K-tuples and Prime Constellations

by

Matthew C Anderson

May 2022

This is a non-fiction about prime numbers.

For My Wife

and

My Father

Prime Numbers

Prime numbers are positive integers that have no factor except 1 and themselves. There are no negative prime numbers. The first few prime numbers are 2, 3, and 5. The number 101 is prime. But the number 1001 is composite. The factors of 1001 are 7, 11, and 13. There are infinitely many prime numbers. There is no largest prime number. However, as of May 22, 2022, the largest prime number known to human kind is the 51st known Mersenne prime number. Mersenne prime numbers have the form 2^n-1. The largest one currently known is $2^{82,589,933}$-1. That number has 24,862,048 digits. So more than 24 million digits. The Great Internet Mersenne Prime Search (GIMPS) project has been going since 1996. You can find out more at Mersenne.org. You can also download software and join the search for a larger prime number. Do it for the sport of it. You could be a prime number hunter.

There are an infinite amount of prime numbers. Euclid proved that there are infinitely many prime. His proof is simple. It is a proof by contradiction. Suppose that there is only a finite number of primes. Give them designation p1, p2, …, pm. Then consider the product of all these prime numbers and add one. So we have p_new = p1*p2*p3*…*pm + 1. Note that p_new is not divisible by p1 or any of the prime numbers up to pm. So we have a new prime number. This is the contradiction. We start with a finite set of primes, and then show that there is another prime. So any finite set of primes does not contain all of the prime numbers. End of proof.

The next topic is the prime counting function. It is written $\pi(x)$. And read pi of x. The $\pi(x)$ function counts the number of primes less than or equal to x. For example,

$\pi(1)$ = 0

$\pi(2)$ = 1 because 2 is the first prime number

$\pi(3)$ = 2

$\pi(4)$ = 2

Similarly, we can see that $\pi(10)$ = 4 because the four primes less than 10 are{2,3,5,7}.

Hopefully this helps you understand about the prime counting function.

2-tuples

Prime numbers are interesting. We list pairs of prime numbers. The Online Encyclopedia of Integer Sequences has lots of fun sequences.

Require that p is a prime number.
Below is a list such that both p and p+n are prime and n is an even number.

2-tuples are pairs of numbers such that both numbers are prime.

In mathematics, a conjecture is a statement that is probably true. This is different from at theorem, which can be proved and is definitely true, based on certain axioms. An axiom is a starting statement. For example, we have an axiom for counting numbers. /C = {1,2,3, …}. This is read script C is the set containing numbers 1, 2, 3, and so on. The counting numbers are a countable infinite set and they go like 1,2,3, … There is no largest counting number.

In number theory, Polignac's conjecture was made by Alphonse de Polignac in 1849 and states: For any positive even number n, there are infinitely many cases of two consecutive prime numbers with difference n.

Numerical evidence suggests that Polignac's conjecture is true for all positive even numbers n. However, it is not a proved theorem. As of 2022, it is still a conjecture.

There is a conjecture that is stronger than Polignac's conjecture. It is called the k-tuple conjecture. If we let k be 2, then we have Polinac's conjecture for all positive even differences n. However k could be 3 or larger. We must understand the concept of an admissible pattern in order to fully state the k-tuple conjecture. The only even prime number is 2. All pairs of prime numbers greater than 2 have a difference of 2 or some multiple of 2. For example, if we choose an inadmissible pattern like (0,1), we will find that the only pair of prime numbers where one number is 1 bigger than the other is {2,3} and there are no others. This is not a long list. It is a list of only one pair. By convention, we make the first number in a pattern zero, and the pattern digits only increase. There are no repeated pattern digits.

For admissible, 3-tuples, we must consider divisibility by 2 and 3. For example, the pattern (0,2,4) is not admissible because all 3 remainders, when divided by 3 are in the pattern. {3,5,7} is the only set of prime numbers in this pattern. However, the admissible 3-tuple pattern (0,2,6) gives way to an apparently possibly infinite list of prime numbers. See online at oeis.org/A022004. Similarly the 3-tuple pattern (0,4,6) is also online at oeis.org/A022005. These two examples have pattern differences as close as possible to each other. If the differences are maximally small for an acceptable k-tuple then the prime numbers will be consecutive and the k-tuple is called a prime constellation.

Another acceptable pattern is (0,2,12). This pattern is recorded online at oeis.org/A046135. The entry title is : Primes p such that p+2 and p+12 are primes. By primes, we mean positive prime numbers. The list starts like this :
5,11,17, and 29. Only the smallest and first entry in this 3-tuple is in the list. Note that {5,7,17} is a 3-tuple that fits this pattern. Also not that 7 and 17 are not consecutive prime numbers. 9,11, and 13 are between 7 and 17. This is okay. I just want to point out that some k-tuples have non-consecutive prime numbers.

OEIS (Online Encyclopedia of Integer Sequences) is a database of many number sequences. Integers are the same as whole numbers. They are not fractions or decimals. Not all integers are positive integers. We also have negative numbers and zero.

Enumeration Counts is length of recorded number list in OEIS as of January 2023

This list below contains OEIS ‘a’ serial number, difference of prime paris, and count of list length.

A001359 p,p+2 enumeration count 100,000 These are known as twin primes.
A023200 p,p+4 enumeration count 10,000 called cousin primes
A023201 p, p+6 enumeration count 10,000 called sexy primes
A023202 p,p+8 enumeration count 10,000 unofficially called octa primes
A023203 p,p+10 enumeration count 10,000 unofficially called deca primes
A046133 p,p+12 enumeration count 1,000
A153417 p,p+14 enumeration count 1,000
A049488 p,p+16 enumeration count 10,000
A153418 p,p+18 enumeration count 1,000
A153419 p,p+20 enumeration count 10,000
A242476 p,p+22 enumeration count 1,000
A033560 p,p+24 enumeration count 1,000
A252089 p,p+26 enumeration count 52
A252090 p,p+28 enumeration count 10,000

A049481 p,p+30 enumeration count 10,000
A049489 p,p+32 enumeration count 10,000
A252091 p,p+34 enumeration count 10,000
A156104 p,p+36 enumeration count 1,000
A271347 p,p+38 enumeration count 10,000
A271981 p,p+40 enumeration count 10,000
A271982 p,p+42 enumeration count 10,000
A272176 p,p+44 enumeration count 10,000
2-tuple pattern for p,p+46 is not in The Online Encyclopedia of Integer sequences as of March 2023

World's largest twin prime pair as of May 26, 2022

The largest known twin prime pair (prime p such that p+2 is also prime) is

2996863034895 • 21290000 ± 1 and was found in 2016 by Tom Greer.

More prime records can be found online at https://pzktupel.de/ktuplets.php

Norman Luhn maintains this website.

Mathematical Group Theory.

The basics of group theory are usually taught in an Abstract Algebra course. Group theory is not that tough, and we will dive right in. We are mostly concerned with finite groups. Every group has a set of elements, and a group operation. Take, for example the integers, and the operation of addition. This set of elements, and group operation form a mathematical group. This group is closed. This means that for any pair of elements in the group, the result from the group operation is another element of the group. For example 4+11 = 15. And 4, 11, and 15 are all integers. There are more properties for a set and a binary operation to be a mathematical group. By binary operation, I mean a function that has two inputs and one output.

See https://www.mathsisfun.com/sets/groups-introduction.html

The simplest group has just one element. This is called the identity element. It is like the number 1 in regular multiplication. We have 1*1=1 and that is it.

The next group has two elements. You can think about e for any even number, and o for any odd number. And the operation is addition. This forms a mathematical group. We have the sum of any even number with any other even number is even. Similarly, the sum of an even with an odd is odd. Also, the sum of an odd with an even is odd. And finally, the sum of an odd with an odd is even. That is it for this group of two elements. An addition table, or multiplication table, or Cayley table makes it easy to check how the elements interact with each other.

group action	e	o
e	e	o
o	o	e

Cayley table for the group of even and odd numbers under addition

The next topic in group theory is cyclic groups. Any group of prime order is a cyclic group. Consider a group of 3 elements. {0,1,2} are our elements. And our operation is addition modulus 3. So for example 2+2 ≡ 1 mod 3. (mod 3 is the remainder you get when you divide by 3.

+ mod 3	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Cyclic group of 3 elements

Much of this material can be found in a text book on group theory. Often times, an introduction to group theory is given in an Abstract Algebra course.

The order of a group is the same as the number of elements in that group. One of the easiest types of groups are the cyclic groups. Every group of prime order is a cyclic group. For example, the cyclic group with 3 elements of C_3 for short is a cyclic group. And its order is 3. That is, $|C_3|=3$. To say that again, if the order of a group is a prime number, then that group is cyclic. I give a reference to validate the truth of this. See - https://groupprops.subwiki.org/wiki/Group_of_prime_order

The next topic to shed some light on is the Euler totient function. The Euler totient is only defined for non-negative integers. The Euler totient function can be written as phi(n). It counts the number of positive integers less than n and relatively prime to n.

For example, phi(5)=4 because 5 is a prime number and 1,2,3,and 4 are relatively prime to 5.

Similarly, phi(6) = 2 because 2*3=6 so only 1 and 5 are relatively prime to 6.

See oeis.org/A10 for a reference.

Maple Code to find prime k-tuples

Maple is a computer language. It has nice built in functions like isprime() and ithprime().

See this code. It prints the smaller of a prime pair such that p and p+44 are both prime.

Note that 44 mod 6 ≡ 2. That is why the counting starting at 5 and increment by 6 works.

Note also all primes > 3 are equal to ±1 mod 6.

<Maple code>

```
 for a from 5 by 6 to 200 do
if isprime(a) and isprime(a+44)
then print(a)
end if end do;
```

< Maple output>

```
5
11
17
29
41
59
71
101
137
179
```

<end Maple output>

The End of notebook 5-26-2022