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A composite number producing polynomial project

Observations on the trinomial $n^2 + n + 41$

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We assume that n is an integer. We consider the composite values of $n^2 + n + 41$. We only consider positive integer values for n in this paper.

The story so far

We consider the behavior of the factorization of integers of the form $h(n) = n^2 + n + 41$ where n is a non-negative integer. It was shown by Legendre, in 1798 that if $0 \le n < 40$ then h(n) is a prime number.

Certain patterns become evident when considering points (x,y) where $h(y) \equiv 0 \bmod x$. These points can be enumerated using a computer tool such as a Computer Algebra System or spreadsheet program. The collection of all such point produces what we are calling a “graph of discrete divisors” for h(n) due to certain self-similar features. From experimental computer data we find that the integer points in this graph lie on a collection of parabolic curves indexed by pairs of relatively prime integers. Each such pair yields (again determined experimentally and by observation of calculations) an integer polynomial $a*z^2 + b*z + c$, and the quartic $h(a*z^2 + b*z + c)$ then factors non-trivially over the integers into two quadratic expressions. A quadratic expression, when graphed forms a parabola.

We call this above statement our "parabola conjecture".

Conjecture is a mathematical term that means possibly true statement.

Certain symmetries in the graph of divisors are due to elementary relationships between pairs of co-prime integers. For instance if m<n are co-prime integers, then there is an observable relationship between the parabola it determines that that formed from (n-m, n).

We conjecture that all composite values of h(n) arise by substituting integer values of z into $h(a*z^2 + b*z + c)$, where this quartic factors algebraically over **Z** for $a*z^2 + b*z + c$ a quadratic polynomial determined by a pair of relatively prime integers.

We name this above statement our "no stray points conjecture" because all the points in the graph of discrete divisors appear to lie on parabolas.

We further conjecture that the minimum x-values for parabolas corresponding to (m, n) with gcd(m, n) = 1 are equal for fixed n. Further, these minimum x-values of parabolas line up at 163*d^2/4 where d = 1, 2, 3, ... The numerical evidence seems to support this.

This statement above is called our "parabolas line up" conjecture.

The notation gcd(m, n) used above is defined here. The greatest common devisor of two integers is the smallest whole number that divides both of those integers.

Theorem 1 - Consider h(n) with n a non negative integer.
h(n) never has a factor less than 41.

We prove Theorem 1 with a modular construction. We make a residue table with all the prime factors less than 41. The fundamental theorem of arithmetic states that any integer greater than one is either a prime number, or can be written as a unique product of prime numbers (ignoring the order). So if h(n) never has a prime factor less than 41, then by extension it never has an integer factor less than 41.

For example, to determine that h(n) is never divisible by 2, note the first column of the residue table. If n is even, then h(n) is odd. Similarly, if n is odd then h(n) is also odd. In either case, h(n) does not have factorization by 2.

Also, for divisibility by 3, there are 3 cases to check. They are n = 0, 1, and 2 mod 3. h(0) mod 3 is 2. h(1) mod 3 is 1. and h(2) mod 3 is 2. Due to these three cases, h(n) is never divisible by 3. This is the second column of the residue table.

The number 0 is first found in the residue table for the cases h(0) mod 41 and h(40) mod 41. This means that if n is congruent to 0 mod 41 then h(n) will be divisible by 41. Similarly, if n is congruent to 40 mod 41 then h(n) is also divisible by 41. After the residue table, we observe a bifurcation graph which has points when h(y) mod x is divisible by x. The points (x,y) can be seen on the bifurcation graph.

see residue table page

Thus we have shown with a proof that h(n) never has a factor less than 41.

Theorem 2

Since h(a) = a^2 + a + 41, we want to show that h(a) = h(-a -1).

Proof of Theorem 2
Because h(a) = a*(a+1) + 41,
Now h(-a -1) = (-a -1)(-a - 1 + 1) + 41.
So h(-a -1) = (-a -1)*(-a) + 41,
And h(-a -1) = h(a)

Which was what we wanted
End Proof of Theorem 2

Corollary 1

Further if h(b) mod c ≡ 0 the h(c -b -1) mod c ≡ 0.

We can observe interesting patterns in the graph of discrete divisors on a following page.

Residue Table

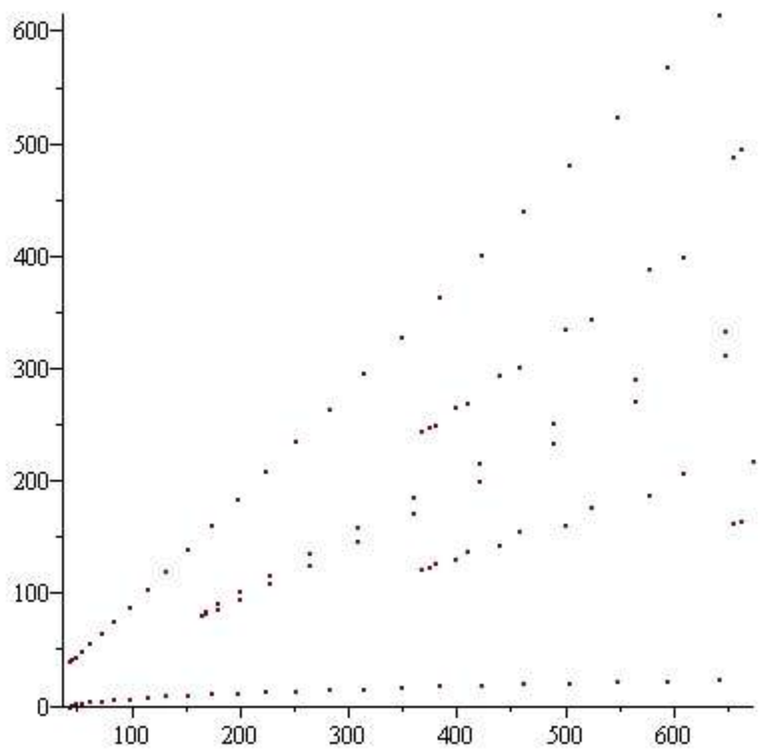
	2	3	5	7	11	13	17	19	23	29	31	37	41	43
0	1	2	1	6	8	2	7	3	18	12	10	4	0	41
1	1	1	3	1	10	4	9	5	20	14	12	6	2	0
2		2	2	5	3	8	13	9	1	18	16	10	6	4
3			3	4	9	1	2	15	7	24	22	16	12	10
4			1	5	6	9	10	4	15	3	30	24	20	18
5				1	5	6	3	14	2	13	9	34	30	28
6				6	6	5	15	7	14	25	21	9	1	40
7					9	6	12	2	5	10	4	23	15	11
8					3	9	11	18	21	26	20	2	31	27
9					10	1	12	17	16	15	7	20	8	2
10					8	8	15	18	13	6	27	3	28	22
11						4	3	2	12	28	18	25	9	1
12						2	10	7	13	23	11	12	33	25
13							2	14	16	20	6	1	18	8
14							13	4	21	19	3	29	5	36
15							9	15	5	20	2	22	35	23
16							7	9	14	23	3	17	26	12
17								5	2	28	6	14	19	3
18								3	15	6	11	13	14	39
19									7	15	18	14	11	34
20									1	26	27	17	10	31
21									20	10	7	22	11	30
22									18	25	20	29	14	31
23										13	4	1	19	34
24										3	21	12	26	39
25										24	9	25	35	3
26										18	30	3	5	12
27										14	22	20	18	23
28										12	16	2	33	36
29											12	23	9	8
30											10	9	28	25
31												34	8	1
32												24	31	22
33												16	15	2
34												10	1	27
35												6	30	11
36												4	20	40
37													12	28
38													6	18
39													2	10
40													0	4
41														0
42														41

The function h(n) which was defined as $n^2 + n + 41$ has interesting properties. Especially when n is restricted to the integers. As we know h(n) is a prime number for as n goes from 0 to39.

h(40) = 41^2, which is a composite number.

Also h(n) can be generated recursively as h(0) = 0 and h(n) = h(n-1) + 2*n.

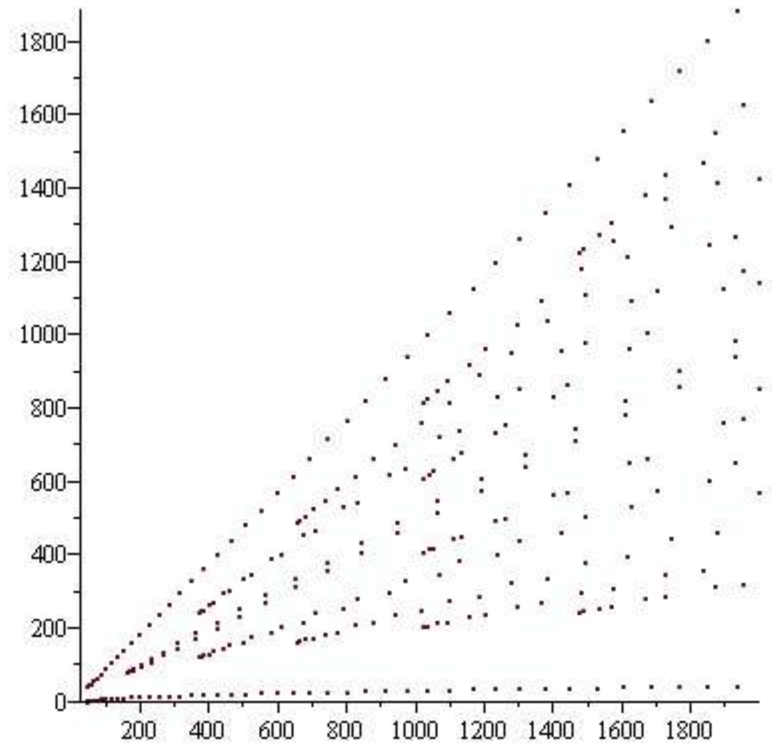
This is a linear recurrence with constant coefficients.



Bifurcation Graph

These are pairs of numbers (x,y) such that h(y) mod x ≡ 0.

And $h(y) = y^2 + y + 41$.



Here is a zoomed out iteration of the same graph as the previous page.

There seems to b an apparent regular structure in this graph of divisibility.

The points give themselves to an exact curve fit of parabolas.

The general form of these parabolas is –

$p(r,c) = c^2x^2 - 2c*r*x*y + r^2y^2 - (c*r + 1)*x + r^2y + 41r^2$. (Equation 1).

p is for parabola, r is for row index, c is for column index, x is the horizontal axis and y is the vertical axis.

This does not include the top and bottom parabolas.

There are also 3 restrictions.

$1<r$

$0<c<r$

$\text{Gcd}(r,c) = 1$.

All the parabolas can be described exactly and algebraically.

The x minimum of p(r,c) is

Pmin = (163*r^2)/4. (expression 2)

This can be found with the Mape Commmand extrema.

To wit –

> # p is for parabola

> $p[r, c] := c^2 \cdot x^2 - r \cdot c \cdot 2 \cdot x \cdot y + r^2 \cdot y^2 - (r \cdot c + 1) \cdot x + r^2 y + 41 \cdot r^2;$

$$p_{r,c} := c^2 x^2 - 2\,r\,c\,x\,y + r^2 y^2 - (c\,r + 1)\,x + r^2\,y + 41\,r^2 \tag{1}$$

> $e2 := extrema(x, p[r, c] = 0, \{x, y\});$

$$e2 := \left\{\frac{163}{4}\,r^2\right\} \tag{2}$$

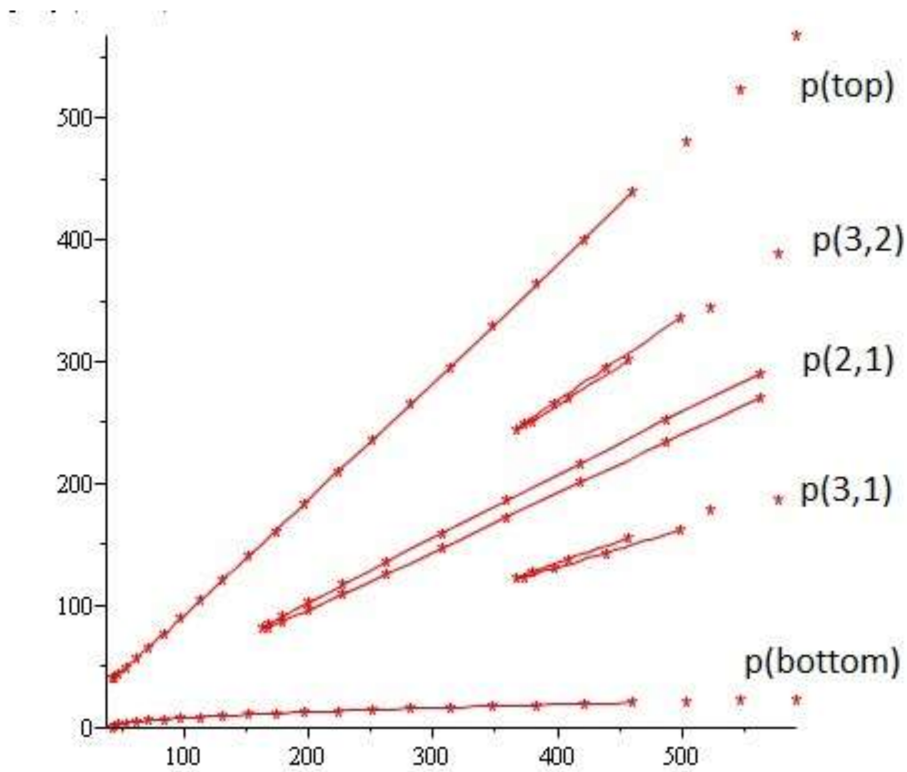
This project is not finished.

Here is some Maple code to show the exact curve fit for the graph of divisors.

```
> # Maple code
> x[bottom] := z^2+z+41; y[bottom] := z;
> p2 := plot([x[bottom], y[bottom], z = 0 .. 20]);
> with(plots);
> x[1, 1, top] := z^2+z+41; y[top] := z^2+40;
> p3 := plot([x[top], y[top], z = 0 .. 20]);
>
> y[2, 1] := 2*z^2+z+81; x[2, 1] := 4*z^2+163;
> p4 := plot([x[2, 1], y[2, 1], z = -10 .. 10]);
>
> y[3, 1] := 3*z^2+2*z+122; x[3, 1] := 9*z^2+3*z+367;
> p5 := plot([x[3, 1], y[3, 1], z = -4 .. 3]);
>
> y[3, 2] := 6*z^2+z+244; x[3, 2] := 9*z^2+3*z+367;
> p6 := plot([x[3, 2], y[3, 2], z = -4 .. 3]);
>
> d1 := display([p2, p3, p4, p5, p6])
> # code for graph of divisors
> xv := Vector[row](89); yv := Vector[row](89); counter := 1;
> for a from 2 to 600 do
    for b from 0 to a-1 do
      if `mod`(b^2+b+41, a) = 0 then
        xv[counter] := a; yv[counter] := b; counter := counter+1
      end if
    end do
  end do;
> counter;
> d2 := plot(xv, yv, style = point, symbol = asterisk);
> display(d1, d2)
> # This produces a graph.
```

The graph of divisors with 5 parabolas appears on the next page.

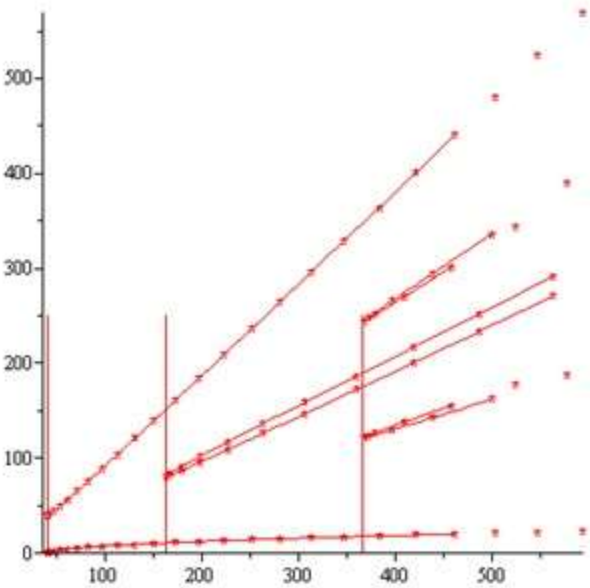
Graph of Divisors with parabolas that exactly fit the points



> "Notice the exact curve fit of parabolas to divisibility points"

Graph of divisibility

Vertical lines at
163*c^2/4



Notice the vertical lines are tangent to the parabolas.

There is still more to be done with this project.