33. Binomial Distribution

Exercise 33.1

26. Question

A man wins a rupee for head and loses a rupee for tail when the coin is tossed. Suppose that he tosses once and quits if he wins but tries once more if he loses on the first toss. Find the probability distribution of the number of rupees the man wins.

Answer

Let X be the number of rupees the man won/lost.

Let n be the number of throws required to get a head.

We have two cases, that is

(i). The man tosses once, head comes up, and he quits. (head means he won)

(ii). The man tosses once; tail comes up then he tosses again, tail comes up. (tail means he lost)

(ii). The man tosses, tail comes up then he tosses again, head comes up. (tail means he lost & head means he won)

In Case (i),

The man tosses once, head comes up, and he quits.

Here, number of throws (n) = 1

Amount won/lost (X) = 1

$$\text{Probability, P(X)} = \frac{1}{2}$$

In Case (ii),

The man tosses once; tail comes up then he tosses again, tail comes up.

Here, number of throws (n) = 2

Amount won/lost (X) = -2

$$\text{Probability, P(X)} = \frac{1}{2} \times \frac{1}{2}$$

$$\Rightarrow \text{Probability, P(X)} = \frac{1}{4}$$

In Case (iii),

The man tosses once, tail comes up then he tosses again, head comes up.

Here, number of throws (n) = 2

Amount won/lost (X) = 0

$$\text{Probability, P(X)} = \frac{1}{2} \times \frac{1}{2}$$

$$\Rightarrow \text{Probability, P(X)} = \frac{1}{4}$$

We have the table:

Number of throws (n):	1	2	2
Amount won/lost (X):	1	0	-2
Probability, P(X):	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Thus, the probability distribution is

X	P(X)
0	$\frac{1}{4}$
1	$\frac{1}{2}$
-2	$\frac{1}{4}$

Hence, the answer is obtained.

27. Question

Five dice are thrown simultaneously. If the occurrence of 3, 4 or 5 in a single die is considered a success, find the probability of at least 3 successes.

Answer

Let p denote the probability of getting 3, 4 or 5 in a throw of dice.

Let us find out the value of p.

We know, a dice has 6 faces numbered 1, 2, 3, 4, 5 and 6.

So, the probability of getting a 3, 4 or 5 is given as,

$$\Rightarrow p = \frac{3}{6}$$

$$\Rightarrow p = \frac{1}{2}$$

If p denotes the probability of getting success, then let q denote the probability of not getting success.

We can say,

p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{2}$$

$$\Rightarrow q = \frac{1}{2}$$

Let X denote the number of successes in the throw of five dice simultaneously.

Let there be total n number of throws of five dice simultaneously.

Then, the probability of getting r successes out of n throws of dice is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Now, substitute the value of p and q in the above equation.

Also, put n = 5 (Since there are 5 dice throw)

$$P(X = r) = {}^5C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r}$$

Now, the probability of getting at least 3 successes is given by

Probability of getting at least 3 successes = P(X = 3) + P(X = 4) + P(X = 5)

Thus,

$$\text{Probability of getting atleast 3 successes} = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3} + {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} + {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{5-5}$$

$$\Rightarrow \text{Probability of getting atleast 3 successes} = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0$$

$$\Rightarrow \text{Probability of getting atleast 3 successes} = \left(\frac{5!}{(5-3)!\,3!}\right)\left(\frac{1}{2}\right)^{3+2} + \left(\frac{5!}{(5-4)!\,4!}\right)\left(\frac{1}{2}\right)^{4+1} + \left(\frac{5!}{(5-5)!\,5!}\right)\left(\frac{1}{2}\right)^{5+0}$$

$\Rightarrow$ Probability of getting atleast 3 successes $= \left(\frac{5!}{2!\,3!}\right)\left(\frac{1}{2}\right)^5 + \left(\frac{5!}{1!\,4!}\right)\left(\frac{1}{2}\right)^5 + \left(\frac{5!}{0!\,5!}\right)\left(\frac{1}{2}\right)^5$

$\Rightarrow$ Probability of getting atleast 3 successes $= \left(\frac{5 \times 4 \times 3!}{2 \times 1 \times 3!}\right)\left(\frac{1}{2}\right)^5 + \left(\frac{5 \times 4!}{4!}\right)\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5$

$\Rightarrow$ Probability of getting atleast 3 successes $= \left(\frac{1}{2}\right)^5 [10 + 5 + 1]$

$\Rightarrow$ Probability of getting atleast 3 successes $= \left(\frac{1}{32}\right) \times 16$

$\Rightarrow$ Probability of getting atleast 3 successes $= \frac{1}{2}$

Thus, the probability of getting 3 successes is 1/2.

28. Question

The items produced by a company contain 10% defective items. Show that the probability of getting 2 defective items in a sample of 8 items is $\frac{28 \times 9^6}{10^8}$.

Answer

Let p be the probability of getting defective items out of 100 items.

And according to the question, the items produced by a company contain 10% defective items.

So, p = 10%

$\Rightarrow p = \frac{10}{100}$

$\Rightarrow p = \frac{1}{10}$

And also, as we know p + q = 1

If p is the probability of getting defective items out of 100, then q is the probability of not getting defective items out of 100.

$\Rightarrow q = 1 - p$

$\Rightarrow q = 1 - \frac{1}{10}$

$\Rightarrow q = \frac{10 - 1}{10}$

$\Rightarrow q = \frac{9}{10}$

Let X be the number of defective items drawn out of 8 items. (Since 8 is the sample size)

Then, the probability of getting r defective items out of 8 items is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

We know that, n = 8

$p = \frac{1}{10}$

$q = \frac{9}{10}$

Put all these values in the previous equation.

$P(X = r) = {}^8C_r \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{8-r}$

Then, the probability of getting two defective items out of a sample of 8 is given by putting r = 2.

Now,

$$P(X=2) = {}^8C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{8-2}$$

$$\Rightarrow P(X=2) = \frac{8!}{(8-2)!\,2!}\times\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^6$$

$$\Rightarrow P(X=2) = \frac{8!}{6!\,2!}\times\frac{1}{10^2}\times\frac{9^6}{10^6}$$

$$\Rightarrow P(X=2) = \frac{8\times7\times6!}{6!\,2!}\times\frac{1}{10^2}\times\frac{9^6}{10^6}$$

$$\Rightarrow P(X=2) = 28\times\frac{9^6}{10^{2+6}}$$

$$\Rightarrow P(X=2) = \frac{28\times9^6}{10^8}$$

Thus, the probability of getting 2 defective items out of a sample of 8 items is $\frac{28\times9^6}{10^8}$.

29. Question

A card is drawn and replaced in an ordinary pack of 52 cards. How many times must a card be drawn so that (i) there is at least an even chance of drawing a heart, (ii) the probability of drawing a heart is greater than 3/4?

Answer

We know that there are 52 cards in a pack of cards.

And there are 13 cards of each suit.

Let p be the probability of drawing a card of heart from a pack of 52 cards.

Then,

$$p = \frac{13}{52}$$

[∵ there are 13 cards of heart in the pack]

$$\Rightarrow p = \frac{1}{4}$$

Also, p + q = 1

Where if p is the probability of getting a heart out of 52 cards, then q is the probability of not getting a heart out of 52 cards.

⇒ q = 1 – p

Putting the value of p in the above equation, we get

$$\Rightarrow q = 1-\frac{1}{4}$$

$$\Rightarrow q = \frac{4-1}{4}$$

$$\Rightarrow q = \frac{3}{4}$$

Now, let the card be drawn n times.

And let X denote the number of hearts drawn out of a pack of 52 cards.

Then, the binomial distribution is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Putting $p=\frac{1}{4}$ and $q=\frac{3}{4}$ above, we get

$$P(X=r) = {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{3}{4}\right)^{n-r}$$ …(A)

Where r = 0, 1, 2, 3… n

(i). We need to find the number of times a card can be drawn so that at least there is an even chance of drawing a heart.

In simple words, since n is the number of times a card is drawn, we need to find the smallest n for which P (X = 0) is less than 1/4 satisfies. $\left(p=\frac{1}{4}\right)$

So,

$$P(X=0)<\frac{1}{4}$$

From equation A, we have

$$\Rightarrow {}^{n}C_{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{n-0}<\frac{1}{4}$$

$$\Rightarrow \left(\frac{n!}{(n-0)!\,0!}\right)\times 1\times\left(\frac{3}{4}\right)^{n}<\frac{1}{4}$$

$$\Rightarrow \left(\frac{n!}{n!}\right)\times 1\times\left(\frac{3}{4}\right)^{n}<\frac{1}{4}$$

$$\Rightarrow \left(\frac{3}{4}\right)^{n}<\frac{1}{4}$$

$$\Rightarrow \left(\frac{3}{4}\right)^{n}<0.25$$

First, put n = 0.

$$\left(\frac{3}{4}\right)^{0}<0.25$$

$\Rightarrow 1<0.25$

But 1 ≮ 0.25

Now, put n = 1.

$$\left(\frac{3}{4}\right)^{1}<0.25$$

⇒ 0.75 < 0.25

But 0.75 ≮ 0.25

Now, put n = 2.

$$\left(\frac{3}{4}\right)^{2}<0.25$$

$$\Rightarrow \frac{9}{16}<0.25$$

⇒ 0.5625 < 0.25

But 0.5626 ≮ 0.25

Now, put n = 3.

$$\left(\frac{3}{4}\right)^{3}<0.25$$

$$\Rightarrow \frac{27}{64}<0.25$$

⇒ 0.42 < 0.25

But 0.42 ≮ 0.25

Now, put n = 4.

$$\left(\frac{3}{4}\right)^{4}<0.25$$

$\Rightarrow \frac{81}{256} < 0.25$

⇒ 0.31 < 0.25

But 0.31 ≮ 0.25

Now, put n = 5.

$\left(\frac{3}{4}\right)^5 < 0.25$

$\Rightarrow \frac{243}{1024} < 0.25$

⇒ 0.23 < 0.25

Thus, smallest n = 5.

∴, we must draw cards at least 5 times.

(ii). We need to find the number of times a card must be drawn so that the probability of drawing a heart is more than 3/4.

If P (X = 0) is the probability of not drawing a heart at all.

Then, 1 – P (X = 0) is the probability of drawing a heart out of 52 cards.

According to the question,

$\text{Probabality of drawing a heart} > \frac{3}{4}$

$\Rightarrow 1 - P(X = 0) > \frac{3}{4}$

$\Rightarrow 1 - \left({}^{n}C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{n-0}\right) > \frac{3}{4}$

$\Rightarrow 1 - \left[\frac{n!}{(n-0)!0!} \times 1 \times \left(\frac{3}{4}\right)^n\right] > \frac{3}{4}$

$\Rightarrow 1 - \left[\frac{n!}{n!} \times 1 \times \left(\frac{3}{4}\right)^n\right] > \frac{3}{4}$

$\Rightarrow 1 - \left(\frac{3}{4}\right)^n > \frac{3}{4}$

$\Rightarrow -\left(\frac{3}{4}\right)^n > \frac{3}{4} - 1$

$\Rightarrow -\left(\frac{3}{4}\right)^n > \frac{3-4}{4}$

$\Rightarrow -\left(\frac{3}{4}\right)^n > -\frac{1}{4}$

$\Rightarrow \left(\frac{3}{4}\right)^n < \frac{1}{4}$

$\Rightarrow \left(\frac{3}{4}\right)^n < 0.25$

First, put n = 0.

$\left(\frac{3}{4}\right)^0 < 0.25$

$\Rightarrow 1 < 0.25$

But 1 ≮ 0.25

Now, put n = 1.

$\left(\frac{3}{4}\right)^1 < 0.25$

⇒ 0.75 < 0.25

But 0.75 ≮ 0.25

Now, put n = 2.

$\left(\frac{3}{4}\right)^2 < 0.25$

$\Rightarrow \frac{9}{16} < 0.25$

⇒ 0.5625 < 0.25

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Now, put n = 5.

$\left(\frac{3}{4}\right)^5 < 0.25$

$\Rightarrow \frac{243}{1024} < 0.25$

⇒ 0.23 < 0.25

Thus, smallest n = 5.

∴, we must draw cards at least 5 times.

30. Question

The mathematics department has 8 graduate assistants who are assigned to the same office. Each assistant is just likely to study at home as in the office. How many desks must there be in the office so that each assistant has a desk at least 90% of the time?

Answer

Let X be the number of a graduate assistants.

⇒ X = 8

Let p be the probability of the assistant studying at the office.

Then, q be the probability of the assistant studying at home.

According to the question, each assistant is just likely to study at home as in the office.

$\Rightarrow \mathrm{p} = \frac{1}{2}$

Also, we know that (p + q) = 1.

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{2}$

$\Rightarrow q = \frac{2-1}{2}$

$\Rightarrow q = \frac{1}{2}$

Now, let there be k number of desks in the office.

We need to find the number of desks in the office so that each assistant has a desk at least 90% of the time.

$\Rightarrow P(X \le k) > \frac{90}{100}$

$\Rightarrow -P(X \le k) < -\frac{90}{100}$

$\Rightarrow 1 - P(X \le k) < 1 - \frac{90}{100}$

$\Rightarrow P(X > k) < \frac{10}{100}$

⇒ P (X > k) < 0.10

Now, we must find a value of k that satisfies the above equation.

Clearly,

P (X > 5) = P (X = 6 or X = 7 or X = 8)

$\Rightarrow P(X > 5) = {}^8C_6 p^6 q^{8-6} + {}^8C_7 p^7 q^{8-7} + {}^8C_8 p^8 q^{8-8}$

$[\because P(X = k) = {}^nC_k p^k q^{n-k}, \text{where n is the total number of entity \& k is the number of successses out of n entity}]$

$\Rightarrow P(X > 5) = {}^8C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^2 + {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^1 + {}^8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^0$

$\Rightarrow P(X > 5) = {}^8C_6 \left(\frac{1}{2}\right)^8 + {}^8C_7 \left(\frac{1}{2}\right)^8 + {}^8C_8 \left(\frac{1}{2}\right)^8$

$\Rightarrow P(X > 5) = \left(\frac{1}{2}\right)^8 \left[\left(\frac{8!}{(8-6)!\,6!}\right) + \left(\frac{8!}{(8-7)!\,7!}\right) + \left(\frac{8!}{(8-8)!\,8!}\right)\right]$

$\Rightarrow P(X > 5) = \left(\frac{1}{2}\right)^8 \left[\left(\frac{8!}{2!\,6!}\right) + \left(\frac{8!}{1!\,7!}\right) + \left(\frac{8!}{0!\,8!}\right)\right]$

$\Rightarrow P(X > 5) = \left(\frac{1}{2}\right)^8 \left[\left(\frac{8 \times 7 \times 6!}{2!\,6!}\right) + \left(\frac{8 \times 7!}{7!}\right) + 1\right]$

$\Rightarrow P(X > 5) = \left(\frac{1}{2}\right)^8 [28 + 8 + 1]$

$\Rightarrow P(X > 5) = \left(\frac{1}{2}\right)^8 \times 37$

$\Rightarrow P(X > 5) = \frac{37}{256}$

⇒ P (X > 5) = 0.14

Also, P (X > 6) = P (X = 7 or X = 8)

$\Rightarrow P(X > 6) = {}^8C_7 p^7 q^{8-7} + {}^8C_8 p^8 q^{8-8}$

$$\Rightarrow P(X>6)={}^8C_7\left(\frac{1}{2}\right)^8+{}^8C_8\left(\frac{1}{2}\right)^8$$

$$\Rightarrow P(X>6)=\left(\frac{1}{2}\right)^8\left[\left(\frac{8!}{(8-7)!\,7!}\right)+\left(\frac{8!}{(8-8)!\,8!}\right)\right]$$

$$\Rightarrow P(X>6)=\left(\frac{1}{2}\right)^8\left[\left(\frac{8!}{1!\,7!}\right)+\left(\frac{8!}{0!\,8!}\right)\right]$$

$$\Rightarrow P(X>6)=\left(\frac{1}{2}\right)^8\left[\left(\frac{8\times 7!}{7!}\right)+1\right]$$

$$\Rightarrow P(X>6)=\left(\frac{1}{2}\right)^8[8+1]$$

$$\Rightarrow P(X>6)=\left(\frac{1}{2}\right)^8\times 9$$

$$\Rightarrow P(X>6)=\frac{9}{256}$$

⇒ $P(X > 6) = 0.035$

∴, $P(X > 6) < 0.10$

Hence, if there are 6 desks, then there is atleast 90% chance for every graduate assistant to get a desk.

31. Question

An unbiased coin is tossed 8 times. Find, by using binomial distribution, the probability of getting at least 6 heads.

Answer

Let x be the number of heads in the tosses.

Let n be the total number of tosses.

Then, binomial distribution is given by

$P(X = x) = {}^nC_x p^x q^{n-x}$

Where x = 1, 2, 3, …, n

Here, p = probability of getting a head.

And q = probability of getting a tail.

$$\Rightarrow p=\frac{1}{2}\ \&\ q=\frac{1}{2}$$

Then,

$$P(x)={}^nC_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x}$$

We need to find the probability of getting at least 6 head.

Then, x = 6, 7, 8 [∵ there are 8 number of tosses]

First, putting n = 8 & x = 6. We get

$$P(6)={}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{8-6}$$

$$\Rightarrow P(6)={}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2$$

$$\Rightarrow P(6)=\left(\frac{8!}{(8-6)!\,6!}\right)\times\left(\frac{1}{2}\right)^8$$

$$\Rightarrow P(6)=\left(\frac{8!}{2!\,6!}\right)\times\left(\frac{1}{2}\right)^8$$

$\Rightarrow P(6) = \left(\frac{8 \times 7 \times 6!}{2!\,6!}\right) \times \left(\frac{1}{2}\right)^8$

$\Rightarrow P(6) = 28 \times \left(\frac{1}{2}\right)^8$ ...(i)

Now, putting n = 8 & x = 7. We get

$P(7) = {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{8-7}$

$\Rightarrow P(7) = {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^1$

$\Rightarrow P(7) = \left(\frac{8!}{(8-7)!\,7!}\right) \times \left(\frac{1}{2}\right)^8$

$\Rightarrow P(7) = \left(\frac{8 \times 7!}{7!}\right) \times \left(\frac{1}{2}\right)^8$

$\Rightarrow P(7) = 8 \times \left(\frac{1}{2}\right)^8$ ...(ii)

Now, putting n = 8 & x = 8. We get

$P(8) = {}^8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{8-8}$

$\Rightarrow P(8) = {}^8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^0$

$\Rightarrow P(8) = {}^8C_8 \left(\frac{1}{2}\right)^8$

$\Rightarrow P(8) = \left(\frac{8!}{(8-8)!\,8!}\right) \times \left(\frac{1}{2}\right)^8$

$\Rightarrow P(8) = \left(\frac{1}{2}\right)^8$ ...(iii)

The probability of getting at least 6 heads is given by

Probability = P(6) + P(7) + P(8)

Substituting values in (i), (ii) & (iii) in above equation. We get

$\Rightarrow \text{Probability} = 28 \times \left(\frac{1}{2}\right)^8 + 8 \times \left(\frac{1}{2}\right)^8 + \left(\frac{1}{2}\right)^8$

$\Rightarrow \text{Probability} = \left(\frac{1}{2}\right)^8 [28 + 8 + 1]$

$\Rightarrow \text{Probability} = \left(\frac{1}{2}\right)^8 \times 37$

$\Rightarrow \text{Probability} = \frac{37}{256}$

Thus, the probability of getting at least 6 heads is $\frac{37}{256}$.

32. Question

Six coins are tossed simultaneously. Find the probability of getting

i. 3 heads

ii. no heads

iii. at least one head

Answer

Let p be the probability of getting a head in a toss.

$\Rightarrow p = \frac{1}{2}$

If p is the probability of getting a head in a toss, then q is the probability of getting a tail in a toss.

We have p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{2}$

$\Rightarrow q = \frac{2-1}{2}$

$\Rightarrow q = \frac{1}{2}$

Let X denote a random variable representing the number of heads in 6 tosses of coin. The probability of getting r heads in n tosses of coins is given by

$P(X = r) = {}^nC_r p^r q^{n-r}$

We know the values of n, p, and q.

n = 6

$p = \frac{1}{2}$

$q = \frac{1}{2}$

We can re-write the binomial distribution as,

$P(X = r) = {}^6C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{6-r}$ ...(A)

(i). We need to find the probability of getting 3 heads.

It is given by,

Probability = P (X = 3)

So, putting r = 3 in equation (A). We get

$\text{Probability} = {}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{6-3}$

$\Rightarrow \text{Probability} = {}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3$

$\Rightarrow \text{Probability} = \left(\frac{6!}{(6-3)!3!}\right)\left(\frac{1}{2}\right)^{3+3}$

$\Rightarrow \text{Probability} = \left(\frac{6!}{3!3!}\right)\left(\frac{1}{2}\right)^6$

$\Rightarrow \text{Probability} = \left(\frac{6 \times 5 \times 4 \times 3!}{3! \times 3 \times 2}\right) \times \left(\frac{1}{2}\right)^6$

$\Rightarrow \text{Probability} = 20 \times \left(\frac{1}{2}\right)^6$

$\Rightarrow \text{Probability} = \frac{20}{64}$

$\Rightarrow \text{Probability} = \frac{5}{16}$

Thus, the probability of getting 3 heads out of tosses of 6 coins is $\frac{5}{16}$.

(ii). We need to find the probability of getting no head.

It is given by,

Probability = P (X = 0)

So, putting r = 0 in equation (A). We get

$$\text{Probability} = {}^6C_0 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0$$

$$\Rightarrow \text{Probability} = \left(\frac{6!}{(6-0)!0!}\right)\left(\frac{1}{2}\right)^6$$

$$\Rightarrow \text{Probability} = \left(\frac{1}{2}\right)^6$$

$$\Rightarrow \text{Probability} = \frac{1}{64}$$

Thus the probability of getting 0 heads out of tosses of 6 coins is $\frac{1}{64}$.

(iii). We need to find the probability of getting at least 1 head.

It is given by,

Probability = P (X = 1) + P (X = 2) + P (X = 3) + P (X = 4) + P (X = 5) + P (X = 6)

Or

Probability = 1 – P (X = 0)

Let us use the shorter formula.

We know from the result of part (ii),

$$P(X = 0) = \frac{1}{64}$$

Using this result, let us find out the probability of getting at least one head.

Then,

$$\text{Probability} = 1 - \frac{1}{64}$$

$$\Rightarrow \text{Probability} = \frac{64-1}{64}$$

$$\Rightarrow \text{Probability} = \frac{63}{64}$$

Thus, the probability of getting at least one head is $\frac{63}{64}$.

33. Question

Suppose that a radio tube inserted into a certain type of set has probability 0.2 of functioning more than 500 hours. If we test 4 tubes at random what is the probability that exactly three of these tubes function for more than 500 hours?

Answer

Let p be the probability of the tube to function for more than 500 hours.

This probability is given as 0.2.

⇒ p = 0.2

$$\Rightarrow p = \frac{2}{10}$$

$$\Rightarrow p = \frac{1}{5}$$

If p is the probability of the tube to function for more than 500 hours, then q is the probability of the tube to not function for more than 500 hours.

⇒ p + q = 1

$\Rightarrow q = 1 - p$

$$\Rightarrow q = 1 - \frac{1}{5}$$

$$\Rightarrow q = \frac{5-1}{5}$$

$$\Rightarrow q = \frac{4}{5}$$

Let X denote a random variable that represents the number of the tube that can function for more than 500 hours out of the total 4 tubes.

And let n denote the total number of tube taken in the sample, that is, 4.

Then binomial distribution for r tube to function more than 500 hours out of 4 tubes is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Putting n = 4, $p = \frac{1}{5}$ and $q = \frac{4}{5}$ above, we get

$$\Rightarrow P(X = r) = {}^4C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{4-r}$$

We need to find the probability that exactly 3 tubes will function for more than 500 hours.

So, put r = 3. We get

$$P(X = 3) = {}^4C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^{4-3}$$

$$\Rightarrow P(X = 3) = {}^4C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)$$

$$\Rightarrow P(X = 3) = \left(\frac{4!}{(4-3)!\,3!}\right) \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)$$

$$\Rightarrow P(X = 3) = \left(\frac{4!}{1!\,3!}\right) \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)$$

$$\Rightarrow P(X = 3) = \left(\frac{4 \times 3!}{3!}\right) \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)$$

$$\Rightarrow P(X = 3) = \frac{4 \times 4}{5 \times 5 \times 5 \times 5}$$

$$\Rightarrow P(X = 3) = \frac{16}{625}$$

Thus, the probability that exactly 3 tubes will function for more than 500 hours is $\frac{16}{625}$.

34. Question

The probability that a certain kind of component will survive a given shock test is 3/4. Find the probability that among the 5 components tested

i. exactly 2 will survive

ii. at most 3 will survive

Answer

Given that, a certain kind of component will survive a given shock $= \frac{3}{4}$

Let p be the probability that component survives the shock test.

Then,

$$p = \frac{3}{4}$$

If p is the probability that component survives the shock test, then q is the probability that the component doesn’t survive the shock test.

⇒ p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{3}{4}$$

$$\Rightarrow q = \frac{4-3}{4}$$

$$\Rightarrow q = \frac{1}{4}$$

Let X denote a random variable that represents the components that survive the shock test out of the 5 components tested.

The probability that r components out of n components survive the shock test is given by the binomial distribution.

P (X = r) = $^nC_r p^r q^{n-r}$

Here, n = 5 (sample components tested)

$$p = \frac{3}{4}$$

And $q = \frac{1}{4}$

We can re-write it as,

$P(X = r) = {}^5C_r \left(\frac{3}{4}\right)^r \left(\frac{1}{4}\right)^{5-r}$ …(A)

(i). We need to find the probability that among 5 components tested exactly 2 will survive.

Then, put r = 2 in equation (A). We have

$$P(X = 2) = {}^5C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^{5-2}$$

$$\Rightarrow P(X = 2) = {}^5C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3$$

$$\Rightarrow P(X = 2) = \left(\frac{5!}{(5-2)!\,2!}\right) \times \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^3$$

$$\Rightarrow P(X = 2) = \left(\frac{5!}{3!\,2!}\right) \times \left(\frac{9}{16}\right) \times \left(\frac{1}{64}\right)$$

$$\Rightarrow P(X = 2) = \left(\frac{5 \times 4 \times 3!}{3! \times 2}\right) \times \left(\frac{9}{1024}\right)$$

$$\Rightarrow P(X = 2) = 10 \times \frac{9}{1024}$$

$$\Rightarrow P(X = 2) = \frac{90}{1024}$$

⇒ P (X = 2) = 0.0879

∴, the probability that exactly 2 components out of 5 will survive the test is 0.0879.

(ii). We need to find the probability that among the 5 components tested at most 3 will survive.

So, this can be give in two ways:

(a). Probability that out of 5 components at most 3 will survive = P (X = 0) + P (X = 1) + P (X = 2) + P (X = 3)

(b). Probability that out of 5 components at most 3 will survive = 1 – [P (X = 4) + P (X = 5)]

Let us solve it by using the formula in (b).

So, let us find out P (X = 4).

Putting r = 4 in equation (A), we get

$$P(X = 4) = {}^5C_4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^{5-4}$$

$$\Rightarrow P(X = 4) = \left(\frac{5!}{(5-4)!\,4!}\right) \times \left(\frac{3}{4}\right)^4 \times \left(\frac{1}{4}\right)$$

$$\Rightarrow P(X = 4) = \left(\frac{5 \times 4!}{1!\,4!}\right) \times \left(\frac{3}{4}\right)^4 \times \left(\frac{1}{4}\right)$$

$$\Rightarrow P(X = 4) = 5 \times \left(\frac{3}{4}\right)^4 \times \left(\frac{1}{4}\right)$$

Now, let us find out P (X = 5).

Putting r = 5 in equation (A), we get

$$P(X = 5) = {}^5C_5\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^{5-5}$$

$$\Rightarrow P(X = 5) = \left(\frac{5!}{(5-5)!\,5!}\right) \times \left(\frac{3}{4}\right)^5 \times \left(\frac{1}{4}\right)^0$$

$$\Rightarrow P(X = 5) = 1 \times \left(\frac{3}{4}\right)^5$$

Using the values of P (X = 4) & P (X = 5) in formula (b), we get

$$\text{Probability} = 1 - \left[\left(5 \times \left(\frac{3}{4}\right)^4 \times \left(\frac{1}{4}\right)\right) + \left(\frac{3}{4}\right)^5\right]$$

$$\Rightarrow \text{Probability} = 1 - \left[\left(\frac{5 \times 3^4}{4^5}\right) + \left(\frac{3}{4}\right)^5\right]$$

$$\Rightarrow \text{Probability} = 1 - \left(\frac{3}{4}\right)^4\left[\left(\frac{5}{4}\right) + \left(\frac{3}{4}\right)\right]$$

$$\Rightarrow \text{Probability} = 1 - \left(\frac{81}{256}\right)\left(\frac{8}{4}\right)$$

$$\Rightarrow \text{Probability} = 1 - \frac{162}{256}$$

$$\Rightarrow \text{Probability} = \frac{256 - 162}{256}$$

$$\Rightarrow \text{Probability} = \frac{94}{256}$$

⇒ Probability = 0.3672

∴, the probability that out of 5 components at most 3 will survive is 0.3672.

35. Question

Assume that the probability that a bomb dropped from an airplane will strike a certain target is 0.2. If 6 bombs are dropped, find the probability that

i. exactly 2 will strike the target.

ii. at least 2 will strike the target.

Answer

We have been given that, the probability that a bomb dropped from an airplane will strike a certain target is 0.2.

Also, that 6 bombs are dropped.

Let p be the probability that a bomb dropped from an airplane will strike a certain target.

Then, q is the probability that a bomb dropped from an airplane will not strike a certain target.

⇒ p = 0.2

$\Rightarrow p = \frac{2}{10}$

$\Rightarrow p = \frac{1}{5}$

We know that, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{5}$

$\Rightarrow q = \frac{5-1}{5}$

$\Rightarrow q = \frac{4}{5}$

Let X be a random variable the represents the number of bombs that strike the target.

Then, the probability that r bombs strike the target out of n bombs is given by,

$P (X = r) = {}^nC_r p^r q^{n-r}$

Where n = 6

Let us put the values of n, p, and q in the above equation.

$\text{Probability} = {}^6C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{6-r}$ …(A)

(i). We need to find the probability that exactly 2 will strike the target out of 6 bombs.

Probability is given by,

Probability = P (X = 2)

So, put r = 2 in equation (A).

$\text{Probability} = {}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2}$

$\Rightarrow \text{Probability} = {}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4$

$\Rightarrow \text{Probability} = \left(\frac{6!}{(6-2)!\,2!}\right) \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4$

$\Rightarrow \text{Probability} = \left(\frac{6!}{4!\,2!}\right) \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4$

$\Rightarrow \text{Probability} = \left(\frac{6 \times 5 \times 4!}{4! \times 2}\right) \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^4$

$\Rightarrow \text{Probability} = \frac{3 \times 5 \times 4^4}{5^2 \times 5^4}$

$\Rightarrow \text{Probability} = \frac{3 \times 4^4}{5^5}$

$\Rightarrow \text{Probability} = \frac{768}{3125}$

⇒ Probability = 0.24576

∴, the probability that exactly 2 will strike the target out of 6 bombs is 0.24576.

(ii). We need to find the probability that at least 2 will strike the target out of 6 bombs.

Probability is given by,

Probability = P (X ≥ 2)

This can also be written as,

Probability = 1 – P (X < 2)

⇒ Probability = 1 – [P (X = 0) + P (X = 1)]

Let us find the value of P (X = 0).

For this, put r = 0 in equation (A).

$$P(X=0) = {}^6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{6-0}$$

$$\Rightarrow P(X=0) = {}^6C_0 \times 1 \times \left(\frac{4}{5}\right)^6$$

$$\Rightarrow P(X=0) = \left(\frac{6!}{(6-0)!\,0!}\right) \times \left(\frac{4}{5}\right)^6$$

$$\Rightarrow P(X=0) = \left(\frac{6!}{6!}\right) \times \left(\frac{4}{5}\right)^6$$

$$\Rightarrow P(X=0) = \left(\frac{4}{5}\right)^6$$

Now, let us find the value of P (X = 1).

For this, put r = 0 in equation (A).

$$P(X=1) = {}^6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{6-1}$$

$$\Rightarrow P(X=1) = {}^6C_1 \left(\frac{1}{5}\right) \left(\frac{4}{5}\right)^5$$

$$\Rightarrow P(X=1) = \left(\frac{6!}{(6-1)!\,1!}\right) \times \left(\frac{4^5}{5^6}\right)$$

$$\Rightarrow P(X=1) = \left(\frac{6!}{5!}\right) \times \left(\frac{4^5}{5^6}\right)$$

$$\Rightarrow P(X=1) = \left(\frac{6 \times 5!}{5!}\right) \times \left(\frac{4^5}{5^6}\right)$$

$$\Rightarrow P(X=1) = 6 \times \left(\frac{4^5}{5^6}\right)$$

Now, putting all these values in 1 – [P (X = 0) + P (X = 1)].

$$\Rightarrow \text{Probability} = 1 - \left[\left(\frac{4}{5}\right)^6 + \left(6 \times \left(\frac{1}{5}\right) \times \left(\frac{4}{5}\right)^5\right)\right]$$

$$\Rightarrow \text{Probability} = 1 - \left(\frac{4}{5}\right)^5 \left[\left(\frac{4}{5}\right) + \left(\frac{6}{5}\right)\right]$$

$$\Rightarrow \text{Probability} = 1 - \left(\frac{1024}{3125}\right)\left[\frac{4+6}{5}\right]$$

$$\Rightarrow \text{Probability} = 1 - \left(\frac{1024}{3125}\right)\left(\frac{10}{5}\right)$$

$$\Rightarrow \text{Probability} = 1 - \frac{2048}{3125}$$

$$\Rightarrow \text{Probability} = \frac{3125 - 2048}{3125}$$

$$\Rightarrow \text{Probability} = \frac{1077}{3125}$$

⇒ Probability = 0.345

∴, the probability that at least 2 will strike the target out of 6 bombs is 0.345.

36. Question

It is known that 60% of mice inoculated with serum are protected from a certain disease. If 5 mice are inoculated, find the probability that

i. none contract the disease

ii. more than 3 contract the disease.

Answer

Given that, the probability that the mice are protected from a certain disease is 60%.

Also, the sample size of mice = 5

Let p be the probability of the mice not contracting with a certain disease.

Then, q is the probability of the mice contracting with a certain disease.

$\Rightarrow$ p = 60%

$\Rightarrow p = \frac{60}{100}$

$\Rightarrow p = \frac{3}{5}$

And we know that,

p + q = 1

$\Rightarrow$ q = 1 – p

$\Rightarrow q = 1 - \frac{3}{5}$

$\Rightarrow q = \frac{5-3}{5}$

$\Rightarrow q = \frac{2}{5}$

Let X be a random variable representing a number of mice contracting with the disease.

Then, the probability of r mice contracting with the disease out of n mice inoculated is given by the following binomial distribution.

$P(X = r) = {}^nC_r q^r p^{n-r}$

Putting the values,

n = 5,

$p = \frac{3}{5}$

& $q = \frac{2}{5}$

We get

$P(X = r) = {}^5C_r \left(\frac{2}{5}\right)^r \left(\frac{3}{5}\right)^{5-r}$ ...(A)

(i). We need to find the probability that none of the mice contract with the disease.

For this, put r = 0.

We get the probability as,

Probability = P (X = 0)

From equation (A),

$P(X = 0) = {}^5C_0 \left(\frac{2}{5}\right)^0 \left(\frac{3}{5}\right)^{5-0}$

$$\Rightarrow P(X=0)=\left(\frac{5!}{(5-0)!0!}\right)\times 1\times\left(\frac{3}{5}\right)^5$$

$$\Rightarrow P(X=0)=\left(\frac{5!}{5!}\right)\times\left(\frac{3}{5}\right)^5$$

$$\Rightarrow P(X=0)=\left(\frac{3}{5}\right)^5$$

$$\Rightarrow P(X=0)=\frac{243}{3125}$$

⇒ P (X = 0) = 0.07776

∴, the probability that none of the mice contracts with the disease is 0.07776.

(ii). We need to find the probability that more than 3 mice contract the disease.

So, probability = P (X = 4) + P (X = 5)

$$\Rightarrow \text{Probability} = {}^5C_4\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^{5-4}+{}^5C_5\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^{5-5}$$

$$\Rightarrow \text{Probability} = {}^5C_4\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)+{}^5C_5\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^0$$

$$\Rightarrow \text{Probability} = \left[\left(\frac{5!}{(5-4)!\,4!}\right)\times\left(\frac{2}{5}\right)^4\times\left(\frac{3}{5}\right)\right]+\left[\left(\frac{5!}{(5-5)!\,5!}\right)\times\left(\frac{2}{5}\right)^5\right]$$

$$\Rightarrow \text{Probability} = \left[\left(\frac{5\times 4!}{1!\,4!}\right)\times\left(\frac{2}{5}\right)^4\times\left(\frac{3}{5}\right)\right]+\left[\left(\frac{5!}{0!\,5!}\right)\times\left(\frac{2}{5}\right)^5\right]$$

$$\Rightarrow \text{Probability} = \left[3\times\left(\frac{2}{5}\right)^4\right]+\left(\frac{2}{5}\right)^5$$

$$\Rightarrow \text{Probability} = \left(\frac{2}{5}\right)^4\left[3+\frac{2}{5}\right]$$

$$\Rightarrow \text{Probability} = \left(\frac{16}{256}\right)\times\left(\frac{15+2}{5}\right)$$

$$\Rightarrow \text{Probability} = \frac{16}{256}\times\frac{17}{5}$$

$$\Rightarrow \text{Probability} = \frac{272}{1280}$$

⇒ Probability = 0.2125

∴, the probability that more than 3 mice contract the disease is 0.2125.

37. Question

An experiment succeeds twice as often as it fails. Find the probability that in the next 6 trials there will be at least 4 successes.

Answer

Let p be the probability of success in the experiment. Then, q is the probability of failure in the experiment.

According to the question,

An experiment succeeds twice as often as it fails.

⇒ p = 2q

But p + q = 1

⇒ 2q + q = 1 [∵ p = 2q]

⇒ 3q = 1

$$\Rightarrow q=\frac{1}{3}$$

Then, p = 2q

$\Rightarrow p = 2 \times \frac{1}{3}$

$\Rightarrow p = \frac{2}{3}$

Let X be a random variable representing the number of successes out of 6 experiments.

Then, the probability of getting r success out of n experiments is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 6,

$p = \frac{2}{3}$

& $q = \frac{1}{3}$

$P(X = r) = {}^6C_r \left(\frac{2}{3}\right)^r \left(\frac{1}{3}\right)^{6-r}$ ...(i)

Then, probability that there will be at least 4 successes out of 6 is given by,

Probability = P (X = 4) + P (X = 5) + P (X = 6)

Put r = 4, 5 and 6 respectively for P (X = 4), P (X = 5) and P (X = 6) in equation (i), we have

$$\text{Probability} = {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^{6-4} + {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^{6-5} + {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^{6-6}$$

$$\Rightarrow \text{Probability} = {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0$$

$$\Rightarrow \text{Probability} = \left(\frac{2}{3}\right)^4 \left[\left(\left(\frac{6!}{(6-4)!\,4!}\right) \times \left(\frac{1}{9}\right)\right) + \left(\left(\frac{6!}{(6-5)!\,5!}\right) \times \left(\frac{2}{3}\right) \times \left(\frac{1}{3}\right)\right) + \left(\left(\frac{6!}{(6-6)!\,6!}\right) \times \left(\frac{2}{3}\right)^2\right)\right]$$

$$\Rightarrow \text{Probability} = \left(\frac{16}{81}\right)\left[\left(\left(\frac{6!}{2!\,4!}\right) \times \left(\frac{1}{9}\right)\right) + \left(\left(\frac{6!}{1!\,5!}\right) \times \left(\frac{2}{9}\right)\right) + \left(\left(\frac{6!}{0!\,6!}\right) \times \left(\frac{4}{9}\right)\right)\right]$$

$$\Rightarrow \text{Probability} = \left(\frac{16}{81}\right)\left[\left(\frac{6 \times 5 \times 4!}{2 \times 4!} \times \frac{1}{9}\right) + \left(\frac{6 \times 5!}{5!} \times \frac{2}{9}\right) + \left(\frac{4}{9}\right)\right]$$

$$\Rightarrow \text{Probability} = \left(\frac{16}{81}\right)\left[\frac{15}{9} + \frac{12}{9} + \frac{4}{9}\right]$$

$$\Rightarrow \text{Probability} = \left(\frac{16}{81}\right) \times \frac{15 + 12 + 4}{9}$$

$$\Rightarrow \text{Probability} = \frac{16}{81} \times \frac{31}{9}$$

$$\Rightarrow \text{Probability} = \frac{496}{729}$$

⇒ Probability = 0.68

∴, the required probability that in the next 6 trials there will be at least 4 successes is 0.68.

38. Question

In a hospital, there are 20 kidney dialysis machines and that the chance of any one of them to be out of service during a day is 0.02. Determine the probability that exactly 3 machines will be out of service on the same day.

Answer

Given that, the chance of any one of the 20 kidney dialysis machines to be out of service during a day is 0.02.

Let p be the probability of a kidney dialysis machine to get out of service.

⇒ p = 0.02

$\Rightarrow p = \frac{2}{100}$

Then, q is the probability of a kidney dialysis machine to not be out of service during a day.

And we know that, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{2}{100}$

$\Rightarrow q = \frac{100 - 2}{100}$

$\Rightarrow q = \frac{98}{100}$

Let X be a random variable that represents a number of machines out of service during a day out of n machines.

Then, the probability of r machines out of total n machines taken in the sample to get out of service is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 20 (as given in the question)

Putting the values of n, p & q in the above formula, we get

$P(X = r) = {}^{20}C_r \left(\frac{2}{100}\right)^r \left(\frac{98}{100}\right)^{20-r}$ …(i)

We need to find the probability that exactly 3 machines will be out of service on the same day.

So, for this, just put r = 3 in the formula (i),

$P(X = 3) = {}^{20}C_3 \left(\frac{2}{100}\right)^3 \left(\frac{98}{100}\right)^{20-18}$

Observe the formula so obtained after substituting the value of r.

The calculation will be huge since the values of p and q are very small.

So, in this case of low probability events, we use Poisson’s distribution rather than Binomial distribution.

Then,

Poisson’s constant can be found out as,

λ = np

where n = 20 & p = 0.02.

⇒ λ = 20 × 0.02

⇒ λ = 0.4

Poisson’s distribution is given as,

$P(X = r) = \frac{(e^{-\lambda} \times \lambda^3)}{r!}$

Put r = 3,

$P(X = 3) = \frac{e^{-0.4} \times 0.4^3}{3!}$

Look up in the table for the value of $e^{-0.4}$.

$P(X = 3) = \frac{0.6703 \times 0.064}{6}$

$\Rightarrow P(X = 3) = \frac{0.0429}{6}$

⇒ P (X = 3) = 0.0071

∴, the required probability that exactly 3 machines will be out of service on one day is 0.0071.

39. Question

The probability that a student entering a university will graduate is 0.4. Find the probability that out of 3 students of the university:

i. none will graduate,

ii. The only one will graduate,

iii. All will graduate.

Answer

Given that, the probability that a student entering a university will graduate is 0.4.

Let p be the probability that a student entering a university will graduate.

⇒ p = 0.4

$\Rightarrow p = \frac{4}{10}$

$\Rightarrow p = \frac{2}{5}$

Then, q is the probability of a student not graduating.

But, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{2}{5}$

$\Rightarrow q = \frac{5-2}{5}$

$\Rightarrow q = \frac{3}{5}$

Let X be a random variable that represents the number of students out of n students graduating after entering a university.

Then, the probability of r students out of n students graduating after entering a university is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, a sample size of students, n = 3

Putting the value of n, p, and q in the above formula, we get

$P(X = r) = {}^3C_r \left(\frac{2}{5}\right)^r \left(\frac{3}{5}\right)^{3-r}$ ...(A)

(i). We need to find the probability that out of 3 students entering a university, none will graduate.

Put r = 0 in equation (A). We get

$P(X = 0) = {}^3C_0 \left(\frac{2}{5}\right)^0 \left(\frac{3}{5}\right)^{3-0}$

$\Rightarrow P(X = 0) = \left(\frac{3!}{(3-0)!0!}\right) \times 1 \times \left(\frac{3}{5}\right)^3$

$\Rightarrow P(X = 0) = \left(\frac{3!}{3!}\right) \times \frac{27}{125}$

$\Rightarrow P(X = 0) = \frac{27}{125}$

$\Rightarrow P(X = 0) = 0.216$

∴, the probability that none will graduate is 0.216.

(ii). We need to find the probability that out of 3 students only 1 will graduate.

So, put r = 1 in equation (A). We get

$$P(X=1) = {}^3C_1 \left(\frac{2}{5}\right)^1 \left(\frac{3}{5}\right)^{3-1}$$

$$\Rightarrow P(X=1) = \left(\frac{3!}{(3-1)!\,1!}\right) \times \left(\frac{2}{5}\right) \times \left(\frac{3}{5}\right)^2$$

$$\Rightarrow P(X=1) = \left(\frac{3!}{2!}\right) \times \frac{2 \times 9}{125}$$

$$\Rightarrow P(X=1) = \left(\frac{3 \times 2!}{2!}\right) \times \frac{18}{125}$$

$$\Rightarrow P(X=1) = 3 \times \frac{18}{125}$$

$$\Rightarrow P(X=1) = \frac{54}{125}$$

⇒ P (X = 1) = 0.432

∴, the probability that exactly one will graduate is 0.432.

(iii) We need to find the probability that out of 3 students all will graduate.

So, put r = 3 in equation (A). We get

$$P(X=3) = {}^3C_3 \left(\frac{2}{5}\right)^3 \left(\frac{3}{5}\right)^{3-3}$$

$$\Rightarrow P(X=3) = \left(\frac{3!}{(3-3)!\,3!}\right) \times \left(\frac{2}{5}\right)^3 \times \left(\frac{3}{5}\right)^0$$

$$\Rightarrow P(X=3) = 1 \times \left(\frac{8}{125}\right) \times 1$$

$$\Rightarrow P(X) = \frac{8}{125}$$

⇒ P (X = 3) = 0.064

∴, the probability that all will graduate is 0.064.

40. Question

Ten eggs are drawn successively, with replacement, from a lot containing 10% defective eggs. Find the probability that there is a least one defective egg.

Answer

Given that, 10 eggs are drawn successively.

Defective eggs in the lot = 10%

Let p be the probability that the eggs drawn from the lot are defective.

⇒ p = 10%

$$\Rightarrow p = \frac{10}{100}$$

$$\Rightarrow p = \frac{1}{10}$$

Then, q is the probability that the eggs drawn from the lot is not defective.

And, p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{10}$$

$\Rightarrow q = \frac{10-1}{10}$

$\Rightarrow q = \frac{9}{10}$

Let X be a random variable that represents defective eggs picked out of n eggs from the lot.

Then, the probability of taking r defective eggs out of n eggs from the lot is given by,

P (X = r) = $^nC_r p^r q^{n-r}$

Here, n = 10

Putting the values of n, p, and q in the above formula, we get

$P(X = r) = {}^{10}C_r \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{10-r}$ ...(i)

We need to find the probability of getting at least one defective egg from the lot.

This is represented as,

Probability = P (X ≥ 1)

This is also written as,

P (X ≥ 1) = P (X = 1) + P (X = 2) + P (X = 3) + P (X = 4) + P (X = 5) + P (X = 6) + P (X = 7) + P (X = 8) + P (X = 9) + P (X = 10)

Or

P (X ≥ 1) = 1 – P (X < 1)

⇒ P (X ≥ 1) = 1 – P (X = 0)

Putting r = 0 in P (X = r) formula in (i), we get

$P(X \geq 1) = 1 - ({}^{10}C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10-0})$

$\Rightarrow P(X \geq 1) = 1 - \left(\left(\frac{10!}{(10-0)!\,0!}\right) \times 1 \times \left(\frac{9}{10}\right)^{10}\right)$

$\Rightarrow P(X \geq 1) = 1 - \left(\frac{9}{10}\right)^{10}$

∴, the probability of getting at least one defective egg from the lot is $1 - \left(\frac{9}{10}\right)^{10}$.

41. Question

In a 20-question true-false examination suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answer ‘true’; if it falls tails, he answers ‘false’. Find the probability that he answers at least 12 questions correctly.

Answer

Given that, there are 20 questions of true-false exam.

If the coin falls head, he answers “true.”

If the coin falls tail, he answers “false.”

Let p be the probability of a correct answer.

That is, p = getting a head and a right answer to be “true” for a question or getting a tail and a right answer to be “false” for a question.

$\Rightarrow p = \frac{1}{2}$

Then, q is the probability of the answer to be incorrect.

And, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{2}$

$\Rightarrow q = \frac{1}{2}$

Let X denote a random variable representing the number of correct answers out of 20 questions.

Then, the probability of getting r correct answers out of n answered questions is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 20 [∵ there are 20 questions in total]

Putting values of n, p, and q in the above equation, we get

$P(X = r) = {}^{20}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{20-r}$

$\Rightarrow P(X = r) = {}^{20}C_r \left(\frac{1}{2}\right)^{r+20-r}$

$\Rightarrow P(X = r) = {}^{20}C_r \left(\frac{1}{2}\right)^{20}$ ...(i)

We need to find the probability that he answers at least 12 questions correctly.

This can be represented as,

Probability = P (X ≥ 12)

It can be written as,

P (X ≥ 12) = P (X = 12) + P (X = 13) + P (X = 14) + P (X = 15) + P (X = 16) + P (X = 17) + P (X = 18) + P (X = 19) + P (X = 20) ...(ii)

Put r = 12, 13, 14, 15, 16, 17, 18, 19, 20 in equation (i) subsequently and substitute in (ii), we get

$P(X \geq 12) = {}^{20}C_{12}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{13}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{14}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{15}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{16}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{17}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{18}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{19}\left(\frac{1}{2}\right)^{20} + {}^{20}C_{20}\left(\frac{1}{2}\right)^{20}$

$\Rightarrow P(X \geq 12) = \left(\frac{1}{2}\right)^{20} \times ({}^{20}C_{12} + {}^{20}C_{13} + {}^{20}C_{14} + {}^{20}C_{15} + {}^{20}C_{16} + {}^{20}C_{17} + {}^{20}C_{18} + {}^{20}C_{19} + {}^{20}C_{20})$

$\Rightarrow P(X \geq 12) = \frac{{}^{20}C_{12} + \dots + {}^{20}C_{20}}{2^{20}}$

∴, we have got the required probability.

42. Question

Suppose X has a binomial distribution with n=6 and p=1/2. Show that X = 3 is the most likely outcome.

Answer

Given that, X has a binomial distribution.

Also, n = 6 & p = 1/2

And as we know that, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{2}$

$\Rightarrow q = \frac{1}{2}$

Let us define a binomial distribution with x as a variable out of other n variables.

$P(X = x) = {}^nC_x p^x q^{n-x}$

Putting values of n, p and q, we get

$$P(X = x) = {}^6C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{6-x}$$

$$\Rightarrow P(X = x) = {}^6C_x \left(\frac{1}{2}\right)^{x+6-x}$$

$$\Rightarrow P(X = x) = {}^6C_x \left(\frac{1}{2}\right)^6$$

From the derived result, it can be seen that P (X = x) will be maximum if ${}^6C_x$ is maximum.

Let us check at what value of x would ${}^6C_x$ be maximum.

Check at x = 0.

$$\Rightarrow {}^6C_0 = \frac{6!}{(6-0)!\,0!}$$

$$\Rightarrow {}^6C_0 = \frac{6!}{6!}$$

$\Rightarrow {}^6C_0 = 1$

Check at x = 6.

$$\Rightarrow {}^6C_6 = \frac{6!}{(6-6)!\,6!}$$

$$\Rightarrow {}^6C_6 = \frac{6!}{0!\,6!}$$

$\Rightarrow {}^6C_6 = 1$

Check at x = 1.

$$\Rightarrow {}^6C_1 = \frac{6!}{(6-1)!\,1!}$$

$$\Rightarrow {}^6C_1 = \frac{6!}{5!}$$

$$\Rightarrow {}^6C_1 = \frac{6 \times 5!}{5!}$$

$\Rightarrow {}^6C_1 = 6$

Check at x = 5.

$$\Rightarrow {}^6C_5 = \frac{6!}{(6-5)!\,5!}$$

$$\Rightarrow {}^6C_5 = \frac{6!}{1!\,5!}$$

$$\Rightarrow {}^6C_5 = \frac{6 \times 5!}{5!}$$

$\Rightarrow {}^6C_5 = 6$

Check at x = 2.

$$\Rightarrow {}^6C_2 = \frac{6!}{(6-2)!\,2!}$$

$$\Rightarrow {}^6C_2 = \frac{6!}{4!\,2!}$$

$$\Rightarrow {}^6C_2 = \frac{6 \times 5 \times 4!}{2 \times 4!}$$

$\Rightarrow {}^6C_2 = 15$

Check at x = 4.

$$\Rightarrow {}^6C_4 = \frac{6!}{(6-4)!\,4!}$$

$$\Rightarrow {}^6C_4 = \frac{6!}{2!\,4!}$$

$$\Rightarrow {}^6C_4 = \frac{6 \times 5 \times 4!}{2 \times 4!}$$

$\Rightarrow {}^6C_4 = 15$

Check at x = 3.

$$\Rightarrow {}^6C_3 = \frac{6!}{(6-3)!\,3!}$$

$$\Rightarrow {}^6C_3 = \frac{6!}{3!\,3!}$$

$$\Rightarrow {}^6C_3 = \frac{6 \times 5 \times 4 \times 3!}{3 \times 2 \times 3!}$$

$\Rightarrow {}^6C_3 = 20$

Note that at x = 3, ${}^6C_x$ is maximum.

Hence, we have shown that x = 3 is the most likely outcome.

43. Question

In a multiple choice examination with three possible answers for each of the five questions out of which only one is correct, what is the probability that a candidate would get four or more correct answers just by guessing?

Answer

We have been given that, there are 3 possible answers of the total 5 questions out of which only 1 answer is correct.

Let p be the probability of getting a correct answer out of the 3 alternative answers.

$$\Rightarrow p = \frac{1}{3}$$

Then, q is the probability of not getting a correct answer out of 3 alternatives.

And, p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{3}$$

$$\Rightarrow q = \frac{3-1}{3}$$

$$\Rightarrow q = \frac{2}{3}$$

Let X be any random variable representing a number of correct answers just be guessing out of 5 questions.

Then, the probability that the candidate would get r answers correct by just guessing out of 5 questions is given by this Binomial distribution.

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 5.

Substitute the value of n, p, and q in the above formula.

$$P(X = r) = {}^5C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{5-r} \ldots(i)$$

We need to find the probability that a candidate would get four or more correct answers just by guessing.

The probability can be expressed as,

Probability = P (X ≥ 4)

This is in turn can be written as,

P (X ≥ 4) = P (X = 4) + P (X = 5)

Put r = 4, 5 subsequently in equation (i) and then substitute in the above formula.

$$P(X \geq 4) = {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^{5-4} + {}^5C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^{5-5}$$

$$\Rightarrow P(X \geq 4) = {}^5C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 + {}^5C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^0$$

$$\Rightarrow P(X \geq 4) = \left[\left(\frac{5!}{(5-4)!\,4!}\right) \times \left(\frac{2}{3^5}\right)\right] + \left[\left(\frac{5!}{(5-5)!\,5!}\right) \times \left(\frac{1}{3^5}\right)\right]$$

$$\Rightarrow P(X \geq 4) = \left[\left(\frac{5 \times 4!}{1!\,4!}\right) \times \frac{2}{3^5}\right] + \left[\left(\frac{5!}{5!}\right) \times \frac{1}{3^5}\right]$$

$$\Rightarrow P(X \geq 4) = \left(\frac{1}{3}\right)^5 \times (10 + 1)$$

$$\Rightarrow P(X \geq 4) = 11 \times \left(\frac{1}{3}\right)^5$$

$$\Rightarrow P(X \geq 4) = \frac{11}{243}$$

⇒ P (X ≥ 4) = 0.0453

Hence, the probability that a candidate would get four or more correct answers just by guessing is 0.0453.

44. Question

A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1/100. What is the probability that he will win a prize

i. at least once

ii. exactly once

iii. at least twice?

Answer

Given that, a person buys a lottery ticket in 50 lotteries.

The probability of winning a prize is 1/100.

Let p be the probability of winning a prize.

Then, $p = \frac{1}{100}$

And q be the probability of not winning a prize.

We can write,

p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{100}$$

$$\Rightarrow q = \frac{100 - 1}{100}$$

$$\Rightarrow q = \frac{99}{100}$$

Let X be a random variable representing the number of times the person wins the lottery out of n lotteries.

Then, the probability of the person winning the lottery r times out of n times is given by this Binomial distribution.

P (X = r) = $^nC_r p^r q^{n-r}$

Here, n = 50

So, putting the value of n, p, and q in the formula of P (X = r), we get

$P(X=r)={}^{50}C_r\left(\frac{1}{100}\right)^r\left(\frac{99}{100}\right)^{50-r}$ ...(A)

(i). We need to find the probability that he will win the prize at least once.

The probability is given by,

Probability = P (X ≥ 1)

Or this can be written as,

P (X ≥ 1) = 1 – P (X < 1)

⇒ P (X ≥ 1) = 1 – P (X = 0) ...(B)

So, put r = 0 in equation (A) and then substitute in equation (B), we get

$$P(X\geq 1)=1-\left({}^{50}C_0\left(\frac{1}{100}\right)^0\left(\frac{99}{100}\right)^{50-0}\right)$$

$$\Rightarrow P(X\geq 1)=1-\left[\left(\frac{50!}{(50-0)!\,0!}\right)\times 1\times\left(\frac{99}{100}\right)^{50}\right]$$

$$\Rightarrow P(X\geq 1)=1-\left[\left(\frac{50!}{50!}\right)\times\left(\frac{99}{100}\right)^{50}\right]$$

$$\Rightarrow P(X\geq 1)=1-\left(\frac{99}{100}\right)^{50}$$

Thus, the probability that he will win the prize at least once is $1-\left(\frac{99}{100}\right)^{50}$.

(ii). We need to find the probability that he will win the prize exactly once.

The probability is given by,

Probability = P (X = 1)

Put r = 1 in equation (A), we get

$$P(X=1)={}^{50}C_1\left(\frac{1}{100}\right)^1\left(\frac{99}{100}\right)^{50-1}$$

$$\Rightarrow P(X=1)=\left(\frac{50!}{(50-1)!\,1!}\right)\times\left(\frac{1}{100}\right)\times\left(\frac{99}{100}\right)^{49}$$

$$\Rightarrow P(X=1)=\left(\frac{50!}{49!}\right)\times\frac{1}{100}\times\left(\frac{99}{100}\right)^{49}$$

$$\Rightarrow P(X=1)=\left(\frac{50\times 49!}{49!}\right)\times\frac{1}{100}\times\left(\frac{99}{100}\right)^{49}$$

$$\Rightarrow P(X=1)=\frac{1}{2}\times\left(\frac{99}{100}\right)^{49}$$

Thus, the probability that he will win the prize exactly once is $\frac{1}{2}\left(\frac{99}{100}\right)^{49}$.

(iii). We need to find the probability that he will win at least twice.

The probability is given by,

Probability = P (X ≥ 2)

Or can be expressed as,

P (X ≥ 2) = 1 – P (X < 2)

⇒ P (X ≥ 2) = 1 – [P (X = 0) + P (X = 1)] ...(B)

Put r = 0 and r = 1 in equation (A) one by one and then substitute it in the equation (B), we get

$$P(X \geq 2) = 1 - [{}^{50}C_0 \left(\frac{1}{100}\right)^0 \left(\frac{99}{100}\right)^{50-0} + {}^{50}C_1 \left(\frac{1}{100}\right)^1 \left(\frac{99}{100}\right)^{50-1}]$$

$$\Rightarrow P(X \geq 2) = 1 - \left[\left(\frac{50!}{(50-0)!\,0!}\right)\left(\frac{99}{100}\right)^{50} + \left(\frac{50!}{(50-1)!\,1!}\right)\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}\right]$$

$$\Rightarrow P(X \geq 2) = 1 - \left[\left(\frac{99}{100}\right)^{50} + \left(\frac{50!}{49!}\right)\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}\right]$$

$$\Rightarrow P(X \geq 2) = 1 - \left[\left(\frac{99}{100}\right)^{50} + \left(\frac{50 \times 49!}{49!}\right)\left(\frac{1}{100}\right)\left(\frac{99}{100}\right)^{49}\right]$$

$$\Rightarrow P(X \geq 2) = 1 - \left(\frac{99}{100}\right)^{49}\left[\frac{99}{100} + \frac{1}{2}\right]$$

$$\Rightarrow P(X \geq 2) = 1 - \left(\frac{99}{100}\right)^{49}\left(\frac{99+50}{100}\right)$$

$$\Rightarrow P(X \geq 2) = 1 - \left(\frac{99}{100}\right)^{49}\left(\frac{149}{100}\right)$$

Thus, the probability that he will win the prize at least twice is $1 - \left(\frac{149}{100}\right)\left(\frac{99}{100}\right)^{49}$.

45. Question

The probability of a shooter hitting a target is 3/4. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Answer

Given that, the probability of a shooter hitting a target is 3/4.

Let p be the probability of hitting a target and q be the probability of not hitting a target.

Then, $p = \frac{3}{4}$

But, we know p + q = 1

⇒ q= 1 – p

$$\Rightarrow q = 1 - \frac{3}{4}$$

$$\Rightarrow q = \frac{4-3}{4}$$

$$\Rightarrow q = \frac{1}{4}$$

Let the shooter shoot n times in total.

Let X be a random variable representing the number of times the shooter hits the target out of total n times.

Then, the probability of hitting the target r times out of total n times is given by Binomial distribution as,

P (X = r) = ${}^{n}C_r p^r q^{n-r}$

Substitute the value of p and q in the above formula, we get

$P(X = r) = {}^{n}C_r \left(\frac{3}{4}\right)^r \left(\frac{1}{4}\right)^{n-r}$ ...(i)

We need to find the minimum number of times the shooter must fire so that the probability of hitting the target at least once is more than 0.99.

It can be represented as,

P(hitting the target atleast once) > 0.99

⇒ P (X ≥ 1) > 0.99

⇒ 1 – P (X < 1) > 0.99

⇒ 1 – P (X = 0) > 0.99

Put r = 0 in equation (i) and then substitute in the above equation, we have

$$1-{}^{n}C_0\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{n-0} > 0.99$$

$$\Rightarrow 1-\left[\left(\frac{n!}{(n-0)!0!}\right)\times 1\times\left(\frac{1}{4}\right)^n\right] > 0.99$$

$$\Rightarrow 1-\left[\left(\frac{n!}{n!}\right)\times\left(\frac{1}{4}\right)^n\right] > 0.99$$

$$\Rightarrow 1-\left(\frac{1}{4}\right)^n > 0.99$$

$$\Rightarrow -\left(\frac{1}{4}\right)^n > 0.99-1$$

$$\Rightarrow -\left(\frac{1}{4}\right)^n > -0.01$$

$$\Rightarrow \left(\frac{1}{4}\right)^n < 0.01$$

$$\Rightarrow \left(\frac{1}{4}\right)^n < \frac{1}{100}$$

$\Rightarrow 4^n > 100$

We need to find the minimum value of n to satisfy this inequality.

Take n = 0.

$\Rightarrow 4^0 > 100$

$\Rightarrow 1 > 100$

But $1 \ngtr 100$.

Take n = 1.

$\Rightarrow 4^1 > 100$

$\Rightarrow 4 > 100$

But $4 \ngtr 100$.

Take n = 2.

$\Rightarrow 4^2 > 100$

$\Rightarrow 16 > 100$

But $16 \ngtr 100$.

Take n = 3.

$\Rightarrow 4^3 > 100$

$\Rightarrow 64 > 100$

But $64 \ngtr 100$.

Take n = 4.

$\Rightarrow 4^4 > 100$

$\Rightarrow 256 > 100$

It is true.

Hence, the minimum value of n to satisfy the inequality is 4.

∴, the shooter must fire 4 times.

46. Question

How many times must a man toss a fair coin so that the probability of having at least one head is more than 90%?

Answer

Let the man toss the coin n times.

Let p be the probability of getting a head in a toss.

Then, q is the probability of getting a tail in a toss.

Since the coin has only two outcomes, so the probability of getting a head = 1/2

$\Rightarrow p = \frac{1}{2}$

Also, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{2}$

$\Rightarrow q = \frac{1}{2}$

Let X be a random variable that represents a number of occurrence of the head in n tosses of a fair coin.

Then, the probability of getting r number of heads out of total n tosses is given by this Binomial distribution.

$P(X = r) = {}^nC_r p^r q^{n-r}$

Substituting the value of p and q in the above equation, we get

$P(X = r) = {}^nC_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n-r}$ ...(i)

We need to find the number of times the man must toss a fair coin so that the probability of having at least one head is more than 90%.

We can represent it as,

P (getting atleast one head) > 90%

⇒ P (X ≥ 1) > 90%

⇒ 1 – P (X < 1) > 90% [∵ P (X ≥ 1) = 1 – P (X < 1)]

⇒ 1 – P (X = 0) > 90% [∵ P (X < 1) = P (X = 0)]

Put r = 0 in equation (i) and then, substituting it in the above equation.

$\Rightarrow 1 - ({}^nC_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{n-0}) > 90\%$

$\Rightarrow 1 - \left(\left(\frac{n!}{(n-0)!0!}\right) \times 1 \times \left(\frac{1}{2}\right)^n\right) > \frac{90}{100}$

$\Rightarrow 1 - \left(\left(\frac{n!}{n!}\right) \times \left(\frac{1}{2}\right)^n\right) > \frac{9}{10}$

$\Rightarrow 1 - \left(\frac{1}{2}\right)^n > \frac{9}{10}$

$\Rightarrow -\left(\frac{1}{2}\right)^n > \frac{9}{10} - 1$

$\Rightarrow -\left(\frac{1}{2}\right)^n > \frac{9-10}{10}$

$\Rightarrow -\left(\frac{1}{2}\right)^n > -\frac{1}{10}$

$\Rightarrow \left(\frac{1}{2}\right)^n < \frac{1}{10}$

$\Rightarrow 2^n > 10$

Now, we need to find the minimum value of n that satisfy this inequality.

Put n = 0.

$\Rightarrow 2^0 > 10$

$\Rightarrow 1 > 10$

But $1 \ngtr 10$.

Put n = 1.

$\Rightarrow 2^1 > 10$

$\Rightarrow 2 > 10$

But $2 \ngtr 10$.

Put n = 2.

$\Rightarrow 2^2 > 10$

$\Rightarrow 4 > 10$

But $4 \ngtr 10$.

Put n = 3.

$\Rightarrow 2^3 > 10$

$\Rightarrow 8 > 10$

But $8 \ngtr 10$.

Put n = 4.

$\Rightarrow 2^4 > 10$

$\Rightarrow 16 > 10$

It is true.

Thus, the minimum n that satisfy this inequality is 4.

Hence, the man should toss the coin 4 or more times.

47. Question

How many times must a man toss a fair coin so that the probability of having at least one head is more than 80%?

Answer

Let the man toss the coin n times.

Let p be the probability of getting a head in a toss.

Then, q is the probability of getting a tail in a toss.

Since the coin has only two outcomes, so the probability of getting a head = 1/2

$\Rightarrow p = \frac{1}{2}$

Also, p + q = 1

$\Rightarrow q = 1 - p$

$\Rightarrow q = 1 - \frac{1}{2}$

$\Rightarrow q = \frac{1}{2}$

Let X be a random variable that represents a number of occurrence of the head in n tosses of a fair coin.

Then, the probability of getting r number of heads out of total n tosses is given by this Binomial distribution.

$P(X = r) = {}^nC_r p^r q^{n-r}$

Substituting the value of p and q in the above equation, we get

$P(X = r) = {}^nC_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n-r}$ ...(i)

We need to find the number of times the man must toss a fair coin so that the probability of having at least one head is more than 80%.

We can represent it as,

P (getting atleast one head) > 80%

⇒ P (X ≥ 1) > 80%

⇒ 1 – P (X < 1) > 80% [∵ P (X ≥ 1) = 1 – P (X < 1)]

⇒ 1 – P (X = 0) > 80% [∵ P (X < 1) = P (X = 0)]

Put r = 0 in equation (i) and then, substituting it in the above equation.

$\Rightarrow 1 - \left({}^nC_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{n-0}\right) > 80\%$

$\Rightarrow 1 - \left(\left(\frac{n!}{(n-0)!0!}\right) \times 1 \times \left(\frac{1}{2}\right)^n\right) > \frac{80}{100}$

$\Rightarrow 1 - \left(\left(\frac{n!}{n!}\right) \times \left(\frac{1}{2}\right)^n\right) > \frac{8}{10}$

$\Rightarrow 1 - \left(\frac{1}{2}\right)^n > \frac{8}{10}$

$\Rightarrow -\left(\frac{1}{2}\right)^n > \frac{8}{10} - 1$

$\Rightarrow -\left(\frac{1}{2}\right)^n > \frac{8-10}{10}$

$\Rightarrow -\left(\frac{1}{2}\right)^n > -\frac{2}{10}$

$\Rightarrow \left(\frac{1}{2}\right)^n < \frac{2}{10}$

$\Rightarrow \left(\frac{1}{2}\right)^n < \frac{1}{5}$

⇒ $2^n > 5$

Now, we need to find the minimum value of n that satisfy this inequality.

Put n = 0.

⇒ $2^0 > 5$

⇒ 1 > 5

But 1 ≯ 5.

Put n = 1.

⇒ $2^1 > 5$

⇒ 2 > 5

But 2 ≯ 5.

Put n = 2.

⇒ $2^2 > 5$

⇒ 4 > 5

But 4 ≯ 5.

Put n = 3.

⇒ $2^3 > 5$

⇒ 8 > 5

It is true.

Thus, the minimum n that satisfy this inequality is 3.

Hence, the man should toss the coin 3 or more times.

48. Question

A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of the number of successes.

Answer

Given that, a pair of dice is thrown 4 times.

And a doublet is considered as a success.

Let p be the probability of getting a doublet in a throw of a pair of dice.

Since, there are 36 possible outcomes in total. {(1, 1), (1, 2), (1, 3), …, (1, 6), …, (2, 6), …, (6 ,6)}

And the 6 possible doublets in 36 outcomes. {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

So,

$$p = \frac{6}{36}$$

$$\Rightarrow p = \frac{1}{6}$$

And also, p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{6}$$

$$\Rightarrow q = \frac{6-1}{6}$$

$$\Rightarrow q = \frac{5}{6}$$

Let X denote a random variable representing a number of doublets (successes) out of 4 throws.

So, Binomial distribution of getting r successes out of 4 throws is given by

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 4.

Now, substituting values of n, p and q in the formula P (X = r). We get

$$P(X = r) = {}^4C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{4-r} \ldots(i)$$

We need to find the probability distribution of the number of successes.

The probability of 0 success in 4 throws is given by,

Probability = P (X = 0)

Put r = 0 in (i),

$$P(X = 0) = {}^4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{4-0}$$

$$\Rightarrow P(X = 0) = \left(\frac{4!}{(4-0)!\,0!}\right) \times 1 \times \left(\frac{5}{6}\right)^4$$

$$\Rightarrow P(X=0)=\left(\frac{4!}{4!}\right)\times\left(\frac{5}{6}\right)^4$$

$$\Rightarrow P(X=0)=\left(\frac{5}{6}\right)^4$$

The probability of 1 success in 4 throws is given by,

Probability = P (X = 1)

Put r = 1 in (i),

$$P(X=1)={}^4C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{4-1}$$

$$\Rightarrow P(X=1)=\left(\frac{4!}{(4-1)!\,1!}\right)\times\left(\frac{1}{6}\right)\times\left(\frac{5}{6}\right)^3$$

$$\Rightarrow P(X=1)=\left(\frac{4!}{3!}\right)\times\frac{1}{6}\times\left(\frac{5}{6}\right)^3$$

$$\Rightarrow P(X=1)=\left(\frac{4\times 3!}{3!}\right)\times\frac{1}{6}\times\left(\frac{5}{6}\right)^3$$

$$\Rightarrow P(X=1)=\frac{2}{3}\left(\frac{5}{6}\right)^3$$

The probability of 2 successes in 4 throws is given by,

Probability = P (X = 2)

Put r = 2 in (i),

$$P(X=2)={}^4C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{4-2}$$

$$\Rightarrow P(X=2)=\left(\frac{4!}{(4-2)!\,2!}\right)\times\left(\frac{1}{6}\right)^2\times\left(\frac{5}{6}\right)^2$$

$$\Rightarrow P(X=2)=\left(\frac{4\times 3\times 2!}{2!\times 2}\right)\times\frac{25}{6^4}$$

$$\Rightarrow P(X=2)=6\times\frac{25}{6^4}$$

$$\Rightarrow P(X=2)=\frac{25}{6^3}$$

The probability of 3 successes in 4 throws is given by,

Probability = P (X = 3)

Put r = 3 in (i),

$$P(X=3)={}^4C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{4-3}$$

$$\Rightarrow P(X=3)=\left(\frac{4!}{(4-3)!\,3!}\right)\times\left(\frac{1}{6}\right)^3\times\left(\frac{5}{6}\right)$$

$$\Rightarrow P(X=3)=\left(\frac{4\times 3!}{3!}\right)\times\frac{5}{6^4}$$

$$\Rightarrow P(X=3)=\frac{20}{6^4}$$

The probability of 4 successes in 4 throws is given by,

Probability = P (X = 4)

Put r = 4 in (i),

$$P(X=4) = {}^4C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{4-4}$$

$$\Rightarrow P(X=4) = \left(\frac{4!}{(4-4)!\,4!}\right)\times\left(\frac{1}{6}\right)^4$$

$$\Rightarrow P(X=4) = \left(\frac{1}{6}\right)^4$$

Thus, the probability distribution is

X	0	1	2	3	4
P(X)	$\left(\frac{5}{6}\right)^4$	$\frac{2}{3}\left(\frac{5}{6}\right)^3$	$\frac{25}{6^3}$	$\frac{20}{6^4}$	$\left(\frac{1}{6}\right)^4$

49. Question

From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Answer

Given that, there are 30 bulbs, which include 6 defective bulbs.

A sample of 4 bulbs is drawn at random with replacement.

Let p be the probability of defective bulbs.

Since 6 bulbs are defective out of 30 bulbs.

$$\Rightarrow p = \frac{6}{30}$$

$$\Rightarrow p = \frac{1}{5}$$

Then, let q be the probability of fine bulbs.

And we know, p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{5}$$

$$\Rightarrow q = \frac{5-1}{5}$$

$$\Rightarrow q = \frac{4}{5}$$

Let X denote a random variable representing a number of defective bulbs out of 4 bulbs drawn at random.

So, Binomial distribution of getting r successes out of 4 bulbs drawn at random is given by

P (X = r) = ${}^nC_r p^r q^{n-r}$

Here, n = 4.

Now, substituting values of n, p and q in the formula P (X = r). We get

$$P(X=r) = {}^4C_r\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{4-r} \ldots(i)$$

We need to find the probability distribution of the number of successes.

The probability of 0 defective bulb in 4 sample bulbs is given by,

Probability = P (X = 0)

Put r = 0 in (i),

$$P(X=0) = {}^4C_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^{4-0}$$

$$\Rightarrow P(X=0)=\left(\frac{4!}{(4-0)!\,0!}\right)\times 1\times\left(\frac{4}{5}\right)^{4}$$

$$\Rightarrow P(X=0)=\left(\frac{4!}{4!}\right)\times\left(\frac{4}{5}\right)^{4}$$

$$\Rightarrow P(X=0)=\left(\frac{4}{5}\right)^{4}$$

The probability of 1 defective bulb in 4 sample bulbs is given by,

Probability = P (X = 1)

Put r = 1 in (i),

$$P(X=1)={}^{4}C_{1}\left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{4-1}$$

$$\Rightarrow P(X=1)=\left(\frac{4!}{(4-1)!\,1!}\right)\times\left(\frac{1}{5}\right)\times\left(\frac{4}{5}\right)^{3}$$

$$\Rightarrow P(X=1)=\left(\frac{4!}{3!}\right)\times\frac{1}{5}\times\left(\frac{4}{5}\right)^{3}$$

$$\Rightarrow P(X=1)=\left(\frac{4\times 3!}{3!}\right)\times\frac{1}{5}\times\left(\frac{4}{5}\right)^{3}$$

$$\Rightarrow P(X=1)=\frac{4}{5}\left(\frac{4}{5}\right)^{3}$$

$$\Rightarrow P(X=1)=\left(\frac{4}{5}\right)^{4}$$

The probability of 2 successes in 4 throws is given by,

Probability = P (X = 2)

Put r = 2 in (i),

$$P(X=2)={}^{4}C_{2}\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{4-2}$$

$$\Rightarrow P(X=2)=\left(\frac{4!}{(4-2)!\,2!}\right)\times\left(\frac{1}{5}\right)^{2}\times\left(\frac{4}{5}\right)^{2}$$

$$\Rightarrow P(X=2)=\left(\frac{4\times 3\times 2!}{2!\times 2}\right)\times 16\times\left(\frac{1}{5}\right)^{4}$$

$$\Rightarrow P(X=2)=6\times 16\times\left(\frac{1}{5}\right)^{4}$$

$$\Rightarrow P(X=2)=96\left(\frac{1}{5}\right)^{4}$$

The probability of 3 successes in 4 throws is given by,

Probability = P (X = 3)

Put r = 3 in (i),

$$P(X=3)={}^{4}C_{3}\left(\frac{1}{5}\right)^{3}\left(\frac{4}{5}\right)^{4-3}$$

$$\Rightarrow P(X=3)=\left(\frac{4!}{(4-3)!\,3!}\right)\times\left(\frac{1}{5}\right)^{3}\times\left(\frac{4}{5}\right)$$

$$\Rightarrow P(X=3)=\left(\frac{4\times 3!}{3!}\right)\times 4\times\left(\frac{1}{5}\right)^{4}$$

$$\Rightarrow P(X=3)=16\left(\frac{1}{5}\right)^{4}$$

The probability of 4 successes in 4 throws is given by,

Probability = P (X = 4)

Put r = 4 in (i),

$$P(X=4) = {}^{4}C_{4}\left(\frac{1}{5}\right)^{4}\left(\frac{4}{5}\right)^{4-4}$$

$$\Rightarrow P(X=4) = \left(\frac{4!}{(4-4)!\,4!}\right) \times \left(\frac{1}{5}\right)^{4}$$

$$\Rightarrow P(X=4) = \left(\frac{1}{5}\right)^{4}$$

Thus, the probability distribution is

X	0	1	2	3	4
P(X)	$\left(\frac{4}{5}\right)^4$	$\left(\frac{4}{5}\right)^4$	$96\left(\frac{1}{5}\right)^4$	$16\left(\frac{1}{5}\right)^4$	$\left(\frac{1}{5}\right)^4$

50. Question

Find the probability that in 10 throws of a fair die a score which is a multiple of 3 will be obtained in at least 8 of the throws.

Answer

Given that, obtaining multiple of 3 in a throw of a die is a success.

There are total 10 throws of a fair die.

Let p be the probability of getting multiple of 3 in a throw of a die.

Since there can be a total 6 outcomes in a throw of a die. That is, {1, 2, 3, 4, 5, 6}.

And a multiple of 3 in a die is {3}, {6}.

$$\Rightarrow p = \frac{2}{6}$$

$$\Rightarrow p = \frac{1}{3}$$

Then, let q be the probability of not getting a multiple of 3 in a throw of a die.

And we know, p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{3}$$

$$\Rightarrow q = \frac{3-1}{3}$$

$$\Rightarrow q = \frac{2}{3}$$

Let X be a random variable representing a number of successes (getting multiple of 3 in die) out of 10 throws of a die.

Then, the probability of getting r successes out of n throws of die is given by,

$P(X = r) = {}^{n}C_{r}p^{r}q^{n-r}$

Here n = 10.

Now, put values of n, p, and q in the above equation.

$$P(X=r) = {}^{10}C_{r}\left(\frac{1}{3}\right)^{r}\left(\frac{2}{3}\right)^{10-r} \ldots(i)$$

We need to find the probability of getting a multiple of 3 in at least 8 of the throws out of 10 throws of a fair die.

It is given,

Probability = P (X ≥ 8)

This can be written as,

P (X ≥ 8) = P (X = 8) + P (X = 9) + P (X = 10)

Just out r = 8, 9, 10 in equation (i) to find the value of P (X = 8), P (X = 9), P (X = 10) respectively, then substitute in the above equation.

$$\Rightarrow P(X \geq 8) = {}^{10}C_8 \left(\frac{1}{3}\right)^8 \left(\frac{2}{3}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{3}\right)^9 \left(\frac{2}{3}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{3}\right)^{10} \left(\frac{2}{3}\right)^{10-10}$$

$$\Rightarrow P(X \geq 8) = {}^{10}C_8 \left(\frac{1}{3}\right)^8 \left(\frac{2}{3}\right)^2 + {}^{10}C_9 \left(\frac{1}{3}\right)^9 \left(\frac{2}{3}\right)^1 + {}^{10}C_{10} \left(\frac{1}{3}\right)^{10} \left(\frac{2}{3}\right)^0$$

$$\Rightarrow P(X \geq 8) = \left[\left(\frac{10!}{(10-8)!8!}\right) \times \left(\frac{2^2}{3^{10}}\right)\right] + \left[\left(\frac{10!}{(10-9)!9!}\right) \times \left(\frac{2}{3^{10}}\right)\right] + \left[\left(\frac{10!}{(10-10)!10!}\right) \times \left(\frac{1}{3}\right)^{10}\right]$$

$$\Rightarrow P(X \geq 8) = \left[\left(\frac{10!}{2!8!}\right) \times 4 \times \left(\frac{1}{3}\right)^{10}\right] + \left[\left(\frac{10!}{9!}\right) \times 2 \times \left(\frac{1}{3}\right)^{10}\right] + \left[\left(\frac{10!}{10!}\right) \times \left(\frac{1}{3}\right)^{10}\right]$$

$$\Rightarrow P(X \geq 8) = \left(\frac{1}{3}\right)^{10} \left[\left(\frac{10 \times 9 \times 8!}{2 \times 8!} \times 4\right) + \left(\frac{10 \times 9!}{9!} \times 2\right) + 1\right]$$

$$\Rightarrow P(X \geq 8) = \left(\frac{1}{3}\right)^{10} [180 + 20 + 1]$$

$$\Rightarrow P(X \geq 8) = 201 \times \left(\frac{1}{3}\right)^{10}$$

$$P(X \geq 8) = \frac{201}{3^{10}}$$

Thus, the probability of getting a multiple of 3 in at least 8 of the throws out of 10 throws of a fair die is $201/3^{10}$.

51. Question

A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.

Answer

Given that, a die is thrown 5 times.

Let p be the probability of getting an odd number in a throw.

Since there are 6 possible outcomes in a throw of a die. That is, {1, 2, 3, 4, 5, 6}

And there are 3 odd numbers out of these 6 outcomes. That is, {1, 3, 5}

$$\Rightarrow p = \frac{3}{6}$$

$$\Rightarrow p = \frac{1}{2}$$

Then, let q be the probability of getting an even number in a throw.

And we know that, p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{2}$$

$$\Rightarrow q = \frac{1}{2}$$

Let X be a random variable representing a number of odd numbers in n throw of a die.

Then, the probability of getting r odd numbers out of n throw of the die can be given as,

$P (X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 5

Putting values of n, p, and q in the above formula. We get

$P(X=r)={}^5C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{5-r}$

$\Rightarrow P(X=r)={}^5C_r\left(\frac{1}{2}\right)^{r+5-r}$

$\Rightarrow P(X=r)={}^5C_r\left(\frac{1}{2}\right)^5$ ...(i)

We need to find the probability that an odd number will come up exactly three times.

It is given by,

Probability = P (X = 3)

Put r = 3 in equation (i) to find P (X = 3), we get

$\Rightarrow P(X=3)={}^5C_3\left(\frac{1}{2}\right)^5$

$\Rightarrow P(X=3)=\left(\frac{5!}{(5-3)!3!}\right)\times\left(\frac{1}{2}\right)^5$

$\Rightarrow P(X=3)=\left(\frac{5!}{2!3!}\right)\times\left(\frac{1}{2}\right)^5$

$\Rightarrow P(X=3)=\frac{5\times4\times3!}{2\times3!}\times\left(\frac{1}{2}\right)^5$

$\Rightarrow P(X=3)=5\times2\times\left(\frac{1}{2}\right)^5$

$\Rightarrow P(X=3)=\frac{5}{16}$

Thus, the probability of getting an odd number to come up exactly three times is 5/16.

52. Question

The probability of a man hitting a target is 0.25. He shoots 7 times. What is he probability of his hitting at least twice?

Answer

Given that, the probability of a man hitting a target is 0.25.

And he shoots 7 times in total.

Let p be the probability of hitting the target.

Then,

p = 0.25

$\Rightarrow p=\frac{25}{100}$

$\Rightarrow p=\frac{1}{4}$

Then, q be the probability of not hitting the target.

And we know that,

p + q = 1

⇒ q = 1 – p

$\Rightarrow q=1-\frac{1}{4}$

$\Rightarrow q=\frac{4-1}{4}$

$\Rightarrow q = \frac{3}{4}$

Let X be a random variable representing the number of times the man hits the target out of n shoots.

Then, the probability of hitting the target r times out of n times is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 7

Putting the values of n, p, and q in the above equation, we get

$P(X = r) = {}^7C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{7-r}$ ...(i)

We need to find the probability of hitting the target at least twice.

This can be expressed as,

Probability = P (X ≥ 2)

Or

P (X ≥ 2) = 1 – P (X < 2)

⇒ P (X ≥ 2) = 1 – [P (X = 0) + P (X = 1)]

So, put r = 0, 1 in equation (i) to get P (X = 0) and P (X = 1) respectively and then, substitute in the above formula.

We get

$$P(X \geq 2) = 1 - [{}^7C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{7-0} + {}^7C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{7-1}]$$

$$\Rightarrow P(X \geq 2) = 1 - [{}^7C_0\left(\frac{3}{4}\right)^7 + {}^7C_1\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6]$$

$$\Rightarrow P(X \geq 2) = 1 - \left[\left(\left(\frac{7!}{(7-0)!0!}\right) \times \left(\frac{3}{4}\right)^7\right) + \left(\left(\frac{7!}{(7-1)!1!}\right) \times 3^6 \times \left(\frac{1}{4}\right)^7\right)\right]$$

$$\Rightarrow P(X \geq 2) = 1 - \left[\left(\left(\frac{7!}{7!}\right) \times 3 \times 3^6 \times \left(\frac{1}{4}\right)^7\right) + \left(\left(\frac{7!}{6!}\right) \times 3^6 \times \left(\frac{1}{4}\right)^7\right)\right]$$

$$\Rightarrow P(X \geq 2) = 1 - 3^6\left(\frac{1}{4}\right)^7\left[3 + \left(\frac{7 \times 6!}{6!}\right)\right]$$

$$\Rightarrow P(X \geq 2) = 1 - \left(\frac{729}{16384} \times 10\right)$$

$$\Rightarrow P(X \geq 2) = \frac{16384 - 7290}{16384}$$

$$\Rightarrow P(X \geq 2) = \frac{9094}{16384}$$

$$\Rightarrow P(X \geq 2) = \frac{4547}{8192}$$

Thus, the probability of hitting the target at least twice is 4547/8192.

53. Question

A factory produces bulbs. The probability that one bulb is defective is 1/50, and they are packed in boxes of 10. From a single box, find the probability that

i. none of the bulbs is defective

ii. exactly two bulbs are defective

iii. more than 8 bulbs work properly.

Answer

Given that, the probability that one bulb is defective is 1/50.

The bulbs are packed in boxes of 10.

Let p be the probability of bulb being defective.

Then,

$$p = \frac{1}{50}$$

Let q be the probability of the bulb not being defective.

Also, we know that

p + q = 1

⇒ q = 1 – p

$$\Rightarrow q = 1 - \frac{1}{50}$$

$$\Rightarrow q = \frac{50 - 1}{50}$$

$$\Rightarrow q = \frac{49}{50}$$

Let X be a random variable representing a number of defective bulbs out of n bulbs.

Then, the probability of r bulbs to be defective out of n bulbs is given by,

$P(X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 10

Putting the values of n, p, and q in the above equation. We get

$$P(X = r) = {}^{10}C_r \left(\frac{1}{50}\right)^r \left(\frac{49}{50}\right)^{10-r} \quad \ldots(A)$$

(i). We need to find the probability that none of the bulbs is defective.

The probability is given by,

Probability = P (X = 0)

Put r = 0 in equation (A),

$$P(X = 0) = {}^{10}C_0 \left(\frac{1}{50}\right)^0 \left(\frac{49}{50}\right)^{10-0}$$

$$\Rightarrow P(X = 0) = \left(\frac{10!}{(10-0)!\,0!}\right) \times 1 \times \left(\frac{49}{50}\right)^{10}$$

$$\Rightarrow P(X = 0) = \left(\frac{10!}{10!}\right) \times \left(\frac{49}{50}\right)^{10}$$

$$\Rightarrow P(X = 0) = \left(\frac{49}{50}\right)^{10}$$

∴, the probability that none of the bulbs is defective is $(49/50)^{10}$.

(ii). We need to find the probability that exactly two bulbs are defective.

The probability is given by,

Probability = P (X = 2)

Put r = 2 in equation (A),

$$P(X = 2) = {}^{10}C_2 \left(\frac{1}{50}\right)^2 \left(\frac{49}{50}\right)^{10-2}$$

$$\Rightarrow P(X = 2) = \left(\frac{10!}{(10-2)!\,2!}\right) \times \left(\frac{1}{50}\right)^2 \times \left(\frac{49}{50}\right)^8$$

$\Rightarrow P(X=2)=\left(\frac{10!}{8!\,2!}\right)\times\frac{49^8}{50^{10}}$

$\Rightarrow P(X=2)=\frac{10\times 9\times 8!}{8!\times 2}\times\frac{49^8}{50^{10}}$

$\Rightarrow P(X=2)=45\times\frac{49^8}{50^{10}}$

∴, the probability that none of the bulbs is defective is $45\left(\frac{49^8}{50^{10}}\right)$.

(iii). We need to find the probability of more than 8 bulbs working properly.

This can also be interpreted as, the probability that at most 2 bulbs are defective.

This can be represented as,

Probability = P (X ≤ 2)

Or

P (X ≤ 2) = P (X = 0) + P (X = 1) + P (X = 2)

Put r = 0, 1, 2 to find P (X = 0), P (X = 1) and P (X = 2) and then, substitute in the above equation.

$P(X\le 2)={}^{10}C_0\left(\frac{1}{50}\right)^0\left(\frac{49}{50}\right)^{10-0}+{}^{10}C_1\left(\frac{1}{50}\right)^1\left(\frac{49}{50}\right)^{10-1}+{}^{10}C_2\left(\frac{1}{50}\right)^2\left(\frac{49}{50}\right)^{10-2}$

$$\Rightarrow P(X\le 2)=\left[\left(\frac{10!}{(10-0)!\,0!}\right)\times\left(\frac{49}{50}\right)^{10}\right]+\left[\left(\frac{10!}{(10-1)!\,1!}\right)\times\left(\frac{1}{50}\right)\times\left(\frac{49}{50}\right)^9\right]+\left[\left(\frac{10!}{(10-2)!\,2!}\right)\times\left(\frac{1}{50}\right)^2\times\left(\frac{49}{50}\right)^8\right]$$

$\Rightarrow P(X\le 2)=\left(\frac{49}{50}\right)^{10}+\left[\left(\frac{10!}{9!}\right)\times 49^9\times\left(\frac{1}{50}\right)^{10}\right]+\left[\left(\frac{10!}{8!\,2!}\right)\times 49^8\times\left(\frac{1}{50}\right)^{10}\right]$

$\Rightarrow P(X\le 2)=\frac{49^8}{50^{10}}\times\left[49^2+\left(\frac{10\times 9!}{9!}\right)\times 49+\left(\frac{10\times 9\times 8!}{8!\times 2}\right)\right]$

$\Rightarrow P(X\le 2)=\frac{49^8}{50^{10}}\times[49^2+490+45]$

$\Rightarrow P(X\le 2)=\frac{49^8}{50^{10}}\times[2401+490+45]$

$\Rightarrow P(X\le 2)=2936\left(\frac{49^8}{50^{10}}\right)$

∴, the probability of more than 8 bulbs working properly is $2936\left(\frac{49^8}{50^{10}}\right)$.

54. Question

A box has 20pens of which 2 are defective. Calculate the probability that out of 5 pens drawn one by one with replacement, at most 2 are defective.

Answer

Given that, a box has 20 pens out of which 2 are defective.

Let p be the probability of a number of pens being defective out of 20 pens.

$\Rightarrow p=\frac{2}{20}$

$\Rightarrow p=\frac{1}{10}$

Then, q be the probability of a number of pens not being defective.

Also, p + q = 1

⇒ q = 1 – p

$\Rightarrow q = 1 - \frac{1}{10}$

$\Rightarrow q = \frac{10-1}{10}$

$\Rightarrow q = \frac{9}{10}$

Let X be a random variable representing a number of defective pens out of 5 pens.

Then, the probability of getting r defective pens out of n pens is given by,

$P (X = r) = {}^nC_r p^r q^{n-r}$

Here, n = 5

Putting all the values of n, p, and q in the above equation, we get

$P(X = r) = {}^5C_r \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{5-r}$ ...(i)

We need to find the probability that at most 2 pens are defective out of 5 pens.

This can be represented as,

Probability = $P (X \le 2)$

Or

$P (X \le 2) = P (X = 0) + P (X = 1) + P (X = 2)$

Put r = 0, 1, 2 in equation (i) to find P (X = 0), P (X = 1) and P (X = 2), and then substitute in the above equation. We get,

$$P(X \le 2) = {}^5C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{5-0} + {}^5C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{5-1} + {}^5C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{5-2}$$

$$\Rightarrow P(X \le 2) = \left[\left(\frac{5!}{(5-0)!\,0!}\right) \times \left(\frac{9}{10}\right)^5\right] + \left[\left(\frac{5!}{(5-1)!\,1!}\right) \times \left(\frac{1}{10}\right) \times \left(\frac{9}{10}\right)^4\right] + \left[\left(\frac{5!}{(5-2)!\,2!}\right) \times \left(\frac{1}{10}\right)^2 \times \left(\frac{9}{10}\right)^3\right]$$

$$\Rightarrow P(X \le 2) = \left(\frac{9}{10}\right)^5 + \left[\left(\frac{5!}{4!}\right) \times 9^4 \times \left(\frac{1}{10}\right)^5\right] + \left[\left(\frac{5!}{3!\,2!}\right) \times 9^3 \times \left(\frac{1}{10}\right)^5\right]$$

$$\Rightarrow P(X \le 2) = \frac{9^3}{10^5} \times \left[9^2 + \left(\frac{5 \times 4!}{4!}\right) \times 9 + \left(\frac{5 \times 4 \times 3!}{3! \times 2}\right)\right]$$

$$\Rightarrow P(X \le 2) = \frac{9^3}{10^5} \times [81 + 45 + 10]$$

$$\Rightarrow P(X \le 2) = 136\left(\frac{9^3}{10^5}\right)$$

Thus, the probability that at most 2 pens are defective out of 5 pens is $136\left(\frac{9^3}{10^5}\right)$.

Exercise 33.2

1. Question

Can the mean of a binomial distribution be less than its variance?

Answer

Let X be a binomial variate with parameters in and p.

Mean = np

Variance = npq

Mean - Variance = np – npq

= np(1 - q)

$= np^2 > 0$

So, Mean – Variance > 0

⇒ Mean >Variance

So, mean can never be less than variance.

2. Question

Determine the binomial distribution whose mean is 9 and variance 9/4.

Answer

Let X denote the variance with parameters n and p

p + q = 1 ⇒ q = 1 - p

$\text{Given, mean} = np = 9 \text{ and variance} = npq = \frac{9}{4}$

$npq = \frac{9}{4}$

$\Rightarrow 9q = \frac{9}{4}$

$\Rightarrow q = \frac{1}{4}$

$\Rightarrow 1 - p = \frac{1}{4}$

$\Rightarrow p = \frac{3}{4}$

Now, np = 9

$\Rightarrow n\frac{3}{4} = 9$

⇒ n = 12

$\therefore n = 12 \text{ and } p = \frac{3}{4}$

The required binomial distribution is given by,

$$P(X = r) = 12Cr\left(\frac{3}{4}\right)^r\left(\frac{1}{4}\right)^{12-r}, \text{for } r = 0,1,2,\dots\dots,12$$

3. Question

If the mean and variance of a binomial distribution are respectively 9 and 6, find the distribution.

Answer

Let X denote the variance with parameters n and p

p + q = 1 ⇒ q = 1 - p

Given, mean = np = 9 and variance = npq = 6

npq = 6

⇒9q = 6

$\Rightarrow q = \frac{2}{3}$

$\Rightarrow 1 - p = \frac{2}{3}$

$\Rightarrow p = \frac{1}{3}$

Now, np = 9

$\Rightarrow n\frac{1}{3} = 9$

⇒ n = 27

$\therefore\ n = 27 \text{ and } p = \frac{1}{3}$

The required binomial distribution is given by,

$P(X = r) = 27Cr\left(\frac{1}{3}\right)^r\left(\frac{3}{4}\right)^{27-r}, \text{for } r = 0,1,2,\dots\dots,27$

4. Question

Find the binomial distribution when the sum of its mean and variance for 5 trials is 4.8.

Answer

Given that,

n = 5

Also, Mean+ Variance = 4.8

⇒np + npq = 4.8

⇒np(1+q) = 4.8

⇒5p(1+q) = 4.8

⇒5(1 - q)(1+q) = 4.8 [Since, p + q = 1]

⇒5(1 - q^2) = 4.8

$\Rightarrow 1 - q^2 = \frac{4.8}{5}$

$\Rightarrow q^2 = 1 - \frac{4.8}{5}$

$\Rightarrow q^2 = \frac{0.2}{5}$

$\Rightarrow q^2 = \frac{1}{25}$

$\Rightarrow q = \frac{1}{5}$

$\therefore p = 1 - q = \frac{4}{5}$

$\text{So}, n = 5, p = \frac{4}{5}, q = \frac{1}{5}$

The required binomial distribution is given by,

$P(X = r) = 5Cr\left(\frac{4}{5}\right)^r\left(\frac{1}{5}\right)^{12-r}, \text{for } r = 0,1,2,\dots,5$

5. Question

Determine the binomial distribution whose mean is 20 and variance 16.

Answer

Let X denote the variance with parameters n and p

p + q = 1 ⇒ q = 1 - p

Given, mean = np = 20 and variance = npq = 16

npq = 16

⇒20q = 16

$\Rightarrow q = \frac{4}{5}$

$\Rightarrow 1 - p = \frac{4}{5}$

$\Rightarrow p = \frac{1}{5}$

Now, np = 20

$\Rightarrow n\frac{1}{5} = 20$

⇒ n = 100

∴ n = 100 and $p = \frac{1}{5}$

The required binomial distribution is given by,

$$P(X = r) = 100Cr\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{100-r}, \text{for } r = 0,1,2,\ldots\ldots,100$$

6. Question

In a binomial distribution, the sum and product of the mean and the variance are 25/3 and 50/3 respectively. Find the distribution.

Answer

Let n and p be the parameters of the required binomial distribution. So,

1 - p = q

$\text{Mean} + \text{Variance} = \frac{25}{3}$

$\Rightarrow np + npq = \frac{25}{3}$

$\Rightarrow np(1 + q) = \frac{25}{3}$

$\Rightarrow np = \frac{25}{3(1 + q)}$

Also,

$\text{mean} \times \text{variance} = \frac{50}{3}$

$\Rightarrow np \times npq = \frac{50}{3}$

$\Rightarrow n^2p^2q = \frac{50}{3}$

$\Rightarrow \left[\frac{25}{3(1 + q)}\right]^2 q = \frac{50}{3}$

$\Rightarrow 625q = \frac{50}{3}[9(1 + q)^2]$

$\Rightarrow 625q = 150[(1+q)^2]$

$\Rightarrow 25q = 6[(1+q)^2]$

$\Rightarrow 6 + 6q^2 + 12q – 25q = 0$

$\Rightarrow 6q^2 – 13q + 6 = 0$

$\Rightarrow 6q^2 – 9q - 4q + 6 = 0$

$\Rightarrow 3q(2q - 3) - 2(2q - 3) = 0$

$\Rightarrow(2q - 3)(3q - 2) = 0$

$\Rightarrow(2q - 3) = 0$ or $(3q - 2) = 0$

$\Rightarrow q = \frac{3}{2}, \frac{2}{3}$

As, $q \leq 1$

$\therefore q = \frac{2}{3}$

Now, $p = 1 - q$

$= 1 - \frac{2}{3}$

$p = \frac{1}{3}$

$\text{As}, np(1 + q) = \frac{25}{3}$

$\Rightarrow n\left(\frac{1}{3}\right)\left(1 + \frac{2}{3}\right) = \frac{25}{3}$

$\Rightarrow n\left(\frac{1}{3}\right)\left(\frac{5}{3}\right) = \frac{25}{3}$

$\Rightarrow n = 15$

The required binomial distribution is given by,

$P(X = r) = 15Cr\left(\frac{1}{3}\right)^{r}\left(\frac{2}{3}\right)^{15-r}, \text{for } r = 0,1,2,\ldots\ldots,15$

7. Question

The mean of a binomial distribution is 20, and the standard deviation 4. Calculate the parameters of the binomial distribution.

Answer

Standard deviation = $\sqrt{npq} = 4$

$\Rightarrow npq = 16$

Therefore, $np = 20$, $npq = 16$

Let X denote the variance with parameters n and p

$p + q = 1 \Rightarrow q = 1 - p$

mean = np = 20 and variance = npq = 16

$npq = 16$

$\Rightarrow 20\,q = 16$

$\Rightarrow q = \frac{4}{5}$

$\Rightarrow 1 - p = \frac{4}{5}$

$\Rightarrow p = \frac{1}{5}$

Now, $np = 20$

$\Rightarrow n\frac{1}{5} = 20$

$\Rightarrow n = 100$

$\therefore n = 100 \text{ and } p = \frac{1}{5}$

8. Question

If the probability of a defective bolt is 0.1, find the (i) mean and (ii) standard deviation for the distribution of bolts in a total of 400 bolts.

Answer

Let p = probability of selecting defective bolt, so

p = 0.1

$\Rightarrow p = \frac{1}{10}$

q = 1 – p

$\Rightarrow q = 1 - \frac{1}{10}$

$\Rightarrow q = \frac{9}{10}$

Given, n = 400,

(i) mean = np

$= 400 \times \frac{1}{10}$

$\Rightarrow \text{mean} = 40$

(ii) Standard deviation = $\sqrt{npq}$

$= \sqrt{400 \times \frac{1}{10} \times \frac{9}{10}}$

$= \sqrt{36}$

∴ Standard deviation = 6

So, mean = 40, standard deviation = 6

9. Question

Find the binomial distribution whose mean is 5 and variance 10/3.

Answer

Let X denote the variance with parameters n and p

p + q = 1 ⇒ q = 1 - p

$\text{Given, mean} = np = 5 \text{ and variance} = npq = \frac{10}{3}$

$npq = \frac{10}{3}$

$\Rightarrow 5q = \frac{10}{3}$

$\Rightarrow q = \frac{2}{3}$

$\Rightarrow 1 - p = \frac{2}{3}$

$\Rightarrow p = \frac{1}{3}$

Now, np = 5

$\Rightarrow n\frac{1}{3} = 5$

⇒ n = 15

$\therefore$ $n = 15$ and $p = \frac{1}{3}$

The required binomial distribution is given by,

$$P(X = r) = 15Cr\left(\frac{1}{3}\right)^{r}\left(\frac{2}{3}\right)^{15-r}, \text{for } r = 0,1,2,\dots\dots,15$$

10. Question

If on an average 9 ships out of 10 arrive safely to ports, find the mean and S.D. of ships returning safely out of a total of 500 ships.

Answer

Let p be the probability of a ship returning safely to ports, so

$$p = \frac{9}{10}$$

$$q = 1 - \frac{9}{10}$$

$$\Rightarrow q = \frac{1}{10}$$

Given, n = 500

Mean = np

$= 500 \times \frac{9}{10}$

= 450

∴ Mean = 450

Standard deviation = $\sqrt{npq}$

$$= \sqrt{500 \times \frac{9}{10} \times \frac{1}{10}}$$

$= \sqrt{45}$

= 6.71

So, mean = 450, standard deviation = 6.71

11. Question

The mean and variance of binomial variate with parameters n and p are 16 and 8 respectively. Find P (X = 0), P (X = 1) and P (X ≥ 2).

Answer

Given that parameters for binomial distribution are n, p

Also, mean $= np = 16$ and variance $= npq = 8$

$npq = 8$

$\Rightarrow 16q = 8$

$$\Rightarrow q = \frac{1}{2}$$

$$\Rightarrow 1 - p = \frac{1}{2}$$

$$\Rightarrow p = \frac{1}{2}$$

Now, np = 16

$$\Rightarrow n\frac{1}{2} = 16$$

⇒ n = 32

$\therefore\ n = 32 \text{ and } p = \frac{1}{2}$

The required binomial distribution is given by,

$P(X = r) = 32Cr\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{32-r}, \text{for } r = 0,1,2,\dots\dots,32$

$P(X = 0) = 32C0\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{32-0}$

$= \left(\frac{1}{2}\right)^{32}$

$P(X = 1) = 32C1\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{32-1}$

$= 32\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{31}$

$= \left(\frac{1}{2}\right)^{27}$

P(X≥2) = 1 - [P(X = 0) + P(X = 1)]

$= 1 - \left[\left(\frac{1}{2}\right)^{32} + \left(\frac{1}{2}\right)^{27}\right]$

$= 1 - \left(\frac{1}{2}\right)^{27}\left(\frac{1}{32} + 1\right)$

$= 1 - \frac{33}{2^{32}}$

So,

$P(X = 0) = \left(\frac{1}{2}\right)^{32}, P(X = 1) = \left(\frac{1}{2}\right)^{27}, P(X \geq 2) = 1 - \frac{33}{2^{32}}$

12. Question

In eight throws of a die 5 or 6 is considered a success, find the mean number of successes and the standard deviation.

Answer

Let p be the probability of success

∴ the p = probability of getting 5 or 6

$\Rightarrow p = \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

Given, n = 8

Mean = np

$= 8 \times \frac{1}{3}$

= 2.66

Standard deviation = $\sqrt{npq}$

$= \sqrt{8 \times \frac{1}{3} \times \frac{2}{3}}$

$= \frac{4}{3}$

= 1.33

Therefore, mean = 2.66, standard deviation = 1.33.

13. Question

Find the expected number of boys in a family with 8 children, assuming the sex distribution to be equally probable.

Answer

Let n and p be the parameters of the binomial distribution.

Let, p = probability of having a boy in the family

Given, p = q

Since, p + q = 1

⇒ p + p = 1

⇒ 2p = 1

$\Rightarrow p = \frac{1}{2}$

$\Rightarrow q = \frac{1}{2}$

Given, n = 8

The expected number of boys = np

$= 8 \times \frac{1}{2}$

= 4

The expected number of boys = 4

14. Question

The probability is 0.02 that an item produced by a factory is defective. A shipment of 10,000 items is sent to its warehouse. Find the expected number of defective items and the standard deviation.

Answer

Let p be the probability of a defective item produced in the factory, so,

p = 0.02

$\Rightarrow p = \frac{2}{100}$

$\Rightarrow p = \frac{1}{50}$

As, p + q = 1

$\therefore q = 1 - \frac{1}{50}$

$\Rightarrow q = \frac{49}{50}$

Given, n = 10,000

Expected number of defective items = np

$= 10000 \times \frac{1}{50}$

= 200

Standard deviation = $\sqrt{npq}$

$= \sqrt{10000 \times \frac{1}{50} \times \frac{49}{50}}$

= 14

So, the expected number of defective items = 200

Standard deviation = 14

15. Question

A die is thrown thrice. A success is 1 or 6 in a throw. Find the mean and variance of the number of successes.

Answer

Let p be the probability of success

∴ the p = probability of getting 1 or 6

$\Rightarrow p = \frac{2}{6} = \frac{1}{3}$

$q = 1 - \frac{1}{3} = \frac{2}{3}$

Given, n = 3

Mean = np

$= 3 \times \frac{1}{3}$

= 1

Variance = npq

$= 3 \times \frac{1}{3} \times \frac{2}{3}$

$= \frac{2}{3}$

= 0.66

Therefore, mean = 1, standard deviation = 0.66.

16. Question

If a random variable X follows a binomial distribution with mean 3 and variance 3/2, find P(X ≤ 5).

Answer

Let parameters for binomial distribution be n, p

$\text{So, mean} = np = 3 \text{ and variance} = npq = \frac{3}{2}$

$npq = \frac{3}{2}$

$\Rightarrow 3q = \frac{3}{2}$

$\Rightarrow q = \frac{1}{2}$

$\Rightarrow 1 - p = \frac{1}{2}$

$\Rightarrow p = \frac{1}{2}$

Now, np = 3

$\Rightarrow n\frac{1}{2} = 3$

⇒ n = 6

$\therefore\ n = 6 \text{ and } p = \frac{1}{2}$

The required binomial distribution is given by,

$P(X = r) = 6Cr\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{6-r}, \text{for } r = 0,1,2,\ldots\ldots,6$

Now, P(X≤5) = 1 - P(X = 6)

$\Rightarrow P(X \leq 5) = 1 - 6C6\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{6-6}$

$= 1 - \left(\frac{1}{2}\right)^{6}$

$= 1 - \frac{1}{64}$

$= \frac{63}{64}$

$\text{Therefore, } P(X \leq 5) = \frac{63}{64}$

17. Question

If X follows a binomial distribution with mean 4 and variance 2, find P(X ≥ 5).

Answer

Given that parameters for binomial distribution are n, p

$\text{Also, mean} = np = 4 \text{ and variance} = npq = 2$

$npq = 2$

$\Rightarrow 4q = 2$

$\Rightarrow q = \frac{1}{2}$

$\Rightarrow 1 - p = \frac{1}{2}$

$\Rightarrow p = \frac{1}{2}$

Now, np = 4

$\Rightarrow n\frac{1}{2} = 4$

⇒ n = 8

$\therefore\ n = 8 \text{ and } p = \frac{1}{2}$

The required binomial distribution is given by,

$P(X = r) = 8Cr\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{8-r}, \text{for } r = 0,1,2,\ldots\ldots,8$

Now, P(X≥5) = P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)

$= 8C5\left(\frac{1}{2}\right)^{5}\left(\frac{1}{2}\right)^{3} + 8C6\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{2} + 8C7\left(\frac{1}{2}\right)^{7}\left(\frac{1}{2}\right) + 8C8\left(\frac{1}{2}\right)^{8}$

$= \frac{8 \times 7 \times 6}{3 \times 2}\left(\frac{1}{2}\right)^{8} + \frac{8 \times 7}{2}\left(\frac{1}{2}\right)^{8} + 8\left(\frac{1}{2}\right)^{8} + \left(\frac{1}{2}\right)^{8}$

$\left(\frac{1}{2}\right)^8 [56 + 28 + 8 + 1]$

$= \frac{93}{256}$

$\text{So}, P(X \geq 5) = \frac{93}{256}$

18. Question

The mean and variance of a binomial distribution are 4/3 and 8/9 respectively. Find P (X ≥ 1).

Answer

Given that parameters for binomial distribution are n, p

$\text{Also, mean} = np = \frac{4}{3} \text{ and variance} = npq = \frac{8}{9}$

$npq = \frac{8}{9}$

$\Rightarrow \frac{4}{3} q = \frac{8}{9}$

$\Rightarrow q = \frac{2}{3}$

$\Rightarrow 1 - p = \frac{2}{3}$

$\Rightarrow p = \frac{1}{3}$

$\text{Now}, np = \frac{4}{3}$

$\Rightarrow n \frac{1}{3} = \frac{4}{3}$

⇒ n = 4

$\therefore n = 4 \text{ and } p = \frac{1}{3}$

The required binomial distribution is given by,

$P(X = r) = 4Cr \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{4-r}, \text{for } r = 0,1,2,3,4$

Now, P(X ≥ 1) = 1 - P(X = 0)

$= 1 - \left(\frac{2}{3}\right)^4$

$= 1 - \frac{16}{81}$

$= \frac{65}{81}$

$\text{Hence}, P(X \geq 1) = \frac{65}{81}$

19. Question

If the sum of the mean and variance of a binomial distribution for 6 trials is 10/3, find the distribution.

Answer

Let, n and p be the parameters of the binomial distribution,

Given, n = 6

$\text{Mean} + \text{Variance} = \frac{10}{3}$

$\Rightarrow np + npq = \frac{10}{3}$

$\Rightarrow 6p + 6pq = \frac{10}{3}$

$\Rightarrow 6p(1+q) = \frac{10}{3}$

$\Rightarrow 6(1-q)(1+q) = \frac{10}{3}$

$\Rightarrow 6(1-q^2) = \frac{10}{3}$

$\Rightarrow 1-q^2 = \frac{10}{18}$

$\Rightarrow q^2 = \frac{4}{9}$

$\Rightarrow q = \frac{2}{3}$

Now, p = 1 - q

$\Rightarrow p = 1-\frac{2}{3} = \frac{1}{3}$

The required binomial distribution is given by,

$P(X = r) = 6Cr\left(\frac{1}{3}\right)^r\left(\frac{2}{3}\right)^{6-r}, \text{for } r = 0,1,2,\ldots\ldots,6$

20. Question

A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of a number of successes and hence find its mean.

Answer

Throwing doublet: {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

Total outcomes = 6×6 = 36

Let p = probability of success

$\therefore p = \frac{6}{36} = \frac{1}{6}$

So, q = 1 - p

$\Rightarrow q = \frac{5}{6}$

Given, n = 4

The required binomial distribution is given by,

$P(X = r) = 4Cr\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{4-r}, \text{for } r = 0,1,2,3,4$

Let, X, be the random variable getting doublet, so,

$P(X = 0) = 4C0p^0q^4 = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}$

$P(X = 1) = 4C1p^1q^3 = 4\times\frac{1}{6}\times\left(\frac{5}{6}\right)^3 = \frac{500}{1296}$

$P(X = 2) = 4C2p^2q^2 = \frac{4.3}{2}\times\left(\frac{1}{6}\right)^2\times\left(\frac{5}{6}\right)^2 = \frac{150}{1296}$

$P(X = 3) = 4C3p^3q^1 = 4\times\left(\frac{1}{6}\right)^3\times\left(\frac{5}{6}\right) = \frac{20}{1296}$

$P(X = 4) = 4C4p^4q^0 = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}$

$\text{Mean}, m = \sum_{i=1}^{4} X_iP(X_i)$

$= 0.\left(\frac{625}{1296}\right) + 1.\left(\frac{500}{1296}\right) + 2.\left(\frac{150}{1296}\right) + 3.\left(\frac{20}{1296}\right) + 4.\left(\frac{1}{1296}\right)$

$= \frac{500 + 300 + 60 + 4}{1296}$

$= \frac{864}{1296}$

$= \frac{2}{3}$

$\text{Hence, mean} = \frac{2}{3}$

21. Question

Find the probability distribution of the number of doublets in three throws of a pair of dice and hence find its mean.

Answer

Throwing doublet: {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)}

Total outcomes = 6×6 = 36

Let p = probability of success

$\therefore p = \frac{6}{36} = \frac{1}{6}$

So, q = 1 - p

$\Rightarrow q = \frac{5}{6}$

Given, n = 3

The required binomial distribution is given by,

$P(X = r) = 3Cr\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{4-r}, \text{for } r = 0,1,2,3$

Let, X, be the random variable getting doublet, so,

$P(X = 0) = 3C0p^0q^3 = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$

$P(X = 1) = 3C1p^1q^2 = 3\times\frac{1}{6}\times\left(\frac{5}{6}\right)^2 = \frac{75}{216}$

$P(X = 2) = 3C2p^2q^1 = 3\times\left(\frac{1}{6}\right)^2\times\left(\frac{5}{6}\right)^1 = \frac{15}{216}$

$P(X = 3) = 3C3p^3q^0 = 1\times\left(\frac{1}{6}\right)^3 = \frac{1}{216}$

$\text{Mean}, m = \sum_{i=1}^{3} X_iP(X_i)$

$= 0.\left(\frac{125}{216}\right) + 1.\left(\frac{75}{216}\right) + 2.\left(\frac{15}{216}\right) + 3.\left(\frac{1}{216}\right)$

$= \frac{75+30+3}{216}$

$= \frac{108}{216}$

$= \frac{1}{2}$

$\text{Hence, mean } = \frac{1}{2}$

22. Question

From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of a number of defective bulbs. Hence, find the mean of the distribution.

Answer

Out of 15 bulbs, five are defective.

Let p = probability that drawn bulb is defective

$\text{So, } p = \frac{5}{15} = \frac{1}{3}$

Also, the q = probability that drawn bulb is not defective.

$\text{So, } q = 1 - \frac{1}{3} = \frac{2}{3}$

Let X be the probability that the drawn bulb is defective out of 4.

Then, X follows a binomial distribution with,

$n = 4, p = \frac{1}{3}, q = \frac{2}{3}, \text{such that,}$

The required binomial distribution is given by,

$P(X = r) = 4Cr\left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{4-r}, \text{for } r = 0,1,2,3,4$

Let, X, be the random variable getting defective ball, so,

$P(X = 0) = 4C0p^0q^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

$P(X = 1) = 4C1p^1q^3 = 4 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^3 = \frac{32}{81}$

$P(X = 2) = 4C2p^2q^2 = \frac{4.3}{2} \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^2 = \frac{24}{81}$

$P(X = 3) = 4C3p^3q^1 = 4 \times \left(\frac{1}{3}\right)^3 \times \left(\frac{2}{3}\right) = \frac{8}{81}$

$P(X = 4) = 4C4p^4q^0 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$

$\text{Mean, } m = \sum_{i=1}^{4} X_i P(X_i)$

$= 0.\left(\frac{16}{81}\right) + 1.\left(\frac{32}{81}\right) + 2.\left(\frac{24}{81}\right) + 3.\left(\frac{8}{81}\right) + 4.\left(\frac{1}{81}\right)$

$= \frac{32+48+24+4}{81}$

$= \frac{108}{81}$

$= \frac{4}{3}$

$\text{Hence, mean } = \frac{4}{3}$

23. Question

A die is thrown three times. Let X be the number of two's seen. Find the expectation of X.

Answer

Let p = probability of getting a 2 when a die is thrown.

$\text{So, p } = \frac{1}{6}$

X follows a binomial distribution with,

$n = 3, p = \frac{1}{6}$

$\text{Expectation } = E(X) = np = 3 \times \frac{1}{6} = \frac{1}{2}$

24. Question

A die is tossed twice. A 'success' is getting an even number on a toss. Find the variance of some successes.

Answer

Let p be the probability of getting an even number on the toss when a die is thrown.

Let q be the probability of not getting an even number on the toss when a die is thrown.

$\text{Then, p } = \frac{3}{6} = \frac{1}{2} \text{ and q } = 1 - \frac{1}{2} = \frac{1}{2}$

As X follows a binomial distribution with,

$n = 2, p = \frac{1}{2}$

∴ Variance = npq

$\Rightarrow \text{variance } = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

25. Question

Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spades. Hence, find the mean of the distribution.

Answer

Let p be the probability of getting a spade card.

Let q be the probability of getting a spade card.

$\text{Then, p } = \frac{1}{4} \text{ And q } = \frac{3}{4}$

X follows a binomial distribution with,

$n = 3, p = \frac{1}{4} \text{ And q } = \frac{3}{4}$

The probability distribution is given by,

The required binomial distribution is given by,

$P(X = r) = 3Cr\left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{3-r}, \text{for r } = 0,1,2,3$

So, mean = np

$= 3 \times \frac{1}{4} = \frac{3}{4}$

26. Question

An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Also, find the mean and variance of the distribution.

Answer

Let p = probability of drawing a red ball

$$\text{Then, p } = \frac{6}{9} = \frac{2}{3}$$

$$\text{Also, q } = \frac{1}{3}$$

Let number of red balls drawn be X.

X follows a binomial distribution with,

$$n = 4, p = \frac{2}{3}, q = \frac{1}{3}$$

The required binomial distribution is given by,

$$P(X = r) = 4Cr\left(\frac{2}{3}\right)^r\left(\frac{1}{3}\right)^{4-r}, \text{for } r = 0,1,2,3,4$$

So, mean = np

$$= 4 \times \frac{2}{3} = \frac{8}{3}$$

27. Question

Five bad oranges are accidentally mixed with 20 good ones. If four oranges are drawn one by one successively with replacement, then find the probability distribution of number of bad oranges drawn. Hence, find the mean and variance of the distribution.

Answer

Total oranges = 5+20 = 25

Let p = probability of drawing a bad orange

$$\text{Then, p } = \frac{5}{25} = \frac{1}{5}$$

$$\text{Also, q } = \frac{4}{5}$$

Let some bad oranges drawn be X.

X follows a binomial distribution with,

$$n = 4, p = \frac{1}{5}, q = \frac{4}{5}$$

The required binomial distribution is given by,

$$P(X = r) = 4Cr\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{4-r}, \text{for } r = 0,1,2,3,4$$

So, mean = np

$$= 4 \times \frac{1}{5} = \frac{4}{5}$$

28. Question

Three cards are drawn successively with replacement from a well - shuffled pack of 52 cards. Find the mean and variance of some red cards.

Answer

Let p = probability of drawing a red card

$$\text{Then, p } = \frac{26}{52} = \frac{1}{2}$$

$\text{Also}, q = \frac{1}{2}$

Let a number of the red cards drawn be X.

X follows a binomial distribution with,

$n = 3, p = \frac{1}{2}, q = \frac{1}{2}$

The required binomial distribution is given by,

$P(X = r) = 3Cr\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{3-r}, \text{for } r = 0,1,2,3$

So, mean = np

$= 3 \times \frac{1}{2} = \frac{3}{2}$

Very Short Answer

1. Question

In a binomial distribution, if n = 20, q = 0.75, then write its mean.

Answer

Given:

n=20

q=0.75

p=?

q=(1-p)

=0.75

p=1-0.75

=0.25

Mean = np

= 20(0.75)

= 15

2. Question

If in a binomial distribution mean is 5 and variance is 4, write the number of trials.

Answer

Given:

Mean=5=np

Variance=4

=np(1-p)

n=?

5(1-p)=4

$1 - p = \frac{4}{5}$

P= $\frac{1}{5}$

np=5

n= 25

3. Question

In a group of 200 items, if the probability of getting a defective item is 0.2, write the mean of the distribution.

Answer

Given:

n=200

p=0.2

mean=?

Mean=np

= 200(0.2)

=40

4. Question

If the mean of a binomial distribution is 20 and its standard deviation is 4, find p.

Answer

Given:

Mean=20

Standard deviation=$\sqrt{}$np(1-p)

=4

P=?

$\sqrt{}$20(1-p)=4

$$(1-\text{p}) = \frac{16}{20}$$

$$\text{p} = \frac{4}{5}$$

5. Question

The mean of a binomial distribution is 10 and its standard deviation is 2, write the value of q.

Answer

Given:

Mean=10=np

Standard deviation=2

=$\sqrt{}$npq

q=?

$$\sqrt{\text{npq}} = 2$$

10q=4

$$\text{q} = \frac{2}{5}$$

6. Question

If the mean and variance of a random variable X having a binomial distribution are 4 and 2 respectively, find P (X = 1).

Answer

Given:

Mean=4

Variance=2

P(x=1)=?

P(x=r)= $^{n}C_{r}\ p^{r}\ q^{n-r}$

$P(x=1) = {}^nC_1\, pq^{n-1}$

$= n \times p \times q^{n-1}$

np(1-p)=2

4(1-p)=2

$(1-p) = \frac{1}{2}$

=q

$p = \frac{1}{2}$

np=4

n=8

$P(x=1) = 8 \times \frac{1}{2} \times \left(\frac{1}{2}\right)^7$

$= \frac{1}{32}$

7. Question

If the mean and variance of a binomial variate X are 2 and 1 respectively, find P(X > 1).

Answer

Given:

Mean= 2

Variance=1

P(x>1)=?

$P(x>1)=1-p(x \leq 1)$

= 1-[p(x=0)+p(x=1)]

$=1-[\, 1 \times 1 \times q^n + n \times p \times q^{n-1}]$

2(1-p)=1

$(1-p) = \frac{1}{2}$

=q

$p = \frac{1}{2}$

np=2

n=4

$P(x > 1) = 1 - [1 \times 1 \times \left(\frac{1}{2}\right)^4 + 4 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^3]$

$= \frac{15}{16}$

8. Question

If in a binomial distribution n = 4 and $P(X=0) = \frac{16}{81}$, find q.

Answer

Given:

n = 4

$p(x=0)=\frac{16}{81}$

q=?

$P(x=r)= {}^nC_r\, p^r\, q^{n-r}$

$p(x=0)= 1\times1\times q^4$

$=\frac{16}{81}$

$q^4 = \left(\frac{2}{3}\right)^4$

$q = \frac{2}{3}$

9. Question

If the mean and variance of a binomial distribution are 4 and 3 respectively, find the probability of no-success.

Answer

Given:

Mean=4

Variance=3

P(x=0)=?

$P(x=r)= {}^nC_r\, p^r\, q^{n-r}$

$p(x=0)= 1\times1\times q^n$

4(1-p)=3

$1-p=\frac{3}{4}$

=q

$p=\frac{1}{4}$

np=4

n=16

$p(x=0) = \left(\frac{3}{4}\right)^{16}$

10. Question

If for a binomial distribution P (X = 1) = P (X = 2) = α, write P (X= 4) in terms of α.

Answer

Given:

P (X = 1) = P (X = 2) = α

P (X= 4)=?

$P(x=r)= {}^nC_r\, p^r\, q^{n-r}$

$P(x=1)= {}^nC_1\, p^1\, q^{n-1}$

$=n\times p\times q^{n-1}$ (1)

$P(x=2)= {}^nC_2\, p^2\, q^{n-2}$

$=\frac{n(n-1)}{2}\times p^2\, q^{n-2}$ (2)

Equating both equations ,we have:

$$n \times p \times q^{n-1} = \frac{n(n-1)}{2} \times p^2 q^{n-2}$$

$$1 = \frac{(n-1)}{2} \times p q^{-1}$$

2q = (n-1) p

$4q^2 = (n-1)^2 p^2$

$P(x=4) = {}^nC_4 p^4 q^{n-4}$

$$= \frac{n(n-1)(n-2)(n-3)}{4!} \times p^2 p^2 q^{n-2} q^{-2}$$

$$= \frac{n(n-1)}{2} \times p^2 q^{n-2} \times \frac{(n-2)(n-3)}{4 \times 3} \times p^2 q^{-2}$$

$$= \frac{\alpha}{3} \times \frac{(n-2)(n-3)}{4q^2} \times \frac{4q^2}{(n-1)^2}$$

$$= \frac{\alpha}{3} \times \frac{n^2\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{n^2\left(1-\frac{1}{n}\right)}$$

For large n; $n \to \infty$

$\frac{1}{n} \to 0$

So, $P(x = 4) = \frac{\alpha}{3}$

11. Question

An unbiased coin is tossed 4 times. Find the mean and variance of the number of heads obtained.

Answer

Given:

n=4

p(getting atleast one head)=?

• 0 head : HHHH

• 1 head : HTTT

THTT

TTHT

TTTH

• 2 heads :HHTT

HTHT

HTTH

THHT

THTH

TTHH

• 3 heads: HHHT

HHTH

HTHH

THHH

• 4 heads : HHHH

Total outcomes = 16

Outcomes containing at least 1 head = 15

P(getting at least 1 head)= $\frac{15}{16}$

12. Question

If X follows binomial distribution with parameters n = 5, p and P(X = 2) = 9 P(X = 3), then find the value of p.

Answer

Given:

n=5

p(x=2)=9p(x=3)

p=?

$P(x=r)= {}^nC_r\, p^r\, q^{n-r}$

$P(x=1)= {}^5C_1\, p^1\, q^4$

$= 5p\, q^4$

$P(x=3)= {}^5C_3\, p^3\, q^2$

$= 20\, p^3\, q^2$

Put values in equation, we have:

$5p\, q^4= 9(20\, p^3\, q^2)$

$q^2= 9(4\, p^2)$

$(1-p)^2= 36p^2$

$1+p^2-2p= 36p^2$

$35p^2+2p-1=0$

$P = \frac{1}{5}, \frac{1}{7}$

MCQ

1. Question

Mark the correct alternative in the following:

In a box containing 100 bulbs, 10 are defective. What is the probability that out of a sample of 5 bulbs, none is defective.

A. $\left(\frac{9}{10}\right)^5$

B. $\frac{9}{10}$

C. 10^{-5}

D. $\left(\frac{1}{2}\right)^2$

Answer

Given:

Total bulbs=100

Defective bulb=10

n=5

P(non-defective bulbs)=?

P(defective bulbs)= $\frac{10}{100}$

$= \frac{1}{10}$

=p

$q = \frac{9}{10}$

P(non-defective bulbs)=p(x=0)

= $^5C_0\ p^0\ q^5$

$= \left(\frac{9}{10}\right)^5$

2. Question

Mark the correct alternative in the following:

If in a binomial distribution n = 4, P(X = 0) = $\frac{16}{81}$, then P(X = 4) equals.

A. $\frac{1}{16}$

B. $\frac{1}{81}$

C. $\frac{1}{27}$

D. $\frac{1}{8}$

Answer

Given:

n = 4

$p(x = 0) = \frac{16}{81}$

p(x=4)=?

P(x=r)= $^nC_r\ p^r\ q^{n-r}$

p(x=0)= $1 \times 1 \times q^4$

$= \frac{16}{81}$

$q^4 = \left(\frac{2}{3}\right)^4$

$q = \frac{2}{3}$

p=1-q

$= \frac{1}{3}$

P(x=4)= $^4C_4\ p^4\ q^0$

$=1 \times 1 \times p^4$

$= \left(\frac{1}{3}\right)^4$

$= \frac{1}{81}$

3. Question

Mark the correct alternative in the following:

A rifleman is firing at a distant target and has only 10% chance o hitting it. The least number of rounds, he must fire in order to have more than 50% chance of hitting it at least once is

A. 11

B.9

C. 7

D. 5

Answer

Given:

Chances of hitting a target in a trial=10%=$\frac{1}{10}$

Chances of not hitting the target = 90%=$\frac{9}{10}$

Total probability = 50%=5

Let number of trials=n

So number of trials missed=n-1 to hit on nth trial

Total probability= $\frac{1}{10} + \frac{9}{10} \times \frac{1}{10} + \left(\frac{9}{10}\right)^2 \times \frac{1}{10} + \cdots . + \left(\frac{9}{10}\right)^{n-1} \times \frac{1}{10}$

= 5

$\Rightarrow \frac{1}{10}\left\{1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \left(\frac{9}{10}\right)^{n-1}\right\} = 5$

$\Rightarrow \frac{\left(\frac{9}{10}\right)^n}{\left(1 - \frac{9}{10}\right)} = 5$

$\Rightarrow \left(\frac{9}{10}\right)^n = 1/2$

Take log on both sides

$n \log \frac{9}{10} = \log \frac{1}{2}$

$n = \frac{\log 0.5}{\log 0.9}$

n=6.58

n$\sim$7

So one must hit it seven times.

4. Question

Mark the correct alternative in the following:

A fair coin is tossed a fixed number of times. If the probability of getting seven heads is equal to that of getting nine heads, the probability of getting two heads is

A. $15/2^8$

B. 2/15

C. $15/2^{13}$

D. none of these

Answer

Given:

P(getting a head)=$\frac{1}{2}$

Probability of getting 7 heads = $^{n}C_{7}$ $(1/2)^{n}$

Probability of getting 9 heads = $^{n}C_{9}$ $(1/2)^{n}$

Both probabilities are equal if n=9+7=16

So probability of getting 4 heads =$^{16}C_{4}\left(\frac{1}{2}\right)^{16}$

5. Question

Mark the correct alternative in the following:

A fair coin is tossed 100 times. The probability that on the tenth throw the fourth six appears is

A. 1/2

B. 1/8

C. 3/8

D. none of these

Ans. A

Answer

Given:

P(getting a head)=p(getting a tail)= $\frac{1}{2}$

$$P(\text{getting 6 heads}) = \frac{\text{number of outcomes having 6}}{\text{total number of outcomes}}$$

$$= \frac{\frac{10!}{6!\,(10-6)!}}{10!}$$

$$= \frac{210}{1024}$$

=0.20

6. Question

Mark the correct alternative in the following:

A fair die is thrown twenty times. The probability that on the tenth throw the fourth six appears is

A. $\frac{^{20}C_{10} \times 5^{6}}{6^{20}}$

B. $\frac{20 \times 5^{7}}{6^{10}}$

C. $\frac{84 \times 5^{6}}{6^{10}}$

D. none of these

Ans. C

Answer

Given:

Total outcomes= 6^{10}

Fourth 6 is at 10^{th} place.

So other three 6 have to be placed at any of the remaining 9 places=$^{9}C_{3} \times 5^{6}$

For remaining 5 places we have 1,2,3,4&5

So total number of favorable cases $= \frac{9!\times 5^6}{3!\times 6!\times (6)^{10}}$

$= \frac{84\times 5^6}{6^{10}}$

7. Question

Mark the correct alternative in the following:

If X is a binomial variate with parameters n and p, where 0 < p < 1 such that $\frac{P(X=r)}{P(X=n-r)}$ is independent of n and r, then p equals.

A. 1/2

B. 1/3

C. 1/4

D. none of these

Answer

Given:

$\frac{P(X=r)}{P(X=n-r)}$

P=?

$P(x=r)= {}^nC_r\, p^r\, q^{n-r}$

$= \frac{n!}{r!(n-r)!}\times p^r\times q^{n-r}$

$P(x=n-r)= {}^nC_{n-r}\, p^{n-r}\, q^{n-(n-r)}$

$= {}^nC_r\, p^{n-r}\, q^r$

$= \frac{n!}{r!(n-r)!}\times p^{n-r}\times q^r$

Put values in equation, we have:

$= \frac{\frac{n!}{r!\,(n-r)!}\times p^r\times q^{n-r}}{\frac{n!}{r!\,(n-r)!}\times p^{n-r}\times q^r}$

$= \frac{p^{2r-n}}{q^{2r-n}}$

$= \left(\frac{p}{q}\right)^{2r-n}$

$= \left(\frac{p}{1-p}\right)^{2r-n}$

1-p=p

1=2p

P=1/2

8. Question

Mark the correct alternative in the following:

Let X denote the number of times heads occur in n tosses of a fair coin. If P (X = 4), P(X = 5) and P(X = 6) are in AP; the value of n is

A. 7, 14

B. 10, 14

C. 12, 7

D. 14, 12

Answer

Given:

P(H)=1/2=p

q=1/2

P(x=r)= $^nC_r\ p^r\ q^n$

P(x=4)= $^nC_4\ p^4\ q^{n-4} = {}^nC_4\ p^n$

P(x=5)= $^nC_5\ p^5\ q^{n-5} = {}^nC_5\ p^n$

P(x=6)= $^nC_6\ p^6\ q^{n-6} = {}^nC_6\ p^n$

Since they are in AP;

So,

p(x=5)= $\frac{p(x=4)+p(x=6)}{2}$

2 p(x=5) = p(x=4)+ p(x=5)

$2\times{}^nC_5\ p^n = {}^nC_4\ p^n + {}^nC_6\ p^n$

$$2\times\frac{n!}{5!\,(n-5)!}=\frac{n!}{4!\,(n-4)!}+\frac{n!}{6!\,(n-6)!}$$

$$2\times\frac{n(n-1)(n-2)n-3)(n-4)}{60}=\frac{n(n-1)(n-2)(n-3)}{12}+\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{120}$$

$$2\times\frac{n(n-1)(n-2)n-3)(n-4)}{60}=\frac{n(n-1)(n-2)(n-3)}{12}\left\{(n-4)+\frac{(n-5)}{10}\right\}$$

$$2\times\frac{(n-4)}{5}=\left\{\frac{(10n-40+n-5)}{10}\right\}$$

$$2\times\frac{(n-4)}{5}=\left\{\frac{(11n-45)}{2}\right\}$$

4n – 16 = 55n – 225

51n = 209

n= 4

9. Question

Mark the correct alternative in the following:

One hundred identical coins, each with probability p of showing heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, the value of p is

A. 1/2

B. 51/101

C. 49/101

D. none of these

Answer

Given:

n=100

p(H)=P(T)=1/2

P(x=r)= ${}^nC_r\ p^r\ q^{n-r}$

P(x=50)= ${}^{100}C_{50}\ p^{50}\ q^{50}$

$= \frac{100!}{50! \times 50!} \times p^{50}(1-p)^{50}$

$= \frac{100\times99\times.....\times52\times51}{2\times3\times....\times49\times50} \times p^{50}(1-p)^{50}$ (1)

P(x=51)= ${}^{100}C_{51}\ p^{51}\ q^{49}$

$= \frac{100!}{51! \times 49!} \times p^{51}(1-p)^{49}$

$= \frac{100\times99\times.....\times53\times52}{2\times3\times....\times48\times49} \times p^{51}(1-p)^{49}$ (2)

Equate both equations;

$$\frac{100 \times 99 \times .....\times 52 \times 51}{2 \times 3 \times .....\times 49 \times 50} \times p^{50}(1-p)^{50} = \frac{100 \times 99 \times .....\times 53 \times 52}{2 \times 3 \times .....\times 48 \times 49} \times p^{51}(1-p)^{49}$$

$\frac{51}{50} \times (1-p) = p$

51-51p = 50 p

101 p = 51

$P = \frac{51}{101}$

10. Question

Mark the correct alternative in the following:

A fair coin is tossed 99 times. If X is the number of times heads occur, then P(X = r) is maximum when r is

A. 49, 50

B. 50, 51

C. 51, 52

D. none of these

Answer

Given:

n=99

p(H)=P(T)=1/2

P(x=r)= ${}^nC_r\ p^r\ q^{n-r}$

P(x=r)= ${}^nC_r\ p^n$

${}^nC_r = \frac{n!}{r!(n-r)!}$

Its maximum value is closest near n/2

Since n is an odd number, so it have two values:

i.e. $\frac{n-1}{2}, \frac{n+1}{2}$

S0, r= 49,50

11. Question

Mark the correct alternative in the following:

The least number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.8, is

A. 7

B. 6

C. 5

D. 3

Answer

Given:

P(atleast 1 head) = 0.8

n=?

p(H)=p(T) = 1/2

Required probability = p= 1- $\left(\frac{1}{2}\right)^n$

P$\geq$0.8

$$1-\left(\frac{1}{2}\right)^n \geq 0.8$$

$$0.2 \geq \left(\frac{1}{2}\right)^n$$

$$\frac{1}{5} \geq \left(\frac{1}{2}\right)^n$$

For n= 2

$$\left(\frac{1}{2}\right)^n = 0.25$$

For n=3

$$\left(\frac{1}{2}\right)^n = 0.125$$

So, n=3

12. Question

Mark the correct alternative in the following:

If the mean and variance of a binomial variate X are 2 and 1 respectively, then the probability that X takes a value greater than 1 is

A. 2/3

B. 4/5

C. 7/8

D. 15/16

Answer

Given:

Mean= 2

Variance=1

P(x>1)=?

P(x>1)=1-p(x$\leq$1)

= 1-[p(x=0)+p(x=1)]

=1-[$1\times1\times q^n + n\times p\times q^{n-1}$]

2(1-p)=1

$(1-p)=\frac{1}{2}$

=q

$p=\frac{1}{2}$

np=2

n=4

$P(x>1)=\ 1-\left[1\times 1\times\left(\frac{1}{2}\right)^4+4\times\left(\frac{1}{2}\right)^1\times\left(\frac{1}{2}\right)^3\right]$

$=\frac{15}{16}$

13. Question

Mark the correct alternative in the following:

A biased coin with probability p, 0 < p < 1, of heads is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, then p equals.

A. 1/3

B. 2/3

C. 5/5

D. 3/5

Answer

Given:

Probability of getting head to appear first time(r=1), n should be even.

So we have to check probability for n=2,4,6…

$P(x=r)= {}^nC_r\, p^r\, q^{n-r} = 2/5$

For n=2;

P(X=1)= pq

For n=4

$P(X=1) = pq^3$

Required probability = $pq + pq^3 + pq^5 + \ldots = 2/5$

$2/5= pq(1+ q^2 +q^4 + \ldots)$

$\frac{2}{5}=pq\left(\frac{1}{1-q^2}\right)$

$\frac{2}{5}\ =\frac{p(1-p)}{1-(1-p)^2}$

$\frac{2}{5}\ =\frac{1-p}{2-p}$

4-2p=5-5p

3p= 1

P =1/3

14. Question

Mark the correct alternative in the following:

If X follows a binomial distribution with parameters n = 8 and p = 1/2, then P (|X – 4)| ≤ 2) equals.

A. $\frac{118}{128}$

B. $\frac{119}{128}$

C. $\frac{117}{128}$

D. none of these

Answer

Given:

n=8

p= 1/2

P (|X – 4)| ≤ 2)=?

Since |X – 4)| ≤ 2

i.e. (x-4) ≤ 2

OR (4-X) ≤ 2

x ≤ 6

OR 2 ≤ x

P (|X – 4)| ≤ 2)= P (2≤x≤ 6)

=p(x=2)+ p(x=3)+ p(x=4)+ p(x=5)+ p(x=6)

P(x=r)= $^nC_r\ p^r\ q^{n-r}$

P (2≤x≤ 6) = $^8C_2\ p^2\ q^6 + {}^8C_3\ p^3\ q^5 + {}^8C_4\ p^4\ q^4 + {}^8C_5\ p^5\ q^3 + {}^8C_6\ p^6\ q^2$

$=p^8\ [{}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6\ ]$

$=\frac{1}{2^8}\ [28\ +\ 56+70+56+28\,]$

$=\frac{238}{256}$

$=\frac{119}{128}$

15. Question

Mark the correct alternative in the following:

If X follows a binomial distribution with parameters n = 100 and p = 1/3, then P (X = r) is

A. 32

B. 34

C. 33

D. 31

Answer

Given:

n=100

p = 1/3

P(x=r)= $^nC_r\ p^r\ q^{n-r}$

Its maximum value is $\frac{n+1}{p}$

$= \frac{100}{3}$

Since $\frac{n+1}{p}$ is not an integer ; its value is

= 33.67

i.e. 33

16. Question

Mark the correct alternative in the following:

A fair die is tossed eight times. The probability that a third six is observed in the eight throw is

A. $\frac{{}^7C_2 \times 5^5}{6^7}$

B. $\frac{{}^7C_2 \times 5^5}{6^8}$

C. $\frac{{}^7C_2 \times 5^5}{6^6}$

D. none of these

Answer

Given:

Total outcomes= 6^8

Fix Third 6 is at 8^{th} row.

So other two 6 have to be placed at any of the remaining 7 places=${}^7C_2$

For remaining 5 places we have 1,2,3,4&5

So total number of favourable cases= $\frac{{}^7C_2 \times 5^5}{6^7}$

17. Question

Mark the correct alternative in the following:

Fifteen coupons are numbered 1 to 15. Seven coupons are selected at random, one at a time with replacement. The probability that the largest umber appearing on a selected coupon is 9, is

A. $\left(\frac{3}{5}\right)^7$

B. $\left(\frac{1}{15}\right)^7$

C. $\left(\frac{8}{15}\right)^7$

D. none of these

Answer

Given:

P(selected coupon bears number $\leq$ 9)= p

p=9/15

= 3/5

P(selected coupon bears number $\leq$ 9) = p(all coupons bear number$\leq$ 9) – p(selected coupon bears number $\leq$ 8)

n=7

p(that all coupon bear number $\leq$ 9)

P(x=7)= ${}^7C_7\, p^7\, q^0$

= p^7

$= \left(\frac{3}{5}\right)^7$

P(that the selected coupon bears number $\leq$ 9)

P= 8/15

P(x=7)= ${}^7C_7\, p^7\, q^0$

= p^7

$= \left(\frac{8}{15}\right)^7$

Required probability $= \left(\frac{3}{5}\right)^7 - \left(\frac{8}{15}\right)^7$

18. Question

Mark the correct alternative in the following:

A five-digit number is written down at random. The probability that the number is divisible by 5 and no two consecutive digits are identical, is

A. $\frac{1}{5}$

B. $\frac{1}{5}\left(\frac{9}{10}\right)^3$

C. $\left(\frac{3}{5}\right)^4$

D. none of these

Answer

Given:

A 5-digit number is divisible by 5 if it ends with 0 0r 5.

So at ones place we have two digits out of 10$\rightarrow$ $\frac{2}{10}$ (one at a time)

Remaining places we have 9 digits out of 10 $\rightarrow$ $\frac{9}{10}\times\frac{9}{10}\times\frac{9}{10}$

For first place ; we have only 8 digits out of 9 because 0 cannot be at first place$\rightarrow$ $\frac{8}{9}$

So probability $= \frac{8}{9}\times\frac{9}{10}\times\frac{9}{10}\times\frac{9}{10}\times\frac{2}{10}$

$= \left(\frac{3}{5}\right)^4$

19. Question

Mark the correct alternative in the following:

A coin is tossed 10 times. The probability of getting exactly six heads is

A. $\frac{512}{513}$

B. $\frac{105}{512}$

C. $\frac{100}{153}$

D. $^{10}C_6$

Answer

Given:

P(H)=p(T)=1/2

n=10

p(x=6)=?

P(x=r)= $^nC_r\ p^r\ q^n$

P(x=6)= $10_{C_6}\ \left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4$

$=\frac{10!}{6!\times 4!}\times\left(\frac{1}{2}\right)^{10}$

$=\frac{210}{1024}$

$=\frac{105}{512}$

20. Question

Mark the correct alternative in the following:

The mean and variance of a binomial distribution are 4 and 3 respectively, then the probability of getting exactly six successes in this distribution, is

A. $^{16}C_6\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^6$

B. $^{16}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10}$

C. $^{12}C_6\left(\frac{1}{20}\right)\left(\frac{3}{4}\right)^6$

D. $^{12}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^6$

Answer

Given:

Mean=4

Variance=3

P(x=6)=?

P(x=r)= $^nC_r\ p^r\ q^{n-r}$

4(1-p)=3

$1-p=\frac{3}{4}$

=q

$p=\frac{1}{4}$

np=4

n=16

P(x=r)= $^{n}C_r\ p^r\ q^{n-r}$

P(x=6)= $^{16}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10}$

21. Question

Mark the correct alternative in the following:

In a binomial distribution, the probability of getting success is 1/4 and standard deviation is 3. Then, its means is

A. 6

B. 8

C. 12

D. 10

Answer

Given:

P(x=1)=1/4=p

q=3/4

Standard deviation =3

Mean=?

$\sqrt{np(1-p)}$= 3

$n\times\frac{1}{4}\times\frac{3}{4} = 9$

$n = \frac{144}{3}$

mean = np

= 12

22. Question

Mark the correct alternative in the following:

A coin is tossed 4 times. The probability that at least one head turns up, is

A. $\frac{1}{6}$

B. $\frac{2}{16}$

C. $\frac{14}{16}$

D. $\frac{15}{16}$

Answer

Given:

n=4

p(getting atleast one head)=?

• 0 head : HHHH

• 1 head : HTTT

THTT

TTHT

TTTH

• 2 heads :HHTT

HTHT

HTTH

THHT

THTH

TTHH

• 3 heads: HHHT

HHTH

HTHH

THHH

• 4 heads : HHHH

Total ouctomes = 16

Outcomes containing atleast 1 head = 15

P(getting atleast 1 head)= $\frac{15}{16}$

23. Question

Mark the correct alternative in the following:

For a binomial variate X, if n = 3 and P (X = 1) = 8 P(X = 3), then p =

A. 4/5

B. 1/5

C. 1/3

D. 2/3

Answer

Given:

n=3

P (X = 1) = 8 P (X = 3)

P =?

P(x=r)= $^nC_r\ p^r\ q^{n-r}$

P(x=1)= $^nC_1\ p^1\ q^{n-1}$

$=3\times p\times q^{2}$ (1)

P(x=3)= $^3C_3\ p^3\ q^0$

$=1\times p^{3}$(2)

Put values in the equations ,we have:

$3\times p\times q^2 = 8\ p^3$

$3\ (1-p)^2 = 8p^2$

$3+3p^2-6p = 6p^2$

$5p^2+6p-3=0$

$$P = \frac{-6 \pm \sqrt{36+60}}{10}$$

$$P = \frac{-6 \pm \sqrt{96}}{10}$$

$$P = \frac{-6 \pm 4\sqrt{6}}{10}$$

$$P = \frac{-3 \pm 2\sqrt{6}}{5}$$

24. Question

Mark the correct alternative in the following:

A coin is tossed n times. The probability of getting at least once is greater than 0.8. Then, the least value of n, is

A. 2

B.3

C. 4

D. 5

Answer

Given:

P(H)=P(T)=1/2

P(H)= $H^1T^{n-1}+H^2T^{n-2}+H^3T^{n-3}+.....+H^{n-1}T^1+H^n$

Required probability = p= 1- $\left(\frac{1}{2}\right)^n$

P≥0.8

$$1 - \left(\frac{1}{2}\right)^n \geq 0.8$$

$$0.2 \geq \left(\frac{1}{2}\right)^n$$

$$\frac{1}{5} \geq \left(\frac{1}{2}\right)^n$$

For n= 2

$$\left(\frac{1}{2}\right)^n = 0.25$$

For n=3

$$\left(\frac{1}{2}\right)^n = 0.125$$

So, n=3

25. Question

Mark the correct alternative in the following:

The probability of selecting a male or a female is same. If the probability that in an office of n persons (n – 1) males being selected is $\frac{3}{2^{10}}$, the value of n is

A. 5

B. 3

C. 10

D. 12

Answer

Given:

P(females)=p(males)=1/2

p(x=n-1) = $\frac{3}{2^{10}}$

n=?

p(x=n-1) = ${}^{n}C_{n-1}\, p^{n-1}\, q^{1}$

Since p=q

p(x=n-1)= n $p^{n-1}q$

=n $p^{n-1}p$

=np^{n}

=n$\left(\frac{1}{2}\right)^{n}$

$$n\left(\frac{1}{2}\right)^{n} = \frac{3}{2^{10}}$$

$$n\left(\frac{1}{2}\right)^{n} = 3 \times \frac{2^{2}}{2^{12}}$$

$$n\left(\frac{1}{2}\right)^{n} = 12\left(\frac{1}{2}\right)^{12}$$

comparing LHS anr RHS

we have,

n=12

26. Question

Mark the correct alternative in the following:

A box contains 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

A. $\left(\frac{9}{10}\right)^{5}$

B. $\frac{1}{2}\left(\frac{9}{10}\right)^{4}$

C. $\frac{1}{2}\left(\frac{9}{10}\right)^{5}$

D. $\left(\frac{9}{10}\right)^{5} + \frac{1}{2}\left(\frac{9}{10}\right)^{4}$

Answer

Given:

Total pens=100

Defective pens=10

n=5

P(atmost one non-defective pen)=?

P(defective pens) $= \frac{10}{100}$

$= \frac{1}{10}$

=p

$q = \frac{9}{10}$

P(atmost one non-defective pen)=p(x<10)

=p(x=0)+p(x=1)

$= {}^5C_0\, p^0\, q^5 + 5pq^4$

$= \left(\frac{9}{10}\right)^5 + 5\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^4$

$= \left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4$

27. Question

Mark the correct alternative in the following:

Suppose a random variable X follows the binomial distribution with parameters n and p, where $0 < p < 1$. If $\frac{P(X=r)}{P(X=n-r)}$ is independent of n and r, then p equals.

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{5}$

D. $\frac{1}{7}$

Answer

Given:

$\frac{P(X=r)}{P(X=n-r)}$

P=?

$P(x=r) = {}^nC_r\, p^r\, q^{n-r}$

$= \frac{n!}{r!(n-r)!} \times p^r \times q^{n-r}$

$P(x=n-r) = {}^nC_{n-r}\, p^{n-r}\, q^{n-(n-r)}$

$= {}^nC_r\, p^{n-r}\, q^r$

$= \frac{n!}{r!(n-r)!} \times p^{n-r} \times q^r$

Put values in equation, we have:

$= \frac{\frac{n!}{r!(n-r)!} \times p^r \times q^{n-r}}{\frac{n!}{r!(n-r)!} \times p^{n-r} \times q^r}$

$= \frac{p^{2r-n}}{q^{2r-n}}$

$= \left(\frac{p}{q}\right)^{2r-n}$

$= \left(\frac{p}{1-p}\right)^{2r-n}$

1-p=p

1=2p

P=1/2

28. Question

Mark the correct alternative in the following:

The probability that is person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

A. ${}^5C_4\ (0.7)^4\ (0.3)$

B. ${}^5C_1\ (0.7)\ (0.3)^4$

C. ${}^5C_4\ (0.7)\ (0.3)^4$

D. $(0.7)^4\ (0.3)$

Answer

Given:

n=5

q=0.3

P(x=4)=?

P=1-0.3=0.7

$P(x=r)= {}^nC_r\ p^r\ q^{n-r}$

$P(x=4)= {}^5C_4\ p^4\ q$

$={}^5C_4\ (0.7)^4\ (0.3)$

29. Question

Mark the correct alternative in the following:

Which one is not a requirement of a binomial distribution?

A. There are 2 outcomes for each trial

B. There is a fixed number of trials

C. The outcomes must be dependent on each other.

D. The probability of success must be the same for all the trials.

Answer

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30. Question

Mark the correct alternative in the following:

The probability of guessing correctly at least 8 out of 10 answers of a true false type examination is

A. $\frac{7}{64}$

B. $\frac{7}{128}$

C. $\frac{45}{1024}$

D. $\frac{7}{41}$

Answer

Given:

n=10

p(true)=p(false)=1/2

$p(x \geq 8) = p(x=8) + p(x=9) + p(x=10)$

$= {}^{10}C_8\, p^8\, q^2 + {}^{10}C_9\, p^9\, q^1 + {}^{10}C_{10}\, p^{10}\, q^0$

$= p^{10}\,(45+ 10+1)$

$= \frac{51}{2^{10}}$

$= \frac{7}{128}$