9. Continuity

Exercise 9.1

1. Question

Test the continuity of the following function at the origin :

$$f(x)=\begin{cases}\dfrac{x}{|x|} & x\neq 0\\ 1 & x=0\end{cases}$$

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x), we can say that if we plot the coordinates (x, f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$$

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means:–

Limiting the value of the left neighbourhood of x = c also called left–hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds, function is continuous else it is discontinuous.

2. The idea of modulus function |x |: You can think this function as a machine in which you can give it any real no. as an input and it returns its absolute value i.e. if positive is entered it returns the same no and if negative is entered it returns the corresponding positive no.

Eg:– |2| = 2 ; |–2| = –(–2) = 2

Similarly, we can define it for variable x, if x ≥ 0 |x| = x

If x < 0 |x| = (–x)

$$\therefore |x| = \begin{cases}-x, x<0\\ x, x\geq 0\end{cases}$$

Now we are ready to solve the question –

We need to check the continuity at the origin (0,0) i.e. we will check it at x=0.

So, we need to see whether at x=0,

IF, LHL = RHL = f(0)

i.e. $\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

For this question c = 0

f(x) can be rewritten using the concept of modulus function as :

$$f(x) = \begin{cases} \frac{x}{x} = 1 \; x > 0 \\ \frac{x}{-x} = -1 \; x < 0 \\ \text{Indeterminate form } x = 0 \end{cases}$$ ...... Equation 1

NOTE : $\left(\frac{0}{0}\right)$ is called indeterminate form as its exact value can't be determined

Now we have three different expressions for different conditions of x.

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \frac{-h}{-(-h)} = \frac{-h}{h} = -1$ using eqn 1

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \frac{h}{h} = 1$ using eqn 1

LHL ≠ RHL so we even don't need to check for f(0)

∴ We can easily say that f(x) is discontinuous at the origin.

2. Question

A function f(x) is defined as $f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & ; \text{if } x \neq 3 \\ 5 & ; \text{if } x = 3 \end{cases}$ Show that f(x) is continuous at x = 3.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x), we can say that if we plot the coordinates (x, f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means:–

Limiting the value of the left neighborhood of x = c also called left–hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighborhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Let's solve :

To prove function is continuous at x=3 we need to show LHL = RHL = f(c) As continuity is to be checked at x = 3, therefore c=3. (in equation 1)

$$f(x) = \begin{cases} \frac{x^2-x-6}{x-3} & ; \text{if} x \neq 3 \\ 5 & ; \text{if} x = 3 \end{cases}$$ ......Eqn 2

∴ f(3) = 5 using eqn 2

LHL = $\lim\limits_{h\to 0} f(3-h)$

Using equation 2 –

$$= \lim_{h\to 0}\frac{(3-h)^2-(3-h)-6}{3-h-3} = \lim_{h\to 0}\frac{9+h^2-6h-3+h-6}{-h} = \lim_{h\to 0}\frac{h^2-5h}{-h}$$

$$= \lim_{h\to 0}\frac{h(h-5)}{-h} = \lim_{h\to 0}(5-h) = 5-0 = 5$$

RHL = $\lim\limits_{h\to 0} f(3+h)$

$$= \lim_{h\to 0}\frac{(3+h)^2-(3+h)-6}{3+h-3} = \lim_{h\to 0}\frac{9+h^2+6h-3-h-6}{h} = \lim_{h\to 0}\frac{h^2+5h}{h}$$

$$= \lim_{h\to 0}\frac{h(h+5)}{+h} = \lim_{h\to 0}(5+h) = 5+0 = 5$$

Clearly, LHL = RHL = f(3) = 5

∴ f(x) is continuous at x=3

3. Question

A function f(x) is defined as $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & ; \text{if } x \neq 3 \\ 6 & ; \text{if } x = 3 \end{cases}$

Show that f(x) is continuous at x = 3.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x), we can say that if we plot the coordinates (x, f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim\limits_{h\to 0} f(c-h) = \lim\limits_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means:–

Limiting the value of the left neighborhood of x = c also called left–hand limit LHL$\{\text{i.e } \lim\limits_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighborhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim\limits_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds, function is continuous else it is discontinuous.

Let’s solve :

To prove function is continuous at x=3, we need to show LHL = RHL = f(c) As continuity is to be checked at x = 3 therefore c=3 (in equation 1)

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & ;\text{if} x \neq 3 \\ 6 & ;\text{if} x = 3 \end{cases}$$ ...... eqn 2

From eqn 2 :

f(3) = 6

LHL = $\lim_{h\to 0} f(3-h)$

Using equation 2 –

$$= \lim_{h\to 0} \frac{(3-h)^2-9}{3-h-3} = \lim_{h\to 0} \frac{9+h^2-6h-9}{-h} = \lim_{h\to 0} \frac{h^2-6h}{-h}$$

$$= \lim_{h\to 0} \frac{h(h-6)}{-h} = \lim_{h\to 0}(6-h) = 6-0 = 6$$

RHL = $\lim_{h\to 0} f(3+h)$

$$= \lim_{h\to 0} \frac{(3+h)^2-9}{3+h-3} = \lim_{h\to 0} \frac{9+h^2+6h-9}{h} = \lim_{h\to 0} \frac{h^2+6h}{h}$$

$$= \lim_{h\to 0} \frac{h(h+6)}{+h} = \lim_{h\to 0}(6+h) = 6+0 = 6$$

Clearly, LHL = RHL = f(3) = 6

∴ f(x) is continuous at x=3

4. Question

If $f(x) = \begin{cases} \frac{x^2-1}{x-1} & ;\text{ for } x \neq 1 \\ 2 & ;\text{ for } x = 1 \end{cases}$. Find whether f(x) is continuous at x = 1.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x), we can say that if we plot the coordinates (x, f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds, function is continuous else it is discontinuous.

Let’s solve :

To check whether function is continuous at x=3 we need to check whether LHL = RHL = f(c)

As continuity is to be checked at x = 1 therefore c=1 (in equation 1)

As, $f(x) = \begin{cases} \frac{x^2-1}{x-1} & ; \text{for } x \neq 1 \\ 2 & ; \text{for } x = 1 \end{cases}$ ....eqn 2

From eqn 2 :

f(1) = 2

LHL = $\lim_{h\to 0} f(1-h)$

Using equation 2 –

$$= \lim_{h\to 0} \frac{(1-h)^2-1}{1-h-1} = \lim_{h\to 0} \frac{1+h^2-2h-1}{-h} = \lim_{h\to 0} \frac{h^2-2h}{-h}$$

$$= \lim_{h\to 0} \frac{h(h-2)}{-h} = \lim_{h\to 0}(2-h) = 2-0 = 2$$

RHL = $\lim_{h\to 0} f(1+h)$

$$= \lim_{h\to 0} \frac{(1+h)^2-1}{1+h-1} = \lim_{h\to 0} \frac{1+h^2+2h-1}{h} = \lim_{h\to 0} \frac{h^2+2h}{h}$$

$$= \lim_{h\to 0} \frac{h(h+2)}{+h} = \lim_{h\to 0}(2+h) = 2+0 = 2$$

Clearly, LHL = RHL = f(1) = 2

∴ f(x) is continuous at x=1

5. Question

If $f(x) = \begin{cases} \frac{\sin 3x}{x} & ; \text{when } x \neq 0 \\ 1 & ; \text{when } x = 0 \end{cases}$. Find whether f(x) is continuous at x = 0.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x), we can say that if we plot the coordinates (x, f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means:–

Limiting the value of the left neighbourhood of x = c also called left–hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\{\text{i.e } \lim_{h\to 0} f(c+h)\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds, function is continuous else it is discontinuous.

2. The idea of sandwich theorem – This theorem also known as squeeze theorem that you may have encountered in your class 11 in limits chapter suggests that

If I be an interval having a point as a limit point. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have{\displaystyle \lim _{x→ a}g(x)=\lim _{x→ a}h(x)=L.}{\displaystyle \lim _{x→ a}f(x)=L.}

g(x) ≤ f(x) ≤ h(x) and also

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = K(say)$

Then , $\lim_{x\to a} f(x) = K$ We say that f(x) is squeezed between g(x) and h(x) or you can assume it like sandwich.

$\frac{\sin x}{x}$ is also squeezed between 1 when $x \to 0$

$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$ equation 2

NOTE : denominator in the above limit should be exactly same as that of content in sine function

Eg : $\lim_{x\to 3} \frac{\sin(3-x)}{3-x} = 1$

Let's solve :

To check whether function is continuous at x=0 we need to check whether LHL = RHL = f(c)

As continuity is to be checked at x = 0 therefore c=0.

As, $f(x) = \begin{cases} \frac{\sin 3x}{x} & ;\text{when} x \neq 0 \\ 1 & ;\text{when} x = 0 \end{cases}$. Equation 3

Clearly, f(0) = 1 using eqn 3

LHL = $\lim_{h\to 0} f(0-h)$

Using equation 3 –

$= \lim_{h\to 0} \frac{\sin 3(0-h)}{0-h} = \lim_{h\to 0} \frac{\sin(-3h)}{-h} = \lim_{h\to 0} \frac{\sin 3h}{h}$ [∵ sin(–θ) = – (sin θ)]

To find its limit we need to think of Sandwich theorem as the form looks similar but term in denominator is not exactly same as that in the content of sine function.

Ok no problem lets bring it there but if we put 3 in denominator we need to put a 3 in numerator too so that both can be cancelled.

Let's do it:

$\therefore$ LHL $= \lim_{h\to 0} 3 * \frac{\sin 3h}{3h} = 3 * \lim_{h\to 0} \frac{\sin 3h}{3h} = 3 * 1 = 3$

RHL = $\lim_{h\to 0} f(0+h)$

Using equation 3 –

$= \lim_{h\to 0} \frac{\sin 3(0+h)}{0+h} = \lim_{h\to 0} \frac{\sin(3h)}{h} = \lim_{h\to 0} \frac{\sin 3h}{h}$

Again using sandwich theorem as we used while finding LHL

$\therefore$ RHL $= \lim_{h\to 0} 3 * \frac{\sin 3h}{3h} = 3 * \lim_{h\to 0} \frac{\sin 3h}{3h} = 3 * 1 = 3$

Thus,

LHL = RHL = 3

But, f(0) = 1 by definition of f(x) i.e. equation 3

∴ LHL = RHL ≠ f(0)

∴ condition for function to be continuous is not satisfied

∴ f(x) is discontinuous at x = 0

6. Question

If $f\left(x\right)=\begin{cases} e^{1/x} & ;\text{ if } x \neq 0 \\ 1 & ;\text{ if } x = 0 \end{cases}$. Find whether f is continuous at x = 0.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

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If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Let's solve :

To check whether function is continuous at x=0 we need to check whether LHL = RHL = f(c)

As continuity is to be checked at x = 0 therefore c=0.

As, $f(x) = \begin{cases} e^{1/x} & ;\text{if} x \neq 0 \\ 1 & ;\text{if} x = 0 \end{cases}$. ...... say it equation 2

Clearly, f(0) = 1 using eqn 3

LHL = $\lim_{h\to 0} f(0-h)$

Using equation 3 –

$= \lim_{h\to 0} e^{\frac{1}{(0-h)}} = \lim_{h\to 0} e^{\frac{-1}{h}} = e^{-\infty} = 0$ [∵ h is very small , $\frac{-1}{h}$ is very large '–ve']

RHL = $\lim_{h\to 0} f(0+h)$

Using equation 3 –

$= \lim_{h\to 0} e^{\frac{1}{(0+h)}} = \lim_{h\to 0} e^{\frac{1}{h}} = e^{\infty} = \infty$ [∵ h is very small , $\frac{1}{h}$ is very large '+ve']

Clearly, LHL ≠ RHL ≠ f(0)

∴ We can easily say that f(x) is discontinuous at origin.

7. Question

Let $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2} & ; \text{when } x \neq 0 \\ 1 & ; \text{when } x = 0\end{cases}$. Show that f(x) is discontinuous at x = 0.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

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where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{$i.e $\lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

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g(x) ≤ f(x) ≤ h(x) and also

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = K(\text{say})$

Then , $\lim_{x\to a} f(x) = K$ We say that f(x) is squeezed between g(x) and h(x) or you can assume it like sandwich.

$\frac{\sin x}{x}$ is also squeezed between 1 when $x \to 0$

$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$ equation 2

NOTE : denominator in the above limit should be exactly same as that of content in sine function

Eg : $\lim_{x\to 3} \frac{\sin(3-x)}{3-x} = 1$

Let’s solve :

To check whether function is continuous at x=0 we need to check whether LHL = RHL = f(c)

As continuity is to be checked at x = 0 therefore c=0.

As, $f(x)=\begin{cases}\frac{1-\cos x}{x^2} & ;\text{when} x\neq 0\\ 1 & ;\text{when} x=0\end{cases}$. ...... Equation 3

Clearly, f(0) = 1 using eqn 3

LHL = $\lim_{h\to 0} f(0-h)$

Using equation 3 –

$= \lim_{h\to 0}\frac{1-\cos(0-h)}{(0-h)^2} = \lim_{h\to 0}\frac{1-\cos h}{h^2}$ [∵ cos(–θ) = (cosθ)]

Whenever you see trigonometric term in numerator and corresponding or similar content of term in numerator try to apply sandwich theorem in finding apply if you are unable to do so think of alternative.

Here we have cos term but to apply the theorem we need to have sin term in numerator.

Can we bring it ?

Yes, we know that :

(1 – cos θ) = 2$\sin^2(\theta/2)$

We have,

LHL $=\lim_{h\to 0}\frac{2\sin^2\left(\frac{h}{2}\right)}{h^2}$ = 2$\lim_{h\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{h}\right]^2$

To find its limit we need to think of Sandwich theorem as the form looks similar but term in denominator is not exactly same as that in the content of sine function.

Ok no problem, let's bring it there but if we multiply 1/2 in denominator we need to multiply 1/2 in numerator too so that both can be cancelled.

Let's do it:

∴ LHL = $2\lim_{h\to 0}\left[\frac{\frac{1}{2}*\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]^2$

$=\frac{1}{4}*2\lim_{h\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]^2=\frac{1}{2}$

RHL = $\lim_{h\to 0} f(0+h)$

Using equation 3 –

$= \lim_{h\to 0}\frac{1+\cos(0+h)}{(0+h)^2} = \lim_{h\to 0}\frac{1+\cos h}{h^2}$

It is not taking $\frac{0}{0}$ or any other indeterminate form.

So, RHL = $\lim_{h\to 0}\frac{1+\cos h}{h^2} = \left(\frac{1+\cos 0}{0}\right) = \frac{2}{0} = \infty$

Thus,

LHL ≠ RHL

∴ LHL ≠ RHL ≠ f(0)

∴ condition for function to be continuous is not satisfied

∴ f(x) is discontinuous at x = 0

8. Question

Show that $f(x)=\begin{cases}\frac{x-|x|}{2} & , \text{when } x\neq 0\\ 2 & , \text{when } x=0\end{cases}$ is discontinuous at x = 0.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

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If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ..Equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

2. Idea of modulus function |x| : You can think this function as a machine in which you can give it any real no. as an input it returns it absolute value i.e if positive is entered it returns the same no and if negative is entered it returns the corresponding positive no.

Eg:– |2| = 2 ; |–2| = –(–2) = 2

Similarly, we can define it for variable x, if x ≥ 0 |x| = x

If x < 0 |x| = (–x)

$\therefore |x| = \begin{cases}-x, x<0\\ x, x\geq 0\end{cases}$

Now we are ready to solve the question –

We need to check the continuity at origin (0,0) i.e we will check it at x=0.

So, we need to see whether at x=0,

LHL = RHL = f(0)

i.e. $\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

For this question c = 0

f(x) can be rewritten using the concept of modulus function as :

$f(x) = \begin{cases}\frac{x-x}{2} = 0\,;\, x>0\\ \frac{x-(-x)}{2} = \frac{2x}{2} = x\,;\, x<0\\ 2\,;\, x=0\end{cases}$ ...... Equation 2

Now we have three different expressions for different conditions of x.

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}(-h) = 0$ using eqn 2

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \frac{h-h}{2} = 0$ using eqn 2

f(0) = 2

Clearly, LHL = RHL ≠ f(0)

∴ We can easily say that f(x) is discontinuous at origin.

9. Question

Show that $f(x) = \begin{cases} \frac{|x-a|}{x-a} & , \text{when } x \neq a \\ 1 & , \text{when } x = a \end{cases}$ is discontinuous at x = a.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ......Equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e} \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\{\text{i.e} \lim_{h\to 0} f(c+h)\}$ and both must be equal to the value of f(x) at x=c f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

2. Idea of modulus function |x| : You can think this function as a machine in which you can give it any real no. as an input it returns it absolute value i.e if positive is entered it returns the same no and if negative is entered it returns the corresponding positive no.

Eg:– |2| = 2 ; |–2| = –(–2) = 2

Similarly, we can define it for variable x, if x ≥ 0 |x| = x

If x < 0 |x| = (–x)

∴ $|x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$

Now we are ready to solve the question –

We need to check the continuity at x = a

So, we need to see whether at x=a,

LHL = RHL = f(a)

i.e. $\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$

For this question c = a

We have:

$$f(x) = \begin{cases} \frac{|x-a|}{x-a} & \text{,when } x \neq a \\ 1 & \text{,when } x = a \end{cases}$$

Using the concept of modulus function we can redefine the above equation as given below :

$$f(x) = \begin{cases} \frac{x-a}{x-a}\ ;\text{when } (x-a) > 0 \text{ or } x > a \\ \frac{-(x-a)}{x-a}\ ;\text{when } (x-a) < 0 \text{ or } x < a \\ 1\ ;\text{when } x = a \end{cases} \text{......Equation 2}$$

Clearly,

f(a) = 1

LHL = $\lim_{h\to a} f(a-h) = \lim_{h\to a}\left(\frac{-(a-h-a)}{a-h-a}\right) = \lim_{h\to a}\frac{h}{-h} = \frac{a}{-a} = -1$ using eqn 2

RHL = $\lim_{h\to a} f(a+h) = \lim_{h\to a} f\left(\frac{a+h-a}{a+h-a}\right) = \lim_{h\to a}\frac{h}{h} = \frac{a}{a} = 1$ using eqn 2

Clearly, LHL ≠ RHL

∴ We can easily say that f(x) is discontinuous at origin.

10 A. Question

Discuss the continuity of the following functions at the indicated point(s).

$$f(x) = \begin{cases} |x|\cos\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases} \text{ at } x = 0$$

Answer

Idea to solve problem :

Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ......Equation 1

where h is a very small positive no (can assume h = 0.00000000001 or even smaller)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Let’s Solve the problem now:

we need to check continuity at x = 0 for given function

$f(x) = \begin{cases} |x|\cos\left(\frac{1}{x}\right) & ,x \neq 0 \\ 0 & ,x = 0 \end{cases}$ ......equation 2

∴ we need to check RHL ,LHL and value of function at x = 0

Clearly,

f(0) = 0 [from equation 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} |-h|\cos\left(\frac{1}{-h}\right) = \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$ [∵ cos (–θ) = cos θ]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} |h|\cos\left(\frac{1}{h}\right) = \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$ [from eqn 2]

Why $\lim_{h\to 0} h\cos\left(\frac{1}{h}\right) = 0$?

Since whatever is value of h, cos(1/h) is goimg to range from –1 to 1

As h→ 0 i.e. h is approximately 0

∴ 0*some finite quantity is equal to 0.

Clearly, LHL = RHL = f(0)

∴ f(x) is continuous at x = 0

10 B. Question

Discuss the continuity of the following functions at the indicated point(s).

$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ at x = 0

Answer

In this problem we need to check continuity at x = 0

Given function is

$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & ,x \neq 0 \\ 0 & ,x = 0 \end{cases}$ at x = 0 ...... Equation 2

∴ we need to check LHL, RHL and value of function at x = 0 (for idea and meaning of continuity refer to Q10(i))

Clearly,

f(0) = 0 [from equation 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} (-h)^2\sin\left(\frac{1}{-h}\right) = \lim_{h\to 0} -(h^2)\sin\left(\frac{1}{h}\right) = 0$ [∵ sin(–θ) = sinθ]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} (h)^2\sin\left(\frac{1}{h}\right) = \lim_{h\to 0} (h^2)\sin\left(\frac{1}{h}\right) = 0$ [using eqn 2]

Why $\lim_{h\to 0} h^2\sin\left(\frac{1}{h}\right) = 0$?

Since whatever is value of h, sin(1/h) is going to range from –1 to 1

As h→ 0 i.e. h is approximately 0

∴ 0*some finite quantity is equal to 0.

Clearly, LHL = RHL = f(0)

∴ f(x) is continuous at x = 0

10 C. Question

Discuss the continuity of the following functions at the indicated point(s).

$$f(x)=\begin{cases}(x-a)\sin\left(\dfrac{1}{x-a}\right) & , x\neq a\\ 0 & , x=a\end{cases}$$ atx = a

Answer

In this problem we need to check continuity at x = a

Given function is

$f(x)=\begin{cases}(x-a)\sin\left(\frac{1}{x-a}\right) & ,x\neq a\\ 0 & ,x=a\end{cases}$ at x = a ...... Equation 2

∴ we need to check LHL, RHL and value of function at x = a (for idea and meaning of continuity refer to Q10(i)

Clearly,

f(a) = 0 [from equation 2]

LHL = $\lim_{h\to 0} f(a-h) = \lim_{h\to 0}(a-h-a)\sin\left(\frac{1}{a-h-a}\right) = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$ [∵ sin(–θ) = sinθ]

RHL = $\lim_{h\to 0} f(a+h) = \lim_{h\to 0}(a+h-a)\sin\left(\frac{1}{a+h-a}\right) = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$ [from eqn 2]

Why $\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$?

Since whatever is value of h, sin(1/h) is going to range from –1 to 1

As h→ 0 ,i.e. approximately 0

∴ 0*some finite quantity is equal to 0.

Clearly, LHL = RHL = f(a)

∴ f(x) is continuous at x = 0

10 D. Question

Discuss the continuity of the following functions at the indicated point(s).

$$f(x)=\begin{cases}\dfrac{x^x-1}{\log(1+2x)} & , \text{if } x\neq 0\\ 7 & , \text{if } x=0\end{cases}$$ at x = 0

Answer

In this problem we need to check continuity at x = 0

Given function is

$f(x)=\begin{cases}\frac{e^x-1}{\log(1+2x)} & ,\text{if } x\neq 0\\ 7 & ,\text{if } x=0\end{cases}$ at x = 0

∴ we need to check LHL, RHL and value of function at x = 0 (for idea and meaning of continuity refer to Q10(i))

1. NOTE : Idea of logarithmic limit and exponential limit –

$\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$ ......equation 1

$\lim_{x\to 0}\frac{e^x-1}{x} = 1$ ...... equation 2

You must have read such limits in class 11. You can verify these by expanding log(1+x) and e^x in its taylor form.

Numerator and denominator conditions also hold for this limit like sandwich theorem.

E.g : $\lim_{x\to 0}\frac{\log(1+2x)}{2x} = 1$

But , $\lim_{x\to 0}\frac{\log(1+2x)}{x} \neq 1$ as denominator does not have 2x

Now we are ready to solve the problem.

Given function is

$f(x) = \begin{cases} \frac{e^x-1}{\log(1+2x)} & ,\text{if} x \neq 0 \\ 7 & ,\text{if} x = 0 \end{cases}$ at x = 0 ...... Equation 3

Clearly,

f(0) = 7 [from equation 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$ [putting x = –h in equation 3]

$= \lim_{h\to 0}\frac{e^{-h}-1}{\log 1+2(-h)} = \lim_{h\to 0}\frac{e^{-h}-1}{\log(1-2h)}$

Using logarithmic and exponential limit as explained above, we have:

LHL = $\frac{1}{2}\lim_{h\to 0}\frac{\frac{(e^{-h}-1)}{-h}}{\frac{\log(1-2h)}{-2h}} = \frac{1}{2}$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$ [putting x = h in equation 3]

$= \lim_{h\to 0}\frac{e^{h}-1}{\log 1+2h} = \lim_{h\to 0}\frac{e^{h}-1}{\log(1+2h)}$

Using logarithmic and exponential limit as explained above, we have:

RHL = $\frac{1}{2}\lim_{h\to 0}\frac{\frac{(e^{h}-1)}{h}}{\frac{\log(1+2h)}{2h}} = \frac{1}{2}$

Thus, LHL = RHL ≠ f(0)

∴ f(x) is discontinuous at x = 0

10 E. Question

Discuss the continuity of the following functions at the indicated point(s).

$f(x) = \begin{cases} \frac{1-x^n}{1-x} & , x \neq 1 \\ n-1 & , x = 1 \end{cases}$ $n \in N$ at x = 1

Answer

In this problem we need to check continuity at x = a

Given function is

$f(x) = \begin{cases} \frac{1-x^n}{1-x} & ,x \neq 1 \\ n-1 & ,x = 1 \end{cases} n \in N$ at x = 1 ...... Equation 2

∴ we need to check LHL, RHL and value of function at x = a (for idea and meaning of continuity refer to Q10(i))

Clearly,

f(1) = n–1 [from equation 2]

LHL = $\lim_{h \to 0} f(1-h) = \lim_{h \to 0} \frac{1-(1-h)^n}{1-(1-h)} = \lim_{h \to 0} \frac{1-(1-h)^n}{h}$

Using binomial theorem – $(1-h)^n = \sum_{k=0}^{n} \binom{n}{k} (-h)^k 1^{n-k}$

$(1-h)^n = 1 - nh + \binom{n}{2}h^2 - \ldots\ldots$

LHL = $\lim_{h \to 0} \frac{1-1+nh-\binom{n}{2}h^2+\cdots \text{higher deg terms}}{h} = \lim_{h \to 0}\{n - \binom{n}{2}h + \binom{n}{3}h^2 - \cdots \text{higher deg terms}\}$

Putting h=0 we get,

LHL = n

RHL = $\lim_{h \to 0} f(1+h) = \lim_{h \to 0} \frac{1-(1+h)^n}{1-(1+h)} = \lim_{h \to 0} \frac{1-(1+h)^n}{-h}$

Using binomial expansion as used above we get the following expression

Similarly,

RHL = $\lim_{h \to 0} \frac{1-1-nh-\binom{n}{2}h^2-\cdots \text{higher deg terms}}{-h} = \lim_{h \to 0}\{n + \binom{n}{2}h + \binom{n}{3}h^2 - \cdots \text{higher deg terms}\}$

Putting h=0 we get,

RHL = n

Thus RHL = LHL ≠ f(1)

∴ f(x) is discontinuous at x=1

10 F. Question

Discuss the continuity of the following functions at the indicated point(s).

$f(x) = \begin{cases} \frac{|x^2-1|}{x-1} & , \text{for } x \neq 1 \\ 2 & , \text{for } x = 1 \end{cases}$ at x = 1

Answer

In this problem we need to check continuity at x = 1

Given function is

$f(x) = \begin{cases} \frac{|x^2-1|}{x-1} & ,\text{for } x \neq 1 \\ 2 & ,\text{for } x = 1 \end{cases}$ at x = 1 ...... Equation 2

∴ we need to check LHL, RHL and value of function at x = 1 (for approaching idea and meaning of continuity refer to Q10(i))

Clearly,

f(1) = 2 [from equation 2]

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0}\frac{|(1-h)^2-1|}{1-h-1} = \lim_{h\to 0}\frac{|1+h^2-2h-1|}{-h} = \lim_{h\to 0}\frac{|h(h-2)|}{-h}$

Since h is positive no which is very close to 0

∴ (h–2) is negative and hence h(h–2) is also negative.

∴ |h(h–2)| = –h(h–2)

∴LHL = $\lim_{h\to 0}\frac{-h(h-2)}{-h} = \lim_{h\to 0}(h-2) = -2$

RHL = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0}\frac{|(1+h)^2-1|}{1+h-1} = \lim_{h\to 0}\frac{|1+h^2+2h-1|}{h} = \lim_{h\to 0}\frac{|h(h+2)|}{h}$

Since h is a positive no which is very close to 0

∴ (h+2) is positive and hence h(h–2) is also positive.

∴ |h(h+2)| = h(h+2)

∴ RHL = $\lim_{h\to 0}\frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 2$

Clearly, LHL ≠ RHL

∴ f(x) is discontinuous at x=1

10 G. Question

Discuss the continuity of the following functions at the indicated point(s).

$$f(x) = \begin{cases} \frac{2|x|+x^2}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$ at x = 0

Answer

In this problem we need to check continuity at x = 1

Given function is

$f(x) = \begin{cases} \frac{2|x|+x^2}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ at x = 0 ...... Equation 2

∴ we need to check LHL, RHL and value of function at x =0 (for approaching idea and meaning of continuity refer to Q10(i))

Clearly,

f(0) = 0 [from equation 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\frac{2|-h|+(-h)^2}{-h}$ [by putting x = –h in eqn 2]

$= \lim_{h\to 0}\frac{2h+h^2}{-h} = \lim_{h\to 0}(-2-h) = -2$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}\frac{2|h|+(h)^2}{h}$ [by putting x = h in eqn 2]

$= \lim_{h\to 0}\frac{2h+h^2}{h} = \lim_{h\to 0}(2+h) = 2$

Clearly, LHL ≠ RHL ≠ f(0)

∴ f(x) is discontinuous at x=0

10 H. Question

Discuss the continuity of the following functions at the indicated point(s).

$$f(x)=\begin{cases}|x-a|\sin\left(\frac{1}{x-a}\right) & \text{, for } x\neq a\\ 0 & \text{, for } x=a\end{cases}$$ at x = a

Answer

In this problem we need to check continuity at x = a

Given function is

$$f(x)=\begin{cases}|(x-a)|\sin\left(\frac{1}{x-a}\right) & ,x\neq a\\ 0 & ,x=a\end{cases}$$ at x = a ...... Equation 2

∴ we need to check LHL, RHL and value of function at x = a (for idea and meaning of continuity refer to Q10(i))

Clearly,

f(a) = 0 [from equation 2]

LHL = $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} |(a-h-a)|\sin\left(\frac{1}{a-h-a}\right)$ [putting x = (a–h) in eqn 2]

$= \lim_{h\to 0}|-h|\sin\left(\frac{1}{-h}\right) = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$

RHL = $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} |a+h-a|\sin\left(\frac{1}{a+h-a}\right) = \lim_{h\to 0}|h|\sin\left(\frac{1}{h}\right)$

$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$

Why $\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$?

Since whatever is value of h, sin(1/h) is going to range from –1 to 1

As h→ 0 ,i.e. approximately 0

∴ 0*some finite quantity is equal to 0.

Clearly, LHL = RHL = f(a)

∴ f(x) is continuous at x = 0

11. Question

Show that $f(x)=\begin{cases}1+x^2 & \text{, if } 0\leq x\leq 1\\ 2-x & \text{, if } x>1\end{cases}$ is discontinuous at x = 1.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ......Equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e} \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e} \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

In this problem we need to check continuity at x = 1

Given function is

$f(x) = \begin{cases} 1+x^2 & ,\text{if } 0 \le x \le 1 \\ 2-x & ,\text{if } x > 1 \end{cases}$ ...... Equation 2

∴ we need to check LHL, RHL and value of function at x = 1

Clearly,

$f(1) = 1+1^2 = 2$ [using equation 2]

LHL = $\lim_{h\to 0} f(1-h)$

since 1–h is less than 1

∴ we will use the first expression from equation 2 i.e. $f(x) = 1+x^2$

LHL = $\lim_{h\to 0} 1+(1-h)^2 = \lim_{h\to 0}(1+1+h^2-2h) = 2$

RHL = $\lim_{h\to 0} f(1+h)$

since 1+h is greater than 1

∴ we will use the second expression from equation 2 i.e. f(x) = 2–x

RHL = $\lim_{h\to 0} 2-(1+h) = 1$

Clearly, LHL ≠ RHL

∴ f(x) is discontinuous at x=1

12. Question

Show that $f(x) = \begin{cases} \dfrac{\sin 3x}{\tan 2x} & ,\text{ if } x<0 \\ \dfrac{3}{2} & ,\text{ if } x=0 \\ \dfrac{\log(1+3x)}{e^{2x}-1} & ,\text{if } x>0 \end{cases}$ is continuous at x = 0

Answer

<u>Ideas required to solve the problem:</u>

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{$i. e $\lim_{h\to 0} f(c-h)$ $\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i. e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

2. Idea of sandwich theorem – This theorem also known as squeeze theorem that you may have encountered in your class 11 in limits chapter suggests that

If I be an interval having the point a as a limit point. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have{\displaystyle \lim _{x→ a}g(x)=\lim _{x→ a}h(x)=L.}{\displaystyle \lim _{x→ a}f(x)=L.}

g(x) ≤ f(x) ≤ h(x) and also

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = K(\text{say})$

Then , $\lim_{x\to a} f(x) = K$ We say that f(x) is squeezed between g(x) and h(x) or you can assume it like sandwich.

$\frac{\sin x}{x}$ is also squeezed between 1 when $x \to 0$

$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$ …… equation 2

$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\cos x \sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1$ …… equation 3

NOTE : denominator in the above limits should be exactly same as that of content in sine function

Eg : $\lim_{x\to 3} \frac{\sin(3-x)}{3-x} = 1$

3. Idea of logarithmic limit and exponential limit –

$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ ……equation 4

$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ …… equation 5

You must have read such limits in class 11. You can verify these by expanding log(1+x) and e^x in its taylor form.

Numerator and denominator conditions also hold for this limit like sandwich theorem.

E.g : $\lim_{x\to 0} \frac{\log(1+2x)}{2x} = 1$

But, $\lim_{x\to 0} \frac{\log(1+2x)}{x} \neq 1$ as denominator does not have 2x

Now we are ready to solve this problem:

Given function is :

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan 2x} & ,\text{if} x < 0 \\ \frac{3}{2} & ,\text{if} x = 0 \\ \frac{\log(1+3x)}{e^{2x}-1} & ,\text{if} x > 0 \end{cases}$$ ...... equation 6

As we need to check continuity at x=0 , so we need to check value of f(x) at x = 0,left hand and right hand limits. If all 3 comes out to be equal we can say that f(x) is continuous at x=0 else it is discontinuous.

Clearly,

$f(0) = \frac{3}{2}$

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$

Putting x = –h in equation 6, we have

LHL = $\lim_{h\to 0} \frac{\sin(-3h)}{\tan(-2h)} = \lim_{h\to 0} \frac{\sin 3h}{\tan 2h}$ [∵ sin(–θ) = sinθ and also tan(–θ) = tanθ]

$= \lim_{h\to 0} \frac{\frac{\sin 3h}{3h}*3}{\frac{\tan 2h}{2h}*2} = \frac{3}{2}\lim_{h\to 0} \frac{\frac{\sin 3h}{3h}}{\frac{\tan 2h}{2h}} = \frac{3}{2}$ [using equations 2 and 3 to apply sandwich theorem]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

Putting x = h in equation 6, we have

RHL = $\lim_{h\to 0} \frac{\log(1+3h)}{e^{2h}-1} = \lim_{h\to 0} \frac{\frac{\log(1+3h)}{3h}*3}{\frac{e^{2h}-1}{2h}*2}$ [trying to apply logarithmic and exponential limit using equation 4 and 5]

$= \frac{3}{2}\lim_{h\to 0} \frac{\frac{\log(1+3h)}{3h}}{\frac{e^{2h}-1}{2h}} = \frac{3}{2}$

Clearly, LHL = RHL = f(0) = $\frac{3}{2}$

∴ f(x) is continuous at x = 0.

13. Question

Find the value of ‘a’ for which the function f defined by $f(x) = \begin{cases} a \sin\frac{\pi}{2}(x+1) & , x \le 0 \\ \frac{\tan x - \sin x}{x^3} & , x > 0 \end{cases}$ is continuous at x = 0.

Answer

<u>Ideas required to solve the problem:</u>

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i. e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i. e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

2. Idea of sandwich theorem – This theorem also known as squeeze theorem that you may have encountered in your class 11 in limits chapter suggests that

If I be an interval having the point a as a limit point. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have{\displaystyle \lim _{x→ a}g(x)=\lim _{x→ a}h(x)=L.}{\displaystyle \lim _{x→ a}f(x)=L.}

g(x) ≤ f(x) ≤ h(x) and also

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = K(\text{say})$$

Then, $\lim_{x\to a} f(x) = K$ We say that f(x) is squeezed between g(x) and h(x) or you can assume it like sandwich.

$\frac{\sin x}{x}$ is also squeezed between 1 when $x \to 0$

$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$ ...... equation 2

$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\cos x \sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1$ ...... equation 3

NOTE : denominator in the above limits should be exactly same as that of content in sine function

Eg : $\lim_{x\to 3} \frac{\sin(3-x)}{3-x} = 1$

Let’s Solve :

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1) & ,x \le 0 \\ \frac{\tan x - \sin x}{x^3} & ,x > 0 \end{cases}$$ ...... equation 4

Given that f(x) is continuous at x=0

∴ LHL = RHL = f(0)

$\therefore \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$

From eqn 4:

f(0) = $a\sin\frac{\pi}{2}(0+1) = a$

LHL = $\lim_{h\to 0} f(-h) = \lim_{h\to 0} a\sin\frac{\pi}{2}(-h+1) = a$

RHL = $\lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{\tan h - \sin h}{h^3}$

$= \lim_{h\to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$

$= \lim_{h\to 0} \frac{\sin h(1-\cos h)}{h^3 \cos h}$ [∵ sinθ = 2sin(θ/2)cos(θ/2) and (1–cosθ) = 2sin^2(θ/2)]

$= \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos(\frac{h}{2})2\sin^2\frac{h}{2}}{h^3 \cos h}$

$= \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos(\frac{0}{2})2\sin^2\frac{h}{2}}{h^3\cos 0}$

$= 4\lim_{h\to 0} \frac{\sin^3\frac{h}{2}}{\frac{h^3}{8}*8} = \frac{4}{8}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^3$ [using sandwich theorem as explained … eqn 2]

$= \frac{1}{2} * 1 = \frac{1}{2}$

As LHL = RHL

∴ a = $\frac{1}{2}$

∴ for f(x) to be continuous at x=0

a = $\frac{1}{2}$

14. Question

Examine the continuity of the function $f(x) = \begin{cases} 3x-2 & , \text{if } x \le 0 \\ x+1 & , \text{if } x > 0 \end{cases}$ at x = 0. Also sketch the graph of this function.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e } \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Let’s Solve now:

Given function is

$f(x) = \begin{cases} 3x-2 & ,\text{if } x \le 0 \\ x+1 & ,\text{if } x > 0 \end{cases}$ …… Equation 2

We need to check whether f(x) is continuous at x=0 or not

For this we need to check LHL, RHL and value of function at x=0

Clearly,

f(0) = 3*0 – 2 = –2 [from equation 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0}\{3(-h)-2\} = -2$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}\{h+1\} = 0+1 = 1$

As, LHL ≠ RHL

∴ f(x) is discontinuous at x = 0

This can also be proved by plotting f(x) on cartesian plane.

For x >0 ,we need to plot

y = x + 1

put y=0, we get x=–1 and for second point we put x=0 and thus get y=1

two points are enough to plot the straight line.

Two coordinates are (–1,0) and (0,1)

For x≤0, we need to plot

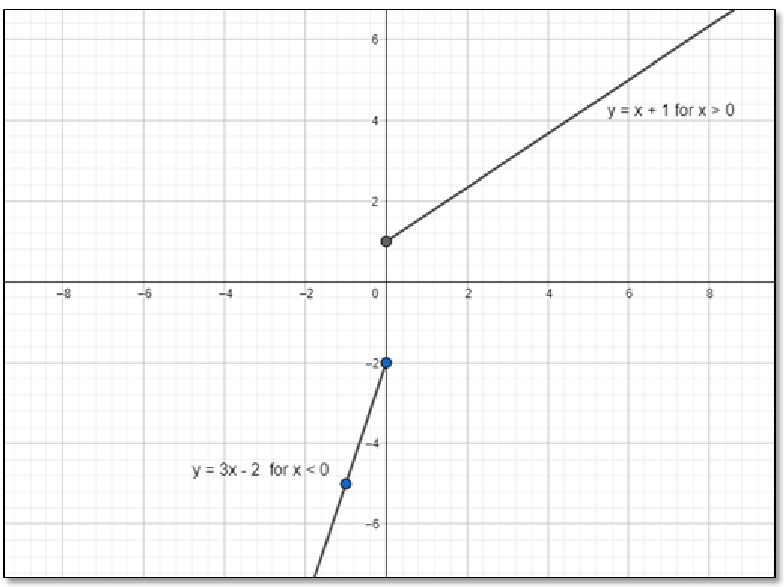
y = 3x – 2

put x = 0 then y = –2

on putting y=0 we get x = 2/3

two coordinates are (0,–2) and $(\frac{2}{3}, 0)$

Graph:



It can be seen from graph that there is breakage in curve at (0,0)

Thus, it is discontinuous at x = 0

15. Question

Discuss the continuity of the function $f(x) = \begin{cases} x & , x > 0 \\ 1 & , x = 0 \\ -x & , x < 0 \end{cases}$ at the point x = 0.

Answer

<u>Ideas required to solve the problem:</u>

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we

can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ...... equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{$i. e $\lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i. e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Given,

$f(x) = \begin{cases} x & ,x > 0 \\ 1 & ,x = 0 \\ -x & ,x < 0 \end{cases}$ ...... equation 2

we are asked to check its continuity at x=0

∴ we need to check LHL, RHL and value of function at x = 0, if all comes out to be equal we can say f(x) is continuous at x=0 else it is discontinuous.

Clearly,

f(0) = 1 [from eqn 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} -(-h) = \lim_{h\to 0} h = 0$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0}(h) = \lim_{h\to 0} h = 0$

Thus, LHL = RHL ≠ f(0)

∴ f(x) is discontinuous at x = 0

16. Question

Discuss the continuity of the function $f(x) = \begin{cases} x & , \quad 0 \le x < 1/2 \\ 1/2 & , \quad x = 1/2 \\ 1-x & , \quad 1/2 < x \le 1 \end{cases}$ at the point x = 1/2.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ...... equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{$i.e $\lim_{h\to 0} f(c-h)$ $\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e } \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Given,

$f(x) = \begin{cases} x & , \quad 0 \le x < 1/2 \\ 1/2 & , \quad x = 1/2 \\ 1-x & , \quad 1/2 < x \le 1 \end{cases}$ ...... equation 2

we are asked to check its continuity at x=1/2

∴ we need to check LHL ,RHL and value of function at x = 1/2 ,if all comes out to be equal we can say f(x) is continuous at x=1/2 else it is discontinuous.

Clearly,

$f(\frac{1}{2}) = \frac{1}{2}$ [from eqn 2]

LHL = $\lim_{h\to 0} f\left(\frac{1}{2}-h\right) = \lim_{h\to 0} \left(\frac{1}{2}-h\right) = \frac{1}{2} - 0 = \frac{1}{2}$

RHL = $\lim_{h\to 0} f\left(\frac{1}{2}+h\right) = \lim_{h\to 0} \left(1-(\frac{1}{2}+h)\right) = \frac{1}{2} - 0 = \frac{1}{2}$

Thus, LHL = RHL = f(0)

∴ f(x) is continuous at x = $\frac{1}{2}$

17. Question

Discuss the continuity of $f(x) = \begin{cases} 2x-1 & , x < 0 \\ 2x+1 & , x \ge 0 \end{cases}$ at x = 0.

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ...... equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e} \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e} \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c i.e. f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Given,

$f(x) = \begin{cases} 2x-1 & ,x<0 \\ 2x+1 & ,x\geq 0 \end{cases}$ ...... equation 2

we are asked to check its continuity at x=0

∴ we need to check LHL ,RHL and value of function at x = 0 ,if all comes out to be equal we can say f(x) is continuous at x=0 else it is discontinuous.

Clearly,

f(0) = 2*0+1 = 1 [from eqn 2]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} 2(-h) - 1 = -1$

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} 2h + 1 = 2 * 0 + 1 = 1$

Thus, LHL ≠ RHL

∴ f(x) is discontinuous at x = 0

18. Question

For what value of k is the function $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & , x \neq 1 \\ k & , x = 1 \end{cases}$ continuous atx = 1?

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

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A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ ...... equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

Limiting value of the left neighbourhood of x = c also called left hand limit LHL$\{\text{i.e} \lim_{h\to 0} f(c-h)\}$ must be equal to limiting value of right neighbourhood of x= c called right hand limit RHL $\left\{\text{i.e} \lim_{h\to 0} f(c+h)\right\}$ and both must be equal to the value of f(x) at x=c f(c).

Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Given,

$f(x) = \begin{cases} \frac{x^2-1}{x-1} & , x \neq 1 \\ k & , x = 1 \end{cases}$ …… equation 2

We need to find the value of k such that f(x) is continuous at x = 1

Since f(x) is continuous at x = 1

∴ (LHL as x tends to 1) = (RHL as x tends to 1) = f(1)

$\therefore \lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$

As, f(1) = k [from equation 2]

We can find either LHL or RHL to equate with f(1)

Let's find RHL but if you want you can proceed with LHL also.

RHL $= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{(1+h)^2-1}{1+h-1} = \lim_{h\to 0} \frac{1+h^2+2h-1}{h}$

$= \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0} \frac{h(h+2)}{h} = \lim_{h\to 0}(h+2) = 0+2 = 2$

As, f(x) is continuous

∴ RHL = f(1)

∴ k = 2

19. Question

Determine the value of the constant k so that the function $f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-1} & , \text{if } x \neq 1 \\ k & , \text{if } x = 1 \end{cases}$ is continuous at x = 1.

Answer

<u>Ideas required to solve the problem:</u>

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

Mathematically we define the same thing as given below:

A function f(x) is said to be continuous at x = c where c is x–coordinate of the point at which continuity is to be checked

If:–

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Thus, it is the necessary condition for a function to be continuous

So, whenever we check continuity we try to check above equality if it holds true, function is continuous else it is discontinuous.

Given,

$f(x) = \begin{cases} \frac{x^2-3x+2}{x-1} & ,\text{if } x \neq 1 \\ k & ,\text{if } x = 1 \end{cases}$ ....equation 2

We need to find the value of k such that f(x) is continuous at x = 1

Since f(x) is continuous at x = 1

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As, f(1) = k [from equation 2]

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Let’s find RHL,you can find LHL also.

RHL $= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{(1+h)^2-3(1+h)+2}{1+h-1} = \lim_{h\to 0} \frac{1+h^2+2h-3-3h+2}{h}$

$= \lim_{h\to 0} \frac{h^2-h}{h} = \lim_{h\to 0} \frac{h(h-1)}{h} = \lim_{h\to 0}(h-1) = 0-1 = -1$

As, f(x) is continuous

∴ RHL = f(1)

∴ k = –1

20. Question

For what value of k is the function $f(x) = \begin{cases} \frac{\sin 5x}{3x} & , \text{if } x \neq 0 \\ k & , \text{if } x = 0 \end{cases}$ continuous atx = 0?

Answer

Ideas required to solve the problem:

1. Meaning of continuity of function – If we talk about a general meaning of continuity of a function f(x) , we can say that if we plot the coordinates (x , f(x)) and try to join all those points in the specified region, we can do so without picking our pen i.e you will put your pen/pencil on graph paper and you can draw the curve without any breakage.

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If:–

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ equation 1

where h is a very small positive no (can assume h = 0.00000000001 like this)

It means :–

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g(x) ≤ f(x) ≤ h(x) and also

$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = K(say)$

Then , $\lim_{x\to a} f(x) = K$ We say that f(x) is squeezed between g(x) and h(x) or you can assume it like sandwich.

$\frac{\sin x}{x}$ is also squeezed between 1 when $x \to 0$

$\therefore \lim_{x\to 0} \frac{\sin x}{x} = 1$ ...... equation 2

NOTE : denominator in the above limit should be exactly same as that of content in sine function

Eg : $\lim_{x\to 3} \frac{\sin(3-x)}{3-x} = 1$

Given,

$f(x) = \begin{cases} \frac{\sin 5x}{3x} & ,\text{if } x \neq 0 \\ k & ,\text{if } x = 0 \end{cases}$ ....equation 3

We need to find the value of k such that f(x) is continuous at x = 0

Since f(x) is continuous at x = 0

∴ (LHL as x tends to 0) = (RHL as x tends to 0) = f(0)

$\therefore \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$

As, f(0) = k [from equation 2]

We can find either LHL or RHL to equate with f(1)

Let’s find RHL, you can proceed with LHL also.

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{\sin 5h}{3h} = \frac{1}{3}\lim_{h\to 0} \frac{5\sin 5h}{5h}$

$= \frac{5}{3}\lim_{h\to 0} \frac{\sin 5h}{5h} = \frac{5}{3} * 1 = \frac{5}{3}$

As, f(x) is continuous

∴ RHL = f(0)

$\therefore k = \frac{5}{3}$

21. Question

Determine the value of the constant k so that the function $f(x) = \begin{cases} kx^2 & , \text{if } x \le 2 \\ 3 & , \text{if } x > 2 \end{cases}$ is continuous at x = 2.

Answer

Given:

It is clear that when x<2 and x>2, the given function is continuous at x = 2.

So, at x = 2

$\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} f(x)\ kx^2$

$= \lim\limits_{x\to2^-} f(x)\ k\times(2^2)$

= 4k

$\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+} 3 = 3$

We know that,

If f is continuous at x = c, then The Left–hand limit, the Right–hand limit and the value of the function at x = c exist and are equal to each other.

$\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$

$\therefore \lim\limits_{x\to2^-} f(x) \lim\limits_{x\to2^+} f(x) = f(2)$

⇒ 4k = 3

$\Rightarrow$ k = $\frac{3}{4}$

Therefore, the required value of k is $\frac{3}{4}$

22. Question

Determine the value of the constant k so that the function $f(x)=\begin{cases}\dfrac{\sin 2x}{5x} & ,\text{ if } x\neq 0\\ k & ,\text{ if } x=0\end{cases}$ is continuous at x = 0.

Answer

Given:

The function f is continuous at x = 0,Therefore,

$\lim\limits_{x\to0} f(x) = f(c)$

We know that,

If f is continuous at x = c, then The Left–hand limit, the Right–hand limit and the value of the function at x = c exist and are equal to each other.

$\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$

$\Rightarrow \lim\limits_{x\to0} \frac{\sin2x}{5x} = k$

$\Rightarrow \lim\limits_{x\to0} \frac{\sin2x}{2x} \times \frac{2x}{5x} = k$

$\Rightarrow$ k = $1\times\frac{2x}{5x}$

$\therefore \lim\limits_{x\to0} \frac{\sin x}{x} = 1$

$\Rightarrow$ $\frac{2}{5}$ = k

$\therefore k = \frac{2}{5}$

Thus, f is continuous at x = 0 if k = $\frac{2}{5}$

23. Question

Find the values of a so that the function $f(x)=\begin{cases} ax+5 & , \text{if } x\le 2 \\ x-1 & , \text{if } x>2 \end{cases}$ is continuous at x = 2.

Answer

Given:

The function f is continuous at x = 2

We know that,

If f is continuous at x = c, then The Left–hand limit, the Right–hand limit and the value of the function at x = c exist and are equal to each other.

$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

$\lim_{x\to 2^-} ax + 5$

$\Rightarrow \lim_{x\to 2^-} a\times 2 + 5$

⇒ 2a + 5

$\lim_{x\to 2^+} x - 1$

$\Rightarrow \lim_{x\to 2^+} 2 - 1$

$\Rightarrow \lim_{x\to 2^+} 1 = 1$

since f is continuous at x = 2,

$\lim_{x\to 2^-} ax + 5 = \lim_{x\to 2^+} x - 1 = f(2)$

∴ 2a + 5 = 1

⇒ 2a = 1 – 5

⇒ 2a = –4

⇒ a = –2

Thus, f is continuous at x = 2 if a = –2.

24. Question

Prove that the function $f(x)=\begin{cases} \dfrac{x}{|x|+2x^2} & , x\ne 0 \\ k & , x=0 \end{cases}$ remains discontinuous at x = 0, regardless of the choice of k.

Answer

To prove given f(x) is discontinuous at x = 0, we have to show that left–hand limit(LHL) and right–hand limit(RHL) is unequal.

LHL = $\lim_{x\to c^-} f(x) = \lim_{h\to 0} f(c-h)$, since (c–h)<c

RHL = = $\lim_{x\to c^+} f(x) = \lim_{h\to 0} f(c+h)$, since (c + h)>c

LHL = $\lim_{x\to 0^-} \frac{x}{|x|+2x^2}$

$\Rightarrow \lim_{h\to 0} \frac{0-h}{|0-h| + 2(0-h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{-h}{|-h| + 2(-h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{-h}{h + 2h^2}$

$\Rightarrow \lim_{h\to 0}\frac{h(-1)}{h(1 + 2h)}$

$\Rightarrow \lim_{h\to 0}\frac{(-1)}{(1 + 2h)}$

$\Rightarrow \frac{(-1)}{(1)}$

$\Rightarrow$ -1

RHL = $\lim_{x\to 0^+}\frac{x}{|x| + 2x^2}$

$\Rightarrow \lim_{h\to 0}\frac{0 + h}{|0 + h| + 2(0 + h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{h}{|h| + 2(h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{h}{h + 2h^2}$

$\Rightarrow \lim_{h\to 0}\frac{h(1)}{h(1 + 2h)}$

$\Rightarrow \lim_{h\to 0}\frac{(1)}{(1 + 2h)}$

$\Rightarrow \frac{(1)}{(1)}$

$\Rightarrow$ 1

since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

The function f(x)remains discontinuous at x = 0, regardless the choice of k.

25. Question

Find the value of k if f(x) is continuous at x = π/2, where $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x} & , \text{if } x \neq \pi/2 \\ 3 & , \text{if } x = \pi/2 \end{cases}$

Answer

Given:

f(x) is continuous at x = $\frac{\pi}{2}$

$\lim_{x\to c^+} f(x) = \lim_{h\to 0} f(c + h)$, since (c + h)>c

$\lim_{x\to\frac{\pi}{2}}\frac{k\cos x}{\pi - 2x}$

$\Rightarrow \lim_{h\to 0}\frac{k\cos(\frac{\pi}{2} + h)}{\pi - 2(\frac{\pi}{2} + h)}$

$$\Rightarrow \lim_{h\to 0}\frac{k(-\sin h)}{\pi-(2\times\frac{\pi}{2}+2\times h)}$$

$$\Rightarrow \lim_{h\to 0}\frac{-k(\sin h)}{\pi-2\times\frac{\pi}{2}-2\times h)}$$

$$\Rightarrow \lim_{h\to 0}\frac{-k\sin h}{\pi-\pi-2h}$$

$$\Rightarrow \lim_{h\to 0}\frac{-k\sin h}{-2h}$$

$$\Rightarrow \lim_{h\to 0}\frac{k\sin h}{2h}$$

$$\Rightarrow \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h}$$

$$\therefore \lim_{h\to 0}\frac{\sin h}{h} = 1$$

$$\Rightarrow \frac{k}{2}\times 1$$

$$\Rightarrow \frac{k}{2}$$

Also, given $f(\frac{\pi}{2}) = 3$

since f(x) is continuous at $x = \frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}} f(x) = f(\frac{\pi}{2})$$

i.e, $\frac{k}{2} = 3$

$\Rightarrow$ k = 6

The value of k is 6 when f(x) is continuous at $x = \frac{\pi}{2}$

26. Question

Determine the values of a, b, c for which the function $f(x)=\begin{cases}\frac{\sin(a+1)x+\sin x}{x}, & \text{for } x<0\\ x, & \text{for } x=0\\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & \text{for } x>0\end{cases}$ is continuous at x = 0.

Answer

Given:

f(x) is continuous at x = 0

For f(x) to be continuous at x = 0, $f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x\to 0}\frac{\sin(a+1)x+\sin x}{x}$

$$\Rightarrow \lim_{h\to 0}\frac{\sin(a+1)h+\sin h}{h}$$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sin(a + 1)h}{h} + \lim\limits_{h\to 0}\frac{\sin h}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sin(a + 1)h}{h}\times\frac{(a + 1)}{(a + 1)} + \lim\limits_{h\to 0}\frac{\sin h}{h}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sin(a + 1)h}{(a + 1)h}\times\frac{(a + 1)}{1} + \lim\limits_{h\to 0}\frac{\sin h}{h}$

$\lim\limits_{h\to 0}\frac{\sin(a + 1)h}{(a + 1)h} = 1$

$\lim\limits_{h\to 0}\frac{\sin h}{h} = 1$

$\Rightarrow 1\times (a + 1) + 1$

$\Rightarrow (a + 1) + 1$

$f(0)^{-} \Rightarrow a + 2$ ...... (1)

RHL = $f(0^{+}) = \lim\limits_{x\to 0}\frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{\frac{3}{2}}}$

$\Rightarrow \lim\limits_{x\to 0}\frac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{\frac{3}{2}}}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sqrt{h + bh^2} - \sqrt{h}}{bh^{\frac{3}{2}}}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sqrt{h + bh^2} - \sqrt{h}}{b\times h\times h^{\frac{1}{2}}}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sqrt{h + bh^2} - \sqrt{h}}{b\times h\times \sqrt{h}}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sqrt{h(1 + bh)} - \sqrt{h}}{b\times h\times \sqrt{h}}$

$\Rightarrow \lim\limits_{h\to 0}\frac{\sqrt{h}(\sqrt{1 + bh} - \sqrt{1})}{bh\times \sqrt{h}}$

$\Rightarrow \lim\limits_{h\to 0}\frac{(\sqrt{1 + bh} - \sqrt{1})}{bh}$

Take the complex conjugate of

$(\sqrt{1 + bh} - \sqrt{1})$,

i.e, $(\sqrt{1 + bh} + \sqrt{1})$ and multiply it with numerator and denominator.

$\Rightarrow \lim\limits_{h\to 0}\frac{(\sqrt{1 + bh} - \sqrt{1})}{bh}\times\frac{(\sqrt{1 + bh} + \sqrt{1})}{(\sqrt{1 + bh} + \sqrt{1})}$

$\Rightarrow \lim\limits_{h\to 0}\frac{(\sqrt{1 + bh})^2 - (\sqrt{1})^2}{bh}$

$\therefore (a + b)(a-b) = a^2 - b^2$

$\Rightarrow \lim\limits_{h\to 0}\frac{(1 + bh - 1)}{bh(\sqrt{1 + bh} + \sqrt{1})}$

$\Rightarrow \lim_{h\to 0} \frac{(bh)}{bh(\sqrt{1 + bh} + \sqrt{1})}$

$\Rightarrow \lim_{h\to 0} \frac{1}{(\sqrt{1 + bh} + \sqrt{1})}$

$\Rightarrow \frac{1}{(\sqrt{1 + b \times 0} + \sqrt{1})}$

$\Rightarrow \frac{1}{1 + 1}$

$f(0)^+ \Rightarrow \frac{1}{2}$ ...... (2)

since, f(x) is continuous at x = 0,From (1) & (2),we get,

$\Rightarrow a + 2 = \frac{1}{2}$

$\Rightarrow a = \frac{1}{2} - 2$

$\Rightarrow a = \frac{-3}{2}$

Also,

$f(0)^- = f(0)^+ = f(0)$

$\Rightarrow$ f(0) = c

$\Rightarrow$ $c = a + 2 = \frac{1}{2}$

$\Rightarrow$ $c = \frac{1}{2}$

So the values of $a = \frac{-3}{2}$,$c = \frac{1}{2}$and b = R–{o}(any real number except 0)

27. Question

If $f(x) = \begin{cases} \frac{1-\cos kx}{x \sin x} & , x \neq 0 \\ \frac{1}{2} & , x = 0 \end{cases}$ is continuous at x = 0, find k.

Answer

Given:

f(x) is continuous at x = 0 & $f(0) = \frac{1}{2}$

For f(x) to be continuous at x = 0,$f(0)^- = f(0)^+ = f(0)$

$LHL = f(0)^- = \lim_{x\to 0} \frac{1-\cos kx}{x \sin x}$

$\Rightarrow \lim_{h\to 0} \frac{1 - \cos k(0-h)}{(0-h)\sin(0-h)}$

$\Rightarrow \lim_{h\to 0} \frac{1 - \cos k(-h)}{(-h)\sin(-h)}$

$\therefore \cos(-x) = \cos x$

$\Rightarrow \lim_{h\to 0} \frac{1 - \cos k(h)}{(-h) \times -\sin(h)}$

$$\Rightarrow \lim_{h\to 0}\frac{1-\cos k(h)}{(h)\sin(h)}$$

$\cos 2x = 1-2\sin^2 x$

$1-\cos 2x = 2\sin^2 x$

$1-\cos x = 2\sin^2\frac{x}{2}$

$$\Rightarrow \lim_{h\to 0}\frac{\sin^2\frac{kh}{2}}{(h)\sin(h)}$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin^2\frac{kh}{2}}{(h)2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin^2\frac{kh}{2}}{(h)2\sin\frac{h}{2}\cos\frac{h}{2}}$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin^2\frac{kh}{2}}{\left(\frac{kh}{2}\right)^2}\times\frac{\left(\frac{kh}{2}\right)^2}{2h\sin\frac{h}{2}\cos\frac{h}{2}}$$

$\lim_{h\to 0}\frac{\cosh}{2} = \frac{1}{2}, \cos 0 = 1$

$$\Rightarrow \lim_{h\to 0}\left\{\frac{\sin^2\frac{kh}{2}}{\left(\frac{kh}{2}\right)^2}\right\}\times\frac{\left(\frac{kh}{2}\right)^2}{2h\times\lim_{h\to 0}\left\{\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right\}\times\frac{1}{2}\times\frac{h}{2}}$$

$$\Rightarrow 1\times\frac{\left(\frac{kh}{2}\right)^2}{2h\times 1\times\frac{1}{2}\times\frac{h}{2}}$$

$$\Rightarrow \frac{\left(\frac{k^2h^2}{2^2}\right)}{h^2\times\frac{1}{2}}$$

$$\Rightarrow \frac{\left(\frac{k^2}{2}\right)}{1}$$

$f(0)^- \Rightarrow \frac{k^2}{2}$

Since ,f(x) is continuous at x = 0 & $f(0) = \frac{1}{2}$

$\therefore \frac{k^2}{2} = \frac{1}{2}$

$\Rightarrow k^2 = 1$

$\Rightarrow k = \pm 1$

28. Question

If $f(x)=\begin{cases}\frac{x-4}{|x-4|}+a & , \text{if} \quad x<4 \\ a+b & , \text{if} \quad x=4 \\ \frac{x-4}{|x-4|} & , \text{if} \quad x>4\end{cases}$ is continuous at x = 4, find a, b.

Answer

Given:

f(x) is continuous at x = 4 & f(4) = a + b

For f(x) to be continuous at x = 4, $f(4)^{-} = f(4)^{+} = f(4)$

LHL = $f(4)^{-} = \lim_{x\to 4} \frac{x-4}{|x-4|} + a$

$\Rightarrow \lim_{h\to 0}\frac{(4-h)-4}{|(4-h)-4|} + a$

$\Rightarrow \lim_{h\to 0}\frac{(4-h-4)}{|(4-h-4|} + a$

$\Rightarrow \lim_{h\to 0}\frac{(-h)}{|(-h)|} + a$

$\Rightarrow \lim_{h\to 0}\frac{(-h)}{h} + a$

$\Rightarrow$ a – 1

LHL = $f(4)^{+} = \lim_{x\to 4} \frac{x-4}{|x-4|}$

$\Rightarrow \lim_{h\to 0}\frac{(4+h)-4}{|(4+h)-4|}$

$\Rightarrow \lim_{h\to 0}\frac{4+h-4}{|4+h-4|}$

$\Rightarrow \lim_{h\to 0}\frac{h}{|h|}$

$\Rightarrow \lim_{h\to 0}\frac{1}{|1|}$

$\Rightarrow$ 1

Since ,f(x) is continuous at x = 4 & f(4) = a + b

$f(4)^{-}=f(4)^{+}=f(4)$

$\therefore$a – 1 = a + b = 1

$\Rightarrow$ a – 1 = 1

$\Rightarrow$ a = 2

a + b = 1

$\Rightarrow$ 2 + b = 1

$\Rightarrow$ b = 1–2

$\Rightarrow$ b = –1

29. Question

For what value of k is the function $f(x)=\begin{cases}\frac{\sin 2x}{x} & , x \neq 0 \\ k & , x=0\end{cases}$ continuous at x = 0?

Answer

For f(x) to be continuous at x = 0, $f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^-$ = $\lim_{x \to 0} \frac{\sin 2x}{x}$

$\Rightarrow \lim_{x \to 0} \frac{\sin 2x}{x}$

$\Rightarrow \lim_{h \to 0} \frac{\sin 2(0-h)}{0-h}$

$\Rightarrow \lim_{h \to 0} \frac{-\sin 2h}{-h}$

$\Rightarrow \lim_{h \to 0} \frac{\sin 2h}{h}$

$\Rightarrow \lim_{h \to 0} \frac{\sin 2h}{2h} \times 2$

$\therefore \lim_{h \to 0} \frac{\sin 2h}{2h} = 1$

$\Rightarrow 1 \times 2$

$\Rightarrow 2$

Since,

$f(0)^- = f(0)^+ = f(0)$

k = 2

30. Question

Let $f(x)=\frac{\log\left(1+\frac{x}{a}\right)-\log\left(1-\frac{x}{b}\right)}{x}$, x ≠ 0. Find the value of f at x = 0 so that f becomes continuous at x = 0.

Answer

For f(x) to be continuous at x = 0, $f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^-$ = $\lim_{x \to 0} \frac{\log\left(1+\frac{x}{a}\right)-\log\left(1-\frac{x}{b}\right)}{x}$

$\Rightarrow \lim_{h \to 0} \frac{\log\left(1+\frac{(0-h)}{a}\right)-\log\left(1-\frac{(0-h)}{b}\right)}{(0-h)}$

$\Rightarrow \lim_{h \to 0} \frac{\log\left(1+\frac{(-h)}{a}\right)-\log\left(1-\frac{(-h)}{b}\right)}{-h}$

$\Rightarrow \lim_{h \to 0} \frac{\log\left(1+\frac{(-h)}{a}\right)-\log\left(1+\frac{h}{b}\right)}{-h}$

$$\Rightarrow \frac{\lim\limits_{h\to 0}\log\left(1+\frac{(-h)}{a}\right)}{-h} - \frac{\lim\limits_{h\to 0}\log(1+\frac{h}{b})}{-h}$$

$$\Rightarrow \frac{\lim\limits_{h\to 0}\log\left(1+\frac{(-h)}{a}\right)}{-\frac{h}{a}\times a} - \frac{\lim\limits_{h\to 0}\log(1+\frac{h}{b})}{-\frac{h}{b}\times b}$$

$$\Rightarrow \frac{\lim\limits_{h\to 0}\log\left(1+\frac{(-h)}{a}\right)}{-\frac{h}{a}\times a} + \frac{\lim\limits_{h\to 0}\log(1+\frac{h}{b})}{\frac{h}{b}\times b}$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}$$

$$\Rightarrow \frac{a+b}{ab}$$

hence,f(0) = $\frac{a+b}{ab}$

31. Question

If $f(x)=\begin{cases}\frac{2^{x+2}-16}{4^x-16} & \text{, if } x\neq 2\\ k & \text{, if } x=2\end{cases}$ is continuous at x = 2, find k.

Answer

Given:

f(x) is continuous at x = 2 & f(2) = k

If f(x) to be continuous at x = 2,then,

$f(2)^- = f(2)^+ = f(2\ )$

LHL = $f(2)^-$ = $\lim\limits_{x\to 2}\frac{2^{x+2}-16}{4^x-16}$

$$\Rightarrow \lim_{h\to 0}\frac{2^{(2-h)+2}-16}{4^{(2-h)}-16}$$

$$\Rightarrow \lim_{h\to 0}\frac{2^{(4-h)}-16}{4^{(2-h)}-16}$$

$$\Rightarrow \lim_{h\to 0}\frac{2^4\times 2^{-h}-16}{4^2\times 4^{-h}-16}$$

$$\Rightarrow \lim_{h\to 0}\frac{16\times(2^{-h}-1)}{16\times(4^{-h}-1)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2^{-h}-1)}{(4^{-h}-1)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2^{-h}-1)}{((2^2)^{-h}-1^2)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2^{-h}-1)}{((2)^{-2h}-1^2)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(2^{-h}-1)}{((2)^{-h}-1)((2)^{-h}+1)}$$

$\therefore$ (a + b)(a – b) = $a^2 - b^2$

$\Rightarrow \lim_{h \to 0} \frac{1}{((2)^{-h} + 1)}$

$\Rightarrow \frac{1}{((2)^{-0} + 1)}$

$\therefore 2^0 = 1$

$\Rightarrow \frac{1}{((2)^{-0} + 1)}$

$\Rightarrow \frac{1}{(1 + 1)}$

$\Rightarrow \frac{1}{2}$

Since, f(x) is continuous at x = 2 and f(2) = k,then

$k = \frac{1}{2}$

32. Question

If $f(x) = \begin{cases} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1} & , x \neq 0 \\ k & , x = 0 \end{cases}$ is continuous at x = 0, find k.

Answer

Given:

f(x) is continuous at x = 0 & f(0) = k

If f(x) to be continuous at x = 0,then

$f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x \to 0} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1}$

$\lim_{h \to 0} \frac{\cos^2(0-h) - \sin^2(0-h) - 1}{\sqrt{(0-h)^2 + 1} - 1}$

$\lim_{h \to 0} \frac{\cos^2(-h) - \sin^2(-h) - 1}{\sqrt{(-h)^2 + 1} - 1}$

$\lim_{h \to 0} \frac{\cos^2 h - (\sin^2 h) - 1}{\sqrt{h^2 + 1} - 1}$

$\lim_{h \to 0} \frac{\cos^2 h - \sin^2 h - 1}{\sqrt{h^2 + 1} - 1}$

$\lim_{h \to 0} \frac{\cos 2h - 1}{\sqrt{h^2 + 1} - 1}$

$\lim_{h \to 0} \frac{-\sin 2h}{\frac{h}{\sqrt{h^2+1}} - 1}$

[By Applying L – Hospital Rule.]

Hence, L.H.L = 0

As f(x) is continuous at x = 0.

Then, k = L.H.L

K = 0

33. Question

Extend the definition of the following by continuity $f(x)=\frac{1-\cos 7(x-\pi)}{5(x-\pi)^2}$ at the point x = π.

Answer

Given:

f(x) is continuous at x = π,

If f(x) to be continuous at x = π,then

$f(\pi)^{-}=f(\pi)^{+}=f(\pi)$

LHL = $f(\pi)^{-} = \lim_{x\to\pi}\frac{1-\cos 7(x-\pi)}{5(x-\pi)^2}$

$$\Rightarrow \lim_{h\to 0}\frac{1-\cos 7(\pi-h-\pi)}{5(\pi-h)-\pi)^2}$$

$$\Rightarrow \lim_{h\to 0}\frac{1-\cos 7(-h)}{5(-h)^2}$$

$$\Rightarrow \lim_{h\to 0}\frac{1-\cos 7h}{5h^2}$$

$$\Rightarrow \lim_{h\to 0}\frac{2\sin^2\frac{7h}{2}}{5h^2}$$

$$\Rightarrow \lim_{h\to 0}\frac{2}{5}\times\left(\frac{\sin\frac{7h}{2}}{\frac{7h}{2}}\right)^2\times\left(\frac{7}{2}\right)^2$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{\sin\frac{7h}{2}}{\frac{7h}{2}}\right)^2\times\frac{2}{5}\times\left(\frac{7}{2}\right)^2$$

$$\therefore \lim_{h\to 0}\left(\frac{\sin\frac{7h}{2}}{\frac{7h}{2}}\right) = 1$$

$$\Rightarrow 1^2\times\frac{2}{5}\times\left(\frac{49}{4}\right)$$

$$\Rightarrow \frac{49}{10}$$

$\therefore f(\pi) = \frac{49}{10}$

34. Question

If $f(x)=\frac{2x+3\sin x}{3x+2\sin x}, x\neq 0$ is continuous at x = 0, then find f(0).

Answer

Given:

f(x) is continuous at x = 0

If f(x) to be continuous at x = 0,then

$f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x\to 0}\frac{2x + 3\sin x}{3x + 2\sin x}$

$\Rightarrow \lim_{h\to 0}\frac{2(0-h) + 3\sin(0-h)}{3(0-h) + 2\sin(0-h)}$

$\Rightarrow \lim_{h\to 0}\frac{2(-h) + 3\sin(-h)}{3(-h) + 2\sin(-h)}$

$\Rightarrow \lim_{h\to 0}\frac{-2h - 3\sinh}{-3h - 2\sinh}$

$\Rightarrow \lim_{h\to 0}\frac{-(2h + 3\sinh)}{-(3h + 2\sinh)}$

$\Rightarrow \lim_{h\to 0}\frac{(2h + 3\sinh)}{(3h + 2\sinh)}$

$\Rightarrow \lim_{h\to 0}\frac{h(2 + \frac{3\sinh}{h})}{h(3 + \frac{2\sinh}{h})}$

$\Rightarrow \lim_{h\to 0}\frac{(2 + \frac{3\sinh}{h})}{(3 + \frac{2\sinh}{h})}$

$\Rightarrow \frac{(2 + 3\times\lim_{h\to 0}\frac{\sinh}{h})}{(3 + 2\times\lim_{h\to 0}\frac{\sinh}{h})}$

$\therefore \lim_{h\to 0}\frac{\sinh}{h} = 1$

$\Rightarrow \frac{(2 + 3\times 1)}{(3 + 2\times 1)}$

$\Rightarrow \frac{5}{5}$

$\Rightarrow 1$

Hence,f(0) = 1

35. Question

Find the value of k for which $f(x) = \begin{cases} \frac{1-\cos 4x}{8x^2} & ,\text{ when } x \neq 0 \\ k & ,\text{ when } x = 0 \end{cases}$ is continuous at x = 0.

Answer

Given:

f(x) is continuous at x = 0 & f(0) = k

If f(x) to be continuous at x = 0,then,

$f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x\to 0} \frac{1-\cos 4x}{8x^2}$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos 4(0-h)}{8(0-h)^2}$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos 4(-h)}{8(-h)^2}$

$\Rightarrow \lim_{x\to 0} \frac{1-\cos 4h}{8h^2}$

$\cos 2x = 1-2\sin^2 x$ $1-\cos 2x = 2\sin^2 x$ $1-\cos 4x = 2\sin^2 2x$

$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 2h}{8h^2}$

$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 2h}{2\times 4h^2}$

$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 2h}{2\times (2h)^2}$

$\Rightarrow \lim_{x\to 0} \left(\frac{\sin 2h}{2h}\right)^2$

$\Rightarrow 1$

Since f(x) is continuous at x = 0 & f(0) = k,then

k = 1

36 A. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$$f(x) = \begin{cases} \frac{1-\cos 2kx}{x^2} & \text{, if } x \neq 0 \\ 8 & \text{, if } x = 0 \end{cases} \text{ at } x = 0$$

Answer

Given:

f(x) is continuous at x = 0 & f(0) = 8

If f(x) to be continuous at x = 0,then,

$f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x\to 0} \frac{1-\cos 2kx}{x^2}$

$\Rightarrow \lim_{h\to 0} \frac{1-\cos 2k(0-h)}{(0-h)^2}$

$\Rightarrow \lim_{h\to 0} \frac{1-\cos 2k(-h)}{(-h)^2}$

$\cos(-x) = \cos x$

$\cos 2x = 1-2\sin^2 x$

1–cos2x = 2sin^{2}x

1–cos2x = 2sin^{2}x

$\Rightarrow \lim_{h\to 0}\frac{1-\cos 2kh}{h^2}$

$\Rightarrow \lim_{h\to 0}\frac{2\sin^2 kh}{h^2}$

$\Rightarrow \lim_{h\to 0}\frac{2\sin^2 h}{(kh)^2}\times k^2$

$\Rightarrow 2k^2\lim_{h\to 0}\left(\frac{\sinh}{h}\right)^2$

$\Rightarrow 2k^2$

Since f(x) is continuous at x = 0 & f(0) = 8,then

2 k^2 = 8

⇒ k^2 = 4

⇒ k = ±2

36 B. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$f(x)=\begin{cases}(x-1)\tan\frac{\pi x}{2} & , \text{if } x\neq 1\\ k & , \text{if } x=1\end{cases}$ at x = 1

Answer

Given:

f(x) is continuous at x = 1 & f(1) = k

If f(x) to be continuous at x = 0,then, f(1)$^-$ = f(1)$^+$ = f(1)

LHL = f(1)$^-$ = $\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$

$\Rightarrow \lim_{h\to 0}((1-h)-1)\tan\frac{\pi(1-h)}{2}$

$\Rightarrow \lim_{h\to 0}(-h)\tan\frac{\pi(1-h)}{2}$

$\Rightarrow \lim_{h\to 0}(-h)\tan\frac{\pi(1-h)}{2}$

$\Rightarrow \lim_{h\to 0}(-h)\tan\left(\frac{\pi}{2}-\frac{\pi h}{2}\right)$

∴ tan($\frac{\pi}{2}$–x) = cotx

$\Rightarrow \lim_{h\to 0}(-h)\cot\left(\frac{\pi h}{2}\right)$

$\Rightarrow \lim_{h\to 0}(-h)\frac{\cos(\frac{\pi h}{2})}{\sin(\frac{\pi h}{2})}$

$$\Rightarrow \lim_{h\to 0}(-h)\frac{\cos(\frac{\pi h}{2})}{\frac{\sin(\frac{\pi h}{2})}{\frac{\pi h}{2}}\times\frac{\pi h}{2}}$$

$$\Rightarrow (-h)\frac{\lim_{h\to 0}\cos(\frac{\pi h}{2})}{\lim_{h\to 0}\left(\frac{\sin(\frac{\pi h}{2})}{\frac{\pi h}{2}}\right)\times\frac{\pi h}{2}}$$

$$\therefore \lim_{h\to 0}\left(\frac{\sin(\frac{\pi h}{2})}{\frac{\pi h}{2}}\right) = 1$$

$$\Rightarrow \lim_{h\to 0}(-1)\frac{\cos(\frac{\pi h}{2})}{1\times\frac{\pi}{2}}$$

$\therefore$ cos(0) = 1

$$\Rightarrow \frac{-1\times 1}{\frac{\pi}{2}}$$

$$\frac{-2}{\pi}$$

36 C. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$$f(x)=\begin{cases} k(x^2-2x) & \text{, if } x<0 \\ \cos x & \text{, if } x\geq 0 \end{cases}$$ at x = 0

Answer

Given:

f(x) is continuous at x = 0

If f(x) to be continuous at x = 0,then,$f(0)^- = f(0)^+ = f(0)$

RHL = $f(0)^- = \lim_{x\to 0}\cos x$

$\Rightarrow \lim_{h\to 0}\cos(0-h)$

$\Rightarrow \lim_{h\to 0}\cos(-h)$

$\Rightarrow \lim_{h\to 0}\cos(h)$

$\therefore$cos(0) = 1

$f(0)^+ \Rightarrow 1$

LHL = $f(0)^- = \lim_{x\to 0}k(x^2-2x)$

$\Rightarrow \lim_{h\to 0}k((0-h)^2-2(0-h))$

$\Rightarrow \lim_{h\to 0}k((-h)^2-2(-h))$

$\Rightarrow \lim_{h \to 0} k(h^2 + 2h)$

$\Rightarrow \lim_{h \to 0} kh(h + 2)$

$\Rightarrow k \times 0$

$\Rightarrow 0$

Hence, the value of k = 0.

36 D. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$f(x) = \begin{cases} kx + 1 & , \text{if } x \le 5 \\ 3x - 5 & , \text{if } x > 5 \end{cases}$ at x = 5

Answer

Given:

f(x) is continuous at x = 5

If f(x) to be continuous at x = 5,then,$f(5)^- = f(5)^+ = f(5)$

LHL = $f(5)^- = \lim_{x \to 5} kx + 1$

$\Rightarrow \lim_{h \to 0} k(5 - h) + 1$

$\Rightarrow 5k + 1$

RHL = $f(5)^+ = \lim_{x \to 0} 3x - 5$

$\Rightarrow \lim_{h \to 0} 3(5 + h) - 5$

$\Rightarrow \lim_{h \to 0} 15 + 3h - 5$

$\Rightarrow \lim_{h \to 0} 10 + 3h$

$\Rightarrow 10$

Since, f(x) is continuous at x = 5

we have,5k + 1 = 10

$\Rightarrow 5k = 9$

$\Rightarrow k = \frac{9}{5}$

The value of k is $\frac{9}{5}$

36 E. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5} & , x \ne 5 \\ k & , x = 5 \end{cases}$ at x = 5

Answer

Given:

f(x) is continuous at x = 5 & f(5) = k

If f(x) to be continuous at x = 5,then,$f(5)^{-} = f(5)^{+} = f(5)$

LHL = $f(5)^{-} = \lim\limits_{x\to5}\frac{x^2-25}{x-5}$

$\Rightarrow \lim\limits_{h\to0}\frac{(5-h)^2-25}{(5-h)-5}$

$\therefore (a-b)^2 = a^2 - 2ab + b^2$

$\Rightarrow \lim\limits_{h\to0}\frac{25-10h+h^2-25}{5-h-5}$

$\Rightarrow \lim\limits_{h\to0}\frac{-10h+h^2}{-h}$

$\Rightarrow \lim\limits_{h\to0}\frac{10h-h^2}{h}$

$\Rightarrow \lim\limits_{h\to0}\frac{h(10-h)}{h}$

$\Rightarrow \lim\limits_{h\to0}\frac{(10-h)}{1}$

$\Rightarrow \lim\limits_{h\to0}(10-h)$

$\Rightarrow 10$

Since ,f(x) is continuous at x = 5 & f(5) = k

k = 10

36 F. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$f(x)=\begin{cases} kx^2 & , x\geq 1 \\ 4 & , x<1 \end{cases}$ at x = 1

Answer

Given:

f(x) is continuous at x = 1

If f(x) to be continuous at x = 1,then,$f(1)^{-}=f(1)^{+}=f(1)$

LHL = $f(1)^{-} = \lim\limits_{x\to1}4$

$\Rightarrow$ 4 ....(1)

RHL = $f(1)^{+} = \lim\limits_{x\to1}kx^2$

$\Rightarrow \lim\limits_{h\to0}k(1-h)^2$

$\Rightarrow k(1-0)^2$

$\Rightarrow$ k ....(2)

Since, f(x) is continuous at x = 1 & also

from (1) & (2)

$\therefore$ k = 4

36 G. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$$f(x) = \begin{cases} k(x^2+2) & , \text{if } x \le 0 \\ 3x+1 & , \text{if } x > 0 \end{cases}$$ at x = 0

Answer

Given:

f(x) is continuous at x = 0

If f(x) to be continuous at x = 0,then,$f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x\to 0} k(x^2 + 2)$

$\Rightarrow \lim_{h\to 0} k((0-h)^2 + 2)$

$\Rightarrow \lim_{h\to 0} k((-h)^2 + 2)$

$\Rightarrow \lim_{h\to 0} k(h^2 + 2)$

$\Rightarrow$ k(0 + 2)

$\Rightarrow$ 2k ...(1)

RHL = $f(0)^+ = \lim_{x\to 0} 3x + 1$

$\Rightarrow \lim_{h\to 0} 3(0 + h) + 1$

$\Rightarrow \lim_{h\to 0} 3h + 1$

$\Rightarrow$ 1 ...(2)

Since, f(x) is continuous at x = 0,From (1) & (2),we get,

2k = 1

$\Rightarrow k = \frac{1}{2}$

36 H. Question

In each of the following, find the value of the constant k so that the given function is continuous at the indicated point :

$$f(x) = \begin{cases} \frac{x^3+x^2-16x+20}{(x-2)^2} & , x \ne 2 \\ k & , x = 2 \end{cases}$$ at x = 2.

Answer

Given:

f(x) is continuous at x = 2 & f(2) = k

If f(x) to be continuous at x = 2,then,$f(2)^- = f(2)^+ = f(2)$ LHL = $f(2)^- = \lim_{x\to 2} \frac{x^3+x^2-16x+20}{(x-2)^2}$

$\Rightarrow \lim_{h\to 0}\frac{(2-h)^3 + (2-h)^2 - 16(2-h) + 20}{((2-h)-2)^2}$

$\Rightarrow \lim_{h\to 0}\frac{[2^3 - h^3 - 3\times 2^2\times(-h) + 3\times 2\times h^2] + [2^2 - 2\times 2\times h + h^2] - 32 + 16h + 20}{(-h)^2}$

$\therefore (a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$

$(a-b)^2 = a^2 - 2ab + b^2$

$\Rightarrow \lim_{h\to 0}\frac{[8 - h^3 - 12h + 6h^2] + [4 - 4h + h^2] - 32 + 16h + 20}{(h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{8 - h^3 - 12h + 6h^2 + 4 - 4h + h^2 - 32 + 16h + 20}{(h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{-h^3 + (6h^2 + h^2) + (16h - 12h - 4h) + (8 + 4 + 20 - 32)}{(h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{-h^3 + 7h^2}{(h)^2}$

$\Rightarrow \lim_{h\to 0}\frac{h^2(-h + 7)}{(h)^2}$

$\Rightarrow \lim_{h\to 0}(-h + 7)$

$\Rightarrow$ 7

Since ,f(x) is continuous at x = 2 & f(2) = k

k = 7

37. Question

Find the values of a and b so that the function f given by $f(x) = \begin{cases} 1 & , \text{if} \quad x \le 3 \\ ax + b & , \text{if} \quad x < x < 5 \\ 7 & , \text{if} \quad x \ge 5 \end{cases}$ is continuous at x = 3 and x = 5

Answer

Given:

f(x) is continuous at x = 3 & x = 5

If f(x) to be continuous at x = 3,then,$f(3)^- = f(3)^+ = f(3)$

LHL = $f(3)^- = \lim_{x\to 3} 1$

= 1 ...(1)

RHL = $f(3)^+ = \lim_{x\to 3} ax + b$

$\Rightarrow \lim_{h\to 0} a(3 + h) + b$

⇒ a(3 + 0) + b

⇒ 3a + b ...(2)

Since ,f(x) is continuous at x = 3 and From (1) & (2),we get

3a + b = 1 ...(3)

Similarly ,f(x) is continuous at x = 5

If f(x) to be continuous at x = 5,then, $f(5)^{-} = f(5)^{+} = f(5)$

LHL = $f(5)^{-} = \lim_{x\to 5} ax + b$

$\Rightarrow \lim_{h\to 0} a(5-h) + b$

⇒ a(5–0) + b

⇒ 5a + b ...(4)

RHL = $f(5)^{+} = \lim_{x\to 5} 7$

⇒ 7 ...(5)

Since , f(x) is continuous at x = 5 and From (4) & (5),we get,

5a + b = 7 ...(6)

Now equate (3) & (6)

$$\begin{array}{l} 3a + b = 1 \\ 5a + b = 7 \\ \hline -2a = -6 \end{array}$$

⇒ a = 3

Now Substitute a = 3 in any one of above equation(3) & (6) ,

3a + b = 1

⇒ 3(3) + b = 1

⇒ 9 + b = 1

⇒ b = –8

38. Question

If $f(x) = \begin{cases} \frac{x^2}{2} & ,\text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \frac{3}{2} & ,\text{if } 1 < x \le 2 \end{cases}$. Show that f is continuous at x = 1.

Answer

Given:

For f(x) is continuous at x = 1

If f(x) to be continuous at x = 1,we have to show, $f(1)^{-} = f(1)^{+} = f(1)$

LHL = $f(1)^{-} = \lim_{x\to 1}(\frac{x^2}{2})$

$\Rightarrow \lim_{h\to 0}(\frac{(1-h)^2}{2})$

$\Rightarrow \lim_{h\to 0}(\frac{(1^2 - 2\times 1\times h + h^2)}{2})$

$\therefore (a-b)^2 = a^2 - 2ab + b^2$

$\Rightarrow \lim_{h\to 0}(\frac{1-2h + h^2}{2})$

$\Rightarrow \dfrac{1-2\times 0 + 0^2}{2}$

$\Rightarrow \frac{1}{2}$ ...(1)

RHL = f(1) $^+$ = $\lim\limits_{x\to 1}(2x^2 - 3x + \frac{3}{2})$

$\Rightarrow \lim\limits_{h\to 0}(2(1+h)^2 - 3(1+h) + \dfrac{3}{2})$ $\therefore (a+b)^2 = a^2 + 2ab + b^2$

$\Rightarrow \lim\limits_{h\to 0} 2(1^2 + 2\times 1\times h + h^2) - 3(1+h) + \dfrac{3}{2}$

$\Rightarrow \lim\limits_{h\to 0} 2(1 + 2h + h^2) - 3(1+h) + \dfrac{3}{2}$

$\Rightarrow \lim\limits_{h\to 0}(2 + 4h + 2h^2 - 3 - 3h + \dfrac{3}{2})$

$\Rightarrow \lim\limits_{h\to 0}(2h^2 + 4h - 3h + 2 - 3 + \dfrac{3}{2})$

$\Rightarrow \lim\limits_{h\to 0}(2h^2 + h + \dfrac{1}{2})$

$\Rightarrow 2\times 0^2 + 0 + \frac{1}{2}$

$\Rightarrow \frac{1}{2}$ ...(2)

From (1) & (2),we get f(1)$^-$ = f(1) $^+$

Hence ,f(x) is continuous at x = 1

39 A. Question

Discuss the continuity of the f(x) at the indicated points :

f(x) = |x| + |x – 1| at x = 0, 1.

Answer

To prove whether f(x) is continuous at x = 0 & 1

If f(x) to be continuous at x = 0,we have to show that f(0)$^-$=f(0) $^+$ = f(0)

LHL = f(0)$^-$ = $\lim\limits_{x\to 0}(|x| + |x-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(0-h)| + |(0-h)-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(-h)| + |(-h)-1|)$

⇒ |(–0)| + |(–0)–1|

∴–|x| = |x| = x

⇒ |–1|

⇒ 1 ...(1)

RHL = f(0) $^+$ = $\lim\limits_{x\to 0}(|x| + |x-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(0+h)| + |(0+h)-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(h)| + |(h)-1|)$

⇒ |0| + |0–1|

∴–|x| = |x| = x

⇒ |–1|

⇒ 1 ...(2)

From (1) & (2),we get $f(0)^{-}=f(0)^{+}$

Hence ,f(x) is continuous at x = 0

If f(x) to be continuous at x = 1,we have to show, $f(1)^{-}=f(1)^{+}=f(1)$

LHL = $f(1)^{-} = \lim\limits_{x\to 1}(|x| + |x-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(1-h)| + |(1-h)-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(1-h)| + |(-h)|)$

⇒ |(1–0) + (–0)|

⇒ |(1)|

⇒ 1 ...(3)

RHL = $f(1)^{+} = \lim\limits_{x\to 1}(|x| + |x-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(1+h)| + |(1+h)-1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(1+h)| + |(h)|)$

⇒ |(1 + 0)| + |0|

∴–|x| = |x| = x

⇒ |1|

⇒ 1 ...(4)

From (3) & (4),we get $f(1)^{-} = f(1)^{+}$

Hence ,f(x) is continuous at x = 1

39 B. Question

Discuss the continuity of the f(x) at the indicated points :

f(x) = |x – 1| + |x + 1| at x = – 1, 1.

Answer

To prove whether f(x) is continuous at –1 & 1

If f(x) to be continuous at x = –1,we have to show, $f(-1)^{-}=f(-1)^{+}=f(-1)$

LHL = $f(-1)^{-} = \lim\limits_{x\to -1}(|x-1| + |x+1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(-1-h)-1| + |(-1-h)+1|)$

$\Rightarrow \lim\limits_{h\to 0}(|(-2-h)| + |(-h)|)$

⇒ |(–2–0)| + |–0|

⇒ |–2|

∴|–x| = |x| = x

⇒ 2 ...(1)

RHL = f(–1) $^{+}$ = $\lim_{x \to -1}(|x-1| + |x+1|)$

$\Rightarrow \lim_{h \to 0}(|(-1+h-1)| + |(-1+h)+1|)$

$\Rightarrow \lim_{h \to 0}(|(-2+h)| + |(h)|)$

⇒ |(–2 + 0)| + |0|

∴|-x| = |x| = x

⇒ |-2|

⇒ 2 ...(2)

From (1) & (2),we get $f(-1)^{-}=f(-1)^{+}$

Hence ,f(x) is continuous at x = –1

If f(x) to be continuous at x = 1,we have to show, $f(1)^{-} = f(1)^{+} = f(1)$

LHL = $f(1)^{-} = \lim_{x \to 1}(|x-1| + |x+1|)$

$\Rightarrow \lim_{h \to 0}(|(1-h-1)| + |(1-h)+1|))$

$\Rightarrow \lim_{h \to 0}(|(-h)| + |(2-h)|)$

⇒ |–0| + |2–0|

⇒ |2|

⇒ 2 ...(3)

RHL = $f(1)^{+} = \lim_{x \to 1}(|x-1| + |x+1|)$

$\Rightarrow \lim_{h \to 0}(|(1+h-1)| + (|1+h+1|))$

$\Rightarrow \lim_{h \to 0}(|(h)| + |(2+h)|)$

⇒ |0| + |2 + 0|

∴|-x| = |x| = x

⇒ |2|

⇒ 2 ...(4)

From (3) & (4),we get $f(1)^{-}=f(1)^{+}$

Hence ,f(x) is continuous at x = 1

40. Question

Prove that $f(x) = \begin{cases} \dfrac{x-|x|}{x} & , x \neq 0 \\ 2 & , x = 0 \end{cases}$ is discontinuous at

x = 0.

Answer

Given:

f(0) = 2

we have to prove that f(x) is discontinuous at x = 0

If f(x) to be discontinuous at x = 0,then $f(0)^{-} \neq f(0)^{+}$

LHL = $f(0)^{-}$ = $\lim_{x\to0}\frac{x-|x|}{x}$

$\Rightarrow \lim_{h\to0}\frac{(0-h)-|0-h|}{(0-h)}$

$\Rightarrow \lim_{h\to0}\frac{(-h)-|-h|}{(0-h)}$

$\therefore |-x| = |x| = x$

$\Rightarrow \lim_{h\to0}\frac{-h-h}{-h}$

$\Rightarrow \lim_{h\to0}\frac{-2h}{-h}$

$\Rightarrow \lim_{h\to0} 2$

$\Rightarrow$ 2 ...(1)

RHL = $f(0)^{+}$ = $\lim_{x\to0}\frac{x-|x|}{x}$

$\Rightarrow \lim_{h\to0}\frac{(0\ +\ h)-|0\ +\ h|}{(0\ +\ h)}$

$\Rightarrow \lim_{h\to0}\frac{(h)-|h|}{(0-h)}$

$\Rightarrow \lim_{h\to0}\frac{h-h}{-h}$

$\Rightarrow \lim_{h\to0} 0$

$\Rightarrow$ 0 ...(2)

From (1) & (2),we know that,

$f(0)^{-} \neq f(0)^{+}$

Hence, f(x) is discontinuous at x = 0

41. Question

If $f(x)=\begin{cases} 2x^2+k & ,\text{ if } x\geq 0 \\ -2x^2+k & ,\text{ if } x<0 \end{cases}$, then what should be the value of k so that f(x) is continuous at x = 0.

Answer

we have to find the value of 'k' such that f(x) is continuous at x = 0

If f(x) is be continuous at x = 0,then, $f(0)^{-}=f(0)^{+} = f(0)$

LHL = $f(0)^{-}=\lim_{x\to0} -2x^2\ +\ k$

$\Rightarrow \lim_{h\to0} -2(0-h)^2\ +\ k$

$\Rightarrow \lim_{h\to0} -2(-h)^2\ +\ k$

$\Rightarrow \lim_{h\to0} -2h^2\ +\ k$

$\Rightarrow -2\times0^2 + k$

$\Rightarrow$ k ...(1)

RHL = f(0) $^{+}$ =$\lim\limits_{x\to 0} 2x^2 + k$

⇒ $\lim\limits_{h\to 0} 2(0 + h)^2 + k$

⇒ $\lim\limits_{h\to 0} 2(h)^2 + k$

⇒ $\lim\limits_{h\to 0} 2h^2 + k$

⇒ $2\times 0^2 + k$

⇒ k ...(2)

From (1) & (2),we get , $f(0)^{-} = f(0)^{+}$

So the value of k can be any real number(R),so that f(x) is continuous at x = 0

42. Question

For what value of is the function $f(x) = \begin{cases} \lambda(x^2 - 2x) & , \text{if } x \le 0 \\ 4x + 1 & , \text{if } x > 0 \end{cases}$ continuous at x = 0? What about continuity at x = ± 1?

Answer

we have to find the value of 'λ' such that f(x) is continuous at x = 0

If f(x) is be continuous at x = 0,then, $f(0)^{-}=f(0)^{+}=f(0)$

LHL = $f(0)^{-}$= $\lim\limits_{x\to 0} \lambda(x^2 - 2x)$

⇒ $\lim\limits_{h\to 0} \lambda((0 - h)^2 - 2(0 - h))$

⇒ $\lim\limits_{h\to 0} \lambda((-h)^2 - 2(-h))$

⇒ $\lim\limits_{h\to 0} \lambda(h^2 + 2h)$

⇒ $\lambda(0^2 + 2\times 0)$

⇒ 0 ...(1)

RHL = f(0) $^{+}$ = $\lim\limits_{x\to 0}(4x + 1)$

⇒ $\lim\limits_{h\to 0} 4(0 + h) + 1)$

⇒ $\lim\limits_{h\to 0} 4(h) + 1)$

⇒ 4(0) + 1

⇒ 1 ...(2)

From (1) & (2),we get $f(0)^{-}=f(0)^{+}$,

Hence f(x) is not continuous at x = 0

we also have to find out the continuity at point ± 1

For f(x) is be continuous at x = 1,

then ,$f(0)^{-}=f(0)^{+}=f(0)$

LHL = f(1) $^{+}$ = $\lim\limits_{x\to 1} \lambda(x^2 - 2x)$

$\Rightarrow \lim\limits_{h\to 0} \lambda((1-h)^2 - 2(1-h))$

$\Rightarrow \lim\limits_{h\to 0} \lambda(1^2 - 2h + h^2 - 2 + 2h)$

$\Rightarrow \lim\limits_{h\to 0} \lambda(h^2 - 1)$

$\Rightarrow \lambda(0^2-1)$

$\Rightarrow -\lambda$ ...(1)

RHL = $f(1)^+ = \lim\limits_{x\to 1}(4x + 1)$

$\Rightarrow \lim\limits_{h\to 0} 4(1 + h) + 1)$

$\Rightarrow \lim\limits_{h\to 0}(4 + 4h + 1)$

$\Rightarrow (5 + 4\times0)$

$\Rightarrow 5$ ...(2)

From (1) & (2),we get $f(0)^- = f(0)^+$,

i.e, $-\lambda = 5$

$\Rightarrow \lambda = -5$

Hence f(x) is continuous at x = 1,when $\lambda = -5$

Similarly, For f(x) is be continuous at x = –1,

then ,$f(-1)^- = f(-1)^+ = f(-1)$

LHL = $f(-1)^- = \lim\limits_{x\to -1} \lambda(x^2 - 2x)$

$\Rightarrow \lim\limits_{h\to 0} \lambda((-1-h)^2 - 2(-1-h))$

$\Rightarrow \lim\limits_{h\to 0} -\lambda((1 + h)^2 + 2(1 + h))$

$\Rightarrow \lim\limits_{h\to 0} -\lambda(1^2 + 2h + h^2 + 2 + 2h)$

$\Rightarrow \lim\limits_{h\to 0} -\lambda(h^2 + 4h + 3)$

$\Rightarrow -\lambda(0^2 + 4\times0 + 3)$

$\Rightarrow -3\lambda$ ...(3)

RHL = $f(-1)^+ = \lim\limits_{x\to -1}(4x + 1)$

$\Rightarrow \lim\limits_{h\to 0} 4(-1 + h) + 1)$

$\Rightarrow \lim\limits_{h\to 0}(-4 + 4h + 1)$

$\Rightarrow (-3 + 4\times0)$

$\Rightarrow -3$ ...(2)

From (1) & (2),we get, $f(-1)^- = f(-1)^+$

i.e, $-3\lambda = -3$

$\Rightarrow \lambda = 1$

Hence f(x) is continuous at x = 1,when $\lambda = 1$

43. Question

For what value of k is the following function continuous at x = 2?

$$f(x)=\begin{cases}2x+1 & ; \quad \text{if } x<2\\ k & ; \quad x=2\\ 3x-1 & ; \quad x>2\end{cases}$$

Answer

Given:

For f(x) is continuous at x = 2 & f(2) = k

If f(x) to be continuous at x = 2,we have to show, $f(2)^{-}=f(2)^{+}=f(2)$

LHL = $f(2)^{-} = \lim_{x\to 2}(2x + 1)$

$\Rightarrow \lim_{h\to 0}(2(2-h) + 1)$

$\Rightarrow \lim_{h\to 0}(4-4h + 1)$

$\Rightarrow \lim_{h\to 0}(5-4h)$

$\Rightarrow (5-4\times 0)$

$\Rightarrow 5$ ...(1)

RHL = $f(2)^{+} = \lim_{x\to 2}(3x-1)$

$\Rightarrow \lim_{h\to 0}(3(2 + h)-1)$

$\Rightarrow \lim_{h\to 0}(6 + 3h-1)$

$\Rightarrow \lim_{h\to 0}(5 + 3h)$

$\Rightarrow (5 - 3 \times 0)$

$\Rightarrow 5$ ...(2)

Since , f(x) is continuous at x = 2 & f(2) = k

k = 5

44. Question

Let $f(x)=\begin{cases}\dfrac{1-\sin^3 x}{3\cos^2 x} & , \text{if} \quad x<\dfrac{\pi}{2}\\ a & , \text{if} \quad x=\dfrac{\pi}{2}\\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & , \text{if} \quad x>\dfrac{\pi}{2}\end{cases}$. If f(x) is continuous at $\frac{\pi}{2}$ find a and b.

Answer

Given:

f(x) is continuous at x = $\frac{\pi}{2}$ & $f(\frac{\pi}{2})$ = a,

If f(x) to be continuous at x = $\frac{\pi}{2}$,we have to show, $f(\frac{\pi}{2})^{-} = f(\frac{\pi}{2})^{+} = f(\frac{\pi}{2})$

LHL = $f(\frac{\pi}{2})^- = \lim_{x\to\frac{\pi}{2}}(\frac{1-\sin^3x}{3\cos^2x})$

$$\Rightarrow \lim_{h\to 0}(\frac{1-\sin^3(\frac{\pi}{2}-h)}{3\cos^2(\frac{\pi}{2}-h)})$$

$\therefore \sin(\frac{\pi}{2}-x) = \cos x$

$\therefore \cos(\frac{\pi}{2}-x) = \sin x$

$$\Rightarrow \lim_{h\to 0}(\frac{1-\cos^3h}{3\sin^2h})$$

$$\Rightarrow \lim_{h\to 0}\frac{1^3-\cos^3h}{3\sin^2h}$$

$$\Rightarrow \lim_{h\to 0}\frac{(1-\cosh)(1^2 + \cosh + \cos^2h)}{3(1-\cos^2h)}$$

$\therefore (a^3-b^3) = (a-b)(a^2 + ab + b^2)$

$$\Rightarrow \lim_{h\to 0}\frac{(1-\cosh)(1^2 + \cosh + \cos^2h)}{3(1^2-\cos^2h)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(1-\cosh)(1^2 + \cosh + \cos^2h)}{3(1-\cosh)(1 + \cosh)}$$

$$\Rightarrow \lim_{h\to 0}\frac{(1^2 + \cosh + \cos^2h)}{3(1 + \cosh)}$$

$$\Rightarrow \frac{(1^2 + \cos 0 + \cos^2 0)}{3(1 + \cos 0)}$$

$\therefore \cos(0) = 1$

$$\Rightarrow \frac{(1 + 1 + 1)}{3(1 + 1)}$$

$$\Rightarrow \frac{(3)}{3(2)}$$

$\Rightarrow \frac{1}{2}$ ...(1)

LHL = $f(\frac{\pi}{2})^+ = \lim_{x\to\frac{\pi}{2}}(\frac{b(1-\sin x)}{(\pi-2x)^2})$

$$\Rightarrow \lim_{h\to 0}\left\{\frac{b\left[1-\sin\left(\frac{\pi}{2} + h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2} + h\right)\right]^2}\right\}$$

$$\therefore \sin\left(\frac{\pi}{2} + x\right) = \cos x$$

$$\Rightarrow \lim_{h\to 0}\left\{\frac{b[1-\cosh]}{\left[\pi-2\times\frac{\pi}{2}-2\times h\right]^2}\right\}$$

$$\Rightarrow \lim_{h\to 0}\left\{\frac{b[1-\cosh]}{[\pi-\pi-2h]^2}\right\}$$

$$\Rightarrow \lim_{h\to 0}\left\{\frac{b[1-\cosh]}{[-2h]^2}\right\}$$

$\Rightarrow \lim_{h\to 0}\left\{\frac{b\times 2\sin^2\frac{h}{2}}{4h^2}\right\}$

$\Rightarrow \lim_{h\to 0}\left\{\frac{b\times 2\sin^2\frac{h}{2}}{4\times\left(\frac{h^2}{4}\right)}\times\frac{1}{4}\right\}$

$\cos 2x = 1-2\sin^2 x$
$1-\cos 2x = 2\sin^2 x$
$1-\cos x = 2\sin^2\frac{x}{2}$

$\Rightarrow \frac{b}{8}\lim_{h\to 0}\left\{\frac{\sin^2\frac{h}{2}}{\left(\frac{h^2}{4}\right)}\right\}$

$\Rightarrow \frac{b}{8}\times 1$

$\Rightarrow \frac{b}{8}$ ...(2)

f(x) is continuous at $x = \frac{\pi}{2}$ & $f(\frac{\pi}{2}) = a$,and from (1) & (2),we get

$f(\frac{\pi}{2})^- = f(\frac{\pi}{2})^+ = f(\frac{\pi}{2})$

$\Rightarrow \frac{1}{2} = \frac{b}{8} = a$

$\Rightarrow \therefore a = \frac{1}{2}$

$\Rightarrow \frac{b}{8} = \frac{1}{2}$

$\Rightarrow b = 4$

Hence ,$a = \frac{1}{2}$ & $b = 4$

45. Question

If the function f(x), defined below is continuous at x = 0, find the value of k:

$$f(x)=\begin{cases}\dfrac{1-\cos 2x}{2x^2} & , \quad x<0\\ k & , \quad x=0\\ \dfrac{x}{|x|} & , \quad x>0\end{cases}$$

Answer

we have to find the value of 'k'

Given:

f(x) is continuous at x = 0 & f(0) = k

If f(x) is be continuous at x = 0,then,

$f(0)^- = f(0)^+ = f(0)$

LHL = $f(0)^- = \lim_{x\to 0}\left(\frac{1-\cos 2x}{2x^2}\right)$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{1-\cos 2(0-h)}{2(0-h)^2}$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{1-\cos 2(-h)}{2(-h)^2}$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{1-\cos 2h}{2h^2}$

$\therefore \cos(0) = 1$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{2\sin^2 h}{2h^2}$

$\cos 2x = 1-2\sin^2 x$ $1-\cos 2x = 2\sin^2 x$

$\Rightarrow \lim\limits_{h\to 0}\dfrac{\sin^2 h}{h^2}$

$\Rightarrow \lim\limits_{h\to 0}\left(\dfrac{\sin h}{h}\right)^2$

$\therefore \lim\limits_{h\to 0}\dfrac{\sin h}{h} = 1$

$\Rightarrow (1)^2$

$\Rightarrow 1$

RHL = $f(0)^{+} = \lim\limits_{x\to 0}\left(\frac{x}{|x|}\right)$

$\Rightarrow \lim\limits_{h\to 0}\left(\dfrac{0+h}{|0+h|}\right)$

$\Rightarrow \lim\limits_{h\to 0}\left(\dfrac{h}{|h|}\right)$

$\Rightarrow \lim\limits_{h\to 0}\left(\dfrac{1}{|1|}\right)$

$\Rightarrow \lim\limits_{h\to 0}(1)$

$\Rightarrow 1$

Since , f(x) is continuous at x = 0 & f(0) = k

And also , $f(0)^{-} = f(0)^{+} = f(0)$

So ,k = 1

46. Question

Find the relationship between 'a' and 'b' so that the function 'f' defined by $f(x) = \begin{cases} ax+1 & , \text{if } x \le 3 \\ bx+3 & , \text{if } x > 3 \end{cases}$ is continuous at x = 3.

Answer

we have to find the value of 'a' & 'b'

Given:

f(x) is continuous at x = 3

If f(x) is be continuous at x = 3,then,$f(3)^{-} = f(3)^{+} = f(3)$

LHL = $f(3)^{-} = \lim\limits_{x \to 3}(ax + 1)$

$\Rightarrow \lim\limits_{h \to 0}(a(3-h) + 1)$

$\Rightarrow \lim\limits_{h \to 0}(3a - ha + 1)$

⇒ 3a – 0 × a + 1

⇒ 3a + 1 .....(1)

LHL = $f(3)^{+} = \lim\limits_{x \to 3}(bx + 3)$

$\Rightarrow \lim\limits_{h \to 0} b(3 + h) + 3$

$\Rightarrow \lim\limits_{h \to 0} 3b + hb + 3$

⇒ 3b – 0 × b + 3

⇒ 3b + 3 ...(2)

Since ,f(x) is continuous at x = 3 and From (1) & (2),we get

3a + 1 = 3b + 3

⇒ 3a + 3b = 3 – 1

⇒ 3a + 3b = 2

⇒ 3(a + b) = 2

⇒ (a + b) = $\frac{2}{3}$

Exercise 9.2

1. Question

Prove that the function $f(x) = \begin{cases} \frac{\sin x}{x} & , x < 0 \\ x + 1 & , x \geq 0 \end{cases}$ is everywhere continuous.

Answer

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim\limits_{h \to 0} f(c-h) = \lim\limits_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of the limit from class 11 we can summarise it as

A function is continuous at x = c if :

$\lim\limits_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} \frac{\sin x}{x} & ,x < 0 \\ x + 1 & ,x \geq 0 \end{cases}$ ....equation 1

To prove it everywhere continuous we need to show that at every point in the domain of f(x) [domain is nothing but a set of real numbers for which function is defined]

$\lim\limits_{x \to c} f(x) = f(c)$,where c is any random point from domain of f

Clearly from definition of f(x) {see from equation 1}, f(x) is defined for all real numbers.

∴ we need to check continuity for all real numbers.

Let c is any random number such that c < 0 [thus c being a random number, it can include all negative numbers]

$f(c) = \frac{\sin c}{c}$ [using eqn 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{\sin x}{x} = \frac{\sin c}{c}$$

Clearly, $\lim_{x\to c} f(x) = f(c) = \frac{\sin c}{c}$

∴ We can say that f(x) is continuous for all x < 0

Now, let m be any random number from the domain of f such that m > 0

thus m being a random number, it can include all positive numbers]

f(m) = m+1 [using eqn 1]

$$\lim_{x\to m} f(x) = \lim_{x\to m} x + 1 = m + 1$$

Clearly, $\lim_{x\to c} f(x) = f(c) = m + 1$

∴ We can say that f(x) is continuous for all x > 0

As zero is a point at which function is changing its nature so we need to check LHL, RHL separately

f(0) = 0+1 = 1 [using eqn 1]

LHL = $\lim_{h\to 0} f(0 - h) = \lim_{h\to 0} \frac{\sin(-h)}{-h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$

[∵ sin – θ = – sin θ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$]

RHL = $\lim_{h\to 0} f(0 + h) = \lim_{h\to 0} h + 1 = 1$

Thus LHL = RHL = f(0).

∴ f(x) is continuous at x = 0

Hence, we proved that f is continuous for x < 0 ; x > 0 and x = 0

Thus f(x) is continuous everywhere.

Hence, proved.

2. Question

Discuss the continuity of the function $f(x) = \begin{cases} \frac{x}{|x|} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$.

Answer

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c - h) = \lim_{h\to 0} f(c + h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of the limit from class 11 we can summarise it as

A function is continuous at x = c if :

$$\lim_{x \to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{x}{|x|} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$ ....equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 0, So we need to check its continuity at x = 0 first.

NOTE: Definition of mod function: $|x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$

LHL = $\lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h) = \frac{-h}{-(-h)} = \frac{-h}{h} = -1$

[using eqn 1 and idea of mod fn]

RHL = $\lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h) = \frac{h}{h} = 1$

[using eqn 1 and idea of mod fn]

f(0) = 0

[using eqn 1]

Clearly, LHL ≠ RHL ≠ f(0)

∴ function is discontinuous at x = 0

Let c be any real number such that c > 0

∴ f(c) = $\frac{c}{|c|} = \frac{c}{c} = 1$

[using eqn 1]

And, $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{c}{|c|} = \lim_{x \to c} \frac{c}{c} = 1$

Thus, $\lim_{x \to c} f(x) = f(c)$

∴ f(x) is continuous everywhere for x > 0.

Let c be any real number such that c < 0

∴ f(c) = $\frac{c}{|c|} = \frac{c}{-c} = -1$

[using eqn 1 and idea of mod fn]

And, $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{c}{|c|} = \lim_{x \to c} \frac{c}{-c} = -1$

Thus, $\lim_{x \to c} f(x) = f(c)$

∴ f(x) is continuous everywhere for x < 0.

Hence, We can conclude by stating that f(x) is continuous for all Real numbers except zero.

3 A. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} x^3 - x^2 + 2x - 2 & , \text{ if } x \neq 1 \\ 4 & , \text{ if } x = 1 \end{cases}$$

Answer

Basic Concept:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} x^3 - x^2 + 2x - 2 & ,\text{if } x \neq 1 \\ 4 & ,\text{if } x = 1 \end{cases}$ ....equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 1, So we need to check its continuity at x = 1 first.

Clearly,

f(1) = 4 [using eqn 1]

$\lim_{x \to 1} f(x) = \lim_{x \to 1}(x^3 - x^2 + 2x - 2) = 1^3 - 1^2 + 2 * 1 - 2 = 0$

Clearly, $\lim_{x \to c} f(x) \neq f(c)$

∴ f(x) is discontinuous at x = 1.

Let c be any real number such that c ≠ 0

f(c) = c^3 – c^2 + 2c – 2 [using eqn 1]

$\lim_{x \to c} f(x) = \lim_{x \to c}(x^3 - x^2 + 2x - 2) = c^3 - c^2 + 2c - 2$

Clearly, $\lim_{x \to c} f(x) = f(c)$

∴ f(x) is continuous for all real x except x =1

3 B. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \dfrac{x^4 - 16}{x - 2} & , \text{if } x \neq 2 \\ 16 & , \text{if } x = 2 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{x^4 - 16}{x - 2} = \frac{(x^2+4)(x-2)(x+2)}{(x-2)} = (x^2+4)(x+2) & ,\text{if } x \neq 2 \\ 16 & ,\text{if } x = 2 \end{cases}$$

…Equation 1

Note: [for changing the expression used identity:– (a^2–b^2) = (a+b)(a–b)]

Note: x – 2 is cancelled from numerator and denominator only because x ≠ 2, else we can't cancel them

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 2, So we need to check its continuity at x = 2 first.

Clearly,

f(2) = 16 [using eqn 1]

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^4 - 16}{x - 2} = \lim_{x \to 2} \frac{(x^2+4)(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x^2+4)(x+2)$$
$$= 16$$

Note: (x – 2) is cancelled as x ≠ 2 but x → 2

Clearly, $\lim_{x \to c} f(x) = f(c)$

∴ f(x) is continuous at x = 2.

Let c be any real number such that c ≠ 0

f(c) = $(c^2+4)(c+2)$ [using eqn 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2+4)(x+2) = (c^2+4)(c+2)$$

Clearly, $\lim_{x \to c} f(x) = f(c)$

∴ f(x) is continuous for all real x

3 C. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \frac{\sin x}{x} & , \text{if } x < 0 \\ 2x + 3 & , \text{if } x \geq 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} \frac{\sin x}{x} & ,\text{if } x < 0 \\ 2x + 3 & ,\text{if } x \geq 0 \end{cases}$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < 0 [thus c being a random number, it can include all negative numbers]

$f(c) = \frac{\sin c}{c}$ [using eqn 1]

$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{\sin x}{x} = \frac{\sin c}{c}$

Clearly, $\lim_{x\to c} f(x) = f(c) = \frac{\sin c}{c}$

∴ We can say that f(x) is continuous for all x < 0

Now, let m be any random number from the domain of f such that m > 0

thus m being a random number, it can include all positive numbers]

f(m) = 2m + 3 [using eqn 1]

$\lim_{x\to m} f(x) = \lim_{x\to m} 2x + 3 = 2m + 3$

Clearly, $\lim_{x\to c} f(x) = f(c) = 2m + 3$

∴ We can say that f(x) is continuous for all x > 0

As zero is a point at which function is changing its nature so we need to check LHL, RHL separately

f(0) = 2×0+3 = 3 [using eqn 1]

LHL = $\lim_{h\to 0} f(0 - h) = \lim_{h\to 0} \frac{\sin -h}{-h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$

[∵ sin –θ = – sin θ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$]

RHL = $\lim_{h\to 0} f(0 + h) = \lim_{h\to 0} 2h + 3 = 3$

Thus LHL ≠ RHL

∴ f(x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0.

3 D. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \frac{\sin 3x}{x} & , \text{if } x \neq 0 \\ 4 & , \text{if } x = 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c - h) = \lim_{h\to 0} f(c + h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} \frac{\sin 3x}{x} & \text{,if } x \neq 0 \\ 4 & \text{,if } x = 0 \end{cases}$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0]

f(c) = $\frac{\sin 3c}{c}$ [using eqn 1]

$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{\sin 3x}{x} = \frac{\sin 3c}{c}$

Clearly, $\lim_{x\to c} f(x) = f(c) = \frac{\sin 3c}{c}$

∴ We can say that f(x) is continuous for all x ≠ 0

As zero is a point at which function is changing its nature, so we need to check the continuity here.

f(0) = 4 [using eqn 1]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sin -3h}{-h} = 3\lim_{h\to 0} \frac{\sin 3h}{3h} = 3$

[∵ sin –θ = – sin θ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{\sin 3h}{h} = 3\lim_{h\to 0} \frac{\sin 3h}{3h} = 3$

Thus LHL = RHL ≠ f(0)

∴ f(x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0.

3 E. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & \text{, if } x \neq 0 \\ 5 & \text{, if } x = 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & ,\text{if } x \neq 0 \\ 5 & ,\text{if } x = 0 \end{cases}$$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0]

f(c) = $\frac{\sin c}{c} + \cos c$ [using eqn 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{\sin x}{x} + \cos x\right) = \frac{\sin c}{c} + \cos c$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x ≠ 0

As zero is a point at which function is changing its nature, so we need to check the continuity here.

f(0) = 5 [using eqn 1]

and,

$$\lim_{x\to 0}\left(\frac{\sin x}{x} + \cos x\right) = \lim_{x\to 0}\frac{\sin x}{x} + \lim_{x\to 0}\cos x = 1 + \cos 0 = 2 \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Thus $\lim_{x\to c} f(x) \neq f(c)$

∴ f(x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0.

3 F. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \frac{x^4 + x^3 + 2x^2}{\tan^{-1} x} & , \text{if } x \neq 0 \\ 10 & , \text{if } x = 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{x^4+x^3+2x^2}{\tan^{-1} x} & ,\text{if } x \neq 0 \\ 10 & ,\text{if } x = 0 \end{cases}$$ ...Equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0]

f(c) = $\frac{c^4+c^3+2c^2}{\tan^{-1} c}$ [using eqn 1]

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(\frac{x^4+x^3+2x^2}{\tan^{-1} x}\right) = \frac{c^4+c^3+2c^2}{\tan^{-1} c}$$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x ≠ 0

As zero is a point at which function is changing its nature so we need to check the continuity here.

f(0) = 10 [using eqn 1]

and,

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x^4+x^3+2x^2}{\tan^{-1} x}\right)$$

or, $\lim_{x\to 0}\left(\frac{x^3+x^2+2x}{\frac{\tan^{-1}x}{x}}\right) = \frac{\lim_{x\to 0}(x^3+x^2+2x)}{\lim_{x\to 0}\frac{\tan^{-1}x}{x}} = \frac{0}{1} = 0$ [∵ using $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$]

Thus $\lim_{x\to c} f(x) \neq f(c)$

∴ f(x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0

3 G. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \dfrac{e^x - 1}{\log_e(1+2x)} & , \text{if } x \neq 0 \\ 7 & , \text{if } x = 0 \\ |x-3| & , \text{if } x < 1 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} \frac{e^x-1}{\log_e(1+2x)} & ,\text{if } x \neq 0 \\ 7 & ,\text{if } x = 0 \end{cases} \quad \text{...Equation 1}$$

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being random number, it is able to include all numbers except 0]

$f(c) = \frac{e^c - 1}{\log_e(1+2c)}$ [using eqn 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left(\frac{e^x - 1}{\log_e(1+2x)}\right) = \frac{e^c - 1}{\log_e(1+2c)}$$

Clearly, $\lim_{x \to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x ≠ 0

As x = 0 is a point at which function is changing its nature so we need to check the continuity here.

Since, f(0) = 7 [using eqn 1]

NOTE : Idea of logarithmic limit and exponential limit –

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1$$

You must have read such limits in class 11. You can verify these by expanding log(1+x) and e^x in its taylor form.

Numerator and denominator conditions also hold for this limit like sandwich theorem.

E.g : $\lim_{x \to 0} \frac{\log(1+2x)}{2x} = 1$

But, $\lim_{x \to 0} \frac{\log(1+2x)}{x} \neq 1$ as denominator does not have 2x

and,

$\lim_{x \to 0} f(x)$

$= \lim_{x \to 0} \frac{e^x - 1}{\log 1+2x}$ [Using logarithmic and exponential limit as explained above, we have:]

$= \frac{1}{2} \lim_{x \to 0} \frac{\frac{(e^x - 1)}{x}}{\frac{\log(1+2x)}{2x}} = \frac{1}{2}$

Thus, $\lim_{x \to c} f(x) \neq f(c)$

∴ f(x) is discontinuous at x = 0

Hence, f is continuous for all x ≠ 0 but discontinuous at x = 0

3 H. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} |x-3| & , \text{if } x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & , \text{if } x < 1 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as $x \to c$ (LHL) = right hand limit as $x \to c$ (RHL) = value of function at $x = c$.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at $x = c$ if :

$$\lim_{x \to c} f(x) = f(c)$$

NOTE: Idea of modulus function |x| : You can think this function as a machine in which you can give it any real no. as an input and it returns its absolute value i.e. if positive is entered it returns the same no and if negative is entered it returns the corresponding positive no.

Eg:– $|2| = 2$; $|-2| = -(-2) = 2$

Similarly, we can define it for variable x, if $x \geq 0$ $|x| = x$

If $x < 0$ $|x| = (-x)$

$$\therefore |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$$

Here we have,

$$f(x) = \begin{cases} |x-3| & ,\text{if } x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & ,\text{if } x < 1 \end{cases}$$

Applying the idea of mod function, f(x) can be rewritten as:

$$f(x) = \begin{cases} \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, \text{if } x < 1 \\ x-3, \text{if } x \geq 3 \\ -(x-3), \text{if } 1 \leq x < 3 \end{cases} \text{......equation 1}$$

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that $c < 1$ [thus c being a random number, it can include all numbers less than 1]

$f(c) = \frac{c^2}{4} - \frac{3c}{2} + \frac{13}{4}$ [using eqn 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c}\left(\frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}\right) = \frac{c^2}{4} - \frac{3c}{2} + \frac{13}{4}$$

Clearly, $\lim_{x \to c} f(x) = f(c)$

$\therefore$ We can say that f(x) is continuous for all $x < 1$

As $x = 1$ is a point at which function is changing its nature, so we need to check the continuity here.

$f(1) = |1 - 3| = 2$ [using eqn 1]

$$\text{LHL} = \lim_{h \to 0} f(1-h) = \lim_{h \to 0}\left(\frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4}\right) = \frac{1^2}{4} - \frac{3}{2} + \frac{13}{4} = 2$$

$$\text{RHL} = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} |1+h-3| = |-2| = 2$$

Thus LHL = RHL = f(1)

$\therefore$ f(x) is continuous at $x = 1$

Now, again f(x) is changing its nature at $x = 3$, so we need to check continuity at $x = 3$

$f(3) = 3 - 3 = 0$ [using eqn 1]

$$\text{LHL} = \lim_{h \to 0} f(3-h) = \lim_{h \to 0} -(3-h-3) = 0$$

$$\text{RHL} = \lim_{h \to 0} f(3+h) = \lim_{h \to 0} 3+h-3 = 0$$

Thus LHL = RHL = f(3)

∴ f(x) is continuous at x = 3

For x > 3 ; f(x) = x–3 whose plot is linear, so it is continuous for all x > 3

You can verify it by checking limits.

Similarly, for 1 < x < 3, f(x) = 3–x whose plot is again a straight line and thus continuous for all point in this range.

Hence, f(x) is continuous for all real x.

3 I. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x)=\begin{cases} |x|+3 & , \text{if } x \le -3 \\ -2x & , \text{if } -3 < x < 3 \\ 6x+2 & , \text{if } x > 3 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$$\lim_{x\to c} f(x) = f(c)$$

NOTE: Idea of modulus function |x| : You can think this function as a machine in which you can give it any real no. as an input and it returns its absolute value i.e. if positive is entered it returns the same no and if negative is entered it returns the corresponding positive no.

Eg:– |2| = 2 ; |–2| = –(–2) = 2

Similarly, we can define it for variable x, if x ≥ 0 |x| = x

If x < 0 |x| = (–x)

$$\therefore |x| = \begin{cases} -x, x < 0 \\ x, x \ge 0 \end{cases}$$

Here we have,

$$f(x) = \begin{cases} |x| + 3 & ,\text{if } x \le -3 \\ -2x & ,\text{if } -3 < x < 3 \\ 6x + 2 & ,\text{if } x \ge 3 \end{cases}$$

Applying the idea of mod function, f(x) can be rewritten as:

$$f(x) = \begin{cases} 3 - x & ,\text{if } x \le -3 \\ -2x & ,\text{if } -3 < x < 3 \\ 6x + 2 & ,\text{if } x \ge 3 \end{cases}$$ ......equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < –3 [thus c being random number, it is able to include all numbers less than –3]

f(c) = $3 - c$ [using eqn 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}(3 - x) = 3 - c$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x < –3

As x = –3 is a point at which function is changing its nature so we need to check the continuity here.

f(–3) = 3–(–3) = 6 [using eqn 1]

LHL = $\lim_{h\to 0} f(-3 - h) = \lim_{h\to 0}(3 - (-3 - h)) = 6$

RHL = $\lim_{h\to 0} f(-3 + h) = \lim_{h\to 0} -2(-3 + h) = 6$

Thus LHL = RHL = f(–3)

∴ f(x) is continuous at x = –3

Let c is any random number such that –3 < m < 3 [thus c being random number, it is able to include all numbers between –3 and 3]

f(c) = $-2m$ [using eqn 1]

and, $\lim_{x\to m} f(x) = \lim_{x\to m}(-2x) = -2m$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all –3 < x < 3

Now, again f(x) is changing its nature at x = 3,so we need to check continuity at x = 3

f(3) = 6*3+2 = 20 [using eqn 1]

LHL = $\lim_{h\to 0} f(3 - h) = \lim_{h\to 0} -2 * (3 - h) = -6$

RHL = $\lim_{h\to 0} f(3 + h) = \lim_{h\to 0} 6(3 + h) + 2 = 20$

Thus LHL ≠ RHL

∴ f(x) is discontinuous at x = 3

For x > 3 ; f(x) = 6x + 2 whose plot is linear, so it is continuous for all x > 3

You can verify it by checking limits.

Hence, f(x) is continuous for all real x except x = 3

There is only one point of discontinuity at x = 3

3 J. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} x^{10} & , \text{if } x \le 1 \\ x^2 & , \text{if } x > 1 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c - h) = \lim_{h\to 0} f(c + h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} x^{10} & ,\text{if } x \le 1 \\ x^2 & ,\text{if } x > 1 \end{cases}$ ......equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < 1 [thus c being random number, it is able to include all numbers less than 1]

f(c) = c^{10} [using eqn 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}(x^{10}) = c^{10}$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x < 1

As x = 1 is a point at which function is changing its nature so we need to check the continuity here.

f(1) = 1^{10} = 1 [using eqn 1]

LHL = $\lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h)^{10} = 1$

RHL = $\lim_{h\to 0} f(1+h) = \lim_{h\to 0}(1+h)^2 = 1$

Thus LHL = RHL = f(1)

∴ f(x) is continuous at x = 1

Let m is any random number such that m > 1 [thus m being random number, it is able to include all numbers greater than 1]

f(m) = m^2 [using eqn 1]

and, $\lim_{x\to m} f(x) = \lim_{x\to m}(x^2) = m^2$

Clearly, $\lim_{x\to m} f(x) = f(m)$

∴ We can say that f(x) is continuous for all m > 1

Hence, f(x) is continuous for all real x

There no point of discontinuity. It is everywhere continuous

3 K. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} 2x & ,\text{ if } x < 0 \\ 0 & ,\text{ if } 0 \le x \le 1 \\ 4x & ,\text{ if } x > 1 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} 2x & ,\text{if } x < 0 \\ 0 & ,\text{if } 0 \le x \le 1 \\ 4x & ,\text{if } x > 1 \end{cases}$ ……equation 1

The function is defined for all real numbers, so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c < 0 [thus c being a random number, it can include all numbers less than 0]

f(c) = $2c$ [using eqn 1]

$\lim_{x \to c} f(x) = \lim_{x \to c} (2x) = 2c$

Clearly, $\lim_{x \to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x < 0

As x = 0 is a point at which function is changing its nature, so we need to check the continuity here.

f(0) = 0

[using eqn 1]

LHL = $\lim_{h \to 0} f(0-h) = \lim_{h \to 0} -2h = 0$

RHL = $\lim_{h \to 0} f(0+h) = \lim_{h \to 0} 0 = 0$

Thus LHL = RHL = f(0)

∴ f(x) is continuous at x = 0

Let m is any random number such that 0 < m < 1 [thus m being a random number, it can include all numbers greater than 0 and less than 1]

f(m) = 0 [using eqn 1]

$\text{and, } \lim_{x \to m} f(x) = \lim_{x \to m} (0) = 0$

Clearly, $\lim_{x \to m} f(x) = f(m)$

∴ We can say that f(x) is continuous for all 0 < x < 1

As x = 1 is again a point at which function is changing its nature, so we need to check the continuity here.

f(1) = 0

[using eqn 1]

LHL = $\lim_{h \to 0} f(1-h) = \lim_{h \to 0} 0 = 0$

RHL = $\lim_{h \to 0} f(1+h) = \lim_{h \to 0} 4(1+h) = 4$

Thus LHL ≠ RHL

∴ f(x) is discontinuous at x = 1

Let k is any random number such that k > 1 [thus k being a random number, it can include all numbers greater than 1]

f(k) = $4k$ [using eqn 1]

and, $\lim_{x\to k} f(x) = \lim_{x\to k} 4x = 4k$

Clearly, $\lim_{x\to k} f(x) = f(k)$

∴ We can say that f(x) is continuous for all x > 1

Hence, f(x) is continuous for all real value of x, except x =1

There is a single point of discontinuity at x = 1

3 L. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} \sin x - \cos x & , \text{if } x \neq 0 \\ -1 & , \text{if } x = 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} \sin x - \cos x & ,\text{if } x \neq 0 \\ -1 & ,\text{if } x = 0 \end{cases}$ ...Equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Let c is any random number such that c ≠ 0 [thus c being a random number, it can include all numbers except 0]

f(c) = $\sin c - \cos c$ [using eqn 1]

$\lim_{x\to c} f(x) = \lim_{x\to c}(\sin x - \cos x) = \sin c - \cos c$

Clearly, $\lim_{x\to c} f(x) = f(c)$

∴ We can say that f(x) is continuous for all x ≠ 0

As zero is a point at which function is changing its nature, so we need to check the continuity here.

f(0) = –1 [using eqn 1]

and,

$\lim_{x\to 0}(\sin x - \cos x) = \lim_{x\to 0} \sin x - \lim_{x\to 0} \cos x = 0 - \cos 0 = -1$

Thus $\lim_{x\to c} f(x) = f(c)$

∴ f(x) is continuous at x = 0

Hence, f is continuous for all x.

f(x) is continuous everywhere.

No point of discontinuity.

3 M. Question

Find the points of discontinuity, if any, of the following functions :

$$f(x) = \begin{cases} -2 & , \text{if } x \le -1 \\ 2x & , \text{if } -1 < x < 1 \\ 2 & , \text{if } x \ge 1 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$$f(x) = \begin{cases} -2 & , \text{if } x \le -1 \\ 2x & , \text{if } -1 < x < 1 \\ 2 & , \text{if } x \ge 1 \end{cases}$$ ....equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

For x < –1, f(x) is having a constant value, so the curve is going to be straight line parallel to x–axis.

So, it is everywhere continuous for x < –1.

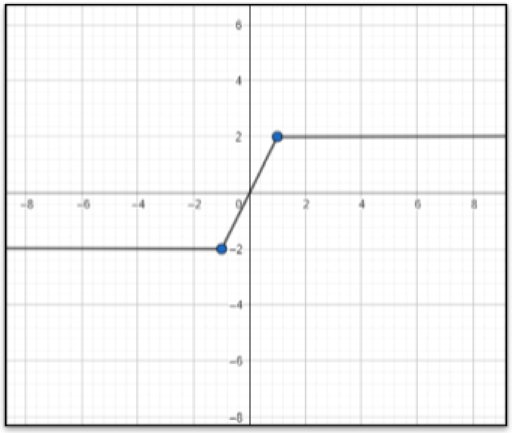
It can be verified using limits as discussed in previous problems

Similarly for –1 < x < 1, plot on X–Y plane is a straight line passing through origin.

So, it is everywhere continuous for –1 < x < 1.

And similarly for x > 1, plot is going to be again a straight line parallel to x–axis

∴ it is also everywhere continuous for x > 1



From graph it is clear that function is continuous everywhere but let’s verify it with limits also.

As x = –1 is a point at which function is changing its nature so we need to check the continuity here.

f(–1) = –2 [using eqn 1]

LHL = $\lim_{h \to 0} f(-1-h) = \lim_{h \to 0} -2 = -2$

RHL = $\lim_{h \to 0} f(-1+h) = \lim_{h \to 0} 2(-1+h) = -2$

Thus LHL = RHL = f(–1)

∴ f(x) is continuous at x = –1

Also at x = 1 function is changing its nature so we need to check the continuity here too.

f(1) = 2 [using eqn 1]

LHL = $\lim_{h \to 0} f(1-h) = \lim_{h \to 0} 2(1-h) = 2$

RHL = $\lim_{h \to 0} f(1+h) = \lim_{h \to 0} 2 = 2$

Thus LHL = RHL = f(1)

∴ f(x) is continuous at x = 1

Thus, f(x) is continuous everywhere and there is no point of discontinuity.

4 A. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{5x} & , \text{ if } x \neq 0 \\ 3k & , \text{ if } x = 0 \end{cases}$$

Answer

Basic Concept:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small ‘+ve’ no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$$f(x)=\begin{cases}\frac{\sin 2x}{5x} & ,\text{if } x\neq 0\\ 3k & ,\text{if } x=0\end{cases}$$ Equation 1

Function is defined for all real numbers and we need to find the value of k so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

As, for x ≠ 0 it is just a combination of trigonometric and linear polynomial both of which are continuous everywhere. It can be verified using limits and also by plotting curves. Since we are given that function is continuous everywhere so don't need to bother about that.

As x = 0 is only point at which function is changing its nature so it needs to be continuous here.

f(0) = 3k [using eqn 1]

and,

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{1}{5}\lim_{x\to 0}2*\frac{\sin 2x}{2x}=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}=\frac{2}{5}\,[\because \lim_{x\to 0}\frac{\sin x}{x}=1-\text{sandwich theorem}]$$

∵ f(x) is continuous everywhere [given in question]

$\therefore \lim_{x\to c}f(x)= f(c)$

$\therefore 3k=\frac{2}{5}$

$\therefore k=\frac{2}{15}$

4 B. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x)=\begin{cases}kx+5 & ,\text{ if } x\le 2\\ x-1 & ,\text{ if } x>2\end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0}f(c-h) = \lim_{h\to 0}f(c+h)=f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$$\lim_{x\to c}f(x)=f(c)$$

Here we have,

$$f(x)=\begin{cases}kx+5 & ,\text{if } x\le 2\\ x-1 & ,\text{if } x>2\end{cases}$$ ....equation 1

Function is defined for all real numbers and we need to find the value of k so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = 2

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 2} f(x) = f(2)$

$\lim_{h \to 0} f(2-h) = \lim_{h \to 0} f(2+h) = f(2)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f(2+h) = f(2)$ [considering RHL as RHL will give expression independent of k]

$\lim_{h \to 0} 2 + h - 1 = 2k + 5$ [using equation 1]

$\therefore$ 2k + 5 = 1

2k = –4

k = $\frac{-4}{2}$ = –2

4 C. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x) = \begin{cases} k(x^2 + 3x) & , \text{if } x < 0 \\ \cos 2x & , \text{if } x \geq 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} k(x^2 + 3x) & ,\text{if } x < 0 \\ \cos 2x & ,\text{if } x \geq 0 \end{cases}$ ....equation 1

Function is defined for all real numbers and we need to find the value of k so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = 0

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 0} f(x) = f(0)$

$\lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) = f(0)$

[using basic ideas of limits and continuity]

$\lim_{h \to 0} f(-h) = f(0)$

[considering LHL as LHL will give expression dependent of k]

$\lim_{h \to 0} k\{ (-h)^2 + 3(-h) \} = \cos 0$ [using equation 1]

$\because$ k * 0 = 1

As above equality never holds true for any value of k

k = not defined

No such value of k is possible for which f(x) is continuous everywhere.

F(x) will always have a discontinuity at x = 0

4 D. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x) = \begin{cases} 2 & , \text{if} \quad x \le 3 \\ ax + b & , \text{if} \quad 3 < x < 5 \\ 9 & , \text{if} \quad x \ge 5 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} 2 & ,\text{if} \quad x \le 3 \\ ax + b & ,\text{if} \quad 3 < x < 5 \\ 9 & ,\text{if} \quad x \ge 5 \end{cases}$ ....equation 1

Function is defined for all real numbers and we need to find the value of a & b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = 3

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 3} f(x) = f(3)$

$\lim_{h \to 0} f(3-h) = \lim_{h \to 0} f(3+h) = f(3)$ [using basic ideas of limits and continuity]

$\lim_{h\to 0} f(3+h) = f(3)$ [considering RHL as RHL will give expression inclusive of a & b]

$\lim_{h\to 0}\{a(3+h)+b\} = 2$ [using equation 1]

∴ 3a + b = 2 ....Equation 2

Also from equation 1 ,it is clear that f(x) is also changing its expression at x = 5

Given,

f(x) is continuous everywhere

∴ $\lim_{x\to 5} f(x) = f(3)$

$\lim_{h\to 0} f(5-h) = \lim_{h\to 0} f(5+h) = f(5)$

[using basic ideas of limits and continuity]

$\lim_{h\to 0} f(5-h) = f(5)$

[considering LHL as LHL will give expression inclusive of a & b]

$\lim_{h\to 0}\{a(5-h)+b\} = 9$ [using equation 1]

∴ 5a + b = 9 ....Equation 3

As , b = 9 – 5a

Putting value of b in equation 2:

3a + 9 – 5a = 2

2a = 7

a = $\frac{7}{2}$

∴ b = 9 - 5($\frac{7}{2}$) = $-\frac{17}{2}$

∴ a = $\frac{7}{2}$ and b = $-\frac{17}{2}$

4 E. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x) = \begin{cases} 4 & , \text{if} \quad x \le -1 \\ ax^2 + b & , \text{if} \quad -1 < x < 0 \\ \cos x & , \text{if} \quad x \ge 0 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

we have,

$$f(x) = \begin{cases} 4 & ,\text{if} \quad x \le -1 \\ ax^2 + b & ,\text{if} \quad -1 < x < 0 \\ \cos x & ,\text{if} \quad x \ge 0 \end{cases}$$ ....equation 1

Function is defined for all real numbers and we need to find the value of a & b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = –1

Given,

f(x) is continuous everywhere

$\therefore \lim_{x\to -1} f(x) = f(-1)$

$\lim_{h\to 0} f(-1-h) = \lim_{h\to 0} f(-1+h) = f(-1)$

[using basic ideas of limits and continuity]

$\lim_{h\to 0} f(-1+h) = f(-1)$

[considering RHL as RHL will give expression inclusive of a & b]

$\lim_{h\to 0}\{a(-1+h)^2 + b\} = 4$ [using equation 1]

∴ a + b = 4 ....Equation 2

Also from equation 1, it is clear that f(x) is also changing its expression at x = 0

Given,

f(x) is continuous everywhere

$\therefore \lim_{x\to 0} f(x) = f(0)$

$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$

[using basic ideas of limits and continuity]

$\lim_{h\to 0} f(-h) = f(0)$

[considering LHL as LHL will give expression inclusive of a & b]

$\lim_{h\to 0}\{a(-h)^2 + b\} = \cos 0 = 1$ [using equation 1]

∴ b = 1 ....Equation 3

Putting value of b in equation 2:

a + 1 = 4

a = 3

∴ a = 3 and b = 1

4 F. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x)=\begin{cases}\dfrac{\sqrt{1+px}-\sqrt{1-px}}{x} & \text{, if } -1 \le x < 0\\ \dfrac{2x+1}{x-2} & \text{, if } 0 \le x \le 1\end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim\limits_{h\to 0} f(c-h) = \lim\limits_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim\limits_{x\to c} f(x) = f(c)$

Here we have,

$$f(x)=\begin{cases}\frac{\sqrt{1+px}-\sqrt{1-px}}{x} & \text{,if } -1 \le x < 0\\ \frac{2x+1}{x-2} & \text{,if } 0 \le x \le 1\end{cases}$$ ....equation 1

Function is defined for all real numbers and we need to find the value of p so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = 0

Given,

f(x) is continuous everywhere

$\therefore \lim\limits_{x\to 0} f(x) = f(0)$

$\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(0+h) = f(0)$ [using basic ideas of limits and continuity]

$\lim\limits_{h\to 0} f(-h) = f(0)$ [considering LHL as LHL will give expression inclusive of p]

$\lim\limits_{h\to 0}\left\{\frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}\right\} = \frac{2*0+1}{0-2} = -\frac{1}{2}$ [using equation 1]

$$\operatorname*{Lim}_{h\to 0}\left\{\left(\frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}\right)\left(\frac{\sqrt{1-ph}+\sqrt{1+ph}}{\sqrt{1-ph}+\sqrt{1+ph}}\right)\right\} = -\frac{1}{2}$$

$$\operatorname*{Lim}_{h\to 0}\left\{\left(\frac{1-ph-1-ph}{-h}\right)\left(\frac{1}{\sqrt{1-ph}+\sqrt{1+ph}}\right)\right\} = -\frac{1}{2}$$

$$\operatorname*{Lim}_{h\to 0}\left\{\frac{-2ph}{(-h)(\sqrt{1-ph}+\sqrt{1+ph})}\right\} = -\frac{1}{2}$$

$$\operatorname*{Lim}_{h\to 0}\left\{\frac{2p}{(\sqrt{1-ph}+\sqrt{1+ph})}\right\} = -\frac{1}{2}$$

$$\frac{2p}{2} = p = -\frac{1}{2}$$

$\therefore$ p = $\frac{-1}{2}$

4 G. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x) = \begin{cases} 5 & , \text{if} \quad x \le 2 \\ ax + b & , \text{if} \quad 2 < x < 10 \\ 21 & , \text{if} \quad x \ge 10 \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$$\lim_{x\to c} f(x) = f(c)$$

Here we have,

$f(x) = \begin{cases} 5 & ,\text{if} \quad x \le 2 \\ ax + b & ,\text{if} \quad 2 < x < 10 \\ 21 & ,\text{if} \quad x \ge 10 \end{cases}$ ....equation 1

Function is defined for all real numbers and we need to find the value of a & b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = 2

Given,

f(x) is continuous everywhere

$\therefore \lim_{x\to 2} f(x) = f(2)$

$$\lim_{h\to 0} f(2-h) = \lim_{h\to 0} f(2+h) = f(2)$$

[using basic ideas of limits and continuity]

$$\lim_{h\to 0} f(2+h) = f(2)$$

[considering RHL as RHL will give expression inclusive of a & b]

$\lim_{h\to 0}\{a(2+h)+b\} = 5$ [using equation 1]

$\therefore$ 2a + b = 5 ....Equation 2

Also from equation 1 ,it is clear that f(x) is also changing its expression at x = 10

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 10} f(x) = f(10)$

$\lim_{h \to 0} f(10-h) = \lim_{h \to 0} f(10+h) = f(10)$

[using basic ideas of limits and continuity]

$\lim_{h \to 0} f(10-h) = f(10)$

[considering LHL as LHL will give expression inclusive of a & b]

$\lim_{h \to 0} \{a(10-h)+b\} = 21$

[using equation 1]

$\therefore$ 10a + b = 21 ……………….Equation 3

As , b = 21 – 10a

Putting value of b in equation 2:

2a + 21 – 10a = 5

8a = 16

a = $\frac{16}{8} = 2$

$\therefore$ b = 21 – 10×2 $= 1$

$\therefore$ a = 2 and b = 1

4 H. Question

In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x} & , \quad x < \frac{\pi}{2} \\ 3 & , \quad x = \frac{\pi}{2} \\ \frac{3 \tan 2x}{2x - \pi} & , \quad x > \frac{\pi}{2} \end{cases}$$

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x} & , \quad x < \frac{\pi}{2} \\ 3 & , \quad x = \frac{\pi}{2} \\ \frac{3\tan 2x}{2x - \pi} & , \quad x > \frac{\pi}{2} \end{cases}$$ ....equation 1

Function is defined for all real numbers and we need to find the value of k so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = π/2

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to \pi/2} f(x) = f(\pi/2)$

$\lim_{h \to 0} f(\pi/2 - h) = \lim_{h \to 0} f(\pi/2 + h) = f(\pi/2)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f(\pi/2 - h) = f(\pi/2)$ [considering LHL as LHL will give expression inclusive of k]

$\lim_{h \to 0} \left\{ \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} \right\} = 3$ [using equation 1]

$\lim_{h \to 0} \left\{ \frac{k\sin h}{2h} \right\} = \frac{k}{2} \lim_{h \to 0} \left\{ \frac{\sin h}{h} \right\} = \frac{k}{2} = 3$

[$\because \lim_{x \to 0} \frac{\sin x}{x} = 1$ (sandwich theorem)]

$\therefore$ k = 3×2 = 6

$\therefore$ k = 6

5. Question

The function $f(x) = \begin{cases} \frac{x^2}{a} & ,\text{if} \quad 0 \le x < 1 \\ a & ,\text{if} \quad 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^2} & ,\text{if} \quad \sqrt{2} \le x < \infty \end{cases}$ is continuous on [0, ∞]. Find the most suitable values of a and b.

Answer

Basic Concept:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c - h) = \lim_{h \to 0} f(c + h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} \frac{x^2}{a} & ,\text{if} \quad 0 \le x < 1 \\ a & ,\text{if} \quad 1 \le x < \sqrt{2} \\ \frac{2b^2-4b}{x^2} & ,\text{if} \quad \sqrt{2} \le x < \infty \end{cases}$$ ....equation 1

The function is defined for [0,∞] and we need to find the value of a and b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1 ,it is clear that f(x) is changing its expression at x = 1

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 1} f(x) = f(1)$

$\lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h) = f(1)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f(1-h) = f(1)$ [considering LHL as LHL will give expression inclusive of a]

$\lim_{h \to 0} \left\{\frac{(1-h)^2}{a}\right\} = a$ [using equation 1]

$\therefore \frac{1}{a} = a => a^2 = 1$

∴ a = ± 1 .... equation 2

Also from equation 1 ,it is clear that f(x) is also changing its expression at x = √2

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to \sqrt{2}} f(x) = f(\sqrt{2})$

$\lim_{h \to 0} f\left(\sqrt{2}-h\right) = \lim_{h \to 0} f\left(\sqrt{2}+h\right) = f\left(\sqrt{2}\right)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f\left(\sqrt{2}-h\right) = f\left(\sqrt{2}\right)$ [considering LHL as LHL will give expression inclusive of a & b]

$$\lim_{h \to 0} a = a = \frac{2b^2 - 4b}{(\sqrt{2})^2} = b^2 - 2b$$

[using equation 1]

∴ b^2 – 2b = a ....Equation 3

From equation 2, a = –1

b^2 – 2b = –1

⇒ b^2 – 2b + 1 = 0

⇒ $(b – 1)^2$ = 0

∴ b = 1 when a = –1

Putting a = 1 in equation 3:

$b^2 - 2b = 1$

$\Rightarrow b^2 - 2b - 1 = 0$

$\Rightarrow b = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

Thus,

For a = –1 ; b = 1

For a = 1 ; b = 1 ± √2

6. Question

Find the values of a and b so that the function f(x) defined by

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & ,\text{if} \quad 0 \le x < \pi/4 \\ 2x \cot x + b & ,\text{if} \quad \pi/4 \le x < \pi/2 \\ a\cos 2x - b\sin x & ,\text{if} \quad \pi/2 \le x \le \pi \end{cases}$$

becomes continuous on [0, π].

Answer

Basic Concept:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & ,\text{if} \quad 0 \le x < \pi/4 \\ 2x \cot x + b & ,\text{if} \quad \pi/4 \le x < \pi/2 \\ a\cos 2x - b\sin x & ,\text{if} \quad \pi/2 \le x \le \pi \end{cases}$$ ....equation 1

Function is defined for [0,π] and we need to find the value of a and b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = π/4

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to \pi/4} f(x) = f(\pi/4)$

$\lim_{h \to 0} f(\pi/4 - h) = \lim_{h \to 0} f(\pi/4 + h) = f(\pi/4)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f\left(\frac{\pi}{4} - h\right) = f(\pi/4)$ [considering LHL as LHL will give expression inclusive of a & b]

$\lim_{h\to 0}\left\{\left(\frac{\pi}{4}-h\right)-a\sqrt{2}\sin\left(\frac{\pi}{4}-h\right)\right\}=\frac{2\pi}{4}\cot\frac{\pi}{4}+b$ [using equation 1]

$\therefore \frac{\pi}{4}-a=\frac{\pi}{2}+b$

∴ a + b = – π/4 .... equation 2

Also from equation 1 ,it is clear that f(x) is also changing its expression at x = π/2

Given,

f(x) is continuous everywhere

$\therefore \lim_{x\to\pi/2} f(x)= f(\pi/2)$

$\lim_{h\to 0} f(\pi/2-h) = \lim_{h\to 0} f(\pi/2+h) = f(\pi/2)$ [using basic ideas of limits and continuity]

$\lim_{h\to 0} f(\pi/2-h) = f(\pi/2)$ [considering LHL as LHL will give expression inclusive of a & b]

$\lim_{h\to 0} 2\left(\frac{\pi}{2}-h\right)\cot\left(\frac{\pi}{2}-h\right)+b = a\cos\pi - b\sin\pi/2$

[using equation 1]

$\lim_{h\to 0} 2\left(\frac{\pi}{2}-h\right)\tan h + b = -a-b$

b = – a – b

∴ a = –2b ....Equation 3

Putting value of a from equation 3 to equation 2

∴ –2b + b = – π/4

⇒ b = π/4

∴ a = –2×(π/4)

= –π/2

Thus, a = –π/2 and b = π/4

7. Question

The function f(x) is defined by $f(x)=\begin{cases} x^2+ax+b & , \quad 0\le x<2 \\ 3x+2 & , \quad 2\le x\le 4 \\ 2ax+5b & , \quad 4<x\le 8 \end{cases}$ If f is continuous on [0, 8], find the values of a and b.

Answer

Basic Concept:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

Here we have,

$$f(x) = \begin{cases} x^2 + ax + b & , \quad 0 \le x < 2 \\ 3x + 2 & , \quad 2 \le x \le 4 \\ 2ax + 5b & , \quad 4 < x \le 8 \end{cases}$$ ....equation 1

Function is defined for [0,8] and we need to find the value of a and b so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

To find the value of constants always try to check continuity at the values of x for which f(x) is changing its expression.

As most of the time discontinuities are here only, if we make the function continuous here, it will automatically become continuous everywhere

From equation 1, it is clear that f(x) is changing its expression at x = 2

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 2} f(x) = f(2)$

$\lim_{h \to 0} f(2-h) = \lim_{h \to 0} f(2+h) = f(2)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f(2-h) = f(2)$ [considering LHL as LHL will give expression inclusive of a & b]

$\lim_{h \to 0} \{(2-h)^2 + a\,(2-h) + b\} = 3 * 2 + 2 = 10$

[using equation 1]

4+2a + b = 8

∴ 2a + b = 4

∴ b = 4 – 2a .... equation 2

Also from equation 1 ,it is clear that f(x) is also changing its expression at x = 4

Given,

f(x) is continuous everywhere

$\therefore \lim_{x \to 4} f(x) = f(4)$

$\lim_{h \to 0} f(4-h) = \lim_{h \to 0} f(4+h) = f(4)$ [using basic ideas of limits and continuity]

$\lim_{h \to 0} f(4+h) = f(4)$ [considering RHL as RHL will give expression inclusive of a & b]

$\lim_{h \to 0} 2a(4+h) + 5b = 3 \times 4 + 2$ [using equation 1]

∴ 8a + 5b = 14 ....Equation 3

Putting value of a from equation 2 to equation 3

∴ 8a + 5(4–2a) = 14

⇒ 2a = 6

∴ a = 6/2

= 3

∴ b = 4 – 2×3 = –2

Thus, a = 3 and b = –2

8. Question

If $f(x)=\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$ for $x \neq \frac{\pi}{4}$, find the value which can be assigned to f(x) at x = π/4 so that the function f(x) becomes continuous every where in [0, π/2].

Answer

Basic Concept:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Function is defined for [0,π] and we need to find the value of f(x) so that it is continuous everywhere in its domain (domain = set of numbers for which f is defined)

As we have expression for x ≠ π/4, which is continuous everywhere in [0,π],so

If we make it continuous at x = π/4 it is continuous everywhere in its domain.

Given,

$f(x) = \frac{tan\left(\frac{\pi}{4}-x\right)}{cot\,2x}\ for\ x\ \neq\ \pi/4$ ....equation 1

Let f(x) is continuous for x = π/4

∴ $\lim_{x \to \pi/4} f(x) = f(\pi/4)$

$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x \to \pi/4} f(x)$

$= \lim_{x \to \pi/4} \frac{tan\left(\frac{\pi}{4}-x\right)}{cot\,2x} = \lim_{x \to \pi/4} \frac{tan\left(\frac{\pi}{4}-x\right)}{tan\left(\frac{\pi}{2}-2x\right)}$ [∵ tan (π/2–θ) = cot θ]

$= \lim_{x \to \pi/4} \frac{\frac{tan\left(\frac{\pi}{4}-x\right)}{\frac{\pi}{4}-x}}{\frac{tan\left(\frac{\pi}{2}-2x\right)}{\frac{\pi}{2}-2x}} * \frac{\frac{\pi}{4}-x}{\frac{\pi}{2}-2x}$ [multiplying and dividing by π/4–x and π/2–2x to apply sandwich theorem]

$= \frac{\lim_{x \to \pi/4} \frac{tan\left(\frac{\pi}{4}-x\right)}{\frac{\pi}{4}-x}}{\lim_{x \to \pi/4} \frac{tan\left(\frac{\pi}{2}-2x\right)}{\frac{\pi}{2}-2x}} * \frac{1}{2} * \lim_{x \to \frac{\pi}{4}} \frac{\pi-4x}{\pi-4x} = \frac{1}{2}$

[$\because \lim_{x \to 0} \frac{tan\,x}{x} = 1\ (sandwich\ theorem)$]

∴ value that can be assigned to f(x) at x = π/4 is $\frac{1}{2}$

9. Question

Discuss the continuity of the function $f(x)=\begin{cases} 2x-1 & , \text{if } x<2 \\ \frac{3x}{2} & , \text{if } x \geq 2 \end{cases}$.

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} 2x-1 & \text{,if } x < 2 \\ \frac{3x}{2} & \text{,if } x \geq 2 \end{cases}$.......equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 2, So we need to check its continuity at x = 2 first.

LHL = $\lim_{h \to 0} f(2-h) = \lim_{h \to 0} 2(2-h) - 1 = 4 - 1 = 3$ [using eqn 1]

RHL = $\lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{3*(2+h)}{2} = \frac{3*2}{2} = 3$ [using eqn 1]

f(2) = $\frac{3*2}{2} = 3$

[using eqn 1]

Clearly, LHL = RHL = f(2)

∴ function is continuous at x = 2

Let c be any real number such that c > 2

∴ f(c) = $\frac{3c}{2}$ [using eqn 1]

And, $\lim_{x \to c} f(x) = \lim_{x \to c} \frac{3x}{2} = \frac{3c}{2}$

Thus, $\lim_{x \to c} f(x) = f(c)$

∴ f(x) is continuous everywhere for x > 2.

Let m be any real number such that m < 2

∴ f(m) = $2m - 1$ [using eqn 1]

And, $\lim_{x \to m} f(x) = \lim_{x \to m} 2m - 1 = 2m - 1$

Thus, $\lim_{x \to m} f(x) = f(m)$

∴ f(x) is continuous everywhere for x < 2.

Hence, We can conclude by stating that f(x) is continuous for all Real numbers

10. Question

Discuss the continuity of f(x) = sin |x|.

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of the limit from class 11 we can summarise it as a function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

NOTE: $Definition\ of\ mod\ function$: $|x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$

Here we are given with

f(x) = sin |x|

$f(x) = \begin{cases} sin\ (-x) = -sin\ x, x < 0 \\ sin\ x, x \geq 0 \end{cases}$ ....equation 1

Function is defined for all real numbers so we need to comment about its continuity for all numbers in its domain (domain = set of numbers for which f is defined)

Function is changing its nature (or expression) at x = 0, So we need to check its continuity at x = 0 first.

LHL = $\lim_{h \to 0} f(0-h)$ = $\lim_{h \to 0} -sin\ (0-h) = \lim_{h \to 0} sin\ h = 0$

[using eqn 1]

RHL = $\lim_{h \to 0} f(0+h) = \lim_{h \to 0} sin\ h = 0$ [using eqn 1]

f(0) = $sin\ 0 = 0$

[using eqn 1]

Clearly, LHL = RHL = f(0)

∴ function is continuous at x = 0

For all x ≠ 0, f(x) is simply a trigonometric function which is everywhere continuous which can be verified by seeing its plot in X–Y plane or even can be verified using limit.

∴ f(x) is everywhere continuous in its domain i.e. it is continuous for all real values of x

11. Question

Prove that $f(x) = \begin{cases} \dfrac{\sin x}{x} & , \text{if } x < 0 \\ x + 1 & , \text{if } x \geq 0 \end{cases}$ is everywhere continuous.

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Here we have,

$f(x) = \begin{cases} \frac{\sin x}{x} & , x < 0 \\ x+1 & , x \geq 0 \end{cases}$ ....equation 1

To prove it everywhere continuous we need to show that at every point in domain of f(x) [domain is nothing but a set of real numbers for which function is defined]

$\lim_{x \to c} f(x) = f(c)$, $where\ c\ is\ any\ random\ point\ from\ domain\ of\ f$

Clearly from definition of f(x) { see from equation 1}, f(x) is defined for all real numbers.

∴ we need to check continuity for all real numbers.

Let c is any random number such that c < 0 [thus c being a random number, it can include all negative numbers]

f(c) = $\frac{\sin c}{c}$

[using eqn 1]

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{\sin x}{x} = \frac{\sin c}{c}$$

Clearly, $\lim_{x \to c} f(x) = f(c) = \frac{\sin c}{c}$

∴ We can say that f(x) is continuous for all x < 0

Now, let m be any random number from domain of f such that m > 0

thus m being a random number, it can include all positive numbers]

f(m) = m+1 [using eqn 1]

$$\lim_{x \to m} f(x) = \lim_{x \to m} x + 1 = m + 1$$

Clearly, $\lim_{x \to c} f(x) = f(c) = m + 1$

∴ We can say that f(x) is continuous for all x > 0

As zero is a point at which function is changing its nature so we need to check LHL, RHL separately

f(0) = 0+1 = 1 [using eqn 1]

LHL = $\lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{\sin -h}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$ [∵ sin –θ = sin θ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$]

RHL = $\lim_{h \to 0} f(0+h) = \lim_{h \to 0} h + 1 = 1$

Thus LHL = RHL = f(0).

∴ f(x) is continuous at x = 0

Hence, we proved that f is continuous for x < 0 ; x > 0 and x = 0

Thus f(x) is continuous everywhere.

Hence, proved.

12. Question

Show that the function g(x) = x – [x] is discontinuous at all integral points. Here [x] denotes the greatest integer function.

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x \to c} f(x) = f(c)$

Given,

g(x) = x – [x] ....equation 1

We need to prove that g(x) is discontinuous at every integral point.

Note: Idea of greatest integer function [x] –

Greatest integer function can be seen as an input output machine in which if you enter a number, It returns the greatest integer that is just less than number x.

For example : [2.5] = 2 ; [9.99998] = 9 ; [–3.899] = –4 ; [4] = 4

To prove g(x) discontinuous at integral points we need to show that

$\lim_{x \to c} f(x) = f(c)$

Where c is any integer.

Let c is any integer

∴ g(c) = c – [c] = c – c =0 [using eqn 1]

LHL = $\lim_{h \to 0} g(c-h) = \lim_{h \to 0} c - h - [c-h] = c - (c-1) = 1$

[∵ c is integer and h is very small positive no, so (c–h) is a number less than integer c ∴ greatest integer less than (c –h) = (c–1)]

RHL = $\lim_{h \to 0} g(c+h) = \lim_{h \to 0} c + h - [c+h] = c - c = 0$ [using eqn 1 and idea of gif function]

Thus, LHL ≠ RHL

∴ g(x) is discontinuous at every integral point.

13 A. Question

Discuss the continuity of the following functions :

f(x) = sin x + cos x

Answer

Idea : If f and g are two functions whose domains are same and both f and g are everywhere continuous then :

i) f + g is also everywhere continuous

ii) f – g is also everywhere continuous

iii) f*g is also everywhere continuous

∵ f(x) = sin x + cos x

It is a purely trigonometric function

As sin x is continuous everywhere and cos x is also continuous everywhere for all real values of x

As f(x) is nothing but sum of two everywhere continuous function

∴ f(x) is also everywhere continuous.

We can see this through its graph which shows no point of discontinuity.

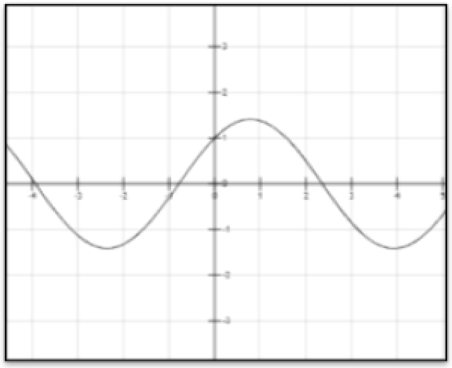


Fig : plot of sin x + cos x

13 B. Question

Discuss the continuity of the following functions :

f(x) = sin x – cos x

Answer

Idea : If f and g are two functions whose domains are same and both f and g are everywhere continuous then :

i) f + g is also everywhere continuous

ii) f – g is also everywhere continuous

iii) f*g is also everywhere continuous

∵ f(x) = sin x – cos x

It is a purely trigonometric function

As sin x is continuous everywhere and cos x is also continuous everywhere for all real values of x

As f(x) is nothing but difference of two everywhere continuous function

∴ f(x) is also everywhere continuous.

We can see this through its graph which shows no point of discontinuity.

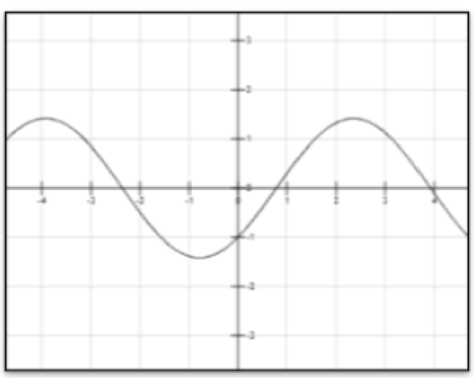


Fig : plot of sin x + cos x

13 C. Question

Discuss the continuity of the following functions :

f(x) = sin x cos x

Answer

Idea : If f and g are two functions whose domains are same and both f and g are everywhere continuous then :

i) f + g is also everywhere continuous

ii) f – g is also everywhere continuous

iii) f*g is also everywhere continuous

iv) f/g is also everywhere continuous for all R except point at which g(x) = 0

∵ f(x) = sin x × cos x

It is a purely trigonometric function

As sin x is continuous everywhere and cos x is also continuous everywhere for all real values of x

As f(x) is nothing but product of two everywhere continuous function

∴ f(x) is also everywhere continuous.

We can see this through its graph which shows no point of discontinuity.

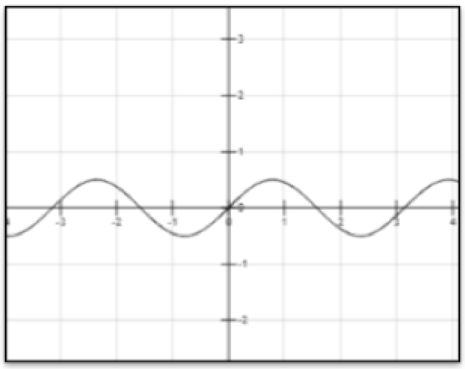


Fig : plot of sin x × cos x

14. Question

Show that f(x) = cos x^2 is a continuous function.

Answer

Idea: Such problems can be solved easily using the idea of the continuity of composite function.

If we not go to very strict mathematical meaning of composite function you can think it as it is a function of function.

Let, g(x) = cos x

And h(x) = x^2

Then g(h(x)) = g(x^2) = cos x^2

We write g(h(x)) as (goh)(x) and this is what we called composite function/function composition.

We have a theorem regarding composition of function in continuity which lets us to solve problems easily.

Theorem: If f and g are real valued function such that (fog) is defined at c, and g is continuous at c and f is continuous at g(c) then (fog) is continuous at x = c

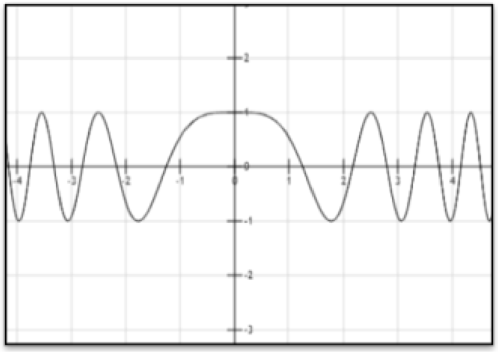
For our problem:

Let, g(x) = cos x

and h(x) = x^2

Given: f(x) = cos x^2 = g(h(x)) = (goh)(x)

Clearly, h(x) is a polynomial function, which is everywhere continuous



And g(x) being cosine function, it is also everywhere continuous.

FIG : Plot of cos x^2

∴ goh(x) = f(x) is also everywhere continuous. [using above explained theorem]

15. Question

Show that f(x) = |cos x| is a continuous function.

Answer

Idea : Such problems can be solved easily using idea of the continuity of composite function.

If we not go to very strict mathematical meaning of composite function you can think it as it is a function of function.

Let, g(x) = cos x

And h(x) = x^2

Then g(h(x)) = g(x^2) = cos x^2

We write g(h(x)) as (goh)(x) and this is what we called composite function/function composition.

We have a theorem regarding composition of function in continuity which lets us to solve problems easily.

Theorem: If f and g are real valued function such that (fog) is defined at c, and g is continuous at c and f is continuous at g(c) then (fog) is continuous at x = c

For our problem:

Let, g(x) = |x|

and h(x) = cos x

Given: f(x) = |cos x| = g(h(x)) = (goh)(x)

Clearly, h(x) is a cosine(trigonometric) function, which is everywhere continuous

And g(x) being mod function ,it is also everywhere continuous.

∴ goh(x) = f(x) is also everywhere continuous. [using above explained theorem]

16. Question

Find all the points of discontinuity of f defined by

f(x) = |x| – |x + 1|.

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h \to 0} f(c-h) = \lim_{h \to 0} f(c+h) = f(c)$ where h is a very small ‘+ve’ no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of the limit from class 11 we can summarise it as a function is continuous at x = c if :

$$\lim_{x\to c} f(x) = f(c)$$

NOTE: $Definition\ of\ mod\ function: |x| = \begin{cases} -x, x < 0 \\ x, x \geq 0 \end{cases}$

Here we have,

f(x) = |x| – |x + 1|

f(x) rewritten using idea of mod function:

$$f(x) = \begin{cases} -x - \{-(x+1)\} = 1\ , x \leq -1 \\ -x - (x+1) = -2x - 1\ , -1 < x \ \leq 0 \\ x - x - 1 = -1\ , x > 0 \end{cases}$$ ....equation 1

Clearly for x < –1 , f(x) = constant and also for x > 1 f(x) is constant

∴ in these regions f(x) is everywhere continuous.

For –1 < x < 0, plot of graph is a straight line as in this region f(x) is given by linear polynomial

∴ it is also continuous here.

∵ function is changing its expression at x = –1 and x = 0, so we need to check continuities at these points.

At x = –1 :

f(–1) = 1 [using equation 1]

LHL = $\lim_{h\to 0} f(-1-h) = \lim_{h\to 0} 1 = 1$ [using equation 1]

RHL = $\lim_{h\to 0} f(-1+h) = \lim_{h\to 0} -2(-1+h) - 1 = 2 - 1 = 1$

[using equation 1]

Clearly,

LHL = RHL = f(–1)

∴ it is continuous at x = –1

At x = 0 :

f(0) = –2*0–1 = –1 [using equation 1]

LHL = $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} -2(-h) - 1 = 0 - 1 = -1$

[using equation 1]

RHL = $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} (-1) = -1$

[using equation 1]

Clearly,

LHL = RHL = f(0)

∴ it is continuous at x = 0

Hence,

f(x) is continuous everywhere in its domain.

17. Question

Is $f(x)=\begin{cases} x^2\sin\frac{1}{x} & , x\neq 0 \\ 0 & , x=0 \end{cases}$ a continuous function?

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of the limit from class 11 we can summarise it as a function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

Given :

$f(x)=\begin{cases} x^2\sin\frac{1}{x} & ,x\neq 0 \\ 0 & ,x=0 \end{cases}$ ....Equation 1

As for x ≠ 0, f(x) is just a product of two everywhere continuous function

∴ it is continuous for all x ≠ 0.

∵ f(x) is changing its nature at x = 0, So we need to check continuity at x = 0

f(0) = 0 [using equation 1]

and $\lim_{x\to 0}(x^2\sin\frac{1}{x})$ = 0

[∵ sin(1/0) is also going to be a value between [–1,1] ,so its product with 0 = 0]

Thus,

$\lim_{x\to 0} f(x) = f(0)$

∴ It is continuous at x = 0

Hence, it is everywhere continuous.

18. Question

Given the function $f(x)=\frac{1}{x+2}$. Find the points of discontinuity of the function f(f(x)).

Answer

Basic Idea:

A real function f is said to be continuous at x = c, where c is any point in the domain of f if :

$\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$ where h is a very small '+ve' no.

i.e. left hand limit as x → c (LHL) = right hand limit as x → c (RHL) = value of function at x = c.

This is very precise, using our fundamental idea of limit from class 11 we can summarise it as, A function is continuous at x = c if :

$\lim_{x\to c} f(x) = f(c)$

NOTE: If f and g are two functions whose domains are same and both f and g are everywhere continuous then f/g is also everywhere continuous for all R except point at which g(x) = 0

As, f(x) = $\frac{1}{x+2}$

Domain of f = { all Real numbers except 2 } = R – {–2}

Clearly it is not defined at x = –2, for rest of values it is continuous everywhere

Because 1 is everywhere continuous and x + 2 is also everywhere continuous

∴ f(x) is everywhere continuous except at x = –2

f(f(x)) = $f\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2}+2} = \frac{x+2}{2x+5}$

Domain of f(f(x)) = R – {–2 , ($\frac{-5}{2}$)}

For rest of values it just a fraction of two everywhere continuous function

∴ at all other points it is everywhere continuous.

Hence,

f(f(x)) is discontinuous at x = –2 and x = –5/2

19. Question

Find all point of discontinuity of the function $f(t) = \frac{1}{t^2+t-2}$, where $t = \frac{1}{x-1}$.

Answer

Clearly,

$t = \frac{1}{x-1}$ is discontinuous at x = 1 as t is not defined at this point.

∴ f(t) being a composition of function involving t, it is also discontinuous at x = 1

$\because f(t) = \frac{1}{t^2+t-2} = \frac{1}{(t+2)(t-1)}$

by observing f(t) we can say that f(t) is not defined at

t = – 2 and t = 1.

∴ this will also contribute to discontinuity

Hence,

$t = -2 = \frac{1}{x-1}$.

⇒ x – 1 = –1/2

⇒ x = 1 – 1/2

⇒ x = 1/2

also other point of discontinuity is obtained by :

$t = 1 = \frac{1}{x-1}$

⇒ x – 1 = 1

⇒ x = 2

Hence points at which f(t) is discontinuous are x = { 1/2,1,2}

At all other point it is continuous.

MCQ

1. Question

Mark the correct alternative in the following:

The function $f(x)=\frac{4-x^2}{4x-x^3}$

A. discontinuous at only one point

B. discontinuous exactly at two points

C. discontinuous exactly at three points

D. none of these

Answer

Formula:- (i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$f(x) = \frac{4-x^2}{4x-x^3}$$

$$f(x) = \frac{1}{x}$$

$$x \neq 0, \pm 2$$

f(x)is discontinuous for three point

2. Question

Mark the correct alternative in the following:

If f(x) = |x – a| ϕ(x), where ϕ(x) is continuous function, then

A. f’ (a^+) = ϕ(a)

B. f’ (a^-) = – ϕ(a)

C. f’ (a^+) = f’(a^-)

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

(ii) $\lim_{h\to 0}\{\frac{f(a+h)-f(a)}{h}\} = f'(a^+)$ for right hand derivative

$\lim_{h\to 0}\{\frac{f(a-h)-f(a)}{h}\} = f'(a^-)$ for left hand derivative

Given:-

f(x) = |x – a| ϕ(x)

using formula (ii)

$$\lim_{h\to 0}\{\frac{f(a+h)-f(a)}{h}\} = \lim_{h\to 0}\{\frac{|h+a-a|\emptyset(a+h)-|a-a|\emptyset(a)}{h}\}$$

$$= \lim_{h\to 0}\frac{h\emptyset(a+h)}{h} = \lim_{h\to 0}\emptyset(a+h) = \emptyset(a) = f'(a^+)$$

For L.H.L

$$\lim_{h\to 0}\{\frac{f(a-h)-f(a)}{h}\}=\lim_{h\to 0}\{\frac{|a-h-a|\emptyset(a-h)-|a-a|\emptyset(a)}{h}\}$$

$$=\lim_{h\to 0}\frac{|-h|\emptyset(a-h)}{h}=\lim_{h\to 0}-\emptyset(a-h)=-\emptyset(a)=f'(a^{-})$$

3. Question

Mark the correct alternative in the following:

If f(x) = $|\log_{10} x|$, then at x = 1

A. f(x) is continuous and f' (1^{+}) = $\log_{10}$e

B. f(x) is continuous and f' (1^{+}) = $-\log_{10}$ e

C. f(x) is continuous and f' (1^{-}) = $\log_{10}$ e

D. f(x) is continuous and f' (1^{-}) = $-\log_{10}$ e

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x)=f(a)$

(ii) $\lim_{h\to 0}\{\frac{f(a+h)-f(a)}{h}\}=f'(a^{+})$ for right hand derivative

$\lim_{h\to 0}\{\frac{f(a-h)-f(a)}{h}\}=f'(a^{-})$ for left hand derivative

Given:-

f(x) = $|\log_{10} x|$

$$=\left|\frac{\log x}{\log_e 10}\right|=|\log_e x.\log_{10} e|$$

Using limit at x=1

$$\lim_{h\to 0}\{\frac{f(1+h)-f(1)}{h}\}=\lim_{h\to 0}\{\frac{|\log_{10} e.\log(1+h)|-|\log_{10} e.\log 1|}{h}\}$$

$$=\lim_{h\to 0}\frac{\log_{10} e\,|\log(1+h)|}{h}=\log_{10} e=f'(1^{+1})$$

For left hand limit

$$\lim_{h\to 0}\{\frac{f(1-h)-f(1)}{h}\}$$

$$=\lim_{h\to 0}\{\frac{|\log_{10} e.\log(1-h)|-|\log_{10} e.\log 1|}{h}\}$$

$$=\lim_{h\to 0}\frac{-\log_{10} e\,|\log(1+h)|}{h}$$

$$=-\log_{10} e=f'(1^{-1})$$

4. Question

Mark the correct alternative in the following:

If $f(x)=\begin{cases}\dfrac{36^x-9^x-4^x+1}{\sqrt{2}-\sqrt{1+\cos x}}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at x = 0, then k equals

A. $16\sqrt{2}\log 2\log 3$

B. $16\sqrt{2}\ln 6$

C. $16\sqrt{2}\ln 2\ln 3$

D. none of these

Answer

Formula:- (i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x)=f(a)$

(ii) $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\}=\frac{l}{m}$, where $\lim_{x\to a} f(x)=l$, $\lim_{x\to a} g(x)=m$

(iii) $\lim_{x\to 0}\frac{\sin x}{x}=1$

Given:-

$$f(x)=\begin{cases}\dfrac{36^x-9^x-4^x+1}{\sqrt{2}-\sqrt{1+\cos x}}, x\neq 0\\ k, x=0\end{cases}$$

Function f(x) is continuous at x=0

$$\lim_{x\to 0} f(x)=f(0)$$

$$\lim_{x\to 0}\left(\frac{36^x-9^x-4^x+1}{\sqrt{2}-\sqrt{1+\cos x}}\right)=k$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{(9^x-1)(4^x-1)}{\sqrt{2}-\sqrt{1+\cos x}}\right)=k$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{(9^x-1)(4^x-1)}{\sqrt{2}-\sqrt{2}\cos\left(\frac{x}{2}\right)}\right)=k$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{(9^x-1)(4^x-1)}{\sqrt{2}\{2\sin^2(\frac{x}{4})\}}\right)=k$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{8(9^x-1)(4^x-1)}{\sqrt{2}x^2\left\{\frac{2\sin^2\left(\frac{x}{4}\right)}{\frac{x^2}{16}}\right\}}\right)=k$$

Using formula (ii) and (iii)

$$\frac{8.\ln 9.\ln 4}{\sqrt{2}}=k$$

$$\Rightarrow \frac{32.\ln 3.\ln 2}{\sqrt{2}}=k$$

$\Rightarrow k = 16\sqrt{2}.\ln 2.\ln 3$

5. Question

Mark the correct alternative in the following:

If f(x) defined by $f(x)=\begin{cases}\frac{|x^2-x|}{x^2-x}, & x\neq 0,\quad 1\\ 1 & , x=0\\ -1 & , x=1\end{cases}$ then f(x) is continuous for all

A. x

B. x except at x = 0

C. x except at x =1

D. x except at x = 0 and x =1

Answer

Formula:-

(i) $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ then f(x) is discontinuous at x=0

(ii) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ then f(x) is continuous at x=0

(iii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

Given:-

$$F(x)=\begin{cases}\frac{|x^2-x|}{x^2-x}, x\neq 0,1\\ 1, x=0\\ -1, x=1\end{cases} = \begin{cases}1, x>1\\ 1, x\leq 0\\ -1, 0\leq x\leq 1\end{cases}$$

Using R.H.L

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h) = -1$

Using L.H.L

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(h) = 1$

f(x) is discontinuous at x=0

Again using R.H.L

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = 1$

Using L.H.L

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = -1$

f(x) is discontinuous at x=1

Therefore, f(x) is continuous for all except at x=0 and x=1

6. Question

Mark the correct alternative in the following:

If $\begin{cases} \dfrac{1-\sin x}{(\pi-2x)^2}\cdot\dfrac{\log\sin x}{\left(\log\left(1+\pi^2-4\pi x+4x^2\right)\right)} & , \quad x \neq \dfrac{\pi}{2} \\ k & , \quad x = \dfrac{\pi}{2} \end{cases}$ is continuous at x = π/2, then k =

A. $-\dfrac{1}{16}$

B. $-\dfrac{1}{32}$

C. $-\dfrac{1}{64}$

D. $-\dfrac{1}{28}$

Answer

Formula:- (i) $\lim_{x\to 0}\frac{\log(1-x)}{x} = 1$ and $\lim_{x\to 0}\frac{\sin x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(iii) $\lim_{x\to a}\{f(x).g(x)\} = l.m$, where $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$

Given:-

$$f(x) = \begin{cases} \dfrac{1-\sin x}{(\pi-2x)^2}\cdot\dfrac{\log x \sin x}{(\log(1+\pi^2-4\pi x+4x^2))}, x \neq \dfrac{\pi}{2} \\ k, x = \dfrac{\pi}{2} \end{cases}$$

Function f(x) is continuous at x=$\frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

Using substitution method

If $\frac{\pi}{2} - x = t$, then

$$\lim_{t\to 0} f\left(\frac{\pi}{2}-t\right) = f\left(\frac{\pi}{2}\right)$$

$$\lim_{t\to 0}\left\{\frac{1-\sin\left(\frac{\pi}{2}-t\right)}{4t^2}\cdot\frac{\log\sin\left(\frac{\pi}{2}-t\right)}{\log\left(1+\pi^2-4\pi\left(\frac{\pi}{2}-t\right)+4\left(\frac{\pi}{2}-t\right)^2\right)}\right\} = k$$

$$\frac{1}{8}\lim_{t\to 0}\left\{\frac{\sin^2\frac{t}{2}}{(\frac{t}{2})^2}\cdot\frac{\log\sqrt{1-\sin^2 t}}{\frac{\log(1+4t^2)}{4t^2}}\right\} = k$$

$$\frac{1}{64}\lim_{t\to 0}\left\{\frac{\sin^2\frac{t}{2}}{(\frac{t}{2})^2}\cdot\frac{\frac{\log(1-\sin^2 t)}{8t^2}}{\frac{\log(1+4t^2)}{4t^2}}\right\} = k$$

Using formula (iii)

$$-\frac{1}{64}\lim_{t\to 0}\left(\frac{\sin t}{t}\right)^2 . \lim_{t\to 0}\log\frac{(1-\sin^2 t)}{-\sin^2 t} = k$$

Using standard limit formula (i)

$$k = -\frac{1}{64}$$

7. Question

Mark the correct alternative in the following:

If f(x) = $(x + 1)^{\cot x}$ be continuous at x = 0, then f(0) is equal to

A. 0

B. $\frac{1}{e}$

C. e

D. none of these

Answer

Formula:- (i)standard limit $\lim_{x\to 0}\frac{\log(1-x)}{x} = 1$ and $\lim_{x\to 0}\frac{\tan x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

(iii) $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{m}$, where $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$

Given:-

f(x) = $(x + 1)^{\cot x}$

log f(x)=(cotx)(log(x+1))..................................taking log both sides

$$\lim_{x\to 0}\log f(x) = \lim_{x\to 0}(\cot x)\big(\log(x+1)\big)$$

$$\lim_{x\to 0}\log f(x) = \lim_{x\to 0}\left(\frac{\frac{\log(x+1)}{x}}{\frac{\tan x}{x}}\right)$$

Using formula (iii)

$$\lim_{x\to 0}\log f(x) = \frac{\lim_{x\to 0}\frac{\log(x+1)}{x}}{\lim_{x\to 0}\frac{\tan x}{x}}$$

Using standard limit formula (i)

$\lim_{x\to 0} f(x) = e$

f(0)=e

8. Question

Mark the correct alternative in the following:

If $f(x)=\begin{cases}\dfrac{\log(1+ax)-\log(1-bx)}{x} & , x\neq 0\\ k & , x=0\end{cases}$ and f(x) is continuous at x = 0, then the value of k is

A. a – b

B. a + b

C. log a + log b

D. none of these

Answer

Formula:-

standard limit $\lim_{x\to 0}\frac{\log(1-x)}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(iii) $\lim_{x\to a}\{f(x)\pm g(x)\} = l \pm m$, where $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$

Given:-

f(x) $=\begin{cases}\frac{\log(1+ax)-\log(1-bx)}{x} & , x\neq 0\\ k, x=0\end{cases}$

And f(x) is continuous at x = 0

$$\lim_{x\to 0}\frac{\log(1+ax)-\log(1-bx)}{x} = k$$

Using formula (ii)

$$a\lim_{x\to 0}\frac{\log(1+ax)}{ax} - b\lim_{x\to 0}\frac{\log(1-bx)}{bx} = k$$

Using formula (i)

a+b=k

9. Question

Mark the correct alternative in the following:

The function $f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1} & , x\neq 0\\ 0 & , x=0\end{cases}$

A. is continuous at x = 0

B. is not continuous at x = 0

C. is not continuous at x = 0, but can be made continuous at x = 0

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{1/x} + 1}, x \neq 0 \\ 0, x = 0 \end{cases}$$

Using substitution method

Let $e^{\frac{1}{x}} = t$ so, $x \to 0, t \to \infty$

$$\lim_{t\to\infty} f(x) = \lim_{t\to\infty}\left(\frac{t-1}{t+1}\right)$$

$$= \frac{1-0}{1+0} = 1$$

and

F(0)=0

Therefore,

$$\lim_{x\to} f(x) \neq f(0)$$

Hence, f(x) is discontinuous at x=0

10. Question

Mark the correct alternative in the following:

Let $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a & , x < 4 \\ a + b & , x = 4 \\ \dfrac{x-4}{|x-4|} + b & , x > 4 \end{cases}$. Then f(x) is continuous at x = 4 when

A. a = 0, b = 0

B. a = 1, b = 1

C. a = –1, b = 1

D. a = 1, b = –1

Answer

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

(i)$f(x) = \begin{cases} \frac{x-4}{|x-4|} + a, x < 4 \\ a + b, x = 4 \\ \frac{x-4}{|x-4|} + b, x > 4 \end{cases}$

(ii) f(x) is continuous at x = 4

Using R.H.L

$\lim\limits_{x \to 4^+} f(x) = \lim\limits_{h \to 0} f(4+h)$

$= \lim\limits_{h \to \infty} \left(\frac{h}{|h|} + b\right) = 1 + b$

Using L.H.L

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{h \to 0} f(4-h)$

$= \lim\limits_{h \to \infty} \left(\frac{-h}{|h|} + a\right) = a - 1$

f(x) is continuous at x = 4

$\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^-} f(x) = f(4)$

a-1=b+1=a+b

b=-1 and a=1

11. Question

Mark the correct alternative in the following:

If the function $f(x) \begin{cases} (\cos x)^{1/x} & , x \neq 0 \\ k & , x = 0 \end{cases}$ is continuous at x = 0, then the value of k is

A. 0

B. 1

C. –1

D. e

Answer

Formula:-

(i) $\lim\limits_{x \to 0} f(x)^{g(x)} = e^{\lim\limits_{x \to 0}(f(x)-1).g(x)}$ where $\lim\limits_{x \to 0} f(x) = 1$ and $\lim\limits_{x \to 0} g(x) = 0$

(ii) $\lim\limits_{x \to 0} \{\frac{(\cos x - 1)}{x}\} = 0$

(iii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim\limits_{x \to a} f(x) = f(a)$

$\lim\limits_{x \to a^+} f(a+h) = \lim\limits_{x \to a^-} f(a-h) = f(a)$

Given:-

(i)$f(x) = \begin{cases} (\cos x)^{1/x}, x \neq 0 \\ k, x = 0 \end{cases}$

(ii) f(x) is continuous at x = 0

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} (\cos x)^{1/x} = k$

Using formula (i)

$\lim_{x \to 0} f(x)^{g(x)} = e^{\lim_{x \to 0} \{\frac{(\cos x - 1)}{x}\}} = k$

$e^0 = k$

K=1

12. Question

Mark the correct alternative in the following:

Let f(x) = |x| + |x – 1|, then

A. f(x) is continuous at x = 0, as well as at x=1

B. f(x) is continuous at x = 0, but not at x = 1

C. f(x) is continuous at x = 1, but not at x =

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$

Given:-

(i) f(x) = |x| + |x – 1|

Both the function are continuous everywhere

According to option

f(x) is continuous at x = 0, as well as at x=1

13. Question

Mark the correct alternative in the following:

Let $f(x) \begin{cases} \dfrac{x^4 - 5x^2 + 4}{(x-1)(x-2)} & , \quad x \neq 1, 2 \\ 6 & , \quad x = 1 \\ 12 & , \quad x = 2 \end{cases}$. Then, f(x) is continuous on the set

A. R

B. R – {1}

C. R – {2}

D. R – {1, 2}

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} \dfrac{x^4 - 5x^2 + 4}{(x-1)(x-2)}, x \neq 1,2 \\ 6, x = 1 \\ 12, x = 2 \end{cases} = \begin{cases} (x+1)(x+2), x < 1 \\ -(x+1)(x+2), 1 < x < 2 \\ (x+1)(x+2), x > 2 \\ 6, x = 1 \\ 12, x = 2 \end{cases}$$

Using R.H.L at x=1

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h)$$

$$= -(2).(3) = -6$$

Using L.H.L at x=1

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h)$$

$$= (2)(3) = 6$$

And Using R.H.L at x=2

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(1+h) = 12$$

Using L.H.L at x=2

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(1-h) = -12$$

F(x) is discontinuous at x=1 and x=2

For f(x) continuous at R-{1,2}

14. Question

Mark the correct alternative in the following:

If $f(x) \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} &, \quad x < 0 \\ c &, \quad x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx\sqrt{x}} &, \quad x > 0 \end{cases}$ is continuous at x = 0, then

A. $a = -\frac{3}{2}, b = 0, c = \frac{1}{2}$

B.. $a = -\frac{3}{2}, b = 1, c = -\frac{1}{2}$.

C. $a = -\frac{3}{2}, b \in R - \{0\}, c = \frac{1}{2}$

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(ii) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Given:- (i) f(x) continuous at x = 0

(ii) $f(x) = \begin{cases} \frac{\sin(a+1)+\sin x}{x}, x < 0 \\ c, x = 0 \\ \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, x > 0 \end{cases}$

Using R.H.L

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$

$= \lim_{h\to 0} \frac{\sqrt{1+bh}-\sqrt{1}}{bh} = \lim_{h\to 0} \frac{1}{\sqrt{1+bh}-\sqrt{1}} = \frac{1}{2}$

Using L.H.L

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$= \lim_{h\to 0} \frac{-\sin(a+1)h + \sin(-h)}{x} = -a-1$

Function f(x) is continuous at x=0

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$-a-1 = \frac{1}{2} = c$

$A = -\frac{3}{2} \text{and } c = \frac{1}{2}$

From

Case iii

b∊R-{0}

15. Question

Mark the correct alternative in the following:

If $f(x) = \begin{cases} mx+1 & , x \le \frac{\pi}{2} \\ \sin x + n & , x > \frac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$, then

A. m = 1, n = 0

B. $m = \frac{n\pi}{2} + 1$

C. $n = \frac{m\pi}{2}$

D. $m = n = \frac{\pi}{2}$

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$

Given:-$\begin{cases} mx+1, x \le \frac{\pi}{2} \\ \sin x + n, x > \frac{\pi}{2} \end{cases}$

Using R.H.L

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right)$$

$$= \lim_{h \to 0} \left\{\sin\left(\frac{\pi}{2} + h\right) + n\right\} = n + 1$$

Using L.H.L

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right)$$

$$= \lim_{h \to 0} m\left(\frac{\pi}{2} - h\right) + 1 = \frac{m\pi}{2} + 1$$

Function f(x) is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}^{++}} f(x) = \lim_{x \to \frac{\pi}{2}^-} f(x)$$

$$\frac{m\pi}{2} + 1 = n + 1$$

$$\frac{m\pi}{2} = n$$

16. Question

Mark the correct alternative in the following:

The value of f(0), so that the function $f(x) = \frac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a+x} - \sqrt{a-x}}$ becomes continuous for all x, given by

A. $a^{3/2}$

B. $a^{1/2}$

C. $-a^{1/2}$

D. $-a^{3/2}$

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

$$f(x) = \frac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a+x} - \sqrt{a-x}}$$

Using rationalization method

$$\lim_{x\to 0} f(x) = \frac{(a^2 - ax + x^2) - (a^2 + ax + x^2)}{\left(\sqrt{a+x} - \sqrt{a-x}\right)\left(\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2}\right)}$$

$$= \lim_{x\to 0} \frac{-2ax\left(\sqrt{a+x} + \sqrt{a-x}\right)}{2x\left(\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2}\right)}$$

$$= \lim_{x\to 0} \frac{-a\left(\sqrt{a+x} + \sqrt{a-x}\right)}{\left(\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2}\right)}$$

$$= \lim_{x\to 0} \frac{-2a\left(\sqrt{a}\right)}{(2a)} = -\sqrt{a}$$

F(0)= $-\sqrt{a}$

17. Question

Mark the correct alternative in the following:

The function $f(x) = \begin{cases} 1 & , \quad |X| \geq 1 \\ \frac{1}{n^2} & , \quad \frac{1}{n} < |x| < \frac{1}{n-1}, n = 2, 3, \ldots \\ 0 & , \quad x = 0 \end{cases}$

A. is discontinuous at finitely many points

B. is continuous everywhere

C. is discontinuous only at $x = \pm\frac{1}{n}, n \in Z - \{0\}$ and x = 0

D. none of these

Answer

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} 1, |x| \geq 1 \\ \frac{1}{n^2}, \frac{1}{n} < |x| < \frac{1}{n-1}, n = 2, 3, \ldots \ldots \ldots \\ 0, x = 0 \end{cases}$$

$$= \begin{cases} 1, -1 \leq x \leq 1 \\ \frac{1}{n^2}, \frac{1}{n} < |x| < \frac{1}{n-1}, n = 2, 3, \ldots \ldots \ldots \\ 0, x = 0 \end{cases}$$

Using R.H.L

$$\lim_{x\to \frac{1}{n}^+} f(x) = \lim_{h\to 0} f\left(\frac{1}{n} + h\right) = \left(\frac{1}{n}\right)^2$$

Using L.H.L

$$\lim_{x \to \frac{1}{n}^-} f(x) = \lim_{h \to 0} f\left(\frac{1}{n} - h\right) = \left(\frac{1}{n-1}\right)^2$$

$$\lim_{x \to \frac{1}{n}^+} f(x) \neq \lim_{x \to \frac{1}{n}^-} f(x)$$

Therefore f(x) is discontinuous only at $x = \pm\frac{1}{n}. n\epsilon Z - \{0\}$ and x = 0

18. Question

Mark the correct alternative in the following:

The value of f(0), so that the function $f(x) = \frac{(27-2x)^{1/3} - 3}{9 - 3(243+5x)^{1/5}} (x \neq 0)$ is continuous, is given by

A. $\frac{2}{5}$

B. 6

C. 2

D. 4

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$$

(ii) $\lim_{x \to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{1}{m}$, where $\lim_{x \to a} f(x) = l, \lim_{x \to a} g(x) = m$

Given:-

$$f(x) = \frac{(27-2x)^{\frac{1}{3}} - 3}{9 - 3(243+5x)^{\frac{1}{5}}}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(27-2x)^{\frac{1}{3}} - 3}{9 - 3(243+5x)^{\frac{1}{5}}}$$

Using factorization method

$$= \frac{2}{15} \lim_{x \to 0} \frac{\frac{(27-2x)^{\frac{1}{3}} - 27^{\frac{1}{3}}}{27 - 2x - 27}}{\frac{(243+5x)^{\frac{1}{5}} - 243^{\frac{1}{5}}}{243 + 5x - 243}}$$

$$= \left(\frac{2}{15}\right) . \left(\frac{1}{3}\right) . \left(\frac{1}{27^{\frac{2}{3}}}\right) . (5) . (243)^{\frac{4}{5}}$$

=2

19. Question

Mark the correct alternative in the following:

The value f(0) so that the function $f(x)=\frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}$, $x \neq 0$ is continuous everywhere, is given by

A. –1

B. 1

C. 26

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

(ii) $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{m}$, where $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$

Given:-

$$f(x) = \frac{2-(256-7x)^{\frac{1}{8}}}{(5x+32)^{\frac{1}{5}}-2}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{2-(256-7x)^{\frac{1}{8}}}{(5x+32)^{\frac{1}{5}}-2}$$

Using factorization method

$$=\frac{7}{5}\lim_{x\to 0}\frac{\frac{(256-7x)^{\frac{1}{8}}-256^{\frac{1}{8}}}{256-7x-256}}{\frac{(5x+32)^{\frac{1}{5}}-32^{\frac{1}{5}}}{5x+32-32}}$$

$$=\frac{7}{5}\cdot\left(\frac{1}{8}\right)\cdot\left(\frac{2^4}{2^7}\right).5$$

$$=\frac{7}{64}$$

20. Question

Mark the correct alternative in the following:

$$f(x)=\begin{cases}\frac{\sqrt{1+px}=\sqrt{1-px}}{x} & ,-1\le x<0\\ \frac{2x+1}{x-2} & ,0\le x\le 1\end{cases}$$ is continuous in the interval [–1, 1], then p is equal to

A. -1

B. $-\frac{1}{2}$

C. $\frac{1}{2}$

D. 1

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} \frac{\sqrt{1+px}-\sqrt{1-px}}{x}, -1 \le x < 0 \\ \frac{2x+1}{x-2}, 0 \le x \le 1 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sqrt{1-ph}-\sqrt{1+ph}}{-h}$$

Using rationalization method

$$\lim_{h\to 0} \frac{\left(\sqrt{1-ph}-\sqrt{1+ph}\right).\left(\sqrt{1-ph}+\sqrt{1+ph}\right)}{-h\left(\sqrt{1-ph}+\sqrt{1+ph}\right)} = \lim_{h\to 0} \frac{2p}{\sqrt{1-ph}-\sqrt{1+ph}}$$

We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h)$$

$$\Rightarrow \lim_{h\to 0} \frac{2p}{\sqrt{1-ph}-\sqrt{1+ph}} = \lim_{h\to 0} \left(\frac{2h+1}{h-2}\right)$$

$$\Rightarrow \frac{2p}{2} = -\frac{1}{2}$$

$$\Rightarrow p = -\frac{1}{2}$$

21. Question

Mark the correct alternative in the following:

The function $f(x) = \begin{cases} x^2 a & , \quad 0 \le x < 1 \\ a & , \quad 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^2} & , \quad \sqrt{2} \le x < \infty \end{cases}$ is continuous for 0 ≤ x < ∞, then the most suitable values of a and b are

A. a = 1, b = –1

B. a = –1, b = 1 + √2

C. a = –1, b = 1

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} \frac{x^2}{a}, 0 \le x < 1 \\ a, 1 \le x < \sqrt{2} \\ \frac{2b^2 - 4b}{x^2}, \sqrt{2} \le x < \infty \end{cases}$$

Case (i)&(ii)

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \frac{(1-h)^2}{a}$$

Now, $\lim_{x \to 1^-} f(x) = f(1)$

$$\Rightarrow \frac{1}{a} = a$$

$\Rightarrow +1 = a$ -------------------- (iv)

Case (ii)&(iii)

$$\lim_{x \to \sqrt{2}^-} f(x) = \lim_{h \to 0} f\left(\sqrt{2} - h\right)$$

$$= \lim_{h \to 0} a = a$$

Now, $\lim_{x \to \sqrt{2}^-} f\left(\sqrt{2} - h\right) = f\left(\sqrt{2}\right)$

$$\Rightarrow \ a = b^2 - 2b$$

$$\Rightarrow b^2 - 2b - a = 0$$

Using value a from (IV)

For a=-1

$b = 1$

22. Question

Mark the correct alternative in the following:

If $f(x) = \frac{1 - \sin x}{(\pi - 2x)^2}$, when $x \ne \frac{\pi}{2}$ and $f\left(\frac{\pi}{2}\right) = \lambda$, then f(x) will be continuous function at x = π/2, where λ =

A. $\frac{1}{8}$

B. $\frac{1}{4}$

C. $\frac{1}{2}$

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(ii) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Given:-

$f(x) = \frac{1-\sin x}{(\pi-2x)^2}$

$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{1-\sin x}{(\pi-2x)^2}$

Using substitution method

Let $\frac{\pi}{2} - x = t$

$= \lim_{t\to 0} \frac{1-\sin(\frac{\pi}{2}-t)}{(2t)^2}$

$= \frac{1}{8} \lim_{t\to 0} \frac{\sin^2 \frac{t}{2}}{\left(\frac{t}{2}\right)^2} = \frac{1}{8}$

Now we know that

$f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x)$

$\Rightarrow \lambda = \frac{1}{8}$

23. Question

Mark the correct alternative in the following:

The value of a for which the function may be continuous at x = 0 is

A. 1

B. 2 $f(x) = \begin{cases} \dfrac{\left(4^x - 1\right)^3}{\sin(ax)\log\left\{\left(1+x^2 3\right)\right\}} & , x \neq 0 \\ 12(\log 4)^3 & , x = 0 \end{cases}$

C. 3

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

(ii) $\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(iii) $\lim_{x\to 0}\frac{\sin x}{x} = 1$

(iv) $\lim_{x\to 0}\frac{a^x - 1}{x} = \log a$

(v) $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

(vi) $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{m}$, where $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$

Given:-

$$f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin\left(\frac{x}{a}\right)\log\left\{\left(1+\frac{x^2}{3}\right)\right\}}, x \neq 0 \\ 12(\log 4)^3, x = 0 \end{cases}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{(4^x - 1)^3}{\sin\left(\frac{x}{a}\right)\log\left\{\left(1+\frac{x^2}{3}\right)\right\}}$$

$$= 3a\lim_{x\to 0}\frac{\left(\frac{4^x - 1}{x}\right)^3}{\frac{\sin\frac{x}{a}}{\frac{x}{a}}.\frac{\log\left\{\left(1+\frac{x^2}{3}\right)\right\}}{\frac{x^2}{3}}}$$

$$= 3a(\log 4)^3$$

Now we know that

$f(0) = \lim_{x\to 0} f(x)$

$12(\log 4)^3 = 3a(\log 4)^3$

a=4

24. Question

Mark the correct alternative in the following:

The function f(x) = tan x is discontinuous on the set

A. {n π: n ϵ Z}

B. {2n π : n ϵ Z}

C. $\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

D. $\left\{\frac{n\pi}{2} : n \in Z\right\}$

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

Given:-

Tan x is discontinuous at $\left\{(2n+1)\frac{\pi}{2} : n\epsilon Z\right\}$

25. Question

Mark the correct alternative in the following:

The function $f(x)=\begin{cases}\dfrac{\sin 3x}{x} & , x \neq 0 \\ \dfrac{k}{2} & , x=0\end{cases}$ is continuous at x = 0, then k =

A. 3

B. 6

C. 9

D. 12

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(ii)$\lim_{x\to 0}\frac{\sin x}{x} = 1$

Given:-

$$f(x) = \begin{cases}\frac{\sin 3x}{x}, x = 0 \\ \frac{k}{2}, x = 0\end{cases}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin 3x}{x}$$

$$= 3\lim_{x\to 0}\frac{\sin 3x}{x} = 3$$

Now we know that

$$f(0) = \lim_{x\to 0} f(x)$$

$$\frac{k}{2} = 3$$

K=6

26. Question

Mark the correct alternative in the following:

If the function $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$ is continuous at each point of its domain, then the value of f(0) is

A. 2

B. $\frac{1}{3}$

C. $-\frac{1}{3}$

D. $\frac{2}{3}$

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(ii)$\lim_{x\to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x\to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{m}$, where $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$

Given:-

$$f(x) = \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}\right) = \lim_{x\to 0}\left(\frac{2 - \frac{\sin^{-1} x}{x}}{2 + \frac{\tan^{-1} x}{x}}\right) = \frac{1}{3}$$

Now we know that

$$f(0) = \lim_{x\to 0} f(x)$$

$$\Rightarrow f(0) = \frac{1}{3}$$

27. Question

Mark the correct alternative in the following:

The value of b for which the function $f(x) = \begin{cases} 5x - 4 & , x < x \le 1 \\ 4x^2 + 3bx & , 1 < x < 2 \end{cases}$ is continuous at every point of its domain, is

A. –1

B. 0

C. $\frac{13}{3}$

D. 1

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

Given:-

$$f(x) = \begin{cases} 5x - 4, 0 < x \le 1 \\ 4x^2 + 3bx, 1 < x < 2 \end{cases}$$

$\lim_{x \to 1^+} f(x) = f(1)$

$\Rightarrow \lim_{h \to 0} f(1+h) = f(1)$

$\Rightarrow \lim_{h \to 0} \left(4(1+h)^2 + b(1+h)\right) = 5 - 4$

$\Rightarrow 4 + 3b = 1$

$\Rightarrow b = -1$

28. Question

Mark the correct alternative in the following:

If $f(x) = \frac{1}{1-x}$, then the set of points discontinuity of the function f(f(f(x))) is

A. {1}

B. {0, 1}

C. {-1, 1}

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$

Given:-

$f(x) = \frac{1}{1-x}$

For

$f\left(f(f(x))\right) = f\left(\frac{x-1}{x}\right) = \frac{1}{1 - \frac{x-1}{x}} = x$

It is $R - \{0,1\}$

It is discontinuous at x=0,1

29. Question

Mark the correct alternative in the following:

Let $f(x) = \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}, x \neq \frac{\pi}{4}.$ The value which should be assigned to f(x) at $x = \frac{\pi}{4}$, so that it is continuous everywhere is

A. 1

B. $\frac{1}{2}$

C. 2

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$

(ii)$\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iii) $\lim_{x \to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{m}$, where $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$

Given:-

$$f(x) = \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$$

Using substitution method

Let $\frac{\pi}{2} - x = t$

$$= \lim_{t \to 0} \frac{\tan t}{\cot 2(\frac{\pi}{4} - t)} = \lim_{t \to} \frac{\frac{\tan t}{t}}{\frac{\tan 2t}{t}}$$

$$= \frac{1}{2} \lim_{t \to} \frac{\frac{\tan t}{t}}{\frac{\tan 2t}{2}}$$

$$= \frac{1}{2}.1 = \frac{1}{2}$$

Now we know that

$$f\left(\frac{\pi}{4}\right) = \lim_{x \to 0} f(x)$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

30. Question

Mark the correct alternative in the following:

The function $f(x) = \frac{x^3 + x^2 - 16x + 20}{x - 2}$ is not defined for x = 2. In order to make f(x) continuous at x = 2, f(2) should be defined as

A. 0

B. 1

C. 2

D. 3

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \frac{x^3+x^2-16x+20}{x-2}$$

Using factorization method

$$f(x) = \frac{(x+5)(x-2)(x-2)}{x-2} = (x+5)(x-2)$$

At point x=2

$$\lim_{x\to 2} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 0}(x+5)(x-2) = f(2)$$

$\Rightarrow 0=f(2)$

31. Question

Mark the correct alternative in the following:

If $f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1) & , x\le 1 \\ \frac{\tan x-\sin x}{x^3} & , x>0 \end{cases}$ is continuous at x = 0, then a equals

A. $\frac{1}{2}$

B. $\frac{1}{3}$

C. $\frac{1}{4}$

D. $\frac{1}{6}$

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

(ii)$\lim_{x\to 0}\frac{\sin x}{x} = 1$, $\lim_{x\to 0}\frac{\tan x}{x} = 1$

(iii) $\lim_{x\to a}\{f(x).g(x)\} = l.m$, where $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$

Given:- $f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), x\le 1 \\ \frac{\tan x-\sin x}{x^3}, x>0 \end{cases}$

Function f(x) is continuous at x=0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(h) = \lim_{h\to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h\to 0} \frac{\sin h(1-\cos h)}{\cos h h^3} = \frac{2}{4} \lim_{h\to 0} \frac{\sin^2 \frac{h}{2} \tan h}{\frac{h^2}{4}.h}$$

$$= \frac{1}{2} \lim_{h\to 0} \frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4}} . \lim_{h\to 0} \frac{\tan h}{h}$$

$$= \frac{1}{2}$$

Now,

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(-h) = \lim_{h\to 0} a\sin\frac{\pi}{2}(-h+1) \ = a\sin\frac{\pi}{2} = a$$

Function f(x) is continuous at x=0

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$$

$$\Rightarrow \frac{1}{2} = a$$

32. Question

Mark the correct alternative in the following:

If $f(x) = \begin{cases} ax^2 + b & , \quad 0 \le x < 1 \\ 4 & , \quad x = 1 \\ x + 3 & , \quad 1 < x \le 2 \end{cases}$,then the value of (a, b) for which f(x) cannot be continuous at x = 1, is

A. (2, 2)

B. (3, 1)

C. (4, 0)

D. (5, 2)

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:- $f(x) = \begin{cases} ax^2 + b \,, 0 \le x < 1 \\ 4, x = 1 \\ x + 3 \,, 1 < x \le 2 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} a(1-h)^2 + b$$

$$= a + b$$

Function f(x) is discontinuous at x=1

$$\lim_{x\to 1^-} f(x) \ne f(1)$$

$\Rightarrow a+b \neq 4$

Checking option

(5,2) is answer

33. Question

Mark the correct alternative in the following:

If the function f(x) defined by $f(x)=\begin{cases}\dfrac{\log(1+3x)-\log(1-2x)}{x} & , x \neq 0 \\ k & , x=0\end{cases}$ is continuous at x = 0, then k =

A. 1

B. 5

C. –1

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(ii) Standard limits $\lim_{x\to 0}\frac{\sin x}{x} = 1$, $\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

(iii) $\lim_{x\to a}\{f(x) \pm g(x)\} = l \pm m$, where $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$

Given:- $f(x) = \begin{cases}\frac{\log(1+3x)-\log(1-2x)}{x}, x \neq 0 \\ k, x=0\end{cases}$

Now,

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{\log(1+3x)-\log(1-2x)}{x}$

Using formula(III)

$= 3\lim_{x\to 0}\dfrac{\log(1+3x)}{3x} + 2\lim_{x\to 0}\dfrac{\log(1-2x)}{-2x}$

$= 5$(Using standard limit)

Function f(x) is continuous at x=0

$\lim_{x\to 0} f(x) = f(0)$

$\Rightarrow 5 = k$

34. Question

Mark the correct alternative in the following:

If $f(x)=\begin{cases}\dfrac{1-\cos 10x}{x^2} & , \quad x<0 \\ a & , \quad x=0 \\ \dfrac{\sqrt{x}}{\sqrt{625+\sqrt{x}}-25} & , \quad x>0\end{cases}$, then the value of a so that f(x) may be continuous at x = 0, is

A. 25

B. 50

C. –25

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

(ii) Standard limits $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Given:-

$$f(x) = \begin{cases}\dfrac{1-\cos 10x}{x^2}, x<0 \\ a, x=0 \\ \dfrac{\sqrt{x}}{\sqrt{625+\sqrt{x}}-25}, x>0\end{cases}$$

Using L.H.L

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(-h)$

$= \lim_{h\to 0} \dfrac{1-\cos(-10h)}{(-h)^2}$

$= 50 \lim_{h\to 0} \dfrac{(\sin 5h)^2}{(5h)^2}$

$= 50$ (Using Standard limits)

Function f(x) is continuous at x=0

$\lim_{x\to 0} f(x) = f(0)$

$\Rightarrow 50 = a$

35. Question

Mark the correct alternative in the following:

If f(x) = $x \sin \dfrac{1}{x}, x \neq 0,$ then the value of the function at x = 0, so that the function is continuous at x = 0, is

A. 0

B. –1

C. 1

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$

(ii) Standard limits $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to a} \{f(x).g(x)\} = l.m$, where $\lim_{x \to a} f(x) = l, \lim_{x \to a} g(x) = m$

Given:-

$f(x) = x \sin \frac{1}{x}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} x. \sin \frac{1}{x}$

Using formula (iii)

$= \lim_{x \to 0} x. \lim_{x \to 0} \sin \frac{1}{x} = 0$

Function f(x) is continuous at x=0

$\lim_{x \to 0} f(x) = f(0)$

$\Rightarrow 0 = f(0)$

36. Question

Mark the correct alternative in the following:

The value of k which makes continuous at x = 0, is

A. 8 $f(x) = \begin{cases} \sin \frac{1}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$

B. 1

C. –1

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$

(ii) Standard limits $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Given:-

$f(x) = \sin\frac{1}{x}$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \sin\frac{1}{x}$

Function f(x) is continuous at x=0

$\lim\limits_{x\to 0} f(x) = f(0)$

$\Rightarrow 0 = f(0)$

$\Rightarrow \lim\limits_{x\to 0} \sin\frac{1}{x} = k$

Value does not exist for function to be continuous

37. Question

Mark the correct alternative in the following:

The values of the constants a, b, and c for which the function

$$f(x) = \begin{cases} (1+ax)^{1/x} & , \quad x < 0 \\ b & , \quad x = 0 \\ \dfrac{(x+c)^{1/3}-1}{(x+1)^{1/2}-1} & , \quad x > 0 \end{cases}$$

May be continuous at x = 0, are

A. $a = \log_e\left(\frac{2}{3}\right), b = -\frac{2}{3}, c = 1$

B. $a = \log_e\left(\frac{2}{3}\right), b = \frac{2}{3}, c = -1$

C. $a = \log_e\left(\frac{2}{3}\right), b = \frac{2}{3}, c = 1$

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim\limits_{x\to a} f(x) = f(a)$

$\lim\limits_{x\to a^+} f(a+h) = \lim\limits_{x\to a^-} f(a-h) = f(a)$

(ii) $\lim\limits_{x\to a}\{f(x).g(x)\} = l.m$, where $\lim\limits_{x\to a} f(x) = l$, $\lim\limits_{x\to a} g(x) = m$

Given:-

$$f(x) = \begin{cases} (1+ax)^{\frac{1}{x}}, x < 0 \\ b, x = 0 \\ \dfrac{(x+c)^{\frac{1}{3}}-1}{(x+1)^{\frac{1}{2}}-1}, x > 0 \end{cases}$$

Now,

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{(h+c)^{\frac{1}{3}} - 1}{(h+1)^{\frac{1}{2}} - 1}$$

Using rationalization method

$$= \lim_{h \to 0} \frac{(h+c)^{\frac{1}{3}} - 1}{(h+1)^{\frac{1}{2}} - 1} \cdot \frac{(h+1)^{\frac{1}{2}} + 1}{(h+1)^{\frac{1}{2}} + 1}$$

Using formula (ii)

$$= \lim_{h \to 0} \frac{(h+c)^{\frac{1}{3}} - 1^{\frac{1}{3}}}{h+c-c} \cdot \lim_{h \to 0} \{(h+1)^{\frac{1}{2}} - 1\}$$

$$= \frac{2}{3}$$

Function f(x) is continuous at x=0

$$\lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow b = \frac{2}{3}$$

Now again,

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (1 - ha)^{\frac{1}{-h}}$$

$$= \lim_{h \to 0} \frac{a\log(1 - ah)}{-ah} = a$$

Function f(x) is continuous at x=0

$$\lim_{x \to 0^-} f(x) = f(0)$$

$$\Rightarrow a = \log b$$

Putting value of b

$$a = \frac{\log 2}{3}$$

38. Question

Mark the correct alternative in the following:

The points of discontinuity of the function is (are)

A. $x = 1, x = \frac{5}{2}$

B. $x = \frac{5}{2}$ $f(x) = \begin{cases} 2\sqrt{x} & , \quad 0 \le x \le 1 \\ 4 - 2x & , \quad 1 < x < \frac{5}{2} \\ 2x - 7 & , \quad \frac{5}{2} \le x \le 4 \end{cases}$

C. $x=1, \frac{5}{2}, 4$

D. x = 0, 4

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} 2\sqrt{x}, 0 \le x < 1 \\ 4-2x, 1 < x < \frac{5}{2} \\ 2x-7, \frac{5}{2} < x \le 4 \end{cases}$$

Now at x=1

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0}\left(4-2(1+h)\right) = 2$$

Again,

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0}\left(2(\sqrt{1-h}\right) = 2$$

$$\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$$

Therefore function f(x) is continuous at x=0

Now again

$$x = \frac{5}{2}$$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f\left(\frac{5}{2}+h\right)$$

$$= \lim_{h\to 0}\left(2(\frac{5}{2}+h)-7\right) = \lim_{h\to 0}(2h-5) = -5$$

For left hand limit

$$\lim_{x\to \frac{5}{2}} f(x) = \lim_{h\to 0} f\left(\frac{5}{2}-h\right)$$

$$= \lim_{h\to 0}\left(4-2(\frac{5}{2}-h)\right) = 2$$

$$\lim_{x\to \frac{5}{2}^+} f(x) = \lim_{x\to \frac{5^-}{2}} f(x)$$

Therefore function f(x) is discontinuous at $x = \frac{5}{2}$

39. Question

Mark the correct alternative in the following:

If $.f(x)=\begin{cases}\dfrac{1-\sin^2 x}{3\cos^2 x} & , \quad x<\dfrac{\pi}{2} \\ a & , \quad x=\dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & , \quad x>\dfrac{\pi}{2}\end{cases}$..Then, f(x) is continuous at $x=\dfrac{\pi}{2}$, if

A. $a=\dfrac{1}{3}, b=2$

B. $a=\dfrac{1}{3}, b=\dfrac{8}{3}$

C. $a=\dfrac{2}{3}, b=\dfrac{8}{3}$

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim\limits_{x\to a} f(x)=f(a)$

$\lim\limits_{x\to a^+} f(a+h)=\lim\limits_{x\to a^-} f(a-h)=f(a)$

(ii) Standard limits $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

Given:-

$$f(x)=\begin{cases}\dfrac{1-\sin^2 x}{3\cos^2 x}, x<\dfrac{\pi}{2} \\ a, x=\dfrac{\pi}{2} \\ \dfrac{b(1-\sin^2 x)}{(\pi-2x)^2}\ x>\dfrac{\pi}{2}\end{cases}$$

Now at x=1

$$\lim_{x\to\frac{\pi}{2}^+} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$

$$=\lim_{h\to 0}\left(\frac{b-b\sin^2\left(\frac{\pi}{2}+h\right)}{\pi^2-2\left(\frac{\pi}{2}+h\right)^2}\right)$$

Using standard limit formula

$$=\frac{b}{8}\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)=\frac{b}{8}$$

Again,

$$\lim_{x\to\frac{\pi}{2}} f(x)=\lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$$

$$= \lim_{h \to 0} \left(\frac{b\left(1 - \sin^2\left(\frac{\pi}{2} - h\right)\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)} \right)$$

$$= \frac{1}{3} \lim_{h \to 0} \left(\frac{\sin^2 h}{\sin^2 h} \right)$$

$$= \frac{1}{3}$$

Function f(x) is continuous at x=$\frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^-} f(x) = f(\frac{\pi}{2})$$

$$\Rightarrow \frac{b}{8} = \frac{1}{3} = a$$

$$\Rightarrow a = \frac{1}{3},$$

$$b = \frac{8}{3}$$

40. Question

Mark the correct alternative in the following:

The points of discontinuity of the

function $f(x) = \begin{cases} \frac{1}{5}(2x^2 + 3) & , \quad x \le 1 \\ 6 - 5x & , \quad 1 < x < 3 \\ x - 3 & , \quad x \ge 3 \end{cases}$ is (are)

A. x = 1

B. x = 3

C. x = 1, 3

D. none of these

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a^+} f(a + h) = \lim_{x \to a^-} f(a - h) = f(a)$$

Given:-

$$f(x) = \begin{cases} \frac{1}{5}(2x^2 + 3), x \le 1 \\ 6 - 5x, 1 < x < 3 \\ x - 3, x \ge 3 \end{cases}$$

Now at x=1

$$\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1 + h)$$

$= \lim_{h\to 0}\{6-5(1+h)\} = 1$

For left hand limit

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$

$= \lim_{h\to 0} \frac{1}{5}(2(1-h)^2+3) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$

Therefore continuous at x=1

Now at x=3

$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h)$

$= \lim_{h\to 0}\{(3+h)-3\} = 0$

For left hand limit

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h)$

$= \lim_{h\to 0} \left(6-5(3-h)\right) = -9$

We have,

$\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$

Therefore discontinuous at x=3

41. Question

Mark the correct alternative in the following:

The value of a for which the function $f(x) = \begin{cases} 5x-4 & , \text{if } 0<x\le 1 \\ 4x^2+3ax & , \text{if } 1<x<2 \end{cases}$ is continuous at every point of its domain, is

A. $\frac{13}{3}$

B. 1

C. 0

D. –1

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$

Given:-

$f(x) = \begin{cases} 5x-4\,, 0<x\le 1 \\ 4x^2+3ax\,, 1<x<2 \end{cases}$

Now at x=1

$$\lim_{x\to1^+} f(x) = \lim_{h\to0} f(1+h) = \lim_{h\to0}\{4(1+h)^2 + 3a(1+h)\} = 4 + 3a$$

$$\lim_{x\to1^-} f(x) = \lim_{h\to0} f(1-h)$$

$$= \lim_{h\to0} \{5(1-h) - 4\} = 1$$

Function f(x) continuous at x=1

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x) = f(1)$$

$$\Rightarrow 4 + 3a = 1$$

$$\Rightarrow a = -1$$

Now at x=3

$$\lim_{x\to3^+} f(x) = \lim_{h\to0} f(3+h)$$

$$= \lim_{h\to0}\{(3+h) - 3\} = 0$$

Using left hand limit

$$\lim_{x\to3^-} f(x) = \lim_{h\to0} f(3-h)$$

$$= \lim_{h\to0} \left(6 - 5(3-h)\right) = -9$$

$$\lim_{x\to3^+} f(x) \neq \lim_{x\to3^-} f(x)$$

Therefore discontinuous at x=3

42. Question

Mark the correct alternative in the following:

If $f(x) = \begin{cases} \dfrac{\sin(\cos x) - \cos x}{(\pi - 2x)^2} & , x \neq \dfrac{\pi}{2} \\ k & , x = \dfrac{\pi}{2} \end{cases}$ is

Continuous at $x = \frac{\pi}{2}$, then k is equal to

A. 0

B. $\frac{1}{2}$

C. 1

D.–1

Answer

Formula:-

(i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

(ii) Standard limits $\lim_{x\to0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x\to a}\{f(x).g(x)\} = l.m$, where $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$

Given:-

$$f(x) = \begin{cases} \dfrac{\sin(\cos x) - \cos x}{(\pi - 2x)^2}, x \neq \dfrac{\pi}{2} \\ k, x = \dfrac{\pi}{2} \end{cases}$$

Now at x=$\frac{\pi}{2}$

$$\lim_{x\to\frac{\pi}{2}} f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$$

$$f(x) = \frac{\sin(\cos x) - \cos x}{(\pi - 2x)^2} = k$$

Using substitution method

Let $\frac{\pi}{2} - x = t$

$$\Rightarrow \lim_{t\to 0} \frac{\sin(\cos\left(\frac{\pi}{2} - t\right)) - \cos\left(\frac{\pi}{2} - t\right)}{(2t)^2} = k$$

Using formula (iii)

$$\Rightarrow \frac{1}{4}\lim_{t\to 0}\left(\frac{\sin t}{t} - 1\right).\lim_{t\to 0}\left(\frac{\sin\left(\frac{\sin t - t}{2}\right)}{\frac{\sin t - t}{2}}\right).\lim_{t\to 0}\left(\frac{\cos\left(\frac{\sin t + t}{2}\right)}{t}\right) = k$$

$$\Rightarrow 0 = k$$

Very short answer

1. Question

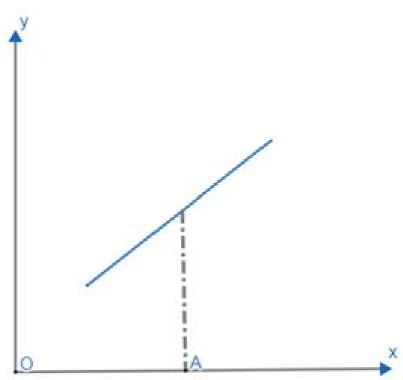
Define continuity of a function at a point.

Answer

A function f(x) is said to be continuous at a point x=a of its domain, iff

$$\lim_{x\to a} f(x) = f(a)$$

If a function is continuous at x=a, then graph of f(x) at the corresponding pint (a,f(a)) will not be broken .



2. Question

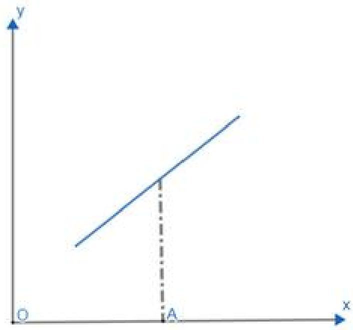
What happens to a function f(x) at x = a, if $\lim_{x\to a} f(x) = f(a)$?

Answer

If f(x) is a function defined in its domain such that

$\lim_{x \to a} f(x) = f(a)$

f(x) become continuous at x=a



3. Question

Find f (0), so that $f(x) = \frac{x}{1-\sqrt{1-x}}$ becomes continuous at x = 0.

Answer

Formula:- (i)If f(x) is continuous at x=0 then, $\lim_{x \to a} f(x) = f(a)$

Given:-

$$f(x) = \frac{x}{1-\sqrt{1-x}}$$

using rationalization method with $1+\sqrt{1-x}$

$$f(x) = \frac{x.(1+\sqrt{1-x})}{(1-\sqrt{1-x}).(1+\sqrt{1-x})}$$

$$\Rightarrow f(x) = \frac{x.(1+\sqrt{1-x})}{1-(1-x)}$$

$$\Rightarrow f(x) = 1+\sqrt{1-x}$$

For function to be continuous at x=0

$$\lim_{x \to 0}(1+\sqrt{1-x}) = f(0)$$

f(0)=2

the function f(x) become continuous at x=0

4. Question

If $f(x) = \begin{cases} \frac{x}{\sin 3x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at x = 0, then write the value of k.

Answer

Formula:- (i)$\lim_{x \to 0} \frac{\sin x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$\lim_{x\to a} f(x) = f(0)$

Given:-$f(x) = \begin{cases} \dfrac{x}{\sin 3x}, \text{if } x \neq 0 \\ k, x = 0 \end{cases}$

$\lim_{x\to a} f(x) = f(0)$

$\lim_{x\to 0} \dfrac{x}{\sin 3x} = k$

Using standard limit

$\lim_{x\to 0} \left(\dfrac{3x}{3\sin 3x}\right) = k$

$\dfrac{1}{3} = k$

5. Question

If the function $f(x) = \dfrac{\sin 10x}{x}$, x ≠ 0 is continuous at x = 0, find f(0).

Answer

Formula:- (i)$\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

Given:- $f(x) = \frac{\sin 10x}{x}$

F(x) function is continuous

$\lim_{x\to a} f(x) = f(0)$

$\Rightarrow \lim_{x\to 0} \dfrac{\sin 10x}{x} = f(0)$

Using standard limit

$\Rightarrow \lim_{x\to 0} 10. \dfrac{\sin 10x}{10x} = f(0)$

f(0)=10

6. Question

If $f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4}, & \text{if } x \neq 4 \\ k, & \text{if } x = 4 \end{cases}$ is continuous at x = 4, find k.

Answer

Formula:- (i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4}, \text{if } x \neq 4 \\ k, x = 4 \end{cases}$

$\lim_{x\to a} f(x) = f(4)$

$\lim_{x\to 4}\left(\frac{x^2-16}{x-4}\right) = k$

Using factorization

$\Rightarrow \lim_{x\to 4}\left(\frac{(x+4)(x-4)}{x-4}\right) = k$

$\Rightarrow \lim_{x\to 4}(x+4) = k$

⇒K=8

7. Question

Determine whether $f(x) = \begin{cases} \frac{\sin x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at x = 0 or not.

Answer

Formula:- (i)$\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

(iii) $\lim_{x\to a}\{f(x).g(x)\} = l.m$, where $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$

$f(x) = \begin{cases} \frac{\sin x^2}{x}, \text{if } x \neq 0 \\ 0, x = 0 \end{cases}$

$\lim_{x\to 0}\frac{x\sin x^2}{x^2} = f(x)$

using formula (ii)

$\lim_{x\to 0}\frac{\sin x^2}{x^2}.\lim_{x\to 0} x = f(x)$

Using standard limit

=0

=f(0)

$\lim_{x\to 0} f(x) = f(0)$

Hence f(x) is continuous at x=0

8. Question

If $f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at x = 0, find k.

Answer

Formula:- (i)$\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, \text{if } x \neq 0 \\ k, x = 0 \end{cases}$$

Now,

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = k$$

$$\Rightarrow \frac{1}{2} \lim_{x \to 0} \frac{\left[\sin\left(\frac{x}{2}\right)\right]^2}{\left(\frac{x}{2}\right)^2} = k$$

$$\Rightarrow \frac{1}{2} = k$$

9. Question

If $f(x) = \begin{cases} \frac{\sin^{-1} x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at x = 0, write the value of k.

Answer

Formula:- standard limit (i)$\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

(ii) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$f(x) = \begin{cases} \frac{\sin^{-1} x}{x}, \text{if } x \neq 0 \\ k, x = 0 \end{cases}$$

Now,

$$\lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin^{-1} x}{x} = f(0)$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin^{-1} x}{x} = k$$

Using standard limit

⇒K=1

10. Question

Write the value of b for which $f(x) = \begin{cases} 5x - 4 & 0 < x \leq 1 \\ 4x^2 + 3bx & 1 < x < 2 \end{cases}$ is continuous at x = 1.

Answer

Formula:- (i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x \to a} f(x) = f(a)$

$$\lim_{x \to a^+} f(a+h) = \lim_{x \to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} 5x - 4, 0 < x \le 1 \\ 4x^2 + 3bx, 1 < x < 2 \end{cases}$$

Function f(x) is continuous at x=1

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to0} f(1-h)$$

$$\Rightarrow \lim_{x\to0} f(1-h) = \lim_{x\to0} 5(1-h) - 4 = 1$$

Again at right hand limit

$$\Rightarrow \lim_{x\to1^+} f(x) = \lim_{x\to0} f(1+h)$$

$$\lim_{x\to0} f(1+h) = \lim_{x\to0} 4(1+h)^2 + 3b(1+h) = 4 + 3b$$

11. Question

Determine the value of constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & x < 0 \\ 3, & x \ge 0 \end{cases}$ is continuous as x = 0.

Answer

Formula:- (i) A function f(x) is said to be continuous at a point x=a of its domain, iff $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to a^+} f(a+h) = \lim_{x\to a^-} f(a-h) = f(a)$$

Given:-

$$f(x) = \begin{cases} \frac{kx}{|x|}, x < 0 \\ 3, x \ge 0 \end{cases}$$

f(x) is continuous at x = 0

$$\lim_{x\to a^-} f(a-h) = f(a)$$

$$\Rightarrow \lim_{x\to0^-} \frac{k(0-h)}{|0-h|} = 3$$

$$\Rightarrow \lim_{h\to0} \frac{k(-h)}{|-h|} = 3$$

$$\Rightarrow \lim_{h\to0} (-k) = 3$$

⇒K=-3

12. Question

Find the value of k for which the function $f(x) = \begin{cases} \frac{x^2 + 3x - 10}{x - 2}, & x \ne 0 \\ k, & x^2 \end{cases}$ is continuous at x = 2.

Answer