The Concise Handbook of Useful Concepts, Strategies, Algorithms and Formulae for Secondary Mathematics Students

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by John D. Butler, Ph.D., PE
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Introduction

This concise math primer is designed as a desktop reference for the high school mathematics student. The five mathematical content strands: (1) Number sense, properties, and operations; (2) Measurement; (3) Geometry and spatial sense; (4) Data analysis, statistics, and probability; and, (5) Algebra and functions (NCES: National Center for Educational Statistics), serve as the framework for this primer.

The depth and breadth of high school mathematics content, coupled with the number of students that enter high school with significant skill gaps in mathematics, precludes covering all areas of need in great depth. However, this primer contains explanations and example problems of several key topics in each of the five strands. A comprehensive collection of basic math facts that every high school student should be familiar with is included.

The primary goals for this primer are:

- To support student success in mathematic by helping to close skill gaps;
- To reinforce knowledge by review of basic math concepts and facts;
- To provide useful information for both students and teachers across all grades and content areas where math may be integrated;
- To provide a resource for preparation for state math assessments

John D. Butler, Ph.D., PE
Director of Teaching, Learning, and Research
Village Green Charter School

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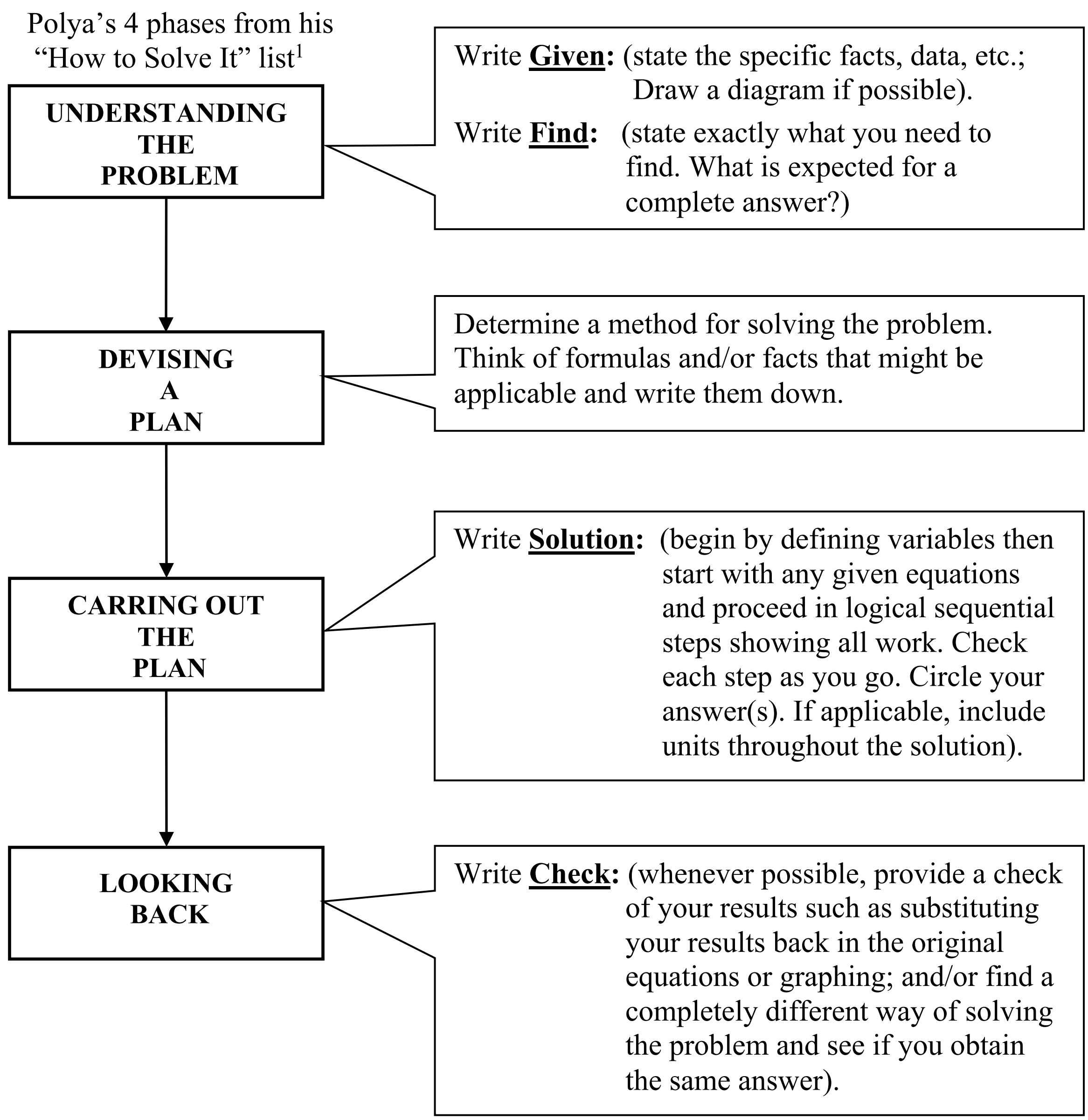
PROBLEM SOLVING: A Few Words about Problem Solving

The primary use of mathematics is to solve problems! But, what is a *problem*?

A **problem** is a task that involves a situation in which there is a goal to be reached but no direct route to the goal. This primer begins with a framework for approaching all problems, guidelines for submitted work, along with strategies for solving problems and taking tests.

Given, Find, Solution, Check: The problem-solving framework

(Adapted from Polya's "How to Solve It" List)



[1]Polya, G. (2004). *How to solve it*. Princeton, NJ: Princeton University Press. (Original work published 1945)

Considerations and Expectations for Hand-Written Mathematical Communication

1. Always do neat work – present your best work - consider that each page is a piece of art...
2. Unless otherwise specified, always use standard 8.5 x 11 inch paper, lined or plain.
3. Problem solving should be done with pencil to facilitate erasing if needed.
4. Crossed-out work should be avoided; but if necessary, neatly draw a single line through the pertinent work. Use scratch paper to perform manual calculations.
5. The first page of work should always have the full name of the author – neatly printed (first and last name) along with the date at the top right. If the problem(s) are from a text, state the source (the name of the book), page number, and problem numbers. All additional pages should be initialed and dated. All pages should be numbered.
6. Use extra pages if necessary – do not restrict yourself to a confined space unless directed to by specific instructions. Write the words "continued on back" or "continued on next page."
7. Don't write near the margins. Try to stay away from writing at least 1 inch from all four edges (preferably at least 1.25 inches from the left margin for hole punching and binding purposes).
8. When tearing out pages from a spiral notebook or similar, avoid ragged edges – neatly remove excess material with scissors or carefully tear along perforated lines if provided. Do not submit work that has torn corners or ripped holes in the case of three-hole punched paper. And, avoid submitting work with coffee stains or work that your pet urinated on (both of these have happened in the past).
9. When stapling pages with holes – make sure the holes line up (for multiple pages, insert a pencil through the holes to help in lining up the pages before stapling).
10. Show all work – every line should be mathematically valid (i.e., true statements).
11. Print your work – do not use cursive.
12. When working with general formulas, always begin your mathematical work by stating variable assignments (refer to "16"), state the general formula used, and then substitute for variables when appropriate. For example if you need to calculate the perimeter of a rectangle with a length of 4 inches and a width of 3 inches, write:

Let P = perimeter, L = length, W = width

P = 2 L + 2W

P = 2(4) + 2(3)

P = 14 inches

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13. When sketching diagrams, figures or making tables, use a ruler or similar straight-edged instrument if available.

14. When using graph paper, your constructed horizontal and vertical lines should overlay the pre-drawn grid lines of the graph paper (i.e., don't draw lines between the provided guide lines).

15. Use units throughout your work; make sure you are using consistent units throughout your work. For example, if the problem statement uses units in meters for some information and units in centimeters for other information, select one or the other to use, make the appropriate conversion and carry the converted units to the end. Most importantly, make sure your answer includes the appropriate units.

16. It is often wise to use abbreviations in math work such as when assigning variables. For example: Let *C = the total cost of attending college for four years,* Let *B = the cost of books,* let *M = the cost of meals,* and let *L = the cost of lab fees*. Make sure that you provide a key to your abbreviations, variables, etc. somewhere (preferably at the beginning) of your solution section. It is common practice to use the first letter of a key word as your variable letter to aid in remembering what the variable represents. In our example above, *C* was representative of cost, *B* for books, *M* for meals, and *L* for lab.

17. In problem solving, use Polya's four stage method for solving problems: *Understand the problem*, *Develop or create a plan of action*, *Carry out the plan*, and Check*/reflect upon your work and answers*. In your written work, write the words: *Given*, *Find*, *Solution* and *Check* (when feasible).

Some Problem-Solving Strategies

The most important strategy toward becoming an expert problem solver is <u>doing lots of problems</u>!

1. Make use of a scratch pad or scratch paper as you solve problems – this is an important tool!
2. Read the problem statement a few times very carefully. Jot down important facts on your scratch paper as you read. Do you see a strange word in the problem statement? Ask if you can use a dictionary, text book, internet or other resources for researching the meaning.
3. Before doing any calculations, try to make a reasonable estimate of what the answer may be.
4. As you problem-solve, keep in mind Polya's four stage framework for solving problems: *Understanding the Problem*, *Devising a Plan*, *Carrying Out the Plan*, and Looking Back (i.e., check/reflecting upon your work and answers). In your written work, write the words: <u>*Given*</u>, <u>*Find*</u>, <u>*Solution*</u> and <u>*Check*</u> (when feasible).
5. Show all work – every line should be mathematically valid (i.e., true statements).
6. Explain your decisions made in writing as you go (i.e., metacognition). For example: "I will subtract $2x$ from both sides."
7. Some problems are hard to picture in your mind. Try to draw a picture, chart, or table. Sometimes it's helpful to construct a model. If available and permissible, use rulers, protractors, compasses, scissors, paper, cardboard, glue sticks, markers, tape, etc. to help model the situation.
8. It is often wise to use abbreviations in math work such as when assigning variables. For example: Let C = *the total cost of attending college for four years,* Let B = *the cost of books,* let M = *the cost of meals,* and let L = *the cost of lab fees*. Make sure that you provide a key to your abbreviations, variables, etc. somewhere (preferably at the beginning) of your solution section. It is common practice to use the first letter of a key word as your variable letter to aid in remembering what the variable represents. In our example above, C was representative of cost, B for books, M for meals, and L for lab.
9. If you get stuck on a problem and have exhausted your thought process – ask questions – you may get a valuable hint!
10. Use units throughout your work. If the problem statement uses units in meters for some information and units in centimeters for other information, select one or the other to use, make the appropriate conversion and carry the converted units to the end. Most importantly, make sure your answer includes the appropriate units.
11. Make sure you have answered exactly what was asked and that your answer is clearly identified such as circling the answer. If the assignment involves boxes, lines, or bubbles for placing your answer – make sure you use them.

Some Strategies for Paper & Pencil Tests

1. Get to bed early the night before and have a good breakfast in the morning of the test.

2. Get to the testing room early – don't be late.

3. If you have a cell phone - make sure you have turned it fully off and it is invisible.

4. If given a choice for seating, choose an area of the room where you will feel comfortable.

5. Be relaxed, alert, and confident! Maintain a positive attitude. If you get nervous – take a few deep breaths to relax.

6. Bring extra sharpened pencils with erasers (they may not be provided).

7. Make sure you understand what resources will be allowed (e.g., calculators, formula sheets, texts, notes, cheat sheets, dictionaries, internet) and take advantage of anything allowed.

8. Ask for scratch paper and use it!

9. When you are able to first open your test booklet, quickly thumb through it to get an idea as to its length – this may help you pace yourself.

10. Bring a watch or look for a clock in the room. Pace yourself. Do not spend too much time on any one test item.

11. Do not talk to anyone and keep your eyes on your own work as to avoid any appearance of cheating.

12. First, do the test items that you feel are easy – these are probably your best chance of getting correct answers and the best way to build some confidence early on. Skip items you don't know and come back to these later. There may be information in other items that will help you on the ones you skipped.

13. Read the problem a few times before devising a plan (i.e., attempting to solve). Note what is given and what you need to find. Is there more than one part to be answered (i.e., (a), (b), (c))?

14. As you devise a plan toward solution, think of formulas that may apply and jot them down. Draw figures or make tables. Organize data.

15. Make sure you answer the item with exactly what is being asked.

Continued next page

16. Make sure you follow the instructions for items that require filling in bubbles or writing within "only the space provided."

17. Don't worry if others finish before you – that doesn't mean they got all correct answers!

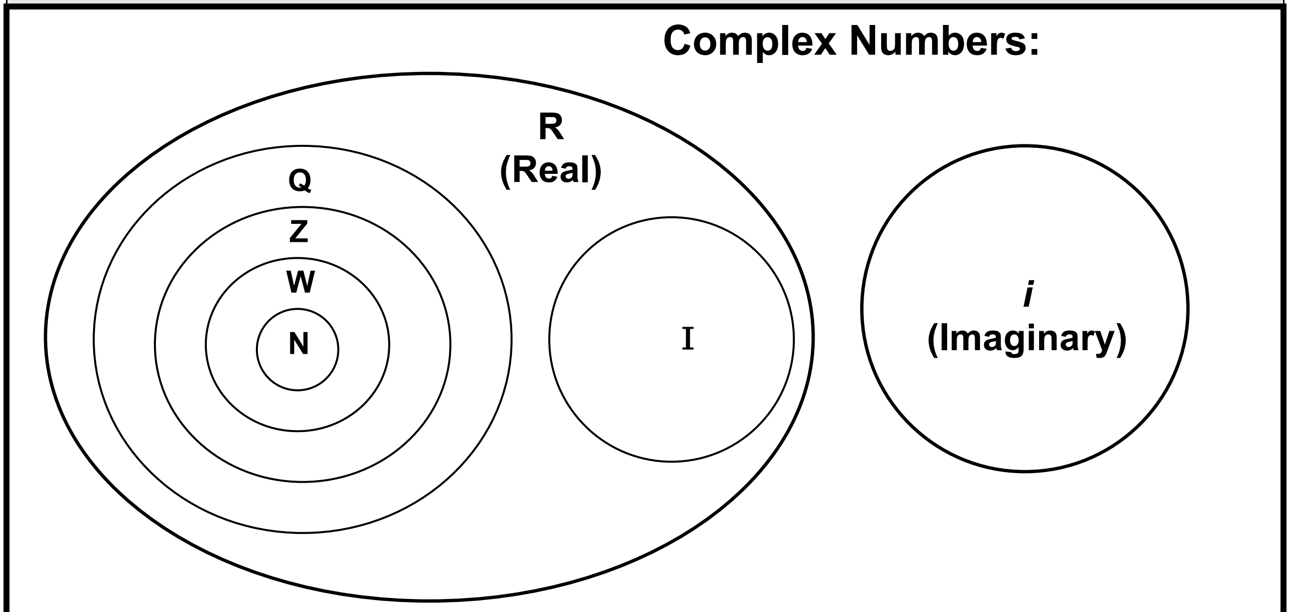
18. With multiple choice items, if your calculated answer does not match one of the possible choices – there are two possibilities: (1) <u>your work is correct **but**</u> your answer is in a different form from the given choices – perhaps you have not simplified or reduced your answer; or, your equation or algebraic expression is arranged differently - make sure you convert improper fractions to mixed numbers and simplify radicals; your answer is a decimal, but the choices are in fractional form – you need to change the decimal to a fraction; (2) <u>your work is incorrect</u> – go back and check.

19. Don't leave items blank always make an attempt.

20. Always write legibly – graders are more apt to mark your work incorrect if they can't follow it.

21. If you finish early, go back and double-check your answers.

A Venn Diagram of Our Complex Number System



N = Natural Numbers { 1, 2, 3, 4, 5, … }
W = Whole Numbers { 0, 1, 2, 3, 4, 5, … }
Z = Integers { …- 5, - 4, - 3, - 2, - 1, 0, 1, 2, 3, 4, 5, … }
Q = Rational Numbers { all numbers that can be written as a ratio in the form: $\frac{a}{b}$ where, *a* and *b* are Integers and, *b* ≠ 0 (*b* not equal to zero) }

Examples: $\frac{1}{2}$, $-\frac{1}{4}$, $\frac{43}{87}$, $\frac{100}{1}$

I = Irrational Numbers { all numbers that <u>cannot</u> be written as a ratio of two integers in the form of $\frac{a}{b}$ or written as a precise decimal }

Examples: **π = 3.141592654…,**
$\sqrt{2}$ **= 1.414213562…,**
$\sqrt{3}$ **= 1.732050808…,**
$\sqrt{5}$ **= 2.236067978…,**

Note: Irrational numbers don't repeat or terminate

R = Real Numbers { Natural Numbers + Whole Numbers + Integers + Rational Numbers + Irrational Numbers }

***i* = Imaginary Numbers** { square roots of negative numbers } based on $i^2 = -1$. where *i* is defined as ***i* =** $\sqrt{-1}$; **Example: 5*i* =** $\sqrt{-25}$

Complex Numbers { Real + Imaginary }

Math Facts to Memorize

Be able to complete a 12 x 12 multiplication table without a calculator!

Memorize the Following Common Fraction ↔ Decimal Equivalents			
$\frac{1}{2} = 0.5$			
$\frac{1}{3} = 0.33\overline{3}$	$\frac{2}{3} = 0.66\overline{6}$		
$\frac{1}{4} = 0.25$	$\frac{3}{4} = 0.75$		
$\frac{1}{5} = 0.2$	$\frac{2}{5} = 0.4$	$\frac{3}{5} = 0.6$	$\frac{4}{5} = 0.8$
$\frac{1}{6} = 0.16\overline{6}$	$\frac{5}{6} = 0..83\overline{3}$		
$\frac{1}{8} = 0.125$	$\frac{3}{8} = 0.375$	$\frac{5}{8} = 0.625$	$\frac{7}{8} = 0.875$
$\frac{1}{10} = 0.1$	$\frac{3}{10} = 0.3$	$\frac{7}{10} = 0.7$	$\frac{9}{10} = 0.9$
$\frac{1}{100} = 0.01$			
$\frac{1}{1000} = 0.001$			

Memorize the First Twenty Perfect Square Numbers[1]										
0	1	4	9	16	25	36	49	64	81	100
121	144	169	196	225	256	289	324	361	400	…

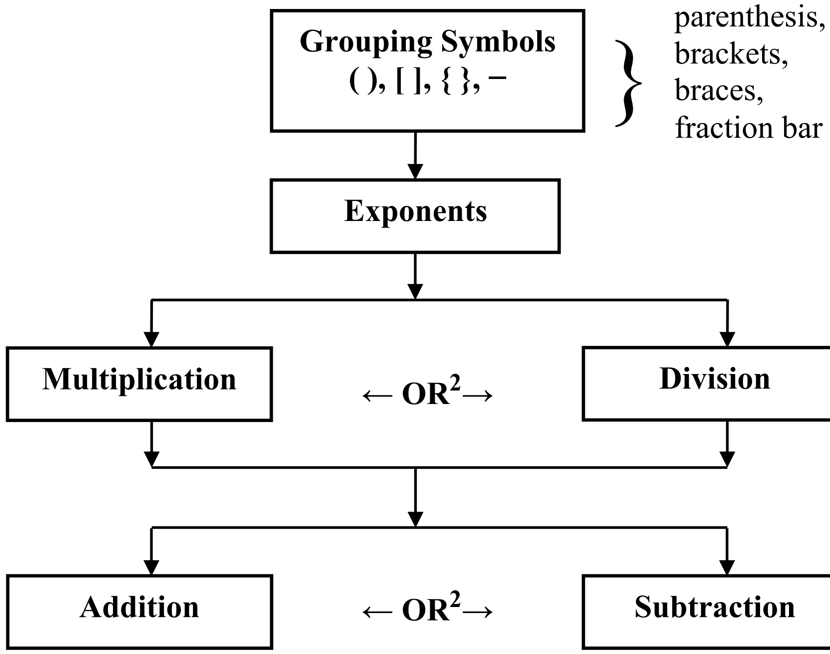
[1]A **perfect square number** (also called a square number) is an integer that is the product of some integer and itself. Perfect square numbers are nonnegative.

Memorize the First Twenty-Five Prime Numbers[1]									
2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	…				

[1]**Prime numbers** are those numbers that are only divisible by 1 and the number itself. The number 2 is the first prime number and the only even prime number. Any numbers not prime (i.e., have three or more factors) are called **composite numbers**.

Math Concepts to Know

"GEMDAS" Order of Operations[1]



[1]Also, the acronym PEMDAS (**P**arenthesis, **E**xponents, **M**ultiplication/**D**ivision, **A**ddition/**S**ubtraction) and the phrase: "**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally" are commonly taught to help students remember the Order of Operations. However, GEMDAS is more correct as it includes all four grouping symbols.

[2]Working left-to-right, do whichever operation comes first.

Operations with Negatives

negative + negative = negative

For: negative + positive or,
negative – positive:
subtract the absolute values and keep the sign of the largest absolute value

---

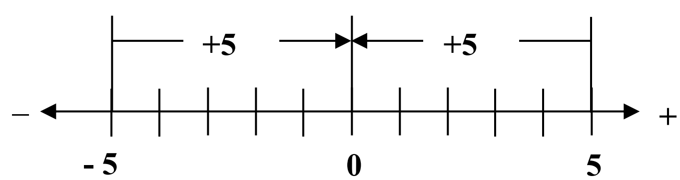
negative x negative = positive
negative x positive = negative
positive x negative = negative

---

negative / negative = positive
negative / positive = negative
positive / negative = negative

Absolute Value

The absolute value of a number *x* is its distance from zero on a real number line (distance is always a positive number). The double bar symbol $|x|$ is used to denote the operation of absolute value.



$|-5| = 5$ **and,** $|5| = 5$

Note: the output is always positive

Inequality Symbols

> greater than	5 > 3	**≥ greater than or equal to**	5 ≥ 5
< less than	3 < 5	**≤ less than or equal to**	5 ≤ 5

More Math Concepts to Know

Scientific Notation

Science involves the use of many very large numbers like the distance to the stars or the speed of light. Scientist also deal with very small numbers like the size of atomic particles or when expressing the size of human cells. **Scientific notation** is a shorthand method of writing very large or very small numbers.

The very large number 186,000,000,000 in scientific notation is written as:

$$\mathbf{1.86 \times 10^{11}}$$

The first number 1.86 is called the **coefficient**. It must be greater than or equal to 1 and less than 10.

The second number is called the **base**. It must always be 10 in scientific notation. The base number 10 is always written in exponent form. In the number 1.86×10^{11} the number 11 is referred to as the exponent or power of ten. The 11 represents the number of decimal places to move to the left from the original 186,000,000,000**.** number to the position between the "1" and the "8" in the scientific notation.

The very small number 0.0000000066 in scientific notation is written as:

$$\mathbf{6.6 \times 10^{-9}}$$

The -9 represents the number of decimal places to move to the right from the original 0.0000000066 number to the position between the first "6" and the second "6" in the scientific notation.

To add, subtract, multiply, or divide numbers in scientific notation, follow the **rules of exponents**.

Simplifying Radicals

For square roots: If possible, factor the number under the radical into the product of the largest perfect square and another factor. Then bring the square root of the perfect square outside the radical:

$$\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$$

$$\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$$

$$\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$$

Percent

Percent means "per hundred."
Percent is a ratio which compares a number to 100. A number followed by the % symbol has a denominator of 100:

Example: Find 25% of 200

$$25\% = \frac{25}{100} = 0.25$$

$$0.25 \times 200 = 50$$

Common Verbal to Algebraic Translations

Verbal phrase	Math operation	Verbal expression	Algebraic expression
added to	addition (+)	x added to 30	$30 + x$
increased by	addition (+)	x increased by 10	$x + 10$
more than	addition (+)	x more than 5	$5 + x$
plus	addition (+)	x plus 20	$x + 20$
sum of	addition (+)	the sum of x and 15	$x + 15$
total	addition (+)	the total of x and 25	$x + 25$

Verbal phrase	Math operation	Verbal expression	Algebraic expression
decreased by	subtraction (-)	x decreased by 9	$x - 9$
difference between	subtraction (-)	The difference between x and 9	$x - 9$
less	subtraction (-)	x less 9	$x - 9$
less than	subtraction (-)	x less than 9	$9 - x$
minus	subtraction (-)	x minus 9	$x - 9$
subtract	subtraction (-)	9 subtract x	$9 - x$

Verbal phrase	Math operation	Verbal expression	Algebraic expression
double	multiplication ($\times$)	double a number	$2n$
multiplied by	multiplication ($\times$)	5 multiplied by a number	$5n$
product of	multiplication ($\times$)	the product of 4 and a number	$4n$
times	multiplication ($\times$)	3 times a number	$3n$
twice	multiplication ($\times$)	twice a number	$2n$

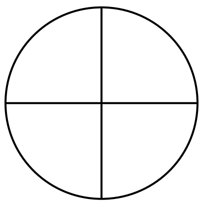
Verbal phrase	Math operation	Verbal expression	Algebraic expression
divided by	division ($\div$) or (/)	g divided by 8	$\frac{g}{8}$
quotient of	division ($\div$) or (/)	the quotient of a and b	$\frac{a}{b}$

Verbal phrase	Math operation	Verbal expression	Algebraic expression
cubed	raising to a power of 3	h cubed	h^3
raised to the power of	raising to a power	x raised to the power of 10	x^{10}
squared	raising to a power of 2	r squared	r^2

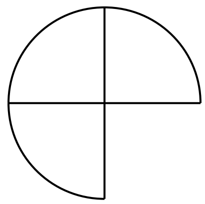
A Refresher on Operations with Fractions

Some Definitions and Facts:

A **fraction** – is a number that stands for **part of the whole** of something.



Whole



Part of the Whole

The word *fraction* is derived from the Latin word *frangere* meaning "to break." There are similar sounding words in the English language that have similar meanings. The word *fracture* is often used to describe a broken bone; the word *fragment* refers to a part of something that has broken away; and, the word *fragile* means easily broken. In mathematics, the concept of fractions was invented to attach a numerical value relative to the size of the altered whole object or objects.

Fractions (i.e., rational numbers) are numbers that can be written in the form $\frac{a}{b}$, where *a* and *b* are integers (except *b* cannot be equal to zero). **Integers** are the set of both positive and negative whole numbers {…- 3, -2, -1, 0, 1, 2, 3…}

Examples: $\frac{2}{3}$, $-\frac{19}{3}$, and $-\frac{3}{22}$

Note, the line between the *a* and *b*, called the ***fraction bar*** or ***fraction line***, means division.

When dealing with negative fractions, it is important to note that the position of the negative sign makes no difference to the value of the fraction. The sign can be attached to the top number, or positioned in the middle next to the fraction bar, or attached to the bottom number:

Example: $\frac{-3}{22} = -\frac{3}{22} = \frac{3}{-22}$

denominator – the bottom value of a fraction which represents the ***divisor***, tells how many equal parts are in the whole or set.

numerator – the top value of a fraction which represents the ***dividend***, tells how many of those parts you're talking about.

$$\frac{a}{b} = \frac{\text{dividend}}{\text{divisor}} = \frac{\text{numerator}}{\text{denominator}} = \frac{\text{How many parts are we talking about?}}{\text{How many equal parts are in the whole?}}$$

Converting an improper fraction to a mixed number:

Note:
On multiple choice questions on standardized tests (e.g., state tests, PSAT, SAT) the choices for answers are almost always stated as mixed numbers not as improper fractions. It is important that you know how to change an improper fraction to its equivalent mixed number form.

Also, it is standard mathematical convention that final answers to problems, if to be stated as fractions, not be left as improper fractions but converted to their mixed number equivalents. We might say:

It's not proper to leave it improper!

Example: $\frac{31}{9}$

We need to ask the question: How many times does 9 go into 31 and,
What is the amount left over?

The answer is that 9 goes into 31 three times with 4 left over:

Let's check: $3 \times 9 = 27$ and $31 - 27 = 4$.

We call the 4 the **remainder.**

We have 3 wholes of something plus 4 parts of the whole as a remainder. The whole had 9 equal parts.

Therefore, the mixed number is written as: $3\frac{4}{9}$

Converting a mixed number to an improper fraction:

Example: $3\frac{4}{9}$

Create the new numerator by multiplying the denominator 9 times 3 and add to the given numerator 4. Then place this new numerator over the original denominator 9:

$$9 \times 3 = 27$$
$$27 + 4 = 31$$

So,

the resulting improper fraction is $\frac{31}{9}$

Adding or subtracting fractions:

First, <u>change any mixed numbers to improper fractions</u>, then:

If the denominators are the same, add or subtract the numerators and keep the same denominator.

Example: $\frac{2}{3} + \frac{4}{3} = \frac{2+4}{3} = \frac{6}{3} = 2$

If the denominators are the different, find a common denominator. Adjust the numerator(s) accordingly then add or subtract the numerators, keeping the same denominator.

Example: $\frac{2}{3} + \frac{4}{5} = \frac{5\times 2}{15} + \frac{3\times 4}{15} = \frac{10+12}{15} = \frac{22}{15}$ or $1\frac{7}{15}$

Multiplying fractions:

First, <u>convert any mixed numbers to improper fractions</u>, then:

multiply the numerators and multiply the denominators.

Example: $\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}$

Dividing simple fractions:

Multiply the fraction in the numerator by the reciprocal of the fraction in the denominator. The term **reciprocal** means flipping the fraction upside down.

Example: $\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{2 \times 5}{3 \times 4} = \frac{10}{12} = \frac{5}{6}$

In this example, $\frac{5}{4}$ is the reciprocal of $\frac{4}{5}$

Reducing fractions down to their simplest form:

Note:
On multiple choice questions on standardized tests (e.g., state tests, PSAT, SAT) the choices for answers are almost always stated in their simplest reduced form. It is important that you know how to reduce a fraction to its simplest form without changing the value of the fraction.

Also, it is standard mathematical convention that final answers to problems, if to be stated as fractions, be reduced down to their simplest form.

Example: $\frac{10}{20}$

Can you find the largest number or at least some number that divides into both the numerator 10 and the denominator 20?

I think 10 is the largest number.

Let's try dividing both the numerator and denominator by 10

$$\frac{10 \div 10}{20 \div 10} = \frac{1}{2}$$

So my original fraction of $\frac{10}{20}$ reduces down to $\frac{1}{2}$ without changing its value.

If you can't think of the largest number that divides into both the numerator and denominator, start with any number that will divide into both, and keep repeating the process until there is no number that will divide into both the top and bottom:

Example: $\frac{216}{360} = \frac{216 \div 2}{360 \div 2} = \frac{108}{180} = \frac{108 \div 2}{180 \div 2} = \frac{54}{90} = \frac{54 \div 2}{90 \div 2} = \frac{27}{45} = \frac{27 \div 9}{45 \div 9} = \frac{3}{5}$

Check:

Our simplest reduced fraction $\frac{3}{5}$ has a value of 0.6,

Method for Estimating Square Roots by Hand-Calculations

Given: $\sqrt{7}$, **Find:** Estimate the primary (positive) square root of $\sqrt{7}$ without a calculator

Step 1. From the set of **perfect squares: { 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 400, …}**, find the two perfect squares that the number you wish to find the square root of, falls between.

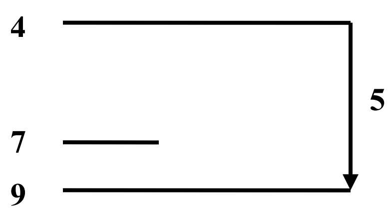
In this example, **7** falls between **4** and **9**. So, the $\sqrt{7}$ must be between the $\sqrt{4}$ and the $\sqrt{9}$.

Therefore, the $\sqrt{\mathbf{7}}$ must be **between 2** and **3**.

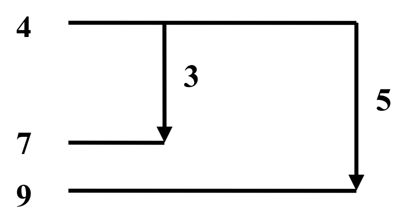
Step 2. Draw two horizontal lines:

Step 3. Place the smaller perfect square top-left and the larger perfect square bottom-left of the two horizontal lines as shown.

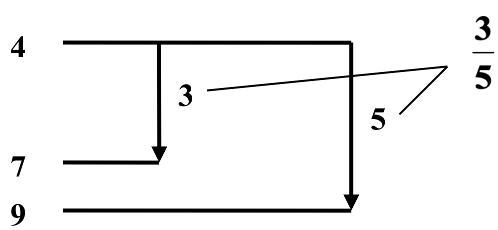
Step 4. Draw a vertical line downward on the right and label it with the difference between the two perfect squares. **9 – 4 = 5**



Step 5. Position the value of the square root you are looking for, **7**, between the two perfect squares on the left and draw a short horizontal line.



Step 6. Draw a vertical line downward to the short horizontal line and label it with the difference between the value of the square root you are looking for, **7**, and the smaller perfect square: **7 – 4 = 3**



Step 7. Form a fraction: the value next to the inside downward arrow is the numerator (i.e., top); the value next to the outside downward arrow is the denominator (i.e., bottom).

Step 8.

From Step 1. we know that the $\sqrt{7}$ must be between 2 and 3. So, we must add $\frac{\mathbf{3}}{\mathbf{5}}$ to **2** to get our answer for the $\sqrt{7}$ as $\mathbf{2\frac{3}{5}}$.

Representing Relations

A **set** is a collection of objects or things. The symbols **{ }**, called braces, are used to enclose the objects or things in a set.

An **ordered pair** is the method of naming the exact address for plotting a point on a rectangular coordinate system and is written in the form of (x, y). The order is important!

A **relation** is a set of ordered pairs For example:

{ (1, 4) , (3, 5) , (-8, 4) , (6, 2) , (2, -3) }

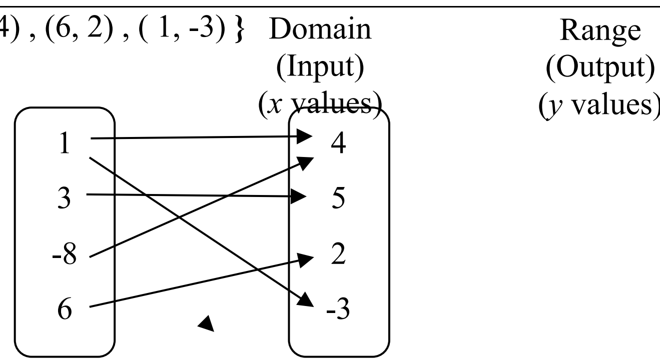
The set of all unique (i.e., different) numbers in the first position in the ordered pairs is called the **domain**. A member of the domain represents the "input" of the relation.

The set of all unique (i.e., different) numbers in the second position in the ordered pairs is called the **range**. A member of the range represents the "output" of the relation.

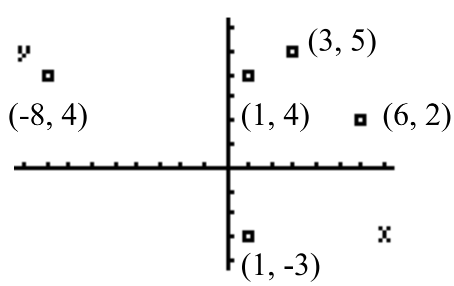
Consider the relation: **{** (1, 4) , (3, 5) , (-8, 4) , (6, 2) , (1, -3) **}**
This relation represented by a **table** is:

x	***y***
1	4
3	5
-8	4
6	2
1	-3

Consider the relation: **{** (1, 4) , (3, 5) , (-8, 4) , (6, 2) , (1, -3) **}**
This relation represented by a **mapping is:**



Consider the relation: **{** (1, 4) , (3, 5) , (-8, 4) , (6, 2) , (1, -3) **}**
This relation represented by a **graph** is:



Functions

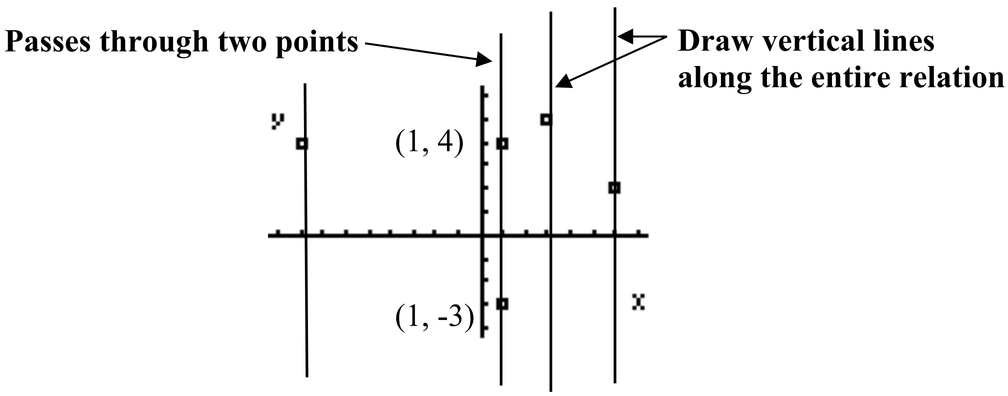
A **function** is a relation that assigns exactly one member of the range to each member of the domain (i.e., exactly one output for each input).

Every function is a relation but not every relation is a function!

Using either a graph or a mapping of a relation, we can visually determine whether the relation is a function.

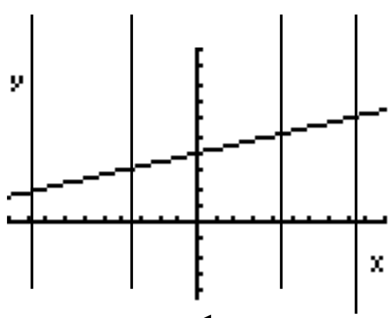
The Vertical Line Test for a Function

Again, consider the graph of our relation: **{** (1, 4) , (3, 5) , (-8, 4) , (6, 2) , (1, -3) **}**



If any of the vertical lines passes through more than one point, the relation is not a function. Since a vertical line passes through both (1. 4) and (1, -3), this is <u>not a function</u>.
There are two outputs, the “4” and the “-3” that are assigned to the input “1.”

However, the graph of a line generated by ***y* = 2x + 4** <u>is a function</u>:

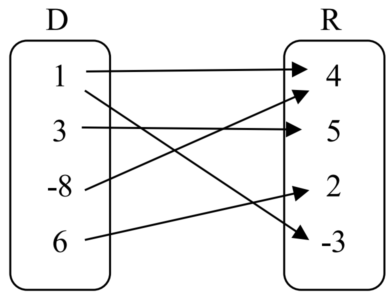


Graph of $y = \frac{1}{4}x + 4$

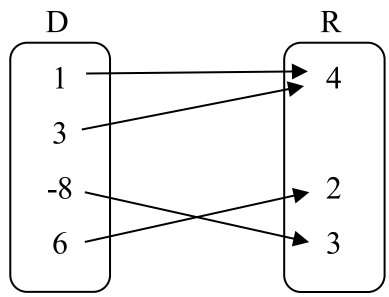
Vertical lines drawn through the relation along the entire relation only pass through one point.

The Mapping Test for a Function

If more than one line leaves the domain from any specific member as from"1" in the following mapping, the relation is not a function. Here, both "4" and "-3" are assigned to the input "1" which contradicts the definition of a function.



However, the following mapping is a function because exactly one member of the range is assigned to each member of the domain. It is okay that two arrows point to a member of the range but two arrows cannot leave the same member of the domain and go to different members of the range as in the previous mapping.



Function Notation *f*(*x*)

In many areas of mathematics, "*y*" is replaced by the notation *f*(*x*) which is read as "f of x." Just think of *f*(*x*) as the same as *y*. Sometimes, we are dealing with more than one function in the same problem. To avoid confusion with the different functions all beginning with "y =," each function can be identified differently by using ***f*(*x*) =, *g*(*x*) =, *h*(*x*) =, etc. They all mean the same as "y =."**

Instead of writing the equation of the line: $y = \frac{1}{4}x + 4$

In function notation we would write: $f(x) = \frac{1}{4}x + 4$

Where *x* is the input and *f*(x) is the output.

To solve for *f*(*-2*) given $f(x) = \frac{1}{4}x + 4$ we substitute -2 for *x* in the equation and solve:

$$f(-2) = \frac{1}{4}(-2) + 4$$

$$f(-2) = 3\frac{1}{2}$$

In other words, an **input of -2** into the function **yields** an **output of** $3\frac{1}{2}$

Quadratic Functions

Some Definitions and Facts

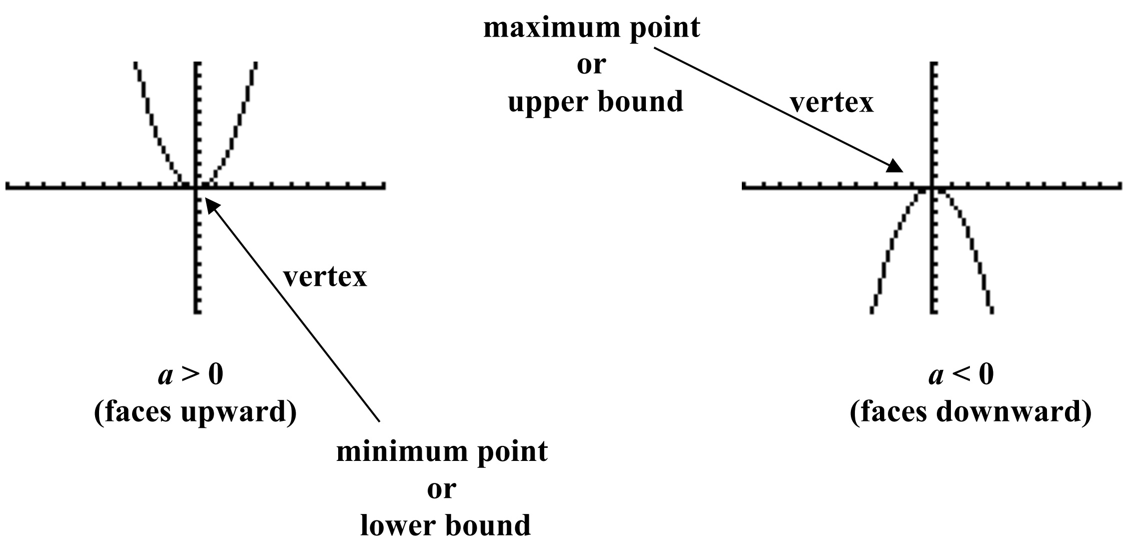
An equation in the form of

$$f(x) = ax^2 + bx + c \,,$$

where, $a \neq 0$,

is called a **quadratic function**. It is a second-degree polynomial in one variable meaning that the polynomial has only one variable, in this case the *x*, and that the highest exponent of the variable *x* is 2.

The graph of this polynomial function is called a **parabola**. Given a quadratic function in terms of *x*, the parabola can either face upward or downward depending on the sign of the coefficient of the x^2 term in the polynomial:



All quadratic functions will have this same general parabolic shape that is symmetric about some line called the **axis or line of symmetry**. However, the parabolas may be wider, narrower, taller, shorter, and/or located at different positions on the rectangular coordinate plane. The intersection between the function and its axis of symmetry is called the **vertex**.

When the parabola faces upward ($a > 0$), the vertex is a **minimum point** which is also referred to as the **lower bound.**

When the parabola faces downward ($a < 0$), the vertex is a **maximum point** which is also referred to as the **upper bound**.

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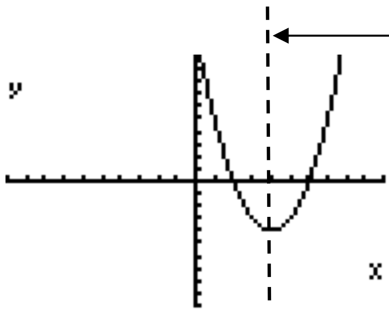
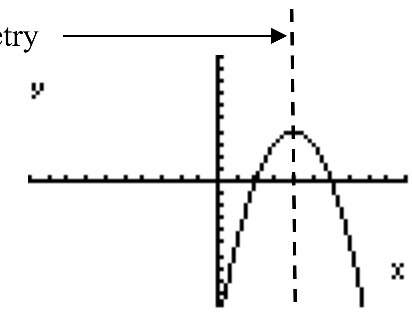
The method for determining the axis of symmetry uses the *a* and *b* values from the standard quadratic equation form, $\boldsymbol{ax^2 + bx + c}$, is as follows:





Graph of: $f(x) = x^2 - 8x + 12$
Where, $a = 1$, $b = -8$, and $c = +12$

$$\boldsymbol{x = -\frac{b}{2a}}$$

$$\boldsymbol{x = -\left(\frac{-8}{2(1)}\right)}$$

So, the axis of symmetry is:

$$\boldsymbol{x = +4}$$

Graph of: $f(x) = -x^2 + 8x - 12$
Where, $a = -1$, $b = +8$, and $c = -12$

$$\boldsymbol{x = -\frac{b}{2a}}$$

$$\boldsymbol{x = -\left(\frac{8}{2(-1)}\right)}$$

So, the axis of symmetry is:

$$\boldsymbol{x = +4}$$

The method for determining the coordinates (*x*, *y*) of the vertex is as follows:

Let *x* equal the *x*-coordinate of the axis of symmetry and substitute it in the given quadratic function $f(x) = ax^2 + bx + c$

Example:

<u>Given</u>: $f(x) = x^2 - 8x + 12$ and the axis of symmetry passing through $x = 4$

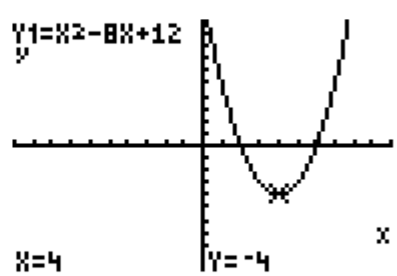
<u>Find</u>: the (x, y) coordinates of the vertex

<u>Solution</u>:

$$f(x) = x^2 - 8x + 12$$
$$f(4) = (4)^2 - 8(4) + 12$$
$$f(4) = 16 - 32 + 12$$
$$f(4) = -4$$

So, the (*x*, *y*) coordinates of the vertex are **(4, – 4)**

The graph of this function is:



Common Factoring Techniques Used on Polynomials

	Any Number of Terms
Greatest Common Factor (GFC)	$a^3b^2 + 2a^2b - 4ab^2 = ab(a^2b + 2a - 4b)$
	Two Terms
Difference of Two Squares	$a^2 - b^2 = (a+b)(a-b)$
Sum of Two Cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
Difference of Two Cubes	$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
	Three Terms
Perfect Square Trinomials	$a^2 + 2ab + b^2 = (a+b)^2$ $a^2 - 2ab + b^2 = (a-b)^2$
General Trinomials	$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$
	Four or More Terms
Grouping	$ra + rb + sa + sb = r(a + b) + s(a + b)$ $= (r + s)(a + b)$

Finding Patterns in Sequences

A **sequence** is a set of numbers in a specific order.

Example: 1, 3, 5, 7, 9,11

To look for patterns in sequences, the first step is always to take the difference between successive terms to see if there is a pattern.

Example 1: 1, 5, 9, 13, 17, …

5 – 1 = 4; 9 – 5 = 4; 13 – 9 = 4; and 17 – 13 = 4 so, *d* = 4

This is also called taking the **"first difference."** If the difference is always the same as in this example, it's called a "**common difference"**

Note: if *d* increases or decreases at a constant rate, the sequence is called an **arithmetic sequence.**

If *d* is not a constant, see if there is some recognizable pattern formed by the successive *d*'s

Example 2: 1, 5, 10, 16, 23, …

5 – 1 = 4; 10 – 5 = 5; 16 – 10 = 6; 23 – 16 = 7 so, *d* increases by 1 each successive term d = 4, 5, 6, 7, …

The next term after 23 will be 23 + 8 = **31**

Common Geometry Symbols

Symbol	Meaning	Example(s)	In Words…
$\overline{AB}$	Line Segment 'AB'	$\overline{AB}$	The line between points A and B
AB	Length of 'AB'	AB	The length from points A to B
$\overleftrightarrow{AB}$	Line 'AB'	$\overleftrightarrow{AB}$	The infinite line that includes points A and B
$\overrightarrow{AB}$	Ray 'AB'	$\overrightarrow{AB}$	The line that starts at A, goes through B, and continues on
$\overset{\frown}{AB}$	Minor arc 'AB'	$\overset{\frown}{AB}$	The minor arc that includes points A and B
$\overset{\frown}{ABC}$	Major arc 'ABC'	$\overset{\frown}{ABC}$	The major arc that includes points A, B, and C
$\angle ABC$	Angle 'ABC'	$\angle ABC$ $\angle D$	The angle with vertex B and rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ The angle with vertex D
m	Measure of… (always indicated numerical value)	$m\angle DEF = 30°$ $m\overset{\frown}{ABC} = 100°$ $m\overset{\frown}{AB} = 43°$	The measure of angle DEF is 30 degrees The measure of major arc ABC is 100 degrees The measure of minor arc AB is 43 degrees
Δ	Triangle	ΔABC	Triangle ABC
▱	Parallelogram	▱ $ABCD$	Parallelogram $ABCD$
⊙	Circle	⊙D	Circle with center point D
$\cong$	Congruent	$\angle C \cong \angle D$ $\Delta ABC \cong \Delta XYZ$	Angle C is congruent to angle D Triangle ABC is congruent to triangle XYZ
$\sim$	Similar	$\Delta ABC \sim \Delta XYZ$	Triangle ABC is similar to triangle XYZ
$\perp$	Perpendicular	$\overline{CD} \perp \overline{FG}$	Segment CD is perpendicular to FG
$\parallel$	Parallel	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	Line AB is parallel to CD
$\therefore$	Therefore	$\therefore m\angle ABC = 90°$	Therefore, the measure of angle ABC is equal to 90 degrees
$\frac{a}{b}$, $a{:}b$	Ratio of a to b	$a:b = 3:4$ $\frac{a}{b} = \frac{3}{4}$	The ratio of a to b is equivalent to the ratio of 3 to 4
sin	Sine	$\sin(x)$	The sine of x
cos	Cosine	$\cos(x)$	The cosine of x
tan	Tangent	$\tan(x)$	The tangent of x
$\sin^{-1}$	Inverse Sine	$\sin^{-1}(x)$	The inverse sine of x
$\cos^{-1}$	Inverse Cosine	$\cos^{-1}(x)$	The inverse cosine of x
$\tan^{-1}$	Inverse Tangent	$\tan^{-1}(x)$	The inverse tangent of x
sec	Secant	$\sec(x)$	The secant of x
cot	Cotangent	$\cot(x)$	The cotangent of x
csc	Cosecant	$\csc(x)$	The cosecant of x

Meet the "Quad" Family
(Properties of Special Quadrilaterals)

Quadrilateral

Def. **<u>Polygon with four sides</u>**
- Angle measures have a sum of 360°

Trapezoid

Def. **<u>Exactly one pair of parallel sides</u>**
- 2 pairs of consecutive
- angles are supplementary

Isosceles Trapezoid

Def. **<u>Exactly one pair of opposite congruent sides</u>**
- Base angles are congruent
- Diagonals are congruent

Parallelogram

Def. **<u>Two pairs of opposite sides are parallel</u>**
- Opposite sides are congruent
- Opposite angles are congruent
- Consecutive angles are supplementary
- Diagonals bisect each other

Kite

Def. **<u>Two pairs of consecutive sides are congruent</u>**
- Exactly one pair of angles are congruent
- Diagonals are perpendicular

Rhombus

Def. **<u>All sides are congruent</u>**
- Diagonals bisect opposite angles
- Diagonals are perpendicular

Rectangle

Def. **<u>All four angles are congruent</u>**
- Diagonals are congruent

Square

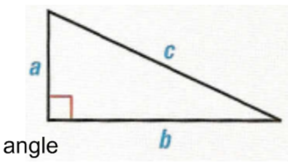
Def. **<u>All sides are congruent</u>**
<u>and</u>
<u>All angles are congruent</u>

The Pythagorean Theorem

Pythagorean Theorem

In a right triangle, $a^2 + b^2 = c^2$

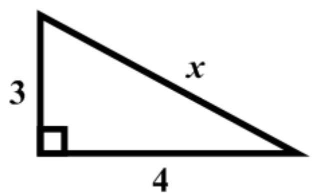
where *a* and *b* are legs, and *c* is the hypotenuse
c is always the longest side and opposite the right angle



The Pythagorean Theorem is used any time that you have a right triangle, know the lengths of two of the sides, and need to know the length of the third. In high school geometry, the Pythagorean Theorem is often used as a "stepping stone" to answer a greater problem. For example, finding the area of a triangle sometimes requires for you to first use the Pythagorean Theorem to find the base or height of the triangle.

Example A:

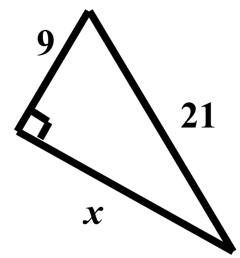
Find value of *x*



Step	Work	Reason
1	$3^2 + 4^2 = x^2$	Substitute into the Pythagorean Theorem. Notice that *x* is the hypotenuse.
2	$9 + 16 = x^2$	Find the values of 3^2 and 4^2
3	$25 = x^2$	Combine like terms by adding 9 and 16
4	$\mathbf{5 = x}$	Take the square root of both sides. The square root of x^2 is just *x*, and the square root of 25 is 5.

Example B:

Find value of *x*



Step	Work	Reason
1	$9^2 + x^2 = 21^2$	Substitute into the Pythagorean Theorem. Notice that 21 is the hypotenuse.
2	$81 + x^2 = 441$	Find the values of 9^2 and 21^2
3	$x^2 = 360$	To get *x* alone, subtract 81 from both sides
4	$\mathbf{x = 18.97}$	Take the square root of both sides. The square root of x^2 is just *x*, and the square root of 360 is approximately 18.97.

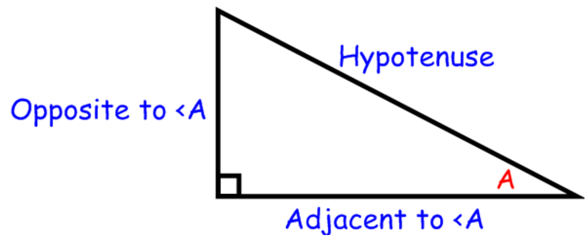
Triangle Trigonometry

Triangle Trigonometry is commonly remembered by the mnemonic **SohCahToa**. What this means is shown below. When solving these problems, remember that the opposite and adjacent sides are relevant to the non-right angle that you either know or are trying to find. Triangle Trigonometry can only be used when you have a right triangle.

Sine $\quad \text{Sin}(A) = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$ = Soh

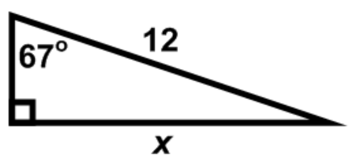
Cosine $\quad \text{Cos}(A) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$ = Cah

Tangent $\quad \text{Tan}(A) = \dfrac{\text{Opposite}}{\text{Adjacent}}$ = Toa



Example A:

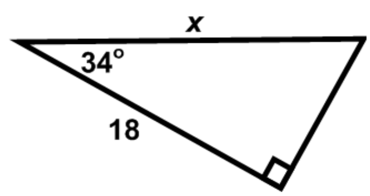
Find value of *x*



Step	Work	Reason
1	$\sin(67^o) = \dfrac{x}{12}$	Since x is opposite to the 67° angle and 12 is the hypotenuse of the triangle, we must use the sine equation (see above).
2	$12 \cdot \sin(67^o) = x$	Multiply both sides by 12 to get x alone.
3	$12 \cdot 0.9205 = x$	Find the decimal for the sin(67°).
4	$\mathbf{x = 11.0461}$	Multiply the decimal by 12 to get the length of the missing side (x).

Example B:

Find value of *x*



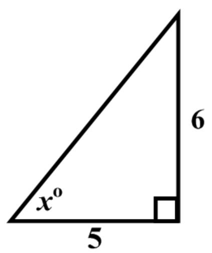
Step	Work	Reason
1	$\cos(34^o) = \dfrac{18}{x}$	Since x is adjacent to the 34° angle and x is the hypotenuse of the triangle, we must use the cosine equation (see above).
2	$x = \dfrac{18}{\cos(34^\circ)}$	So that you can get x alone, switch places with cos(34°).
3	$x = \dfrac{18}{0.829}$	Find the decimal for the cos(34°).
4	$\mathbf{x = 21.7119}$	Divide 18 by the decimal to get the length of the missing side (x).

Triangle Trigonometry Continued

To find the measure of an angle in a triangle, you must know the lengths of two sides. Once you solve the equation for the trigonometric function, you must use the inverse function (i.e. $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$)

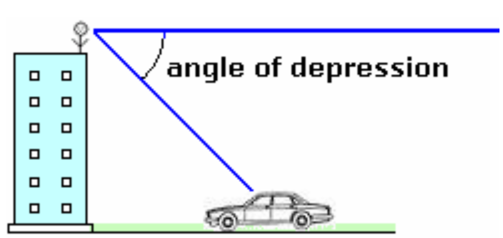
Example C:

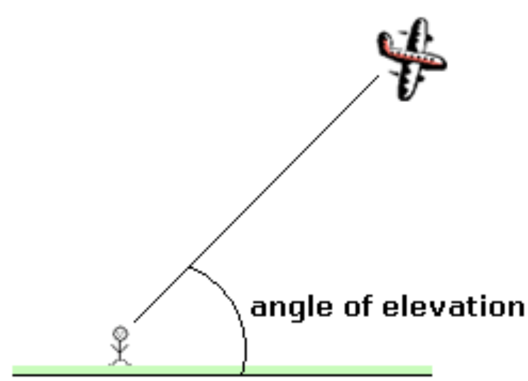
What is the measurement of angle *x*?



Step	Work	Reason
1	$\tan(x^o) = \frac{6}{5}$	Since 5 is adjacent to angle *x,* and 6 is opposite to angle *x*, we must use the tangent equation (see above).
2	$\tan(x^o) = 1.2$	Find the decimal for 6 divided by 5. Whenever possible, try not to round.
3	$x^o = \tan^{-1}(1.2)$	Take the inverse tangent function to each side of the equation
4	$\boldsymbol{x = 50.1944^o}$	Use your calculator to find the angle measure. Be sure you use the 2^{nd} button to get the inverse trig function.

In Pythagorean Theorem and Triangle Trigonometry word problems, you will often see the terms **Angle of Depression** and **Angle of Elevation**. Angles of depression and elevation commonly start at a platform, surface, ground, or building and then point down or up, respectively.





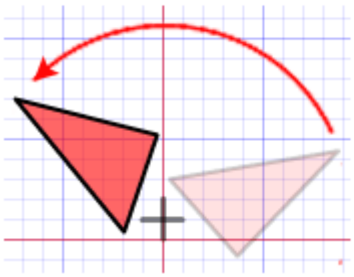
Transformations

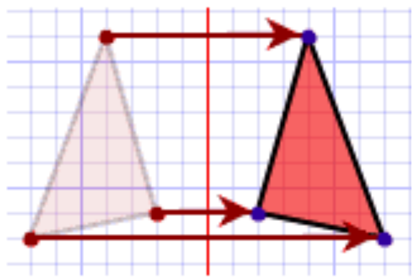
In geometry, a **transformation** is a change made to a point, line, or shape on or off the coordinate plane. A translation can be a **rotation**, **reflection**, and/or a **translation**.

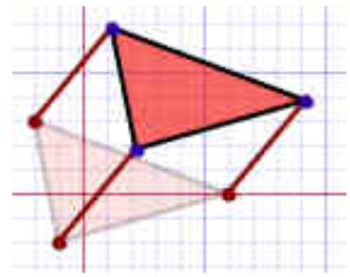
Rotation is a <u>turn</u> around a point. Think of a game spinner.

Reflection is a <u>flip</u> over a line. Think of a fold in the paper.

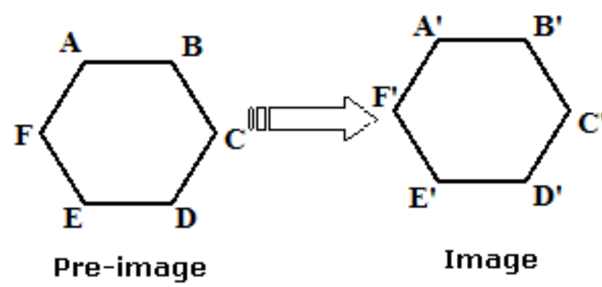
Translation is a <u>slide</u> to another spot on the graph







The original figure is called the **pre-image**; the new (copied) picture is called the **image** of the transformation. The pre-image is identified by an apostrophe called a 'prime' after each letter. In the drawing to the right hexagon ABCDEF is translated to hexagon A'B'C'D'E'F'.



Common Reflections of Points

Reflection over x-axis: $(x, y) \rightarrow (x, -y)$

Reflection over y-axis: $(x, y) \rightarrow (-x, y)$

Reflection over line $y = x$: $(x, y) \rightarrow (y, x)$

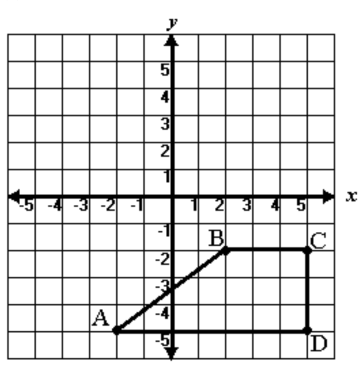
Common Rotations of Points

Rotation of 90° Counter-Clockwise: $(x, y) \rightarrow (-y, x)$

Rotation of 180° Counter-Clockwise: $(x, y) \rightarrow (-x, -y)$

Rotation of 270° Counter-Clockwise: $(x, y) \rightarrow (y, -x)$

Example:
The trapezoid below is rotated 90° clockwise about point *A*. What will be the coordinates of point *C'* ?



Answer:
In the graph, the coordinate for point C is located at (5, -2). The question asks for a clockwise rotation of 90°. Notice that the properties in the box above are each in counter-clockwise rotations. Since there is 360° in one full rotation around a circle, a 90° clockwise rotation is the same as a 270° counter-clockwise rotation.
Therefore, we will use $(x, y) \rightarrow (y, -x)$.

$(5, -2) \rightarrow (-2, -5)$

So, the coordinates for C' is (-2, -5)

Similarity

Similarity means that all of the sides are proportional (reduce down to the same fraction) and each of the angles is the same measure. If you see a similarity statement like…

$$\Delta ABC \sim \Delta DEF$$

It means that triangle *ABC* is similar to triangle *DEF* and each of the following will be true:

$$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

This is true for all polygons, not just triangles.

Determining Similarity in Triangles:

If the following conditions are met, everything to the left will be true for the triangles

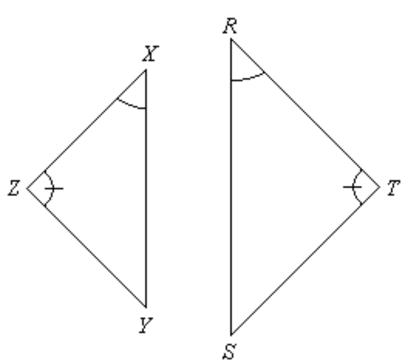
Angle-Angle Similarity – If two angles pairs are congruent in a pair of triangles, then the triangles are similar.

Side-Angle-Side Similarity – If two pairs of sides are proportional and the angle included angle pair is congruent, then the triangles are similar.

Side-Side-Side Similarity – If each of the pairs of sides in the triangles are proportional, then the triangles are similar.

Example 1:

The triangles below are similar. If $\overline{XY} = 15.4$ *inches,* $\overline{XZ} = 11$ *inches, and* $\overline{RT} = 13.75$ *inches, then*



What is the length of $\overline{RS}$ ***?***

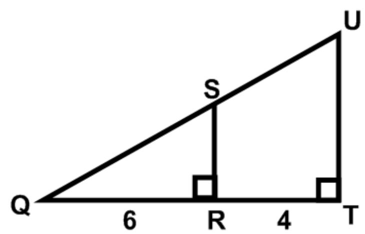
Step	Work	Reason
1	$\frac{15.4}{x} = \frac{11}{13.75}$	We know that the triangles are similar because of AA Similarity. From this, we can determine $\frac{XY}{RS} = \frac{XZ}{RT}$. From here, we substitute what we know into the proportion.
2	$11x = 15.4 \cdot 13.75$	Cross multiply
3	$11x = 211.75$	Simplify the $15.4 \cdot 13.75$
4	$x = 19.25$ inches	Divide both sides by 11 to get x alone

Example 2:

In the figure below $\Delta QRS \sim \Delta QTU$ and the length of the segment QU is 15.

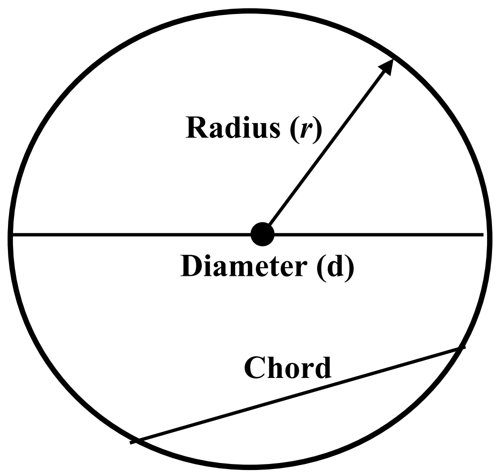
What is the length of $\overline{QS}$?

Tip with drawings like these, first draw the triangles out separately.



Step	Work	Reason
1	$\frac{6}{6+4} = \frac{x}{15}$	Since we know the triangles are similar because of $\Delta QRS \sim \Delta QTU$ we know that $\frac{QR}{QT} = \frac{QS}{QU}$. From here, we substitute what we know into the proportion.
2	$\frac{6}{10} = \frac{x}{15}$	Simplify the 6+4
3	$10x = 6 \cdot 15$	Cross multiply
4	$10x = 90$	Simplify the $6 \cdot 15$
5	$x = 9$	Divide both sides by 10 to get x alone

Circles and Arcs



Circle – a collection of an infinite number of points equidistant from a center point.

Chord – a line segment that joins two points on a curve.

Diameter – the length of a chord that passes through the center of a circle. It is the longest chord possible and its length is twice that of the radius.

Radius – the length of a ray that extends from the center of a circle to the curve. All radii of a given circle are of equal length. The radius of a circle is one-half the length of the diameter.

Circumference – the distance around the circle (the perimeter) and is always equal to π (3.1416) times the length of the diameter. The special relationship between the diameter and circumference generates a constant number named pi (pronounced pie) and is designated by the Greek symbol (**π**).

$$C = 2\pi r \quad \text{or} \quad C = \pi d$$

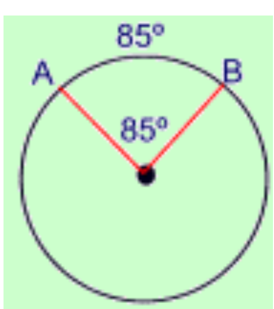
Where, C is the circumference, *r* is the radius, and *d* is the diameter

Area – is the radius squared, multiplied by π (3.1416).

$$A = \pi r^2 \quad \text{or} \quad A = (\pi d^2)/4$$

Where, A is the area, *r* is the radius, and *d* is the diameter

Arc Length –



In a circle, the ***degree measure of an arc*** is equal to the measure of the central angle that intercepts the arc.

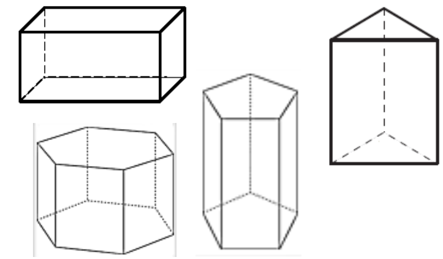
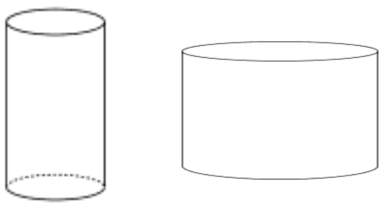
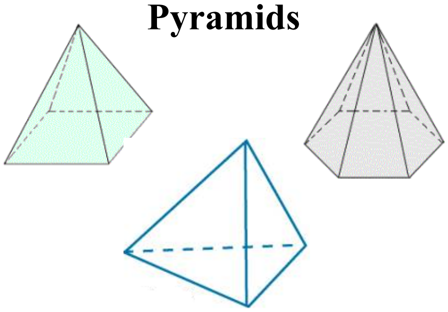
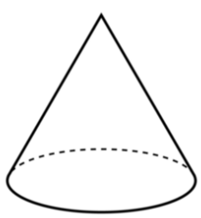
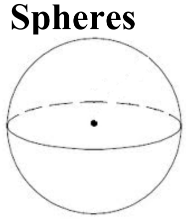
In the same circle, arcs with ***congruent central angles*** have congruent arcs.

In a circle, the ***length of an arc*** is a portion of the circumference.

The arc length can be calculated with the following proportion:

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\text{arc measure}}{360^\circ}$$

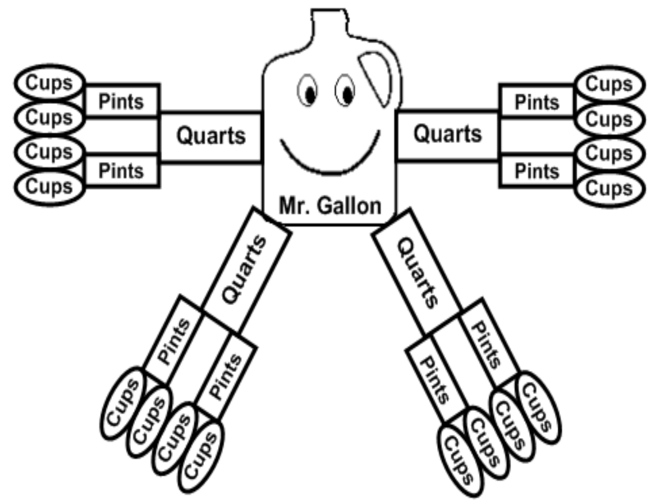
Volume and Surface Area Formulas

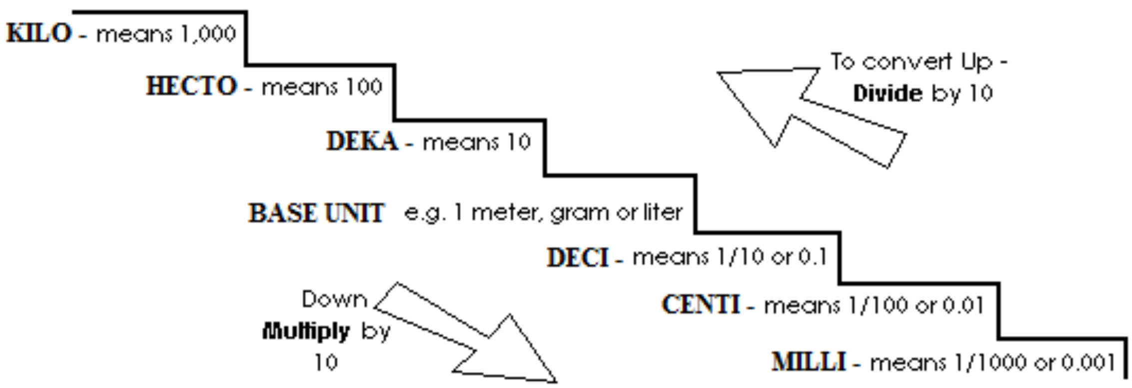
Object	Volume	Surface Area
Prisms 	V = (Area of Base) · (Height)	SA = 2 · (Area of Base) + (Perimeter of Base) · (Height)
Cylinders 	V = (Area of Circle) · (Height)	SA = 2 · (Area of Circle) + (Circumference) · (Height)
Pyramids 	$V = \frac{1}{3}$ (Area of Base) · (Height)	*Surface Area of a Regular Pyramid* $SA = \frac{1}{2}$ (Perimeter of Base) · (Slant Height) + (Area of Base)
Cone 	$V = \frac{1}{3}$ (Area of Circle) · (Height)	$SA = \frac{1}{2}$ (Circumference) · (Slant Height) + (Area of Circle)
Spheres 	$V = \frac{4}{3}\pi r^3$	$SA = 4\pi r^2$

Converting Unit Measures

Common Conversions

Time	60 seconds	=	1 minute
	60 minutes	=	1 hour
	24 hours	=	1 day
	7 days	=	1 week
	365 days	=	1 year
Length	12 inches	=	1 foot
	3 feet	=	1 yard
	5280 feet	=	1 mile
Mass	16 ounces	=	1 pound
	2,000 pounds	=	1 ton





Example:
The football coach has been timing his players running up and down the field. The field is 100 yards long, and the players have an average pace of 6 feet per second. What is their average pace in yards per hour?

Step	Work	Reason
1	$\frac{6\text{ ft}}{1\text{ sec}}\cdot\frac{1\text{ yd}}{3\text{ ft}}\cdot\frac{60\text{ sec}}{1\text{ min}}\cdot\frac{60\text{ min}}{1\text{ hr}}$	Three conversions are necessary for this problem. Notice that everything cancels out except yards and hours.
2	$\frac{6\cdot1\cdot60\cdot60\text{ yd}}{1\cdot3\cdot1\cdot1\text{ hr}}$	Simplify
3	$\frac{21,600\text{ yd}}{3\text{ hr}}=7,200\ \text{yd}/\text{hr}$	Simplify and then divide to find the average yards per hour.

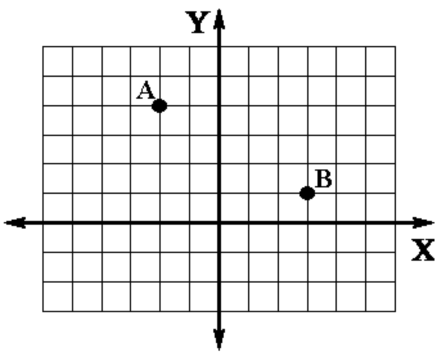
Distance and Midpoint Formulas

For the points (x_1, y_1) and (x_2, y_2) the distance can be calculated with the **Distance Formula:**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1:

Find the distance between points A (–2 , 4) and B (3 , 1).



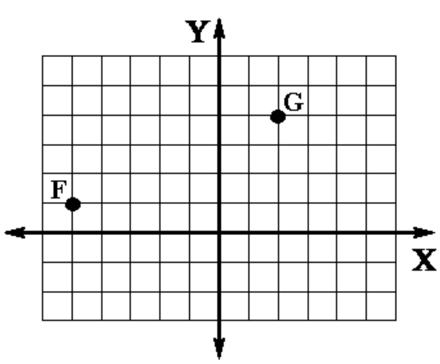
Step	Work	Reason
1	$d = \sqrt{(3-(-2))^2 + (1-4)^2}$	Point *A* is (-2, 4) and Point *B* is (3, 1). Substitute the coordinates into the formula.
2	$d = \sqrt{(3+2)^2 + (1-4)^2}$	Simplify. Remember that a double negative is the same as just one positive.
3	$d = \sqrt{(5)^2 + (-3)^2}$	Simpify
4	$d = \sqrt{25+9}$	Simplify. Remember when you square a negative number, the answer will always be positive.
5	$d = \sqrt{36}$	Simplify
6	$d = 6$ units	Note that you answer will not always be a whole number.

For endpoints (x_1, y_1) and (x_2, y_2) the coordinates for the midpoint can be calculated with the

Midpoint Formula: $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right)$

Example 2:

Find the coordinates of the midpoint between endpoints F(–5 , 1) and G (2 , 4).



Step	Work	Reason
1	$\left(\dfrac{2+(-5)}{2}, \dfrac{4+1}{2} \right)$	Given Point F (-5, 1) and Point G (2, 4), substitute the coordinates into the formula.
2	$\left(\dfrac{2-5}{2}, \dfrac{4+1}{2} \right)$	Simplify
3	$\left(\dfrac{-3}{2}, \dfrac{5}{2} \right)$	Simpify
4	$(-1.5, 2.5)$	For ease of plotting the midpoint, converting to decimals may be helpful.

Mean, Median, and Mode: Measures of central tendency

Mean – the **arithmetic average** of a set of data .

<u>Given</u>: data set D = { 2, 2, 1, 3, 6, 6, 10, 6, 6 }

<u>Find</u>: the **mean** of data set D

<u>Solution</u>: $\textbf{Mean} = \dfrac{\textbf{sum of all values}}{\textbf{total number of values}}$

$$\textbf{Mean} = \frac{\mathbf{2+2+1+3+6+6+10+6+6}}{\mathbf{9}} = \frac{\mathbf{42}}{\mathbf{9}} = \mathbf{4.7}$$

Median – the exact **<u>middle value</u>** <u>after</u> arranging all values from lowest to highest (ascending) or, from highest to lowest (descending). If there is an even number of values, then the median is the mean of the two middle values.

<u>Given</u>: data set D = { 2, 2, 1, 3, 6, 6, 10, 6, 6 }

<u>Find</u>: the **median** of data set D

<u>Solution</u>: D = { 2, 2, 1, 3, 6, 6, 10, 6, 6 }

After placing the values in ascending order, the middle value **6 is the median:**

1, 2, 2, 3, **6**, 6, 6, 6, 10

Mode – the value that appears the most. It's possible to have more than one mode. A tie of two values is called "bimodal" A tie of three or more is called "multi-modal."

<u>Given</u>: data set D = { 2, 2, 1, 3, 6, 6, 10, 6, 6 }

<u>Find</u>: the **mode** of data set D

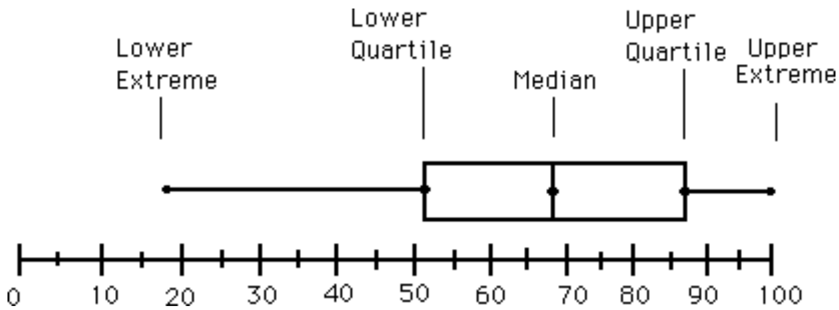
<u>Solution</u>: D = { 2, 2, 1, 3, **6**, **6**, 10, **6**, **6** }

Since the 6 appears the most often, **6 is the mode**.

Recommended Guidelines for when to use Mean, Median, and Mode:

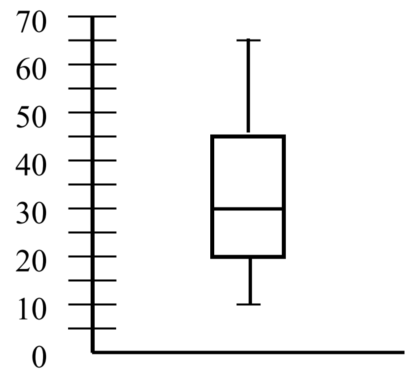
- For sets of data with no unusually high or low numbers, Mean is a good measure to use.
- For sets of data with some points that are much higher or lower than others, Median may work well.
- For sets of data with many data points that are the same, Mode may be the most useful measure.

Box–and–Whisker Plots



The five-number summary of a box-and-whisker plot

A "**box-and-whisker plot**" (also called a box-and-whisker diagram, box plot, or boxplot) is a graphical method of quickly identifying the median (i.e., the center), spread, and overall range of the data without listing all of the data. The plots can be drawn horizontally or vertical.



Procedure for Constructing a Box-and-Whisker Plot

Computer software such as Microsoft EXCEL and technology such as graphing calculators can be used to create box-and-whisker plots. However, such technology is not always available, so the following is a simple procedure for manually creating the plot.

We need to determine five numbers in order to construct the "box-and-whisker plot": the median, the lower quartile, the upper quartile, the lower extreme, and the upper extreme. These five numbers comprise the five-number summary shown in the figure above.

Example 1

Given: data set = **{** 80, 84, 87, 102, 95, 93, 89, 16, 25, 32, 50, 52, 57, 59, 68 **}**

Find: Create a box-and-whisker plot for the given data

Solution:

Step 1. Arrange the data in ascending (increasing order) if not already done.

16, 25, 32, 50, 52, 57, 59, 68, 80, 84, 87, 89, 93, 95, 102

Step 2. Find the median (the middle value). This is the 50th percentile. The median is often labeled as **Q2** on box-and-whisker plots.

16, 25, 32, 50, 52, 57, 59, **68**, 80, 84, 87, 89, 93, 95, 102

↑

The median = 68

Step 3. List only the numbers to the left of the median 68 from Step 2 and find the median of these numbers. This number is called the **lower quartile** (commonly labeled **Q1**) and always corresponds to the 25th percentile.

16, 25, 32, **50**, 52, 57, 59

↑

lower quartile = 50 also

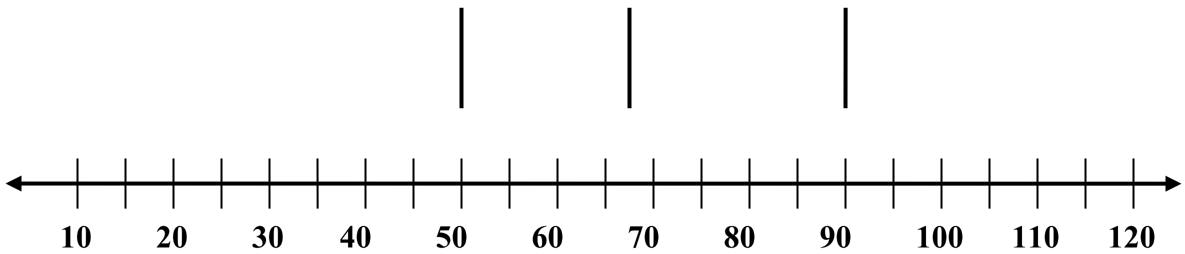
Step 4. Next**,** list only the numbers to the right of the median 68 from Step 2 and find the median of these numbers. This number is called the **upper quartile** (commonly labeled **Q3**) and always corresponds to the 75th percentile.

80, 84, 87, **89**, 93, 95, 102

↑

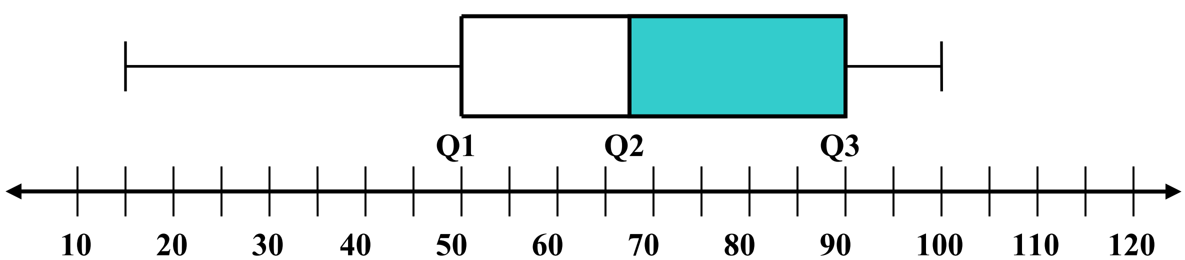
upper quartile = 89

Step 5. Draw a number line and make short vertical lines above the median (68), lower quartile (50), and upper quartile (89).



Step 6. Draw horizontal lines to form the box.

Step 7. Finally, add the whiskers which extend to the smallest value (16) on the left and to the largest value (102) on the right.



The **range** of the data is from the **lower extreme** (i.e., minimum data value) equal to 16 to the **upper extreme** (i.e., maximum data value) equal to 102: 102 – 16 = **86**

The three medians calculated (Steps 2, 3, and 4) split the data into four equal parts called **quartiles** (i.e., quarters) with three values within each quarter:

- One quarter contained the data less than 50 (Check: {16, 25, 32}
- One quarter contained the data between 50 and 68 (Check: {52, 57, 59}
- One quarter contained the data between 68 and 89 (Check: {80, 84, 87}
- One quarter contained the data greater than 89 (Check: {93, 95, 102}

This means that half the data (50%) is represented by the box (from 50 to 89). The difference between the upper quartile and the lower quartile is called the **interquartile range** (IQR). In this case IRQ = 89 – 50 = 39. In other words, 50% of the data falls between 50 and 89.

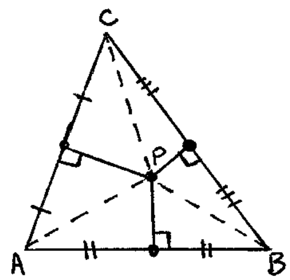
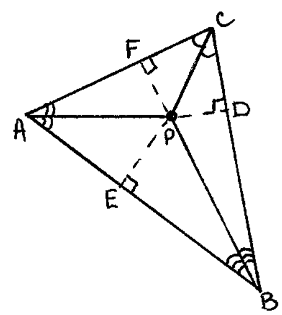
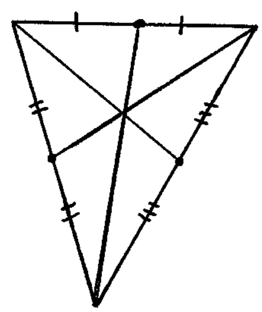
The quarter of the data between 68 and 89 (indicated by the shading in Figure 4-5) is spread out more than the data between 50 and 68.

Determining Outliers:

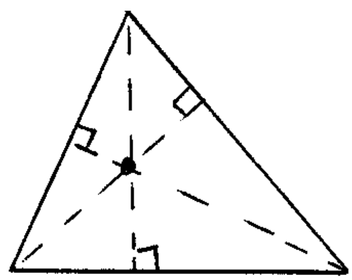
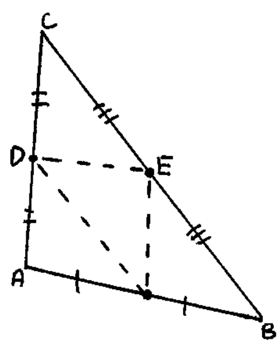
When data points are distinctly different from the rest of the data, they may be considered outliers and eliminated from overall data set for statistical calculations. One method for determining whether the data point is an outlier is to multiply 1.5 times the IRQ. If the data point is more than 1.5IRQ below the lower quartile (i.e., Q1) or above the third quartile (i.e., Q3) then the data point is considered an outlier. In our example above, the IRQ = 39; Q1 = 50; and, Q3 = 89.

Q1 – 1.5IRQ = 50 – 1.5(39) = - 8.5
Q3 + 1.5IRQ = 89 + 1.5(39) = 147.5
Since, none of the data is below – 8.5 or above 89, **there are no outliers.**

APPENDIX

Points in a Triangle				
Point of Concurrency	**How it is Found**	**What it Looks Like**	**Theorems Tell Me…**	**Other Important Information**
Circumcenter	Point of intersection of the **<u>Perpendicular Bisectors</u>** of each side of the triangle		**The distances from the circumcenter to each vertex of the triangle are all equal** PA = PB = PC	<u>Acute Δ</u> - Circumcenter lies **inside** Δ <u>Right Δ</u> - Circumcenter lies on the **hypotenuse**
Incenter	Point of intersection of the **<u>Angle Bisectors</u>** of each angle of the triangle		**The distances from the incenter to each side of the triangle are all equal** PD = PE = PF	The incenter will **always** lie inside the triangle
Centroid	The point of intersection of the **<u>medians</u>** of the triangle A <u>median</u> is a segment drawn from the midpoint of a		**Vertex to centroid is ⅔ the length of the median** **Midpoint of the side to the centroid is ⅓ the length of**	The centroid of a triangle can be used as its balancing point The centroid will **always** lie inside the triangle

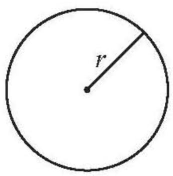
Points in a Triangle
(table continued)

Point of Concurrency	How it is Found	What it Looks Like	Theorems Tell Me…	Other Important Information
Orthocenter	The point of intersection of the **altitudes** of the triangle An **altitude** is a perpendicular segment from a vertex to the opposite side (or line containing the opposite side)		**The altitudes of a triangle intersect**	Acute Δ - Orthocenter lies **inside** Δ Right Δ - Orthocenter lies on the **vertex of the right angle** Obtuse Δ - Orthocenter lies **outside** Δ
Special Segment: **Midsegment**	A **midsegment** connects the midpoints of two sides of a triangle		**A midsegment of a triangle is always parallel to the unconnected side and is half as long as that side** DE = ½ AB	To find coordinates of midpoints of a side use the **midpoint formula** To find the length of a side or midsegment use the **distance formula**

Mathematics Reference Sheet – Grade 11
New England Common Assessment Program (NECAP)

Use the information below as needed to answer questions on the mathematics test.

Circle

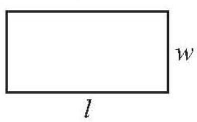


Area = πr^2

Circumference = $2\pi r$

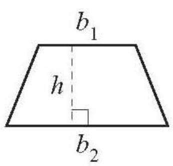
$\pi \approx 3.14$

Rectangle



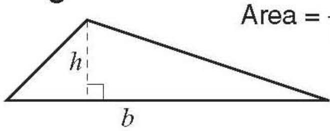
Area = lw

Trapezoid



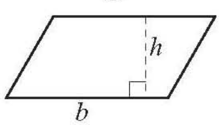
Area = $\frac{1}{2}h(b_1+b_2)$

Triangle



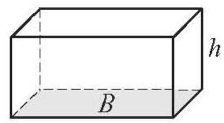
Area = $\frac{1}{2}bh$

Parallelogram



Area = bh

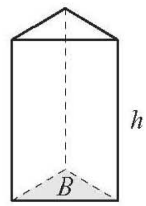
Rectangular Prism



Volume = area of the base • height

= Bh

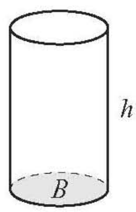
Triangular Prism



Volume = area of the base • height

= Bh

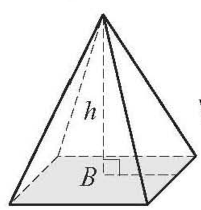
Cylinder



Volume = area of the base • height

= Bh

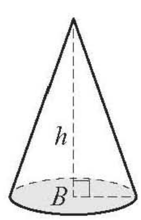
Pyramid



Volume = $\frac{1}{3}$ • area of the base • height

= $\frac{1}{3}Bh$

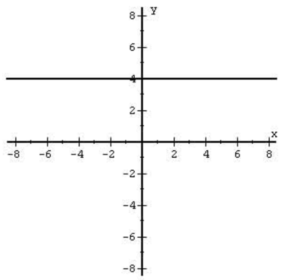
Cone



Volume = $\frac{1}{3}$ • area of the base • height

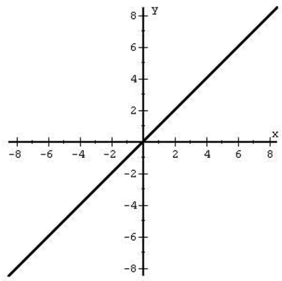
= $\frac{1}{3}Bh$

Parent Functions



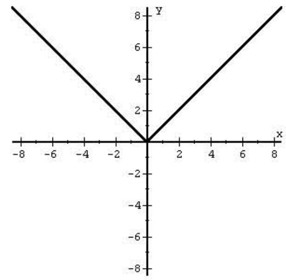
$f(x) = a$

Constant



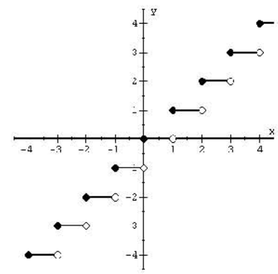
$f(x) = x$

Linear



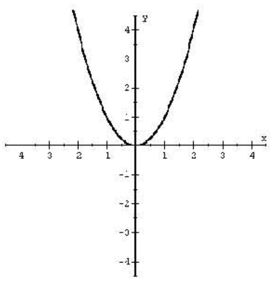
$f(x) = |x|$

Absolute Value



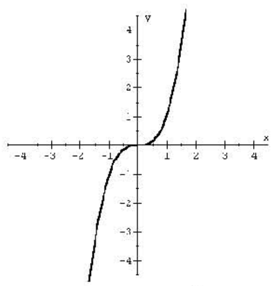
$f(x) = \text{int}(x) = [x]$

Greatest Integer



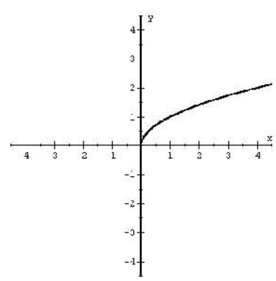
$f(x) = x^2$

Quadratic



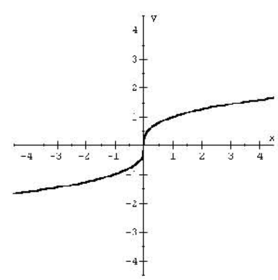
$f(x) = x^3$

Cubic



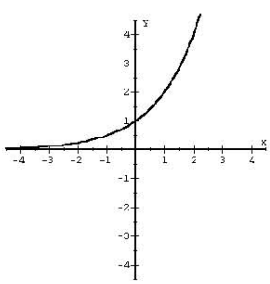
$f(x) = \sqrt{x}$

Square Root



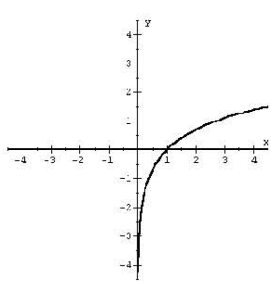
$f(x) = \sqrt[3]{x}$

Cube Root



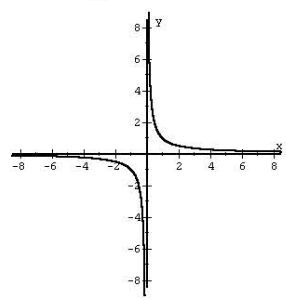
$f(x) = a^x$

Exponential



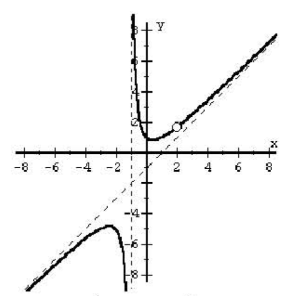
$f(x) = \log_a x$

Logarithmic



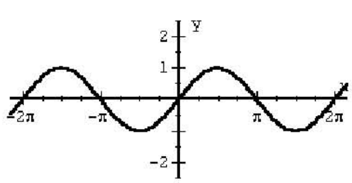
$f(x) = \dfrac{1}{x}$

Reciprocal

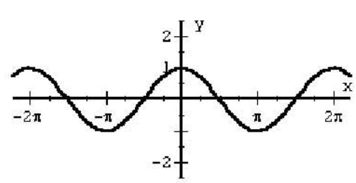


$f(x) = \dfrac{(x^2 + 1)(x - 2)}{(x + 1)(x - 2)}$

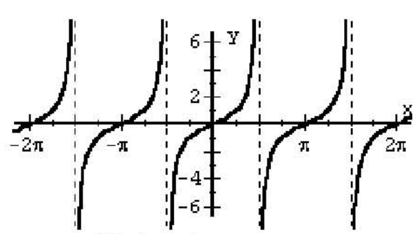
Rational



$f(x) = \sin x$



$f(x) = \cos x$



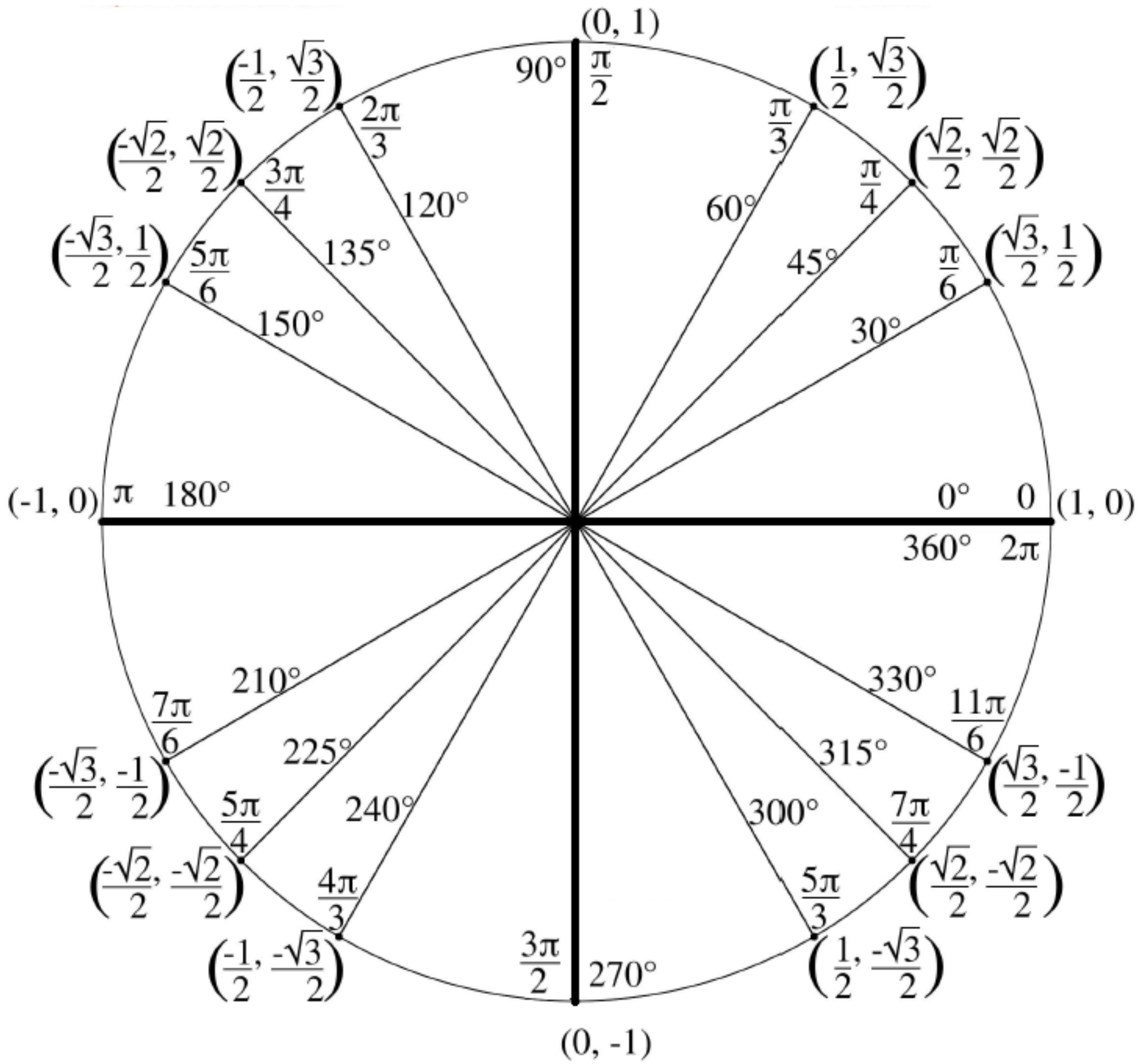
$f(x) = \tan x$

Trigonometric Functions

The Unit Circle

Quadrant II
Positive: sin, csc
Negative: cos, tan, sec, cot

Quadrant I
Positive: sin, cos, tan, sec, csc, cot
Negative: none



Quadrant III
Positive: tan, cot
Negative: sin, cos, sec, csc

Quadrant IV
Positve: cos, sec
Negative: sin, tan, csc, cot