Chapter 0
Vectors

Introduction of Vector

Physical quantities having magnitude, direction and obeying laws of vector algebra are called vectors.

Example : Displacement, velocity, acceleration, momentum, force, impulse, weight, thrust, torque, angular momentum, angular velocity *etc.*

If a physical quantity has magnitude and direction both, then it does not always imply that it is a vector. For it to be a vector the third condition of obeying laws of vector algebra has to be satisfied.

Example : The physical quantity current has both magnitude and direction but is still a scalar as it disobeys the laws of vector algebra.

Types of Vector

(1) **Equal vectors :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal when they have equal magnitudes and same direction.

(2) **Parallel vector :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be parallel when

(i) Both have same direction.

(ii) One vector is scalar (positive) non-zero multiple of another vector.

(3) **Anti-parallel vectors :** Two vectors $\vec{A}$ and $\vec{B}$ are said to be anti-parallel when

(i) Both have opposite direction.

(ii) One vector is scalar non-zero negative multiple of another vector.

(4) **Collinear vectors :** When the vectors under consideration can share the same support or have a common support then the considered vectors are collinear.

(5) **Zero vector** $(\vec{0})$**:** A vector having zero magnitude and arbitrary direction (not known to us) is a zero vector.

(6) **Unit vector :** A vector divided by its magnitude is a unit vector. Unit vector for $\vec{A}$ is $\hat{A}$ (read as A cap or A hat).

Since, $\hat{A} = \frac{\vec{A}}{A} \Rightarrow \vec{A} = A\hat{A}$.

Thus, we can say that unit vector gives us the direction.

(7) **Orthogonal unit vectors** $\hat{i}, \hat{j}$ and $\hat{k}$ are called orthogonal unit vectors. These vectors must form a Right Handed Triad (It is a coordinate system such that when we Curl the fingers of right hand from x to y then we must get the direction of z along thumb). The

$$\hat{i} = \frac{\vec{x}}{x}, \hat{j} = \frac{\vec{y}}{y}, \hat{k} = \frac{\vec{z}}{z}$$

$$\therefore \ \vec{x} = x\hat{i}, \ \vec{y} = y\hat{j}, \ \vec{z} = z\hat{k}$$

(8) **Polar vectors :** These have starting point or point of application . Example displacement and force *etc.*

(9) **Axial Vectors :** These represent rotational effects and are always along the axis of rotation in accordance with right hand screw rule. Angular velocity, torque and angular momentum, etc., are example of physical quantities of this type.

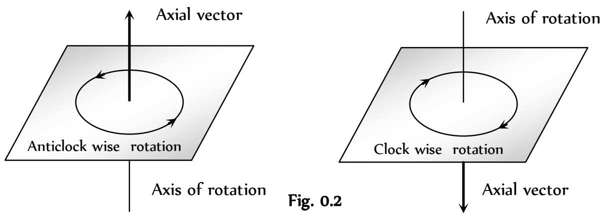

Fig. 0.2

(10) **Coplanar vector** : Three (or more) vectors are called coplanar vector if they lie in the same plane. Two (free) vectors are always coplanar.

Triangle Law of Vector Addition of Two Vectors

If two non zero vectors are represented by the two sides of a triangle taken in same order then the resultant is given by the closing side of triangle in opposite order. *i.e.* $\vec{R} = \vec{A} + \vec{B}$

$$\because \ \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$

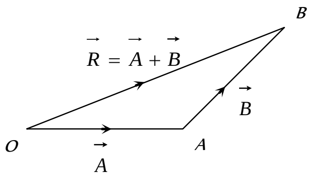

Fig. 0.3

(1) **Magnitude of resultant vector**

In ΔABN, $\cos\theta = \dfrac{AN}{B}$ $\therefore AN = B\cos\theta$

$\sin\theta = \dfrac{BN}{B}$ $\therefore BN = B\sin\theta$

In ΔOBN, we have $OB^2 = ON^2 + BN^2$

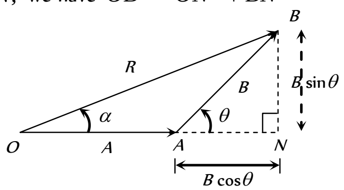

Fig. 0.4

$$\Rightarrow R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$\Rightarrow R^2 = A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\Rightarrow R^2 = A^2 + B^2(\cos^2\theta + \sin^2\theta) + 2AB\cos\theta$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

(2) **Direction of resultant vectors :** If θ is angle between $\vec{A}$ and $\vec{B}$, then

$$|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

If $\vec{R}$ makes an angle α with $\vec{A}$, then in ΔOBN,

$$\tan\alpha = \frac{BN}{ON} = \frac{BN}{OA + AN}$$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

Parallelogram Law of Vector Addition

If two non zero vectors are represented by the two adjacent sides of a parallelogram then the resultant is given by the diagonal of the parallelogram passing through the point of intersection of the two vectors.

(1) **Magnitude**

Since, $R^2 = ON^2 + CN^2$

$$\Rightarrow R^2 = (OA + AN)^2 + CN^2$$

$$\Rightarrow R^2 = A^2 + B^2 + 2AB\cos\theta$$

$$\therefore \quad R = |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

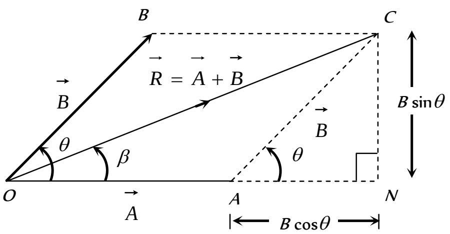

Fig. 0.5

Special cases : $R = A + B$ when $\theta = 0$

$R = A - B$ when $\theta = 180$

$R = \sqrt{A^2 + B^2}$ when $\theta = 90$

(2) **Direction**

$$\tan\beta = \frac{CN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}$$

Polygon Law of Vector Addition

If a number of non zero vectors are represented by the $(n - 1)$ sides of an n-sided polygon then the resultant is given by the closing side or the n side of the polygon taken in opposite order. So,

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$$

$$\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$$

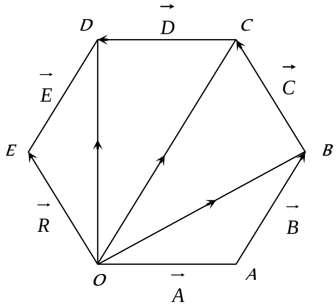

Fig. 0.6

Note : ❑ Resultant of two unequal vectors can not be zero.

❑ Resultant of three co-planar vectors may or may not be zero

❑ Resultant of three non co- planar vectors can not be zero.

Subtraction of vectors

Since, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$ and

$$|\vec{A} + \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

$$\Rightarrow |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 + 2AB\cos(180^o - \theta)}$$

Since, $\cos(180 - \theta) = -\cos\theta$

$$\Rightarrow |\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$$

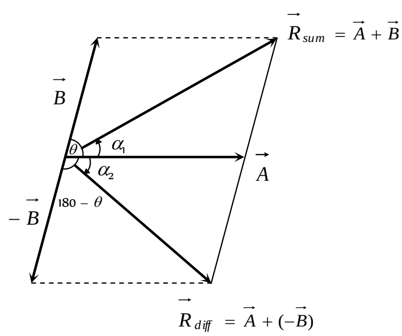

Fig. 0.7

$$\tan\alpha_1 = \frac{B\sin\theta}{A + B\cos\theta}$$

and $\tan\alpha_2 = \dfrac{B\sin(180 - \theta)}{A + B\cos(180 - \theta)}$

But $\sin(180-\theta)=\sin\theta$ and $\cos(180-\theta)=-\cos\theta$

$$\Rightarrow \tan\alpha_2 = \frac{B\sin\theta}{A-B\cos\theta}$$

Resolution of Vector Into Components

Consider a vector $\vec{R}$ in *X*-*Y* plane as shown in fig. If we draw orthogonal vectors $\vec{R}_x$ and $\vec{R}_y$ along *x* and *y* axes respectively, by law of vector addition, $\vec{R}=\vec{R}_x+\vec{R}_y$

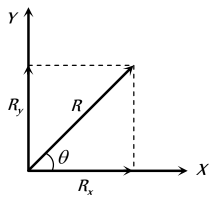

Fig. 0.8

Now as for any vector $\vec{A}=A\hat{n}$ so, $\vec{R}_x=\hat{i}R_x$ and $\vec{R}_y=\hat{j}R_y$

so $\vec{R}=\hat{i}R_x+\hat{j}R_y$ ...(i)

But from figure $R_x = R\cos\theta$ ...(ii)

and $R_y = R\sin\theta$ ...(iii)

Since R and θ are usually known, Equation (ii) and (iii) give the magnitude of the components of $\vec{R}$ along *x* and *y*-axes respectively.

Here it is worthy to note once a vector is resolved into its components, the components themselves can be used to specify the vector as

(1) The magnitude of the vector $\vec{R}$ is obtained by squaring and adding equation (ii) and (iii), *i.e.*

$$R=\sqrt{R_x^2+R_y^2}$$

(2) The direction of the vector $\vec{R}$ is obtained by dividing equation (iii) by (ii), *i.e.*

$$\tan\theta = (R_y/R_x) \text{ or } \theta=\tan^{-1}(R_y/R_x)$$

Rectangular Components of 3-D Vector

$$\vec{R}=\vec{R}_x+\vec{R}_y+\vec{R}_z q \text{ or } \vec{R}=R_x\hat{i}+R_y\hat{j}+R_z\hat{k}$$

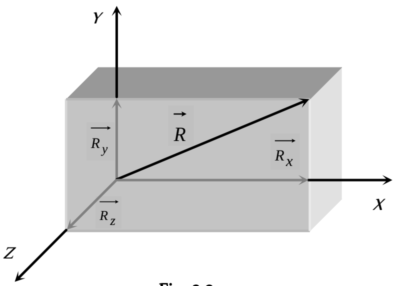

Fig. 0.9

If $\vec{R}$ makes an angle α with *x* axis, β with *y* axis and γ with *z* axis, then

$$\Rightarrow \cos\alpha=\frac{R_x}{R}=\frac{R_x}{\sqrt{R_x^2+R_y^2+R_z^2}}=l$$

$$\Rightarrow \cos\beta=\frac{R_y}{R}=\frac{R_y}{\sqrt{R_x^2+R_y^2+R_z^2}}=m$$

$$\Rightarrow \cos\gamma=\frac{R_z}{R}=\frac{R_z}{\sqrt{R_x^2+R_y^2+R_z^2}}=n$$

Where *l, m, n* are called Direction Cosines of the vector $\vec{R}$ and

$$l^2+m^2+n^2=\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{R_x^2+R_y^2+R_z^2}{R_x^2+R_y^2+R_z^2}=1$$

Note : ❑ When a point *P* have coordinate (*x, y, z*) then its position vector $\overrightarrow{OP}=x\hat{i}+y\hat{j}+z\hat{k}$

❑ When a particle moves from point (*x, y, z*) to (*x, y, z*) then its displacement vector

$$\vec{r}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$

Scalar Product of Two Vectors

(1) **Definition :** The scalar product (or dot product) of two vectors is defined as the product of the magnitude of two vectors with cosine of angle between them.

Thus if there are two vectors $\vec{A}$ and $\vec{B}$ having angle θ between them, then their scalar product written as $\vec{A}.\vec{B}$ is defined as $\vec{A}.\vec{B}$ $=AB\cos\theta$

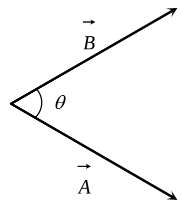

Fig. 0.10

(2) **Properties :** (i) It is always a scalar which is positive if angle between the vectors is acute (*i.e.*, < 90°) and negative if angle between them is obtuse (*i.e.* 90°<θ < 180°).

(ii) It is commutative, *i.e.* $\vec{A}.\vec{B}=\vec{B}.\vec{A}$

(iii) It is distributive, *i.e.* $\vec{A}.(\vec{B}+\vec{C})=\vec{A}.\vec{B}+\vec{A}.\vec{C}$

(iv) As by definition $\vec{A}.\vec{B}=AB\cos\theta$

The angle between the vectors $\theta=\cos^{-1}\left[\frac{\vec{A}.\vec{B}}{AB}\right]$

(v) Scalar product of two vectors will be maximum when $\cos\theta=\max=1$, *i.e.* $\theta=0^o$, *i.e.*, vectors are parallel

$$(\vec{A}.\vec{B})_{\max}=AB$$

(vi) Scalar product of two vectors will be minimum when $|\cos\theta|=\min=0$, *i.e.* $\theta=90^o$

$$(\vec{A}.\vec{B})_{\min}=0$$

i.e. if the scalar product of two nonzero vectors vanishes the vectors are orthogonal.

(vii) The scalar product of a vector by itself is termed as self dot product and is given by $(\vec{A})^2=\vec{A}.\vec{A}=AA\cos\theta=A^2$

i.e. $A = \sqrt{\vec{A}.\vec{A}}$

(viii) In case of unit vector $\hat{n}$

$\hat{n}.\hat{n} = 1 \times 1 \times \cos 0 = 1$ so $\hat{n}.\hat{n} = \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

(ix) In case of orthogonal unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 1 \times 1 \cos 90° = 0$

(x) In terms of components

$$\vec{A}.\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z).(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$$

(3) **Example :** (i) Work W : In physics for constant force work is defined as, $W = Fs\cos\theta$ ...(i)

But by definition of scalar product of two vectors, $\vec{F}.\vec{s} = Fs\cos\theta$ ...(ii)

So from *eq* (i) and (ii) $W = \vec{F}.\vec{s}$ *i.e.* work is the scalar product of force with displacement.

(ii) Power P :

As $W = \vec{F}.\vec{s}$ or $\frac{dW}{dt} = \vec{F}.\frac{d\vec{s}}{dt}$ [As $\vec{F}$ is constant]

or $P = \vec{F}.\vec{v}$ *i.e.*, power is the scalar product of force with velocity. $\left[\text{As} \frac{dW}{dt} = P \text{ and } \frac{d\vec{s}}{dt} = \vec{v}\right]$

(iii) Magnetic Flux ϕ :

Magnetic flux through an area is given by $d\phi = B\,ds\cos\theta$ ...(i)

But by definition of scalar product $\vec{B}.d\vec{s} = Bds\cos\theta$ ...(ii)

So from *eq* (i) and (ii) we have

$$d\phi = \vec{B}.d\vec{s} \quad \text{or} \quad \phi = \int \vec{B}.d\vec{s}$$

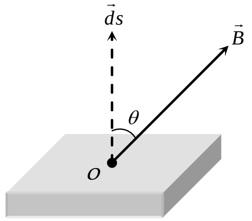

Fig. 0.11

(iv) Potential energy of a dipole U : If an electric dipole of moment $\vec{p}$ is situated in an electric field $\vec{E}$ or a magnetic dipole of moment $\vec{M}$ in a field of induction $\vec{B}$, the potential energy of the dipole is given by :

$$U_E = -\vec{p}.\vec{E} \text{ and } U_B = -\vec{M}.\vec{B}$$

Vector Product of Two Vectors

(1) **Definition** : The vector product or cross product of two vectors is defined as a vector having a magnitude equal to the product of the magnitudes of two vectors with the sine of angle between them, and direction perpendicular to the plane containing the two vectors in accordance with right hand screw rule.

$$\vec{C} = \vec{A} \times \vec{B}$$

Thus, if $\vec{A}$ and $\vec{B}$ are two vectors, then their vector product written as $\vec{A} \times \vec{B}$ is a vector $\vec{C}$ defined by

$$\vec{C} = \vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$$

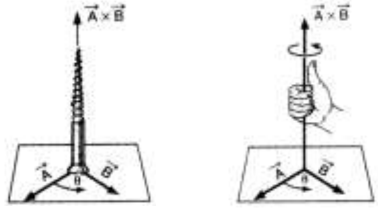

Fig. 0.12

The direction of $\vec{A} \times \vec{B}$, *i.e.* $\vec{C}$ is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and in the sense of advance of a right handed screw rotated from $\vec{A}$ (first vector) to $\vec{B}$ (second vector) through the smaller angle between them. Thus, if a right handed screw whose axis is perpendicular to the plane framed by $\vec{A}$ and $\vec{B}$ is rotated from $\vec{A}$ to $\vec{B}$ through the smaller angle between them, then the direction of advancement of the screw gives the direction of $\vec{A} \times \vec{B}$ *i.e.* $\vec{C}$

(2) **Properties**

(i) Vector product of any two vectors is always a vector perpendicular to the plane containing these two vectors, *i.e.*, orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

(ii) Vector product of two vectors is not commutative, *i.e.*, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ [but $= -\vec{B} \times \vec{A}$]

Here it is worthy to note that

$|\vec{A} \times \vec{B}| = |\vec{B} \times \vec{A}| = AB\sin\theta$

i.e. in case of vector $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ magnitudes are equal but directions are opposite.

(iii) The vector product is distributive when the order of the vectors is strictly maintained, *i.e.*

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iv) The vector product of two vectors will be maximum when $\sin\theta = \max = 1$, *i.e.*, $\theta = 90^o$

$[\vec{A} \times \vec{B}]_{\max} = AB\,\hat{n}$

i.e. vector product is maximum if the vectors are orthogonal.

(v) The vector product of two non- zero vectors will be minimum when $|\sin\theta| =$ minimum = 0, *i.e.*, $\theta = 0^o$ or 180^o

$[\vec{A} \times \vec{B}]_{\min} = 0$

i.e. if the vector product of two non-zero vectors vanishes, the vectors are collinear.

(vi) The self cross product, *i.e.*, product of a vector by itself vanishes, *i.e.*, is null vector $\vec{A} \times \vec{A} = AA\sin 0^o\,\hat{n} = \vec{0}$

(vii) In case of unit vector $\hat{n} \times \hat{n} = \vec{0}$ so that $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

(viii) In case of orthogonal unit vectors, $\hat{i}, \hat{j}, \hat{k}$ in accordance with right hand screw rule :

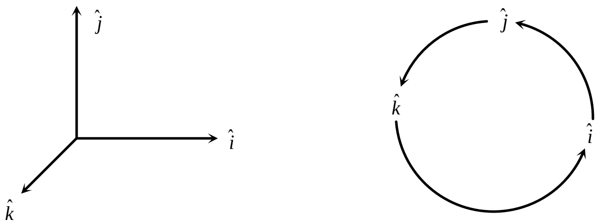

Fig. 0.13

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$

And as cross product is not commutative,

$\hat{j} \times \hat{i} = -\hat{k}$, $\hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = -\hat{j}$

(x) In terms of components

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}(A_y B_z - A_z B_y) + \hat{j}(A_z B_x - A_x B_z) + \hat{k}(A_x B_y - A_y B_x)$$

(3) **Example :** Since vector product of two vectors is a vector, vector physical quantities (particularly representing rotational effects) like torque, angular momentum, velocity and force on a moving charge in a magnetic field and can be expressed as the vector product of two vectors. It is well – established in physics that :

(i) Torque $\vec{\tau} = \vec{r} \times \vec{F}$

(ii) Angular momentum $\vec{L} = \vec{r} \times \vec{p}$

(iii) Velocity $\vec{v} = \vec{\omega} \times \vec{r}$

(iv) Force on a charged particle q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by $\vec{F} = q(\vec{v} \times \vec{B})$

(v) Torque on a dipole in a field $\vec{\tau}_E = \vec{p} \times \vec{E}$ and $\vec{\tau}_B = \vec{M} \times \vec{B}$

Lami's Theorem

In any ΔABC with sides $\vec{a}, \vec{b}, \vec{c}$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

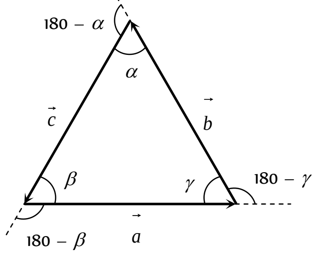

Fig. 0.14

i.e. for any triangle the ratio of the sine of the angle containing the side to the length of the side is a constant.

For a triangle whose three sides are in the same order we establish the Lami's theorem in the following manner. For the triangle shown

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$ [All three sides are taken in order] ...(i)

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$ ...(ii)

Pre-multiplying both sides by $\vec{a}$

$\vec{a} \times (\vec{a} + \vec{b}) = -\vec{a} \times \vec{c} \Rightarrow \vec{0} + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$

$\Rightarrow \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(iii)

Pre-multiplying both sides of (ii) by $\vec{b}$

$\vec{b} \times (\vec{a} + \vec{b}) = -\vec{b} \times \vec{c} \Rightarrow \vec{b} \times \vec{a} + \vec{b} \times \vec{b} = -\vec{b} \times \vec{c}$

$\Rightarrow -\vec{a} \times \vec{b} = -\vec{b} \times \vec{c} \Rightarrow \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ...(iv)

From (iii) and (iv), we get $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Taking magnitude, we get $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

$\Rightarrow ab\sin(180 - \gamma) = bc\sin(180 - \alpha) = ca\sin(180 - \beta)$

$\Rightarrow ab\sin\gamma = bc\sin\alpha = ca\sin\beta$

Dividing through out by *abc*, we have

$$\Rightarrow \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Relative Velocity

(1) **Introduction :** When we consider the motion of a particle, we assume a fixed point relative to which the given particle is in motion. For example, if we say that water is flowing or wind is blowing or a person is running with a speed *v*, we mean that these all are relative to the earth (which we have assumed to be fixed).

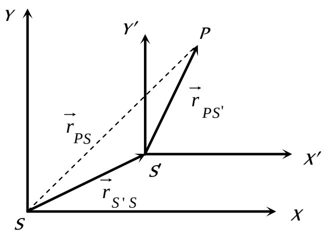

Fig. 0.15

Now to find the velocity of a moving object relative to another moving object, consider a particle *P* whose position relative to frame *S* is $\vec{r}_{PS}$ while relative to S' is $\vec{r}_{PS'}$.

If the position of frames S' relative to *S* at any time is $\vec{r}_{S'S}$ then from figure, $\vec{r}_{PS} = \vec{r}_{PS'} + \vec{r}_{S'S}$

Differentiating this equation with respect to time

$$\frac{d\vec{r}_{PS}}{dt} = \frac{d\vec{r}_{PS'}}{dt} + \frac{d\vec{r}_{S'S}}{dt}$$

or $\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}$ [as $\vec{v} = d\vec{r}/dt$]

or $\vec{v}_{PS'} = \vec{v}_{PS} - \vec{v}_{S'S}$

(2) **General Formula :** The relative velocity of a particle P_1 moving with velocity $\vec{v}_1$ with respect to another particle P_2 moving with velocity $\vec{v}_2$ is given by, $\vec{v}_{r_{12}} = \vec{v}_1 - \vec{v}_2$

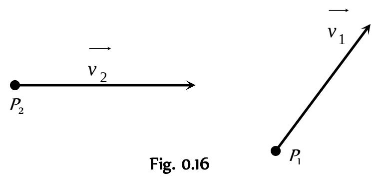

Fig. 0.16

(i) If both the particles are moving in the same direction then :

$$\upsilon_{r_{12}} = \upsilon_1 - \upsilon_2$$

(ii) If the two particles are moving in the opposite direction, then :

$$\upsilon_{r_{12}} = \upsilon_1 + \upsilon_2$$

(iii) If the two particles are moving in the mutually perpendicular directions, then:

$$\upsilon_{r_{12}} = \sqrt{\upsilon_1^2 + \upsilon_2^2}$$

(iv) If the angle between $\vec{\upsilon}_1$ and $\vec{\upsilon}_2$ be θ, then $\upsilon_{r_{12}} = \left[\upsilon_1^2 + \upsilon_2^2 - 2\upsilon_1\upsilon_2\cos\theta\right]^{1/2}$.

(3) **Relative velocity of satellite :** If a satellite is moving in equatorial plane with velocity $\vec{v}_s$ and a point on the surface of earth with $\vec{v}_e$ relative to the centre of earth, the velocity of satellite relative to the surface of earth

$$\vec{v}_{se} = \vec{v}_s - \vec{v}_e$$

So if the satellite moves form west to east (in the direction of rotation of earth on its axis) its velocity relative to earth's surface will be $v_{se} = v_s - v_e$

And if the satellite moves from east to west, *i.e.,* opposite to the motion of earth, $v_{se} = v_s - (-v_e) = v_s + v_e$

(4) **Relative velocity of rain :** If rain is falling vertically with a velocity $\vec{v}_R$ and an observer is moving horizontally with speed $\vec{v}_M$ the velocity of rain relative to observer will be $\vec{v}_{RM} = \vec{v}_R - \vec{v}_M$

which by law of vector addition has magnitude

$$v_{RM} = \sqrt{v_R^2 + v_M^2}$$

direction $\theta = \tan^{-1}(v_M / v_R)$ with the vertical as shown in fig.

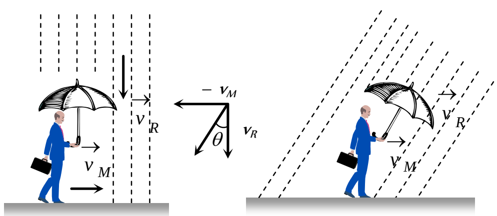

Fig. 0.17

(5) **Relative velocity of swimmer :** If a man can swim relative to water with velocity $\vec{v}$ and water is flowing relative to ground with velocity $\vec{v}_R$ velocity of man relative to ground $\vec{v}_M$ will be given by:

$$\vec{v} = \vec{v}_M - \vec{v}_R \text{ , i.e., } \vec{v}_M = \vec{v} + \vec{v}_R$$

So if the swimming is in the direction of flow of water, $v_M = v + v_R$

And if the swimming is opposite to the flow of water, $v_M = v - v_R$

(6) **Crossing the river :** Suppose, the river is flowing with velocity $\vec{\upsilon}_r$. A man can swim in still water with velocity $\vec{\upsilon}_m$. He is standing on one bank of the river and wants to cross the river, two cases arise.

(i) To cross the river over shortest distance : That is to cross the river straight, the man should swim making angle θ with the upstream as shown.

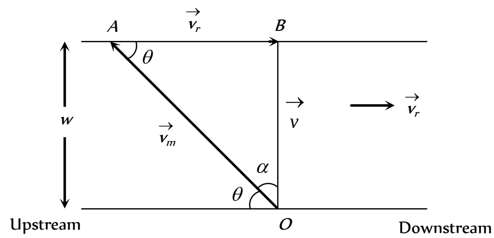

Fig. 0.18

Here OAB is the triangle of vectors, in which $\vec{OA} = \vec{v}_m, \vec{AB} = \vec{\upsilon}_r$.

Their resultant is given by $\vec{OB} = \vec{\upsilon}$. The direction of swimming makes angle θ with upstream. From the triangle OBA, we find,

$$\cos\theta = \frac{\upsilon_r}{\upsilon_m} \quad \text{Also } \sin\alpha = \frac{\upsilon_r}{\upsilon_m}$$

Where α is the angle made by the direction of swimming with the shortest distance (OB) across the river.

Time taken to cross the river : If w be the width of the river, then time taken to cross the river will be given by

$$t_1 = \frac{w}{\upsilon} = \frac{w}{\sqrt{\upsilon_m^2 - \upsilon_r^2}}$$

(ii) To cross the river in shortest possible time : The man should swim perpendicular to the bank.

The time taken to cross the river will be:

$$t_2 = \frac{w}{\upsilon_m}$$

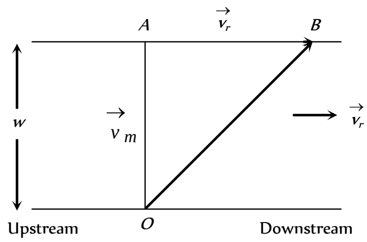

Fig. 0.19

In this case, the man will touch the opposite bank at a distance AB down stream. This distance will be given by:

$$AB = \upsilon_r t_2 = \upsilon_r \frac{w}{\upsilon_m} \quad \text{or} \quad AB = \frac{\upsilon_r}{\upsilon_m} w$$

Tips & Tricks

- All physical quantities having direction are not vectors. For example, the electric current possesses direction but it is a scalar quantity because it can not be added or multiplied according to the rules of vector algebra.
- A vector can have only two rectangular components in plane and only three rectangular components in space.
- A vector can have any number, even infinite components. (minimum 2 components)
- Following quantities are neither vectors nor scalars : Relative density, density, viscosity, frequency, pressure, stress, strain, modulus of elasticity, poisson's ratio, moment of inertia, specific heat, latent heat, spring constant loudness, resistance, conductance, reactance, impedance, permittivity, dielectric constant, permeability, susceptibility, refractive index, focal length, power of lens, Boltzman constant, Stefan's constant, Gas constant, Gravitational constant, Rydberg constant, Planck's constant etc.
- Distance covered is a scalar quantity.
- The displacement is a vector quantity.
- Scalars are added, subtracted or divided algebraically.
- Vectors are added and subtracted geometrically.
- Division of vectors is not allowed as directions cannot be divided.
- Unit vector gives the direction of vector.
- Magnitude of unit vector is 1.
- Unit vector has no unit. For example, velocity of an object is 5 ms^{-1} due East.

i.e. $\vec{v} = 5ms^{-1}$ due east.

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{5ms^{-1}(\text{East})}{5ms^{-1}} = \text{East}$$

So unit vector $\hat{v}$ has no unit as East is not a physical quantity.

- Unit vector has no dimensions.
- $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$
- $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$
- $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$
- $\vec{A}\times\vec{A} = \vec{0}$. Also $\vec{A}-\vec{A} = \vec{0}$ But $\vec{A}\times\vec{A} \neq \vec{A}-\vec{A}$

Because $\vec{A}\times\vec{A} \perp \vec{A}$ and $\vec{A}-\vec{A}$ is collinear with $\vec{A}$

- Multiplication of a vector with –1 reverses its direction.

If $\vec{A} = \vec{B}$, then $A = B$ and $\hat{A} = \hat{B}$.

If $\vec{A} + \vec{B} = \vec{0}$, then $A = B$ but $\hat{A} = -\hat{B}$.

- Minimum number of collinear vectors whose resultant can be zero is two.
- Minimum number of coplaner vectors whose resultant is zero is three.
- Minimum number of non coplaner vectors whose resultant is zero is four.
- Two vectors are perpendicular to each other if $\vec{A}.\vec{B} = 0$.
- Two vectors are parallel to each other if $\vec{A}\times\vec{B} = 0$.
- Displacement, velocity, linear momentum and force are polar vectors.
- Angular velocity, angular acceleration, torque and angular momentum are axial vectors.
- Division with a vector is not defined because it is not possible to divide with a direction.
- Distance covered is always positive quantity.
- The components of a vectors can have magnitude than that of the vector itself.
- The rectangular components cannot have magnitude greater than that of the vector itself.
- When we multiply a vector with 0 the product becomes a null vector.
- The resultant of two vectors of unequal magnitude can never be a null vector.
- Three vectors not lying in a plane can never add up to give a null vector.
- A quantity having magnitude and direction is not necessarily a vector. For example, time and electric current. These quantities have magnitude and direction but they are scalar. This is because they do not obey the laws of vector addition.
- A physical quantity which has different values in different directions is called a tensor. For example : Moment of inertia has different values in different directions. Hence moment of inertia is a tensor. Other examples of tensor are refractive index, stress, strain, density etc.
- The magnitude of rectangular components of a vector is always less than the magnitude of the vector
- If $\vec{A} = \vec{B}$, then $A_x = B_x$, $A_y = B_y$ and $A_z = B_z$.
- If $\vec{A} + \vec{B} = \vec{C}$. Or if $\vec{A} + \vec{B} + \vec{C} = \vec{0}$, then $\vec{A}$, $\vec{B}$ and $\vec{C}$ lie in one plane.
- If $\vec{A}\times\vec{B} = \vec{C}$, then $\vec{C}$ is perpendicular to $\vec{A}$ as well as $\vec{B}$.
- If $|\vec{A}\times\vec{B}| = |\vec{A}-\vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ is 90°.
- Resultant of two vectors will be maximum when θ = 0° *i.e.* vectors

are parallel.

$$R_{max} = \sqrt{P^2 + Q^2 + 2PQ\cos 0^\circ} = |P + Q|$$

✍ Resultant of two vectors will be minimum when θ = 180° *i.e.* vectors are anti-parallel.

$$R_{min} = \sqrt{P^2 + Q^2 + 2PQ\cos 180^\circ} = |P - Q|$$

Thus, minimum value of the resultant of two vectors is equal to the difference of their magnitude.

✍ Thus, maximum value of the resultant of two vectors is equal to the sum of their magnitude.

When the magnitudes of two vectors are unequal, then

$$R_{min} = P - Q \neq 0$$

$$[\because |\vec{P}| \neq |\vec{Q}|]$$

Thus, two vectors $\vec{P}$ and $\vec{Q}$ having different magnitudes can never be combined to give zero resultant. From here, we conclude that the minimum number of vectors of unequal magnitude whose resultant can be zero is three. On the other hand, the minimum number of vectors of equal magnitude whose resultant can be zero is two.

✍ Angle between two vectors $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}$$

✍ Projection of a vector $\vec{A}$ in the direction of vector $\vec{B}$

$$= \frac{\vec{A}.\vec{B}}{|\vec{B}|}$$

✍ Projection of a vector $\vec{B}$ in the direction of vector $\vec{A}$

$$= \frac{\vec{A}.\vec{B}}{|\vec{A}|}$$

✍ If vectors $\vec{A}, \vec{B}$ and $\vec{C}$ are represented by three sides *ab, bc* and *ca* respectively taken in a order, then

$$\frac{|\vec{A}|}{ab} = \frac{|\vec{B}|}{bc} = \frac{|\vec{C}|}{ca}$$

✍ The vectors $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the coordinate axes at an angle of 54.74 degrees.

✍ If $\vec{A} \pm \vec{B} = \vec{C}$, then $\vec{A}.\vec{B} \times \vec{C} = 0$.

✍ If $\vec{A}.\vec{B} \times \vec{C} = 0$, then $\vec{A}.\vec{B}$ and $\vec{C}$ are coplanar.

✍ If angle between $\vec{A}$ and $\vec{B}$ is 45°,

then $\vec{A}.\vec{B} = |\vec{A} \times \vec{B}|$

✍ If $\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + ... + \vec{A}_n = \vec{0}$ and $A_1 = A_2 = A_3 = ... = A_n$ then the adjacent vector are inclined to each other at angle $2\pi / n$.

✍ If $\vec{A} + \vec{B} = \vec{C}$ and $A^2 + B^2 = C^2$, then the angle between $\vec{A}$ and $\vec{B}$ is 90°. Also *A, B* and *C* can have the following values.

(i) $A = 3$, $B = 4$, $C = 5$

(ii) $A = 5$, $B = 12$, $C = 13$

(iii) $A = 8$, $B = 15$, $C = 17$.