3} + \frac{y}{4} + \frac{z}{-2} = 1$$

We know that, the equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

So by comparing a = 3, b = 4, c = – 2

So the intercepts are 3, 4, – 2

2 B. Question

Reduce the equations of the following planes in the intercept form and find its intercepts on the coordinate axes:

2x + 3y – z = 6

Answer

given equation is 2x + 3y – z = 6

Now let us divide both sides by 6

We get,

$$\frac{2x + 3y - z}{6} = \frac{6}{6}$$

$$\frac{x}{3} + \frac{y}{2} - \frac{z}{6} = 1$$

$$\frac{x}{3} + \frac{y}{2} + \frac{z}{-6} = 1$$

We know that, the equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

So by comparing a = 3, b = 2, c = – 6

So the intercepts are 3, 2, – 6.

2 C. Question

Reduce the equations of the following planes in the intercept form and find its intercepts on the coordinate axes:

2x – y + z = 5

Answer

given equation is 2x – y + z = 5

Now let us divide both sides by 5

We get,

$$\frac{2x - y + z}{5} = \frac{5}{5}$$

$$\frac{x}{\frac{5}{2}} + \frac{y}{-5} + \frac{z}{5} = 1$$

We know that, the equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

So by comparing $a = \frac{5}{2}, b = -5, c = 5$

So the intercepts are. $= \frac{5}{2}, -5, 5$

3. Question

Find the equation of a plane which meets the axes in A, B and C, given that the centroid of the triangle ABC is the point (α, β, γ).

Answer

It is given in the question that the plane meets the axes in A, B, C

So now let us assume that A = (a, 0, 0), B = (0, b, 0) and C = (0, 0, c)

Given that the centroid is (α, β, γ) and we know the formula of the centroid of

$$\Delta ABC = \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}$$

So,

$$(\alpha, \beta, \gamma) = \frac{a + 0 + 0}{3}, \frac{0 + b + 0}{3}, \frac{0 + 0 + c}{3}$$

$(\alpha, \beta, \gamma) = \frac{a}{3}, \frac{b}{3}, \frac{c}{3}$

So we have now

$\frac{a}{3} = \alpha, a = 3\alpha$

$\frac{b}{3} = \beta, b = 3\beta$

$\frac{c}{3} = \gamma, c = 3\gamma$

We know that, The equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Now substitute a, b, c

We get $\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1$

$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$

so this is the equation.

4. Question

Find the equation of the plane passing through the point (2, 4, 6) and making equal intercepts of the coordinate axes.

Answer

It is given that untercepts on the axes are equal

We know that, the equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Now we have a = b = c

So now let us substitute a in place of b and c

So we get $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1$

$x + y + z = a$

And it is given that (2, 4, 6) is on the plane so by substituting it we get

2 + 4 + 6 = a

A = 12

So the equation is x + y + z = 12

5. Question

A plane meets the coordinate axes at A, B and C respectively such that the centroid of the triangle ABC is (1, – 2, 3). Find the equation of the plane.

Answer

It is given in the question that the plane meets the axes in A, B, C

And the centroid is (1, – 2, 3)

We know that, the equation of the plane whose intercepts on the coordinate axes a, b, c is given by the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\text{centroid of } \Delta ABC = \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}$$

$$(1, -2, 3) = \frac{a + 0 + 0}{3}, \frac{0 + b + 0}{3}, \frac{0 + 0 + c}{3}$$

$$(1, -2, 3) = \frac{a}{3}, \frac{b}{3}, \frac{c}{3}$$

So we have now

$$\frac{a}{3} = 1, a = 3$$

$$\frac{b}{3} = -2, b = -6$$

$$\frac{c}{3} = 3, c = 9$$

Now substitute a, b, c

We get

$$\frac{x}{3} + \frac{y}{-6} + \frac{z}{9} = 1$$

$$\frac{6x - 3y + 2z}{18} = 1$$

$$6x - 3y + 2z = 18$$

Hence this is the equation.

Exercise 29.3

1. Question

Find the vector equation of a plane passing through a point having position vector $2\hat{i} - \hat{j} + \hat{k}$ and perpendicular to the vector $4\hat{i} + 2\hat{j} - 3\hat{k}$.

Answer

Given: Position vector of the point - $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

Point is perpendicular to the vector $\vec{n} = 4\hat{i} + 2\hat{j} - 3\hat{k}$

To find: the vector equation of a plane passing through a point

We know that, vector equation of a plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is given by

$$(\vec{r} - \vec{a}).\vec{n} = 0 \ldots\ldots (i)$$

Substituting the values from given criteria, we get

$$\left(\vec{r} - (2\hat{i} - \hat{j} + \hat{k})\right).(4\hat{i} + 2\hat{j} - 3\hat{k}) = 0$$

$$\Rightarrow \vec{r}.(4\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}).(4\hat{i} + 2\hat{j} - 3\hat{k}) = 0$$

$\Rightarrow \vec{r}.(4\hat{i} + 2\hat{j} - 3\hat{k}) - [(2)(4) + (-1)(2) + (1)(-3)] = 0$ (by multiplying the two vectors using the formula

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.\left(4\hat{i}+2\hat{j}-3\hat{k}\right)-[8-2-3]=0$

$\Rightarrow \vec{r}.\left(4\hat{i}+2\hat{j}-3\hat{k}\right)=3$ is the vector equation of a plane passing through the given point.

2. Question

Find the Cartesian form of the equation of a plane whose vector equation is

i. $\vec{r}\cdot\left(12\hat{i}-3\hat{j}+4\hat{k}\right)+5=0$

ii. $\vec{r}\cdot\left(-\hat{i}+\hat{j}+2\hat{k}\right)=9$

Answer

i. $\vec{r}.\left(12\hat{i}-3\hat{j}+4\hat{k}\right)+5=0$

Given the vector equation of a plane,

$\vec{r}.\left(12\hat{i}-3\hat{j}+4\hat{k}\right)+5=0$

Let $\vec{r}=\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

Then, the given vector equation becomes,

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(12\hat{i}-3\hat{j}+4\hat{k}\right)+5=0$

Now multiplying the two vectors using the formula $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(12)+(y)(-3)+(z)(4)+5=0$

$\Rightarrow 12x-3y+4z+5=0$

This is the Cartesian form of equation of a plane whose vector equation is $\vec{r}.\left(12\hat{i}-3\hat{j}+4\hat{k}\right)+5=0$

ii. $\vec{r}.\left(-\hat{i}+\hat{j}+2\hat{k}\right)=9$

Given the vector equation of a plane,

$\vec{r}.\left(-\hat{i}+\hat{j}+2\hat{k}\right)=9$

Let $\vec{r}=\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$

Then, the given vector equation becomes,

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(-\hat{i}+\hat{j}+2\hat{k}\right)=9$

Now multiplying the two vectors using the formula $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(-1)+(y)(1)+(z)(2)=9$

$\Rightarrow -x+y+2z=9$

This is the Cartesian form of equation of a plane whose vector equation is $\vec{r}.\left(-\hat{i}+\hat{j}+2\hat{k}\right)=9$

3. Question

Find the vector equation of the coordinates planes.

Answer

Here we need to find the vector equation of the xy-plane, xz-plane and yz-plane.

For xy-plane

We know the xy-plane passes through the point i.e., origin and is perpendicular to the z-axis, so

Let $\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ and $\vec{n} = \hat{k}$ ....(i)

We know that, vector equation of a plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from equation (i), we get

$\left(\vec{r} - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)\right).\hat{k} = 0$

$\Rightarrow \vec{r}.\left(\hat{k}\right) - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right).\left(\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\hat{k} = 0 \ldots (ii)$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

For xz-plane

We know the xz-plane passes through the point i.e., origin and is perpendicular to the y-axis, so

Let $\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ and $\vec{n} = \hat{j}$ ....(iii)

We know that, vector equation of a plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from equation (iii), we get

$\left(\vec{r} - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)\right).\hat{j} = 0$

$\Rightarrow \vec{r}.(\hat{j}) - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right).(\hat{j}) = 0$

$\Rightarrow \vec{r}.\hat{j} = 0 \ldots (iv)$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

For yz-plane

We know the yz-plane passes through the point i.e., origin and is perpendicular to the x-axis, so

Let $\vec{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$ and $\vec{n} = \hat{i}$ ....(v)

We know that, vector equation of a plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from equation (v), we get

$\left(\vec{r} - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)\right).\hat{i} = 0$

$\Rightarrow \vec{r}.(\hat{i}) - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right).(\hat{i}) = 0$

$\Rightarrow \vec{r}.\hat{i} = 0 \ldots (vi)$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

Hence from equation (ii), (iv) and (vi), we get

The equation of xy, yz and yz plane as

$\vec{r}.\hat{k} = 0, \vec{r}.\hat{j} = 0, \vec{r}.\hat{i} = 0$

4. Question

Find the vector equation of each one of the following planes:

i. 2x – y + 2z = 8

ii. x + y – z = 5

iii. x + y = 3

Answer

i. The given equation of the plane is 2x – y + 2z = 8

This is in Cartesian form, to convert this to vector form, this can be done as shown below:

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} - \hat{j} + 2\hat{k}) = 8$

We know $x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$

Hence the above equation becomes,

$\vec{r}.(2\hat{i} - \hat{j} + 2\hat{k}) = 8$

The vector equation of the plane whose Cartesian form 2x – y + 2z = 8 is given is $\vec{r}.(2\hat{i} - \hat{j} + 2\hat{k}) = 8$

ii. x + y – z = 5

This is in Cartesian form, to convert this to vector form, this can be done as shown below:

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j} - \hat{k}) = 5$

We know $x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$

Hence the above equation becomes,

$\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 5$

The vector equation of the plane whose Cartesian form x + y – z = 5 is given is $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 5$

iii. x + y = 3

This is in Cartesian form, to convert this to vector form, this can be done as shown below:

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + \hat{j}) = 3$

We know $x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$

Hence the above equation becomes,

$\vec{r}.(\hat{i} + \hat{j}) = 3$

The vector equation of the plane whose Cartesian form x + y =3 is given is $\vec{r}.(\hat{i} + \hat{j}) = 3$

5. Question

Find the vector and Cartesian equation of a plane passing through the point (1, -1, 1) and normal to the line joining the point (1, 2, 5) and (-1, 3, 1).

Answer

The plane is passing through the point (1, -1, 1). Let the position vector of this point be

$\vec{a} = \hat{i} - \hat{j} + \hat{k} \ldots \ldots (i)$

And it is also given the plane is normal to the line joining the points A(1, 2, 5) and B(-1, 3, 1).

Then $\vec{n} = \overrightarrow{AB}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{B}$ - position vector of $\vec{A}$

$\Rightarrow \vec{n} = \left(-\hat{i} + 3\hat{j} + \hat{k}\right) - \left(\hat{i} + 2\hat{j} + 5\hat{k}\right)$

$\Rightarrow \vec{n} = -2\hat{i} + \hat{j} - 4\hat{k} \ldots\ldots (ii)$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - \left(\hat{i} - \hat{j} + \hat{k}\right)\right).\left(-2\hat{i} + \hat{j} - 4\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(-2\hat{i} + \hat{j} - 4\hat{k}\right) - \left(\hat{i} - \hat{j} + \hat{k}\right).\left(-2\hat{i} + \hat{j} - 4\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(-2\hat{i} + \hat{j} - 4\hat{k}\right) - [(1)(-2) + (-1)(1) + (1)(-4)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.\left(-2\hat{i} + \hat{j} - 4\hat{k}\right) - [-2 - 1 - 4] = 0$

$\Rightarrow \vec{r}.\left(-2\hat{i} + \hat{j} - 4\hat{k}\right) + 7 = 0$

Multiplying by (-1) on both sides we get,

$\underline{\Rightarrow \vec{\mathbf{r}}.\left(\mathbf{2\hat{i} - \hat{j} + 4\hat{k}}\right) = \mathbf{7}}$ is the vector equation of a plane passing through the point (1, -1, 1) and normal to the line joining the point (1, 2, 5) and (-1, 3, 1).

Let $\vec{r} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

Then, the above vector equation of the plane becomes,

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} - \hat{j} + 4\hat{k}\right) = 7$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(2) + (y)(-1) + (z)(4) = 7$

$\underline{\Rightarrow \mathbf{2x - y + 4z = 7}}$

This is the Cartesian form of equation of a plane passing through the point (1, -1, 1) and normal to the line joining the point (1, 2, 5) and (-1, 3, 1).

6. Question

If $\vec{n}$ is a vector of magnitude √3 is equally inclined with an acute with the coordinate axes. Find the vector and Cartesian forms of the equation of a plane which passes through (2, 1, -1) and is normal to $\vec{n}$.

Answer

Given: $\vec{n} = \sqrt{3}$ and $\vec{n}$ is equally inclined with an acute with the coordinate axes

To find: the vector and Cartesian forms of the equation of a plane which passes through (2, 1, -1) and is normal to $\vec{n}$

Let $\vec{n}$ has direction cosines as l, m and n and it makes an angle of α, β and γ with the coordinate axes. So as per the given condition

α=β=γ

⇒ cos α =cos β =cos γ

⇒ l=m=n=p (let assume)

We know that,

$l^2+m^2+n^2=1$

$\Rightarrow p^2+p^2+p^2=1$

$\Rightarrow 3p^2=1$

$\Rightarrow p = \pm\frac{1}{\sqrt{3}}$

So, $\Rightarrow l = m = n = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma = \left(\pm\frac{1}{\sqrt{3}}\right)$

For the negative value of cos the angles are obtuse so that we will neglect it

So we have

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma = \left(\frac{1}{\sqrt{3}}\right)$

Hence $l = m = n = \frac{1}{\sqrt{3}}$

So the vector equation of the normal becomes,

$\vec{n} = |\vec{n}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right)$

$\Rightarrow \vec{n} = \sqrt{3}\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$

$\Rightarrow \vec{n} = \hat{i} + \hat{j} + \hat{k} \ldots \ldots (i)$

The plane is passing through the point (2, 1, -1). Let the position vector of this point be

$\vec{a} = 2\hat{i} + \hat{j} - \hat{k} \ldots \ldots (ii)$

We know that vector equation of a plane passing through point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - \left(2\hat{i} + \hat{j} - \hat{k}\right)\right).\left(\hat{i} + \hat{j} + \hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) - \left(2\hat{i} + \hat{j} - \hat{k}\right).\left(\hat{i} + \hat{j} + \hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) - [(2)(1) + (1)(1) + (-1)(1)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) - [2 + 1 - 1] = 0$

$\Rightarrow \vec{\mathbf{r}}.\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right) = \mathbf{2}$ is the vector and Cartesian forms of the equation of a plane which passes through (2, 1, -1) and is normal to $\vec{n}$.

Let $\vec{r} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

Then, the above vector equation of the plane becomes,

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$

Now multiplying the two vectors using the formula $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$, we get

$\Rightarrow (x)(1) + (y)(1) + (z)(1) = 2$

$\underline{\Rightarrow \mathbf{x + y + z = 2}}$

This is the Cartesian form of the equation of a plane which passes through (2, 1, -1) and is normal to $\vec{n}$.

7. Question

The coordinates of the foot of the perpendicular drawn from the origin to a plane are (12, -4, 3). Find the equation of the plane.

Answer

Given: the coordinates of the foot of the perpendicular drawn from the origin to a plane are (12, -4, 3)

To find: the equation of the plane

As it is given that the foot of the perpendicular drawn from origin O to the plane is P(12, -4, 3)

This means that the required plane is passing through P(12, -4, 3) and is perpendicular to OP. Let the position vector of this point P be

$\vec{a} = 12\hat{i} - 4\hat{j} + 3\hat{k} \ldots \ldots (i)$

And it is also given the plane is normal to the line joining the points O(0,0,0) and P(12, -4, 3).

Then $\vec{n} = \overrightarrow{OP}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{P}$ - position vector of $\vec{O}$

$\Rightarrow \vec{n} = (12\hat{i} - 4\hat{j} + 3\hat{k}) - (0\hat{i} + 0\hat{j} + 0\hat{k})$

$\Rightarrow \vec{n} = 12\hat{i} - 4\hat{j} + 3\hat{k} \ldots \ldots (ii)$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (12\hat{i} - 4\hat{j} + 3\hat{k})\right).(12\hat{i} - 4\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \vec{r}.(12\hat{i} - 4\hat{j} + 3\hat{k}) - (12\hat{i} - 4\hat{j} + 3\hat{k}).(12\hat{i} - 4\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \vec{r}.(12\hat{i} - 4\hat{j} + 3\hat{k}) - [(12)(12) + (-4)(-4) + (3)(3)] = 0$ (by multiplying the two vectors using the formula $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$)

$\Rightarrow \vec{r}.(12\hat{i} - 4\hat{j} + 3\hat{k}) - [144 + 16 + 9] = 0$

$\Rightarrow \vec{r}.(12\hat{i} - 4\hat{j} + 3\hat{k}) - 169 = 0$

$\underline{\Rightarrow \vec{\mathbf{r}}.(\mathbf{12\hat{i} - 4\hat{j} + 3\hat{k}}) = \mathbf{169}}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(12\hat{i} - 4\hat{j} + 3\hat{k}) = 169$

Now multiplying the two vectors using the formula $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$, we get

$\Rightarrow (x)(12) + (y)(-4) + (z)(3) = 169$

$\Rightarrow \mathbf{12x - 4y + 3z = 169}$

This is the Cartesian form of the equation of the required plane.

8. Question

Find the equation of the plane passing through the point (2, 3, 1) given that the direction ratios of normal to the plane are proportional to 5, 3, 2.

Answer

Given: The plane is passing through P(2, 3, 1) and perpendicular to the line having 5, 3, 2 as the direction ratios.

To find: the equation of the plane

Let the position vector of this point P be

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k} \ldots \ldots (i)$

And it is also given the plane is normal having 5, 3, 2 as the direction ratios.

Then

$\Rightarrow \vec{n} = 5\hat{i} + 3\hat{j} + 2\hat{k} \ldots \ldots (ii)$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (2\hat{i} + 3\hat{j} + \hat{k})\right).(5\hat{i} + 3\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \vec{r}.(5\hat{i} + 3\hat{j} + 2\hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k}).(5\hat{i} + 3\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \vec{r}.(5\hat{i} + 3\hat{j} + 2\hat{k}) - [(2)(5) + (3)(3) + (1)(2)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.(5\hat{i} + 3\hat{j} + 2\hat{k}) - [10 + 9 + 2] = 0$

$\Rightarrow \vec{r}.(5\hat{i} + 3\hat{j} + 2\hat{k}) - 21 = 0$

$\Rightarrow \vec{\mathbf{r}}.(\mathbf{5\hat{i} + 3\hat{j} + 2\hat{k}}) = \mathbf{21}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(5\hat{i} + 3\hat{j} + 2\hat{k}) = 21$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(5) + (y)(3) + (z)(2) = 21$

$\Rightarrow \mathbf{5x + 3y + 2z = 21}$

This is the Cartesian form of the equation of the required plane.

9. Question

If the axes are rectangular and P is the point (2, 3, -1), find the equation of the plane through P at right angles to OP.

Answer

Given: P is the point (2, 3, -1) and the required plane is passing through P at right angles to OP.

To find: the equation of the plane.

As per the given criteria, it means that the plane is passing through P and OP is the vector normal to the plane

Let the position vector of this point P be

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k} \ldots \ldots (i)$

And it is also given the plane is normal to the line joining the points O(0,0,0) and P(2, 3, -1).

Then $\vec{n} = \overrightarrow{OP}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{P}$ - position vector of $\vec{O}$

$\Rightarrow \vec{n} = (2\hat{i} + 3\hat{j} - \hat{k}) - (0\hat{i} + 0\hat{j} + 0\hat{k})$

$\Rightarrow \vec{n} = 2\hat{i} + 3\hat{j} - \hat{k} \ldots \ldots (ii)$

We know that vector equation of a plane passing through point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (2\hat{i} + 3\hat{j} - \hat{k})\right).(2\hat{i} + 3\hat{j} - \hat{k}) = 0$

$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) - (2\hat{i} + 3\hat{j} - \hat{k}).(2\hat{i} + 3\hat{j} - \hat{k}) = 0$

$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) - [(2)(2) + (3)(3) + (-1)(-1)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) - [4 + 9 + 1] = 0$

$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) - 14 = 0$

$\underline{\Rightarrow \vec{\mathbf{r}}.(\mathbf{2\hat{i} + 3\hat{j} - \hat{k}}) = \mathbf{14}}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 3\hat{j} - \hat{k}) = 14$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(2) + (y)(3) + (z)(-1) = 14$

$\underline{\Rightarrow \mathbf{2x + 3y - z = 14}}$

This is the Cartesian form of equation of the required plane.

10. Question

Find the intercepts made on the coordinate axes by the plane 2x + y – 2z = 3 and also find the direction cosines of the normal to the plane.

Answer

The given equation of the plane is 2x + y – 2z = 3

Dividing by 3 on both the sides, we get

$$\frac{2x}{3}+\frac{y}{3}-\frac{2z}{3}=\frac{3}{3}$$

$$\Rightarrow \frac{x}{\frac{3}{2}}+\frac{y}{3}-\frac{z}{\frac{3}{2}}=1 \dots \dots (i)$$

We know that, if a, b, c are the intercepts by the plane on the coordinate axes, new equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \dots \dots \dots (ii)$$

Comparing the equation (i) and (ii), we get

$$a=\frac{3}{2}, b=3, c=-\frac{3}{2}$$

Again the given equation of the plane is

2x+y-2z=3

Writing this in the vector form, we get

$$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})(2\hat{i}+\hat{j}-2\hat{k})=3$$

$$\Rightarrow \vec{r}.(2\hat{i}+\hat{j}-2\hat{k})=3$$

So vector normal to the plane is given by

$$\vec{n}=2\hat{i}+\hat{j}-2\hat{k}$$

$$\Rightarrow |\vec{n}|=\sqrt{(2)^2+(1)^2+(-2)^2}$$

$$\Rightarrow |\vec{n}|=\sqrt{4+1+4}$$

$$\Rightarrow |\vec{n}|=3$$

Direction vector of $\vec{n}=2,1,-2$

Direction vector of $\vec{n}=\frac{2}{|\vec{n}|},\frac{1}{|\vec{n}|},\frac{-2}{|\vec{n}|}\Rightarrow \vec{n}=\frac{2}{3},\frac{1}{3},\frac{-2}{3}$

So,

Intercepts by the plane on the coordinate axes are $=\frac{3}{2},3,-\frac{3}{2}$

Direction cosines of normal to the plane are $=\frac{2}{3},\frac{1}{3},-\frac{2}{3}$

11. Question

A plane passes through the point (1, -2, 5) and is perpendicular to the line joining the origin to the point $3\hat{i}+\hat{j}-\hat{k}$. Find the vector and Cartesian forms of the equation of the plane.

Answer

As per the given criteria the required plane is passing through Q (1, -2, 5) and is perpendicular to OP, where point O is the origin and position vector of point P is $3\hat{i}+\hat{j}-\hat{k}$. Let the position vector of this point Q be

$$\vec{a}=\hat{i}-2\hat{j}+5\hat{k} \dots \dots (i)$$

And it is also given the plane is normal to the line joining the points O(0,0,0) and position vector of point P is $3\hat{i}+\hat{j}-\hat{k}$

Then $\vec{n} = \overrightarrow{OP}$

$\Rightarrow \vec{n}$ = Position vector of $\vec{P}$ - position vector of $\vec{O}$

$\Rightarrow \vec{n} = \left(3\hat{i} + \hat{j} - \hat{k}\right) - \left(0\hat{i} + 0\hat{j} + 0\hat{k}\right)$

$\Rightarrow \vec{n} = 3\hat{i} + \hat{j} - \hat{k} \ldots\ldots (ii)$

We know that vector equation of a plane passing through point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - \left(\hat{i} - 2\hat{j} + 5\hat{k}\right)\right).\left(3\hat{i} + \hat{j} - \hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(3\hat{i} + \hat{j} - \hat{k}\right) - \left(\hat{i} - 2\hat{j} + 5\hat{k}\right).\left(3\hat{i} + \hat{j} - \hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(3\hat{i} + \hat{j} - \hat{k}\right) - [(1)(3) + (-2)(1) + (5)(-1)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.\left(3\hat{i} + \hat{j} - \hat{k}\right) - [3 - 2 - 5] = 0$

$\Rightarrow \vec{r}.\left(3\hat{i} + \hat{j} - \hat{k}\right) + 4 = 0$

$\underline{\boldsymbol{\Rightarrow \vec{r}.\left(3\hat{i} + \hat{j} - \hat{k}\right) + 4 = 0}}$ is the vector equation of a required plane.

Let $\vec{r} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

Then, the above vector equation of the plane becomes,

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(3\hat{i} + \hat{j} - \hat{k}\right) - 4 = 0$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(3) + (y)(1) + (z)(-1) - 4 = 0$

$\underline{\boldsymbol{\Rightarrow 3x + y - z - 4 = 0}}$

This is the Cartesian form of equation of the required plane.

12. Question

Find the equation of the plane that bisects the line segment joining points (1, 2, 3) and (3, 4, 5) and is at right angle to it.

Answer

The given plane bisects the line segment joining points A(1, 2, 3) and B(3, 4, 5) and is at right angle to it.

This means the plane passes through the midpoint of the line AB

Therefore,

$$\vec{a} = \frac{\text{position vector ofA} + \text{Position vector of B}}{2}$$

$$\Rightarrow \vec{a} = \frac{\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \left(3\hat{i} + 4\hat{j} + 5\hat{k}\right)}{2}$$

$$\Rightarrow \vec{a} = \frac{4\hat{i} + 6\hat{j} + 8\hat{k}}{2}$$

$\Rightarrow \vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k} \ldots\ldots (i)$

And it is also given the plane is normal to the line joining the points A(1, 2, 3) and B(3, 4, 5)

Then $\vec{n} = \overrightarrow{AB}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{B}$ - position vector of $\vec{A}$

$\Rightarrow \vec{n} = (3\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \vec{n} = 2\hat{i} + 2\hat{j} + 2\hat{k} \ldots\ldots (ii)$

We know that vector equation of a plane passing through point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (2\hat{i} + 3\hat{j} + 4\hat{k})\right).(2\hat{i} + 2\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \vec{r}.(2\hat{i} + 2\hat{j} + 2\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}).(2\hat{i} + 2\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \vec{r}.(2\hat{i} + 2\hat{j} + 2\hat{k}) - [(2)(2) + (3)(2) + (4)(2)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.(2\hat{i} + 2\hat{j} + 2\hat{k}) - [4 + 6 + 8] = 0$

$\Rightarrow \vec{r}.(2\hat{i} + 2\hat{j} + 2\hat{k}) - 18 = 0$

$\underline{\Rightarrow \vec{\mathbf{r}}.(\mathbf{2\hat{i}} + \mathbf{2\hat{j}} + \mathbf{2\hat{k}}) = \mathbf{18}}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} + 2\hat{j} + 2\hat{k}) = 18$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(2) + (y)(2) + (z)(2) = 18$

$\underline{\Rightarrow \mathbf{2x + 2y + 2z = 18}}$

This is the Cartesian form of equation of the required plane.

13. Question

Show that the normal to the following pairs of planes are perpendicular to each other:

i. x – y + z – 2 =0 and 3x + 2y – z +4=0

ii. $\vec{r} \cdot \left(2\hat{i} - \hat{j} + 3\hat{k}\right) = 5$ and $\vec{r}.\left(2\hat{i} - 2\hat{j} - 2\hat{k}\right) = 5$

Answer

i. The vector equation of the plane x-y+z-2=0 can be written as

$(x\hat{i} + y\hat{j} + z\hat{k})(\hat{i} - \hat{j} + \hat{k}) = 2$

$\Rightarrow \vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 2$

The normal to this plane is $\overrightarrow{n_1} = \hat{i} - \hat{j} + \hat{k} \ldots \ldots . . (i)$

The vector equation of the plane 3x + 2y – z +4=0 can be written as

$$(x\hat{i} + y\hat{j} + z\hat{k})(3\hat{i} + 2\hat{j} - \hat{k}) = -4$$

$$\Rightarrow \vec{r}.(3\hat{i} + 2\hat{j} - \hat{k}) = -4$$

The normal to this plane is $\overrightarrow{n_2} = 3\hat{i} + 2\hat{j} - \hat{k} \ldots \ldots . . (ii)$

Now

$$\overrightarrow{n_1}.\overrightarrow{n_2} = (\hat{i} - \hat{j} + \hat{k})(3\hat{i} + 2\hat{j} - \hat{k})$$

$$\Rightarrow \overrightarrow{n_1}.\overrightarrow{n_2} = (1)(3) + (-1)(2) + (1)(-1) = 0$$

Hence $\overrightarrow{n_1}$ is perpendicular to $\overrightarrow{n_2}$

Therefore, the normal to the given pairs of planes are perpendicular to each other.

ii. The equation of the first plane is

$$\vec{r}.(2\hat{i} - \hat{j} + 3\hat{k}) = 5$$

The normal to this plane is $\overrightarrow{n_1} = 2\hat{i} - \hat{j} + 3\hat{k} \ldots \ldots . . (i)$

The equation of the first plane is

$$\vec{r}.(2\hat{i} - 2\hat{j} - 2\hat{k}) = 5$$

The normal to this plane is $\overrightarrow{n_2} = 2\hat{i} - 2\hat{j} - 2\hat{k} \ldots \ldots . . (ii)$

Now

$$\overrightarrow{n_1}.\overrightarrow{n_2} = (2\hat{i} - \hat{j} + 3\hat{k})(2\hat{i} - 2\hat{j} - 2\hat{k})$$

$$\Rightarrow \overrightarrow{n_1}.\overrightarrow{n_2} = (2)(2) + (-1)(-2) + (3)(-2) = 0$$

Hence $\overrightarrow{n_1}$ is perpendicular to $\overrightarrow{n_2}$

Therefore, the normal to the given pairs of planes are perpendicular to each other.

14. Question

Show that the normal vector to the plane 2x + 2y + 2z = 3 is equally inclined with the coordinate axes.

Answer

The vector equation of the plane 2x + 2y + 2z = 3 can be written as

$$(x\hat{i} + y\hat{j} + z\hat{k})(2\hat{i} + 2\hat{j} + 2\hat{k}) = 3$$

$$\Rightarrow \vec{r}.(2\hat{i} + 2\hat{j} + 2\hat{k}) = 3$$

The normal to this plane is $\vec{n} = 2\hat{i} + 2\hat{j} + 2\hat{k} \ldots \ldots . . (i)$

Direction ratio of $\vec{n} = 2, 2, 2$

Direction cosine of $\vec{n} = \frac{2}{|\vec{n}|}, \frac{2}{|\vec{n}|}, \frac{2}{|\vec{n}|}$

$$|\vec{n}| = \sqrt{(2)^2 + (2)^2 + (2)^2}$$

$$\Rightarrow |\vec{n}| = \sqrt{4 + 4 + 4} = 2\sqrt{3}$$

Direction cosine of $|\vec{n}| = \frac{2}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

So, $l = \frac{1}{\sqrt{3}}, m = \frac{1}{\sqrt{3}}, n = \frac{1}{\sqrt{3}}$

Let α, β, γ be the angle that normal $\vec{n}$ makes with the coordinate axes respectively

$$l = \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \ldots\ldots (ii)$$

Similarly,

$$\Rightarrow \beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \ldots\ldots (iii)$$

$$\Rightarrow \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \ldots\ldots (iv)$$

Hence $\alpha = \beta = \gamma$

So the normal vector to the plane 2x + 2y + 2z = 3 is equally inclined with the coordinate axes.

15. Question

Find a vector of magnitude 26 units normal to the plane 12x – 3y + 4z = 1.

Answer

The vector equation of the plane 12x – 3y + 4z = 1 can be written as

$$\left(x\hat{i} + y\hat{j} + z\hat{k}\right)\left(12\hat{i} - 3\hat{j} + 4\hat{k}\right) = 1$$

$$\Rightarrow \vec{r}.\left(12\hat{i} - 3\hat{j} + 4\hat{k}\right) = 1$$

The normal to this plane is $\vec{n} = 12\hat{i} - 3\hat{j} + 4\hat{k} \ldots\ldots\ldots (i)$

Its magnitude is

$$|\vec{n}| = \sqrt{(12)^2 + (-3)^2 + (4)^2}$$

$$\Rightarrow |\vec{n}| = \sqrt{144 + 9 + 16}$$

$$\Rightarrow |\vec{n}| = \sqrt{169} = 13$$

The unit vector becomes $\hat{n} = \frac{12\hat{i} - 3\hat{j} + 4\hat{k}}{13} = \frac{12}{13}\hat{i} - \frac{3}{13}\hat{j} + \frac{4}{13}\hat{k}$

Now a vector normal to the plane with the magnitude 26 will be

$$= 26\hat{n}$$

$$\Rightarrow = 26\left(\frac{12}{13}\hat{i} - \frac{3}{13}\hat{j} + \frac{4}{13}\hat{k}\right)$$

$$\Rightarrow = 24\hat{i} - 6\hat{j} + 8\hat{k}$$

Therefore, a vector of magnitude 26 units normal to the plane 12x – 3y + 4z = 1 is $24\hat{i} - 6\hat{j} + 8\hat{k}$

16. Question

If the line drawn from (4, -1, 2) meets a plane at right angles at the point (-10,5,4), find the equation of the plane.

Answer

It means the plane passes through the point B (-10, 5, 4). Therefore the position vector of this point is,

$\Rightarrow \vec{a} = -10\hat{i} + 5\hat{j} + 4\hat{k} \dots\dots (i)$

And also given the line segment joining points A(4, -1, 2) and B (-10, 5, 4) and is at right angle to it.

Then $\vec{n} = \overrightarrow{AB}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{B}$ - position vector of $\vec{A}$

$\Rightarrow \vec{n} = (-10\hat{i} + 5\hat{j} + 4\hat{k}) - (4\hat{i} - \hat{j} + 2\hat{k})$

$\Rightarrow \vec{n} = -14\hat{i} + 6\hat{j} + 2\hat{k} \dots\dots (ii)$

We know that vector equation of a plane passing through point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (-10\hat{i} + 5\hat{j} + 4\hat{k})\right).(-14\hat{i} + 6\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \vec{r}.(-14\hat{i} + 6\hat{j} + 2\hat{k}) - (-10\hat{i} + 5\hat{j} + 4\hat{k}).(-14\hat{i} + 6\hat{j} + 2\hat{k}) = 0$

$\Rightarrow \vec{r}.(-14\hat{i} + 6\hat{j} + 2\hat{k}) - [(-10)(-14) + (5)(6) + (4)(2)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.(-14\hat{i} + 6\hat{j} + 2\hat{k}) - [140 + 30 + 8] = 0$

$\Rightarrow \vec{r}.(-14\hat{i} + 6\hat{j} + 2\hat{k}) - 178 = 0$

$\boldsymbol{\Rightarrow \vec{r}.(-14\hat{i} + 6\hat{j} + 2\hat{k}) = 178}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(-14\hat{i} + 6\hat{j} + 2\hat{k}) = 178$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(-14) + (y)(6) + (z)(2) = 178$

$\boldsymbol{\Rightarrow -14x + 6y + 2z = 178}$

This is the Cartesian form of the equation of the required plane.

17. Question

Find the equation of the plane which bisects the line segment joining the points (-1, 2, 3) and (3, -5, 6) at right angles.

Answer

The given plane bisects the line segment joining points A(-1, 2, 3) and B(3, -5, 6) and is at a right angle to it.

This means the plane passes through the midpoint of the line AB

Therefore,

$\vec{a} = \frac{\text{position vector of} A + \text{Position vector of B}}{2}$

$\Rightarrow \vec{a} = \frac{(-\hat{i} + 2\hat{j} + 3\hat{k}) + (3\hat{i} - 5\hat{j} + 6\hat{k})}{2}$

$\Rightarrow \vec{a} = \frac{2\hat{i} - 3\hat{j} + 9\hat{k}}{2}$

$\Rightarrow \vec{a} = \hat{i} - \frac{3}{2}\hat{j} + \frac{9}{2}\hat{k} \ldots\ldots (i)$

And it is also given the plane is normal to the line joining the points A(-1, 2, 3) and B(3, -5, 6)

Then $\vec{n} = \overrightarrow{AB}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{B}$ - position vector of $\vec{A}$

$\Rightarrow \vec{n} = (3\hat{i} - 5\hat{j} + 6\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \vec{n} = 4\hat{i} - 7\hat{j} + 3\hat{k} \ldots\ldots (ii)$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - \left(\hat{i} - \frac{3}{2}\hat{j} + \frac{9}{2}\hat{k}\right)\right).(4\hat{i} - 7\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \vec{r}.(4\hat{i} - 7\hat{j} + 3\hat{k}) - \left(\hat{i} - \frac{3}{2}\hat{j} + \frac{9}{2}\hat{k}\right).(4\hat{i} - 7\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \vec{r}.(4\hat{i} - 7\hat{j} + 3\hat{k}) - \left[(1)(4) + \left(-\frac{3}{2}\right)(-7) + \left(\frac{9}{2}\right)(3)\right] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.(4\hat{i} - 7\hat{j} + 3\hat{k}) - \left[4 + \frac{21}{2} + \frac{27}{2}\right] = 0$

$\Rightarrow \vec{r}.(4\hat{i} - 7\hat{j} + 3\hat{k}) - \left[\frac{8 + 21 + 27}{2}\right] = 0$

$\Rightarrow \vec{r}.(4\hat{i} - 7\hat{j} + 3\hat{k}) - 28 = 0$

$\underline{\boldsymbol{\Rightarrow \vec{r}.(4\hat{i} - 7\hat{j} + 3\hat{k}) = 28}}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(4\hat{i} - 7\hat{j} + 3\hat{k}) = 28$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(4) + (y)(-7) + (z)(3) = 28$

$\underline{\boldsymbol{\Rightarrow 4x - 7y + 3z = 28}}$

This is the Cartesian form of equation of the required plane.

18. Question

Find the vector and Cartesian equation of the plane which passes through the point (5, 2, -4) and perpendicular to the line with direction ratios 2, 3, -1.

Answer

Given: The plane is passing through P(5, 2, -4) and perpendicular to the line having 2, 3, -1 as the direction ratios.

To find: the equation of the plane

Let the position vector of this point P be

$\vec{a} = 5\hat{i} + 2\hat{j} - 4\hat{k} \ldots \ldots (i)$

And it is also given the plane is normal having 2, 3, -1 as the direction ratios.

Then

$\Rightarrow \vec{n} = 2\hat{i} + 3\hat{j} - \hat{k} \ldots \ldots (ii)$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - \left(5\hat{i} + 2\hat{j} - 4\hat{k}\right)\right).\left(2\hat{i} + 3\hat{j} - \hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(2\hat{i} + 3\hat{j} - \hat{k}\right) - \left(5\hat{i} + 2\hat{j} - 4\hat{k}\right).\left(2\hat{i} + 3\hat{j} - \hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(2\hat{i} + 3\hat{j} - \hat{k}\right) - [(2)(5) + (2)(3) + (-4)(-1)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.\left(2\hat{i} + 3\hat{j} - \hat{k}\right) - [10 + 6 + 4] = 0$

$\Rightarrow \vec{r}.\left(2\hat{i} + 3\hat{j} - \hat{k}\right) - 20 = 0$

$\underline{\Rightarrow \vec{\mathbf{r}}.\left(\mathbf{2\hat{i} + 3\hat{j} - \hat{k}}\right) = \mathbf{20}}$ is the vector equation of a required plane.

Let $\vec{r} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

Then, the above vector equation of the plane becomes,

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} + 3\hat{j} - \hat{k}\right) = 20$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(2) + (y)(3) + (z)(-1) = 20$

2x + 3y - z = 20

This is the Cartesian form of the equation of the required plane.

19. Question

If O be the origin and the coordinates of P be (1, 2, -3), then find the equation of the plane passing through P and perpendicular to OP.

Answer

As it is given that the required plane is passing through P(1, 2, -3) and is perpendicular to OP. Let the position vector of this point P be

$\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k} \ldots \ldots (i)$

And it is also given the plane is normal to the line joining the points O(0,0,0) and P(1, 2, -3).

Then $\vec{n} = \overrightarrow{OP}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{P}$ - position vector of $\vec{O}$

$\Rightarrow \vec{n} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (0\hat{i} + 0\hat{j} + 0\hat{k})$

$\Rightarrow \vec{n} = \hat{i} + 2\hat{j} - 3\hat{k} \ldots \ldots (ii)$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (\hat{i} + 2\hat{j} - 3\hat{k})\right).(\hat{i} + 2\hat{j} - 3\hat{k}) = 0$

$\Rightarrow \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k}).(\hat{i} + 2\hat{j} - 3\hat{k}) = 0$

$\Rightarrow \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) - [(1)(1) + (2)(2) + (-3)(-3)] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\Rightarrow \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) - [1 + 4 + 9] = 0$

$\Rightarrow \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) - 14 = 0$

$\underline{\boldsymbol{\Rightarrow \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) = 14}}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + 2\hat{j} - 3\hat{k}) = 14$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(1) + (y)(2) + (z)(-3) = 14$

$\underline{\boldsymbol{\Rightarrow x + 2y - 3z = 14}}$

This is the Cartesian form of the equation of the required plane.

20. Question

If O is the origin and the coordinates of A are (a, b, c). Find the direction cosines of OA and the equation of the plane through A at right angles to OA.

Answer

As it is given that the required plane is passing through A(a, b, c) and is perpendicular to OA. Let the position vector of this point A be

$\vec{a} = a\hat{i} + b\hat{j} + c\hat{k} \ldots \ldots (i)$

And it is also given the plane is normal to the line joining the points O(0,0,0) and A(a, b, c)

Then $\vec{n} = \overrightarrow{OA}$

$\Rightarrow \vec{n} =$ Position vector of $\vec{A}$ - position vector of $\vec{O}$

$\Rightarrow \vec{n} = (a\hat{i} + b\hat{j} + c\hat{k}) - (0\hat{i} + 0\hat{j} + 0\hat{k})$

$\Rightarrow \vec{n} = a\hat{i} + b\hat{j} + c\hat{k} \ldots..(ii)$

Therefore the direction ratios of OA are proportional to a, b, c

Hence the direction cosines are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

We know that the vector equation of a plane passing through the point $\vec{a}$ and perpendicular/normal to the vector $\vec{n}$ is given by

$(\vec{r} - \vec{a}).\vec{n} = 0$

Substituting the values from eqn(i) and eqn(ii) in the above equation, we get

$\left(\vec{r} - (a\hat{i} + b\hat{j} + c\hat{k})\right).(a\hat{i} + b\hat{j} + c\hat{k}) = 0$

$\Rightarrow \vec{r}.(a\hat{i} + b\hat{j} + c\hat{k}) - (a\hat{i} + b\hat{j} + c\hat{k}).(a\hat{i} + b\hat{j} + c\hat{k}) = 0$

$\Rightarrow \vec{r}.(a\hat{i} + b\hat{j} + c\hat{k}) - [a^2 + b^2 + c^2] = 0$ (by multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$)

$\underline{\Rightarrow \mathbf{\vec{r}.(a\hat{i} + b\hat{j} + c\hat{k}) = a^2 + b^2 + c^2}}$ is the vector equation of a required plane.

Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

Then, the above vector equation of the plane becomes,

$(x\hat{i} + y\hat{j} + z\hat{k}).(a\hat{i} + b\hat{j} + c\hat{k}) = a^2 + b^2 + c^2$

Now multiplying the two vectors using the formula $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$, we get

$\Rightarrow (x)(a) + (y)(b) + (z)(c) = a^2 + b^2 + c^2$

$\underline{\Rightarrow \mathbf{ax + by + cz = a^2 + b^2 + c^2}}$

This is the Cartesian form of equation of the required plane.

21. Question

Find the vector equation of the plane with intercepts 3, -4 and 2 on x, y and z axes respectively.

Answer

Let the equation of the plane be

Ax + By + Cz + D=0...........(i) (where D≠0)

As per the given criteria, the plane makes 3, -4, 2 intercepts on x, y, z axes respectively.

Hence the plane meets the x, y, z axes (3, 0, 0), (0, -4, 0) and (0, 0, 2) respectively.

Therefore by putting (0, 0, 2), we get

$A(0) + B(0) + C(2) + D = 0 \Rightarrow C = -\frac{D}{2}$

Similarly by putting (0, -4, 0) we get

$A(0) + B(-4) + C(0) + D = 0 \Rightarrow B = \frac{D}{4}$

And by putting (3, 0, 0) we get

$$A(3) + B(0) + C(0) + D = 0 \Rightarrow A = -\frac{D}{3}$$

Substituting the values of A, B, C in equation (i), we get

by putting B(0, -4, 0) we get

$$\left(-\frac{D}{3}\right)x + \frac{D}{4}y + \left(-\frac{D}{2}\right)z + D = 0$$

$$\Rightarrow \frac{-4Dx + 3Dy - 6Dz + 12D}{12} = 0$$

$$\Rightarrow (-D)(4x - 3y + 6z - 12) = 0$$

$$\Rightarrow 4x - 3y + 6z = 12$$

This is the Cartesian form of equation of the required plane

Now the vector equation of the plane 4x - 3y + 6z = 12 can be written as

$$(x\hat{i} + y\hat{j} + z\hat{k})(4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$$

$$\Rightarrow \vec{r}.(4\hat{i} - 3\hat{j} + 6\hat{k}) = 12$$

This is the required vector equation of the plane with intercepts 3, -4 and 2 on x, y and z axes respectively.

Exercise 29.4

1. Question

Find the vector equation of a plane which is at a distance of 3 units from the origin and has $\hat{k}$ as the unit vector normal to it.

Answer

Given: Normal vector, $\vec{n}$ = i

Now, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{k}}{|\hat{k}|} = \frac{\hat{k}}{1} = k$

The equation of a plane in normal form is $\vec{r}.\hat{n}$ = d (where d is the distance of the plane from the origin)

Substituting n = k and d = 3 in the relation, we get

$\vec{r}.\hat{k}$ = 3

2. Question

Find the vector equation of a plane which is at a distance of 5 units from the origin and which is normal to the vector $\hat{i} - 2\hat{j} - 2\hat{k}$.

Answer

It is given that the normal vector $\vec{n} = \hat{i} - 2\hat{j} - 2\hat{k}$

Now, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i}-2\hat{j}-2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i}-2\hat{j}-2\hat{k}}{\sqrt{9}} = \frac{\hat{i}-2\hat{j}-2\hat{k}}{3} = \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$

The equation of a plane in normal form is $\vec{r}.\hat{n}$ = d (where d is the distance of the plane from the origin)

Substituting $\hat{n} = \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$ and d = 5.

We get,

$\vec{r}.\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{2}{3}\hat{k}=5$

3. Question

Reduce the equation 2x – 3y – 6z = 14 to the normal form and hence find the length of the perpendicular from the origin to the plane. Also, find the direction cosines of the normal to the plane.

Answer

The given equation of the plane is

2x – 3y – 6z = 14 ......(i)

Now, $\sqrt{2^2+(-3)^2+(-6)^2}=\sqrt{4+9+36}=\sqrt{49}=7$

Dividing (i) by 7, we get

$\frac{2}{7}x-\frac{3}{7}y-\frac{6}{7}z=\frac{14}{7}=2$ ......(ii)

The Cartesian Equation of the normal form of a plane is

lx + my + nz = p ......(iii)

where l, m and n are direction cosines of normal to the plane and p is the length of the perpendicular from the origin to the plane.

Comparing (ii) and (iii), we get

Direction cosine: l = $\frac{2}{7}$, m = $\frac{-3}{7}$, n = $\frac{-6}{7}$ and

Length of the perpendicular from the origin to the plane: p = 2.

4. Question

Reduce the equation $\vec{r}.\left(\hat{i}-2\hat{j}+2\hat{k}\right)+6=0$ to normal form and hence find the length of perpendicular from the origin to the plane.

Answer

The given equation of the plane is

$\vec{r}.\left(\hat{i}-2\hat{j}+2\hat{k}\right)+6=0$

$\vec{r}.\left(\hat{i}-2\hat{j}+2\hat{k}\right)=-6$ or $\vec{r}.\vec{n}=-6$, where $\vec{n}=\left(\hat{i}-2\hat{j}+2\hat{k}\right)$

$|\vec{n}|=\sqrt{1+4+4}=3$

For reducing the given equation to normal form, we need to divide it by $|\vec{n}|$.

Then, we get,

$\vec{r}.\frac{\vec{n}}{|\vec{n}|}=\frac{-6}{|\vec{n}|}$

$\vec{r}.\left(\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}\right)=\frac{-6}{3}$

$\vec{r}.\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)=-2$

Dividing both sides by – 1, we get

$\vec{r}.\left(-\frac{1}{3}\hat{i}+\frac{2}{3}\hat{j}-\frac{2}{3}\hat{k}\right)=2$ ......(i)

The equation of a plane in normal form is

$\vec{r}.\hat{n} = d$ ......(ii)

Where d is the distance of the plane from the origin

Comparing (i) and (ii)

Length of the perpendicular from the origin to the plane = d = 2 units.

5. Question

Write the normal form of the equation of the plane

2x–3y + 6z + 14 = 0.

Answer

The given equation of the plane

2x–3y + 6z + 14 = 0

2x–3y + 6z = – 14 ......(i)

Now, $\sqrt{2^2 + (-3)^2 + (6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

Dividing (i) by 7, we get,

$$\frac{2}{7}x - \frac{3}{7}y + \frac{6}{7}z = \frac{-14}{7} = -2$$

Multiplying both sides by – 1, we get

$$-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$$

This is the normal form of the given equation of the plane.

6. Question

The direction ratios of the perpendicular from the origin to a plane are 12, – 3, 4 and the length of the perpendicular is 5. Find the equation of the plane.

Answer

It is given that the direction ratios of the normal vector $\vec{n}$ is 12, – 3, and 4.

So, $\vec{n} = 12\hat{i} - 3\hat{j} + 4\hat{k}$

$|\vec{n}| = \sqrt{12^2 + (-3)^2 + (4)^2} = \sqrt{144 + 9 + 16} = \sqrt{169} = 13$

Now, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{12\hat{i} - 3\hat{j} + 4\hat{k}}{13} = \frac{12}{13}\hat{i} - \frac{3}{13}\hat{j} + \frac{4}{13}\hat{k}$

Length of the perpendicular from the origin to the plane, d = 5

The equation of the plane in normal form is

$\vec{r}.\hat{n} = d$

$$\vec{r}.\left(\frac{12}{13}\hat{i} - \frac{3}{13}\hat{j} + \frac{4}{13}\hat{k}\right) = 5$$

7. Question

Find a normal unit vector to the plane x + 2y + 3z–6 = 0.

Answer

The given equation of the plane is x + 2y + 3z–6 = 0

x + 2y + 3z = 6

$\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) = 6$

or, $\vec{r}.\vec{n} = 6,$

where $\vec{n} = \hat{i} + 2\hat{j} + 3\hat{k}$ ......(i)

Now, $|\vec{n}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

Unit vector to the plane, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$

8. Question

Find the equation of a plane which is at a distance of 3√3 units from the origin and the normal to which is equally inclined with the coordinate axes.

Answer

Let α, β and γ be the angles made by $\vec{n}$ with x, y and z - axes respectively.

It is given that

$\alpha = \beta = \gamma$

$\cos \alpha = \cos \beta = \cos \gamma$

l = m = n, where l, m, n are direction cosines of $\vec{n}$.

But $l^2 + m^2 + n^2 = 1$

Or, $l^2 + l^2 + l^2 = 1$

Or, $3 l^2 = 1$

Or, $l^2 = \frac{1}{3}$

Or, $l = \frac{1}{\sqrt{3}}$

So, $l = m = n = \frac{1}{\sqrt{3}}$

It is given that the length of the perpendicular of the plane from the origin, $p = 3\sqrt{3}$.

The normal form of the plane is lx + my + nz = p.

$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = 3\sqrt{3}$

$x + y + z = 3\sqrt{3} \times \sqrt{3}$

$x + y + z = 9.$

9. Question

Find the equation of the plane passing through the point (1,2,1) and perpendicular to the line joining the points (1,4,2) and (2,3,5). Find also the perpendicular distance of the origin from this plane.

Answer

We know that the vector equation of the plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is

$(\vec{r} - \vec{a})\vec{n} = 0$ ......(i)

Here,

$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

And, $\vec{n} = \overrightarrow{BC}$

$\vec{n} = (2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + 4\hat{j} + 2\hat{k})$

$\vec{n} = 2i + 3j + 5k - i + 4j + 2k$

$\vec{n} = (\hat{i} - \hat{j} + 3\hat{k})$

Putting the value of $\vec{n}$ and $\vec{a}$ in (i)

$[\vec{r} - (\hat{i} - 2\hat{j} + \hat{k})].(\hat{i} - \hat{j} + 3\hat{k}) = 0.$

$[\vec{r}.(\hat{i} - \hat{j} + 3\hat{k})] - [(\hat{i} - 2\hat{j} + \hat{k}).(\hat{i} - \hat{j} + 3\hat{k})] = 0.$

$[\vec{r}.(\hat{i} - \hat{j} + 3\hat{k})] - [(1 \times 1) + (2 \times -1) + (1 \times 3)] = 0.$

$[\vec{r}.(\hat{i} - \hat{j} + 3\hat{k})] - [1 - 2 + 3] = 0.$

$[\vec{r}.(\hat{i} - \hat{j} + 3\hat{k})] - 2 = 0$ ......(ii)

$|\vec{n}| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{1 + 1 + 9} = \sqrt{11}$

Dividing (ii) by $\sqrt{11}$

$\vec{r}.\left(\frac{1}{\sqrt{11}}\hat{i} - \frac{1}{\sqrt{11}}\hat{j} + \frac{3}{\sqrt{11}}\hat{k}\right) - \frac{2}{\sqrt{11}} = 0$

$\vec{r}.\left(\frac{1}{\sqrt{11}}\hat{i} - \frac{1}{\sqrt{11}}\hat{j} + \frac{3}{\sqrt{11}}\hat{k}\right) = \frac{2}{\sqrt{11}}$

$\vec{r}.\hat{n} = d$

So the perpendicular distance of plane from origin = $\frac{2}{\sqrt{11}}$ units

Equation of plane: $\vec{r}.(\hat{i} - \hat{j} + 3\hat{k}) = 2$

Equation of plane : x – y + 3z – 2 = 0.

10. Question

Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also, find its Cartesian form.

Answer

Given, normal vector $\vec{n} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

Now, $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{2^2 + (-3)^2 + 4^2}} = \frac{2\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{4 + 9 + 16}} = \frac{2\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{29}} = \frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}$

The equation of the plane in normal form is

$\vec{r}.\hat{n} = d$ ...... (i)

(where d is the distance of the plane from the origin)

Substituting $\hat{n} = \frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}$ and d = $\frac{6}{\sqrt{29}}$ in (i)

$\vec{r}.\left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}$ ......(ii)

Cartesian Form

For Cartesian Form, substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (ii), we get

$$(x\hat{i} + y\hat{j} + z\hat{k}).\left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}$$

$$\frac{2x - 3y + 4z}{\sqrt{29}} = \frac{6}{\sqrt{29}}$$

So, 2x – 3y + 4z = 6, is the Cartesian form

11. Question

Find the distance of the plane 2x – 3y + 4z – 6 = 0 from the origin.

Answer

The given equation of the plane is

2x – 3y + 4z – 6 = 0

Or, 2x – 3y + 4z = 6 ...... (i)

Now, $\sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$

Dividing (i) by $\sqrt{29}$, we get

$\left(\frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z\right) = \frac{6}{\sqrt{29}}$, which is the normal form of the plane (i).

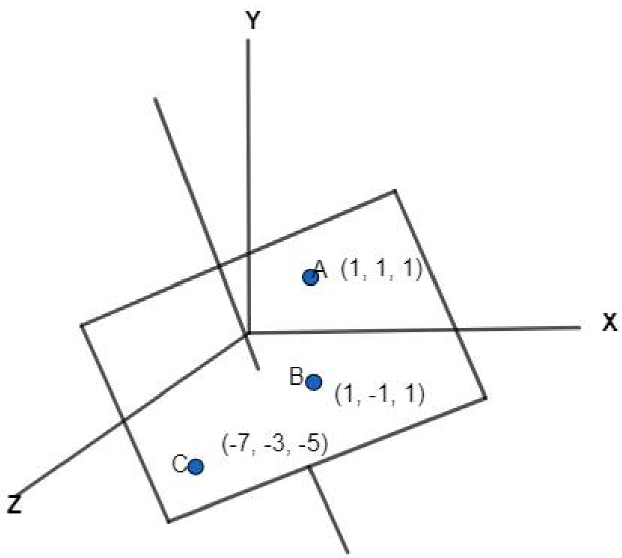
So, the length of the perpendicular from the origin to the plane $= \frac{6}{\sqrt{29}}$.

Exercise 29.5

1. Question

Find the vector equation of the plane passing through the points (1, 1, 1), (1, – 1, 1) and (– 7, – 3, – 5).

Answer



We know that the vector equation of the plane passing through three points having position vectors $\vec{a}, \vec{b},$ and $\vec{c}$ is

$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$ ...... (i)

According to the question, $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{c} = -7\hat{i} - 3\hat{j} - 5\hat{k}$

$(\vec{b}-\vec{a}) = -2\hat{j}, (\vec{c}-\vec{a}) = -8\hat{i}-4\hat{j}-6\hat{k}$

$$[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 0 \\ -8 & -4 & -6 \end{vmatrix} = 0$$

$= -2(-6\hat{i} + 8\hat{k})$

$= 12\hat{i} - 16\hat{k}$

From (i), the vector equation of the required plane is

$[\vec{r} - (\hat{i} + \hat{j} + \hat{k})].(12\hat{i} - 16\hat{k}) = 0$

or, $\vec{r}(12\hat{i} - 16\hat{k}) - [(\hat{i} + \hat{j} + \hat{k}).(12\hat{i} - 16\hat{k})] = 0$

or, $\vec{r}(12\hat{i} - 16\hat{k}) = [(\hat{i} + \hat{j} + \hat{k}).(12\hat{i} - 16\hat{k})]$

or, $\vec{r}(12\hat{i} - 16\hat{k}) = 12 - 16$

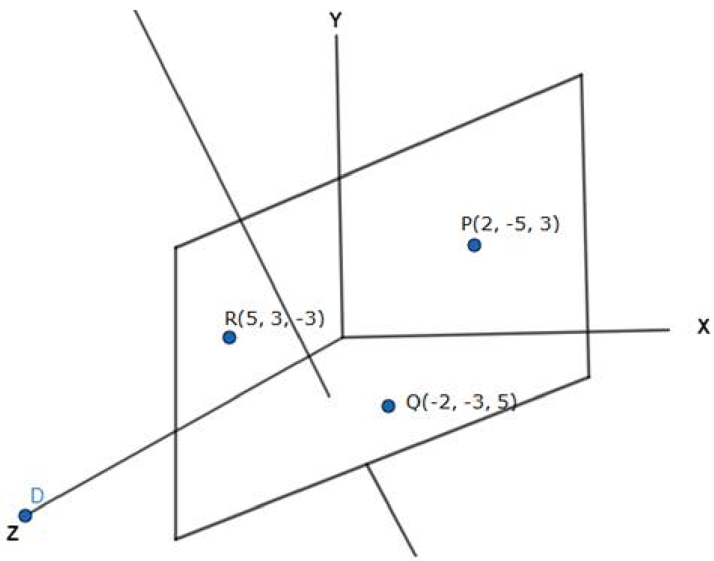
or, $\vec{r}(3\hat{i} - 4\hat{k}) = -1$

or, $\vec{r}(3\hat{i} - 4\hat{k}) + 1 = 0$

2. Question

Find the vector equation of the plane passing through the points P(2, 5, – 3), Q(– 2, – 3, 5) and R(5, 3, – 3).

Answer



The required plane passes through the point P(2, 5, – 3) whose position vector is $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and is normal to the vector $\vec{n}$ given by

$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

Clearly,

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (-2\hat{i} - 3\hat{j} + 5\hat{k}) - (2\hat{i} + 5\hat{j} - 3\hat{k}) = -4\hat{i} - 8\hat{j} + 8\hat{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (5\hat{i} + 3\hat{j} - 3\hat{k}) - (2\hat{i} + 5\hat{j} - 3\hat{k}) = 3\hat{i} - 2\hat{j} - 0\hat{k}$

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 16\hat{i} + 24\hat{j} + 32\hat{k}$$

The vector equation of the required plane is,

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

or, $\vec{r}.\left(16\hat{i} + 24\hat{j} + 32\hat{k}\right) = \left(2\hat{i} + 5\hat{j} - 3\hat{k}\right).\left(16\hat{i} + 24\hat{j} + 32\hat{k}\right)$

or, $\vec{r}.\left[8\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)\right] = 32 + 120 - 96$

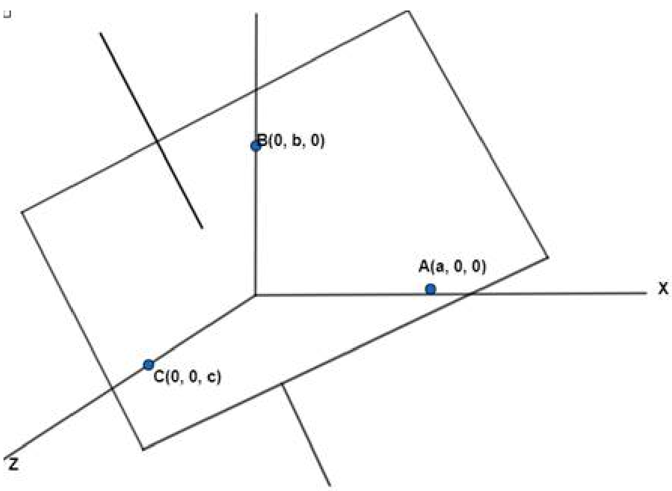
or, $\vec{r}.\left[8\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)\right] = 56$

or, $\vec{r}.\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) = 7$

3. Question

Find the vector equation of the plane passing through point A(a, 0, 0), B(0, b, 0) and C(0, 0, c). Reduce it to normal form. If plane ABC is at a distance p from the origin, prove that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

Answer



The required plane passes through the point A(a,0,0) whose position vector is $\vec{a} = a\hat{i} + 0\hat{j} + 0\hat{k}$ and is normal to the vector $\vec{n}$ given by

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

Clearly,

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(0\hat{i} + b\hat{j} + 0\hat{k}\right) - \left(a\hat{i} + 0\hat{j} - 0\hat{k}\right) = -a\hat{i} - b\hat{j} + 0\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \left(0\hat{i} + 0\hat{j} - 3c\right) - \left(a\hat{i} + 0\hat{j} - 0\hat{k}\right) = -a\hat{i} + 0\hat{j} + c\hat{k}$

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = bc\hat{i} + ac\hat{j} + ab\hat{k}$

The vector equation of the required plane is,

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

or, $\vec{r}.\left(bc\hat{i} + ac\hat{j} + ab\hat{k}\right) = \left(a\hat{i} + 0\hat{j} + 0\hat{k}\right).\left(bc\hat{i} + ac\hat{j} + ab\hat{k}\right)$

or, $\vec{r}.\left(bc\hat{i} + ac\hat{j} + ab\hat{k}\right) = abc + 0$

or, $\vec{r}.\left(bc\hat{i} + ac\hat{j} + ab\hat{k}\right) = abc \ldots\ldots$ (i)

Now, $|\vec{n}| = \sqrt{(bc)^2 + (ac)^2 + (ab)^2} = \sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

For reducing (i) to normal form, we need to divide both sides of (i) by $\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

Then, we get,

$\vec{r}.\left(\frac{(bc\hat{i} + ac\hat{j} + ab\hat{k})}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\right) = \left(\frac{abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\right)$, which is the normal form of plane (i)

So, the distance of the plane (i) from the origin is,

$$p = \left(\frac{abc}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}\right)$$

$$\text{or, } \frac{1}{p} = \frac{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}{abc} \text{ \{reciprocal\}}$$

$$\text{or, } \frac{1}{p^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2} \text{ [squaring both sides]}$$

$$\text{or, } \frac{1}{p^2} = \frac{b^2c^2}{a^2b^2c^2} + \frac{b^2c^2}{a^2b^2c^2} + \frac{b^2c^2}{a^2b^2c^2}$$

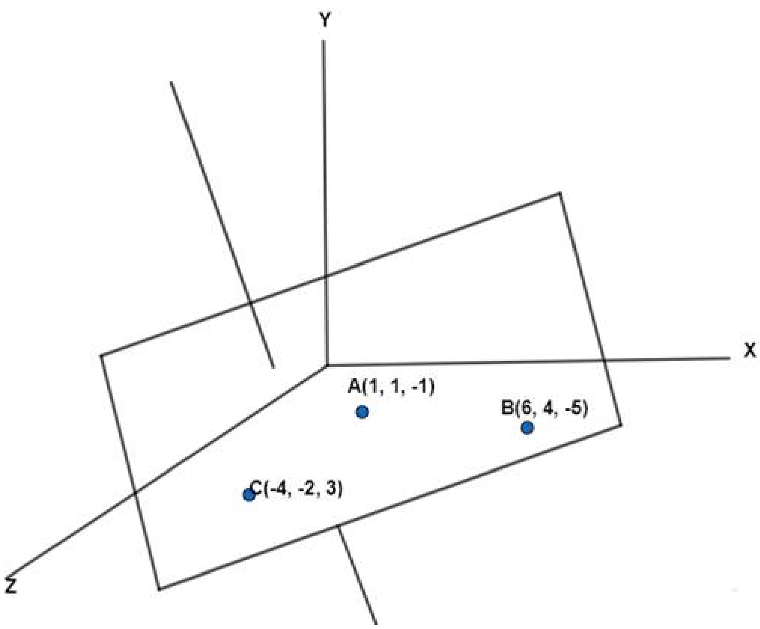
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{p^2}$$

Hence, Proved.

4. Question

Find the vector equation of the plane passing through the points (1, 1, – 1), (6, 4, – 5) and (– 4, – 2, 3).

Answer



Let A(1,1, – 1), B(6,4, – 5), C(– 4, – 2 – 3).

The required plane passes through the point A(1,1, – 1), whose position vector is $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and is normal to the vector $\vec{n}$ given by

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

Clearly,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (6\hat{i} + 4\hat{j} - 5\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) = 5\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (-4\hat{i} - 2\hat{j} - 3c) - (\hat{i} + \hat{j} - \hat{k}) = -5\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & -4 \\ -5 & -3 & 4 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

So, the given points are collinear.

Thus there will be an infinite number of planes passing through these points.

Their equations (passing through (1,1, – 1) are given by,

a(x – 1) + b(y – 1) + c(z + 1) = 0 ......(i)

Since this passes through B(6,4, – 5),

a(6 – 1) + b(4 – 1) + c(– 5 + 1) = 0

or, 5a + 3b – 4c = 0 ......(ii)

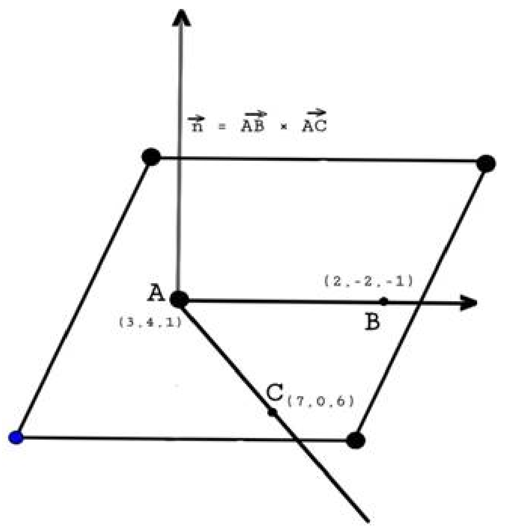
From (i) and (ii), the equations of the infinite planes are

a(x – 1) + b(y – 1) + c(z + 1) = 0, where5a + 3b – 4c = 0.

5. Question

Find the vector equation of the plane passing through the points $3\hat{i}+4\hat{j}+2\hat{k}$, $2\hat{i}+2\hat{j}-\hat{k}$, and $7\hat{i}+6\hat{k}$.

Answer



Let A(3,4,2), B(2, – 2, – 1), C(7,0,6).

The required plane passes through the point A(3,4,2), whose position vector is $\vec{a} = 3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$ and is normal to the vector $\vec{n}$ given by

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

Clearly,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(-2\hat{\imath} - 2\hat{\jmath} - 1\hat{k}\right) - \left(\hat{3\imath} + 4\hat{\jmath} + 2\hat{k}\right) = -\hat{\imath} - 6\hat{\jmath} - 3\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (7\hat{\imath} + 0\hat{\jmath} - 3c) - \left(\hat{3\imath} + 4\hat{\jmath} + 2\hat{k}\right) = 4\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & -6 & -3 \\ 4 & -4 & 4 \end{vmatrix} = -36\hat{\imath} - 8\hat{\jmath} + 28\hat{k}$$

The vector equation of the required plane is,

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

or, $\vec{r}.\left(-36\hat{\imath} - 8\hat{\jmath} + 28\hat{k}\right) = \left(3\hat{\imath} + 4\hat{\jmath} + 2\hat{k}\right).\left(-36\hat{\imath} - 8\hat{\jmath} + 28\hat{k}\right)$

or, $\vec{r}.\left[-4\left(9\hat{\imath} - 2\hat{\jmath} + 4\hat{k}\right)\right] = -108 - 32 + 56$

or, $\vec{r}.\left[-4\left(9\hat{\imath} - 2\hat{\jmath} + 4\hat{k}\right)\right] = -84$

or, $\vec{r}.\left(9\hat{i}-2\hat{j}+4\hat{k}\right) = 21$

Exercise 29.6

1 A. Question

Find the angle between the planes :

$\vec{r}\cdot\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=1$ and $\vec{r}\cdot\left(-\hat{i}+\hat{j}\right)=4$

Answer

Given planes, $\vec{r}.\left(2\hat{i}-3\hat{j}+4\hat{k}\right) = 1$ and $\vec{r}.(-\hat{i}+\hat{j}) = 1$

$\vec{r}.\left(2\hat{i}-3\hat{j}+4\hat{k}\right) = 1$ ......(a)

$\vec{r}.(-\hat{i}+\hat{j}) = 1$ ......(b)

We know that the angle between two planes,

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here we have $\vec{n}_1 = 2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{n}_2 = -\hat{i}+\hat{j}$

$$\therefore \cos\theta = \frac{\left(\left(2\hat{i}-3\hat{j}+4\hat{k}\right).(-\hat{i}+\hat{j})\right)}{|2\hat{i}-3\hat{j}+4\hat{k}||-\hat{i}+\hat{j}|}$$

$$= \frac{\left((2)(-1)+(-3)(1)+(4)(0)\right)}{\sqrt{2^2+3^2+4^2}\sqrt{1^2+1^2}}$$

$$= \frac{-2-3+0}{\sqrt{29}\sqrt{2}}$$

$$= -\frac{5}{\sqrt{58}}$$

Now, as

$$\cos\theta = -\frac{5}{\sqrt{58}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{5}{\sqrt{58}}\right)$$

Hence the angle between the two planes is $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{58}}\right)$

1 B. Question

Find the angle between the planes :

$\vec{r}\cdot\left(2\hat{i}-\hat{j}+2\hat{k}\right)=6$ and $\vec{r}\cdot\left(3\hat{i}+6\hat{j}-2\hat{k}\right)=9$

Answer

Given planes, $\vec{r}.\left(2\hat{i}-\hat{j}+2\hat{k}\right) = 6$ and $\vec{r}.\left(3\hat{i}+6\hat{j}-2\hat{k}\right) = 9$

$\vec{r}.\left(2\hat{i}-\hat{j}+2\hat{k}\right) = 6$ ......(a)

$\vec{r}.\left(3\hat{i}+6\hat{j}-2\hat{k}\right) = 9$ ......(b)

We know that the angle between two planes,

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here we have $\vec{n}_1 = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{n}_2 = 3\hat{i} + 6\hat{j} - 2\hat{k}$

$$\therefore \cos\theta = \frac{\left((2\hat{i} - \hat{j} + 2\hat{k}).(3\hat{i} + 6\hat{j} - 2\hat{k})\right)}{|2\hat{i} - \hat{j} + 2\hat{k}||3\hat{i} + 6\hat{j} - 2\hat{k}|}$$

$$= \frac{\left((2)(3) + (-1)(6) + (2)(-2)\right)}{\sqrt{2^2 + 1^2 + 2^2}\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{6 - 6 - 4}{\sqrt{9}\sqrt{49}}$$

$$= -\frac{4}{3 \times 7} = -\frac{4}{21}$$

Now, as

$$\cos\theta = -\frac{4}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{4}{21}\right)$$

Hence the angle between the two planes is $\theta = \cos^{-1}\left(-\frac{4}{21}\right)$

1 C. Question

Find the angle between the planes :

$$\vec{r}\cdot\left(2\hat{i} + 3\hat{j} - 6\hat{k}\right) = 5 \text{ and } \vec{r}\cdot\left(\hat{i} - 2\hat{j} + 2\hat{k}\right) = 9$$

Answer

Given planes, $\vec{r}.(2\hat{i} + 3\hat{j} - 6\hat{k}) = 6$ and $\vec{r}.(\hat{i} - 2\hat{j} + 2\hat{k}) = 9$

$\vec{r}.(2\hat{i} + 3\hat{j} - 6\hat{k}) = 5$ ......(a)

$\vec{r}.(\hat{i} - 2\hat{j} + 2\hat{k}) = 9$ ......(b)

We know that the angle between two planes,

$\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Here we have $\vec{n}_1 = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{n}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$

$$\therefore \cos\theta = \frac{\left((2\hat{i} + 3\hat{j} - 6\hat{k}).(\hat{i} - 2\hat{j} + 2\hat{k})\right)}{|2\hat{i} + 3\hat{j} - 6\hat{k}||\hat{i} - 2\hat{j} + 2\hat{k}|}$$

$$= \frac{\left((2)(1) + (3)(-2) + (-6)(2)\right)}{\sqrt{2^2 + 3^2 + 6^2}\sqrt{1^2 + 2^2 + 2^2}}$$

$$= \frac{2 - 6 - 12}{\sqrt{49}\sqrt{9}}$$

$$= -\frac{16}{3 \times 7} = -\frac{16}{21}$$

Now, as

$$\cos\theta = -\frac{16}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{16}{21}\right)$$

Hence the angle between the two planes is $\theta = \cos^{-1}\left(-\frac{16}{21}\right)$

2 A. Question

Find the angle between the planes :

2x – y + z = 4 and x + y + 2z = 3

Answer

Given planes are 2x – y + z = 4 and x + y + 2z = 3

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here we have,

$a_1 = 2, b_1 = -1, c_1 = 1$

$a_2 = 1, b_2 = 1, c_2 = 2$

$$\therefore \cos\theta = \frac{(2)(1) + (-1)(1) + (1)(2)}{\sqrt{2^2 + 1^2 + 1^2}\sqrt{1^2 + 1^2 + 2^2}}$$

$$= \frac{2 - 1 + 2}{\sqrt{6}\sqrt{6}}$$

$$= \frac{3}{6} = \frac{1}{2}$$

$$\therefore \cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Hence, the angle between planes 2x – y + z = 4 and x + y + 2z = 3 is $\frac{\pi}{3}$.

2 B. Question

Find the angle between the planes :

x + y – 2z = 3 and 2x – 2y + z = 5

Answer

Given planes are x + y – 2z = 3 and 2x – 2y + z = 5

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here we have,

$a_1 = 1$, $b_1 = 1$, $c_1 = -2$

$a_2 = 2$, $b_2 = -2$, $c_2 = 1$

$$\therefore \cos\theta = \frac{(1)(2) + (1)(-2) + (-2)(1)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + 2^2 + 1^2}}$$

$$= \frac{2 - 2 - 2}{\sqrt{6}\sqrt{9}}$$

$$= -\frac{2}{\sqrt{6} \times 3} = -\frac{2}{3\sqrt{6}}$$

$$\therefore \cos\theta = -\frac{2}{3\sqrt{6}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$$

Hence, the angle between planes x + y – 2z = 3 and 2x – 2y + z = 5 is $\theta = \cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$.

2 C. Question

Find the angle between the planes :

x – y + z = 5 and x + 2y + z = 9

Answer

Given planes are x – y + z = 5 and x + 2y + z = 9

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here we have,

$a_1 = 1$, $b_1 = -1$, $c_1 = 1$

$a_2 = 1$, $b_2 = 2$, $c_2 = 1$

$$\therefore \cos\theta = \frac{(1)(1) + (-1)(2) + (1)(1)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + 2^2 + 1^2}}$$

$$= \frac{1 - 2 + 1}{\sqrt{3}\sqrt{4}}$$

$$= \frac{0}{\sqrt{3} \times 2} = 0$$

$$\therefore \cos\theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

Hence, the angle between planes x – y + z = 5 and x + 2y + z = 9 is $\theta = \frac{\pi}{2}$.

2 D. Question

Find the angle between the planes :

2x – 3y + 4z = 1 and –x + y = 4

Answer

Given planes are 2x – 3y + 4z = 1 and – x + y = 4

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here we have,

$a_1 = 2$, $b_1 = -3$, $c_1 = 4$

$a_2 = -1$, $b_2 = 1$, $c_2 = 0$

$$\therefore \cos\theta = \frac{(2)(-1) + (-3)(1) + (4)(0)}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{1^2 + 1^2 + 0^2}}$$

$$= \frac{-2 - 3 + 0}{\sqrt{29}\sqrt{2}}$$

$$= \frac{-5}{\sqrt{58}}$$

$$\therefore \cos\theta = \frac{-5}{\sqrt{58}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$$

Hence, the angle between planes 2x – 3y + 4z = 1 and – x + y = 4 is $\theta = \cos^{-1}\left(\frac{-5}{\sqrt{58}}\right)$

2 E. Question

Find the angle between the planes :

2x + y – 2z = 5 and 3x – 6y – 2z = 7

Answer

Given planes are 2x + y – 2z = 5 and 3x – 6y – 2z = 7

We know that angle between two planes,

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$ is given as

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here we have,

$a_1 = 2$, $b_1 = 1$, $c_1 = -2$

$a_2 = 3, b_2 = -6, c_2 = -2$

$$\therefore \cos\theta = \frac{(2)(3) + (1)(-6) + (-2)(-2)}{\sqrt{2^2 + 1^2 + 2^2}\sqrt{3^2 + 6^2 + 2^2}}$$

$$= \frac{6 - 6 + 4}{\sqrt{9}\sqrt{49}}$$

$$= \frac{4}{3 \times 7} = \frac{4}{21}$$

$$\therefore \cos\theta = \frac{4}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Hence, the angle between planes 2x + y – 2z = 5 and 3x – 6y – 2z = 7 is $\theta = \cos^{-1}\left(\frac{4}{21}\right)$.

3 A. Question

Show that the following planes are at right angles :

$\vec{r}\cdot\left(2\hat{i} - \hat{j} + \hat{k}\right) = 5$ and $\vec{r}\cdot\left(-\hat{i} - \hat{j} + \hat{k}\right) = 3$

Answer

Given planes, $\vec{r}.\left(2\hat{\imath} - \hat{\jmath} + \hat{k}\right) = 5$ and $\vec{r}.\left(-\hat{\imath} - \hat{\jmath} + \hat{k}\right) = 3$

We know that planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$

We have $\vec{n}_1 = 2\hat{\imath} - \hat{\jmath} + \hat{k}$ and $\vec{n}_2 = -\hat{\imath} - \hat{\jmath} + \hat{k}$

Now, $\vec{n}_1.\vec{n}_2 = \left(2\hat{\imath} - \hat{\jmath} + \hat{k}\right).\left(-\hat{\imath} - \hat{\jmath} + \hat{k}\right)$

$= (-2 + 1 + 1) = 0$

Hence, the two given planes are perpendicular.

3 B. Question

Show that the following planes are at right angles :

x – 2y + 4z = 10 and 18x + 17y + 4z = 49

Answer

Given planes, x – 2y + 4z = 10 and 18x + 17y + 4z = 49

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (a)

We have, $a_1 = 1, b_1 = -2, c_1 = 4$ and $a_2 = 18, b_2 = 17, c_2 = 4$

Using (a) we have,

$a_1a_2 + b_1b_2 + c_1c_2 = (1)(18) + (-2)(17) + (4)(4)$

= 18 – 34 + 16 = 0

Hence, the planes are at right angle to each other.

4 A. Question

Determine the value of λ for which the following planes are perpendicular to each other.

$\vec{r}\cdot\left(\hat{i}+2\hat{j}+3\hat{k}\right)=7$ and $\vec{r}\cdot\left(\lambda\hat{i}+2\hat{j}-7\hat{k}\right)=26$

Answer

Given planes, $\vec{r}.\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 7$ and $\vec{r}.\left(\lambda\hat{i} + 2\hat{j} - 7\hat{k}\right) = 26$

We know that planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ are perpendicular if $\vec{n}_1.\vec{n}_2 = 0$

We have $\vec{n}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{n}_2 = \lambda\hat{i} + 2\hat{j} - 7\hat{k}$

Now, $\vec{n}_1.\vec{n}_2 = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(\lambda\hat{i} + 2\hat{j} - 7\hat{k}\right) = 0$

$\Rightarrow(\lambda + 4 - 21) = 0$

$\Rightarrow\lambda = 21 - 4 = 17$

Hence, for $\lambda = 17$ the given planes are perpendicular.

4 B. Question

Determine the value of λ for which the following planes are perpendicular to each other.

$2x - 4y + 3z = 5$ and $x + 2y + \lambda z = 5$

Answer

Given planes, $2x - 4y + 3z = 5$ and $x + 2y + \lambda z = 5$

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(a)

We have, $a_1 = 2$, $b_1 = -4$, $c_1 = 3$ and $a_2 = 1$, $b_2 = 2$, $c_2 = \lambda$

Using (a) we have,

$a_1a_2 + b_1b_2 + c_1c_2 = (2)(1) + (-4)(2) + (3)(\lambda) = 0$

$\Rightarrow 2 - 8 + 3\lambda = 0$

$\Rightarrow 6 = 3\lambda$

$\Rightarrow 2 = \lambda$

Hence, for $\lambda = 2$ the given planes are perpendicular.

4 C. Question

Determine the value of λ for which the following planes are perpendicular to each other.

$3x - 6y - 2z = 7$ and $2x + y - \lambda z = 5$

Answer

Given planes, $3x - 6y - 2z = 7$ and $2x + y - \lambda z = 5$

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(a)

We have, $a_1 = 3$, $b_1 = -6$, $c_1 = -2$ and $a_2 = 2$, $b_2 = 1$, $c_2 = -\lambda$

Using (a) we have,

$a_1a_2 + b_1b_2 + c_1c_2 = (3)(2) + (-6)(1) + (-2)(-\lambda) = 0$

$\Rightarrow 6 - 6 + 2\lambda = 0$

$\Rightarrow 0 = -2\lambda$

$\Rightarrow 0 = \lambda$

∴ For $\lambda = 0$ the given planes are perpendicular to each other.

5. Question

Find the equation of a plane passing through the point (– 1, – 1, 2) and perpendicular to the planes 3x + 2y – 3z = 1 and 5x – 4y + z = 5.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as -

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (– 1, – 1,2), so the equation of plane is

$a(x + 1) + b(y + 1) + c(z - 2) = 0$

$\Rightarrow ax + by + cz = 2c - a - b$ ...... (1)

Now, the required plane is also perpendicular to the planes,

$3x + 2y - 3z = 1$ and $5x - 4y + z = 5$

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...... (a)

Using (a) we have,

$3a + 2b - 3c = 0$ ...... (b)

$5a - 4b + c = 0$ ...... (c)

Solving (b) and (c) we get,

$$\frac{a}{(2)(1)-(-3)(-4)} = \frac{b}{(5)(-3)-(3)(1)} = \frac{c}{(3)(-4)-(2)(5)}$$

$$\Rightarrow \frac{a}{2-12} = \frac{b}{-15-3} = \frac{c}{-12-10}$$

$$\Rightarrow \frac{a}{-10} = \frac{b}{-18} = \frac{c}{-22} = \lambda(\text{say})$$

∴$a = -10\lambda$, $b = -18\lambda$, $c = -22\lambda$

Putting values of a,b,c in equation (1) we get,

$(-10\lambda)x + (-18\lambda)y + (-22\lambda)z = 2(-22)\lambda - (-10\lambda) - (-18\lambda)$

$\Rightarrow -10\lambda x - 18\lambda y - 22\lambda z = -44\lambda + 10\lambda + 18\lambda$

$\Rightarrow -10\lambda x - 18\lambda y - 22\lambda z = -16\lambda$

Dividing both sides by (-2λ) we get

$5x + 9y + 11z = 8$

So, the equation of the required planes is $5x + 9y + 11z = 8$

6. Question

Obtain the equation of the plane passing through the point (1, – 3, – 2) and perpendicular to the planes x + 2y + 2z = 5 and 3x + 3y + 2z = 8.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as –

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (1, – 3, – 2), so the equation of the plane is

$a(x - 1) + b(y + 3) + c(z + 2) = 0$

$\Rightarrow ax + by + cz = a - 3b - 2c$ …… (1)

Now, the required plane is also perpendicular to the planes,

$x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (a)

Using (a) we have,

$a + 2b + 2c = 0$ …… (b)

$3a + 3b + 2c = 0$ …… (c)

Solving (b) and (c) we get,

$$\frac{a}{(2)(2)-(3)(2)} = \frac{b}{(3)(2)-(1)(2)} = \frac{c}{(1)(3)-(2)(3)}$$

$$\Rightarrow \frac{a}{4-6} = \frac{b}{6-2} = \frac{c}{3-6}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{4} = \frac{c}{-3} = \lambda(\text{say})$$

$\therefore a = -2\lambda, b = 4\lambda, c = -3\lambda$

Putting values of a,b,c in equation (1) we get,

$(-2\lambda)x + (4\lambda)y + (-3\lambda)z = (-2)\lambda - 3(4\lambda) - 2(-3\lambda)$

$\Rightarrow -2\lambda x + 4\lambda y - 3\lambda z = -2\lambda - 12\lambda + 6\lambda$

$\Rightarrow -2\lambda x + 4\lambda y - 3\lambda z = -8\lambda$

Dividing both sides by $(-\lambda)$ we get

$2x - 4y + 3z = 8$

So, the equation of the required planes is $2x - 4y + 3z = 8$

7. Question

Find the equation of the plane passing through the origin and perpendicular to each of the planes $x + 2y - z = 1$ and $3x - 4y + z = 5$.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as -

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (0,0,0), so the equation of plane is

$a(x - 0) + b(y - 0) + c(z - 0) = 0$

$\Rightarrow ax + by + cz = 0$ …… (1)

Now, the required plane is also perpendicular to the planes,

$x + 2y - z = 1$ and $3x - 4y + z = 5$

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (a)

Using (a) we have,

a + 2b – c = 0 ...... (b)

3a – 4b + c = 0 ...... (c)

Solving (b) and (c) we get,

$$\frac{a}{(2)(1)-(-4)(-1)} = \frac{b}{(3)(-1)-(1)(1)} = \frac{c}{(1)(-4)-(2)(3)}$$

$$\Rightarrow \frac{a}{2-4} = \frac{b}{-3-1} = \frac{c}{-4-6}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{-4} = \frac{c}{-10} = \lambda(\text{say})$$

∴a = – 2λ, b = – 4λ, c = – 10λ

Putting values of a,b,c in equation (1) we get,

(– 2λ)x + (– 4λ)y + (– 10λ)z = 0

Dividing both sides by (– 2λ) we get

x + 2y + 5z = 0

So, the equation of the required planes is x + 2y + 5z = 0

8. Question

Find the equation of the plane passing through the point (1, – 1, 2) and (2, – 2, 2) and which is perpendicular to the plane 6x – 2y + 2z = 9.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as -

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (1, – 1, 2), so the equation of plane is

a(x – 1) + b(y + 1) + c(z – 2) = 0 ...... (i)

Plane (i) is also passing through (2, – 2, 2), so(2, – 2, 2) must satisfy the equation of plane, so we have

a(2 – 1) + b(– 2 + 1) + c(2 – 2) = 0

⇒a – b = 0 ...... (ii)

Plane 6x – 2y + 2z = 9 is perpendicular to the required plane

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...... (a)

Using (a) we have,

a(6) + b(– 2) + c(2) = 0

⇒6a – 2b + 2c = 0 ...... (iii)

Solving (ii) and (iii) we get,

$$\frac{a}{(-1)(2)-(-2)(0)} = \frac{b}{(6)(0)-(1)(2)} = \frac{c}{(1)(-2)-(6)(-1)}$$

$$\Rightarrow \frac{a}{-2-0} = \frac{b}{0-2} = \frac{c}{-2+6}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{-2} = \frac{c}{4} = \lambda(\text{say})$$

$\therefore a = -2\lambda, b = -2\lambda, c = 4\lambda$

Putting values of a,b,c in equation (i) we get,

$(-2\lambda)(x - 1) + (-2\lambda)(y + 1) + (4\lambda)(z - 2) = 0$

$\Rightarrow -2\lambda x + 2\lambda - 2\lambda y - 2\lambda + 4\lambda z - 8\lambda = 0$

$\Rightarrow -2\lambda x - 2\lambda y + 4\lambda z - 8\lambda = 0$

Dividing by -2λ we get,

$x + y - 2z + 4 = 0$

So, the required plane is $x + y - 2z + 4 = 0$

9. Question

Find the equation of the plane passing through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 1$.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as –

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (2,2,1), so the equation of the plane is

$a(x - 2) + b(y - 2) + c(z - 1) = 0$ …… (i)

Plane (i) is also passing through (9,3,6), so(9,3,6) must satisfy the equation of plane, so we have

$a(9 - 2) + b(3 - 2) + c(6 - 1) = 0$

$\Rightarrow 7a + b + 5c = 0$ …… (ii)

Plane $2x + 6y + 6z = 1$ is perpendicular to the required plane

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (a)

Using (a) we have,

$a(2) + b(6) + c(6) = 0$

$\Rightarrow 2a + 6b + 6c = 0$ …… (iii)

Solving (ii) and (iii) we get,

$$\frac{a}{(1)(6)-(5)(6)} = \frac{b}{(2)(5)-(7)(6)} = \frac{c}{(7)(6)-(2)(1)}$$

$$\Rightarrow \frac{a}{6-30} = \frac{b}{10-42} = \frac{c}{42-2}$$

$$\Rightarrow \frac{a}{-24} = \frac{b}{-32} = \frac{c}{-40} = \lambda(\text{say})$$

$\therefore a = -24\lambda, b = -32\lambda, c = -40\lambda$

Putting values of a,b,c in equation (i) we get,

$(-24\lambda)(x - 2) + (-32\lambda)(y - 2) + (-40\lambda)(z - 1) = 0$

$\Rightarrow -24\lambda x + 48\lambda - 32\lambda y + 64\lambda - 40\lambda z + 40\lambda = 0$

$\Rightarrow -24\lambda x - 32\lambda y - 40\lambda z + 152\lambda = 0$

Dividing by -8λ we get,

$3x + 4y + 5z - 19 = 0$

So, the required plane is $3x + 4y + 5z = 19$

10. Question

Find the equation of the plane passing through the points whose coordinates are (– 1, 1, 1) and (1, – 1, 1) and perpendicular to the plane $x + 2y + 2z = 5$.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as -

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (– 1,1,1), so the equation of plane is

$a(x + 1) + b(y - 1) + c(z - 1) = 0$ ...... (i)

Plane (i) is also passing through (1, – 1,1), so(1, – 1,1) must satisfy the equation of plane, so we have

$a(1 + 1) + b(- 1 - 1) + c(1 - 1) = 0$

$\Rightarrow 2a - 2b = 0$ ...... (ii)

Plane $x + 2y + 2z = 5$ is perpendicular to the required plane

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...... (a)

Using (a) we have,

$a(1) + b(2) + c(2) = 0$

$\Rightarrow a + 2b + 2c = 0$ ...... (iii)

Solving (ii) and (iii) we get,

$$\frac{a}{(-2)(2)-(2)(0)} = \frac{b}{(1)(0)-(2)(2)} = \frac{c}{(2)(2)-(1)(-2)}$$

$$\Rightarrow \frac{a}{-4-0} = \frac{b}{0-4} = \frac{c}{4+2}$$

$$\Rightarrow \frac{a}{-4} = \frac{b}{-4} = \frac{c}{6} = \lambda(\text{say})$$

$\therefore a = -4\lambda, b = -4\lambda, c = 6\lambda$

Putting values of a,b,c in equation (i) we get,

$(-4\lambda)(x + 1) + (-4\lambda)(y - 1) + (6\lambda)(z - 1) = 0$

$\Rightarrow -4\lambda x - 4\lambda - 4\lambda y + 4\lambda + 6\lambda z - 6\lambda = 0$

$\Rightarrow -4\lambda x - 4\lambda y + 6\lambda z - 6\lambda = 0$

Dividing by -2λ we get,

$2x + 2y - 3z + 3 = 0$

So, the required plane is $2x + 2y - 3z + 3 = 0$

11. Question

Find the equation of the plane with intercept 3 on the y - axis and parallel to ZOX plane.

Answer

We know that the equation of ZOX plane is $y = 0$ so a plane parallel to plane ZOX will have the equation y = constant

Now, it is given that the plane makes an intercept of 3 on y - axis so the value of constant is equal to 3.

Therefore, the equation of the required plane is y = 3.

12. Question

Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to each of the planes 2x + 3y–2z = 5and x + 2y - 3z = 8.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as -

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (1, – 1,2), so the equation of plane is

$a(x - 1) + b(y + 1) + c(z - 2) = 0$

$\Rightarrow ax + by + cz = a - b + 2c$ ...... (1)

Now, the required plane is also perpendicular to the planes,

2x + 3y – 2z = 5 and x + 2y – 3z = 8

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...... (a)

Using (a) we have,

2a + 3b – 2c = 0 ...... (b)

a + 2b – 3c = 0 ...... (c)

Solving (b) and (c) we get,

$$\frac{a}{(3)(-3)-(2)(-2)} = \frac{b}{(1)(-2)-(2)(-3)} = \frac{c}{(2)(2)-(1)(3)}$$

$$\Rightarrow \frac{a}{-9+4} = \frac{b}{-2+6} = \frac{c}{4-3}$$

$$\Rightarrow \frac{a}{-5} = \frac{b}{4} = \frac{c}{1} = \lambda(\text{say})$$

$\therefore a = -5\lambda,\ b = 4\lambda,\ c = \lambda$

Putting values of a,b,c in equation (1) we get,

$(-5\lambda)x + (4\lambda)y + (\lambda)z = -5\lambda - 4\lambda + 2\lambda$

$\Rightarrow -5\lambda x + 4\lambda y + \lambda z = -7\lambda$

Dividing both sides by (– λ) we get

5x – 4y – z = 7

So, the equation of the required planes is 5x - 4y – z = 7

13. Question

Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 2$.

Answer

The required plane is parallel to the plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2$

Any plane parallel to $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2$ is given as $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = k$

Further, it is given that the plane is passing through (a,b,c). So, point (a,b,c) should satisfy the equation of the plane,

∴ we have

$$(a\hat{i} + b\hat{j} + c\hat{k}).(\hat{i} + \hat{j} + \hat{k}) = k$$

$\Rightarrow a + b + c = k$

Hence, the equation of the required plane is

$$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = a + b + c. \text{ or } x + y = z = a + b + c$$

14. Question

Find the equation of the plane passing through the point (– 1, 3, 2) and perpendicular to each of planes x + 2y + 3x = 5 and 3x + 3y + z = 0.

Answer

We know that solution of a plane passing through (x_1,y_1,z_1) is given as -

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

The required plane passes through (– 1,3,2), so the equation of plane is

a(x + 1) + b(y – 3) + c(z – 2) = 0

⇒ ax + by + cz = 3b + 2c – a …… (1)

Now, the required plane is also perpendicular to the planes,

x + 2y + 3z = 5 and 3x + 3y + z = 0

We know that planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles if,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (a)

Using (a) we have,

a + 2b + 3c = 0 …… (b)

3a + 3b + c = 0 …… (c)

Solving (b) and (c) we get,

$$\frac{a}{(2)(1)-(3)(3)} = \frac{b}{(3)(3)-(1)(1)} = \frac{c}{(1)(3)-(3)(2)}$$

$$\Rightarrow \frac{a}{2-9} = \frac{b}{9-1} = \frac{c}{3-6}$$

$$\Rightarrow \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = \lambda(\text{say})$$

∴a = – 7λ, b = 8λ, c = – 3λ

Putting values of a,b,c in equation (1) we get,

(– 7λ)x + (8λ)y + (– 3λ)z = 3(8λ) + 2(– 3λ) + 7λ

⇒ – 7λx + 8λy – 3λz = 24λ – 6λ + 7λ

⇒ – 7λx + 8λy – 3λz = 25λ

Dividing both sides by (– λ) we get

7x – 8y + 3z – 25 = 0

So, the equation of required planes is 7x – 8y + 3z – 25 = 0

15. Question

Find the vector equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4) and perpendicular to the plane

x – 2y + 4z = 10.

Answer

Vector equation of a plane is given as $(\vec{r}-\vec{a}).\vec{n} = 0$

Where $\vec{a}$ is any point on the plane and $\vec{n}$ is a vector perpendicular to the plane.

Now, the given plane x – 2y + 4z = 10 is perpendicular to required plane. So, the normal vector of x – 2y + 4z = 10 will be parallel to the required plane. Hence, $\hat{\imath}-2\hat{\jmath}+4\hat{k}$ is parallel to the required plane.

Points say A(2,1, – 1) and B(– 1,3,4) are on the plane hence the vector $\overrightarrow{AB}$ is also parallel to the required plane so,

$(-1-2)\hat{\imath}+(3-1)\hat{\jmath}+\left(4-(-1)\right)\hat{k} = -3\hat{\imath}+2\hat{\jmath}+5\hat{k}$ is parallel to the required plane.

Hence as both $\hat{\imath}-2\hat{\jmath}+4\hat{k}$ and $-3\hat{\imath}+2\hat{\jmath}+5\hat{k}$ are parallel to the plane so the direction of $\vec{n}$ is the cross product of the two vectors.

$\therefore \vec{n} = \left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)\times\left(-3\hat{\imath}+2\hat{\jmath}+5\hat{k}\right) = -18\hat{\imath}-17\hat{\jmath}-4\hat{k}$

So, the equation of required plane is,

$\left(\vec{r}-\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right)\right).\left(-18\hat{\imath}-17\hat{\jmath}-4\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(-18\hat{\imath}-17\hat{\jmath}-4\hat{k}\right)-\left(\hat{\imath}-2\hat{\jmath}+4\hat{k}\right).\left(-18\hat{\imath}-17\hat{\jmath}-4\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(-18\hat{\imath}-17\hat{\jmath}-4\hat{k}\right)-(-18+34-16) = 0$

$\Rightarrow \vec{r}.\left(-18\hat{\imath}-17\hat{\jmath}-4\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(18\hat{\imath}+17\hat{\jmath}+4\hat{k}\right) = 0$

Hence, the vector equation of required plane is

$\vec{r}.\left(18\hat{\imath}+17\hat{\jmath}+4\hat{k}\right) = 0.$

Exercise 29.7

1 A. Question

Find the vector equation of the following planes in scalar product form

$\vec{r}=\left(2\hat{i}-\hat{k}\right)+\lambda\hat{i}+\mu\left(\hat{i}-2\hat{j}-\hat{k}\right)$

Answer

Here, $\vec{r} = \left(2\hat{\imath}-\hat{k}\right)+\lambda\hat{\imath}+\mu(\hat{\imath}-2\hat{\jmath}-\hat{k})$

We know that $\vec{r} = \vec{a}+\lambda\vec{b}+\mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = 2\hat{\imath}-\hat{k}, \vec{b} = \hat{\imath}, \vec{c} = \hat{\imath}-2\hat{\jmath}-\hat{k}.$

Now, the plane is perpendicular to $\vec{b}\times\vec{c}$,

Hence $\vec{n} = \vec{b}\times\vec{c} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & 0 & 0\\ 1 & -2 & -1\end{vmatrix}$

$\Rightarrow \vec{n} = \hat{\imath}(0-0)-\hat{\jmath}(-1-0)+\hat{k}(-2-0)$

$\Rightarrow \vec{n} = \hat{\jmath}-2\hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ … … (a)

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.(\hat{j} - 2\hat{k}) = (2\hat{i} - \hat{k}).(\hat{j} - 2\hat{k})$

$\Rightarrow \vec{r}.(\hat{j} - 2\hat{k}) = (2)(0) + (0)(1) + (-1)(-2)$

$\Rightarrow \vec{r}.(\hat{j} - 2\hat{k}) = 2$

Hence, the required equation is $\vec{r}.(\hat{j} - 2\hat{k}) = 2$.

1 B. Question

Find the vector equation of the following planes in scalar product form

$$\vec{r} = (1+s-t)\hat{i} + (2-s)\hat{j} + (3-2s+2t)\hat{k}$$

Answer

We have, $\vec{r} = (1 + s - t)\hat{i} + (2 - s)\hat{j} + (3 - 2s + 2t)\hat{k}$

$\Rightarrow \vec{r} = \hat{i} + s\hat{i} - t\hat{i} + 2\hat{j} - s\hat{j} + 3\hat{k} - 2s\hat{k} + 2t\hat{k}$

$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + s\hat{i} - s\hat{j} - 2s\hat{k} - t\hat{i} + 2t\hat{k}$

$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + (\hat{i} - \hat{j} - 2\hat{k})s + (-\hat{i} + 2\hat{k})t$

Now, we know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} - \hat{j} - 2\hat{k}, \vec{c} = -\hat{i} + 2\hat{k}$.

Now, the plane is perpendicular to $\vec{b} \times \vec{c}$,

Hence $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 0 & 2 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}(-2 - 0) - \hat{j}(2 - 2) + \hat{k}(0 - 1)$

$\Rightarrow \vec{n} = -2\hat{i} - \hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ … … (a)

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.(-2\hat{i} - \hat{k}) = (\hat{i} + 2\hat{j} + 3\hat{k}).(-2\hat{i} - \hat{k})$

$\Rightarrow \vec{r}.(-2\hat{i} - \hat{k}) = (1)(-2) + (2)(0) + (3)(-1)$

$\Rightarrow \vec{r}.(-2\hat{i} - \hat{k}) = -5$

$\Rightarrow \vec{r}.(2\hat{i} + \hat{k}) = 5$

Hence, the required equation is $\vec{r}.(2\hat{i} + \hat{k}) = 5$.

1 C. Question

Find the vector equation of the following planes in scalar product form

$$\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$$

Answer

Here, $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$

We know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = \hat{i} + \hat{j}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}, \vec{c} = -\hat{i} + \hat{j} - 2\hat{k}$.

Now, the plane is perpendicular to $\vec{b} \times \vec{c}$,

Hence $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}(-4 + 1) - \hat{j}(-2 - 1) + \hat{k}(1 + 2)$

$\Rightarrow \vec{n} = -3\hat{i} + 3\hat{j} + 3\hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ ......(a)

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.\left(-3\hat{i} + 3\hat{j} + 3\hat{k}\right) = (\hat{i} + \hat{j}).\left(-3\hat{i} + 3\hat{j} + 3\hat{k}\right)$

$\Rightarrow \vec{r}.\left(-3\hat{i} + 3\hat{j} + 3\hat{k}\right) = (1)(-3) + (1)(3) + (0)(3)$

$\Rightarrow \vec{r}.\left(-3\hat{i} + 3\hat{j} + 3\hat{k}\right) = 0$

$\Rightarrow \vec{r}.\left(\hat{i} - \hat{j} - \hat{k}\right) = 0$

Hence, the required equation is $\vec{r}.\left(\hat{i} - \hat{j} - \hat{k}\right) = 0$.

1 D. Question

Find the vector equation of the following planes in scalar product form

$$\vec{r} = \hat{i} - \hat{j} + \lambda\left(\hat{i} + \hat{j} + \hat{k}\right) + \mu\left(4\hat{i} - 2\hat{j} + 3\hat{k}\right)$$

Answer

Here, $\vec{r} = (\hat{i} - \hat{j}) + \lambda\left(\hat{i} + \hat{j} + \hat{k}\right) + \mu\left(4\hat{i} - 2\hat{j} + 3\hat{k}\right)$

We know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = \hat{i} - \hat{j}, \vec{b} = \hat{i} + \hat{j} + \hat{k}, \vec{c} = 4\hat{i} - 2\hat{j} + 3\hat{k}$.

Now, the plane is perpendicular to $\vec{b} \times \vec{c}$,

Hence $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 4 & -2 & 3 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}\left(3 - (-2)\right) - \hat{j}(3 - 4) + \hat{k}(-2 - 4)$

$\Rightarrow \vec{n} = 5\hat{i} + \hat{j} - 6\hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ ......(a)

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.\left(5\hat{i} + \hat{j} - 6\hat{k}\right) = (\hat{i} - \hat{j}).\left(5\hat{i} + \hat{j} - 6\hat{k}\right)$

$\Rightarrow \vec{r}.\left(5\hat{i} + \hat{j} - 6\hat{k}\right) = (1)(5) + (-1)(1) + (0)(-6)$

$\Rightarrow \vec{r}.\left(5\hat{i} + \hat{j} - 6\hat{k}\right) = 4$

Hence, the required equation is $\vec{r}.\left(5\hat{i} + \hat{j} - 6\hat{k}\right) = 4$

2 A. Question

Find the Cartesian form of the equation of the following planes :

$$\vec{r} = \left(\hat{i} - \hat{j}\right) + s\left(-\hat{i} + \hat{j} + 2\hat{k}\right) + t\left(\hat{i} + 2\hat{j} + \hat{k}\right)$$

Answer

(i) We have, $\vec{r} = (\hat{i} - \hat{j}) + \left(-\hat{i} + \hat{j} + 2\hat{k}\right)s + \left(\hat{i} + 2\hat{j} + \hat{k}\right)t$

Now, we know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + \hat{j} + 2\hat{k}, \vec{c} = \hat{i} + 2\hat{j} + \hat{k}$.

Now, the plane is perpendicular to $\vec{b} \times \vec{c}$,

Hence $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}(1 - 4) - \hat{j}(-1 - 2) + \hat{k}(-2 - 1)$

$\Rightarrow \vec{n} = -3\hat{i} + 3\hat{j} - 3\hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ ......(a)

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.\left(-3\hat{i} + 3\hat{j} - 3\hat{k}\right) = (\hat{i} - \hat{j}).\left(-3\hat{i} + 3\hat{j} - 3\hat{k}\right)$

$\Rightarrow \vec{r}.\left(-3\hat{i} + 3\hat{j} - 3\hat{k}\right) = (1)(-3) + (-1)(3) + (0)(-3)$

$\Rightarrow \vec{r}.\left(-3\hat{i} + 3\hat{j} - 3\hat{k}\right) = -6$

$\Rightarrow \vec{r}.\left(\hat{i} - \hat{j} + \hat{k}\right) = 2$

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore$ We have, $\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\hat{i} - \hat{j} + \hat{k}\right) = 2$

$\Rightarrow x - y + z = 2$

Hence, the required equation is $x - y + z = 2$

2 B. Question

Find the Cartesian form of the equation of the following planes :

$$\vec{r} = (1 + s + t)\hat{i} + (2 - s + t)\hat{j} + (3 - 2s + 2t)\hat{k}$$

Answer

We have, $\vec{r} = (1 + s + t)\hat{i} + (2 - s + t)\hat{j} + (3 - 2s + 2t)\hat{k}$

$\Rightarrow \vec{r} = \hat{i} + s\hat{i} + t\hat{i} + 2\hat{j} - s\hat{j} + t\hat{j} + 3\hat{k} - 2s\hat{k} + 2t\hat{k}$

$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + s\hat{i} - s\hat{j} - 2s\hat{k} + t\hat{i} + t\hat{j} + 2t\hat{k}$

$\Rightarrow \vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \left(\hat{i} - \hat{j} - 2\hat{k}\right)s + \left(\hat{i} + \hat{j} + 2\hat{k}\right)t$

Now, we know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = \hat{i} - \hat{j} - 2\hat{k}, \vec{c} = \hat{i} + \hat{j} + 2\hat{k}$.

Now, the plane is perpendicular to $\vec{b} \times \vec{c}$,

Hence $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ 1 & 1 & 2 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}(-2 + 2) - \hat{j}(2 + 2) + \hat{k}(1 + 1)$

$\Rightarrow \vec{n} = -4\hat{j} + 2\hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n} \ldots \ldots (a)$

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.\left(-4\hat{j} + 2\hat{k}\right) = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(-4\hat{j} + 2\hat{k}\right)$

$\Rightarrow \vec{r}.\left(-4\hat{j} + 2\hat{k}\right) = (1)(0) + (2)(-4) + (3)(2)$

$\Rightarrow \vec{r}.\left(-4\hat{j} + 2\hat{k}\right) = -2$

$\Rightarrow \vec{r}.\left(2\hat{j} - \hat{k}\right) = 1$

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

∴ We have, $\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{j} - \hat{k}\right) = 1$

$\Rightarrow 2y - z = 1$

Hence, the required equation of plane is $2y - z = 1$

3 A. Question

Find the vector equation of the following planes in non – parametric form :

$$\vec{r} = (\lambda - 2\mu)\hat{i} + (3 - \mu)\hat{j} + (2\lambda + \mu)\hat{k}$$

Answer

Given equation of plane is $\vec{r} = (\lambda - 2\mu)\hat{i} + (3 - \mu)\hat{j} + (2\lambda + \mu)\hat{k}$

$\Rightarrow \vec{r} = \lambda\hat{i} - 2\mu\hat{i} + 3\hat{j} - \mu\hat{j} + 2\lambda\hat{k} + \mu\hat{k}$

$\Rightarrow \vec{r} = 3\hat{j} + \lambda\hat{i} + 2\lambda\hat{k} - 2\mu\hat{i} - \mu\hat{j} + \mu\hat{k}$

$\Rightarrow \vec{r} = 3\hat{j} + \lambda\left(\hat{i} + 2\hat{k}\right) + \left(-2\hat{i} - \hat{j} + \hat{k}\right)\mu$

Now, $\vec{r} = 3\hat{j} + \lambda\left(\hat{i} + 2\hat{k}\right) + \left(-2\hat{i} - \hat{j} + \hat{k}\right)\mu$

We know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = 3\hat{j}, \vec{b} = \hat{i} + 2\hat{k}, \vec{c} = -2\hat{i} - \hat{j} + \hat{k}$.

Now, the plane is perpendicular to $\vec{b} \times \vec{c}$,

Hence $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -2 & -1 & 1 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}\left(0 - (-2)\right) - \hat{j}\left(1 - (-4)\right) + \hat{k}(-1 - 0)$

$\Rightarrow \vec{n} = 2\hat{i} - 5\hat{j} - \hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n} \ldots \ldots (a)$

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.\left(2\hat{i}-5\hat{j}-\hat{k}\right) = (3\hat{j}).\left(2\hat{i}-5\hat{j}-\hat{k}\right)$

$\Rightarrow \vec{r}.\left(2\hat{i}-5\hat{j}-\hat{k}\right) = (0)(2) + (3)(-5) + (0)(-1)$

$\Rightarrow \vec{r}.\left(2\hat{i}-5\hat{j}-\hat{k}\right) = -15$

Hence, the required equation is $\vec{r}.\left(2\hat{i}-5\hat{j}-\hat{k}\right) + 15 = 0$

3 B. Question

Find the vector equation of the following planes in non - parametric form :

$$\vec{r}=\left(2\hat{i}+2\hat{j}-\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\mu\left(5\hat{i}-2\hat{j}+7\hat{k}\right)$$

Answer

Here, $\vec{r} = \left(2\hat{i} + 2\hat{j}-\hat{k}\right) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(5\hat{i}-2\hat{j} + 7\hat{k})$

We know that $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to the vectors $\vec{b}$ and $\vec{c}$.

Clearly, $\vec{a} = 2\hat{i} + 2\hat{j}-\hat{k}, \vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{c} = 5\hat{i}-2\hat{j} + 7\hat{k}$.

Now, the plane is perpendicular to $\vec{b}\times\vec{c}$,

Hence $\vec{n} = \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}$

$\Rightarrow \vec{n} = \hat{i}(14 + 6)-\hat{j}(7-15) + \hat{k}(-2-10)$

$\Rightarrow \vec{n} = 20\hat{i} + 8\hat{j}-12\hat{k}$

We know that vector equation of a plane in scalar product form is given as $\vec{r}.\vec{n} = \vec{a}.\vec{n}$ ... ... (a)

Put $\vec{a}$ and $\vec{n}$ in equation (a) we get,

$\vec{r}.\left(20\hat{i} + 8\hat{j}-12\hat{k}\right) = \left(2\hat{i} + 2\hat{j}-\hat{k}\right).\left(20\hat{i} + 8\hat{j}-12\hat{k}\right)$

$\Rightarrow \vec{r}.\left(20\hat{i} + 8\hat{j}-12\hat{k}\right) = (2)(20) + (2)(8) + (-1)(-12)$

$\Rightarrow \vec{r}.\left(20\hat{i} + 8\hat{j}-12\hat{k}\right) = 40 + 16 + 12$

$\Rightarrow \vec{r}.\left(20\hat{i} + 8\hat{j}-12\hat{k}\right) = 68$

$\Rightarrow \vec{r}.\left(5\hat{i} + 2\hat{j}-3\hat{k}\right) = 17$

Hence, the required equation is $\vec{r}.\left(5\hat{i} + 2\hat{j}-3\hat{k}\right) = 17$.

Exercise 29.8

1. Question

Find the equation of plane parallel to 2x – 3y + z = 0 and passing through the point (1, – 1,2)?

Answer

Given Eq. of plane is 2x – 3y + z = 0 ...... (1)

We know that equation of a plane parallel to given plane (1) is

2x – 3y + z + k = 0 ...... (2)

As given that , plane (2) is passing through the point (1, – 1,2) so it satisfy the plane (2),

$2(1) - 3(-1) + (2) + k = 0$

$2 + 3 + 2 + k = 0$

$7 + k = 0$

$k = -7$

put the value of k in equation (2),

$2x - 3y + z - 7 = 0$

So, equation of the required plane is , $2x - 3y + z = 7$

2. Question

Find the equation of the plane through (3, 4, –1) which is parallel to the plane $\vec{r} \cdot \left(2\hat{i} - 3\hat{j} + 5\hat{k}\right) + 2 = 0$.

Answer

given equation of the plane is

$\vec{r}.\left(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) + 2 = 0$ ...... (1)

We know that the equation of a plane parallel to given plane (1) is

$\vec{r}.\left(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) + k = 0$ ...... (2)

As given that, plane (2) is passing through the point $3\hat{\imath} + 4\hat{\jmath} - \hat{k}$ so it satisfy the equation (2),

$3\hat{\imath} + 4\hat{\jmath} - \hat{k}\left(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) + k = 0$

$(3)(2) + (4)(-3) + (-1)(5) + k = 0$

$6 - 12 - 5 + k = 0$

$k = 11$

put the value of k in equation (2),

$\vec{r}.\left(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) + 11 = 0$

So, the equation of the required plane is, $\vec{r}.\left(2\hat{\imath} - 3\hat{\jmath} + 5\hat{k}\right) + 11 = 0$

3. Question

Find the equation of the plane through the intersection of the planes $2x - 7y + 4z - 3 = 0$ and $3x - 5y + 4z + 11 = 0$ and the point (– 2,1,3)?

Answer

we know that, equation of a plane passing through the line of intersection of two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

Given , equation of plane is,

$2x - 7y + 4z - 3 = 0$ and $3x - 5y + 4z + 11 = 0$

So equation of plane passing through the line of intersection of given two planes is

$(2x - 7y + 4z - 3) + k(3x - 5y + 4z + 11) = 0$

$2x - 7y + 4z - 3 + 3kx - 5ky + 4kz + 11k = 0$

$x(2 + 3k) + y(-7 - 5k) + z(4 + 4k) - 3 + 11k = 0$ ......(1)

As given that, plane (1) is passing through the point (– 2,1,3) so it satisfy the equation (1),

(– 2)(2 + 3k) + (1)(– 7 – 5k) + (3)(4 + 4k) – 3 + 11k = 0

– 2 + 12k = 0

12k = 2

$k = \frac{2}{12}$

$k = \frac{1}{6}$

put the value of k in equation (1)

x(2 + 3k) + y(– 7 – 5k) + z(4 + 4k) – 3 + 11k = 0

$x(2 + \frac{3}{6}) + y(-7 - \frac{5}{6}) + z(4 + \frac{4}{6}) - 3 + \frac{11}{6} = 0$

$x(\frac{12+3}{6}) + y(\frac{-42-5}{6}) + z(\frac{24+4}{6}) - \frac{18+11}{6} = 0$

$x(\frac{15}{6}) + y(\frac{-47}{6}) + z(\frac{28}{6}) - \frac{7}{6} = 0$

multiplying by 6 , we get

15x – 47y + 28z – 7 = 0

Therefore , equation of required plane is 15x – 47y + 28z – 7 = 0

4. Question

Find the equation of the plane through the point $2\hat{i} + \hat{j} - \hat{k}$ and passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 0$ and $\vec{r} \cdot (\hat{j} + 2\hat{k}) = 0.$

Answer

we know that, the equation of a plane passing through the line of intersection of two planes

$\vec{r}.\overrightarrow{n_1} = d_1$ and $\vec{r}.\overrightarrow{n_2} = d_2$

is given by $\vec{r}.(\overrightarrow{n_1} + k\overrightarrow{n_2}) = d_1 + kd_2$

So the equation of the plane passing through the line of intersection of given two planes

$\vec{r}.(\hat{i} + 3\hat{j} - \hat{k}) = 0$ and $\vec{r}.(\hat{j} + 2\hat{k}) = 0$ is given by

$\vec{r}.(\hat{i} + 3\hat{j} - \hat{k} + k(\hat{j} + 2\hat{k})) = 0$ ...... (1)

As given that, plane (1) is passing through the point $2\hat{i} + \hat{j} - \hat{k}$ so

$(2\hat{i} + \hat{j} - \hat{k})(\hat{i} + 3\hat{j} - \hat{k}) + k(2\hat{i} + \hat{j} - \hat{k})(\hat{j} + 2\hat{k}) = 0$

(2)(1) + (1)(3) + (– 1)(– 1) + k[(2)(0) + (1)(1) + (– 1)(2)] = 0

(2 + 3 + 1) + k(1 – 2) = 0

6 – k = 0

k = 6

put the value of k in equation (1)

$\vec{r}.(\hat{i} + 3\hat{j} - \hat{k} + k(\hat{j} + 2\hat{k})) = 0$

$\vec{r}.\left(\hat{i} + 3\hat{j} - \hat{k} + 6(\hat{j} + 2\hat{k})\right) = 0$

$\vec{r}.\left(\hat{i} + 3\hat{j} - \hat{k} + 6\hat{j} + 12\hat{k}\right) = 0$

$\vec{r}.\left(\hat{i} + 9\hat{j} + 11\hat{k}\right) = 0$

So, the equation of the required plane is $\vec{r}.\left(\hat{i} + 9\hat{j} + 11\hat{k}\right) = 0$

5. Question

Find the equation of the plane through the line of intersection of the planes 2x – y = 0 and 3z – y = 0 which is perpendicular to the plane 4x + 5y – 3z = 8 ?

Answer

we know that, equation of a plane passing through the line of intersection of two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of given two planes

2x – y = 0 and 3z – y = 0 is

(2x – y) + k(3z – y) = 0

2x – y + 3kz – ky = 0

x(2) + y(– 1 – k) + z(3k) = 0 …… (1)

we know that, two planes are perpendicular if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (2)

given, plane (1) is perpendicular to plane

4x + 5y – 3z = 8 …… (3)

Using (1) and (3) in equation (2)

(2)(4) + (– 1 – k)(5) + (3k)(– 3) = 0

8 – 5 – 5k – 9k = 0

3 – 14k = 0

– 14k = – 3

$k = \frac{3}{14}$

put the value of k in equation (1)

x(2) + y(– 1 – k) + z(3k) = 0

$x(2) + y(-1 - \frac{3}{14}) + z(3(\frac{3}{14})) = 0$

$x(2) + y(\frac{-14-3}{14}) + z(\frac{9}{14})) = 0$

$x(2) + y(\frac{-17}{14}) + z(\frac{9}{14})) = 0$

multiplying with 14 we get

28x – 17y + 9z = 0

Equation of required plane is, 28x – 17y + 9z = 0

6. Question

Find the equation of the plane through the line of intersection of the planes x + 2y + 3z – 4 = 0 and 2x + y – z + 5 = 0 which is perpendicular to the plane 5x + 3y – 6z + 8 = 0?

Answer

we know that, equation of a plane passing through the line of intersection of two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of given two planes

x + 2y + 3z – 4 = 0 and 2x + y – z + 5 = 0 is given by,

(x + 2y + 3z – 4) + k(2x + y – z + 5) = 0

x + 2y + 3z – 4 + 2kx + ky – kz + k5 = 0

x(1 + 2k) + y(2 + k) + z(3 – k) – 4 + 5k = 0 ...... (1)

we know that, two planes are perpendicular if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...... (2)

given, plane (1) is perpendicular to plane,

5x + 3y – 6z + 8 = 0 ...... (3)

Using (1) and (3) in equation (2)

(5)(1 + 2k) + (3)(2 + k) + (– 6)(3 – k) = 0

5 + 10k + 6 + 3k – 18 + 6k = 0

– 7 + 19k = 0

$$k = \frac{7}{19}$$

put the value of k in equation (1)

x(1 + 2k) + y(2 + k) + z(3 – k) – 4 + 5k = 0

$x(1 + \frac{14}{19}) + y(2 + \frac{7}{19}) + z(3 - \frac{7}{19}) - 4 + \frac{35}{19} = 0$

$x(\frac{19+14}{19}) + y(\frac{38+7}{19}) + z(\frac{57-7}{19}) + \frac{-76+35}{19} = 0$

$x(\frac{33}{19}) + y(\frac{45}{19}) + z(\frac{50}{19}) - \frac{41}{19} = 0$

multiplying with 19 we get

33x + 45y + 50z – 41 = 0

Equation of required plane is, 33x + 45y + 50z – 41 = 0

7. Question

Find the equation of the plane through the line of intersection of the planes x + 2y + 3z + 4 = 0 and x – y + z + 3 = 0 and passing through the origin ?

Answer

we know that, equation of a plane passing through the line of intersection of two planes

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of given two planes x + 2y + 3z + 4 = 0 and x – y + z + 3 = 0 is

(x + 2y + 3z + 4) + k(x – y + z + 3) = 0

x(1 + k) + y(2 – k) + z(3 + k) + 4 + 3k = 0 ...... (1)

Equation (1) is passing through origin , so

(0)(1 + k) + (0)(2 – k) + (0)(3 + k) + 4 + 3k = 0

0 + 0 + 0 + 4 + 3k = 0

3k = – 4

$$k = -\frac{4}{3}$$

Put the value of k in equation (1),

x(1 + k) + y(2 – k) + z(3 + k) + 4 + 3k = 0

$x(1-\frac{4}{3}) + y(2 + \frac{4}{3}) + z(3-\frac{4}{3}) + 4-\frac{12}{3} = 0$

$x(\frac{3-4}{3}) + y(\frac{6+4}{3}) + z(\frac{9-4}{3}) + 4-4 = 0$

$$-\frac{x}{3} + \frac{10y}{3} + \frac{5z}{3} = 0$$

Multiplying by 3, we get

– x + 10y + 5z = 0

x – 10y – 5z = 0

the equation of required plane is, x – 10y – 5z = 0

8. Question

Find the equation of the plane through the line of intersection of the planes x – 3y + 2z – 5 = 0 and 2x – y + 3z – 1 = 0 and passing through (1, – 2, 3)?

Answer

We know that equation of plane passing through the line of intersection of planes

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of planes

x – 3y + 2z – 5 = 0 and 2x – y + 3z – 1 = 0 is given by

(x – 3y + 2z – 5) + k(2x – y + 3z – 1) = 0

x(1 – 2k) + y(– 3 – k) + z(2 + 3k) – 5 – k = 0 ...... (1)

plane (1) is passing the through the point(1, – 2, 3) so,

1(1 + 2k) + (– 2)(– 3 – k) + (3)(2 + 3k) – 5 – k = 0

1 + 2k + 6 + 2k + 6 + 9k – 5 – k = 0

8 + 12k = 0

12k = – 8

$$k = -\frac{8}{12}$$

$$k = -\frac{2}{3}$$

Put value of k in eq.(1),

x(1 + 2k) + y(– 3 – k) + z(2 + 3k) – 5 – k = 0

$$x\left(1-\frac{4}{3}\right)+y\left(-3+\frac{2}{3}\right)+z\left(2-\frac{6}{3}\right)-\frac{15+2}{3}=0$$

$$-\frac{1}{3}x-\frac{7}{3}y-\frac{13}{3}=0$$

Multiplying by(– 13),

x + 7y + 13 = 0

$$(x\hat{i}+y\hat{j}+z\hat{k})(\hat{i}+7\hat{j})+13=0$$

$$\vec{r}(\hat{i}+7\hat{j})+13=0$$

Equation of required plane is, $\vec{r}(\hat{i}+7\hat{j})+13=0$

9. Question

Find the equation of the plane through the line of intersection of the planes x – 3y + 2z – 5 = 0 and 2x – y + 3z – 1 = 0 which is perpendicular to the plane 5x + 3y – 6z + 8 = 0 ?

Answer

We know that equation of plane passing through the line of intersection of planes

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of planes

x – 3y + 2z – 5 = 0 and 2x – y + 3z – 1 = 0 is given by

(x – 3y + 2z – 5) + k(2x – y + 3z – 1) = 0

x(1 – 2k) + y(– 3 – k) + z(2 + 3k) – 5 – k = 0 ...... (1)

Given that plane (1) is perpendicular if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(2)

We know that two planes are perpendicular to plane,

5x + 3y + 6z + 8 = 0 ......(3)

Using (1) and (3) in equation (2),

5(1 + 2k) + 3(2 + k) + 6(3 – k) = 0

5 + 10k + 6 + 3k + 18 – 6k = 0

29 + 7k = 0

7k = – 29

$$k=-\frac{29}{7}$$

Put the value of k in equation (1),

$$x(1+2k)+y(2+k)+z(3-k)-4+5k=0$$

$$x\left(1-\frac{58}{7}\right)+y\left(2-\frac{29}{7}\right)+z\left(3+\frac{39}{7}\right)-4-\frac{145}{7}=0$$

$$x\left(-\frac{51}{7}\right)+y\left(-\frac{15}{7}\right)+z\left(\frac{50}{7}\right)-\frac{173}{7}=0$$

51 x + 15 y – 50 z + 173 = 0

10. Question

Find the equation of the plane through the line of intersection of the planes $\vec{r}.\left(\hat{i}+3\hat{j}\right)+6=0$ and $\vec{r}.\left(3\hat{i}-\hat{j}-4\hat{k}\right)=0$, which is at a unit distance from the origin?

Answer

$\vec{r}.(\hat{i} + 3\hat{j}) + 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$

x + 3y + 6 = 0 and 3x – y – 4z = 0

x + 3y + 6 + k(3x – y – 4z) = 0

x(1 + 3k) + y(3 – k) – 4zk + 6 = 0

Distance from origin to plane = $\left|\frac{6}{\sqrt{(1+3k)^2+(3-k)^2+(4k)^2}}\frac{1}{1}\right| = 1$

$36 = (1 + 3k)^2 + (3 – k)^2 + (4k)^2$

$36 = 1 + 6k + 9k^2 + 9 – 6k + k^2 + 16k^2$

$26 = 26k^2$

$k^2 = 1$

$k = \pm 1$

case :1 k = 1

x + 3y + 6 + 1(3x – y – 4z) = 0

4x + 2y – 4z + 6 = 0

Case :2 k = – 1

x + 3y + 6 – 1(3x – y – 4z) = 0

2x – 4y – 4z – 6 = 0

11. Question

Find the equation of the plane through the line of intersection of the planes 2x + 3y – z + 1 = 0 and x + y – 2z + 3 = 0 which is perpendicular to the plane 3x – y – 2z – 4 = 0 ?

Answer

We know that equation of plane passing through the line of intersection of planes

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of planes

2x + 3y – z + 1 = 0 and x + y – 2z + 3 = 0 is

(2x + 3y – z + 1) + k(x + y – 2z + 3) = 0

x(2 + k) + y(3 + k) + z(– 1 – 2k) + 1 + 3k = 0……(1)

Given that plane (1) is perpendicular if

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ …… (2)

We know that two planes are perpendicular to plane,

3x – y – 2z – 4 = 0 …… (3)

Using (1) and (3) in eq. (2),

3(2 + k) + (– 1)(3 + k) + (– 2)(– 1 – 2k) = 0

6 + 3k – 3 – k + 2 + 4k = 0

6k + 5 = 0

6k = – 5

$k = -\frac{5}{6}$

Put the value of k in equation (1),

x(2 + k) + y(3 + k) + z(– 1 – 2k) + 1 + 3k = 0

$x\left(2 - \frac{5}{6}\right) + y\left(3 - \frac{5}{6}\right) + z\left(-1 + \frac{10}{6}\right) + 1 - \frac{15}{6} = 0$

$\frac{7x}{6} + \frac{13y}{6} + \frac{4z}{6} - \frac{9}{6} = 0$

7 x + 13 y + 4 z - 9 = 0

12. Question

Find the equation of the plane through the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0$ and $\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$?

Answer

We know that, the equation of a plane through the line of intersection of the planes

$\vec{r}.\overrightarrow{n_1} - d_1 = 0$ and $\vec{r}.\overrightarrow{n_2} = d_2 = 0$

is given by $\vec{r}.(\overrightarrow{n_1} + k\overrightarrow{n_2}) - d_1 + kd_2 = 0$

So, equation of the plane passing through the line of intersection of the plane $\vec{r}.(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) - 4 = 0$

and $\vec{r}.(2\hat{\imath} + \hat{\jmath} - \hat{k}) + 5 = 0$ is given by

$[\vec{r}.\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right) - 4]$ + k$[\vec{r}.(2\hat{\imath} + \hat{\jmath} - \hat{k}) + 5]$ = 0

$\vec{r}$.$[\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$ + k$[(2\hat{\imath} + \hat{\jmath} - \hat{k})]$ – 4 + 5k = 0 …… (1)

We know that two planes perpendicular if

$\overrightarrow{n1}\cdot\overrightarrow{n2} = 0$

Given that plane (1) is perpendicular to the plane

$\vec{r}.\left(5\hat{\imath} + 3\hat{\jmath} - 6\hat{k}\right) + 8 = 0$

Using (1)and (3) in equation (2),

$[\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$ + k$(2\hat{\imath} + \hat{\jmath} - \hat{k})](5\hat{\imath} + 3\hat{\jmath} - 6\hat{k})$ = 0

(1 + 2k)(5) + (2 + k)(3) + (3 – k)(– 6) = 0

5 + 10k + 6 + 3k – 18 + 6k = 0

19k – 7 = 0

k = $\frac{7}{19}$

Put the value of k in equation (1),

$\vec{r}.[(\hat{i} + 2\hat{j} + 3\hat{k}) + (\frac{14}{19}\hat{i} + \frac{7}{19}\hat{j} - \frac{7}{19}\hat{k})] - 4 + 5(\frac{7}{19}) = 0$

$\vec{r}.[\frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} - \frac{50}{19}\hat{k}] - \frac{76+35}{19} = 0$

$\vec{r}.[\frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} - \frac{50}{19}\hat{k}] - \frac{41}{19} = 0$

Multiplying by 19,

$\vec{r}.(33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$

Equation of required plane is,

$\vec{r}.(33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$

33x + 45y + 50z – 41 = 0

13. Question

Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 6$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})$ = – 5 and the point (1, 1, 1).

Answer

The equation of the plane passing through the intersection of

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$ is

$[\vec{r}.(\hat{i} + \hat{j} + \hat{k}) - 6] + k[\vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) + 5 = 0$

$\vec{r}.[(1 + 2k)\hat{i} + (1 + 3k)\hat{j} + (1 + 4k)\hat{k}] = (6 - 5k)$ ...... (1)

$[x\hat{i} + y\hat{j} + z\hat{k}].[(1 + 2k)\hat{i} + (1 + 3k)\hat{j} + (1 + 4k)\hat{k}] = (6 - 5k)$

$[x(1 + 2k) + (1 + 3k)y + (1 + 4k)z] = (6 - 5k)$ ...... (2)

The required plane also passes through the point (1,1,1)

Substituting x = 1, y = 1, z = 1 in eq. (2),we have,

$1(1 + 2k) + (1 + 3k)1 + (1 + 4k)1$ = (6 – 5k)

1 + 2k + 1 + 3k + 1 + 4k = 6 – 5k

3 + 9k = 6 – 5k

14k = 3

k = $\frac{3}{14}$

Substituting the value k = $\frac{3}{14}$ in equation (1),We have,

$\vec{r}.\left[\frac{20}{14}\hat{i} + \frac{23}{14}\hat{j} + \frac{26}{14}\hat{k}\right] = \frac{69}{14}$

$\vec{r}.[20\hat{i} + 23\hat{j} + 26\hat{k}] = 69$

14. Question

Find the equation of the plane passing through the intersection of the planes

$\vec{r}\cdot\left(2\hat{i}+\hat{j}+3\hat{k}\right)=7, \vec{r}\cdot\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$ and the point (2, 1, 3).

Answer

We know that, the equation of a plane through the line of intersection of the plane

$\vec{r}.\overrightarrow{n_1} - d_1 = 0$ and $\vec{r}.\overrightarrow{n_2} = d_2 = 0$

is given by $\vec{r}.(\overrightarrow{n_1} + k\overrightarrow{n_2}) - d_1 + kd_2 = 0$

So, equation of plane passing through the line of intersection of plane $\vec{r}.(2\hat{i} + \hat{j} + 3\hat{k}) - 7 = 0$ and $\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 = 0$ is given by

$$\left[\vec{r}.\left(2\hat{i} + \hat{j} + 3\hat{k}\right) - 7\right] + k\left[\vec{r}.\left(2\hat{i} + 5\hat{j} + 3\hat{k}\right) - 9\right] = 0$$

$\vec{r}\left[(2 + 2k)\hat{i} + (1 + 5k)\hat{j} + (3 + 3k)\hat{k}\right] - 7 - 9k = 0$ ...... (1)

Given that plane (1) is passing through $2\hat{i} + \hat{j} + 3\hat{k}$ so

$$(2\hat{i} + \hat{j} + 3\hat{k})\left[(2 + 2k)\hat{i} + (1 + 5k)\hat{j} + (3 + 3k)\hat{k}\right] - 7 - 9k = 0$$

(2)(2 + 2k) + (1)(1 + 5k) + (3)(3 + 3k) – 7 – 9k = 0

4 + 4k + 1 + 5k + 9 + 9k – 7 – 9k = 0

9k = – 7

$$k = -\frac{7}{9}$$

Put the value of k in equation (1),

$$\vec{r}\left[(2 + 2k)\hat{i} + (1 + 5k)\hat{j} + (3 + 3k)\hat{k}\right] - 7 - 9k = 0$$

$$\vec{r}\left[\left(2 - \frac{14}{9}\right)\hat{i} + \left(1 - \frac{35}{9}\right)\hat{j} + \left(3 - \frac{21}{9}\right)\hat{k}\right] - 7 + \frac{63}{9} = 0$$

$$r\left[\left(\frac{18 - 14}{9}\right)\hat{i} + \left(\frac{9 - 35}{9}\right)\hat{j} + \left(\frac{27 - 21}{9}\right)\hat{k}\right] - 7 + 7 = 0$$

$$r\left[\left(\frac{4}{9}\right)\hat{i} - \left(\frac{26}{9}\right)\hat{j} + \left(\frac{6}{9}\right)\hat{k}\right] = 0$$

Multiplying by $(\frac{9}{2})$, we get

$$\vec{r}.\left[(2)\hat{i} - (13)\hat{j} + (3)\hat{k}\right] = 0$$

Equation of required plane is , $r\left[(2)\hat{i} - (13)\hat{j} + (3)\hat{k}\right] = 0$

15. Question

Find the equation of family of planes through the line of intersection of the planes 3x – y + 2z = 4 and x + y + z = 2 which is passing through (2, 2, 1)?

Answer

The equation of the family of planes through the line of intersection of planes

3x – y + 2z = 4 and x + y + z = 2 is,

(3x – y + 2z – 4) + k(x + y + z – 2) = 0 ......(1)

If it passes through (2, 2, 1) then,

(6 – 2 + 2 – 4) + k(2 + 2 + 1 – 2) = 0

$k = -\frac{2}{3}$

Substituting $k = -\frac{2}{3}$ in eq.(1) We get,

7x – 5y + 4z = 0 as the equation of the required plane.

16. Question

Find the equation of family of planes through the line of intersection of the planes x + y + z = 1 and 2x + 3y + 4z = 5 which is perpendicular to the plane x – y + z = 0 and parallel to $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=0$?

Answer

The equation of the family of planes through the line of intersection of planes

x + y + z = 1 and 2x + 3y + 4z = 5 is,

(x + y + z – 1) + k(2x + 3y + 4z – 5) = 0 ......(1)

(2k + 1)x + (3k + 1)y + (4k + 1)z = 5k + 1

It is perpendicular to the plane x – y + z = 0

(2k + 1)(1) + (3k + 1)(– 1) + (4k + 1)(1) = 5k + 1

2k + 1 – 3k – 1 + 4k + 1 = 5k + 1

$K = -\frac{1}{3}$

Sustiuting $k = -\frac{1}{3}$ in eq.(1) , We get, x – z + 2 = 0 as the equation of the required plane

And its vector equation is $\vec{r}.\left(\hat{i} - \hat{k}\right) + 2 = 0$

The equation of the family of a plane parallel to $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 0$ is,

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = d$ ...... (1)

If it passes through (a, b, c) then

$(a\hat{i} + b\hat{j} + c\hat{k})(\hat{i} + \hat{j} + \hat{k}) = d$

a + b + c = d

Substituting a + b + c = d in eq.(1), we get,

$\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = a + b + c$

x + y + z = a + b + c as the equation of the required plane.

17. Question

Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r}.\left(\hat{i}+\hat{j}+\hat{k}\right)=2$?

Answer

Given that equation is parallel to $\vec{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 2$ so that

The normal vector to that plane will be $\vec{n} = \hat{i} + \hat{j} + \hat{k}$ ...... (1)

And equation of the plane passing through the point is

$a_1(x - x_1) + b_1(y - y_1) + c_1(z - z_1) = 0$

and point is (a, b, c) so that,

$a_1(x – a) + b_1(y – b) + c_1(z – c) = 0$

by equation (1) $a_1 = b_1 = c_1 = 1$

$x + y + z – a – b – c = 0$

$x + y + z = a + b + c$

18. Question

Find the equation of the plane which contains the line of intersection of the plane x + 2y + 3z – 4 = 0 and 2x + y – z + 5 = 0 and whose x – intercept is twice of z – intercept. Hence, write the equation of the plane passing through the point (2,3, – 1) and parallel to the plane obtained above?

Answer

We know that equation of plane passing through the line of intersection of planes

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of planes

$x + 2y + 3z – 4 = 0$ and $2x + y – z + 5 = 0$ is

$x + 2y + 3z – 4 + k(2x + y – z + 5) = 0$

$x(1 + 2k) + y(2 + k) + z(3 – k) – 4 + 5k = 0$

$$\frac{x}{\frac{4-5k}{1\ +\ 2k}} + \frac{y}{\frac{4-5k}{2\ +\ k}} + \frac{z}{\frac{4-5k}{3-k}} = 1$$

as given that x–intercept is twice of z intercept

so $\frac{4-5k}{1\ +\ 2k} = 2(\frac{4-5k}{3-k})$

$3 – k = 2(1 + 2k)$

$3 – k = 2 + 4k$

$5k = 1$

$$k = \frac{1}{5}$$

Put this value in equation (1)

$x(1 + 2k) + y(2 + k) + z(3 – k) – 4 + 5k = 0$

$x(1 + \frac{2}{5}) + y(2 + \frac{1}{5}) + z(3 – \frac{1}{5}) – 4 + \frac{5}{5} = 0$

$x(\frac{7}{5}) + y(\frac{11}{5}) + z(\frac{14}{5}) – 3 = 0$

multiply by 5

$7x + 11y + 14z = 15$ …… (2)

And equation of the plane passing through the point is

$a_1(x – x_1) + b_1(y – y_1) + c_1(z – z_1) = 0$

and point is (2,3, – 1) so that,

$a_1(x – 2) + b_1(y – 3) + c_1(z + 1) = 0$

by equation (2) $a_1 = 7, b_1 = 11, c_1 = 14$

so $7(x – 2) + 11(y – 3) + 14(z + 1) = 0$

$7x + 11y + 14z – 14 – 33 + 14 = 0$

7x + 11y + 14z – 33 = 0

19. Question

Find the equation of the plane through the line of intersection of the plane x + y + z = 1 and 2x + 3y + 4z = 5 and twice of its y–intercept is equals to the three times its z intercept?

Answer

We know that equation of plane passing through the line of intersection of planes

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So equation of plane passing through the line of intersection of planes

x + y + z = 1 and 2x + 3y + 4z = 5 is

x + y + z – 1 + k(2x + 3y + 4z – 5) = 0

x(1 + 2k) + y(1 + 3k) + z(1 + 4k) – 1 – 5k = 0 ...... (1)

so, $\frac{x}{\frac{1+5k}{1+2k}} + \frac{y}{\frac{1+5k}{1+3k}} + \frac{z}{\frac{1+5k}{1+4k}} = 1$ as given that twice of its y intercept is equals to the three times its z intercept

so $2\left(\frac{1+5k}{1+3k}\right) = 3\left(\frac{1+5k}{1+4k}\right)$

2(1 + 4k) = 3(1 + 3k)

2 + 8k = 3 + 9k

k = – 1

put this in equation (1)

x(1 + 2k) + y(1 + 3k) + z(1 + 4k) – 1 – 5k = 0

x[1 + 2(– 1)] + y[1 + 3(– 1)] + z[1 + 4(– 1)] – 1 – 5(– 1) = 0

– x – 2y – 3z + 4 = 0

x + 2y + 3z = 4

Exercise 29.9

1. Question

Find the distance of the point $2\hat{i} - \hat{j} - 4\hat{k}$ from the plane $\vec{r} \cdot \left(3\hat{i} - 4\hat{j} + 12\hat{k}\right) - 9 = 0.$

Answer

Given:

Point given by the equation: $\vec{a} = 2\hat{i} - \hat{j} - 4\hat{k}$

Plane given by the equation: $\vec{r}.\left(3\hat{i} - 4\hat{j} + 12\hat{k}\right) - 9 = 0$, where the normal vector is: $\vec{n} = 3\hat{i} - 4\hat{j} + 12\hat{k}$

We know, the distance of $\vec{a}$ from the plane $\vec{r}.\vec{n} - d = 0$ is given by:

$$p = \left|\frac{\vec{a}.\vec{n} - d}{|\vec{n}|}\right|$$

Putting the values of $\vec{a}$ and $\vec{n}$:

$$\Longrightarrow p = \left|\frac{(2\hat{i} - \hat{j} - 4\hat{k}).(3\hat{i} - 4\hat{j} + 12\hat{k}) - 9}{|3\hat{i} - 4\hat{j} + 12\hat{k}|}\right|$$

$\Longrightarrow p = \frac{47}{13}$ units the distance of the point $2\hat{i} - \hat{j} - 4\hat{k}$ from the plane

$\vec{r}.\left(3\hat{i} - 4\hat{j} + 12\hat{k}\right) - 9 = 0$ is $\frac{47}{13}$ units

2. Question

Show that the points $\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 3\hat{j} + 3\hat{k}$ are equidistant from the plane $\vec{r} \cdot \left(5\hat{i} + 2\hat{j} - 7\hat{k}\right) + 9 = 0.$

Answer

Given:

* Points given by the equation: $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$; $\vec{b} = 3\hat{i} + 3\hat{j} + 3\hat{k}$

* Plane given by the equation: $\vec{r}.\left(5\hat{i} + 2\hat{j} - 7\hat{k}\right) + 9 = 0$, where the normal vector is: $\vec{n} = 5\hat{i} + 2\hat{j} - 7\hat{k}$

We know, the distance of $\vec{a}$ from the plane $\vec{r}.\vec{n} - d = 0$ is given by:

$$p = \left|\frac{\vec{a}.\vec{n} - d}{|\vec{n}|}\right|$$

$\Longrightarrow$ Distance of $\hat{i} - \hat{j} + 3\hat{k}$ from the plane $= \left|\frac{(\hat{i} - \hat{j} + 3\hat{k}).(5\hat{i} + 2\hat{j} - 7\hat{k}) + 9}{|5\hat{i} + 2\hat{j} - 7\hat{k}|}\right|$

$= \frac{9}{\sqrt{78}}$ units

And,

$\Longrightarrow$ Distance of $3\hat{i} + 3\hat{j} + 3\hat{k}$ from the plane $= \left|\frac{(3\hat{i} + 3\hat{j} + 3\hat{k}).(5\hat{i} + 2\hat{j} - 7\hat{k}) + 9}{|5\hat{i} + 2\hat{j} - 7\hat{k}|}\right|$

$= \frac{9}{\sqrt{78}}$ units

∴ the points $\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 3\hat{j} + 3\hat{k}$ are equidistant from the plane $\vec{r}.\left(5\hat{i} + 2\hat{j} - 7\hat{k}\right) + 9 = 0$.

3. Question

Find the distance of the point (2, 3, –5) from the plane x + 2y – 2z – 9 = 0.

Answer

Given:

* Point : A(2, 3, –5)

* Plane : π = x + 2y – 2z – 9 = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the necessary values

$\Longrightarrow$ Distance of the plane from A $= \left|\frac{(1)(2) + (2)(3) + (-2)(-5) - 9}{\sqrt{(1^2 + 2^2 + (-2)^2}}\right|$

= 3 units

∴ the distance of the point (2, 3, –5) from the plane

x + 2y – 2z – 9 = 0 is 3 units

4. Question

Find the equations of the planes parallel to the plane x + 2y – 2z + 8 = 0 which are at distance of 2 units from the point (2, 1, 1).

Answer

Since the planes are parallel to x + 2y – 2z + 8 = 0, they must be of the form:

x + 2y – 2z + θ = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

According to the question, the distance of the planes from (2, 1, 1) is 2 units.

$\Longrightarrow \left|\frac{(1)(2)+(2)(1)+(-2)(1)+\theta}{\sqrt{1^2+2^2+(-2)^2}}\right| = 2$

$\Longrightarrow \left|\frac{2+\theta}{3}\right| = 2$

$\Longrightarrow \frac{2+\theta}{3} = 2$ or $\frac{2+\theta}{3} = -2$

$\Longrightarrow$ θ = 4 or –8

$\Longrightarrow$ The required planes are:

x + 2y – 2z + 4 = 0 and x + 2y – 2z – 8 = 0

5. Question

Show that the points (1, 1, 1) and (–3, 0, 1) are equidistant from the plane 3x + 4y – 12z + 13 = 0.

Answer

Given:

* Points: A(1, 1, 1) and B(–3, 0, 1)

* Plane: π = 3x + 4y – 12z + 13 = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$\Longrightarrow$ Distance of (1,1,1) from the plane = $\left|\frac{(3)(1)+(4)(1)+(-12)(1)+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$

= $\frac{8}{13}$ units

$\Longrightarrow$ Distance of (–3,0,1) from the plane = $\left|\frac{(3)(-3)+(4)(0)+(-12)(1)+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$

= $\frac{8}{13}$ units

∴ the points (1, 1, 1) and (–3, 0, 1) are equidistant from the plane

3x + 4y – 12z + 13 = 0.

6. Question

Find the equations of the planes parallel to the plane x – 2y + 2z – 3 = 0 and which are at a unit distance from the point (1, 1, 1).

Answer

Since the planes are parallel to x – 2y + 2z – 3 = 0, they must be of the form:

x – 2y + 2z + θ = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

According to the question, the distance of the planes from (1, 1, 1) is 1 unit.

$$\Longrightarrow \left|\frac{(1)(1)+(-2)(1)+(2)(1)+\theta}{\sqrt{1^2+(-2)^2+2^2}}\right| = 1$$

$$\Longrightarrow \left|\frac{1+\theta}{3}\right| = 1$$

$\Longrightarrow \frac{1+\theta}{3} = 1$ or $\frac{1+\theta}{3} = -1$

⟹ θ = 2 or –4

⟹ The required planes are:

x – 2y + 2z + 2 = 0 and x – 2y + 2z – 4 = 0

7. Question

Find the distance of the point (2, 3, 5) from the xy–plane.

Answer

Given:

* Points: A(2, 3, 5)

* Plane: z = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the values

$$\Longrightarrow p = \left|\frac{(0)(2)+(0)(3)+(1)(5)+0}{\sqrt{0^2+0^2+1^2}}\right|$$

⟹ p = 5 units

∴ the distance of the point (2, 3, 5) from the xy–plane is 5 units

8. Question

Find the distance of the point (3, 3, 3) from the plane $\vec{r}\cdot(5i + 2j - 7k) + 9 = 0$

Answer

Given:

* Points: A(3, 3, 3)

* Plane: $\vec{r}.\left(5\hat{\imath} + 2\hat{\jmath} - 7\hat{k}\right) + 9 = 0$, which in cartesian form is:

5x + 2y – 7z + 9 = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the values

$$\Longrightarrow p = \left|\frac{(5)(3)+(2)(3)+(-7)(3)+9}{\sqrt{5^2+2^2+(-7)^2}}\right|$$

$$\Longrightarrow p = \frac{9}{\sqrt{78}} \text{ units}$$

∴ the distance of the point (3, 3, 3) from the plane

$\vec{r}.\left(5\hat{i} + 2\hat{j} - 7\hat{k}\right) + 9 = 0$ is $\frac{9}{\sqrt{78}}$ units.

9. Question

If the product of distances of the point (1, 1, 1) from the origin and the plane x – y + z + λ = 0 be 5, find the value of λ.

Answer

The distance of the point (1, 1, 1) from the origin

We know, distance of (x_1, y_1, z_1) form the origin is :

$$\sqrt{x_1^2 + y_1^2 + z_1^2}$$

Putting values of $x_1, y_1, z_1 = 1$

$\Longrightarrow$ Required Distance = √3

Distance of the point (1, 1, 1) from plane x – y + z + λ = 0

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the necessary values,

$$\Longrightarrow p = \left|\frac{(1)(1)+(-1)(1)+(1)(1)+\lambda}{\sqrt{1^2+(-1)^2+1^2}}\right|$$

$$\Longrightarrow p = \left|\frac{1+\lambda}{\sqrt{3}}\right|$$

According to question, the product of the above two distances is 5

$$\Longrightarrow \left|\frac{1+\lambda}{\sqrt{3}}\right| \times \sqrt{3} = 5$$

$\Longrightarrow |1 + \lambda| = 5$

$\Longrightarrow 1 + \lambda = 5$ or $1 + \lambda = -5$

$\Longrightarrow \lambda = 4$ or $\lambda = -6$

10. Question

Find an equation for the set of all points that are equidistant from the planes 3x – 4y + 12z = 6 and 4x + 3z = 7.

Answer

Let the set of points be denoted by (x_1,y_1,z_1)

Distance of (x_1,y_1,z_1) from 3x – 4y + 12z = 6

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$\Rightarrow p = \left|\frac{(3)x_1+(-4)y_1+(12)z_1-6}{\sqrt{3^2+(-4)^2+12^2}}\right|$$

$$\Rightarrow p = \left|\frac{3x_1-4y_1+12z_1-6}{13}\right|$$

Similarly,

Distance of (x_1,y_1,z_1) from 4x + 3z = 7:

$$q = \left|\frac{(4)x_1 + (0)y_1 + (3)z_1 - 7}{\sqrt{4^2 + 0^2 + 3^2}}\right|$$

$$q = \left|\frac{4x_1 + 3z_1 - 7}{5}\right|$$

According to the question, $p = q$

$$\Rightarrow \left|\frac{3x_1-4y_1+12z_1-6}{13}\right| = \left|\frac{4x_1+3z_1-7}{5}\right|$$

$$\Rightarrow \frac{3x_1-4y_1+12z_1-6}{13} = \pm\frac{4x_1+3z_1-7}{5}$$

$$\Rightarrow \frac{3x_1-4y_1+12z_1-6}{13} = \frac{4x_1+3z_1-7}{5} \text{ or } \frac{3x_1-4y_1+12z_1-6}{13} = \frac{-4x_1-3z_1+7}{5}$$

$\Rightarrow 37x_1 + 20y_1 - 21z_1 - 61 = 0$ or $67x_1 - 20y_1 + 99z_1 - 121 = 0$

∴ Equations of set of points equidistant from planes 3x – 4y + 12z = 6 and 4x + 3z = 7 is $37x_1 + 20y_1 - 21z_1 - 61 = 0$ or $67x_1 - 20y_1 + 99z_1 - 121 = 0$

11. Question

Find the distance between the point (7, 2, 4) and the plane determined by the points A(2, 5, –3), B(–2, –3, 5) and (5, 3, –3).

Answer

The equation of the plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by the following equation:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

According to question,

$(x_1, y_1, z_1) = (2, 5, -3)$

$(x_2, y_2, z_2) = (-2, -3, 5)$

$(x_3, y_3, z_3) = (5, 3, -3)$

Putting these values,

$$\Longrightarrow \begin{vmatrix} x-2 & y-5 & z-(-3) \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

⟹ (x – 2)(16) + (y – 5)(24) + (z + 3)(32) = 0

⟹ 2x + 3y + 4z –7 = 0

Distance of 2x + 3y + 4z –7 = 0 from (7, 2, 4)

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$$\Longrightarrow p = \left|\frac{(2)(7)+(3)(2)+(4)(4)-7}{\sqrt{2^2+3^2+4^2}}\right|$$

⟹ p = √29 units

∴ the distance between the point (7, 2, 4) and the plane determined by the points A(2, 5, –3), B(–2, –3, 5) and (5, 3, –3) is √29 units.

12. Question

A plane makes intercepts –6, 3, 4 respectively on the coordinate axes. Find the length of the perpendicular from the origin on it.

Answer

The equation of the plane which makes intercepts a, b, and c with the x, y , and z axis respectively is :

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Putting the values of a, b and c

Required equation of the plane:

$$\Longrightarrow \frac{x}{-6} + \frac{y}{3} + \frac{z}{4} = 1$$

⟹ –2x + 4y + 3z = 12

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

⟹ Distance from the origin i.e. (0, 0, 0) : $\left|\frac{d}{\sqrt{a^2+b^2+c^2}}\right|$

⟹ Required Distance = $\left|\frac{-12}{\sqrt{(-2)^2+4^2+3^2}}\right|$

⟹ The length of the perpendicular from the origin on the plane = $\frac{12}{\sqrt{29}}$ units

13. Question

Find the distance of the point (1, –2, 4) from a plane passing through the point (1, 2, 2) and perpendicular to the planes x – y + 2z = 3 and 2x – 2y + z + 12 = 0.

Answer

We know, equation of plane passing through (x_1, y_1, z_1):

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

⟹ Equation of plane passing through (1, 2, 2):

$a(x - 1) + b(y - 2) + c(z - 2) = 0$

i.e. $ax + by + cz = a + 2b + 2c$ eq(i)

We know, if two planes $a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular, then:

$a_1.a_2 + b_1.b_2 + c_1.c_2 = 0$

According to question,

⟹ $(1)(a) + (-1)(b) + (2)(c) = 0$

⟹ $(2)(a) + (-2)(b) + (1)(c) = 0$

i.e.

⟹ $a - b + 2c = 0$

⟹ $2a - 2b + c = 0$

Solving the above equations using cross multiplication method:

$$\Longrightarrow \frac{a}{-1+4} = \frac{b}{4-1} = \frac{c}{-2+2} = \theta$$

⟹ $a = 3\theta$, $b = 3\theta$, $c = 0$

Putting this in eq(i)

Equation of plane:

$3\theta(x) + 3\theta(y) + (0)z = 3\theta + 2(3\theta) + 0$

i.e.

$x + y = 3$

Distance of (1, –2, 4) from $x + y = 3$

We know, the distance of point (x_1, y_1, z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the necessary values,

$$\Longrightarrow p = \left|\frac{(1)(1)+(1)(-2)+(0)(4)+0}{\sqrt{1^2+1^2+0^2}}\right|$$

$$\Longrightarrow p = \frac{1}{\sqrt{2}}$$

∴ the distance of the point (1, –2, 4) from plane passing through the point (1, 2, 2) and perpendicular to the planes $x - y + 2z = 3$ and

$2x - 2y + z + 12 = 0$ is $\frac{1}{\sqrt{2}}$ units.

Exercise 29.10

1. Question

Find the distance between the parallel planes $2x - y + 3z - 4 = 0$ and $6x - 3y + 9z + 13 = 0$.

Answer

Let $P(x_1,y_1,z_1)$ be any point on 2x – y + 3z – 4 = 0.

$\Longrightarrow 2x_1 - y_1 + 3z_1 - 4 = 0$

$\Longrightarrow 2x_1 - y_1 + 3z_1 = 4$ eq(i)

Distance between (x_1,y_1,z_1) and the plane

6x – 3y + 9z + 13 = 0:

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the necessary values,

$$\Longrightarrow p = \left|\frac{(6)(x_1)+(-3)(y_1)+(9)(z_1)+13}{\sqrt{6^2+(-3)^2+9^2}}\right|$$

$$\Longrightarrow p = \left|\frac{3(2x_1-y_1+3z_1)+13}{\sqrt{6^2+(-3)^2+9^2}}\right|$$

$\Longrightarrow p = \left|\frac{3(4)+13}{3\sqrt{14}}\right|$ (using eq (i))

$$\Longrightarrow p = \frac{25}{3\sqrt{14}}$$

∴ the distance between the parallel planes 2x – y + 3z – 4 = 0 and

6x – 3y + 9z + 13 = 0 is $\frac{25}{3\sqrt{14}}$ units.

2. Question

Find the equation of the plane which passes through the point (3, 4, –1) and is parallel to the plane 2x – 3y + 5z + 7 = 0. Also, find the distance between the two planes.

Answer

Since the plane is parallel to 2x – 3y + 5z + 7 = 0, it must be of the form:

2x – 3y + 5z + θ = 0

According to question,

The plane passes through (3, 4, –1)

$\Longrightarrow$ 2(3) – 3(4) +5(–1) + θ = 0

$\Longrightarrow$ θ = 11

So, the equation of the plane is as follows:

2x – 3y + 5z + 11 = 0

Distance of the plane 2x – 3y + 5z + 7 = 0 from (3, 4, –1):

We know, the distance of point (x_1,y_1,z_1) from the plane

$\pi: ax + by + cz + d = 0$ is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Putting the necessary values,

$\Longrightarrow p = \left|\frac{(2)(3)+(-3)(4)+(5)(-1)+7}{\sqrt{2^2+(-3)^2+5^2}}\right|$

$\Longrightarrow p = \frac{4}{\sqrt{38}}$

∴ the distance of the plane 2x – 3y + 5z + 7 = 0 from (3, 4, –1) is

$\frac{4}{\sqrt{38}}$ units

3. Question

Find the equation of the plane mid–parallel to the planes 2x – 2y + z + 3 = 0 and 2x – 2y + z + 9 = 0.

Answer

Given:

* Equation of planes: π_1= 2x – 2y + z + 3 = 0π_2= 2x – 2y + z + 9 = 0

Let the equation of the plane mid–parallel to these planes be:

π_3: 2x – 2y + z + θ = 0

Now,

Let P(x_1,y_1,z_1) be any point on this plane,

⟹ 2(x_1) – 2(y_1) + (z_1) + θ = 0 eq(i)

We know, the distance of point (x_1,y_1,z_1) from the plane

π: ax + by + cz + d = 0 is given by:

$$p = \left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

⟹ Distance of P from π_1:

$$p = \left|\frac{(2x_1 - 2y_1 + z_1) + 3}{\sqrt{2^2 + (-2)^2 + 1^2}}\right|$$

$\Longrightarrow p = \left|\frac{(-\theta)+3}{3}\right|$ (using eq(i))

Similarly

⟹ Distance of P from π_2 :

$$q = \left|\frac{(2x_1 - 2y_1 + z_1) + 9}{\sqrt{2^2 + (-2)^2 + 1^2}}\right|$$

$\Longrightarrow q = \left|\frac{-\theta+9}{3}\right|$ (using eq(i))

As π_3 is mid–parallel to π_1 and π_2 :

p = q

$\Longrightarrow \left|\frac{(-\theta)+3}{3}\right| = \left|\frac{-\theta+9}{3}\right|$

Squaring both sides,

$\Longrightarrow \left(\frac{(-\theta)+3}{3}\right)^2 = \left(\frac{-\theta+9}{3}\right)^2$

⟹ $(3 - \theta)^2 = (9 - \theta)^2$

$\Longrightarrow 9 - 6\theta + \theta^2 = 81 - 18\theta + \theta^2$

$\Longrightarrow \theta = 6$

∴ equation of the mid–parallel plane is 2x – 2y + z + 6 = 0

4. Question

Find the distance between the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})+7=0$ and $\vec{r}\cdot(2\hat{i}+4\hat{j}+6\hat{k})$ + 7 = 0.

Answer

Let $\vec{a}$ be the position vector of any point P on the plane

$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})+7=0$:

$\Longrightarrow \vec{a}.(\hat{i}+2\hat{j}+3\hat{k})+7=0$ eq(i)

We know, the distance of $\vec{a}$ from the plane $\vec{r}.\vec{n}-d=0$ is given by:

$$p=\left|\frac{\vec{a}.\vec{n}-d}{|\vec{n}|}\right|$$

Putting the values of $\vec{a}$ and $\vec{n}$:

$\Longrightarrow p=\left|\frac{\vec{a}.(2\hat{i}+4\hat{j}+6\hat{k})+7}{|2\hat{i}+4\hat{j}+6\hat{k}|}\right|$

$\Longrightarrow p=\left|\frac{2\vec{a}.(\hat{i}+2\hat{j}+3\hat{k})+7}{\sqrt{2^2+4^2+6^2}}\right|$

$\Longrightarrow p=\left|\frac{2(-7)+7}{\sqrt{56}}\right|$

$\Longrightarrow p=\frac{7}{\sqrt{56}}$

∴ the distance between the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})+7=0$ and $\vec{r}.(2\hat{i}+4\hat{j}+6\hat{k})+7=0$ is $\frac{7}{\sqrt{56}}$ units.

Exercise 29.11

1. Question

Find the angle between the line $\vec{r}=(2\hat{i}+3\hat{j}+9\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k})$and the plane $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=5$.

Answer

Equation of line is

$\vec{r} = (2\hat{i} + 3\hat{j} + 9\hat{k}) + k(2\hat{i} + 3\hat{j} + 9\hat{k})$

And the equation of the plane is

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 5$

As we know that the angle θ between the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and a plane $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here, $a_1 = 1$, $b_1 = 1$ and $c_1 = 1$

and $a_2 = 2$, $b_2 = 3$ and $c_2 = 4$

The angle between them is given by

$$\sin\theta = \frac{1\times 2 + 1\times 3 + 1\times 4}{\sqrt{(1+1+1)(4+9+16)}} = \frac{2+3+4}{\sqrt{3\times 29}} = \frac{9}{\sqrt{87}}$$

2. Question

Find the angle between the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$ and the plane 2x + y – z = 4.

Answer

As we know that the angle θ between the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and a plane $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \text{ ......(1)}$$

Now, given equation of the line is

$$\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{1}$$

So, $a_1 = 1$, $b_1 = -1$ and $c_1 = 1$

Equation of plane is 2x + y – z – 4 = 0

So, $a_2 = 2$, $b_2 = 1$, $c_2 = -1$ and $d_2 = -4$

$$\therefore \sin\theta = \frac{1\times 2 + (-1)\times 1 + 1\times(-1)}{\sqrt{1^2+(-1)^2+1^2}\sqrt{2^2+1^2+(-1)^2}}$$

$$\Rightarrow \sin\theta = \frac{2-1-1}{\sqrt{1+1+1}\sqrt{4+1+1}}$$

$$\Rightarrow \sin\theta = \frac{0}{\sqrt{1+1+1}\sqrt{4+1+1}} = 0$$

⇒ sinθ = 0

∴ the angle between the plane and the line is 0°

3. Question

Find the angle between the line joining the points (3, – 4, – 2) and (12, 2, 0) and the plane 3x – y + z = 1.

Answer

As we know that the angle θ between the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and a plane $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \text{ ......(1)}$$

Given that the line is passing through A(3, – 4, – 2) and (12, 2, 0)

∴ the direction ratios of line AB are

= (12 – 3, 2 – (– 4), 0 – (– 2))

= (12 – 3, 2 + 4, 0 + 2)

= (9, 6, 2)

So,

$a_1 = 9$, $b_1 = 6$ and $c_1 = 2$ ......(2)

Given equation of plane is 3x – y + z = 1

So,

$a_2 = 3, b_2 = -1$ and $c_2 = 1$ ……(3)

$\therefore \sin\theta = \frac{3\times9 + 6\times-1 + 2\times1}{\sqrt{9^2 + 6^2 + 2^2}\sqrt{3^2 + 1^2 + (-1)^2}}$

$\Rightarrow \sin\theta = \frac{27-6+2}{\sqrt{81+36+4}\sqrt{9+1+1}}$

$\Rightarrow \sin\theta = \frac{23}{\sqrt{121}\sqrt{11}} = \frac{23}{11\sqrt{11}}$

$\Rightarrow \theta = \sin^{-1}(\frac{23}{11\sqrt{11}})$

Therefore the required angle is $\sin^{-1}(\frac{23}{11\sqrt{11}})$

4. Question

The line is parallel to the plane $\vec{r}.(m\hat{i} + 3\hat{j} + \hat{k}) = 4$. Find m.

Answer

We know that line $\vec{r} = \vec{a} + k\vec{b}$ is parallel to the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ …… (1)

Given the equation of the line is

$\vec{r} = \hat{i} + k(2\hat{i} - m\hat{j} - 3\hat{k})$

and the equation of the plane is

$\vec{r}.(m\hat{i} + 3\hat{j} + \hat{k}) = 4$

So, $\vec{b} = (2\hat{i} - m\hat{j} - 3\hat{k})$

$\vec{n} = (m\hat{i} + 3\hat{j} + \hat{k})$

Putting the values in equation (1)

$(2\hat{i} - m\hat{j} - 3\hat{k}).(m\hat{i} + 3\hat{j} + \hat{k}) = 0$

$\Rightarrow 2m - 3m - 3 = 0$

$\Rightarrow -m = 3$

$\Rightarrow m = -3$

5. Question

Show that the line whose vector equation is $\vec{r} = 2\hat{i} + 5\hat{j} + 7\hat{k} + \lambda(\hat{i} + 3\hat{j} + 4\hat{k})$ is parallel to the plane whose vector equation is $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 7$. Also, find the distance between them.

Answer

We know that line $\vec{r} = \vec{a} + k\vec{b}$ and plane $\vec{r}.\vec{n} = d$ is parallel if

$\vec{b}.\vec{n} = 0$ ……(1)

Given, the equation of the line $\vec{r} = (2\hat{i} + 5\hat{j} + 7\hat{k}) + k(\hat{i} + 3\hat{j} + 4\hat{k})$ and equation of plane is the

$\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 7$,

So, $\vec{b} = \hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{n} = \hat{i} + \hat{j} - \hat{k}$

Now, $\vec{b}.\vec{n}$

$= (\hat{i} + 3\hat{j} + 4\hat{k})(\hat{i} + \hat{j} - \hat{k})$

= 1 + 3 – 4 = 0

So, the line and the plane are parallel

We know that the distance (D) of a plane $\vec{r}.\vec{n} = d$ from a point $\vec{a}$ is given by

$$D = \frac{\vec{a}.\vec{n} - d}{n}$$

$$\vec{a} = (2\hat{i} + 5\hat{j} + 7\hat{k})$$

$$D = \frac{(2\hat{i} + 5\hat{j} + 7\hat{k}).(\hat{i} + \hat{j} - \hat{k}) - 7}{\sqrt{1^2 + 1^2 + (-1)^2}}$$

$$D = \frac{2 + 5 - 7 - 7}{\sqrt{1 + 1 + 1}}$$

$$D = -\frac{7}{\sqrt{3}}$$

We take the mod value

So, $D = \frac{7}{\sqrt{3}}$

6. Question

Find the vector equation of the line through the origin which is perpendicular to the plane $\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) = 3.$

Answer

The required line is perpendicular to the plane $\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) = 3$

So line is parallel to normal vector $\vec{n} = (\hat{i} + 2\hat{j} + 3\hat{k})$ of plane.

And it is also passing through $\vec{a} = (\hat{0\imath} + 0\hat{j} + 0\hat{k})$

We know that the equation of the line passing through $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r} = \vec{a} + k\vec{b}$ ……(1)

$$\vec{b} = \vec{n} = (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{r} = (\hat{0\imath} + 0\hat{j} + 0\hat{k}) + k(\hat{i} + 2\hat{j} + 3\hat{k})$$

Hence equation of the required line is

$$\vec{r} = k(\hat{i} + 2\hat{j} + 3\hat{k})$$

7. Question

Find the equation of the plane through (2, 3, – 4) and (1, – 1, 3) and parallel to the x – axis.

Answer

We know that the equation of plane passing through (x_1,y_1,z_1) is given by

$a(x – x_1) + b(y – y_1) + c(z – z_1) = 0$ ……(1)

So, equation of plane passing through (2,3, – 4) is

a(x – 2) + b(y – 3) + c(z + 4) = 0 ……(2)

It also passes through (1, – 1, – 3)

So, equation (2) must satisfy the point (1, – 1, – 3)

$\therefore$ a(1 – 2) + b(– 1 – 3) + c(– 3 + 4) = 0

$\Rightarrow$ – a – 4b + c = 0

$\Rightarrow$ a + 4b – 7c = 0 ......(3)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane

$a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(4)

Here, equation(2) is parallel to x axis,

$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ ......(5)

Using (2) and (5) in equation (4)

a×1 + b×0 + c×0 = 0

$\Rightarrow$ a = 0

Putting the value of a in equation (3)

a – 4b + 7c = 0

$\Rightarrow$ 0 – 4b + 7c = 0

$\Rightarrow$ – 4b = – 7c

$\Rightarrow$ b = $\frac{7c}{4}$

Now, putting the value of a and b in equation (2)

a(x – 2) + b(y – 3) + c(z + 4)

$\Rightarrow$ 0(x – 2) + $\frac{7c}{4}$(y – 3) + c(z + 4) = 0

$\Rightarrow 0 + \frac{7cy}{4} - \frac{21c}{4} + cz + 4c = 0$

$\Rightarrow$ 7cy – 21c + 4cz + 16c = 0

Dividing by c we have,

7y – 21 + 4z + 16 = 0

$\Rightarrow$ 7y + 4z – 5 = 0

Equation of required plane is 7y + 4z – 5 = 0

8. Question

Find the equation of a plane passing through the points (0, 0, 0) and (3, – 1, 2) and parallel to the line

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}.$$

Answer

We know that the equation of plane passing through (x_1,y_1,z_1) is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ......(1)

So, equation of plane passing through (0,0,0) is

a(x – 0) + b(y – 0) + c(z – 0) = 0

ax + by + cz = 0 ......(2)

It also passes through (3, – 1,2)

So, equation (2) must satisfy the point (3, – 1,2)

$\therefore$ 3a – b + 2c = 0 ……(3)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ……(4)

Here, the plane is parallel to line,

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

So,

a×1 + b× – 4 + c×7 = 0

⇒ a – 4b + 7c = 0 ……(5)

Solving equation (3) and (5) by cross multiplication we have,

$$\frac{a}{-1\times 7-(-4)\times 2} = \frac{b}{1\times 2-3\times 7} = \frac{c}{3\times -4-1\times -1}$$

$\Rightarrow \frac{a}{-7+8} = \frac{b}{2-21} = \frac{c}{-12+1}$

$\Rightarrow \frac{a}{1} = \frac{b}{-19} = \frac{c}{-11} = k(\text{let})$

$\therefore$ a = k, b = – – 19k and c = – 11k

Putting the value in equation (2)

ax + by + cz = 0

kx – 19ky – 11kz = 0

Dividing by k we have

x – 19y – 11z = 0

The required equation is x – 19y – 11z = 0

9. Question

Find the vector and Cartesian equations of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+2\hat{k}\right)=6$

Answer

We know that the equation of line passing through (1,2,3) is given by

$\frac{x-1}{a_1} = \frac{y-2}{b_1} = \frac{z-3}{c_1}$ ……(1)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ……(2)

Here, line (1) is parallel to plane,

x – y + 2z = 5

So,

a×1 + b× – 1 + c×2 = 0

⇒ a – b + 2c = 0 ……(3)

Also, line (1) is parallel to plane,

3x + y + z = 6

So,

$a\times3 + b\times1 + c\times1 = 0$

$\Rightarrow 3a + b + c = 0$ ......(4)

Solving equation (3) and (4) by cross multiplication we have,

$$\frac{a}{-1\times1-(1)\times2} = \frac{b}{3\times2-1\times1} = \frac{c}{1\times1-3\times-1}$$

$\Rightarrow \frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3}$

$\Rightarrow \frac{a}{-3} = \frac{b}{5} = \frac{c}{4} = k(\text{let})$

$\therefore$ a = – 3k, b = 5k and c = 4k

Putting the value in equation (1)

$$\frac{x-1}{-3k} = \frac{y-2}{5k} = \frac{z-3}{4k}$$

Multiplying by k we have

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

The required equation is $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$

$\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + k(-3\hat{\imath} + 5\hat{\jmath} + 4\hat{k})$

10. Question

Prove that the line of section of the planes 5x + 2y – 4z + 2 = 0 and 2x + 8y + 2z – 1 = 0 parallel to the plane 4x – 2y – 5z – 2 = 0.

Answer

Let a_1,b_1 and c_1 be the direction ratios of the line 5x + 2y – 4z + 2 = 0 and 2x + 8y + 2z – 1 = 0.

As we know that if two planes are perpendicular with direction ratios as a_1, b_1 and c_1 and a_2 , b_2 and c_2 then

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

Since, line lies in both the planes, so it is perpendicular to both planes

$5a_1 + 2b_1 - 4c_1 = 0$ ......(1)

$2a_1 + 8b_1 + 2c_1 = 0$ ......(2)

Solving equation (1) and (2) by cross multiplication we have,

$$\frac{a_1}{2\times2-(-4)\times8} = \frac{b_1}{2\times-4-5\times2} = \frac{c_1}{5\times8-2\times2}$$

$\Rightarrow \frac{a_1}{4+32} = \frac{b_1}{-8-10} = \frac{c_1}{40-4}$

$\Rightarrow \frac{a_1}{36} = \frac{b_1}{-18} = \frac{c_1}{36} = k(\text{let})$

$\Rightarrow \frac{a_1}{2} = \frac{b_1}{-1} = \frac{c_1}{2}$

$\therefore$ a = 2k, b = – k and c = 2k

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(3)

Here, line with direction ratios a_1,b_1 and c_1 is parallel to plane,

$4x - 2y - 5z = 5$

So,

$2\times4 + (-1)\times -2 + 2\times -5 = 0$

$\Rightarrow 8 + 2 - 10 = 0$

Therefore, the line of section is parallel to the plane.

11. Question

Find the vector equation of the line passing through the point (1, – 1, 2) and perpendicular to the plane 2x – y + 3z – 5 = 0.

Answer

Equation of line passing through $\vec{a}$ and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + k\vec{b}$ ......(1)

Given that the line passes through (1, – 1,2) is

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + k\vec{b}$ ...... (2)

Since , line (1) is perpendicular to the plane 2x – y + 3z – 5 = 0, so normal to plane is parallel to the line.

In vector form,

$\vec{b}$ is parallel to $\vec{n} = 2\hat{i} - \hat{j} + 3\hat{k}$

So, $\vec{b} = l(2\hat{i} - \hat{j} + 3\hat{k})$ as l is any scalar

Thus, the equation of required line,

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + k(l(2\hat{i} - \hat{j} + 3\hat{k})$

$\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + m(2\hat{i} - \hat{j} + 3\hat{k})$

12. Question

Find the equation of the plane through the points (2, 2, – 1) and (3, 4, 2) and parallel to the line whose direction ratios are 7, 0, 6.

Answer

We know that the equation of plane passing through (x_1,y_1,z_1) is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ......(1)

So, equation of plane passing through (2,2, – 1) is

$a(x - 2) + b(y - 2) + c(z + 1) = 0$ ......(2)

It also passes through (3,4,2)

So, equation (2) must satisfy the point (3,4,2)

$\therefore a(3 - 2) + b(4 - 2) + c(2 + 1) = 0$

$\Rightarrow a + 2b + 3c = 0$ ......(3)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(4)

Here, the plane (2) is parallel to line having direction ratios 7,0,6 ,

So,

$a\times7 + b\times0 + c\times6 = 0$

$\Rightarrow 7a + 6c = 0$

$\Rightarrow a = -\frac{6c}{7}$ ......(5)

Putting the value of a in equation (3)

a + 2b + 3c = 0

$\Rightarrow -\frac{6c}{7}$ + 2b + 3c = 0

⇒ – 6c + 14b + 21c = 0

⇒ 14b + 15c = 0

$\Rightarrow a = -\frac{15c}{14}$

Putting the value of a and b in equation (2)

a(x – 2) + b(y – 2) + c(z + 1) = 0

$\Rightarrow -\frac{6c}{7}$ (x – 2) + $\left(-\frac{15c}{14}\right)$(y – 2) + c(z + 1) = 0

$\Rightarrow -\frac{6cx}{7} + \frac{12c}{7} - \frac{15cy}{14} + \frac{30c}{14} + cz + c = 0$

Multiplying by $\frac{14}{c}$ we have,

– 12x + 24 – 15y + 30 + 14z + 14 = 0

⇒ – 12x + 15y + 14z + 68 = 0

⇒ 12x – 15y – 14z – 68 = 0

Equation of required plane is 12x – 15y – 14z – 68 = 0

13. Question

Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$ and the plane 3x + 4y + z + 5 = 0.

Answer

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ is given by

$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$ ......(1)

Now, given equation of line is

$$\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{2}$$

So, a1 = 3 , b1 = – 1 and c1 = 2

Equation of plane is 3x + 4y + z + 5 = 0

So, a2 = 3, b2 = 4, c2 = 1 and d2 = – 5

$\therefore \sin\theta = \frac{3\times3 + (-1)\times4 + 2\times(1)}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{3^2 + 4^2 + (1)^2}}$

$\Rightarrow \sin\theta = \frac{9-4+2}{\sqrt{9+1+4}\sqrt{9+16+1}}$

$\Rightarrow \mathbf{sin\theta} = \frac{7}{\sqrt{14}\sqrt{26}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{7\sqrt{7}}{7\sqrt{52}} = \frac{\sqrt{7}}{\sqrt{52}}$

$\Rightarrow \sin\theta = \frac{\sqrt{7}}{\sqrt{52}} \Rightarrow \theta = \sin^{-1}\left(\frac{\sqrt{7}}{\sqrt{52}}\right)$

∴ the angle between the plane and the line is $\sin^{-1}\left(\frac{\sqrt{7}}{\sqrt{52}}\right)$

14. Question

Find the equation of the plane passing through the intersection of the planes x – 2y + z = 1 and 2x + y + z = 8 and parallel to the line with direction ratios proportional to 1, 2, 1. Find also the perpendicular distance of (1, 1, 1) from this plane.

Answer

We know that equation of plane passing through the intersection of planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So, equation of plane passing through the intersection of planes

x – 2y + z – 1 = 0 and 2x + y + z – 8 = 0 is

(x – 2y + z – 1) + k(2x + y + z – 8) = 0 ......(1)

⇒ x(1 + 2k) + y(– 2 + k) + z(1 + k) + (– 1 – 8k) = 0

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Given the plane is parallel to line with direction ratios 1,2,1

1×(1 + 2k) + 2×(– 2 + k) + 1×(1 + k) = 0

⇒ 1 + 2k – 4 + 2k + 1 + k = 0

⇒ $k = \frac{2}{5}$

Putting the value of k in equation (1)

$$(x - 2y + z - 1) + \frac{2}{5}(2x + y + z - 8) = 0$$

$$\Rightarrow x + \frac{4}{5}x - 2y + \frac{2}{5}y + z + \frac{2}{5}z - 1 - \frac{16}{5} = 0$$

$$\Rightarrow \frac{9x}{5} - \frac{8y}{5} + \frac{7z}{5} - \frac{21}{5} = 0$$

⇒ 9x – 8y + 7z – 21 = 0

We know that the distance (D) of point (x_1,y_1,z_1) from plane ax + by + cz – d = 0 is given by

$$D = \frac{ax_1 + by_1 + cz_1 + d_1}{\sqrt{a^2 + b^2 + c^2}}$$

So, distance of point (1,1,1) from plane (1) is

$$\Rightarrow D = \frac{9\times1 + (-8)\times1 + 7\times1 - 21}{\sqrt{9^2 + (-8)^2 + 7^2}}$$

$$\Rightarrow D = \frac{9 - 8 + 7 - 21}{\sqrt{81 + 64 + 49}}$$

$$\Rightarrow D = \frac{-13}{\sqrt{194}}$$

Taking the mod value we have

$$D = \frac{13}{\sqrt{194}}$$

15. Question

State when the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r}.\vec{n} = d$. Show that the line $\vec{r} = \hat{i} + \hat{j} + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ is parallel to the plane $\vec{r}.(2\hat{j}+\hat{k})=3$. Also, find the distance between the line and the plane.

Answer

We know that line $\vec{r} = \vec{a} + k\vec{b}$ and plane $\vec{r}.\vec{n} = d$ is parallel if

$\vec{b}.\vec{n} = 0$ ……(1)

Given, equation of line $\vec{r} = (\hat{i} + \hat{j}) + k(3\hat{i} - \hat{j} + 2\hat{k})$ and equation of plane is

$\vec{r}.(2\hat{j} + \hat{k}) = 3$,

So, $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{n} = 2\hat{j} + \hat{k}$

Now, $\vec{b}.\vec{n}$

$= (3\hat{i} - \hat{j} + 2\hat{k})(2\hat{j} + \hat{k})$

= – 2 + 2 = 0

So, the line and the plane are parallel

We know that the distance (D) of a plane $\vec{r}.\vec{n} = d$ from a point $\vec{a}$ is given by

$$D = \frac{\vec{a}.\vec{n} - d}{n}$$

$\vec{a} = (\hat{i} + \hat{j})$

$$D = \frac{(\hat{i} + \hat{j}).(2\hat{j} + \hat{k}) - 3}{\sqrt{2^2 + (1)^2}}$$

$$D = \frac{2 - 3}{\sqrt{1 + 4}}$$

$$D = -\frac{1}{\sqrt{5}}$$

We take the mod value

So, $D = \frac{1}{\sqrt{5}}$

16. Question

Show that the plane whose vector equation is $\vec{r}.(\hat{i}+2\hat{j}-\hat{k})=1$ and the line whose vector equation is $\vec{r}=(-\hat{i}+\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ are parallel. Also, find the distance between them.

Answer

We know that line $\vec{r} = \vec{a} + k\vec{b}$ and plane $\vec{r}.\vec{n} = d$ is parallel if

$\vec{b}.\vec{n} = 0$ ……(1)

Given, equation of line $\vec{r} = (-\hat{i} + \hat{j} + \hat{k}) + k(2\hat{i} + \hat{j} + 4\hat{k})$ and equation of plane is

$\vec{r}.(\hat{i} + 2\hat{j} - \hat{k}) = 1$,

So, $\vec{b} = 2\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{n} = \hat{i} + 2\hat{j} - \hat{k}$

Now, $\vec{b}.\vec{n}$

$= (2\hat{i} + \hat{j} + 4\hat{k})(\hat{i} + 2\hat{j} - \hat{k})$

= 2 + 2 – 4 = 0

So, the line and the plane are parallel

We know that the distance (D) of a plane $\vec{r}.\vec{n} = d$ from a point $\vec{a}$ is given by

$$D = \frac{\vec{a}.\vec{n} - d}{n}$$

$$\vec{a} = (-\hat{i} + \hat{j} + \hat{k})$$

$$D = \frac{(-\hat{i} + \hat{j} + \hat{k}).(\hat{i} + 2\hat{j} - \hat{k}) - 1}{\sqrt{-1^2 + 2^2 + (1)^2}}$$

$$D = \frac{-1 + 2 - 1 - 1}{\sqrt{1 + 4 + 1}}$$

$$D = -\frac{1}{\sqrt{6}}$$

We take the mod value

So, $D = \frac{1}{\sqrt{6}}$

17. Question

Find the equation of the plane through the intersection of the planes 3x – 4y + 5z = 10 and 2x + 2y – 3z = 4 and parallel to the line x = 2y = 3z.

Answer

We know that equation of plane passing through the intersection of planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$(a_1x + b_1y + c_1z + d_1) + k(a_2x + b_2y + c_2z + d_2) = 0$

So, equation of plane passing through the intersection of planes

3x – 4y + 5z – 10 = 0 and 2x + 2y – 3z – 4 = 0 is

(3x – 4y + 5z – 10) + k(2x + 2y – 3z – 4) = 0 ......(1)

⇒ x(3 + 2k) + y(– 4 + 2k) + z(5 – 3k) + (– 10 – 4k) = 0

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Given the plane is parallel to line $x = 2y = 3z$ or $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$

6×(3 + 2k) + 3× (– 4 + 2k) + 2×(5 – 3k) = 0

⇒ 18 + 12k – 12 + 6k + 10 – 6k = 0

⇒ $k = -\frac{16}{12} = -\frac{4}{3}$

Putting the value of k in equation (1)

$$(3x - 4y + 5z - 10) - \frac{4}{3}(2x + 2y - 3z - 4) = 0$$

$$\Rightarrow 3x - \frac{8}{3}x - 4y - \frac{8}{3}y + 5z + 4z - 10 + \frac{16}{3} = 0$$

$$\Rightarrow \frac{x}{3} - \frac{20y}{3} + 9z - \frac{14}{3} = 0$$

⇒ x – 20y + 27z – 14 = 0

The required equation is x – 20y + 27z – 14 = 0

18. Question

Find the vector and Cartesian forms of the equation of the plane passing through the point (1, 2, – 4) and parallel to the lines $\vec{r} = \left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$ and $\vec{r} = \left(\hat{i} - 3\hat{j} + 5\hat{k}\right) + \mu\left(\hat{i} + \hat{j} - \hat{k}\right)$. Also, find the distance of the point (9, – 8, – 10) from the plane thus obtained.

Answer

The plane passes through the point (1,2, – 4)

A vector in a direction perpendicular to

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + m(2\hat{i} + 3\hat{j} + 6\hat{k})$ and

$\vec{r} = (\hat{i} - \hat{3}\hat{j} + 5\hat{k}) + n(\hat{i} + \hat{j} - \hat{k})$ is

$\vec{n} = (2\hat{i} + 3\hat{j} + 6\hat{k}) \times (\hat{i} + \hat{j} - \hat{k})$

$$\Rightarrow \vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = -9\hat{i} + 8\hat{j} - \hat{k}$$

Equation of the plane is $(\vec{r} - \vec{a}).\vec{n} = 0$

$\overrightarrow{(r} - (\hat{i} + 2\hat{j} - 4\hat{k})).(-9\hat{i} + 8\hat{j} - \hat{k}) = 0$

$\Rightarrow \vec{r}.(-9\hat{i} + 8\hat{j} - \hat{k}) = 11$

Substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we get the Cartesian form as

– 9x + 8y – z = 11

The distance of the point (9, – 8, – 10) from the plane

$$= \frac{-9 \times 9 + 8 \times -8 - (-10) - 11}{\sqrt{9^2 + 8^2 + 1^2}} = \frac{-81 - 64 + 10 - 11}{\sqrt{81 + 64 + 1}} = \frac{146}{\sqrt{146}} = \sqrt{146}$$

19. Question

Find the equation of the plane passing through the point (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$.

Answer

We know that the equation of plane passing through (x_1,y_1,z_1) is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ...... (1)

So, equation of plane passing through (3,4,1) is

a(x – 3) + b(y – 4) + c(z – 1) = 0 ......(2)

It also passes through (0,1,0)

So, equation (2) must satisfy the point (0,1,0)

∴ a(0 – 3) + b(1 – 4) + c(0 – 1) = 0

⇒ – 3a – 3b – c = 0

⇒ 3a + 3b + c = 0 ......(3)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ...... (4)

So,

a×2 + b×7 + c×5 = 0

⇒ 2a + 7b + 5c = 0 ......(5)

Solving equation (3) and (5) by cross multiplication we have,

$$\frac{a}{3 \times 5 - (7) \times 1} = \frac{b}{2 \times 1 - 3 \times 5} = \frac{c}{3 \times 7 - 2 \times 3}$$

$\Rightarrow \frac{a}{15-7} = \frac{b}{2-15} = \frac{c}{21-6}$

$\Rightarrow \frac{a}{8} = \frac{b}{-13} = \frac{c}{15} = k(\text{let})$

∴ a = 8k, b = – – 13k and c = 15k

Putting the value in equation (2)

8k(x – 3) – 13k(y – 4) + 15k(z – 1) = 0

8kx – 24k – 13ky + 52k + 15kz – 15k = 0

Dividing by k we have

8x – 13y + 15z + 13 = 0

Equation of required plane is 8x – 13y + 15z + 13 = 0

20. Question

Find the coordinates of the point where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ intersects the plane x – y + z – 5 = 0. Also find the angle between the line and the plane.

Answer

Given line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$

Let $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = r$

⇒ x = 3r + 2, y = 4r – 1, z = 2r + 2

Substituting in the equation of the plane x – y + z – 5 = 0

We get , (3r + 2) – (4r – 1) + (2r + 2) – 5 = 0

⇒ 3r + 2 – 4r + 1 + 2r + 2 – 5 = 0

⇒ r = 0

∴ x = 3×0 + 2, y = 4×0 – 1 , z = 2×0 + 2

⇒ x = 2, y = – 1, z = 2

Direction ratios of the line are 3, 4, 2

Direction ratios of the line perpendicular to the plane are 1, – 1, 1

$\therefore \sin\theta = \frac{3 \times 1 + 4 \times -1 + 2 \times (1)}{\sqrt{3^2 + (4)^2 + 2^2}\sqrt{1^2 + -1^2 + (1)^2}}$

$\Rightarrow \sin\theta = \frac{3 - 4 + 2}{\sqrt{9 + 16 + 4}\sqrt{1 + 1 + 1}}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{29}\sqrt{3}} = \frac{1}{\sqrt{87}}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{87}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$

∴ the angle between the plane and the line is $\sin^{-1}(\frac{1}{\sqrt{87}})$

21. Question

Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r}.(\hat{i}+2\hat{j}-5\hat{k})+9=0.$

Answer

We know that equation of line passing through point $\vec{a}$ and parallel to vector $\vec{b}$ is given by

$\vec{r} = \vec{a} + k\vec{b}$................(1)

Given that , the line is passing through (1,2,3)

So, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

It is given that line is perpendicular to plane $\vec{r}(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$

So, normal to plane ($\vec{n}$) is parallel to $\vec{b}$

So, let $\vec{b} = l\vec{n} = l(\hat{i} + 2\hat{j} - 5\hat{k})$

Putting $\vec{a}$ and $\vec{b}$ in (1) , equation of line is

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + k(l(\hat{i} + 2\hat{j} - 5\hat{k}))$

$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + m(\hat{i} + 2\hat{j} - 5\hat{k})$

22. Question

Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane 10x + 2y – 11z = 3.

Answer

Direction ratios of the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ are (2,3,6)

Direction ratio of a line perpendicular to the plane

10x + 2y – 11z = 3 are 10, 2, – 11

As we know that the angle θ between the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and a plane $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If θ is the angle between the line and the plane, then

$\therefore \sin\theta = \frac{2\times10+3\times2+6\times-11}{\sqrt{2^2+(3)^2+6^2}\sqrt{10^2+2^2+(11)^2}}$

$\Rightarrow \sin\theta = \frac{20+6-66}{\sqrt{4+9+36}\sqrt{100+4+121}}$

$\Rightarrow \sin\theta = \frac{-40}{\sqrt{49}\sqrt{225}} = \frac{-40}{7\times15} = -\frac{8}{21}$

$\Rightarrow \sin\theta = -\frac{8}{21} \Rightarrow \theta = \sin^{-1}(-\frac{8}{21})$

∴ the angle between the plane and the line is $\sin^{-1}(-\frac{8}{21})$

23. Question

Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6.$

Answer

We know that the equation of line passing through (1,2,3) is given by

$\frac{x-1}{a1} = \frac{y-2}{b1} = \frac{z-3}{c1}$ ......(1)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ......(2)

Here, line (1) is parallel to plane,

x – y + 2z = 5

So,

a×1 + b× – 1 + c×2 = 0

⇒ a – b + 2c = 0 ......(3)

Also, line (1) is parallel to plane,

3x + y + z = 6

So,

a×3 + b×1 + c×1 = 0

⇒ 3a + b + c = 0 ......(4)

Solving equation (3) and (4) by cross multiplication we have,

$$\frac{a}{-1\times1-(1)\times2} = \frac{b}{3\times2-1\times1} = \frac{c}{1\times1-3\times-1}$$

$\Rightarrow \frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3}$

$\Rightarrow \frac{a}{-3} = \frac{b}{5} = \frac{c}{4} = k(\text{let})$

∴ a = – 3k, b = 5k and c = 4k

Putting the value in equation (1)

$$\frac{x-1}{-3k} = \frac{y-2}{5k} = \frac{z-3}{4k}$$

Multiplying by k we have

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

The required equation is $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$

The vector equation is

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + k\left(-3\hat{i} + 5\hat{j} + 4\hat{k}\right)$$

24. Question

Find the value of k such that the line $\frac{x-2}{6}=\frac{y-1}{\lambda}=\frac{z+5}{-4}$ is perpendicular to the plane 3x – y – 2z = 7.

Answer

Here, given midline $\frac{x-2}{6} = \frac{y-1}{k} = \frac{z+5}{-4}$ is perpendicular to plane 3x – y – 2z = 7.

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is perpendicular to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $\frac{a_1}{a_2} + \frac{b_1}{b_2} + \frac{c_1}{c_2} = 0$

So, normal vector of plane is parallel to line .

So, direction ratios of normal to plane are proportional to the direction ratios of line .

Here, $\frac{6}{3} = \frac{k}{-1} = \frac{-4}{-2}$

By cross multiplying the last two we have

– 2k = 4

⇒ k = – 2

25. Question

Find the equation of the plane passing through the points (– 1, 2, 0), (2, 2, – 1) and parallel to the line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

Answer

We know that the equation of plane passing through (x_1,y_1,z_1) is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ……(1)

So, equation of plane passing through (– 1,2,0) is

a(x + 1) + b(y – 2) + c(z – 0) = 0 ……(2)

It also passes through (2,2, – 1)

So, equation (2) must satisfy the point (2,2, – 1)

∴ a(2 + 1) + b(2 – 2) + c(– 1) = 0

⇒ 3a – c = 0 ……(3)

We know that line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ is parallel to plane $a_2x + b_2y + c_2z + d_2 = 0$ if $a_1a_2 + b_1b_2 + c_1c_2 = 0$ ……(4)

Here, the plane is parallel to line,

$$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$$

So,

a×1 + b×2 + c×1 = 0

⇒ a + 2b + c = 0 ……(5)

Solving equation (3) and (5) by cross multiplication we have,

$$\frac{a}{0-(-1)\times 2} = \frac{b}{1\times -1 - 3\times 1} = \frac{c}{3\times 2 - 0}$$

$\Rightarrow \frac{a}{2} = \frac{b}{-4} = \frac{c}{6}$

$\Rightarrow \frac{a}{1} = \frac{b}{-2} = \frac{c}{3} = k(\text{let})$

∴ a = k, b = – 2k and c = 3k

Putting the value in equation (2)

k(x + 1) – 2k(y – 2) + 3k(z – 0) = 0

kx + k – 2ky + 4k + 3kz = 0

Dividing by k we have

x – 2y + 3z + 5 = 0

The required equation is x – 2y + 3z + 5 = 0

Exercise 29.12

1. Question

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the(a) yz –plane(b) zx – planeAlso, Find the angle which this line makes with these planes?

Answer

(a) Direction ratio of given line are (5 – 3, 1 – 4, 6 – 1) = (2, – 3, 5)Hence, equation of line is $\frac{x-5}{2} = \frac{y-1}{-3} = \frac{z-6}{5} = r$For any point on the yz –plane x = 0x = 2r + 5 = 0r = $-\frac{5}{2}$y = – 3($-\frac{5}{2}$) + 1 = $\frac{17}{2}$z = 5($-\frac{5}{2}$) + 6 = $-\frac{13}{2}$Hence, the coordinates of this point are (0, $\frac{17}{2}$, $-\frac{13}{2}$).

(b): Direction ratio of given line are (5 – 3, 1 – 4, 6 – 1) = (2, – 3, 5)

Hence, equation of line is

$$\frac{x-5}{2} = \frac{y-1}{-3} = \frac{z-6}{5} = r$$

For any point on zx –plane y = 0

y = – 3r + 1 = 0

r = $\frac{1}{3}$

x = 2($\frac{1}{3}$) + 5 = $\frac{17}{3}$

z = 5($\frac{1}{3}$) + 6 = $\frac{23}{3}$

Hence, the coordinates of this point are ($\frac{17}{3}$, 0, $\frac{23}{3}$).

2. Question

Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane 2x + y + z = 7?

Answer

Let the coordinates of the points A and B be (3, – 4, – 5) and (2, – 3, 1) respectively.

The equation of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = r,$ where r is a constant

Thus, the equation of AB is

$$\frac{x-3}{2-3} = \frac{y-(-4)}{(-3)-(-4)} = \frac{z-(-5)}{1-(-5)} = r$$

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = r$$

Any point on the line AB is the form

– r + 3, r – 4, 6r – 5

Let p be the point of intersection of the line AB and the plane 2x + y + z = 7

Thus, we have,

2(– r + 3) + r – 4 + 6r – 5 = 7

⇒ – 2r + 6 + r – 4 + 6r – 5 = 7

⇒ 5r = 10

⇒ r = 2

Substituting the value of r in – r + 3, r – 4, 6r – 5, the coordinates of P are:

(– 2 + 3, 2 – 4, 12 – 5) = (1, – 2, 7)

3. Question

Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line and the $\vec{r}.\left(2\hat{i}-\hat{j}+2\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5.$

Answer

The equation of the given line is

$\vec{r}.(2\hat{i}-\hat{j}+2\hat{k})+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ ...... (1)

The equation of the given plane is

$\vec{r}.(\hat{i}-\hat{j}+\hat{k})=5$ ...... (2)

Substituting the value of from equation (1) in equation (2), We obtain

$[2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})].(\hat{i}-\hat{j}+\hat{k})=5$

(3λ + 2) – (4λ – 1) + (2λ + 2) = 5

λ = 0

Substituting the value of equation (1), We obtain the equation of the line as

$\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})$

This means that the position vector of the point of intersection of the line and the

plane is $\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})$

This shows that the point of intersection of the given plane and line is given by the coordinates, (2, – 1, 2).
The point is (– 1, – 5, – 10) .

The distance d between the points, (2, – 1, 2) and (– 1, – 5, – 10) is

d = $\sqrt{9+16+144}$

d = $\sqrt{169}$

d = 13

4. Question

Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r}=2\hat{i}-4\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-2\hat{j}+\hat{k}\right)=0.$

Answer

To find the point of intersection of the line

$\vec{r}=(2\hat{i}-4\hat{j}+2\hat{k})+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ and the plane

$\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$

We are substituting $\vec{r}$ of line in the plane.

$[2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].(\hat{i} - 2\hat{j} + \hat{k}) = 0$

$\Rightarrow 2 + 3\lambda + 8 - 8\lambda + 2 + 2\lambda = 0$

$\Rightarrow 3\lambda = 12$

$\Rightarrow \lambda = 4$

$\vec{r} = (2\hat{i} - 4\hat{j} + 2\hat{k}) + 4(3\hat{i} + 4\hat{j} + 2\hat{k}) = 14\hat{i} + 12\hat{j} + 10\hat{k}$

Hence, the distance of the point $2\hat{i} + 12\hat{j} + 5\hat{k}$ from $14\hat{i} + 12\hat{j} + 10\hat{k}$ is

$\sqrt{(14-2)^2 + (12-12)^2 + (10-5)^2} = \sqrt{169} = 13$

5. Question

Find the distance of the point P(– 1, – 5, – 10) from the point of intersection of the line joining the points A(2, – 1, 2) and B(5, 3, 4) with the plane x – y + z = 5 ?

Answer

Equation of line through the point A(2, – 1, 2) and B (5, 3, 4) is

$\frac{x-2}{5-2} = \frac{y+1}{3+1} = \frac{z-2}{4-2} = r$

$\Rightarrow \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = r$

$\Rightarrow x = 3r + 2, y = 4r - 1, z = 2r + 2$

Substituting these in the plane equation, We get

(3r + 2) – (4r – 1) + (2r + 2) = 5

⇒ r = 0

⇒ x = 2, y = – 1, z = 2

Distance of (2, – 1, 2) from (– 1, – 5, – 10) is

$= \sqrt{(2-(-1))^2 + (-1-(-5))^2 + (2-(-10))^2} = \sqrt{169}$

= 13

6. Question

Find the distance of the point P (3, 4, 4) from the point, where the line joining the points A(3, – 4, – 5) and B(2, – 3, 1) intersects the plane 2x + y + z = 7?

Answer

Equation of line through the point A (3, – 4, – 5) and B (2, – 3, 1) is

$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = r$

$\Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = r$

$\Rightarrow x = -r + 3, y = r - 4, z = 6r - 5$

Substituting these in the plane equation, We get

2(– r + 3) + (r – 4) + (6r – 5) = 7

⇒ r = 2

⇒ x = 1, y = – 4, z = 7

Distance of (1, – 2, 7) from (3, 4, 4) is

$= \sqrt{(3-1)^2 + (4+2)^2 + (4-7)^2} = \sqrt{49}$

= 7

7. Question

Find the distance of the point (1, – 5, 9) from the plane x – y + z = 5 measured along the line x = y = z ?

Answer

The given plane is x – y + z = 5...... (1)

We have to find the distance of the point (1, – 5, 9) from this plane measured along a line to

x = y = z

So, the direction ratio of the line from the point (1, – 5, 9) to the given plane will be the same as that

of given line.

The equation of line passing through (1, – 5, 9) and having direction ratio is

$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = r$

x = r + 1, y = r – 5, z = r + 9

put in equation (1)

r + 1 – r + 5 + r + 9 = 5

⇒ r + 15 = 5

⇒ r = – 10

Coordinates are (–10 + 1, –10 – 5, –10 + 9) is (–9, –8, –1)

Distance of (–9, –15, –1) from (1, – 5, 9)

$= \sqrt{(1+9)^2 + (-5+15)^2 + (9+1)^2}$

$= \sqrt{300}$

Exercise 29.13

1. Question

Show that the lines $\vec{r} = \left(2\hat{j} - 3\hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$ and $\vec{r} = \left(2\hat{i} + 6\hat{j} + 3\hat{k}\right) + \mu\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)$ are coplanar. Also, find the equation of the plane containing them.

Answer

$\vec{r} = \left(2\hat{j} - 3\hat{k}\right) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$

$\vec{r} = \left(2\hat{i} + 6\hat{j} + 3\hat{k}\right) + u(2\hat{i} + 3\hat{j} + 4\hat{k})$

We know that the lines,

$\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \lambda\overrightarrow{b_2}$ are coplanar if

$\overrightarrow{a_1}.\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right) = \overrightarrow{a_2}.(\overrightarrow{b_1}\times\overrightarrow{b_2})$

And the equation of the plane containing them is

$\vec{r}.\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right) = \overrightarrow{a_1}.(\overrightarrow{b_1}\times\overrightarrow{b_2})$

Here, $\overrightarrow{a_1} = 2\hat{j} - 3\hat{k}, \overrightarrow{b_1} = \hat{i} + 2\hat{j} + 3\hat{k}, \overrightarrow{a_2} = 2\hat{i} + 6\hat{j} + 3\hat{k}, \overrightarrow{b_2} = 2\hat{i} + 3\hat{j} + 4\hat{k})$

$$\therefore \overrightarrow{b_1}\times\overrightarrow{b_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 2 & 3 & 4\end{vmatrix} = \hat{i}(8-9) - \hat{j}(4-6) + \hat{k}(3-4)$$

$\overrightarrow{b_1}\times\overrightarrow{b_2} = -\hat{i} + 2\hat{j} - \hat{k}$

$\overrightarrow{a_1}.\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right) = \left(2\hat{j} - 3\hat{k}\right).\left(-\hat{i} + 2\hat{j} - \hat{k}\right) = 0 + 4 + 3 = 7$

And

$\overrightarrow{a_2}.\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right) = \left(2\hat{i} + 6\hat{j} + 3\hat{k}\right).\left(-\hat{i} + 2\hat{j} - \hat{k}\right) = -2 + 12 - 3 = 7$

Since $\overrightarrow{a_1}.\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right) = \overrightarrow{a_2}.(\overrightarrow{b_1}\times\overrightarrow{b_2})$, the lines are coplanar Now the equation of the plane containing the given lines is

$\vec{r}.\left(\overrightarrow{b_1}\times\overrightarrow{b_2}\right) = \overrightarrow{a_1}.(\overrightarrow{b_1}\times\overrightarrow{b_2})$

$\vec{r}.\left(-\hat{i} + 2\hat{j} - \hat{k}\right) = 7$

$\vec{r}.\left(\hat{i} - 2\hat{j} + \hat{k}\right) = -7$

$\vec{r}.\left(\hat{i} - 2\hat{j} + \hat{k}\right) + 7 = 0$

2. Question

Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ are coplanar. Also, find the equation of the plane containing them.

Answer

we know that line $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix}x_2 - x_1 & y_2 - y_1 & z_2 - z_1\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix} = 0$$

And equation of the plane containing them is

$$\begin{vmatrix}x - x_1 & y - y_1 & z - z_1\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix} = 0$$

Here, equation of lines are

$\frac{(x+1)}{-3} = \frac{(y-3)}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$

So, $x_1 = -1$, $y_1 = 3$, $z_1 = -2$, $l_1 = -3$, $m_1 = 2$, $n_1 = 1$

$x_2 = 0$, $y_2 = 7$, $z_2 = -7$, $l_2 = 1$, $m_2 = -3$, $n_2 = 2$

so,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 0 + 1 & 7 - 3 & -7 + 2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 4 & -5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$

= 1(4 + 3) – 4(– 6 – 1) – 5(9 – 2)

= 7 + 28 – 35

= 0

So, lines are coplanar

Equation of plane containing line is

$$\begin{vmatrix} x + 1 & y - 3 & z + 2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

(x + 1)(4 + 3) – (y – 3)(– 6 – 1) + (z + 2)(9 – 2) = 0

7x + 7 + 7y – 21 + 7z + 14 = 0

7x + 7y + 7z = 0

X + y + z = 0

3. Question

Find the equation of the plane containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and the point (0, 7, –7) and show that the line $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ also lies in the same plane.

Answer

we know that equation of plane passing through (x_1, y_1, z_1) is given by

a(x – x_1) + b(y – y_1) + c(z – z_1) = 0 ...... (1)

Required plane is passing through (0, 7, – 7) so

a(x – 0) + b(y – 7) + c(z + 7) = 0

ax + b(y – 7) + c(z + 7) = 0 ...... (2)

plane (2) also contain line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ so, it passes through point (– 1, 3, – 2),

a(– 1) + b(3 – 7) + c(– 2 + 7) = 0

– a – 4b + 5c = 0 ...... (3)

Also plane (2) will be parallel to line so,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

(a)(– 3) + (b)(2) + (c)(1) = 0

– 3a + 2b + c = 0 ...... (4)

Solution (3) and (4) by cross – multiplication,

$$\frac{a}{(-4)(1)-(5)(2)} = \frac{b}{(-3)(5)-(-1)(1)} = \frac{c}{(-1)(2)-(-4)(-3)}$$

$$\frac{a}{-4-10} = \frac{b}{-15+1} = \frac{c}{-2-12}$$

$$\frac{a}{-14} = \frac{b}{-14} = \frac{c}{14} = \lambda$$

a = – 14λ, b = – 14λ, c = – 14λ

put a, b, c in equation (2),

ax + b(y – 7) + c(z + 7) = 0

(– 14λ)x + (– 14λ)(y – 7) + (– 14λ)(z + 7) = 0

Dividing by (– 14λ) we get

x + y – 7 + z + 7 = 0

x + y + z = 0

so, equation of plane containing the given point and line is x + y + z = 0

the other line is $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$

so, $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(1)(1) + (1)(– 3) + (1)(2) = 0

1 – 3 + 2 = 0

0 = 0

LHS = RHS

So, $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ lie on plane x + y + z = 0

4. Question

Find the equation of the plane which contains two parallel lines $\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$ and $\frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$.

Answer

we know that equation of plane passing through (x_1, y_1, z_1) is given by

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ...... (1)

since, required plane contain lines

$\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$ and $\frac{x-3}{1} = \frac{y+2}{-4} = \frac{z}{5}$

So, required plane passes through (4, 3, 2) and (3, – 2, 0) so equation of required plane is

a(x – 4) + b(y – 3) + c(z – 2) = 0 ...... (2)

plane (2) also passes through (3, – 2, 0), so

a(3 – 4) + b(– 2 – 3) + c(0 – 2)

– a – 5b – 2c = 0

a + 5b + 2c = 0 ...... (3)

now plane (2) is also parallel to line with direction ratios 1, – 4, 5 so,

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

(a)(1) + (b)(– 4) + (c)(5) = 0

a – 4b + 5c = 0 ...... (4)

solving equation (3) and (4) by cross – multiplication,

$$\frac{a}{(5)(5)-(-4)(2)} = \frac{b}{(1)(2)-(1)(5)} = \frac{c}{(1)(-4)-(1)(5)}$$

$$\frac{a}{25 + 8} = \frac{b}{2-5} = \frac{c}{-4-5}$$

$$\frac{a}{33} = \frac{b}{-3} = \frac{c}{-9}$$

Multiplying by 3,

$$\frac{a}{11} = \frac{b}{-1} = \frac{c}{-3} = \lambda$$

a = 11 λ, b = – λ, c = – 3 λ

put a, b, c in equation (2),

a(x – 4) + b(y – 3) + c(z – 2) = 0

(11 λ)(x – 4) + (– λ)(y – 3) + (– 3 λ)(z – 2) = 0

11 λx – 44 λ – λy + 3 λ – 3 λz + 6 λ = 0

11 λx – λy – 3 λz – 35 λ = 0

Dividing by λ,

11x – y – 3z – 35 = 0

So, equation of required plane is 11x – y – 3z – 35 = 0

5. Question

Show that the lines $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ and 3x – 2y + z + 5 = 0 = 2x + 3y + 4z – 4 intersect. Find the equation of the plane in which they lie and also their point of intersection.

Answer

we have, equation of the line is $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2} = \lambda$

General point on the line is given by (3 λ – 4, 5 λ – 6, – 2 λ + 1) ...... (1)

Another equation of line is

3x – 2y + z + 5 = 0

2x + 3y + 4z – 4 = 0

Let a, b, c be the direction ratio of the line so, it will be perpendicular to normal of 3x – 2y + z + 5 = 0 and 2x + 3y + 4z – 4 = 0

So, using a1a2 + b1b2 + c1c2 = 0

(3)(a) + (– 2)(b) + (1)(c) = 0

3a – 2b + c = 0 ...... (2)

Again, (2)(a) + (3)(b) + (4)(c) = 0

2a + 3b + 4c = 0 ...... (3)

Solving (2) and (3) by cross – multiplication,

$$\frac{a}{(-2)(4)-(3)(1)} = \frac{b}{(2)(1)-(3)(4)} = \frac{c}{(3)(3)-(-2)(2)}$$

$$\frac{a}{-8-3} = \frac{b}{2-12} = \frac{c}{9+4}$$

$$\frac{a}{-11} = \frac{b}{-10} = \frac{c}{13}$$

Direction ratios are proportional to – 11, – 10, 13

Let z = 0 so

3x – 2y = – 5 ...... (i)

2x + 3y = 4 ...... (ii)

Solving (i) and (ii) by eliminations method,

$$6x - 4y = -10$$

$$\underline{\pm 6x \pm 9y = \pm 12}$$

– 13y = – 22

$$y = \frac{22}{13}$$

Put y in equation (i)

3x – 2y = – 5

$3x - 2\frac{22}{13} = -5$

$3x - \frac{44}{13} = -5$

$3x = -5 + \frac{44}{13}$

$3x = \frac{-21}{13}$

$x = \frac{-7}{13}$

so, the equation of the line (2) in symmetrical form,

$$\frac{x+\frac{7}{13}}{-11} = \frac{y-\frac{22}{13}}{-10} = \frac{z-0}{13}$$

Put the general point of a line from equation (1)

$$\frac{3\lambda-4+\frac{7}{13}}{-11} = \frac{5\lambda-6-\frac{22}{13}}{-10} = \frac{-2\lambda+1}{13}$$

$$\frac{39\lambda-52+7}{-11\times 13} = \frac{65\lambda-78-22}{-10\times 13} = \frac{-2\lambda+1}{13}$$

$$\frac{39\lambda-45}{-11} = \frac{65\lambda-100}{-10} = \frac{-2\lambda+1}{1}$$

The equation of the plane is 45x – 17y + 25z + 53 = 0

Their point of intersection is (2, 4, – 3)

6. Question

Show that the plane whose vector equation is $\vec{r}.\left(\hat{i}+2\hat{j}-\hat{k}\right)=3$ contains the line whose vector equation is $\vec{r}=\hat{i}+\hat{j}+\lambda\left(2\hat{i}+\hat{j}+4\hat{k}\right)$.

Answer

we know that plane $r.n = d$ contains the line $r = a + \lambda b$ if

$b.n = 0$

$a.n = d$

Given, equation of plane $\vec{r}.(\hat{\imath} + 2\hat{\jmath} - \hat{k}) = 3$ and equation of line $\vec{r} = \hat{\imath} + \hat{\jmath} + \lambda(2\hat{\imath} + \hat{\jmath} + 4\hat{k})$

So $\vec{n} = \hat{\imath} + 2\hat{\jmath} - \hat{k}, \vec{a} = \hat{\imath} + \hat{\jmath}$

d = 3 $\vec{b} = 2\hat{\imath} + \hat{\jmath} + 4\hat{k}$

$\vec{b}.\vec{n} = (2\hat{\imath} + \hat{\jmath} + 4\hat{k})(\hat{\imath} + 2\hat{\jmath} - \hat{k})$

= (2)(1) + (1)(2) + (4)(– 1)

= 2 + 2 – 4

$\vec{b}.\vec{n} = 0$

$\vec{a}.\vec{n} = (\hat{\imath} + \hat{\jmath})(\hat{\imath} + 2\hat{\jmath} - \hat{k})$

= (1)(1) + (1)(2) + (0)(– 1)

= 1 + 2 – 0

= 3

= d

Since, $\vec{b}.\vec{n} = 0$ and $\vec{a}.\vec{n} = d$

So, Given line is on the given plane.

Hence, Proved.

7. Question

Find the equation of the plane determined by the intersection of the lines $\frac{x+3}{3}=\frac{y}{-2}=\frac{z-7}{6}$ and $\frac{x+6}{1}=\frac{y+5}{-3}=\frac{z-1}{2}$,

Answer

Let L_1: $\frac{x+3}{3} = \frac{y}{-2} = \frac{z-7}{6}$ and

L_2 : $\frac{x+6}{1} = \frac{y+5}{-3} = \frac{z-1}{2}$ be the equations of two lines

Let the plane be ax + by + cz + d = 0...... (1)

Given that the required plane through the intersection of the lines L_1 and L_2

Hence the normal to the plane is perpendicular to the lines L_1 and L_2

$\therefore 3a - 2b + 6c = 0$

$a - 3b + 2c = 0$

Using cross multiplication, we get

$$\frac{a}{-4 + 18} = \frac{b}{6 - 6} = \frac{c}{-9 + 2}$$

$$\frac{a}{14} = \frac{b}{0} = \frac{c}{-1}$$

$$\frac{a}{2} = \frac{b}{0} = \frac{c}{-1}$$

8. Question

Find the vector equation of the plane passing through the points (3, 4, 2) and (7, 0, 6) and perpendicular to the plane $2x - 5y - 15 = 0$. Also, show that the plane thus obtained contains the line $\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$.

Answer

Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ...... (1)

Plane is passing through (3, 4, 2) and (7, 0, 6)

$$\frac{3}{a} + \frac{4}{b} + \frac{2}{c} = 1$$

$$\frac{7}{a} + \frac{0}{b} + \frac{6}{c} = 1$$

Required plane is perpendicular to $2x - 5y - 15 = 0$

$$\frac{2}{a} + \frac{-5}{b} + \frac{0}{c} = 0$$

$2b = 5a$

$\therefore b = 2.5a$

$$\frac{3}{a} + \frac{4}{2.5a} + \frac{2}{c} = 1$$

$$\frac{7}{a} + \frac{6}{c} = 1$$

Solving the above equations

$a = 3.4 = \frac{17}{5}$, $b = 8.5 = \frac{17}{2}$ and $c = -\frac{34}{6} = -\frac{17}{3}$

substituting the values in (1)

$$\frac{x}{\frac{17}{5}} + \frac{y}{\frac{17}{2}} + \frac{z}{-\frac{17}{3}} = 1$$

$$\frac{5x}{17} + \frac{2y}{17} - \frac{3z}{17} = 1$$

$5x + 2y - 3z = 17$

$$(x\hat{i} + y\hat{j} + z\hat{k}).(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$$

$$\vec{r}.(5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$$

Vector equation of the plane is $\vec{r}.\left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 17$

The line passes through B(1, 3, – 2)

5(1) + 2(3)—3(– 2) = 17

The point B lies on the plane .

∴ the line $\vec{r} = \hat{i} + 3\hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$ lies on the plane $\vec{r}.\left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 17$

9. Question

If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of the plane containing these lines.

Answer

The direction ratio of the line $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ is r_1 = (– 3, – 2k, 2)

The direction ratio of the line $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ is r_2 = (k, 1, 5)

Since the line $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular so

$r_1.r_2 = 0$

(– 3, – 2k, 2). (k, 1, 5) = 0

– 3k – 2k + 10 = 0

– 5k = – 10

k = 2

∴ the equation of the line are $\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}$ and $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$

The equation of the plane containing the perpendicular lines $\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}$ and $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$ is

$$\begin{vmatrix} x & y & z \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{vmatrix} = d$$

(– 20 – 2)x – y(– 15 – 4) + z(– 3 + 8) = d

– 22x + 19y + 5z = d

The line $\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}$ pass through the point (1, 2, 3) so putting x = 1, y = 2, z = 3 in the equation –22x + 19y + 5z = d we get

– 22(1) + 19(2) + 5(3) = d

d = – 22 + 38 + 15

d = 31

∴ The equation of the plane containing the lines is – 22x + 19y + 5z = 31

10. Question

Find the coordinates of the point where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ intersect the plane x – y + z – 5 = 0. Also, find the angle between the line and the plane.

Answer

Any point on the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = k$ is of the form,

(3k + 2, 4k – 1, 2k + 2)

If the point p(3k + 2, 4k – 1, 2k + 2) lies in the planex – y + z – 5 = 0, we have,

(3k + 2) – (4k – 1) + (2k + 2) – 5 = 0

3k + 2 – 4k + 1 + 2k + 2 – 5 = 0

k = 0

thus, the coordinates of the point of intersection of the line and the plane are :

{3(0) + 2, 4(0) – 1, 2(0) + 2}

P(2, – 1,, 2)

Let θ be the angle between the line and the plane . thus

$\sin\theta = \frac{al + bm + cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{l^2+m^2+n^2}}$, where l, m and n are the direction ratios of the line and a, b and c are the direction ratios of the normal to the plane

Here, l = 3, m = 4, n = 2, a = 1, b = – 1, and c = 1 hence,

$$\sin\theta = \frac{1\times 3 + (-1)\times 4 + 1\times 2}{\sqrt{1^2 + (-1)^2 + 1^2}\sqrt{3^2 + 4^2 + 2^2}}$$

$$\sin\theta = \frac{1}{\sqrt{3}\sqrt{29}}$$

$$\theta = \sin^{-1}\left(\frac{1}{\sqrt{3}\sqrt{29}}\right)$$

11. Question

Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also, find the coordinates of the point of intersection of this plane and the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$.

Answer

Let A, B and C be three point with position vector $\hat{i} + \hat{j} - 2\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$

Thus, $\overrightarrow{AB} = \vec{b} - \vec{a} = \left(2\hat{i} - \hat{j} + \hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right) = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\overrightarrow{AC} = \vec{c} - \vec{a} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right) = \hat{j} + 3\hat{k}$$

As we know that cross product of two vectors gives a perpendicular vector so

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix}$$

$$\vec{n} = \hat{i}(-6-3) - 3\hat{j} + \hat{k} = -9\hat{i} - 3\hat{j} + \hat{k}$$

So, the equation of the required plane is

$$(\vec{r} - \vec{a}).\vec{n} = 0$$

$$(\vec{r}.\vec{n}) = (\vec{a}.\vec{n})$$

$$(\vec{r}.\left(-9\hat{i} - 3\hat{j} + \hat{k}\right)) = \left(\hat{i} + \hat{j} - 2\hat{k}\right).\left(-9\hat{i} - 3\hat{j} + \hat{k}\right)$$

$\vec{r}.\left(-9\hat{i}-3\hat{j}+\hat{k}\right) = 14$

Also we have to find the coordinates of the point of intersection of this plane and the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$

Any point on the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$is of the form, p(3 + 2λ, – 1 – 2λ, – 1 + λ)

Point p(3 + 2λ, – 1 – 2λ, – 1 + λ) lies in the plane,

$\vec{r}.\left(-9\hat{i}-3\hat{j}+\hat{k}\right) = 14$ so,

9(3 + 2λ) – 3(– 1 – 2λ) – (– 1 + λ) = 14

27 + 18λ – 3 – 6λ + 1 – λ = 14

11λ = – 11

λ = – 1

Thus the required point of intersection is

p(3 + 2λ, – 1 – 2λ, – 1 + λ)

put value of λ in this equation

p[3 + 2(– 1), – 1 – 2(– 1), – 1 + (– 1)]

p(1, 1, – 2)

12. Question

Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar.

Answer

$$\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ ......(1)

$$\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-8}{3}$$

$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-8}{3}$ ......(2)

$a_1 = 4$, $b_1 = 4$, $c_1 = -5$

$a_2 = 7$, $b_2 = 1$, $c_2 = 3$

$x_1 = 5$, $y_1 = 7$, $z_1 = -3$

$x_2 = 8$, $y_2 = 4$, $z_2 = 5$

the condition for two line to be coplanar,

$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$= \begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$

= 3(12 + 5) + 3(12 + 35) + 8(4 – 28)

= 3 × 17 + 3 × 47 + 8 × (–24)

= 51 + 141 – 192

= 192 – 192

= 0

∴ The lines are coplanar to each other .

13. Question

Find the equation of a plane which passes through the point (3, 2, 0) and contains the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$.

Answer

Given that, a plane is passes through the point (3, 2, 0) so equation will be

a(x – 3) + b(y – 2) + c(z – 0) = 0

a(x – 3) + b(y – 2) + cz = 0 ……(1)

plane also contains the line $\frac{x-3}{1} = \frac{y-6}{5} = \frac{z-4}{4}$

so it pass through the point (3, 2, 0)

a(3 – 3) + b(6 – 2) + c(4) = 0

4b + 4c = 0 …… (2)

Also plane will be parallel to,

a(1) + b(5) + c(4) = 0

a + 5b + 4c = 0 ……(3)

solving (2) and (3) by cross multiplication,

$$\frac{a}{16-20} = \frac{b}{4-0} = \frac{c}{0-4} = k$$

$$-\frac{a}{4} = \frac{b}{4} = -\frac{c}{4} = k$$

a = – k, b = k, c = – k

put a = – k, b = k, c = – k in equation (i) we get

(– k)(x – 3) + (k)(y – 2) + (– k)z = 0

– x + 3 + y – 2 – z = 0

x – y + z – 1 = 0

14. Question

Show that the lines $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar. Hence, find the equation of the plane containing these lines.

Answer

We know that the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here,

$x_1 = -3$, $x_1 = -1$, $y_1 = 1$, $y_2 = 2$, $z_1 = 5$, $z_2 = 5$

$l_1 = -3$, $l_2 = -1$, $m_1 = 1$, $m_2 = 2$, $n_1 = 5$, $n_2 = 5$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 2(5-10) - 1\big(-15-(-5)\big) + 0\big(-6-(-1)\big)$$

= 2(–5) – 1(–10) = – 10 + 10

= 0

So the given line are coplanar .

The equation of plane contains lines is $\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$

(x + 3)(5 – 10) – (y – 1)(– 15 – (– 5)) + (z – 5)(– 6 – (– 1)) = 0

– 5x – 15 + 10y – 10 – 5z + 25 = 0

– 5x + 10y – 5z = 0

Divided by – 5

x – 2y + z = 0

15. Question

If the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane lx + my – z = 9, then find the value of $l^2 + m^2$?

Answer

We know that the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ lies in plane ax + by + cz + d = 0, then

$ax_1 + by_1 + cz_1 + d = 0$ and $al + bm + cn = 0$

Here,

$x_1 = 3$, $y_1 = -2$, $z_1 = -4$ and l = 2, m = –1, n = 3

a = l, b = m, c = –1, d = –9

i.e, 3l + (– 2)m + (– 4)(– 1) – 9 = 0 and 2l – m – 3 = 0

3l – 2m = 5 and 2l – m = 3

3l – 2m = 5 ...... (1)

2l – m = 3 ......(2)

Multiply eq.(1) by 2 and eq.(2) by 3 and then subtract we get

m = –1

$l = 1$

$l^2 + m^2 = 2$

16. Question

Find the values of λ for which the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z+3}{\lambda^2}$ and $\frac{x-3}{1} = \frac{y-2}{\lambda^2} = \frac{z-1}{2}$ are coplanar.

Answer

We know that the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here,

$x_1 = 1, x_2 = 3, y_1 = 2, y_2 = 2, z_1 = 3, z_2 = 1$

$l_1 = 1, l_2 = 1, m_1 = 2, m_2 = \lambda^2, n_1 = \lambda^2, n_2 = 2$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 0 & -2 \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix} = 0$$

$2(4-\lambda^4) - 0(2-\lambda^2) - 2(\lambda^2-2) = 0$

$8 - 2\lambda^4 - 2\lambda^2 + 4 = 0$

$\lambda^4 + \lambda^2 - 6 = 0$

Let $\lambda^2 = t$ then

$(t + 3)(t - 2) = 0$

Put value of t

$(\lambda^2 + 3)(\lambda^2 - 2) = 0$

$\lambda^2 = -3$ is neglected because direction cosine can not be imaginary

$\therefore \lambda = \pm\sqrt{2}$

17. Question

If the lines $x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar, find the values of α.

Answer

We know that the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here,

$x_1 = 5, x_2 = \alpha, y_1 = 0, y_2 = 0, z_1 = 0, z_2 = 0$

$l_1 = 1, l_2 = 1, m_1 = 3 - \alpha, m_2 = -1, n_1 = -2, n_2 = 2 - \alpha$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} \alpha - 5 & 0 & 0 \\ 1 & 3 - \alpha & -2 \\ 1 & -1 & 2 - \alpha \end{vmatrix} = 0$$

$(\alpha - 5)[(3 - \alpha)(2 - \alpha) - 2] = 0$

$(\alpha - 5)(6 - 3\alpha - 2\alpha + \alpha^2 - 2) = 0$

$\alpha^3 - 10\alpha^2 + 29\alpha - 20 = 0$

$(\alpha - 1)(\alpha - 4)(\alpha - 5) = 0$

$\alpha = 1, 4, 5$

18. Question

If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, find the equations of the planes containing them.

Answer

We know that the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here,

$x_1 = 1, x_2 = -1, y_1 = -1, y_2 = -1, z_1 = 0, z_2 = 0$

$l_1 = 2, l_2 = 5, m_1 = k, m_2 = 2, n_1 = 2, n_2 = k$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0$$

$-2(k^2 - 4) = 0$

$k = \pm 2$

The equation of plane contains lines is $\begin{vmatrix} x - 1 & y + 1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$

when $k = 2$

$(x - 1)(4 - 4) - (y + 1)(4 - 10) + (z)(4 - 10) = 0$

$6y - 6z + 6 = 0$

$y - z + 1 = 0$

The equation of plane contains lines is $\begin{vmatrix} x - 1 & y + 1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$

When k = – 2

$(x - 1)(4 - 4) - (y + 1)(- 4 - 10) + (z)(4 + 10) = 0$

$14y + 14z + 14 = 0$

$y + z + 1 = 0$

Exercise 29.14

1. Question

Find the shortest distance between the lines $\frac{x-2}{-1} = \frac{y-5}{2} = \frac{z-0}{3}$ and $\frac{x-0}{2} = \frac{y+1}{-1} = \frac{z-1}{2}$.

Answer

Let the two lines be l_1 and l_2.

So, $l_1 : \frac{x-2}{-1} = \frac{y-5}{2} = \frac{z-0}{3}$ and $l_2 : \frac{x-0}{2} = \frac{y+1}{-1} = \frac{z-1}{2}$

We need to find the shortest distance between l_1 and l_2.

Recall the shortest distance between the lines: $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by

$$\left| \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \right|$$

Here, $(x_1, y_1, z_1) = (2, 5, 0)$ and $(x_2, y_2, z_2) = (0, -1, 1)$

Also $(a_1, b_1, c_1) = (-1, 2, 3)$ and $(a_2, b_2, c_2) = (2, -1, 2)$

We will evaluate the numerator first.

Let $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = N$

$\Rightarrow N = \begin{vmatrix} 0 - 2 & -1 - 5 & 1 - 0 \\ -1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix}$

$\Rightarrow N = \begin{vmatrix} -2 & -6 & 1 \\ -1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix}$

$\Rightarrow N = (-2)[(2)(2) - (-1)(3)] - (-6)[(-1)(2) - (2)(3)] + (1)[(-1)(-1) - (2)(2)]$

$\Rightarrow N = -2(4 + 3) + 6(-2 - 6) + (1 - 4)$

$\Rightarrow N = -14 - 48 - 3$

$\therefore N = -65$

Now, we will evaluate the denominator.

Let $\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = D$

$b_1c_2 - b_2c_1 = (2)(2) - (-1)(3) = 4 - (-3) = 7$

$c_1a_2 - c_2a_1 = (3)(2) - (2)(-1) = 6 - (-2) = 8$

$a_1b_2 - a_2b_1 = (-1)(-1) - (2)(2) = 1 - 4 = -3$

$\Rightarrow D = \sqrt{7^2 + 8^2 + (-3)^2}$

$\Rightarrow D = \sqrt{49 + 64 + 9}$

$\therefore D = \sqrt{122}$

So, shortest distance = $\left|\frac{-65}{\sqrt{122}}\right| = \frac{65}{\sqrt{122}}$

Thus, the required shortest distance is $\frac{65}{\sqrt{122}}$ units.

2. Question

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.

Answer

Let the two lines be l_1 and l_2.

So, $l_1: \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $l_2: \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

We need to find the shortest distance between l_1 and l_2.

Recall the shortest distance between the lines: $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by

$$\left|\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}\right|$$

Here, $(x_1, y_1, z_1) = (-1, -1, -1)$ and $(x_2, y_2, z_2) = (3, 5, 7)$

Also $(a_1, b_1, c_1) = (7, -6, 1)$ and $(a_2, b_2, c_2) = (1, -2, 1)$

We will evaluate the numerator first.

Let $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = N$

$$\Rightarrow N = \begin{vmatrix} 3-(-1) & 5-(-1) & 7-(-1) \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$\Rightarrow N = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$\Rightarrow N = (4)[(-6)(1) - (-2)(1)] - (6)[(7)(1) - (1)(1)] + (8)[(7)(-2) - (1)(-6)]$

$\Rightarrow N = 4(-6 + 2) - 6(7 - 1) + 8(-14 + 6)$

$\Rightarrow N = -16 - 36 - 64$

$\therefore N = -116$

Now, we will evaluate the denominator.

Let $\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = D$

$b_1c_2 - b_2c_1 = (-6)(1) - (-2)(1) = -6 + 2 = -4$

$c_1a_2 - c_2a_1 = (1)(1) - (1)(7) = 1 - 7 = -6$

$a_1b_2 - a_2b_1 = (7)(-2) - (1)(-6) = -14 + 6 = -8$

$\Rightarrow D = \sqrt{(-4)^2 + (-6)^2 + (-8)^2}$

$\Rightarrow D = \sqrt{16 + 36 + 64}$

$\therefore D = \sqrt{116}$

So, shortest distance = $\left|\frac{-116}{\sqrt{116}}\right| = \sqrt{116} = 2\sqrt{29}$

Thus, the required shortest distance is $2\sqrt{29}$ units.

3. Question

Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z+2}{1}$ and $3x - y - 2z + 4 = 0 = 2x + y + z + 1$.

Answer

Let the two lines be l_1 and l_2.

So, $l_1: \frac{x-1}{2} = \frac{y-3}{4} = \frac{z+2}{1}$ and l_2: $3x - y - 2z + 4 = 0 = 2x + y + z + 1$

We need to find the shortest distance between l_1 and l_2.

The equation of a plane containing the line l_2 is given by

$(3x - y - 2z + 4) + \lambda(2x + y + z + 1) = 0$

$\Rightarrow (3 + 2\lambda)x + (\lambda - 1) y + (\lambda - 2) z + (4 + \lambda) = 0$

Direction ratios of l_1 are 2, 4, 1 and those of the line containing the shortest distance are proportional to $3 + 2\lambda$, $\lambda - 1$ and $\lambda - 2$.

We know that if two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular to each other, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

$\Rightarrow (3 + 2\lambda)(2) + (\lambda - 1)(4) + (\lambda - 2)(1) = 0$

$\Rightarrow 6 + 4\lambda + 4\lambda - 4 + \lambda - 2 = 0$

$\Rightarrow 9\lambda = 0$

$\therefore \lambda = 0$

Thus, the plane containing line l_2 is $3x - y - 2z + 4 = 0$.

We have $l_1: \frac{x-1}{2} = \frac{y-3}{4} = \frac{z+2}{1} = \alpha$ (say)

When $\alpha = 0$, $(x, y, z) = (1, 3, -2)$

So, the point $(1, 3, -2)$ lies on the line l_1.

Hence, the shortest distance between the two lines is same as the distance of the perpendicular from $(1, 3, -2)$ on to the plane $3x - y - 2z + 4 = 0$.

Recall the length of the perpendicular drawn from (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is given by

$$\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Here, $(x_1, y_1, z_1) = (1, 3, -2)$ and $(A, B, C, D) = (3, -1, -2, 4)$

$$\Rightarrow d = \left|\frac{(3)(1) + (-1)(3) + (-2)(-2) + 4}{\sqrt{3^2 + (-1)^2 + (-2)^2}}\right|$$

$$\Rightarrow d = \left|\frac{3-3+4+4}{\sqrt{9+1+4}}\right|$$

$$\Rightarrow d = \left|\frac{8}{\sqrt{14}}\right|$$

$$\therefore d = \frac{8}{\sqrt{14}}$$

Thus, the required shortest distance is $\frac{8}{\sqrt{14}}$ units.

Exercise 29.15

1. Question

Find the image of the point (0, 0, 0) in the plane 3x + 4y – 6z + 1 = 0.

Answer

Let point P = (0, 0, 0) and M be the image of P in the plane 3x + 4y – 6z + 1 = 0.

Direction ratios of PM are proportional to 3, 4, – 6 as PM is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

Here, (x_1, y_1, z_1) = (0, 0, 0) and (l, m, n) = (3, 4, –6)

Hence, the equation of PM is

$$\frac{x-0}{3} = \frac{y-0}{4} = \frac{z-0}{-6}$$

$$\Rightarrow \frac{x}{3} = \frac{y}{4} = \frac{z}{-6} = \alpha \text{ (say)}$$

⇒ x = 3α, y = 4α, z = –6α

Let M = (3α, 4α, –6α).

As M is the image of P in the given plane, the midpoint of PM lies on the plane.

Using the midpoint formula, we have

$$\text{Midpoint of PM} = \left(\frac{0+3\alpha}{2}, \frac{0+4\alpha}{2}, \frac{0-6\alpha}{2}\right)$$

$$\Rightarrow \text{Midpoint of PM} = \left(\frac{3\alpha}{2}, 2\alpha, -3\alpha\right)$$

This point lies on the given plane, which means this point satisfies the plane equation.

$$\Rightarrow 3\left(\frac{3\alpha}{2}\right) + 4(2\alpha) - 6(-3\alpha) + 1 = 0$$

$$\Rightarrow \frac{9\alpha}{2} + 8\alpha + 18\alpha + 1 = 0$$

$$\Rightarrow \frac{9\alpha\ +\ 16\alpha\ +\ 36\alpha}{2} = -1$$

$$\Rightarrow 61\alpha = -2$$

$$\therefore \alpha = -\frac{2}{61}$$

We have M = (3α, 4α, –6α)

$$\Rightarrow \mathrm{M} = \left(3\left(-\frac{2}{61}\right), 4\left(-\frac{2}{61}\right), -6\left(-\frac{2}{61}\right)\right)$$

$$\therefore \mathrm{M} = \left(-\frac{6}{61}, -\frac{8}{61}, \frac{12}{61}\right)$$

Thus, the image of (0, 0, 0) in the plane 3x + 4y – 6z + 1 = 0 is $\left(-\frac{6}{61}, -\frac{8}{61}, \frac{12}{61}\right)$.

2. Question

Find the reflection of the point (1, 2, –1) in the plane 3x – 5y + 4z = 5.

Answer

Let point P = (1, 2, –1) and M be the image of P in the plane 3x – 5y + 4z = 5.

Direction ratios of PM are proportional to 3, –5, 4 as PM is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, (x_1, y_1, z_1) = (1, 2, –1) and (l, m, n) = (3, –5, 4)

Hence, the equation of PM is

$$\frac{x - 1}{3} = \frac{y - 2}{-5} = \frac{z - (-1)}{4}$$

$$\Rightarrow \frac{x - 1}{3} = \frac{y - 2}{-5} = \frac{z + 1}{4} = \alpha \text{ (say)}$$

⇒ x = 3α + 1, y = –5α + 2, z = 4α – 1

Let M = (3α + 1, –5α + 2, 4α – 1).

As M is the image of P in the given plane, the midpoint of PM lies on the plane.

Using the midpoint formula, we have

$$\text{Midpoint of PM} = \left(\frac{1 + (3\alpha + 1)}{2}, \frac{2 + (-5\alpha + 2)}{2}, \frac{-1 + (4\alpha - 1)}{2}\right)$$

$$\Rightarrow \text{Midpoint of PM} = \left(\frac{3\alpha + 2}{2}, \frac{-5\alpha + 4}{2}, 2\alpha - 1\right)$$

This point lies on the given plane, which means this point satisfies the plane equation.

$$\Rightarrow 3\left(\frac{3\alpha + 2}{2}\right) - 5\left(\frac{-5\alpha + 4}{2}\right) + 4(2\alpha - 1) = 5$$

$$\Rightarrow \frac{9\alpha + 6}{2} - \frac{-25\alpha + 20}{2} + 8\alpha - 4 = 5$$

$$\Rightarrow \frac{9\alpha + 6 + 25\alpha - 20 + 16\alpha}{2} = 9$$

$$\Rightarrow 50\alpha - 14 = 18$$

$$\Rightarrow 50\alpha = 32$$

$$\therefore \alpha = \frac{32}{50} = \frac{16}{25}$$

We have M = (3α + 1, –5α + 2, 4α – 1)

$\Rightarrow \text{M} = \left(3\left(\frac{16}{25}\right)+1, -5\left(\frac{16}{25}\right)+2, 4\left(\frac{16}{25}\right)-1\right)$

$\Rightarrow \text{M} = \left(\frac{48}{25}+1, -\frac{16}{5}+2, \frac{64}{25}-1\right)$

$\therefore \text{M} = \left(\frac{73}{25}, -\frac{6}{5}, \frac{39}{25}\right)$

Thus, the image of (1, 2, –1) in the plane 3x – 5y + 4z = 5 is $\left(\frac{73}{25}, -\frac{6}{5}, \frac{39}{25}\right)$.

3. Question

Find the coordinates of the foot of the perpendicular drawn from the point (5, 4, 2) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$. Hence or otherwise deduce the length of the perpendicular.

Answer

Let point P = (5, 4, 2) and Q be the foot of the perpendicular drawn from to P the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$.

Q is a point on the given line. So, for some α, Q is given by

$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = \alpha \text{ (say)}$

⇒ x = 2α – 1, y = 3α + 3, z = –α + 1

Thus, Q = (2α – 1, 3α + 3, –α + 1)

Now, we find the direction ratios of PQ.

Recall the direction ratios of a line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by ($x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$).

Here, (x_1, y_1, z_1) = (5, 4, 2) and (x_2, y_2, z_2) = (2α – 1, 3α + 3, –α + 1)

⇒ Direction Ratios of PQ are ((2α – 1) – (5), (3α + 3) – (4), (–α + 1) – (2))

⇒ Direction Ratios of PQ are (2α – 6, 3α – 1, –α – 1)

PQ is perpendicular to the given line, whose direction ratios are (2, 3, –1).

We know that if two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2) are perpendicular to each other, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

⇒ (2)(2α – 6) + (3)(3α – 1) + (–1)(–α – 1) = 0

⇒ 4α – 12 + 9α – 3 + α + 1 = 0

⇒ 14α – 14 = 0

⇒ 14α = 14

∴ α = 1

We have Q = (2α – 1, 3α + 3, –α + 1)

⇒ Q = (2×1 – 1, 3×1 + 3, –1 + 1)

∴ Q = (1, 6, 0)

Using the distance formula, we have

$\text{PQ} = \sqrt{(1-5)^2 + (6-4)^2 + (0-2)^2}$

$\Rightarrow \text{PQ} = \sqrt{(-4)^2 + 2^2 + (-2)^2}$

$\Rightarrow \text{PQ} = \sqrt{16+4+4}$

$\therefore PQ = \sqrt{24} = 2\sqrt{6}$

Thus, the required foot of perpendicular is (1, 6, 0) and the length of the perpendicular is $2\sqrt{6}$ units.

4. Question

Find the image of the point with position vector $3\hat{i} + \hat{j} + 2\hat{k}$ in the plane $\vec{r} \cdot \left(2\hat{i} - \hat{j} + \hat{k}\right) = 4$. Also, find the position vectors of the foot of the perpendicular and the equation of the perpendicular line through $3\hat{i} + \hat{j} + 2\hat{k}$.

Answer

Let P be the point with position vector $\vec{p} = 3\hat{i} + \hat{j} + 2\hat{k}$ and M be the image of P in the plane $\vec{r}.\left(2\hat{i} - \hat{j} + \hat{k}\right) = 4$.

In addition, let Q be the foot of the perpendicular from P on to the given plane. So, Q is the midpoint of PM.

Direction ratios of PM are proportional to 2, –1, 1 as PM is normal to the plane and parallel to $2\hat{i} - \hat{j} + \hat{k}$.

Recall the vector equation of the line passing through the point with position vector $\vec{r}$ and parallel to vector $\vec{b}$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$

Here, $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$

Hence, the equation of PM is

$\vec{r} = \left(3\hat{i} + \hat{j} + 2\hat{k}\right) + \lambda\left(2\hat{i} - \hat{j} + \hat{k}\right)$

$\therefore \vec{r} = (3 + 2\lambda)\hat{i} + (1 - \lambda)\hat{j} + (2 + \lambda)\hat{k}$

Let the position vector of M be $\vec{m}$. As M is a point on this line, for some scalar α, we have

$\Rightarrow \vec{m} = (3 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (2 + \alpha)\hat{k}$

Now, let us find the position vector of Q, the midpoint of PM.

Let this be $\vec{q}$.

Using the midpoint formula, we have

$$\vec{q} = \frac{\vec{p} + \vec{m}}{2}$$

$$\Rightarrow \vec{q} = \frac{\left[3\hat{i} + \hat{j} + 2\hat{k}\right] + \left[(3 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (2 + \alpha)\hat{k}\right]}{2}$$

$$\Rightarrow \vec{q} = \frac{\left(3 + (3 + 2\alpha)\right)\hat{i} + \left(1 + (1 - \alpha)\right)\hat{j} + \left(2 + (2 + \alpha)\right)\hat{k}}{2}$$

$$\Rightarrow \vec{q} = \frac{(6 + 2\alpha)\hat{i} + (2 - \alpha)\hat{j} + (4 + \alpha)\hat{k}}{2}$$

$$\therefore \vec{q} = (3 + \alpha)\hat{i} + \frac{(2 - \alpha)}{2}\hat{j} + \frac{(4 + \alpha)}{2}\hat{k}$$

This point lies on the given plane, which means this point satisfies the plane equation $\vec{r}.\left(2\hat{i} - \hat{j} + \hat{k}\right) = 4$.

$$\Rightarrow \left[(3 + \alpha)\hat{i} + \frac{(2 - \alpha)}{2}\hat{j} + \frac{(4 + \alpha)}{2}\hat{k}\right].\left(2\hat{i} - \hat{j} + \hat{k}\right) = 4$$

$$\Rightarrow 2(3 + \alpha) - \left(\frac{2 - \alpha}{2}\right)(1) + \left(\frac{4 + \alpha}{2}\right)(1) = 4$$

$\Rightarrow 6 + 2\alpha + \frac{4 + \alpha - (2 - \alpha)}{2} = 4$

$\Rightarrow 2\alpha + (1 + \alpha) = -2$

$\Rightarrow 3\alpha = -3$

$\therefore \alpha = -1$

We have the image $\vec{m} = (3 + 2\alpha)\hat{i} + (1 - \alpha)\hat{j} + (2 + \alpha)\hat{k}$

$\Rightarrow \vec{m} = [3 + 2(-1)]\hat{i} + [1 - (-1)]\hat{j} + [2 + (-1)]\hat{k}$

$\therefore \vec{m} = \hat{i} + 2\hat{j} + \hat{k}$

Therefore, image is (1, 2, 1)

Foot of the perpendicular $\vec{q} = (3 + \alpha)\hat{i} + \frac{(2-\alpha)}{2}\hat{j} + \frac{(4+\alpha)}{2}\hat{k}$

$\Rightarrow \vec{q} = [3 + (-1)]\hat{i} + \left[\frac{2 - (-1)}{2}\right]\hat{j} + \left[\frac{4 + (-1)}{2}\right]\hat{k}$

$\therefore \vec{q} = 2\hat{i} + \frac{3}{2}\hat{j} + \frac{3}{2}\hat{k}$

Thus, the position vector of the image is $\hat{i} + 2\hat{j} + \hat{k}$ and that of the foot of perpendicular is $2\hat{i} + \frac{3}{2}\hat{j} + \frac{3}{2}\hat{k}$.

5. Question

Find the coordinates of the foot of the perpendicular drawn from the point (1, 1, 2) to the plane 2x – 2y + 4z + 5 = 0. Also, find the length of the perpendicular.

Answer

Let point P = (1, 1, 2) and Q be the foot of the perpendicular drawn from P to the plane 2x – 2y + 4z + 5 = 0.

Direction ratios of PQ are proportional to 2, –2, 4 as PQ is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, (x_1, y_1, z_1) = (1, 1, 2) and (l, m, n) = (2, –2, 4)

Hence, the equation of PQ is

$$\frac{x - 1}{2} = \frac{y - 1}{-2} = \frac{z - 2}{4}$$

$$\Rightarrow \frac{x - 1}{2} = \frac{y - 1}{-2} = \frac{z - 2}{4} = \alpha \text{ (say)}$$

⇒ x = 2α + 1, y = –2α + 1, z = 4α + 2

Let Q = (2α + 1, –2α + 1, 4α + 2).

This point lies on the given plane, which means this point satisfies the plane equation.

⇒ 2(2α + 1) – 2(–2α + 1) + 4(4α + 2) + 5 = 0

⇒ 4α + 2 + 4α – 2 + 16α + 8 + 5 = 0

⇒ 24α + 13 = 0

⇒ 24α = –13

$\therefore \alpha = -\frac{13}{24}$

We have Q = (2α + 1, –2α + 1, 4α + 2)

$\Rightarrow Q = \left(2\left(-\frac{13}{24}\right) + 1, -2\left(-\frac{13}{24}\right) + 1, 4\left(-\frac{13}{24}\right) + 2\right)$

$\Rightarrow Q = \left(-\frac{13}{12} + 1, \frac{13}{12} + 1, \frac{-13}{6} + 2\right)$

$\therefore Q = \left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$

Recall the length of the perpendicular drawn from (x_1, y_1, z_1) to the plane Ax + By + Cz + D = 0 is given by

$\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$

Here, (x_1, y_1, z_1) = (1, 1, 2) and (A, B, C, D) = (2, –2, 4, 5)

$\Rightarrow PQ = \left|\frac{(2)(1) + (-2)(1) + (4)(2) + 5}{\sqrt{2^2 + (-2)^2 + 4^2}}\right|$

$\Rightarrow PQ = \left|\frac{2 - 2 + 8 + 5}{\sqrt{4 + 4 + 16}}\right|$

$\Rightarrow PQ = \left|\frac{13}{\sqrt{24}}\right|$

$\therefore PQ = \frac{13}{2\sqrt{6}}$

Thus, the required foot of perpendicular is $\left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$ and the length of the perpendicular is $\frac{13}{2\sqrt{6}}$ units.

6. Question

Find the distance of the point (1, -2, 3) from the plane x – y + z = 5 measured along a line parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Answer

Let point P = (1, –2, 3).

We need to find distance from P to the plane x – y + z = 5 measured along a line parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Let the line drawn from P parallel to the given line meet the plane at Q.

Direction ratios of PQ are proportional to 2, 3, –6 as PQ is parallel to the given line.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

Here, (x_1, y_1, z_1) = (1, –2, 3) and (l, m, n) = (2, 3, –6)

Hence, the equation of PQ is

$\frac{x - 1}{2} = \frac{y - (-2)}{3} = \frac{z - 3}{-6}$

$\Rightarrow \frac{x - 1}{2} = \frac{y + 2}{3} = \frac{z - 3}{-6} = \alpha \text{ (say)}$

$\Rightarrow x = 2\alpha + 1,\ y = 3\alpha - 2,\ z = -6\alpha + 3$

Let Q = $(2\alpha + 1, 3\alpha - 2, -6\alpha + 3)$.

This point lies on the given plane, which means this point satisfies the plane equation.

$\Rightarrow (2\alpha + 1) - (3\alpha - 2) + (-6\alpha + 3) = 5$

$\Rightarrow 2\alpha + 1 - 3\alpha + 2 - 6\alpha + 3 = 5$

$\Rightarrow -7\alpha + 6 = 5$

$\Rightarrow -7\alpha = -1$

$$\therefore \alpha = \frac{1}{7}$$

We have Q = $(2\alpha + 1, 3\alpha - 2, -6\alpha + 3)$

$$\Rightarrow Q = \left(2\left(\frac{1}{7}\right) + 1, 3\left(\frac{1}{7}\right) - 2, -6\left(\frac{1}{7}\right) + 3\right)$$

$$\Rightarrow Q = \left(\frac{2}{7} + 1, \frac{3}{7} - 2, -\frac{6}{7} + 3\right)$$

$$\therefore Q = \left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$$

Using the distance formula, we have

$$PQ = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(-\frac{11}{7} - (-2)\right)^2 + \left(\frac{15}{7} - 3\right)^2}$$

$$\Rightarrow PQ = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2}$$

$$\Rightarrow PQ = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}$$

$$\Rightarrow PQ = \sqrt{\frac{49}{49}}$$

$$\therefore PQ = \sqrt{1} = 1$$

Thus, the required distance is 1 unit.

7. Question

Find the coordinates of the foot of the perpendicular drawn from the point (2, 3, 7) to the plane 3x – y – z = 7. Also, find the length of the perpendicular.

Answer

Let point P = (2, 3, 7) and Q be the foot of the perpendicular drawn from P to the plane 3x – y – z = 7.

Direction ratios of PQ are proportional to 3, –1, –1 as PQ is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, $(x_1, y_1, z_1) = (2, 3, 7)$ and $(l, m, n) = (3, -1, -1)$

Hence, the equation of PQ is

$$\frac{x-2}{3}=\frac{y-3}{-1}=\frac{z-7}{-1}$$

$$\Rightarrow \frac{x-2}{3}=3-y=7-z=\alpha\text{ (say)}$$

⇒ $x = 3\alpha + 2$, $y = 3 - \alpha$, $z = 7 - \alpha$

Let Q = $(3\alpha + 2, 3 - \alpha, 7 - \alpha)$.

This point lies on the given plane, which means this point satisfies the plane equation.

⇒ $3(3\alpha + 2) - (3 - \alpha) - (7 - \alpha) = 7$

⇒ $9\alpha + 6 - 3 + \alpha - 7 + \alpha = 7$

⇒ $11\alpha - 4 = 7$

⇒ $11\alpha = 11$

∴ $\alpha = 1$

We have Q = $(3\alpha + 2, 3 - \alpha, 7 - \alpha)$

⇒ Q = $(3\times1 + 2, 3 - 1, 7 - 1)$

∴ Q = (5, 2, 6)

Using the distance formula, we have

$$PQ=\sqrt{(5-2)^2+(2-3)^2+(6-7)^2}$$

$$\Rightarrow PQ=\sqrt{3^2+(-1)^2+(-1)^2}$$

$$\Rightarrow PQ=\sqrt{9+1+1}$$

$$\therefore PQ=\sqrt{11}$$

Thus, the required foot of perpendicular is (5, 2, 6) and the length of the perpendicular is $\sqrt{11}$ units.

8. Question

Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

Answer

Let point P = (1, 3, 4) and M be the image of P in the plane $2x - y + z + 3 = 0$.

Direction ratios of PM are proportional to 2, –1, 1 as PM is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Here, (x_1, y_1, z_1) = (1, 3, 4) and (l, m, n) = (2, –1, 1)

Hence, the equation of PM is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}$$

$$\Rightarrow \frac{x-1}{2}=3-y=z-4=\alpha\text{ (say)}$$

⇒ $x = 2\alpha + 1$, $y = 3 - \alpha$, $z = \alpha + 4$

Let M = $(2\alpha + 1, 3 - \alpha, \alpha + 4)$.

As M is the image of P in the given plane, the midpoint of PM lies on the plane.

Using the midpoint formula, we have

$$\text{Midpoint of PM} = \left(\frac{1+(2\alpha+1)}{2}, \frac{3+(3-\alpha)}{2}, \frac{4+(\alpha+4)}{2}\right)$$

$$\Rightarrow \text{Midpoint of PM} = \left(\alpha+1, \frac{6-\alpha}{2}, \frac{\alpha+8}{2}\right)$$

This point lies on the given plane, which means this point satisfies the plane equation.

$$\Rightarrow 2(\alpha+1) - \left(\frac{6-\alpha}{2}\right) + \left(\frac{\alpha+8}{2}\right) + 3 = 0$$

$$\Rightarrow 2\alpha + 2 + \frac{\alpha+8-(6-\alpha)}{2} + 3 = 0$$

$$\Rightarrow 2\alpha + (\alpha+1) + 5 = 0$$

$$\Rightarrow 3\alpha = -6$$

$$\therefore \alpha = -\frac{6}{3} = -2$$

We have M = $(2\alpha + 1, 3 - \alpha, \alpha + 4)$

$\Rightarrow$ M = (2(–2) + 1, 3 – (–2), (–2) + 4)

$\therefore$ M = (–3, 5, 2)

Thus, the image of (1, 3, 4) in the plane 2x – y + z + 3 = 0 is (–3, 5, 2).

9. Question

Find the distance of the point with position vector $-\hat{i} - 5\hat{j} - 10\hat{k}$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$ with the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

Answer

Let P be the point with position vector $\vec{p} = -\hat{\imath} - 5\hat{\jmath} - 10\hat{k}$ and Q be the point of intersection of the given line and the plane.

We have the line equation as

$$\vec{r} = (2\hat{\imath} - \hat{\jmath} + 2\hat{k}) + \lambda(3\hat{\imath} + 4\hat{\jmath} + 12\hat{k})$$

$$\therefore \vec{r} = (2+3\lambda)\hat{\imath} + (-1+4\lambda)\hat{\jmath} + (2+12\lambda)\hat{k}$$

Let the position vector of Q be $\vec{q}$. As Q is a point on this line, for some scalar α, we have

$$\Rightarrow \vec{q} = (2+3\alpha)\hat{\imath} + (-1+4\alpha)\hat{\jmath} + (2+12\alpha)\hat{k}$$

This point Q also lies on the given plane, which means this point satisfies the plane equation $\vec{r}.(\hat{\imath} - \hat{\jmath} + \hat{k}) = 5$.

$$\Rightarrow [(2+3\alpha)\hat{\imath} + (-1+4\alpha)\hat{\jmath} + (2+12\alpha)\hat{k}].(\hat{\imath} - \hat{\jmath} + \hat{k}) = 5$$

$\Rightarrow (2 + 3\alpha)(1) + (-1 + 4\alpha)(-1) + (2 + 12\alpha)(1) = 5$

$\Rightarrow 2 + 3\alpha + 1 - 4\alpha + 2 + 12\alpha = 5$

$\Rightarrow 11\alpha + 5 = 5$

$\Rightarrow 11\alpha = 0$

$\therefore \alpha = 0$

We have $\vec{q} = (2+3\alpha)\hat{\imath} + (-1+4\alpha)\hat{\jmath} + (2+12\alpha)\hat{k}$

$$\Rightarrow \vec{q} = [2+3(0)]\hat{\imath} + [-1+4(0)]\hat{\jmath} + [2+12(0)]\hat{k}$$

$\therefore \vec{q} = 2\hat{i} - \hat{j} + 2\hat{k}$

Using the distance formula, we have

$$PQ = \sqrt{(2-(-1))^2 + ((-1)-(-5))^2 + (2-(-10))^2}$$

$\Rightarrow PQ = \sqrt{3^2 + 4^2 + 12^2}$

$\Rightarrow PQ = \sqrt{9 + 16 + 144}$

$\therefore PQ = \sqrt{169} = 13$

Thus, the required distance is 13 units.

10. Question

Find the length and the foot of the perpendicular from the point (1, 1, 2) to the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0.$

Answer

Let point P = (1, 1, 2) and Q be the foot of the perpendicular drawn from to P the plane $\vec{r}.(\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$
.

$\Rightarrow$ Position Vector of $P = \vec{p} = \hat{i} + \hat{j} + 2\hat{k}$

Direction ratios of PQ are proportional to 1, –2, 4 as PQ is normal to the plane and parallel to $\hat{i} - 2\hat{j} + 4\hat{k}$.

Recall the vector equation of the line passing through the point with position vector $\vec{r}$ and parallel to vector $\vec{b}$ is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$

Here, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 4\hat{k}$

Hence, the equation of PQ is

$\vec{r} = (\hat{i} + \hat{j} + 2\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 4\hat{k})$

$\therefore \vec{r} = (1+\lambda)\hat{i} + (1-2\lambda)\hat{j} + (2+4\lambda)\hat{k}$

Let the position vector of Q be $\vec{q}$. As Q is a point on this line, for some scalar α, we have

$\Rightarrow \vec{q} = (1+\alpha)\hat{i} + (1-2\alpha)\hat{j} + (2+4\alpha)\hat{k}$

This point lies on the given plane, which means this point satisfies the plane equation $\vec{r}.(\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$.

$\Rightarrow [(1+\alpha)\hat{i} + (1-2\alpha)\hat{j} + (2+4\alpha)\hat{k}].(\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$

$\Rightarrow (1+\alpha)(1) + (1-2\alpha)(-2) + (2+4\alpha)(4) = -5$

$\Rightarrow (1 + \alpha)(1) + (1 - 2\alpha)(-2) + (2 + 4\alpha)(4) = -5$

$\Rightarrow 1 + \alpha - 2 + 4\alpha + 8 + 16\alpha = -5$

$\Rightarrow 21\alpha + 7 = -5$

$\Rightarrow 21\alpha = -12$

$$\therefore \alpha = -\frac{12}{21} = -\frac{4}{7}$$

$\therefore \vec{m} = 4\hat{i} + 4\hat{j} + 7\hat{k}$

Foot of the perpendicular $\vec{q} = (1+\alpha)\hat{i} + (1-2\alpha)\hat{j} + (2+4\alpha)\hat{k}$

$$\Rightarrow \vec{q} = \left[1 + \left(-\frac{4}{7}\right)\right]\hat{\imath} + \left[1 - 2\left(-\frac{4}{7}\right)\right]\hat{\jmath} + \left[2 + 4\left(-\frac{4}{7}\right)\right]\hat{k}$$

$$\Rightarrow \vec{q} = \left(1 - \frac{4}{7}\right)\hat{\imath} + \left(1 + \frac{8}{7}\right)\hat{\jmath} + \left(2 - \frac{16}{7}\right)\hat{k}$$

$$\therefore \vec{q} = \frac{3}{7}\hat{\imath} + \frac{15}{7}\hat{\jmath} - \frac{2}{7}\hat{k}$$

Thus, $Q = \left(\frac{3}{7}, \frac{15}{7}, -\frac{2}{7}\right)$

Using the distance formula, we have

$$PQ = \sqrt{\left(\frac{3}{7} - 1\right)^2 + \left(\frac{15}{7} - 1\right)^2 + \left(-\frac{2}{7} - 2\right)^2}$$

$$\Rightarrow PQ = \sqrt{\left(-\frac{4}{7}\right)^2 + \left(\frac{8}{7}\right)^2 + \left(-\frac{16}{7}\right)^2}$$

$$\Rightarrow PQ = \sqrt{\frac{16}{49} + \frac{64}{49} + \frac{256}{49}}$$

$$\therefore PQ = \sqrt{\frac{336}{49}} = \frac{4\sqrt{21}}{7}$$

Thus, the required foot of perpendicular is $\left(\frac{3}{7}, \frac{15}{7}, -\frac{2}{7}\right)$ and the length of the perpendicular is $\frac{4\sqrt{21}}{7}$ units.

11. Question

Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point (3, 2, 1) from the plane 2x – y + z + 1 = 0. Find also the image of the point in the plane.

Answer

Let point P = (3, 2, 1) and M be the image of P in the plane 2x – y + z + 1 = 0.

In addition, let Q be the foot of the perpendicular from P on to the given plane so that Q is the midpoint of PM.

Direction ratios of PM are proportional to 2, –1, 1 as PM is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, (x_1, y_1, z_1) = (3, 2, 1) and (l, m, n) = (2, –1, 1)

Hence, the equation of PM is

$$\frac{x - 3}{2} = \frac{y - 2}{-1} = \frac{z - 1}{1}$$

$$\Rightarrow \frac{x - 3}{2} = 2 - y = z - 1 = \alpha \text{ (say)}$$

⇒ x = 2α + 3, y = 2 – α, z = α + 1

Let M = (2α + 3, 2 – α, α + 1).

Now, we will find Q, the midpoint of PM.

Using the midpoint formula, we have

$Q = \left(\frac{3 + (2\alpha + 3)}{2}, \frac{2 + (2 - \alpha)}{2}, \frac{1 + (\alpha + 1)}{2}\right)$

$\Rightarrow Q = \left(\alpha + 3, \frac{4 - \alpha}{2}, \frac{\alpha + 2}{2}\right)$

This point Q lies on the given plane, which means Q satisfies the plane equation 2x – y + z + 1 = 0.

$\Rightarrow 2(\alpha + 3) - \left(\frac{4 - \alpha}{2}\right) + \left(\frac{\alpha + 2}{2}\right) + 1 = 0$

$\Rightarrow 2\alpha + 6 + \frac{\alpha + 2 - (4 - \alpha)}{2} + 1 = 0$

$\Rightarrow 2\alpha + (\alpha - 1) = -7$

$\Rightarrow 3\alpha = -6$

$\therefore \alpha = -2$

We have M = (2α + 3, 2 – α, α + 1)

⇒ M = (2(–2) + 3, 2 – (–2), (–2) + 1)

∴ M = (–1, 4, –1)

We have $Q = \left(\alpha + 3, \frac{4-\alpha}{2}, \frac{\alpha+2}{2}\right)$

$\Rightarrow Q = \left((-2) + 3, \frac{4 - (-2)}{2}, \frac{(-2) + 2}{2}\right)$

$\Rightarrow Q = \left(1, \frac{6}{2}, 0\right)$

∴ Q = (1, 3, 0)

Using the distance formula, we have

$PQ = \sqrt{(1 - 3)^2 + (3 - 2)^2 + (0 - 1)^2}$

$\Rightarrow PQ = \sqrt{(-2)^2 + 1^2 + (-1)^2}$

$\Rightarrow PQ = \sqrt{4 + 1 + 1}$

$\therefore PQ = \sqrt{6}$

Thus, the required foot of perpendicular is (1, 3, 0) and the length of the perpendicular is $\sqrt{6}$ units. Also, the image of the given point is (–1, 4, –1)

12. Question

Find the direction cosines of the unit vector perpendicular to the plane $\vec{r} \cdot \left(6\hat{i} - 3\hat{j} - 2\hat{k}\right)$ + 1 = 0 passing through the origin.

Answer

The given plane equation is $\vec{r}.\left(6\hat{i} - 3\hat{j} - 2\hat{k}\right) + 1 = 0$

$\Rightarrow \vec{r}.\left(6\hat{i} - 3\hat{j} - 2\hat{k}\right) = -1$

$\Rightarrow \vec{r}.\left(-6\hat{i} + 3\hat{j} + 2\hat{k}\right) = 1$

Now, we calculate the magnitude of the vector $-6\hat{i} + 3\hat{j} + 2\hat{k}$.

$\left|-6\hat{i} + 3\hat{j} + 2\hat{k}\right| = \sqrt{(-6)^2 + 3^2 + 2^2}$

$\Rightarrow |-6\hat{i} + 3\hat{j} + 2\hat{k}| = \sqrt{36 + 9 + 4}$

$\therefore |-6\hat{i} + 3\hat{j} + 2\hat{k}| = \sqrt{49} = 7$

On dividing both sides of the plane equation by 7, we get

$\Rightarrow \vec{r}.\frac{(-6\hat{i} + 3\hat{j} + 2\hat{k})}{7} = \frac{1}{7}$

$\Rightarrow \vec{r}.\left(-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) = \frac{1}{7}$

Recall that the equation of the plane in normal form is given by $\vec{r}.\hat{n} = d$ where $\hat{n}$ is a unit vector perpendicular to the plane through the origin.

So, here $\hat{n} = -\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$

This is a unit vector normal to the plane $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$.

Thus, the direction cosines of the unit vector perpendicular to the given plane are $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$.

13. Question

Find the coordinates of the foot of the perpendicular drawn from the origin to the plane 2x – 3y + 4z – 6 = 0.

Answer

Let point P = (0, 0, 0) and Q be the foot of the perpendicular drawn from P to the plane 2x – 3y + 4z – 6 = 0.

Direction ratios of PQ are proportional to 2, –3, 4 as PQ is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, (x_1, y_1, z_1) = (0, 0, 0) and (l, m, n) = (2, –3, 4)

Hence, the equation of PQ is

$$\frac{x - 0}{2} = \frac{y - 0}{-3} = \frac{z - 0}{4}$$

$\Rightarrow \frac{x}{2} = -\frac{y}{3} = \frac{z}{4} = \alpha \text{ (say)}$

⇒ x = 2α, y = –3α, z = 4α

Let Q = (2α, –3α, 4α).

This point lies on the given plane, which means this point satisfies the plane equation.

⇒ 2(2α) – 3(–3α) + 4(4α) – 6 = 0

⇒ 4α + 9α + 16α – 6 = 0

⇒ 29α = 6

$\therefore \alpha = \frac{6}{29}$

We have Q = (2α, –3α, 4α)

$\Rightarrow Q = \left(2\left(\frac{6}{29}\right), -3\left(\frac{6}{29}\right), 4\left(\frac{6}{29}\right)\right)$

$\therefore Q = \left(\frac{12}{29}, -\frac{18}{29}, \frac{24}{29}\right)$

Thus, the required foot of perpendicular is $\left(\frac{12}{29}, -\frac{18}{29}, \frac{24}{29}\right)$.

14. Question

Find the length and the foot of the perpendicular from the point (1, 3/2, 2) to the plane 2x – 2y + 4z + 5 = 0.

Answer

Let point P = (1, 3/2, 2) and Q be the foot of the perpendicular drawn from P to the plane 2x – 2y + 4z + 5 = 0.

Direction ratios of PQ are proportional to 2, –2, 4 as PQ is normal to the plane.

Recall the equation of the line passing through (x_1, y_1, z_1) and having direction ratios proportional to l, m, n is given by

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Here, (x_1, y_1, z_1) = (1, $\frac{3}{2}$, 2) and (l, m, n) = (2, –2, 4)

Hence, the equation of PQ is

$$\frac{x-1}{2} = \frac{y-\frac{3}{2}}{-2} = \frac{z-2}{4}$$

$$\Rightarrow \frac{x-1}{2} = \frac{2y-3}{-4} = \frac{z-2}{4} = \alpha \text{ (say)}$$

$$\Rightarrow x = 2\alpha + 1, y = \frac{-4\alpha + 3}{2}, z = 4\alpha + 2$$

Let $Q = (2\alpha + 1, \frac{-4\alpha+3}{2}, 4\alpha + 2)$.

This point lies on the given plane, which means this point satisfies the plane equation.

$$\Rightarrow 2(2\alpha + 1) - 2\left(\frac{-4\alpha + 3}{2}\right) + 4(4\alpha + 2) + 5 = 0$$

⇒ 4α + 2 – (–4α + 3) + 16α + 8 + 5 = 0

⇒ 20α + 4α – 3 + 15 = 0

⇒ 24α = –12

$$\therefore \alpha = -\frac{1}{2}$$

We have $Q = (2\alpha + 1, \frac{-4\alpha+3}{2}, 4\alpha + 2)$

$$\Rightarrow Q = \left(2\left(-\frac{1}{2}\right) + 1, \frac{-4\left(-\frac{1}{2}\right) + 3}{2}, 4\left(-\frac{1}{2}\right) + 2\right)$$

$$\Rightarrow Q = \left(-1 + 1, \frac{2 + 3}{2}, -2 + 2\right)$$

$$\therefore Q = \left(0, \frac{5}{2}, 0\right)$$

Using the distance formula, we have

$$PQ = \sqrt{(0-1)^2 + \left(\frac{5}{2} - \frac{3}{2}\right)^2 + (0-2)^2}$$

$$\Rightarrow PQ = \sqrt{(-1)^2 + 1^2 + (-2)^2}$$

$\Rightarrow PQ = \sqrt{1+1+4}$

$\therefore PQ = \sqrt{6}$

Thus, the required foot of perpendicular is $\left(0, \frac{5}{2}, 0\right)$ and the length of the perpendicular is $\sqrt{6}$ units.

15. Question

Find the position vector of the foot of the perpendicular and the perpendicular distance from the point P with position vector $2\hat{i}+3\hat{j}+4\hat{k}$ to the plane $\vec{r}\cdot\left(2\hat{i}+\hat{j}+3\hat{k}\right)-26=0$. Also, find the image of P in the plane.

Answer

Let the position vector of P be $\vec{p}$ so that $\vec{p} = 2\hat{i}+3\hat{j}+4\hat{k}$ and M be the image of P in the plane $\vec{r}.\left(2\hat{i}+\hat{j}+3\hat{k}\right)-26=0$.

In addition, let Q be the foot of the perpendicular from P on to the given plane so that Q is the midpoint of PM.

Direction ratios of PM are proportional to 2, 1, 3 as PM is normal to the plane and parallel to $2\hat{i}+\hat{j}+3\hat{k}$.

Recall the vector equation of the line passing through the point with position vector $\vec{r}$ and parallel to vector $\vec{b}$ is given by

$\vec{r} = \vec{a}+\lambda\vec{b}$

Here, $\vec{a} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}+3\hat{k}$

Hence, the equation of PM is

$\vec{r} = \left(2\hat{i}+3\hat{j}+4\hat{k}\right)+\lambda\left(2\hat{i}+\hat{j}+3\hat{k}\right)$

$\therefore \vec{r} = (2+2\lambda)\hat{i}+(3+\lambda)\hat{j}+(4+3\lambda)\hat{k}$

Let the position vector of M be $\vec{m}$. As M is a point on this line, for some scalar α, we have

$\Rightarrow \vec{m} = (2+2\alpha)\hat{i}+(3+\alpha)\hat{j}+(4+3\alpha)\hat{k}$

Now, let us find the position vector of Q, the midpoint of PM.

Let this be $\vec{q}$.

Using the midpoint formula, we have

$$\vec{q} = \frac{\vec{p}+\vec{m}}{2}$$

$$\Rightarrow \vec{q} = \frac{\left[2\hat{i}+3\hat{j}+4\hat{k}\right]+\left[(2+2\alpha)\hat{i}+(3+\alpha)\hat{j}+(4+3\alpha)\hat{k}\right]}{2}$$

$$\Rightarrow \vec{q} = \frac{\left(2+(2+2\alpha)\right)\hat{i}+\left(3+(3+\alpha)\right)\hat{j}+\left(4+(4+3\alpha)\right)\hat{k}}{2}$$

$$\Rightarrow \vec{q} = \frac{(4+2\alpha)\hat{i}+(6+\alpha)\hat{j}+(8+3\alpha)\hat{k}}{2}$$

$$\therefore \vec{q} = (2+\alpha)\hat{i}+\frac{(6+\alpha)}{2}\hat{j}+\frac{(8+3\alpha)}{2}\hat{k}$$

This point lies on the given plane, which means this point satisfies the plane equation $\vec{r}.\left(2\hat{i}+\hat{j}+3\hat{k}\right)-26=0$.

$$\Rightarrow \left[(2+\alpha)\hat{i}+\frac{(6+\alpha)}{2}\hat{j}+\frac{(8+3\alpha)}{2}\hat{k}\right].\left(2\hat{i}+\hat{j}+3\hat{k}\right)-26=0$$

$$\Rightarrow 2(2+\alpha)+\left(\frac{6+\alpha}{2}\right)(1)+\left(\frac{8+3\alpha}{2}\right)(3)=26$$

$$\Rightarrow 4+2\alpha+\frac{6+\alpha+3(8+3\alpha)}{2}=26$$

$$\Rightarrow 2\alpha+\frac{30+10\alpha}{2}=22$$

$$\Rightarrow 14\alpha+30=44$$

$$\Rightarrow 14\alpha=14$$

$$\therefore \alpha=1$$

We have the image $\vec{m}=(2+2\alpha)\hat{\imath}+(3+\alpha)\hat{\jmath}+(4+3\alpha)\hat{k}$

$$\Rightarrow \vec{m}=[2+2(1)]\hat{\imath}+[3+(1)]\hat{\jmath}+[4+3(1)]\hat{k}$$

$$\therefore \vec{m}=4\hat{\imath}+4\hat{\jmath}+7\hat{k}$$

Foot of the perpendicular $\vec{q}=(2+\alpha)\hat{\imath}+\frac{(6+\alpha)}{2}\hat{\jmath}+\frac{(8+3\alpha)}{2}\hat{k}$

$$\Rightarrow \vec{q}=[2+(1)]\hat{\imath}+\left[\frac{6+(1)}{2}\right]\hat{\jmath}+\left[\frac{8+3(1)}{2}\right]\hat{k}$$

$$\therefore \vec{q}=3\hat{\imath}+\frac{7}{2}\hat{\jmath}+\frac{11}{2}\hat{k}$$

Using the distance formula, we have

$$PQ=\sqrt{(3-2)^2+\left(\frac{7}{2}-3\right)^2+\left(\frac{11}{2}-4\right)^2}$$

$$\Rightarrow PQ=\sqrt{1^2+\left(\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2}$$

$$\Rightarrow PQ=\sqrt{1+\frac{1}{4}+\frac{9}{4}}$$

$$\therefore PQ=\sqrt{\frac{14}{4}}=\frac{1}{2}\sqrt{14}$$

Thus, the position vector of the image of the given point is $4\hat{\imath}+4\hat{\jmath}+7\hat{k}$ and that of the foot of perpendicular is $3\hat{\imath}+\frac{7}{2}\hat{\jmath}+\frac{11}{2}\hat{k}$. Also, the length of this perpendicular is $\frac{1}{2}\sqrt{14}$ units.

Very Short Answer

1. Question

Write the equation of the plane parallel to XOY - plane and passing through the point (2, –3, 5)

Answer

Equation of XOY plane is z=0.

Since the required plane should pass through the point (2, - 3, 5).

We know, the vector equation of a plane perpendicular to a given direction$(\vec{n})$ and passing through a given point$(\vec{a})$ is given by,

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

i.e. $\left[(x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}\right]\cdot\hat{k} = 0$

$\Rightarrow \left[(x-2)\hat{i} + (y-(-3))\hat{j} + (z-5)\hat{k}\right]\cdot\hat{k} = 0$

$\Longrightarrow$ (z - 5)=0

$$\begin{bmatrix} \because \text{We know}, \hat{i}\cdot\hat{k} = |\hat{i}||\hat{k}|\cos 90^\circ = 0, \\ \hat{j}\cdot\hat{k} = |\hat{j}||\hat{k}|\cos 90^\circ = 0, \\ \hat{k}\cdot\hat{k} = |\hat{k}||\hat{k}|\cos 0^\circ = 0 \end{bmatrix}$$

$\Longrightarrow$ z=5

here $\vec{n}$ is given by $\hat{k}$, as it is perpendicular to XOY plane, and here

$\vec{a} = 2\hat{i} - 3\hat{j} + 5\hat{k}$

Hence, the desired equation for the plane is <u>z=5</u>.

2. Question

Write the equation of the plane parallel to YOZ - plane and passing through (–4, 1, 0).

Answer

Equation of YOZ plane is x=0.

Since the required plane should pass through the point (- 4, 1, 0).

We know, the vector equation of a plane perpendicular to a given direction$(\vec{n})$ and passing through a given point$(\vec{a})$ is given by,

$(\vec{r}-\vec{a})\cdot\vec{n} = 0$

i.e. $\left[(x-x_1)\hat{i} + (y-y_1)\hat{j} + (z-z_1)\hat{k}\right]\cdot\hat{i} = 0$

$\Rightarrow \left[(x-(-4))\hat{i} + (y-1)\hat{j} + (z-0)\hat{k}\right]\cdot\hat{i} = 0$

$\Longrightarrow$ (x + 4)=0

$$\begin{bmatrix} \because \text{We know}, \hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0^\circ = 1, \\ \hat{j}\cdot\hat{i} = |\hat{j}||\hat{i}|\cos 90^\circ = 0, \\ \hat{k}\cdot\hat{i} = |\hat{k}||\hat{i}|\cos 0^\circ = 0 \end{bmatrix}$$

$\Longrightarrow$ x= - 4

here $\vec{n}$ is given by $\hat{k}$, as it is perpendicular to YOZ plane, and here

$\vec{a} = -4\hat{i} + 1\hat{j} + 0\hat{k}$

Hence, the desired equation for the plane is <u>x= - 4</u>.

3. Question

Write the equation of the plane passing through point (a, 0, 0), (0, b, 0) and (0, 0, c).

Answer

We know that the general equation of a plane is given by,

Ax + By + Cz + D=0, where $D \neq 0$ .... (1)

Here, A, B, C are the co - ordinates of a normal vector to the plane, while (x, y, z) are the co - ordinates of any general point through which the plane passes.

Now let us say, this plane is making intercepts at points P, Q, and R on the x, y, and z - axes respectively at (a, 0, 0), (0, b, 0) and (0, 0, c).

So, the plane cuts the x - axis, y - axis and z - axis at three points P(a, 0, 0), Q(0, b, 0) and R(0, 0, c) respectively.

Since the plane also passes through each of these three points, we can substitute them into equation (1) i.e. general equation of the plane and we have,

(i) Aa + D=0

$\Rightarrow A = -\frac{D}{a}$

(ii) Bb + D=0

$\Rightarrow B = -\frac{D}{b}$

(iii) Cc + D=0

$\Rightarrow C = -\frac{D}{c}$

Substituting these values of A, B, and C in equation (1) of the plane, we shall get the equation of a plane in intercept form, which is given by,

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

if the plane makes intercepts at (a, 0, 0), (0, b, 0) and (0, 0, c) with the x - , y - and z - axes respectively.

4. Question

Write the general equation of a plane parallel to X - axis.

Answer

The required plane is parallel to X - axis i.e. the normal of the plane is perpendicular to X - axis so, the component of the normal vector along X - axis is zero (0).

We know that the general equation of a plane is given by,

Ax + By + Cz + D=0, where $D \neq 0$ .... (1)

Here, A, B, C are the coordinates of a normal vector to the plane, while (x, y, z) are the co - ordinates of any point through which the plane passes.

Putting A=0 [∵ the component of the normal vector along X - axis is zero (0)] in the general equation i.e. in equation (1) of plane we get,

By + Cz + D=0, where $D \neq 0$ .... (2)

Hence, By + Cz + D=0 is the general equation of a plane parallel to X - axis.

5. Question

Write the value of k for which the planes x – 2y + kz = 4 and 2x + 5y – z = 9 are perpendicular.

Answer

Equation of the first plane is given as,

x–2y + kz=4 .... (1)

and the equation of the second plane is given as,

2x + 5y–z=9 .... (2)

So, the normal vector of plane (1) is given by,

$\overrightarrow{n_1} = \hat{i} - 2\hat{j} + k\hat{k}$

Similarly, the normal vector of plane (2) is given by,

$\overrightarrow{n_2} = 2\hat{i} + 5\hat{j} - \hat{k}$

When the two planes are perpendicular to each other, we should have,

$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 0$

$\Rightarrow (\hat{i} - 2\hat{j} + k\hat{k}) \cdot (2\hat{i} + 5\hat{j} - \hat{k}) = 0$

$\Rightarrow (1 \times 2) + ((-2) \times 5) + (k \times (-1)) = 0$

2 - 10 - k=0

k= - 8.

Hence, the planes x–2y + kz=4 and 2x + 5y–z=9 will be perpendicular to each other if k= - 8.

6. Question

Write the intercepts made by the plane 2x – 3y + 4z = 12 on the coordinate axes.

Answer

We know, that the general equation of a plane is given by,

Ax + By + Cz + D=0, where $D \neq 0$ .... (1)

Here, A, B, C are the coordinates of a normal vector to the plane, while (x, y, z) are the co - ordinates of any point through which the plane passes.

Again, we know the intercept form of plane, which is given by,

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Where, $A = -\frac{D}{a}, B = -\frac{D}{b}$ and $C = -\frac{D}{c}$ and the plane makes intercepts at (a, 0, 0), (0, b, 0) and (0, 0, c) with the x - , y - and z - axes respectively.

The equation of the plane is given as,

2x–3y + 4z=12

i.e. 2x–3y + 4z - 12=0 .... (2)

Comparing equation (2) with in the general equation i.e. in equation (1) of plane we get,

A=2, B= - 3 and C=4 and D= - 12.

$\therefore a = -\frac{D}{A}$

$= -\frac{(-12)}{2}$

=6,

$b = -\frac{D}{B}$

$= -\frac{(-12)}{(-3)}$

=−4

$c = -\frac{D}{C}$

$= -\frac{(-12)}{4}$

$= 3$

The given plane (given by equation (2)) makes intercepts at (6, 0, 0), (0, - 4, 0) and (0, 0, 3)with the x - , y - and z - axes respectively.

7. Question

Write the ratio in which the plane 4x + 5y – 3z = 8 divides the line segment joining points (–2, 1, 5) and (3, 3, 2).

Answer

We know that, the ratio in which the plane Ax + By + Cz + D=0 (where $D \neq 0$) divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) then is given as,

$$-\frac{(Ax_1 + By_1 + Cz_1 + D)}{Ax_2 + By_2 + Cz_2 + D}$$

Here, the equation of the given plane is, 4x + 5y–3z=8 i.e. 4x + 5y–3z - 8=0 and the co - ordinates of the two points are (–2, 1, 5) and (3, 3, 2).

Comparing with the general formula, we get,

A=4, B=5, C= - 3, D= - 8, x_1= - 2, y_1=1, z_1=5and x_2=3, y_2=3 and z_2=2.

So, the required ratio is

$$= -\frac{(Ax_1 + By_1 + Cz_1 + D)}{Ax_2 + By_2 + Cz_2 + D}$$

$$= -\frac{\left((4 \times (-2)) + (5 \times 1) + \left((-3) \times 5\right) - 8\right)}{\left((4 \times 3) + (5 \times 3) + \left((-3) \times 2\right) - 8\right)}$$

$$= -\frac{(-8 + 5 - 15 - 8)}{(12 + 15 - 6 - 8)}$$

$$= -\frac{(-26)}{13}$$

$$= \frac{26}{13}$$

$$= \frac{2}{1}$$

Hence, the plane 4x + 5y–3z=8 divides the line segment joining points (–2, 1, 5) and (3, 3, 2) in 2:1 ratio.

8. Question

Write the distance between the parallel planes 2x – y + 3z = 4 and 2x – y + 3z = 18.

Answer

We know that, distance between two parallel planes:

Ax + By + Cz + D_1=0...... (1) and Ax + By + Cz + D_2=0...... (2) is given by,

$$D = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}$$

Here, the two parallel planes are given as,

2x–y + 3z=4 i.e. 2x–y + 3z - 4=0 .... (3)

and 2x–y + 3z=18 i.e. 2x–y + 3z - 18=0 .... (4)

Comparing equation (3) with equation (1) and equation (4) with equation (2) we get,

A=2, B= - 1, C=3, D_1= - 4 and D_2= - 18.

So, the distance between the given two parallel planes are,

$D = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}$

$= \frac{|(-18) - (-4)|}{\sqrt{2^2 + (-1)^2 + 3^2}}$

$= \frac{|-18 + 4|}{\sqrt{4 + 1 + 9}}$

$= \frac{14}{\sqrt{14}}$

$\Rightarrow D = \sqrt{14}$

Hence, the distance between the parallel planes 2x–y + 3z=4 and 2x–y + 3z=18 is $\sqrt{14}$.

9. Question

Write the plane $\vec{r}.\left(2\hat{i}+3\hat{j}-6\hat{k}\right)=14$ in normal form.

Answer

The plane is given as, $\vec{r}\cdot\left(2\hat{i} + 3\hat{j} - 6\hat{k}\right) = 14$.

We can write the equation of the plane in general form as,

2x + 3y - 6z - 14=0 .... (1)

$[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

Now, to get the normal form of a plane given in general form as, Ax + By + Cz + D=0 where $D \neq 0$ ...... (2), we have to divide the equation (1) by $|\vec{n}|$, where $\vec{n}$ is the normal vector given as, $\vec{n} = 2\hat{i} + 3\hat{j} - 6\hat{k}$

Now, $|\vec{n}| = \left|2\hat{i} + 3\hat{j} - 6\hat{k}\right|$

$= \sqrt{2^2 + 3^2 + (-6)^2}$

$= \sqrt{4 + 9 + 36}$

$= \sqrt{49}$

$\Rightarrow |\vec{n}| = 7$

Comparing equation (1) with equation (2), we get, D= - 14

$p = \frac{-D}{|\vec{n}|}$

$= \frac{-(-14)}{7}$

$= \frac{14}{7}$

=2

[where p is the distance between the plane and the origin]

Normal form of the equation is given as,

$\frac{Ax + By + Cz}{|\vec{n}|} = p$

Here, normal form of the given plane is,

$\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$

or, $\vec{r} \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 2 \; [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

10. Question

Write the distance of the plane $\vec{r}.\left(2\hat{i} - \hat{j} + 2\hat{k}\right) = 12$ from the origin.

Answer

The plane is given as, $\vec{r} \cdot \left(2\hat{i} - \hat{j} + 2\hat{k}\right) = 12.$

We can write the equation of the plane in general form as,

2x - y + 2z - 12=0 .... (1)

$[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

Now, to get the normal form of a plane given in general form as, Ax + By + Cz + D=0 where $D \neq 0$ ...... (2), we have to divide the equation (1) by $|\vec{n}|$, where $\vec{n}$ is the normal vector given as, $\vec{n} = 2\hat{i} - \hat{j} + 2\hat{k}$

Now, $|\vec{n}| = \left|2\hat{i} - \hat{j} + 2\hat{k}\right|$

$= \sqrt{2^2 + (-1)^2 + 2^2}$

$= \sqrt{4 + 1 + 4}$

$= \sqrt{9}$

$\Rightarrow |\vec{n}| = 3$

Comparing equation (1) with equation (2), we get, D= - 12

So, the distance between the plane and the origin (using formula)

$= p = \frac{-D}{|\vec{n}|}$

$= \frac{-(-12)}{3}$

$= \frac{12}{3}$

p=4

The distance between the plane and the origin is 4 units.

11. Question

Write the equation of the plane $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ in scalar product form.

Answer

The given plane is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}.$

So, it is clear from the given equation of plane, that the plane passing through a point $(\vec{a})$ and parallel to two vectors $\vec{b}$ and $\vec{c}$.

So, the equation of the vector normal to the plane is given as,

$\vec{n} = (\vec{b} \times \vec{c})$

So, in scalar product form the vector equation of the plane is given as,

$(\vec{r}-\vec{a})\cdot\vec{n}=0$

$\Rightarrow(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$\Rightarrow\vec{r}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot(\vec{b}\times\vec{c})$.

Hence, the equation of the plane $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ in scalar product form is given as, $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$ or, $\vec{r}\cdot(\vec{b}\times\vec{c})=\vec{a}\cdot(\vec{b}\times\vec{c})$.

12. Question

Write a vector normal to the plane $\vec{r}=\ell\vec{b}+m\vec{c}$.

Answer

We have the plane as, $\vec{r}=l\vec{b}+m\vec{c}$

So, it is clear from the given equation of plane, that the plane passing through origin and parallel to two vectors $\vec{b}$ and $\vec{c}$.

Hence, a vector normal to the plane $\vec{r}=l\vec{b}+m\vec{c}$ is $\vec{n}=(\vec{b}\times\vec{c})$

13. Question

Write the equation of the plane passing through (2, –1,1) and parallel to the plane 3x + 2y – z = 7.

Answer

The required plane is parallel to 3x + 2y - z=7, so required plane and the given plane must have the same normal vector.

Vector normal to the plane 3x + 2y - z=7 is $\vec{n}=3\hat{i}+2\hat{j}-\hat{k}$

We know that, equation of plane perpendicular to a given direction $(\vec{n})$ & passing through a given point $(\vec{a})$ is given by,

$(\vec{r}-\vec{a})\cdot\vec{n}=0$

Here, it is given that, the plane passes through (2, - 1, 1) so in this case, in vector form, $\vec{a}$ can be denoted as,

$\vec{a}=2\hat{i}-\hat{j}+\hat{k}$

Equation of the required plane is,

$(\vec{r}-\vec{a})\cdot\vec{n}=0$

$\Rightarrow\left(\vec{r}-(2\hat{i}-\hat{j}+\hat{k})\right)\cdot(3\hat{i}+2\hat{j}-\hat{k})=0$

$\Rightarrow\left((x-2)\hat{i}+(y+1)\hat{j}+(z-1)\hat{k}\right)\cdot(3\hat{i}+2\hat{j}-\hat{k})=0\ [\because\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}]$

3(x - 2) + 2(y + 1) - (z - 1)=0

3x - 6 + 2y + 2 - z + 1=0

3x + 2y - z=3

Hence, the equation of the plane passing through (2, –1, 1) and parallel to the plane 3x + 2y–z=7 is 3x + 2y - z=3.

14. Question

Write the equation of the plane containing the lines $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}=\vec{a}+\mu\vec{c}$.

Answer

The, required plane should contain the lines, $\vec{r}=\vec{a}+\lambda\vec{b}$ and $\vec{r}=\vec{a}+\mu\vec{c}$

So, it is clear from the given equation of lines, that both the lines are passing through a point $(\vec{a})$ and one of the line is parallel to $\vec{b}$ and the other one is parallel to $\vec{c}$.

So, the equation of the vector normal to the plane is given as,

$\vec{n} = (\vec{b} \times \vec{c})$

So, in scalar product form the vector equation of the plane is given as,

$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$

$\Rightarrow \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$.

Hence, the equation of the plane containing the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a} + \mu\vec{c}$ is given as, $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$ or, $\vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$

15. Question

Write the position vector of the point where the line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r}.\vec{n}$.

Answer

Let, the the position vector of the point where the line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r} \cdot \vec{n} = 0$ be $\vec{r_0}$.

As $\vec{r_0}$ is the position vector of the point of intersection of the line and the plane, so it must satisfy both of the equation of line and the equation of plane.

Substituting, $\vec{r_0}$ in place of $\vec{r}$ in both the equations, we

get,

$\vec{r_0} = \vec{a} + \lambda\vec{b}$ .... (1), and

$\vec{r_0} \cdot \vec{n} = 0$ .... (2)

Putting $\vec{r_0} = \vec{a} + \lambda\vec{b}$ in equation (2) we get,

$(\vec{a} + \lambda\vec{b}) \cdot \vec{n} = 0$

$\Rightarrow \vec{a} \cdot \vec{n} + \lambda\vec{b} \cdot \vec{n} = 0$

$\Rightarrow \lambda = -\dfrac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}$

Substituting this value of λ in equation (1) we get,

$\vec{r_0} = \vec{a} + \left(-\dfrac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}\right)\vec{b}$

$\Rightarrow \vec{r_0} = \vec{a} - \left(\dfrac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}\right)\vec{b}$

The position vector of the point where the line $\vec{r} = \vec{a} + \lambda\vec{b}$ meets the plane $\vec{r}\cdot\vec{n} = 0$ is $\vec{a} - \left(\frac{\vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$.

15. Question

Write the value of k for which the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{k}$ is perpendicular to the normal to the plane $\vec{r}\,.\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) = 4$.

Answer

Equation of the line in Cartesian form is given as,

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{k}$$

So, the direction cosines of the line are given as, $\langle 2\ 3\ k\rangle$

The equation of the plane is, $\vec{r}\cdot\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) = 4$ so, we have the vector normal to the plane as,

$$\vec{n} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$$

It is required that, the line should be perpendicular to the normal to the plane $\vec{r}\cdot\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) = 4$, so, we should have,

(2×2) + (3×3) + (k×4)=0

4 + 9 + 4k=0

13 + 4k=0

4k= - 13

$$\Rightarrow k = -\frac{13}{4}$$

Hence, for $k = -\frac{13}{4}$ the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{k}$ will be perpendicular to the normal to the plane

$\vec{r}\cdot\left(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}\right) = 4$.

16. Question

Write the angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ and the plane x + y + 4 = 0.

Answer

We know, the angle between the line $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$ and the plane Ax + By + Cz + D=0 is given as,

$$\varphi = \sin^{-1}\frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2}\cdot\sqrt{l^2 + m^2 + n^2}} \ldots\ldots\ldots\ldots (1)$$

In this case, l=2, m=1, n= - 2, A=1, B=1, C=0 and D=4.

Putting these values in equation (1) we get,

$$\varphi = \sin^{-1}\frac{Al + Bm + Cn}{\sqrt{A^2 + B^2 + C^2}\cdot\sqrt{l^2 + m^2 + n^2}}$$

$$= \sin^{-1}\frac{\left((1\times 2) + (1\times 1) + (0\times(-2))\right)}{\sqrt{1^2 + 1^2 + 0^2}\cdot\sqrt{2^2 + 1^2 + (-2)^2}}$$

$$= \sin^{-1}\frac{(2 + 1 + 0)}{\sqrt{2}\cdot\sqrt{9}}$$

$= \sin^{-1}\frac{3}{\sqrt{2}\cdot 3}$

$= \sin^{-1}\frac{1}{\sqrt{2}}$

=45°

Hence, the angle between the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+3}{-2}$ and the plane x + y + 4=0 is $\underline{45^{\circ}}$

17. Question

Write the intercept cut off by the plane 2x + y – z = 5 on x - axis.

Answer

We know, that the general equation of a plane is given by,

Ax + By + Cz + D=0, where $D \neq 0$ .... (1)

Here, A, B, C are the coordinates of a normal vector to the plane, while (x, y, z) are the co - ordinates of any point through which the plane passes.

Again, we know the intercept form of plane which is given by,

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Where, $A=-\frac{D}{a}, B=-\frac{D}{b}$ and $C=-\frac{D}{c}$ and the plane makes intercepts at (a, 0, 0), (0, b, 0) and (0, 0, c) with the x - , y - and z - axes respectively.

The equation of the plane is given as,

2x + y - z=5

i.e. 2x + y - z - 5=0 .... (2)

Comparing equation (2) with in the general equation i.e. in equation (1) of plane we get,

A=2, B=1 and C= - 1 and D= - 5.

$\therefore a=-\frac{D}{A}$

$=-\frac{(-5)}{2}$

$=\frac{5}{2}$

=2.5

Hence, the intercept cut off by the plane 2x + y–z=5 on x - axis is of <u>2.5 units</u>.

18. Question

Find the length of the perpendicular drawn from the origin to the plane 2x – 3y + 6z + 21 = 0.

Answer

We know the distance of a point (x_0, y_0, z_0) from a plane Ax + By + Cz + D=0 .... (1) is

$=\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

On comparing, the equation of the given plane i.e.

2x - 3y + 6z + 21=0 with equation (1) we get,

A=2, B= - 3, C=6, D=21.

Again, we know that, the co - ordinates of the origin are

(0, 0, 0).

So, the length of the perpendicular drawn from the origin is

$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

$= \frac{\left|\left((2 \times 0) + \left((-3) \times 0\right) + (6 \times 0) + 21\right)\right|}{\sqrt{2^2 + (-3)^2 + 6^2}}$

$[(x_0, y_0, z_0) \approx (0, 0, 0)]$

$= \frac{\left|\left((2 \times 0) + \left((-3) \times 0\right) + (6 \times 0) + 21\right)\right|}{\sqrt{4 + 9 + 36}}$

$= \frac{|21|}{\sqrt{49}}$

$= \frac{21}{7}$

=3

Hence, the length of the perpendicular drawn from the origin to the plane 2x–3y + 6z + 21=0 is = 3 units.

19. Question

Write the vector equation of the line passing through the point (1, –2, –3) and normal to the plane $\vec{r} \cdot \left(2\hat{i} + \hat{j} + 2\hat{k}\right) = 5$.

Answer

Equation of the given plane is, $\vec{r} \cdot \left(2\hat{i} + \hat{j} + 2\hat{k}\right) = 5$

So, the equation of the vector normal to the plane is given as,

$\vec{n} = \left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

As the required line should be normal to the given plane, so, the line should be parallel to the normal vector i.e. $\vec{n}$.

The line should pass through the point (1, –2, –3).

We can write the position vector of the point as,

$\vec{a} = \hat{i} - 2\hat{j} - 3\hat{k}$

So, the vector equation of the line passing through the point (1, –2, –3) and normal to the plane i.e. parallel to $\vec{n} = \left(2\hat{i} + \hat{j} + 2\hat{k}\right)$ is given by,

$\vec{r} = \vec{a} + \lambda\vec{n}$, where λ is a scalar(constant).

$\Rightarrow \vec{r} = \left(\hat{i} - 2\hat{j} - 3\hat{k}\right) + \lambda\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

Hence, equation of the required line is

$\vec{r} = \left(\hat{i} - 2\hat{j} - 3\hat{k}\right) + \lambda\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$where, λ is a scalar(constant)

20. Question

Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) = 2$.

Answer

The required plane is parallel to $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$, so required plane and the given plane must have the same normal vector.

Vector normal to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ is

$\vec{n} = 3\hat{i} + 2\hat{j} - \hat{k}$

The required plane is passing through a given point

(a, b, c), so can write the position vector of the point as $\vec{r_0} = a\hat{i} + b\hat{j} + c\hat{k}$

Now, the equation of the required plane is given by,

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$

$\Rightarrow \left(\vec{r} - (a\hat{i} + b\hat{j} + c\hat{k})\right) \cdot (\hat{i} + \hat{j} + \hat{k}) = 0 \left[\begin{matrix} \because \vec{r_0} = a\hat{i} + b\hat{j} + c\hat{k} \\ \text{and } \vec{n} = 3\hat{i} + 2\hat{j} - \hat{k} \end{matrix}\right]$

$\Rightarrow \left((x-a)\hat{i} + (y-b)\hat{j} + (z-c)\hat{k}\right) \cdot (\hat{i} + \hat{j} + \hat{k}) = 0 \ [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

(x - a) + (y - b) + (z - c)=0

x + y + z - (a + b + c)=0

x + y + z = a + b + c

Hence, the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ is $\left(\vec{r} - (a\hat{i} + b\hat{j} + c\hat{k})\right) \cdot (\hat{i} + \hat{j} + \hat{k}) = 0$ i.e. $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$ (in vector form), or, in general form x + y + z = a + b + c.

21. Question

Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Answer

From the given vector normal to the required plane, we can write the equation of the plane as,

$\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = d \left[\begin{matrix} \because \vec{r_0} = a\hat{i} + b\hat{j} + c\hat{k} \\ \text{and } \vec{n} = 3\hat{i} + 2\hat{j} - \hat{k} \end{matrix}\right]$

[where, d is a constant]

$\Rightarrow 2x - 3y + 6z = d$

$\Rightarrow 2x - 3y + 6z - d = 0$ …………………… (1)

We know, that the distance of a point (x_0, y_0, z_0) from a plane Ax + By + Cz + D=0 ………….. (2) is

$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

On comparing, equation (1) i.e. 2x - 3y + 6z + D=0 with

equation (2) we get,

A=2, B= - 3, C=6, D= - d.

Again, we know that, the co - ordinates of the origin are

(0, 0, 0).

So, the length of the perpendicular drawn from the origin is

$$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Here, it is given that, the plane is at a distance of 5 units from the origin, so, we have,

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} = 5$$

$$\Rightarrow \frac{\left|\left((2 \times 0) + ((-3) \times 0) + (6 \times 0) + D\right)\right|}{\sqrt{2^2 + (-3)^2 + 6^2}} = 5$$

$[(x_0, y_0, z_0) \approx (0, 0, 0)]$

$$\Rightarrow \frac{\left|\left((2 \times 0) + ((-3) \times 0) + (6 \times 0) + D\right)\right|}{\sqrt{4 + 9 + 36}} = 5$$

$$\Rightarrow \frac{|D|}{\sqrt{49}} = 5$$

$$\Rightarrow \frac{|D|}{7} = 5$$

|D|=35

D=±35

∴d=± 35 [∵ D= - d]

Hence, the vector equation of a plane which is at a distance of 5 units from the origin and whose normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$ is, 2x - 3y + 6z - (- 35)=0 i.e. 2x - 3y + 6z + 35=0 or 2x - 3y + 6z - 35=0.

Hence, required equation of the plane, is $\vec{r} \cdot \left(2\hat{i} - 3\hat{j} + 6\hat{k}\right) = -35$ i.e. 2x - 3y + 6z + 35=0 or, $\vec{r} \cdot \left(2\hat{i} - 3\hat{j} + 6\hat{k}\right) = 35$ i.e. 2x - 3y + 6z - 35=0.

22. Question

Write the equation of a plane which is at a distance of $5\sqrt{3}$ units from the origin and the normal to which is equally inclined to coordinate axes.

Answer

Given, the plane is at a distance of $5\sqrt{3}$ units form the origin and the normal to the plane is equally inclined with the co - ordinates axis, so, its direction cosines are

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

We know, for a plane having direction cosines as l, m and n, and p be the distance of the plane from the origin, the equation of the plane is given as, lx + my + nz=p

So, in this problem, the equation of the required equation of the plane is given by,

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = 5\sqrt{3} \text{ [here } p = 5\sqrt{3}]$$

$$\Rightarrow x + y + z = 5\sqrt{3} \times \sqrt{3}$$

=15

Hence, the equation of the required plane which is at a distance of $5\sqrt{3}$ units from the origin and the normal to which is equally inclined to co - ordinate axes is

x + y + z=15.

MCQ

1. Question

Mark the correct alternative in the following:

The plane 2x – (1 + λ)y + 3λz = 0 passes through the intersection of the planes.

A. 2x – y = 0 and y – 3x = 0

B. 2x + 3y = 0 and y = 0

C. 2x – y + 3z = 0 and y – 3z = 0

D. none of these

Answer

The given equation plane is, $2x - (1 + \lambda)y + 3\lambda z = 0$

We can rewrite the equation of the given plane as,

$2x-(1 + \lambda)y + 3\lambda z=0$

$2x - y - \lambda(y - 3z)=0$

So, the given plane passes through the intersection of

the planes 2x - y=0 and y - 3z=0.

2. Question

Mark the correct alternative in the following:

The acute angle between the planes 2x – y + z = 6 and x + y + 2z = 3 is

A. 45°

B. 60°

C. 30°

D. 75°

Answer

We know, the angle between two planes,

$a_1x + b_1y + c_1z + d_1=0$ and $a_2x + b_2y + c_2z + d_2=0$ is,

$$\theta = \cos^{-1}\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Here, $a_1=2$, $b_1= - 1$, $c_1=1$, $d_1= - 6$ and $a_2=1$, $b_2=1$, $c_2=2$,

$d_2= - 3$.

So, the acute angle between the planes 2x–y + z=6 and

x + y + 2z = 3 is

$$\theta = \cos^{-1}\frac{\left((2 \times 1) + \left((-1) \times 1\right) + (1 \times 2)\right)}{\sqrt{2^2 + (-1)^2 + 1^2}\cdot\sqrt{1^2 + 1^2 + 2^2}}$$

$$= \cos^{-1}\frac{(2 - 1 + 2)}{\sqrt{4 + 1 + 1}\cdot\sqrt{1 + 1 + 4}}$$

$$= \cos^{-1}\frac{3}{\sqrt{6}\cdot\sqrt{6}}$$

$= \cos^{-1}\frac{3}{6}$

$= \cos^{-1}\frac{1}{2}$

θ=60°

the acute angle between the planes 2x–y + z=6 and

x + y + 2z = 3 is 60°.

3. Question

Mark the correct alternative in the following:

The equation of the plane through the intersection of the planes x + 2y + 3z = 4 and 2x + y – z = –5 and perpendicular to the plane 5x + 3y + 6z + 8 = 0 is

A. 7x – 2y + 3z + 81 = 0

B. 23x + 14y – 9z + 48 = 0

C. 51x – 15y – 50z + 173 = 0

D. none of these

Answer

The equation of the plane through the intersection of

the planes x + 2y + 3z=4 or, x + 2y + 3z - 4=0 and

2x + y–z=–5 or, 2x + y–z + 5=0 is given as,

(x + 2y + 3z - 4) + λ(2x + y–z + 5)=0

[where λ is a scalar]

x(1 + 2λ) + y(2 + λ) + z(3 - λ) - 4 + 5λ=0

Given, that the required plane is perpendicular to the plane 5x + 3y + 6z + 8=0 so, we should have,

5(1 + 2λ) + 3(2 + λ) + 6(3 - λ)=0

5 + 10λ + 6 + 3λ + 18 - 6λ=0

29 + 7λ=0

$\Rightarrow \lambda = -\frac{29}{7}$

Therefore, the equation of the required plane is,

$(x + 2y + 3z - 4) - \frac{29}{7}(2x + y - z + 5) = 0$

7(x + 2y + 3z - 4) - 29(2x + y–z + 5)=0

7x + 14y + 21z - 28 - 58x - 29y + 29z - 145=0

- 51x - 15y + 50z - 173=0

51x + 15y - 50z + 173=0

4. Question

Mark the correct alternative in the following:

The distance between the planes 2x + 2y – z + 2 = 0 and 4x + 4y – 2z + 5 = 0 is

A. 1/2

B. 1/4

C. 1/6

D. none of these

Answer

We know that, distance between two parallel planes:

Ax + By + Cz + D_1=0...... (1) and Ax + By + Cz + D_2=0...... (2) is given by,

$$D = \frac{|D_2 - D_1|}{\sqrt{A^2 + B^2 + C^2}}$$

Here, the two parallel planes are given as,

2x + 2y - z + 2=0 .... (3)

and 4x + 4y - 2z + 5=0 i.e. 2x + 2y - z + $\frac{5}{2}$=0 .... (4)

Comparing equation (3) with equation (1) and equation (4) with equation (2) we get,

A=2, B=2, C= - 1, D_1=2 and D_2=$\frac{5}{2}$.

So, the distance between the given two parallel planes are,

$$D = \frac{\left|\frac{5}{2} - 2\right|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{\left|\frac{1}{2}\right|}{\sqrt{2^2 + 2^2 + (-1)^2}}$$

$$= \frac{\frac{1}{2}}{\sqrt{4 + 4 + 1}}$$

$$= \frac{1}{2\sqrt{9}}$$

$$= \frac{1}{2 \times 3}$$

$$\Rightarrow D = \frac{1}{6}$$

Hence, the distance between the parallel planes 2x + 2y - z + 2=0 and 4x + 4y - 6z + = is $\frac{1}{6}$.

5. Question

Mark the correct alternative in the following:

The image of the point (1, 3, 4) in the plane 2x – y + z + 3 = 0 is

A. (3, 5, 2)

B. (–3, 5, 2)

C. (3, 5, –2)

D. (3, –5, 2)

Answer

We know, if the image of a point P (x_0, y_0, z_0) on a plane Ax + By + Cz + D=0......... (1) is Q (x_1, y_1, z_1) then,

$$\frac{x_1 - x_0}{A} = \frac{y_1 - y_0}{B} = \frac{z_1 - z_0}{C} = \frac{-2(Ax_0 + By_0 + Cz_0 + D)}{A^2 + B^2 + C^2}$$

The given plane is, 2x–y + z + 3=0 .... (2)

Comparing equation (2) with equation (1) we get,

A=2, B= - 1, C=1, D=3

And, here x_0=1, y_0=3, z_0=4

So, $\frac{x_1 - 1}{2} = \frac{y_1 - 3}{-1} = \frac{z_1 - 4}{1} = \frac{-2\left((2 \times 1) + ((-1) \times 3) + (1 \times 4) + 3\right)}{2^2 + (-1)^2 + 1^2}$

$$= \frac{-2(2 - 3 + 4 + 3)}{4 + 1 + 1}$$

$$= \frac{-2 \times 6}{6}$$

$$= -2$$

$$\Rightarrow \frac{x_1 - 1}{2} = \frac{y_1 - 3}{-1} = \frac{z_1 - 4}{1} = -2$$

x_1=(2×(- 2)) + 1

= - 4 + 1

x_1= - 3,

y_1=((- 1)×(- 2)) + 3

=2 + 3

y_1=5 and

z_1=(1×(- 2)) + 4

= - 2 + 4

z_1=2

So, the image of the point (1, 3, 4) in the plane 2x – y + z + 3 = 0 is (- 3, 5, 2)

6. Question

Mark the correct alternative in the following:

The equation of the plane containing the two lines

$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-0}{3}$ and $\frac{x}{-2} = \frac{y-2}{-3} = \frac{z+1}{-1}$ is

A. 8x + y – 5z – 7 = 0

B. 8x + y + 5z – 7 = 0

C. 8x – y – 5z – 7 = 0

D. none of these

Answer

We know, the two lines given as,

$$\frac{x - x_1}{A_1} = \frac{y - y_1}{B_1} = \frac{z - z_1}{C_1} \ \& \ \frac{x - x_2}{A_2} = \frac{y - y_2}{B_2} = \frac{z - z_2}{C_2}$$

will be co - planar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0$

Here, x_1=1, y_1= - 1, z_1=0, x_2=0, y �$_2$=2, z_2= - 1 and, A_1=2, B �$_1$= - 1, C_1=3, A_2= - 2, B �$_2$= - 3, C_2= - 1.

$$\begin{vmatrix} 0-1 & 2-(-1) & -1-0 \\ 2 & -1 & 3 \\ -2 & -3 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 & -1 \\ 2 & -1 & 3 \\ -2 & -3 & -1 \end{vmatrix}$$

=[{((- 1)×(- 1)×(- 1)) + (3×3×(- 2)) + ((- 1)×2×(- 3))} - {((- 1)×(- 1)×(- 2)) + (3×2×(- 1)) + ((- 1)×(- 3)×3)}]

=[{(- 1) - 18 + 6} - {(- 2) + (- 6) + 9}]

= - 13 - 1

= - 14

≠ 0

Hence, the two lines are not co - planar.

7. Question

Mark the correct alternative in the following:

The equation of the plane $\vec{r} = \hat{i} - \hat{j} + \lambda\left(\hat{i} + \hat{j} + \hat{k}\right) + \mu\left(\hat{i} - 2\hat{j} + 3\hat{k}\right)$ in scalar product from is

A. $\vec{r}.\left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = 7$

B. $\vec{r}.\left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 7$

C. $\vec{r}.\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right) = 7$

D. none of these

Answer

The given plane is $\vec{r} = \hat{\imath} - \hat{\jmath} + \lambda(\hat{\imath} + \hat{\jmath} + \hat{k}) + \mu(\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$.

So, it is clear from the given equation of plane, that the plane passing through a point $(\hat{\imath} - \hat{\jmath})$ and parallel to two vectors $(\hat{\imath} + \hat{\jmath} + \hat{k})$ and $(\hat{\imath} - 2\hat{\jmath} + 3\hat{k})$.

So, the equation of the vector normal to the plane is given as,

$$\vec{n} = \left((\hat{\imath} + \hat{\jmath} + \hat{k}) \times (\hat{\imath} - 2\hat{\jmath} + 3\hat{k})\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \left((1 \times 3) - \left((-2) \times 1\right)\right)\hat{\imath} - \left((1 \times 3) - (1 \times 1)\right)\hat{\jmath} + \left(\left(1 \times (-2)\right) - (1 \times 1)\right)\hat{k}$$

$$= \left(3 - (-2)\right)\hat{\imath} - (3 - 1)\hat{\jmath} + \left((-2) - 1\right)\hat{k}$$

$$= (5\hat{\imath} - 2\hat{\jmath} - 3\hat{k})$$

So, in scalar product form the vector equation of the plane is given as,

$\left(\vec{r} - (\hat{i} - \hat{j})\right) \cdot \vec{n} = 0$

$\Rightarrow \left(\vec{r} - (\hat{i} - \hat{j})\right) \cdot \left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = 0$

$\Rightarrow \vec{r} \cdot \left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = (\hat{i} - \hat{j}) \cdot \left(5\hat{i} - 2\hat{j} - 3\hat{k}\right)$

$\Rightarrow \vec{r} \cdot \left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = 5 + 2$

$\Rightarrow \vec{r} \cdot \left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = 7$

Hence, the equation of the plane $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k})$ in scalar product form is given as,

$\vec{r} \cdot \left(5\hat{i} - 2\hat{j} - 3\hat{k}\right) = 7.$

8. Question

Mark the correct alternative in the following:

The distance of the line $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda\left(\hat{i} + \hat{j} + 4\hat{k}\right)$ from the plane $\vec{r}.\left(\hat{i} - 5\hat{j} + \hat{k}\right) = 5,$ is

A. $\frac{5}{3\sqrt{3}}$

B. $\frac{10}{3\sqrt{3}}$

C. $\frac{25}{3\sqrt{3}}$

D. none of these

Answer

We have the, straight line given as,

$\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + \hat{j} + 4\hat{k})$ and the plane as,

$\vec{r} \cdot (\hat{i} - 5\hat{j} + \hat{k}) = 5$ i.e. x - 5y + z=5 x - 5y + z - 5=0

Let us, check whether the plane and the straight line are parallel using the scalar product between the governing vector of the straight line, $\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + \hat{j} + 4\hat{k})$, and the normal vector of the plane given as, $\vec{n} = (\hat{i} - 5\hat{j} + \hat{k})$. If the straight line and the plane are parallel the scalar product will be zero.

$(\hat{i} + \hat{j} + 4\hat{k}) \cdot (\hat{i} - 5\hat{j} + \hat{k})$

$= 1 + \left(1 \times (-5)\right) + (4 \times 1)$

=1 - 5 + 4

=0

From the given equation of the line, it is clear that, (2, - 2, 3) is a point on the straight line.

Distance from point (2, - 2, 3) to the plane, will be equal to the distance of the line from the plane.

We know, that the distance of a point (x_0, y_0, z_0) from a plane Ax + By + Cz + D=0 .... (2) is

$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

On comparing, equation (1) i.e. x - 5y + z - 5=0 with equation (2) we get,

A=1, B= - 5, C=1, D= - 5.

So, the distance from point (2, - 2, 3) to the plane

$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

$= \frac{\left|\left((1 \times 2) + \left((-5) \times (-2)\right) + (1 \times 3) + (-5)\right)\right|}{\sqrt{1^2 + (-5)^2 + 1^2}}$

$[(x_0, y_0, z_0) \approx (2, -2, 3)]$

$= \frac{|(2 + 10 + 3 - 5)|}{\sqrt{1 + 25 + 1}}$

$= \frac{10}{\sqrt{27}}$

$= \frac{10}{\sqrt{9 \times 3}}$

$= \frac{10}{3\sqrt{3}}$

9. Question

Mark the correct alternative in the following:

The equation of the plane through the line x + y + z + 3 = 0 = 2x – y + 3z + 1 and parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is

A. x – 5y + 3z = 7

B. x – 5y + 3z = –7

C. x + 5y + 3z = 7

D. x + 5y + 3z = –7

Answer

Equation of line passing through the line x + y + z + 3=0 and 2x–y + 3z + 1=0 is given by,

(x + y + z + 3) + k(2x–y + 3z + 1)=0 ……………………(1)

x(1 + 2k) + y(1 - k) + z(1 + 3k) + 3 + k=0 [k is a constant]

Again, the required plane is parallel to the line

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

So, we should have,

[1×(1 + 2k)] + [2×(1 - k)] + [3×(1 + 3k)]=0

1 + 2k + 2 - 2k + 3 + 9k=0

9k= - 6

$\Rightarrow k = -\frac{6}{9}$

$\Rightarrow k = -\frac{2}{3}$

Putting $k = -\frac{2}{3}$ in equation (1) we get,

$(x + y + z + 3) - \frac{2}{3}(2x - y + 3z + 1) = 0$

3(x + y + z + 3) - 2(2x–y + 3z + 1)=0

3x + 3y + 3z + 9 - 4x + 2y - 6z - 2=0

- x + 5y - 3z + 7=0

x - 5y + 3z - 7=0

x - 5y + 3z=7

∴The equation of the plane through the line x + y + z + 3 = 0 = 2x – y + 3z + 1 and parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is x - 5y + 3z=7.

10. Question

Mark the correct alternative in the following:

The vector equation of the plane containing the line $\vec{r} = \left(-2\hat{i} - 3\hat{j} + 4\hat{k}\right) + \lambda\left(3\hat{i} - 2\hat{j} - \hat{k}\right)$ and the point $\hat{i} + 2\hat{j} + 3\hat{k}$ is

A. $\vec{r}.\left(\hat{i} + 3\hat{k}\right) = 10$

B. $\vec{r}.\left(\hat{i} - 3\hat{k}\right) = 10$

C. $\vec{r}.\left(3\hat{i} + \hat{k}\right) = 10$

D. none of these

Answer

The plane contains the line $\vec{r} = (-2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - \hat{k})$ and the point $(\hat{i} + 2\hat{j} + 3\hat{k})$

As, the plane contains the line,

$\vec{r} = (-2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - \hat{k})$ so, the plane contains the $(-2\hat{i} - 3\hat{j} + 4\hat{k})$ point also.

On putting λ=1, we get another point on the plane which is $(-2\hat{i} - 3\hat{j} + 4\hat{k}) + 1 \times (3\hat{i} - 2\hat{j} - \hat{k})$ i.e. $(\hat{i} - 5\hat{j} + 3\hat{k})$

So, we got three points on the plane, they are, $(\hat{i} + 2\hat{j} + 3\hat{k})$, $(-2\hat{i} - 3\hat{j} + 4\hat{k})$ and $(\hat{i} - 5\hat{j} + 3\hat{k})$

Let, $\vec{a} = (\hat{i} - 5\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

and $\vec{b} = (\hat{i} - 5\hat{j} + 3\hat{k}) - (-2\hat{i} - 3\hat{j} + 4\hat{k})$

So, $\vec{a} = -7\hat{j}$ and $\vec{b} = 3\hat{i} - 2\hat{j} - \hat{k}$

Now, the normal of these two vectors i.e. $\vec{a}$ and $\vec{b}$ is,

$\vec{n} = \left((-7\hat{j}) \times (3\hat{i} - 2\hat{j} - \hat{k})\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -7 & 0 \\ 3 & -2 & -1 \end{vmatrix}$$

$= \left(\left(((-7)\times(-1)) - ((-2)\times 0)\right)\hat{\imath} - \left((0\times(-1)) - (0\times 3)\right)\hat{\jmath} + \left((0\times(-2)) - ((-7)\times 3)\right)\hat{k}\right.$

$= (7-0)\hat{\imath} - 0\hat{\jmath} + 21\hat{k}$

$= \left(7\hat{\imath} + 21\hat{k}\right)$

The general equation of plane is,

$\left[(x-1)\hat{\imath} + (y-2)\hat{\jmath} + (z-3)\hat{k}\right]\cdot\vec{n} = 0 \left[\begin{matrix}\because \text{the plane contains} \\ \text{the point}\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)\end{matrix}\right]$

$\Rightarrow \left[(x-1)\hat{\imath} + (y-2)\hat{\jmath} + (z-3)\hat{k}\right]\cdot\left(7\hat{\imath} + 21\hat{k}\right) = 0$

7(x - 1) + 21(z - 3)=0

7x - 7 + 21z - 63=0

7x + 21z=70

x + 3z=10

or, $\vec{r}\cdot\left(\hat{\imath} + 3\hat{k}\right) = 10 \left[\because \vec{r} = \left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)\right]$

Hence, the vector equation of the plane containing the line $\vec{r} = \left(-2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}\right) + \lambda\left(3\hat{\imath} - 2\hat{\jmath} - \hat{k}\right)$ and the point $\left(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$ is $\vec{r}\cdot\left(\hat{\imath} + 3\hat{k}\right) = 10$.

11. Question

Mark the correct alternative in the following:

A plane meets the coordinate axes at A, B, C such that the centroid of ΔABC is the point (a, b, c). If the equation of the plane is then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k$, k =

A. 1

B. 2

C. 3

D. none of these

Answer

A plane meets the co - ordinate axes at A, B, C such that the centroid of ΔABC is the point (a, b, c).

Let, the co - ordinates of the point A (α, 0, 0), B (0, β, 0) and C (0, 0, γ).

According to the centroid formula,

$a = \frac{\alpha + 0 + 0}{3}$

$\Rightarrow \alpha = 3a$

$, b = \frac{0 + \beta + 0}{3}$

$\Rightarrow \beta = 3b$

and $c = \frac{0 + 0 + \gamma}{3}$

$\Rightarrow \gamma = 3c$

We know the intercept form of a plane is given as,

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1$$

if the plane makes intercepts at (p, 0, 0), (0, q, 0) and (0, 0, r) with the x - , y - and z - axes respectively.

Here, p=α=3a, q=β=3b and r=γ=3c

So, the equation of the plane is,

$$\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1$$

$$\Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3 \ldots\ldots\ldots\ldots\ldots(1)$$

Equation of the given plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$$

On comparing, we get, k=3.

12. Question

Mark the correct alternative in the following:

The distance between the point (3, 4, 5) and the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane x + y + z = 17, is

A. 1

B. 2

C. 3

D. none of these

Answer

Let, the point of intersection of the line

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$$

and the plane x + y + z=17 be (x_0, y_0, z_0).

As (x_0, y_0, z_0) is the point of intersection of the line and

the plane, so it must satisfy both of the equation of

line and the equation of plane.

Substituting, (x_0, y_0, z_0) in place of (x, y, z) in both the equations, we get,

$$\frac{x_0-3}{1}=\frac{y_0-4}{2}=\frac{z_0-5}{2}=k(\text{let})$$

$$\Rightarrow \frac{x_0-3}{1}=k,$$

$$\frac{y_0-4}{2}=k \text{ and}$$

$$\frac{z_0-5}{2}=k$$

i.e. x_0=k + 3,

y_0=2k + 4 and

z_0=2k + 5

Putting this values in the equation of plane we get,

$x_0 + y_0 + z_0$=17

(k + 3) + (2k + 4) + (2k + 5)=17

5k + 12=17

5k=5

k=1

∴ x_0=k + 3

=1 + 3

=4

y_0=2k + 4

=(2×1) + 4

=6

z_0=2k + 5

=(2×1) + 5

=7

Hence, the point of intersection is, (4, 6, 7).

Now, the distance between the point (3, 4, 5) and (4, 6, 7) is,

$= \sqrt{(4-3)^2 + (6-4)^2 + (7-5)^2}$

$= \sqrt{1^2 + 2^2 + 2^2}$

$= \sqrt{1 + 4 + 4}$

$= \sqrt{9}$

$= 3$

Hence, the distance between the point (3, 4, 5) the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane x + y + z = 17, is 3 units.

13. Question

Mark the correct alternative in the following:

A vector parallel to the line of intersection of planes $\vec{r}.\left(3\hat{i} - \hat{j} + \hat{k}\right) = 1$ and $\vec{r}.\left(\hat{i} - 4\hat{j} - 2\hat{k}\right) = 2$ is

A. $-2\hat{i} + 7\hat{j} - 13\hat{k}$

B. $2\hat{i}+7\hat{j}-13\hat{k}$

C. $-2\hat{i}-7\hat{j}+13\hat{k}$

D. $2\hat{i}+7\hat{j}+13\hat{k}$

Answer

The two planes are, $\vec{r}\cdot\left(3\hat{i}-\hat{j}+\hat{k}\right)=1$ and

$\vec{r}\cdot\left(\hat{i}-4\hat{j}-2\hat{k}\right)=2$

The line of intersection of planes $\vec{r}\cdot\left(3\hat{i}-\hat{j}+\hat{k}\right)=1$ and

$\vec{r}\cdot\left(\hat{i}-4\hat{j}-2\hat{k}\right)=2$ is parallel to

$\left(3\hat{i}-\hat{j}+\hat{k}\right)\times\left(\hat{i}-4\hat{j}-2\hat{k}\right)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & -4 & -2 \end{vmatrix}$$

$$=\left(\left((-1)\times(-2)\right)-\left((-4)\times 1\right)\right)\hat{i}-\left(\left(3\times(-2)\right)-(1\times 1)\right)\hat{j}+\left(\left(3\times(-4)\right)-\left((-1)\times 1\right)\right)\hat{k}$$

$$=\left(2-(-4)\right)\hat{i}-\left((-6)-1\right)\hat{j}+\left((-12)-(-1)\right)\hat{k}$$

$$=\left(6\hat{i}+7\hat{j}-11\hat{k}\right)$$

The line of intersection of planes $\vec{r}\cdot\left(3\hat{i}-\hat{j}+\hat{k}\right)=1$ and

$\vec{r}\cdot\left(\hat{i}-4\hat{j}-2\hat{k}\right)=2$ is parallel to $\underline{\left(\mathbf{6\hat{i}+7\hat{j}-11\hat{k}}\right)}$

<u>Alternative:</u>

The two planes are, $\vec{r}\cdot\left(3\hat{i}-\hat{j}+\hat{k}\right)=1$ and

$\vec{r}\cdot\left(\hat{i}-4\hat{j}-2\hat{k}\right)=2$

or, 3x - y + z=1 and x - 4y - 2z=2

Putting, z=k, we get,

3x - y + k=1 .... (1)

and x - 4y - 2k=2 .... (2)

Multiplying equation (2) by 3 and then subtracting equation (1) from it, we get,

3(x - 4y - 2k) - (3x - y + k)=(3×2) - 1

3x - 12y - 6k - 3x + y - k=6 - 1

- 11y - 7k=5

$$\Rightarrow k=\frac{5+11y}{-7}\text{ and }y=\frac{5+7k}{-11}$$

Substituting y, in equation (1) we get,

$$3x-\left(\frac{5+7k}{-11}\right)+k=1$$

$$3x+\frac{5+7k}{11}+k=1$$

33x + 5 + 7k + 11k=11

18k=11 - 5 - 33x

$$\Rightarrow k = \frac{6 - 33x}{18} = \frac{2 - 11x}{6}$$

$$\therefore \frac{2 - 11x}{6} = \frac{5 + 11y}{-7} = k = z$$

$$\Rightarrow \frac{x + \frac{2}{-11}}{\frac{6}{-11}} = \frac{y + \frac{5}{11}}{-\frac{7}{11}} = \frac{z}{1} = k$$

$$\Rightarrow \frac{x - \frac{2}{11}}{\frac{6}{-11}} = \frac{y + \frac{5}{11}}{-\frac{7}{11}} = \frac{z}{1} = k$$

$$\Rightarrow x - \frac{2}{11} = \frac{6}{-11}k$$

$$\Rightarrow y + \frac{5}{11} = -\frac{7}{11}k$$

$$\Rightarrow \frac{z}{1} = k$$

Equation of the line of intersection,

$$\left(\frac{2}{11} - \frac{6k}{11}\right)\hat{i} + \left(-\frac{5}{11} - \frac{7k}{11}\right)\hat{j} + k\hat{k} = 0$$

$$\Rightarrow \left(\frac{2}{11}\hat{i} - \frac{5}{11}\hat{j}\right) - k\left(\frac{6}{11}\hat{i} + \frac{7}{11}\hat{j} + \hat{k}\right) = 0$$

$$\Rightarrow (2\hat{i} - 5\hat{j}) - k\left(6\hat{i} + 7\hat{j} + 11\hat{k}\right) = 0$$

The line of intersection of planes $\vec{r} \cdot \left(3\hat{i} - \hat{j} + \hat{k}\right) = 1$ and

$\vec{r} \cdot \left(\hat{i} - 4\hat{j} - 2\hat{k}\right) = 2$ is parallel to $\underline{\left(\mathbf{6\hat{i} + 7\hat{j} - 11\hat{k}}\right)}$.

14. Question

Mark the correct alternative in the following:

If a plane passes through the point (1, 1, 1) and is perpendicular to the line $\frac{x-1}{3} = \frac{y-1}{0} = \frac{z-1}{4}$, then its perpendicular distance from the origin is

A. 3/4

B. 4/3

C. 7/5

D. 1

Answer

Let, the equation of the plane be, Ax + By + Cz + D=0, as the plane is perpendicular to, $\frac{x-1}{3} = \frac{y-1}{0} = \frac{z-1}{4}$ so, we have,

A=3, B=0 and C=4

As the plane passes through (1, 1, 1) we have, (A×1) + (B×1) + (C×1) + D=0

A + B + C + D=0

3 + 0 + 4 + D=0

D= - 7

So, the equation of the plane becomes, 3x + 4z - 7=0

Now, the perpendicular distance of the plane from the origin is

$$= \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|((3 \times 0) + (0 \times 0) + (4 \times 0) - 7)|}{\sqrt{3^2 + 0^2 + 4^2}}$$

$[\because (x_0, y_0, z_0) \approx (0, 0, 0)]$

$$= \frac{|0 - 7|}{\sqrt{9 + 16}}$$

$$= \frac{|-7|}{\sqrt{25}}$$

$$= \frac{7}{5}$$

Hence, the perpendicular distance from the origin to the plane is $= \frac{7}{5}$ units.

15. Question

Mark the correct alternative in the following:

The equation of the plane parallel to the lines x – 1 = 2y – 5 = 2z and 3x = 4y – 11 = 3z – 4 and passing through the point (2, 3, 3) is

A. x – 4y + 2z + 4 = 0

B. x + 4y + 2z + 4 = 0

C. x – 4y + 2z – 4 = 0

D. none of these

Answer

The required plane is parallel to the lines

x–1=2y–5=2z and 3x=4y–11=3z–4.

Equation of the lines can be re - written as,

$$\frac{x-1}{1} = \frac{2y-5}{1} = \frac{2z}{1}$$

$$\Rightarrow \frac{x-1}{1} = \frac{y - \frac{5}{2}}{\frac{1}{2}} = \frac{z}{\frac{1}{2}} = \lambda(\text{let})$$

And,

$$\frac{3x}{1} = \frac{4y-11}{1} = \frac{3z-4}{1}$$

$$\Rightarrow \frac{x}{\frac{1}{3}} = \frac{y - \frac{11}{4}}{\frac{1}{4}} = \frac{z - \frac{4}{3}}{\frac{1}{3}} = \mu(\text{let})$$

So, we have the straight lines as,

$(1 + \lambda)\hat{i} + \left(\frac{5}{2} + \frac{1}{2}\lambda\right)\hat{j} + \frac{1}{2}\lambda\hat{k} = 0$

$\Rightarrow \hat{i} + \frac{5}{2}\hat{j} + \lambda\left(\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) = 0$

And,

$\frac{1}{3}\mu\hat{i} + \left(\frac{11}{4} + \frac{1}{4}\mu\right)\hat{j} + \left(\frac{4}{3} + \frac{1}{3}\mu\right)\hat{k} = 0$

$\Rightarrow \frac{11}{4}\hat{j} + \frac{4}{3}\hat{k} + \mu\left(\frac{1}{3}\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{3}\hat{k}\right) = 0$

We have the normal vector of the plane as,

$\vec{n} = \left(\left(\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) \times \left(\frac{1}{3}\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{3}\hat{k}\right)\right)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{3} \end{vmatrix}$

$= \left(\left(\frac{1}{2} \times \frac{1}{3}\right) - \left(\frac{1}{2} \times \frac{1}{4}\right)\right)\hat{i} - \left(\left(1 \times \frac{1}{3}\right) - \left(\frac{1}{2} \times \frac{1}{3}\right)\right)\hat{j} + \left(\left(1 \times \frac{1}{4}\right) - \left(\frac{1}{2} \times \frac{1}{3}\right)\right)\hat{k}$

$= \left(\frac{1}{6} - \frac{1}{8}\right)\hat{i} - \left(\frac{1}{3} - \frac{1}{6}\right)\hat{j} + \left(\frac{1}{4} - \frac{1}{6}\right)\hat{k}$

$= \left(\frac{1}{24}\hat{i} - \frac{1}{6}\hat{j} + \frac{1}{12}\hat{k}\right)$

So, the equation of plane is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, where

$\vec{a} = 2\hat{i} + 3\hat{j} + 3\hat{k}$ [∵the plane passes through the point (2, 3, 3)]

$\therefore (\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$\Rightarrow \vec{r}.\vec{n} = \vec{a} \cdot \vec{n}$

$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\vec{n} = \left(2\hat{i} + 3\hat{j} + 3\hat{k}\right) \cdot \vec{n}$

$\left[\because \vec{r} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\right]$

$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(\frac{1}{24}\hat{i} - \frac{1}{6}\hat{j} + \frac{1}{12}\hat{k}\right) = \left(2\hat{i} + 3\hat{j} + 3\hat{k}\right) \cdot \left(\frac{1}{24}\hat{i} - \frac{1}{6}\hat{j} + \frac{1}{12}\hat{k}\right)$

$\Rightarrow \frac{1}{24}x - \frac{1}{6}y + \frac{1}{12}z = \frac{1}{12} - \frac{1}{2} + \frac{1}{4}$

$\Rightarrow \frac{1}{24}x - \frac{1}{6}y + \frac{1}{12}z = \frac{-2}{12}$

x - 4y + 2z= - 4

x - 4y + 2z + 4=0

The equation of the plane parallel to the lines

x–1=2y–5=2z and 3x=4y–11=3z–4 and passing through the point (2, 3, 3) is x–4y + 2z + 4=0

16. Question

Mark the correct alternative in the following:

The distance of the point (–1, –5, –10) from the point of intersection of the line

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 12\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$ is

A. 9

B. 13

C. 17

D. none of these

Answer

Let, the point of intersection of the line

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$ and the plane

$\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ be (x_0, y_0, z_0).

As (x_0, y_0, z_0) is the point of intersection of the line and the plane, so the position vector of this point i.e.

$\vec{r_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ must satisfy both of the equation of

line and the equation of plane.

Substituting, $\vec{r_0}$ in place of $\vec{r}$ in both the equations, we

get,

$(x_0\hat{i} + y_0\hat{j} + z_0\hat{k}) = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$

And, $(x_0\hat{i} + y_0\hat{j} + z_0\hat{k}) \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ .... (2)

i.e.x_0 =2 + 3λ

y_0 = - 1 + 4λ

z_0 =2 + 12λ

Substituting, these values in equation (2) we get,

((2 + 3λ)×1) - (1×(- 1 + 4λ)) + (1×(2 + 12λ))=5

2 + 3λ + 1 - 4λ + 2 + 12λ=5

11λ=0

λ=0

∴x_0 =2 + 3λ

=2

y_0= - 1 + 4λ

= - 1

z_0=2 + 12λ

=2

Hence, the point of intersection is, (2, - 2, 2).

Now, the distance between the point (- 1, - 5, - 10) and (2, - 1, 2) is,

$= \sqrt{\left(2-(-1)\right)^2 + \left((-1)-(-5)\right)^2 + \left(2-(-10)\right)^2}$

$= \sqrt{3^2 + 4 + 12^2}$

$= \sqrt{9 + 16 + 144}$

$= \sqrt{169}$

=13

Hence, the required distance between the point (- 1, - 5, - 10) the point where the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 12\hat{k}\right)$ the plane $\vec{r}\cdot\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$, is 13 units.

17. Question

Mark the correct alternative in the following:

The equation of the plane through the intersection of the planes ax + by + cz + d = 0 and lx + my + nz + p = 0 and parallel to the line y = 0, z = 0

A. (bl – am)y + (cl – an) z + dl – ap = 0

B. (am – bl)x + (mc – bm) z + md – bp = 0

C. (na – cl)x + (bm – cm) y + nd – cp = 0

D. none of these

Answer

The equation of the plane through the intersection of

the planes ax + by + cz + d=0 and lx + my + nz + p=0 is given as,

(ax + by + cz + d) + λ(lx + my + nz + p)=0

[where λ is a scalar]

x(a + lλ) + y(b + mλ) + z(c + nλ) + d + pλ=0

Given, that the required plane is parallel to the line y=0, z=0 i.e. x - axis so, we should have,

1(a + lλ) + 0(b + mλ) + 0(c + nλ)=0

a + lλ=0

$\Rightarrow \lambda = -\frac{a}{l}$

Substituting the value of λ we get,

$(ax + by + cz + d) - \frac{a}{l}(lx + my + nz + p) = 0$

(alx + bly + clz + dl) - a(lx + my + nz + p)=0

alx + bly + clz + dl - alx + amy + anz + ap=0

bly + clz + dl - amy - anz - ap=0

(bl - an)y + (cl - an)z + dl - ap=0

Therefore, the equation of the required plane is

(bl–am)y + (cl–an)z + dl–ap=0

18. Question

Mark the correct alternative in the following:

The equation of the plane which cuts equal intercepts of unit length on the coordinate axes is

A. x + y + z = 1

B. x + y + z = 0

C. x + y – z = 0

D. x + y + z = 2

Answer

We know, that the general equation of a plane is given by,

Ax + By + Cz + D=0, where $D \neq 0$ .... (1)

Here, A, B, C are the coordinates of a normal vector to the plane, while (x, y, z) are the co - ordinates of any point through which the plane passes.

Again, we know the intercept form of plane which is given by,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \ldots \ldots \ldots \ldots .. (2)$$

Where, $A = -\frac{D}{a}, B = -\frac{D}{b}$ and $C = -\frac{D}{c}$ and the plane makes intercepts at (a, 0, 0), (0, b, 0) and (0, 0, c) with the x - , y - and z - axes respectively.

Here, a=b=c=1.

Putting, the value of a, b, c in equation (2), we are getting,

$$\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$$

x + y + z=1

Hence, the equation of the plane which cuts equal intercepts of unit length on the coordinate axes is, x + y + z=1.