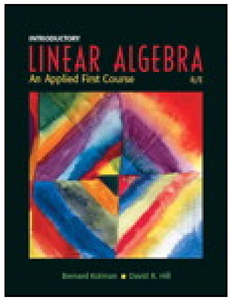
Introductory Linear Algebra: An Applied First Course, 8/E



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Chapter 1

Linear Equations and Matrices

Section 1.1, p. 8

2. $x = 1$, $y = 2$, $z = -2$. 4. No solution.

6. $x = 13 + 10z$, $y = -8 - 8z$, $z =$ any real number.

8. No solution.

10. $x = 2$, $y = -1$.

12. No solution.

14. $x = -1$, $y = 2$, $z = -2$.

16. (c) Yes. (d) Yes.

18. $x = 2$, $y = 1$, $z = 0$.

20. There is no such value of r.

22. Zero, infinitely many, zero.

24. 1.5 tons of regular and 2.5 tons of special plastic.

26. 20 tons of 2-minute developer and a total of 40 tons of 6-minute and 9-minute developer.

28. $7000, $14,000, $3000.

T.1. The same numbers s_j satisfy the system when the pth equation is written in place of the qth equation and vice versa.

T.2. If $s_1, s_2, \ldots, s_n$ is a solution to (2), then the ith equation of (2) is satisfied: $a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i$. Then for any $r \neq 0$, $ra_{i1}s_1 + ra_{i2}s_2 + \cdots + ra_{in}s_n = rb_i$. Hence $s_1, s_2, \ldots, s_n$ is a solution to the new system. Conversely, for any solution $s'_1, s'_2, \ldots, s'_n$ to the new system, $ra_{i1}s'_1 + \cdots + ra_{in}s'_n = rb_i$, and dividing both sides by nonzero r we see that $s'_1, \ldots, s'_n$ must be a solution to the original linear system.

T.3. If $s_1, s_2, \ldots, s_n$ is a solution to (2), then the pth and qth equations are satisfied:

$$\begin{aligned} a_{p1}s_1 + \cdots\ a_{pn} &= b_p \\ a_{q1}s_1 + \cdots\ a_{qn} &= b_q. \end{aligned}$$

Thus, for any real number r,

$$(a_{p1} + ra_{q1})s_1 + \cdots + (a_{pn} + ra_{qn})s_n = b_p + rb_q$$

and so $s_1, \ldots, s_n$ is a solution to the new system. Conversely, any solution to the new system is also a solution to the original system (2).

T.4. Yes; $x = 0$, $y = 0$ is a solution for any values of a, b, c, and d.

Section 1.2, p. 19

2. $a = 3$, $b = 1$, $c = 8$, $d = -2$.

4. (a) $C + E = E + C = \begin{bmatrix} 5 & -5 & 8 \\ 4 & 2 & 9 \\ 5 & 3 & 4 \end{bmatrix}$. (b) Impossible. (c) $\begin{bmatrix} 7 & -7 \\ 0 & 1 \end{bmatrix}$.

   (d) $\begin{bmatrix} -9 & 3 & -9 \\ -12 & -3 & -15 \\ -6 & -3 & -9 \end{bmatrix}$. (e) $\begin{bmatrix} 0 & 10 & -9 \\ 8 & -1 & -2 \\ -5 & -4 & 3 \end{bmatrix}$. (f) Impossible.

6. (a) $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$, $(A^T)^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$. (b) $\begin{bmatrix} 5 & 4 & 5 \\ -5 & 2 & 3 \\ 8 & 9 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} -6 & 10 \\ 11 & 17 \end{bmatrix}$.

   (d) $\begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$. (e) $\begin{bmatrix} 3 & 4 \\ 6 & 3 \\ 9 & 10 \end{bmatrix}$. (f) $\begin{bmatrix} 17 & 2 \\ -16 & 6 \end{bmatrix}$.

8. Yes: $2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

10. $\begin{bmatrix} \lambda - 1 & -2 & -3 \\ -6 & \lambda + 2 & -3 \\ -5 & -2 & \lambda - 4 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (d) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. (e) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

14. $\mathbf{v} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$.

T.1. Let A and B each be diagonal $n \times n$ matrices. Let $C = A + B$, $c_{ij} = a_{ij} + b_{ij}$. For $i \neq j$, a_{ij} and b_{ij} are each 0, so $c_{ij} = 0$. Thus C is diagonal. If $D = A - B$, $d_{ij} = a_{ij} - b_{ij}$, then $d_{ij} = 0$. Therefore D is diagonal.

T.2. Following the notation in the solution of T.1 above, let A and B be scalar matrices, so that $a_{ij} = 0$ and $b_{ij} = 0$ for $i \neq j$, and $a_{ii} = a$, $b_{ii} = b$. If $C = A + B$ and $D = A - B$, then by Exercise T.1, C and D are diagonal matrices. Moreover, $c_{ii} = a_{ii} + b_{ii} = a + b$ and $d_{ii} = a_{ii} - b_{ii} = a - b$, so C and D are scalar matrices.

T.3. (a) $\begin{bmatrix} 0 & b - c & c - e \\ c - b & 0 & 0 \\ e - c & 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 2a & c + b & e + c \\ b + c & 2d & 2e \\ c + e & 2e & 2f \end{bmatrix}$. (c) Same as (b).

T.4. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $C = \begin{bmatrix} c_{ij} \end{bmatrix}$, so $c_{ij} = ka_{ij}$. If $ka_{ij} = 0$, then either $k = 0$ or $a_{ij} = 0$ for all i, j.

T.5. (a) Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be upper triangular matrices, and let $C = A + B$. Then for $i > j$, $c_{ij} = a_{ij} + b_{ij} = 0 + 0 = 0$, and thus C is upper triangular. Similarly, if $D = A - B$, then for $i > j$, $d_{ij} = a_{ij} - b_{ij} = 0 - 0 = 0$, so D is upper triangular.

(b) Proof is similar to that for (a).

(c) Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be both upper and lower triangular. Then $a_{ij} = 0$ for $i > j$ and for $i < j$. Thus, A is a diagonal matrix.

T.6. (a) Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be upper triangular, so that $a_{ij} = 0$ for $i > j$. Since $A^T = \begin{bmatrix} a^T_{ij} \end{bmatrix}$, where $a^T_{ij} = a_{ji}$, we have $a^T_{ij} = 0$ for $j > i$, or $a^T_{ij} = 0$ for $i < j$. Hence A^T is lower triangular.

(b) Proof is similar to that for (a).

T.7. To justify this answer, let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be an $n \times n$ matrix. Then $A^T = \begin{bmatrix} a_{ji} \end{bmatrix}$. Thus, the (i,i)th entry of $A - A^T$ is $a_{ii} - a_{ii} = 0$. Therefore, all entries on the main diagonal of $A - A^T$ are 0.

T.8. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be an n-vector. Then

$$\mathbf{x} + \mathbf{0} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 0 \\ x_2 + 0 \\ \vdots \\ x_n + 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{x}.$$

T.9. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; four.

T.10. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; eight

T.11. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$; sixteen

T.12. Thirty two; 2^n.

T.13. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$
$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$; sixteen

T.14. $2^9 = 512$.

T.15. 2^{n^2}.

T.16. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ so $B = A$ is such that $A + B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

T.17. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$; if $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

T.18. (a) $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$ since $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) Yes; $C + B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(c) Let B be a bit matrix of all 1s. $A + B$ will be a matrix that reverses each state of A.

ML.1. Once you have entered matrices A and B you can use the commands given below to see the items requested in parts (a) and (b).

(a) Commands: **A(2,3)**, **B(3,2)**, **B(1,2)**

(b) For $\text{row}_1(A)$ use command **A(1,:)**
For $\text{col}_3(A)$ use command **A(:,3)**
For $\text{row}_2(B)$ use command **B(2,:)**
(In this context the colon means 'all.')

(c) Matrix B in **format long** is

```
8.00000000000000     0.66666666666667
0.00497512437811    −3.20000000000000
0.00000100000000     4.33333333333333
```

ML.2. (a) Use command **size(H)**

(b) Just type **H**

(c) Type **format rat**, then **H**. (To return to decimal format display type **format**.)

(d) Type **H(:,1:3)**

(e) Type **H(4:5,:)**

Section 1.3, p. 34

2. (a) 4. (b) 0. (c) 1. (d) 1.

4. 1

6. $x = \frac{6}{5}$, $y = \frac{12}{5}$.

8. (a) $\begin{bmatrix} 26 & 42 \\ 34 & 54 \end{bmatrix}$. (b) Same as (a). (c) $\begin{bmatrix} -7 & -12 & 18 \\ 4 & 6 & -8 \end{bmatrix}$.

(d) Same as (c). (e) $\begin{bmatrix} 4 & 8 & -12 \\ -1 & 6 & -7 \end{bmatrix}$.

10. $DI_2 = I_2D = D$.

12. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

14. (a) $\begin{bmatrix} 1 \\ 14 \\ 0 \\ 13 \end{bmatrix}$. (b) $\begin{bmatrix} 0 \\ 18 \\ 3 \\ 13 \end{bmatrix}$.

16. $\text{col}_1(AB) = 1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$; $\text{col}_2(AB) = -1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$.

18. $\begin{bmatrix} -2 & 2 & 3 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

20. $$\begin{aligned} -2x - y + 4w &= 5 \\ -3x + 2y + 7z + 8w &= 3 \\ x + 2w &= 4 \\ 3x + z + 3w &= 6. \end{aligned}$$

22. (a) $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 7 \\ 4 \end{bmatrix}$. (c) $\left[\begin{array}{ccc:c} 3 & -1 & 2 & 4 \\ 2 & 1 & 0 & 2 \\ 0 & 1 & 3 & 7 \\ 4 & 0 & -1 & 4 \end{array}\right]$.

24. (a) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

26. (a) Can say nothing. (b) Can say nothing.

28. There are infinitely many choices. For example, $r = 1$, $s = 0$; or $r = 0$, $s = 2$; or $r = 10$, $s = -18$.

30. $A = \left[\begin{array}{c:c:c} 2\times 2 & 2\times 2 & 2\times 1 \\ \hdashline 2\times 2 & 2\times 2 & 2\times 1 \\ \hdashline 2\times 2 & 2\times 2 & 2\times 1 \end{array}\right]$ and $B = \left[\begin{array}{c:c} 2\times 2 & 2\times 3 \\ \hdashline 2\times 2 & 2\times 3 \\ \hdashline 1\times 2 & 1\times 3 \end{array}\right]$.

$A = \left[\begin{array}{c:c} 3\times 3 & 3\times 2 \\ \hdashline 3\times 3 & 3\times 2 \end{array}\right]$ and $B = \left[\begin{array}{c:c} 3\times 3 & 3\times 2 \\ \hdashline 2\times 3 & 2\times 2 \end{array}\right]$.

$$AB = \begin{bmatrix} 21 & 48 & 41 & 48 & 40 \\ 18 & 26 & 34 & 33 & 5 \\ 24 & 26 & 42 & 47 & 16 \\ 28 & 38 & 54 & 70 & 35 \\ 33 & 33 & 56 & 74 & 42 \\ 34 & 37 & 58 & 79 & 54 \end{bmatrix}$$

32. For each product P or Q, the daily cost of pollution control at plant X or at plant Y.

34. (a) \$103,400. (b) \$16,050.

36. (a) 1. (b) 0.

38. $x = 0$ or $x = 1$.

40. $AB = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

T.1. (a) No. If $\mathbf{x} = (x_1, x_2, \dots, x_n)$, then $\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$.
(b) $\mathbf{x} = \mathbf{0}$.

T.2. Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$, $\mathbf{b} = (b_1, b_2, \dots, b_n)$, and $\mathbf{c} = (c_1, c_2, \dots, c_n)$. Then

(a) $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i$ and $\mathbf{b} \cdot \mathbf{a} = \sum_{i=1}^{n} b_i a_i$, so $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(b) $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \sum_{i=1}^{n}(a_i+b_i)c_i = \sum_{i=1}^{n} a_i c_i + \sum_{i=1}^{n} b_i c_i = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}$.

(c) $(k\mathbf{a})\cdot\mathbf{b} = \sum_{i=1}^{n}(ka_i)b_i = k\sum_{i=1}^{n} a_i b_i = k(\mathbf{a}\cdot\mathbf{b})$.

T.3. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be $m \times p$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be $p \times n$.

(a) Let the ith row of A consist entirely of zeros, so $a_{ik} = 0$ for $k = 1, 2, \ldots, p$. Then the (i, j) entry in AB is

$$\sum_{k=1}^{p} a_{ik} b_{kj} = 0 \qquad \text{for } j = 1, 2, \ldots, n.$$

(b) Let the jth column of B consist entirely of zeros, so $b_{kj} = 0$ for $k = 1, 2, \ldots, p$. Then again the (i, j) entry of AB is 0 for $i = 1, 2, \ldots, m$.

T.4. Let A and B be diagonal matrices, so $a_{ij} = 0$ and $b_{ij} = 0$ for $i \neq j$. Let $C = AB$. Then, if $C = \begin{bmatrix} c_{ij} \end{bmatrix}$, we have

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{1.1}$$

For $i \neq j$ and any value of k, either $k \neq i$ and so $a_{ik} = 0$, or $k \neq j$ and so $b_{kj} = 0$. Thus each term in the summation (1.1) equals 0, and so also $c_{ij} = 0$. This holds for every i, j such that $i \neq j$, so C is a diagonal matrix.

T.5. Let A and B be scalar matrices, so that $a_{ij} = a$ and $b_{ij} = b$ for all $i = j$. If $C = AB$, then by Exercise T.4, $c_{ij} = 0$ for $i \neq j$, and $c_{ii} = a \cdot b = a_{ii} \cdot b_{ii} = c$, so C is a scalar matrix.

T.6. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ be upper triangular matrices.

(a) Let $C = AB$, $c_{ij} = \sum a_{ik} b_{kj}$. If $i > j$, then for each k, either $k > j$ (and so $b_{kj} = 0$), or else $k \leq j < i$ (and so $a_{ik} = 0$). Thus $c_{ij} = 0$.

(b) Proof similar to that for (a).

T.7. Yes. If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ are diagonal matrices, then $C = \begin{bmatrix} c_{ij} \end{bmatrix}$ is diagonal by Exercise T.4. Moreover, $c_{ii} = a_{ii} b_{ii}$. Similarly, if $D = BA$, then $d_{ii} = b_{ii} a_{ii}$. Thus, $C = D$.

T.8. (a) Let $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$. Then

$$\begin{aligned}\mathbf{a}B &= [\ a_1 b_{11} + a_2 b_{21} + \cdots + a_n b_{n1} \quad a_1 b_{12} + a_2 b_{22} + \cdots + a_n b_{n2} \quad \cdots \\ &\qquad\qquad a_1 b_{1p} + a_2 b_{2p} + \cdots + a_n b_{np}\ ] \\ &= a_1 \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \end{bmatrix} + a_2 \begin{bmatrix} b_{21} & b_{22} & \cdots & b_{2p} \end{bmatrix} + \cdots + a_n \begin{bmatrix} b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}.\end{aligned}$$

(b) $1\begin{bmatrix} 2 & 1 & -4 \end{bmatrix} - 2\begin{bmatrix} -3 & -2 & 3 \end{bmatrix} + 3\begin{bmatrix} 4 & 5 & -2 \end{bmatrix}$.

T.9. (a) The jth column of AB is

$$\begin{bmatrix} \sum_k a_{1k} b_{kj} \\ \sum_k a_{2k} b_{kj} \\ \vdots \\ \sum_k a_{mk} b_{kj} \end{bmatrix}.$$

(b) The ith row of AB is

$$\left[\sum_k a_{ik}b_{k1} \quad \sum_k a_{ik}b_{k2} \quad \cdots \quad \sum_k a_{ik}b_{kn}\right].$$

T.10. The i, ith element of the matrix AA^T is

$$\sum_{k=1}^{n} a_{ik}a_{ki}^T = \sum_{k=1}^{n} a_{ik}a_{ik} = \sum_{k=1}^{n}(a_{ik})^2.$$

Thus if $AA^T = O$, then each sum of squares $\sum_{k=1}^{n}(a_{ik})^2$ equals zero, which implies $a_{ik} = 0$ for each i and k. Thus $A = O$.

T.11. (a)
$$\begin{aligned}\sum_{i=1}^{n}(r_i + s_i)a_i &= (r_1 + s_1)a_1 + (r_2 + s_2)a_2 + \cdots + (r_n + s_n)a_n \\ &= r_1a_1 + s_1a_1 + r_2a_2 + s_2a_2 + \cdots + r_na_n + s_na_n \\ &= (r_1a_1 + r_2a_2 + \cdots + r_na_n) + (s_1a_1 + s_2a_2 + \cdots + s_na_n) = \sum_{i=1}^{n} r_ia_i + \sum_{i=1}^{n} s_ia_i\end{aligned}$$

(b) $\sum_{i=1}^{n} c(r_ia_i) = cr_1a_1 + cr_2a_2 + \cdots + cr_na_n = c(r_1a_1 + r_2a_2 + \cdots + r_na_n) = c\sum_{i=1}^{n} r_ia_i.$

T.12.
$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} &= (a_{11} + a_{12} + \cdots + a_{1m}) + (a_{21} + a_{22} + \cdots + a_{2m}) + \cdots + (a_{n1} + a_{n2} + \cdots + a_{nm}) \\ &= (a_{11} + a_{21} + \cdots + a_{n1}) + (a_{12} + a_{22} + \cdots + a_{n2}) + \cdots + (a_{1m} + a_{2m} + \cdots + a_{nm}) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}.\end{aligned}$$

T.13. (a) True. $\sum_{i=1}^{n}(a_i + 1) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} 1 = \sum_{i=1}^{n} a_i + n.$

(b) True. $\sum_{i=1}^{n}\sum_{j=1}^{m} 1 = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} 1\right) = \sum_{i=1}^{n} m = mn.$

(c) True.
$$\begin{aligned}\left[\sum_{i=1}^{n} a_i\right]\left[\sum_{j=1}^{m} b_j\right] &= a_1\sum_{j=1}^{m} b_j + a_2\sum_{j=1}^{m} b_j + \cdots + a_n\sum_{j=1}^{m} b_j \\ &= (a_1 + a_2 + \cdots + a_n)\sum_{j=1}^{m} b_j \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} b_j = \sum_{j=1}^{m}\sum_{i=1}^{n} a_ib_j\end{aligned}$$

T.14. (a) If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_iv_i = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{u}^T\mathbf{v}.$$

(b) If $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{u}\mathbf{v}^T.$$

(c) If $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, then

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i = \mathbf{u}\mathbf{v}.$$

ML1.1 (a) $\mathbf{A} * \mathbf{C}$

```
ans =
     4.5000    2.2500    3.7500
     1.5833    0.9167    1.5000
     0.9667    0.5833    0.9500
```

(b) $\mathbf{A} * \mathbf{B}$

```
???  Error using ===> *
Inner matrix dimensions must agree.
```

(c) $\mathbf{A} + \mathbf{C}'$

```
ans =
     5.0000    1.5000
     1.5833    2.2500
     2.4500    3.1667
```

(d) $\mathbf{B} * \mathbf{A} - \mathbf{C}' * \mathbf{A}$

```
???  Error using ===> *
Inner matrix dimensions must agree.
```

(e) $(\mathbf{2} * \mathbf{C} - \mathbf{6} * \mathbf{A}') * \mathbf{B}'$

```
???  Error using ===> *
Inner matrix dimensions must agree.
```

(f) $\mathbf{A} * \mathbf{C} - \mathbf{C} * \mathbf{A}$

```
???  Error using ===> -
Inner matrix dimensions must agree.
```

(g) $\mathbf{A} * \mathbf{A}' + \mathbf{A}' * \mathbf{C}$

```
ans =
     18.2500    7.4583   12.2833
      7.4583    5.7361    8.9208
     12.2833    8.9208   14.1303
```

ML.2. aug =

```
     2     4     6   -12
     2    -3    -4    15
     3     4     5    -8
```

ML.3. `aug =`

```
 4  -3   2  -1  -5
 2   1  -3   0   7
-1   4   1   2   8
```

ML.4. (a) **R = A(2,:)**

```
R =
    3   2   4
```

C = B(:,3)

```
C =
    -1
    -3
     5
```

V = R ∗ C

```
V =
    11
```

V is the (2,3)-entry of the product **A ∗ B**.

(b) **C = B(:,2)**

```
C =
    0
    3
    2
```

V = A ∗ C

```
V =
     1
    14
     0
    13
```

V is column 2 of the product **A ∗ B**.

(c) **R = A(3,:)**

```
R =
    4  -2   3
```

V = R ∗ B

```
V =
    10   0  17   3
```

V is row 3 of the product **A ∗ B**.

ML.5. (a) **diag([1 2 3 4])**

```
ans =
    1  0  0  0
    0  2  0  0
    0  0  3  0
    0  0  0  4
```

(b) **diag([0 1 1/2 1/3 1/4])**

```
ans =
     0         0         0         0         0
     0    1.0000         0         0         0
     0         0    0.5000         0         0
     0         0         0    0.3333         0
     0         0         0         0    0.2500
```

(c) **diag([5 5 5 5 5 5])**

```
ans =
     5  0  0  0  0  0
     0  5  0  0  0  0
     0  0  5  0  0  0
     0  0  0  5  0  0
     0  0  0  0  5  0
     0  0  0  0  0  5
```

ML.6. (a) (i) **dot(v,w)** $= 15$. (ii) **dot(v,w)** $= 0$. (b) $k = -\frac{2}{3}$.

(b) (i) **dot(v,v)** $= 29$. (ii) **dot(v,v)** $= 127$. (iii) **dot(v,v)** $= 39$.

The sign of each of these dot products is positive since it is a sum of squares. This is not true for the zero vector.

ML.8. 0.

ML.9. (a) **bingen(0,7,3)** $= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$.

(b) $AB = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.

(c) The columns of B which contain an odd number of 1s are dotted with a vector of all 1s (a row of A) hence the result is 1.

ML.10. Here

$$AB = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The columns of B which contain an odd number of 1s are dotted with a vector of all 1s (a row of A) hence the result is 1.

ML.11.

$n = 2 \quad BB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad n = 4, \quad BB = O$

$n = 3, \quad BB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad n = 5, \quad BB =$ matrix of all 1s.

$$BB = \begin{cases} \text{zero matrix} & \text{if } n \text{ is even} \\ \text{matrix of 1s} & \text{if } n \text{ is odd.} \end{cases}$$

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2. $A(BC) = \begin{bmatrix} -2 & 34 \\ 24 & -9 \end{bmatrix}$.

4. $r(sA) = \begin{bmatrix} -48 & -24 \\ -12 & 36 \end{bmatrix}$, $(r+s)A = \begin{bmatrix} 16 & 8 \\ 4 & -12 \end{bmatrix}$, $r(A+B) = \begin{bmatrix} 24 & 24 \\ -18 & 0 \end{bmatrix}$.

6. $(A+B)^T = \begin{bmatrix} 5 & 0 \\ 5 & 2 \\ 1 & 2 \end{bmatrix}$, $(rA)^T = \begin{bmatrix} -4 & -8 \\ -12 & -4 \\ -8 & 12 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 5 & 17 \\ 6 & -5 \end{bmatrix}$. (b) Same as (a).

   (c) $\begin{bmatrix} 1 & 18 & -4 \\ 0 & 11 & -3 \\ -9 & 14 & -12 \end{bmatrix}$. (d) $\begin{bmatrix} 5 & 2 & 4 \\ 2 & 25 & -5 \\ 4 & -5 & 5 \end{bmatrix}$. (e) $\begin{bmatrix} 14 & 8 \\ 8 & 21 \end{bmatrix}$.

14. (a) $\begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} -24 & -30 \\ 60 & 36 \end{bmatrix}$.

16. $k = \pm\sqrt{\frac{1}{6}}$

18. (a) $\begin{bmatrix} \frac{16}{45} \\ \frac{29}{45} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{3}{8} \\ \frac{5}{8} \end{bmatrix}$.

20. (a) After one year: $\begin{bmatrix} \frac{13}{36} \\ \frac{17}{36} \\ \frac{1}{6} \end{bmatrix} \approx \begin{bmatrix} 0.3611 \\ 0.4722 \\ 0.1667 \end{bmatrix}$.

    After 2 years: $\begin{bmatrix} \frac{43}{108} \\ \frac{191}{432} \\ \frac{23}{144} \end{bmatrix} \approx \begin{bmatrix} 0.3981 \\ 0.4421 \\ 0.1597 \end{bmatrix}$.

    (c) S. It will gain approximately 11.95% of the market.

24. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

T.1. (b) The (i,j) entry of $A+(B+C)$ is $a_{ij}+(b_{ij}+c_{ij})$, that of $(A+B)+C$ is $(a_{ij}+b_{ij})+c_{ij}$. These two entries are equal because of the associative law for addition of real numbers.

(d) For each (i,j) let $d_{ij} = -a_{ij}$, $D = \begin{bmatrix} d_{ij} \end{bmatrix}$. Then $A+D = D+A = O$.

T.2. $\displaystyle\sum_{p=1}^{3} a_{ip}\left(\sum_{q=1}^{4} b_{pq}c_{qj}\right) = \sum_{p=1}^{3}\sum_{q=1}^{4} a_{ip}b_{pq}c_{qj} = \sum_{q=1}^{4}\sum_{p=1}^{3} a_{ip}b_{pq}c_{qj} = \sum_{q=1}^{4}\left(\sum_{p=1}^{3} a_{ip}b_{pq}\right)c_{qj}$.

T.3. (b) $\displaystyle\sum_{k=1}^{p} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{p}(a_{ik}b_{kj}+a_{ik}c_{kj}) = \sum_{k=1}^{p} a_{ik}b_{kj} + \sum_{k=1}^{p} a_{ik}c_{kj}$

(c) $\displaystyle\sum_{k=1}^{p}(a_{ik}+b_{ik})c_{kj} = \sum_{k=1}^{p}(a_{ik}c_{kj}+b_{ik}c_{kj}) = \sum_{k=1}^{p} a_{ik}c_{kj} + \sum_{k=1}^{p} b_{ik}c_{kj}$.

T.4. Denote the entries of the identity matrix by d_{ij}, so that

$$d_{ij} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{if } i \neq j. \end{cases}$$

Then for $C = AI_n$, $c_{ij} = \sum_{k=1}^{p} a_{ik} d_{kj} = a_{ij} d_{jj}$(all other d_{kj} are zero) $= a_{ij}$, and thus $C = A$. A similar argument shows that $I_m A = A$.

T.5. $A^p A^q = \underbrace{(A \cdot A \cdots A)}_{p \text{ factors}} \cdot \underbrace{(A \cdot A \cdots A)}_{q \text{ factors}} = A^{p+q}$, $\quad (A^p)^q = \underbrace{A^p \cdot A^p \cdot A^p \cdots A^p}_{q \text{ factors}} = A^{\overbrace{p + p + \cdots + p}^{q \text{ summands}}} = A^{pq}$.

T.6. We are given that $AB = BA$. For $p = 0$, $(AB)^0 = I_n = A^0 B^0$; for $p = 1$, $(AB)^1 = AB = A^1 B^1$; and for $p = 2$, $(AB)(AB) = A(BA)B = A(AB)B = A^2 B^2$. Now assume that for $p = k$, $(AB) = A^k B^k$. Then

$$(AB)^{k+1} = (AB)^k (AB) = A^k B^k \cdot A \cdot B = A^k (B^{k-1} AB) B = A^k (B^{k-2} AB^2) B = \cdots = A^{k+1} B^{k+1}.$$

Thus the result is true also for $p = k + 1$. Hence it is true for all positive integers p.

T.7. From Exercise T.2 in Section 1.2 we know that the product of two diagonal matrices is a diagonal matrix. Let $A = [a_{ij}]$, $B = [b_{ij}]$, $AB = C = [c_{ij}]$ and $BA = D = [d_{ij}]$. Then

$$c_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki} = a_{ii} b_{ii}\,; \qquad d_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki} = b_{ii} a_{ii}$$

so $c_{ii} = d_{ii}$ for $i = 1, 2, \ldots, n$. Hence, $C = D$.

T.8. $B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is such that $AB = BA$. There are infinitely many such matrices B.

T.9. Possible answers: $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Infinitely many.

T.10. (a) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$. (b) $\begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$. (c) $\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$.

(d) The result is true for $p = 2$ and 3 as shown in parts (a) and (b). Assume that it is true for $p = k$. Then

$$\begin{aligned} A^{k+1} = A^k A &= \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos k\theta \cos \theta - \sin k\theta \sin \theta & \cos k\theta \sin \theta + \sin k\theta \cos \theta \\ -\sin k\theta \cos \theta - \cos k\theta \sin \theta & \cos k\theta \cos \theta - \sin k\theta \sin \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}. \end{aligned}$$

Hence, it is true for all positive integers k.

T.11. For $p = 0$, $(cA)^0 = I_n = 1 \cdot I_n = c^0 \cdot A^0$. For $p = 1$, $cA = cA$. Assume the result true for $p = k$: $(cA)^k = c^k A^k$. Then for $p = k + 1$, we have

$$(cA)^{k+1} = (cA)^k (cA) = c^k A^k \cdot cA = c^k (A^k c) A = c^k (cA^k) A = (c^k c)(A^k A) = c^{k+1} A^{k+1}.$$

Therefore the result is true for all positive integers p.

T.12. (a) For $A = \left[a_{ij}\right]$, the (i, j) element of $r(sA)$ is $r(sa_{ij})$, that of $(rs)A$ is $(rs)a_{ij}$, and these are equal by the associative law for multiplication of real numbers.

(b) The (i, j) element of $(r+s)A$ is $(r+s)a_{ij}$, that of $rA + sA$ is $ra_{ij} + sa_{ij}$, and these are equal by the distributive law of real numbers.

(c) $r(a_{ij} + b_{ij}) = ra_{ij} + rb_{ij}$.

(d) $\displaystyle\sum_{k=1}^{p} a_{ik}(rb_{kj}) = r\sum_{k=1}^{p} a_{ik}b_{kj} = \sum_{k=1}^{p}(ra_{ik})b_{kj}$.

T.13. $(-1)a_{ij} = -a_{ij}$ (see Exercise T.1.).

T.14. (a) The i, jth element of $(A^T)^T$ is the j, ith element of A^T, which is the i, jth element of A.

(b) The i, jth element of $(A+B)^T$ is c_{ji}, where $c_{ij} = a_{ij} + b_{ij}$. Thus $c_{ji} = a_{ji} + b_{ji}$. Hence $(A+B)^T = A^T + B^T$.

(c) The i, jth element of $(rA)^T$ is the j, ith element of rA, that is, ra_{ji}. Thus $(rA)^T = rA^T$.

T.15. Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$. Then $A - B = \left[c_{ij}\right]$, where $c_{ij} = a_{ij} - b_{ij}$. Then $(A-B)^T = \left[c_{ij}^T\right]$, so

$$c_{ij}^T = c_{ji} = a_{ji} - b_{ji} = a_{ij}^T - b_{ij}^T = \text{the } i,\ j\text{th entry in } A^T - B^T.$$

T.16. (a) We have $A^2 = AA$, so $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$.

(b) From part (a), $(A^3)^T = (A^2A)^T = A^T(A^2)^T = A^T(A^T)^2 = (A^T)^3$.

(c) The result is true for $p = 2$ and 3 as shown in parts (a) and (b). Assume that it is true for $p = k$. Then

$$(A^{k+1})^T = (AA^k)^T = (A^k)^T A^T = (A^T)^k A^T = (A^T)^{k+1}.$$

Hence, it is true for $k = 4, 5, \ldots$.

T.17. If A is symmetric, then $A^T = A$. Thus $a_{ji} = a_{ij}$ for all i and j. Conversely, if $a_{ji} = a_{ij}$ for all i and j, then $A^T = A$ and A is symmetric.

T.18. Both "A is symmetric" and "A^T is symmetric" are logically equivalent to "$a_{ji} = a_{ij}$ for all i and j."

T.19. If $A\mathbf{x} = \mathbf{0}$ for all $n \times 1$ matrices $\mathbf{x}$, then $A\mathbf{e}_j = \mathbf{0}$, $j = 1, 2, \ldots, n$, where $\mathbf{e}_j =$ column j of I_n. But then

$$A\mathbf{e}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{0}.$$

Hence column j of A is equal to $\mathbf{0}$ for each j and it follows that $A = O$.

T.20. If $A\mathbf{x} = \mathbf{x}$ for all $n \times 1$ matrices $\mathbf{x}$, then $A\mathbf{e}_j = \mathbf{e}_j$, where $\mathbf{e}_j =$ column j of I_n. Since

$$A\mathbf{e}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{e}_j,$$

it follows that $a_{ij} = 1$ if $i = j$ and 0 otherwise. Hence $A = I_n$.

T.21. Given that $AA^T = O$, we have that each entry of AA^T is zero. In particular then, each diagonal entry of AA^T is zero. Hence

$$0 = \text{row}_i(A) \cdot \text{col}_i(A^T) = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = \sum_{j=1}^{n} (a_{ij})^2.$$

(Recall that $\text{col}_i(A^T)$ is $\text{row}_i(A)$ written in column form.) A sum of squares is zero only if each member of the sum is zero, hence $a_{i1} = a_{i2} = \cdots = a_{in} = 0$, which means that $\text{row}_i(A)$ consists of all zeros. The previous argument holds for each diagonal entry, hence each row of A contains all zeros. Thus it follows that $A = O$.

T.22. Suppose that A is a symmetric matrix. By Exercise T.16(c) we have $(A^k)^T = (A^T)^k = A^k$ so A^k is symmetric for $k = 2, 3, \ldots$.

T.23. (a) $(A + B)^T = A^T + B^T = A + B$, so $A + B$ is symmetric.

(b) Suppose that AB is symmetric. Then

$$\begin{aligned} (AB)^T &= AB \\ B^T A^T &= AB \qquad \text{[Thm. 1.4(c)]} \\ BA &= AB \qquad \text{(A and B are each symmetric)} \end{aligned}$$

Thus A and B commute. Conversely, if A and B commute, then $(AB)^T = AB$ and AB is symmetric.

T.24. Suppose A is skew symmetric. Then the j, ith element of A equals $-a_{ij}$. That is, $a_{ij} = -a_{ji}$.

T.25. Let $A = rI_n$. Then $(rI_n)^T = -rI_n$ so $rI_n = -rI_n$. Hence, $r = -r$, which implies that $r = 0$. That is, $A = O$.

T.26.

$$\begin{aligned} (AA^T)^T &= (A^T)^T A^T \qquad \text{[Thm. 1.4(c)]} \\ &= AA^T \qquad \text{[Thm. 1.4(a)]} \end{aligned}$$

Thus AA^T is symmetric. A similar argument applies to $A^T A$.

T.27. (a) $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$.

(b) $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

T.28. Let

$$S = \tfrac{1}{2}(A + A^T) \quad \text{and} \quad K = \tfrac{1}{2}(A - A^T).$$

Then S is symmetric and K is skew symmetric, by Exercise T.15, and $S+K = \frac{1}{2}(A+A^T+A-A^T) = \frac{1}{2}(2A) = A$. Conversely, suppose $A = S + K$ is any decomposition of A into the sum of a symmetric and skew symmetric matrix. Then

$$\begin{aligned} A^T &= (S + K)^T = S^T + K^T = S - K, \\ A + A^T &= (S + K) + (S - K) = 2S, \qquad \Longrightarrow \quad S = \tfrac{1}{2}(A + A^T), \\ A - A^T &= (S + K) - (S - K) = 2K, \qquad \Longrightarrow \quad K = \tfrac{1}{2}(A - A^T). \end{aligned}$$

T.29. If the diagonal entries of A are r, then since $r = r \cdot 1$, $A = rI_n$.

T.30. $I_n = \begin{bmatrix} d_{ij} \end{bmatrix}$, where $d_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$ Then $d_{ji} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$ Thus $I_n^T = I$.

T.31. Suppose $r \neq 0$. The i, jth entry of rA is ra_{ij}. Since $r \neq 0$, $a_{ij} = 0$ for all i and j. Thus $A = O$.

T.32. Let $\mathbf{u}$ and $\mathbf{v}$ be two (distinct) solutions to the linear system $A\mathbf{x} = \mathbf{b}$, and consider $\mathbf{w} = r\mathbf{u} + s\mathbf{v}$ for $r + s = 1$. Then $\mathbf{w}$ is a solution to the system since

$$A\mathbf{w} = A(r\mathbf{u} + s\mathbf{v}) = r(A\mathbf{u}) + s(A\mathbf{v}) = r\mathbf{b} + s\mathbf{b} = (r + s)\mathbf{b} = \mathbf{b}.$$

If $\mathbf{b} = \mathbf{0}$, then at least one of $\mathbf{u}$, $\mathbf{v}$ must be nonzero, say $\mathbf{u}$, and then the infinitely many matrices $r\mathbf{u}$, r a real number, constitute solutions.

If $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ cannot be nontrivial multiples of each other. (If $\mathbf{v} = t\mathbf{u}$, $t \neq 1$, then $A\mathbf{v} = t\mathbf{b} \neq \mathbf{b} = A\mathbf{u}$, a contradiction.) Thus if $r\mathbf{u} + s\mathbf{v} = r'\mathbf{u} + s'\mathbf{v}$ for some r, s, r', s', then

$$(r - r')\mathbf{u} = (s' - s)\mathbf{v},$$

whence $r = r'$ and $s = s'$. Therefore the matrices $\mathbf{w} = r\mathbf{u} + s\mathbf{v}$ are distinct as r ranges over real numbers and $s = 1 - r$.

T.33. Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies $AB = BA$ for any 2×2 matrix B. Choosing $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

which implies $b = c = 0$. Thus $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is diagonal. Next choosing $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ we get

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix},$$

or $a = d$. Thus $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is a scalar matrix.

T.34. Skew symmetric. To show this, let A be a skew symmetric matrix. Then $A^T = -A$. Therefore $(A^T)^T = A = -A^T$. Hence A^T is skew symmetric.

T.35. A symmetric matrix. To show this, let $A_1, \ldots, A_n$ be symmetric matrices and let $c_1, \ldots, c_n$ be scalars. Then $A_1^T = A_1, \ldots, A_n^T = A_n$. Therefore

$$\begin{aligned}(c_1A_1 + \cdots + c_nA_n)^T &= (c_1A_1)^T + \cdots + (c_nA_n)^T \\ &= c_1A_1^T + \cdots + c_nA_n^T \\ &= c_1A_1 + \cdots + c_nA_n.\end{aligned}$$

Hence the linear combination $c_1A_1 + \cdots + c_nA_n$ is symmetric.

T.36. A scalar matrix. To show this, let $A_1, \ldots, A_n$ be scalar matrices and let $r_1, \ldots, r_n$ be scalars. Then $A_i = c_iI_n$ for scalars $c_1, \ldots, c_n$. Therefore

$$r_1A_1 + \cdots + r_nA_n = r_1(c_1I_1) + \cdots + r_n(c_nI_n) = (r_1c_1 + \cdots + r_nc_n)I_n$$

which is the scalar matrix whose diagonal entries are all equal to $r_1c_1 + \cdots + r_nc_n$.

T.37. Let $A = \left[a_{ij}\right]$, $B = \left[b_{ij}\right]$, and $C = AB$. Then

$$c_{ij} = \sum_{\substack{k=1 \\ i \neq k}}^{p} a_{ik}b_{kj} + a_{ii}b_{ij} = rb_{ij}.$$

T.38. For any $m \times n$ bit matrix $A + A = \left[\text{ent}_{ij}(A) + \text{ent}_{ij}(A)\right]$. Since $\text{ent}_{ij}(A) = 0$ or 1 and $0 + 0 = 0$, $1 + 1 = 0$, we have $A + A = O$; hence $-A = A$.

T.39. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a bit matrix. Then

$$A^2 = O \quad \text{provided} \quad \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $(a + d)b = 0$ and $(a + d)c = 0$.

Case $b = 0$. Then $a^2 = 0 \Longrightarrow a = 0$ and $d^2 = 0 \Longrightarrow d = 0$. Hence $c = 0$ or 1. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Case $c = 0$. Then $a^2 = 0 \Longrightarrow a = 0$ and $d^2 = 0 \Longrightarrow d = 0$. Hence $b = 0$ or 1. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Case $a + d = 0$

(i) $a = d = 0 \Longrightarrow bc = 0$ so $b = 0$, $c = 0$ or 1, or $b = 1$, $c = 0$. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(ii) $a = d = 1 \Longrightarrow bc + 1 = 0 \Longrightarrow bc = 1 \Longrightarrow b = c = 1$. So

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

T.40. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a bit matrix. Then

$$A^2 = I_2 \quad \text{provided} \quad \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $(a + d)b = 0$ and $(a + d)c = 0$.

Case $b = 0 \Longrightarrow a^2 = 0 \Longrightarrow a = 0$ and $d^2 = 0 \Longrightarrow d = 0$. Hence $c = 0$ or 1. But we also must have $bc = 1 \Longrightarrow c = 1$ and $b = 1$. So

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Case $c = 0 \Longrightarrow a^2 = 0 \Longrightarrow a = 0$ and $d^2 = 0 \Longrightarrow d = 0$. Hence $b = 0$ or 1. But we also must have $bc = 1 \Longrightarrow b = 1$ and $c = 1$. So

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Case $a + d = 0$

(i) $a = d = 0 \Longrightarrow bc = 1 \Longrightarrow b = c = 1$. So

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(ii) $a = d = 1 \Longrightarrow bc + 1 = 1 \Longrightarrow bc = 0$. Thus if $b = 0$, $c = 0$ or 1 and if $c = 0$, $b = 0$ or 1. So

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

ML.1. (a) **A^2**

```
ans =
     0   1   0
     0   0   1
     1   0   0
```

A^3

```
ans =
     1   0   0
     0   1   0
     0   0   1
```

Thus $k = 3$.

(b) **A^2**

```
ans =
    -1    0   0   0
     0   -1   0   0
     0    0   1   0
     0    0   0   1
```

A^3

```
ans =
     0   -1   0   0
     1    0   0   0
     0    0   0   1
     0    0   1   0
```

A^4

```
ans =
     1   0   0   0
     0   1   0   0
     0   0   1   0
     0   0   0   1
```

Thus $k = 4$.

ML.2. (a) **A = tril(ones(5), − 1)**

A

```
ans =
     0   0   0   0   0
     1   0   0   0   0
     1   1   0   0   0
     1   1   1   0   0
     1   1   1   1   0
```

A^2

```
ans =
     0   0   0   0   0
     0   0   0   0   0
     1   0   0   0   0
     2   1   0   0   0
     3   2   1   0   0
```

A^3

```
ans =
     0   0   0   0   0
     0   0   0   0   0
     0   0   0   0   0
     1   0   0   0   0
     3   1   0   0   0
```

A^4

```
ans =
     0   0   0   0   0
     0   0   0   0   0
     0   0   0   0   0
     0   0   0   0   0
     1   0   0   0   0
```

A^5

```
ans =
     0   0   0   0   0
     0   0   0   0   0
     0   0   0   0   0
     0   0   0   0   0
     0   0   0   0   0
```

Thus $k = 5$.

(b) This exercise uses the random number generator **rand**. The matrix A and the value of k may vary.

A = tril(fix(10 ∗ rand(7)),2)

```
A =
   0   0   0   0   0   2   8
   0   0   0   6   7   9   2
   0   0   0   0   3   7   4
   0   0   0   0   0   7   7
   0   0   0   0   0   0   4
   0   0   0   0   0   0   0
   0   0   0   0   0   0   0
```

Here **A^3** is all zeros, so $k = 5$.

ML.3. (a) Define the vector of coefficients

v = [1 − 1 1 0 2];

then we have

polyvalm(v,A)

```
ans =
     0  -2   4
     4   0  -2
    -2   4   0
```

(b) Define the vector of coefficients

v = [1 − 3 3 0];

then we have

polyvalm(v,A)

```
ans =
    0  0  0
    0  0  0
    0  0  0
```

ML.4. (a) **(A^2 − 7 ∗ A) ∗ (A + 3 ∗ eye(A))**

```
ans =
    -2.8770   -7.1070   -14.0160
    -4.9360   -5.0480   -14.0160
    -6.9090   -7.1070    -9.9840
```

(b) **(A − eye(A))^2 + (A^3 + A)**

```
ans =
    1.3730   0.2430   0.3840
    0.2640   1.3520   0.3840
    0.1410   0.2430   1.6160
```

(c) Computing the powers of A as $\mathbf{A^2}, \mathbf{A^3}, \ldots$ soon gives the impression that the sequence is converging to

```
0.2273   0.2727   0.5000
0.2273   0.2727   0.5000
0.2273   0.2727   0.5000
```

Typing **format rat**, and displaying the preceding matrix gives

```
ans =
    5/22   3/11   1/2
    5/22   3/11   1/2
    5/22   3/11   1/2
```

ML.5. The sequence seems to be converging to

```
1.0000   0.7500
     0        0
```

ML.6. The sequence is converging to the zero matrix.

ML.7. (a) **A′ ∗ A**

`ans =`

2	−3	−1
−3	9	2
−1	2	6

A ∗ A′

`ans =`

6	−1	−3
−1	6	4
−3	4	5

A^TA and AA^T are not equal.

(b) **B = A + A′**

`B =`

2	−3	1
−3	2	4
1	4	2

C = A − A′

`C =`

0	−1	1
1	0	0
−1	0	0

Just observe that $B = B^T$ and that $C^T = -C$.

(c) **B + C**

`ans =`

2	−4	2
−2	2	4
0	4	2

We see that $B + C = 2A$.

ML.8. (a) Use command $B =$ **binrand(3, 3)**. The results will vary.

(b) $B + B = O$, $B + B + B = B$.

(c) If n is even, the result is O; otherwise it is B.

ML.9. $k = 4$.

ML.10. $k = 4$.

ML.11. $k = 8$.

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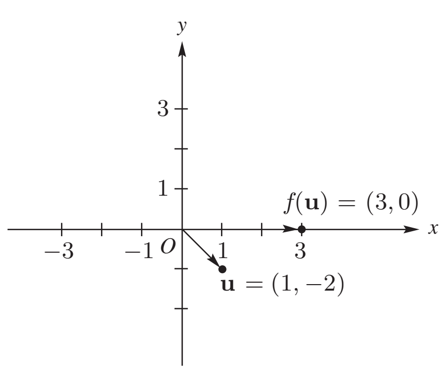
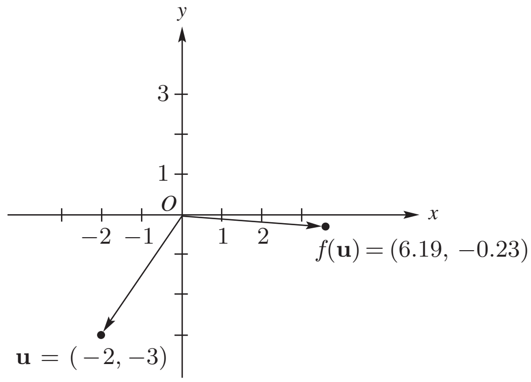
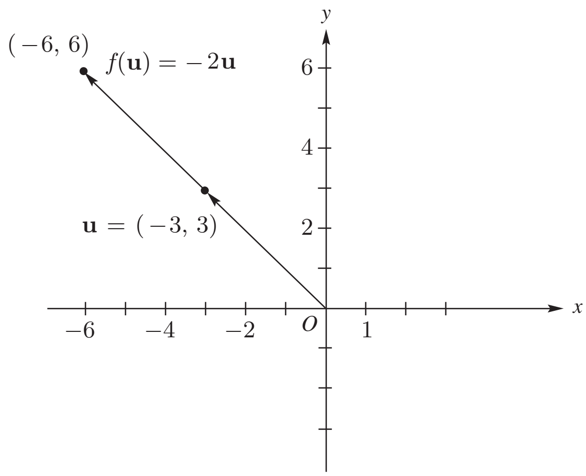
2.



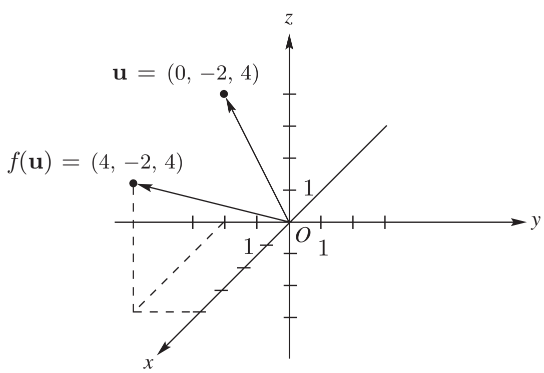
4.



6.



8.



10. Yes.

12. Yes.

14. Yes.

16. (a) Reflection about the line $y = x$.
    (b) Reflection about the line $y = -x$.

18. (a) Possible answers: $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Possible answers: $\begin{bmatrix}0\\4\\4\end{bmatrix}$, $\begin{bmatrix}1\\2\\0\end{bmatrix}$.

T.1. (a) $f(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u}+A\mathbf{v} = f(\mathbf{u})+f(\mathbf{v})$.

(b) $f(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cf(\mathbf{u})$.

(c) $f(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u})+A(c\mathbf{v}) = c(A\mathbf{u})+d(A\mathbf{v}) = cf(\mathbf{u})+df(\mathbf{v})$.

T.2. For any real numbers c and d, we have

$$f(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u}+d\mathbf{v}) = A(c\mathbf{u})+A(d\mathbf{v}) = c(A\mathbf{u})+d(A\mathbf{v}) = cf(\mathbf{u})+df(\mathbf{v}) = c\mathbf{0}+d\mathbf{0} = \mathbf{0}+\mathbf{0} = \mathbf{0}.$$

T.3. (a) $O(\mathbf{u}) = \begin{bmatrix}0 & \cdots & 0\\ & \vdots & \\ 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} = \mathbf{0}$.

(b) $I(\mathbf{u}) = \begin{bmatrix}1 & 0 & \cdots & 0\\ & & \vdots & \\ 0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix} = \mathbf{u}$.

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2. Neither.

4. Neither.

6. Neither.

8. Neither.

10. (a) $\begin{bmatrix}2 & 0 & 4 & 2\\ -1 & 3 & 1 & 1\\ 3 & -2 & 5 & 6\end{bmatrix}$.

(b) $\begin{bmatrix}2 & 0 & 4 & 2\\ -12 & 8 & -20 & -24\\ -1 & 3 & 1 & 1\end{bmatrix}$.

(c) $\begin{bmatrix}0 & 6 & 6 & 4\\ 3 & -2 & 5 & 6\\ -1 & 3 & 1 & 1\end{bmatrix}$.

12. Possible answers:

(a) $\begin{bmatrix}4 & 3 & 7 & 5\\ 2 & 0 & 1 & 4\\ -2 & 4 & -2 & 6\end{bmatrix}$.

(b) $\begin{bmatrix}3 & 5 & 6 & 8\\ -4 & 8 & -4 & 12\\ 2 & 0 & 1 & 4\end{bmatrix}$.

(c) $\begin{bmatrix}4 & 3 & 7 & 5\\ -1 & 2 & -1 & 3\\ 0 & 4 & -1 & 10\end{bmatrix}$.

14. $\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -\frac{2}{7} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

16. $\begin{bmatrix} 1 & -2 & 1 & 4 & -3 \\ 0 & 1 & -\frac{2}{3} & -\frac{7}{3} & \frac{10}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

18. (a) No. (b) Yes. (c) Yes. (d) No.

20. (a) $x = 1$, $y = 2$, $z = -2$. (b) No solution. (c) $x = 1$, $y = 1$, $z = 0$.
(d) $x = 0$, $y = 0$, $z = 0$.

22. (a) $x = 1 - \frac{2}{5}r$, $y = -1 + \frac{1}{5}r$, $z = r$. (b) $x = 1 - r$, $y = 3 + r$, $z = 2 - r$, $w = r$. (c) No solution.
(d) $x = 0$, $y = 0$, $z = 0$.

24. (a) $a = \pm\sqrt{3}$. (b) $a \neq \pm\sqrt{3}$. (c) None.

26. (a) $a = -3$. (b) $a \neq \pm 3$. (c) $a = 3$.

28. (a) $x = r$, $y = -2r$, $z = r$, $r =$ any real number. (b) $x = 1$, $y = 2$, $z = 2$.

30. (a) No solution. (b) $x = 1 - r$, $y = 2 + r$, $z = -1 + r$, $r =$ any real number.

32. $x = -2 + r$, $y = 2 - 2r$, $z = r$, where r is any real number.

34. $c - b - a = 0$

36. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$.

38. $x = 5r$, $y = 6r$, $z = r$, $r =$ any nonzero real number.

40. $-a + b - c = 0$.

42. $\mathbf{x} = \begin{bmatrix} r \\ r \end{bmatrix}$, $r \neq 0$.

44. $\mathbf{x} = \begin{bmatrix} -\frac{1}{2}r \\ \frac{1}{2}r \\ r \end{bmatrix}$, $r \neq 0$.

46. $\mathbf{x} = \begin{bmatrix} \frac{19}{6} \\ -\frac{59}{30} \\ \frac{17}{30} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}r \\ \frac{2}{15}r \\ \frac{19}{15}r \\ r \end{bmatrix}$.

48. $y = \frac{25}{2}x^2 - \frac{61}{2}x + 23$.

50. $y = \frac{2}{3}x^3 + \frac{4}{3}x^2 - \frac{2}{3}x + \frac{2}{3}$.

52. 60 in deluxe binding. If r is the number in bookclub binding, then r is an integer which must satisfy $0 \le r \le 90$ and then the number of paperbacks is $180 - 2r$.

54. $\frac{3}{2}x^2 - x + \frac{1}{2}$.

56. (a) $\begin{bmatrix}1\\0\\1\end{bmatrix}$. (b) $\begin{bmatrix}0\\1\\0\end{bmatrix}$.

58. (a) Inconsistent.

(b) $\begin{bmatrix}0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}$.

T.1. Suppose the leading one of the ith row occurs in the jth column. Since leading ones of rows $i+1, i+2, \ldots$ are to the right of that of the ith row, and in any nonzero row, the leading one is the first nonzero element, all entries in the jth column below the ith row must be zero.

T.2. (a) A is row equivalent to itself: the sequence of operations is the empty sequence.

(b) Each elementary row operation of types (a), (b) or (c) has a corresponding inverse operation of the same type which "undoes" the effect of the original operation. For example, the inverse of the operations "add d times row r of A to row s of A" is "add $-d$ times row r of A to row s of A." Since B is assumed row equivalent to A, there is a sequence of elementary row operations which gets from A to B. Take those operations in the reverse order, and for each operation do its inverse, and that takes B to A. Thus A is row equivalent to B.

(c) Follow the operations which take A to B with those which take B to C.

T.3. The sequence of elementary row operations which takes A to B, when applied to the augmented matrix $\left[A \mid \mathbf{0}\right]$, yields the augmented matrix $\left[B \mid \mathbf{0}\right]$. Thus both systems have the same solutions, by Theorem 1.7.

T.4. A linear system whose augmented matrix has the row

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \qquad (1.2)$$

can have no solution: that row corresponds to the unsolvable equation $0x_1 + 0x_2 + \cdots + 0x_n = 1$. If the augmented matrix of $A\mathbf{x} = \mathbf{b}$ is row equivalent to a matrix with the row (1.2) above, then by Theorem 1.7, $A\mathbf{x} = \mathbf{b}$ can have no solution.

Conversely, assume $A\mathbf{x} = \mathbf{b}$ has no solution. Its augmented matrix is row equivalent to some matrix $\left[C \mid D\right]$ in reduced row echelon form. If $\left[C \mid D\right]$ does not contain the row (1.2) then it has at most m nonzero rows, and the leading entries of those nonzero rows all correspond to unknowns of the system. After assigning values to the free variables — the variables not corresponding to leading entries of rows — one gets a solution to the system by solving for the values of the leading entry variables. This contradicts the assumption that the system had no solution.

T.5. If $ad - bc = 0$, the two rows of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are multiples of one another:

$$c\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} ac & bc \end{bmatrix} \quad \text{and} \quad a\begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} ac & ad \end{bmatrix} \quad \text{and} \quad bc = ad.$$

Any elementary row operation applied to A will produce a matrix with rows that are multiples of each other. In particular, elementary row operations cannot produce I_2, and so I_2 is not row equivalent to A. If $ad - bc \neq 0$, then a and c are not both 0. Suppose $a \neq 0$.

a	b	1	0	Multiply the first row by $\frac{1}{a}$, and add $(-c)$ times the first row to the second row.
c	d	0	1	
1	$\dfrac{b}{a}$	$\dfrac{1}{a}$	0	Multiply the second row by $\frac{a}{ad-bc}$.
0	$d - \dfrac{bc}{a}$	$-\dfrac{c}{a}$	1	
1	$\dfrac{b}{a}$	$\dfrac{1}{a}$	0	Add $\left(-\frac{b}{a}\right)$ times the second row to the first row.
0	1	$\dfrac{-c}{ad-bc}$	$\dfrac{1}{ad-bc}$	
1	0	$\dfrac{d}{ad-bc}$	$\dfrac{-b}{ad-bc}$	
0	1	$\dfrac{-c}{ad-bc}$	$\dfrac{a}{ad-bc}$	

T.6. (a) Since $a(kb) - b(ka) = 0$, it follows from Exercise T.5 that A is not row equivalent to I_2.

(b) Suppose that $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$. Since $0 \cdot b - 0 \cdot c = 0$, it follows from Exercise T.5 that A is not row equivalent to I_2.

T.7. For any angle θ, $\cos\theta$ and $\sin\theta$ are not both zero. Assume that $\cos\theta \neq 0$ and proceed as follows. The row operation $\frac{1}{\cos\theta}$ times row 1 gives

$$\begin{bmatrix} 1 & \dfrac{\sin\theta}{\cos\theta} \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Applying row operation $\sin\theta$ times row 1 added to row 2 we obtain

$$\begin{bmatrix} 1 & \dfrac{\sin\theta}{\cos\theta} \\ 0 & \cos\theta + \dfrac{\sin^2\theta}{\cos\theta} \end{bmatrix}.$$

Simplifying the $(2,2)$-entry we have

$$\cos\theta + \frac{\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} = \frac{1}{\cos\theta}$$

and hence our matrix is

$$\begin{bmatrix} 1 & \dfrac{\sin\theta}{\cos\theta} \\ 0 & \dfrac{1}{\cos\theta} \end{bmatrix}.$$

Applying row operations $\cos\theta$ times row 2 followed by $\left(-\frac{\sin\theta}{\cos\theta}\right)$ times row 2 added to row 1 gives us I_2. Hence the reduced row echelon form is the 2×2 identity matrix. (If $\cos\theta = 0$, then interchange rows and proceed in a similar manner.)

T.8. By Corollary 1.1, A is row equivalent to a matrix B in reduced row echelon form which determines the same solutions as A. The possibilities for the 2×2 matrix B are I_2 and

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}. \tag{1.3}$$

The homogeneous system $I_2\mathbf{x} = \mathbf{0}$ has only the trivial solution. The other three forms (1.3) clearly have nontrivial solutions. Thus $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $ad - bc \neq 0$.

T.9. Let A be in reduced row echelon form and assume $A \neq I_n$. Thus there is at least one row of A without a leading 1. From the definition of reduced row echelon form, this row must be a zero row.

T.10. By Exercise T.8.

T.11. (a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

(b) $A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{0} - \mathbf{0} = \mathbf{0}$.

(c) $A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}$.

(d) $A(r\mathbf{u} + s\mathbf{v}) = r(A\mathbf{u}) + s(A\mathbf{v}) = r\mathbf{0} + s\mathbf{0} = \mathbf{0}$.

T.12. If $A\mathbf{u} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{b}$, then $A(\mathbf{u} - \mathbf{v}) = A\mathbf{u} - A\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

T.13. (a) $A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

(b) Let $\mathbf{x}_p$ be a solution to $A\mathbf{x} = \mathbf{b}$ and let $\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p$. Then $\mathbf{x} = \mathbf{x}_p + (\mathbf{x} - \mathbf{x}_p) = \mathbf{x}_p + \mathbf{x}_h$ and $A\mathbf{x}_h = A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

T.14. Suppose at some point in the process of reducing the augmented matrix to reduced row echelon form we encounter a row whose first n entries are zero but whose $(n+1)$st entry is some number $c \neq 0$. The corresponding linear equation is

$$0 \cdot x_1 + \cdots + 0 \cdot x_n = c \quad \text{or} \quad 0 = c.$$

This equation has no solution, thus the linear system is inconsistent.

ML.1. Enter A into MATLAB and use the following MATLAB commands.

(a) **A(1,:) = (1/4) * A(1,:)**

```
A =
      1.0000    0.5000   0.5000
     -3.0000    1.0000   4.0000
      1.0000         0   3.0000
      5.0000   -1.0000   5.0000
```

(b) **A(2,:) = 3 * A(1,:) + A(2,:)**

```
A =
    1.0000    0.5000   0.5000
         0    2.5000   5.5000
    1.0000         0   3.0000
    5.0000   -1.0000   5.0000
```

(c) **A(3,:) = −1 ∗ A(1,:) + A(3,:)**

```
A =
    1.0000    0.5000   0.5000
         0    2.5000   5.5000
         0   −0.5000   2.5000
    5.0000   −1.0000   5.0000
```

(d) **A(4,:) = −5 ∗ A(1,:) + A(4,:)**

```
A =
    1.0000    0.5000   0.5000
         0    2.5000   5.5000
         0   −0.5000   2.5000
         0   −3.5000   2.5000
```

(e) **temp = A(2,:)**

```
temp =
         0   2.5000   5.5000
```

A(2,:) = A(4,:)

```
A =
    1.0000    0.5000   0.5000
         0   −3.5000   2.5000
         0   −0.5000   2.5000
         0   −3.5000   2.5000
```

A(4,:) = temp

```
A =
    1.0000    0.5000   0.5000
         0   −3.5000   2.5000
         0   −0.5000   2.5000
         0    2.5000   5.5000
```

ML.2. Enter the matrix A into MATLAB and use the following MATLAB commands. We use the **format rat** command to display the matrix A in rational form at each stage.

A = [1/2 1/3 1/4 1/5;1/3 1/4 1/5 1/6;1 1/2 1/3 1/4]

```
A =
    0.5000   0.3333   0.2500   0.2000
    0.3333   0.2500   0.2000   0.1667
    1.0000   0.5000   0.3333   0.2500
```

format rat, A

```
A =
    1/2   1/3   1/4   1/5
    1/3   1/4   1/5   1/6
      1   1/2   1/3   1/4
```

format

(a) **A(1,:) = 2 ∗ A(1,:)**

```
A =
    1.0000   0.6667   0.5000   0.4000
    0.3333   0.2500   0.2000   0.1667
    1.0000   0.5000   0.3333   0.2500
```

format rat, A

```
A =
     1   2/3   1/2   2/5
   1/3   1/4   1/5   1/6
     1   1/2   1/3   1/4
```

format

(b) **A(2,:) = (− 1/3) ∗ A(1,:) + A(2,:)**

```
A =
   1.0000   0.6667   0.5000   0.4000
        0   0.0278   0.0333   0.0333
   1.0000   0.5000   0.3333   0.2500
```

format rat, A

```
A =
   1    2/3    1/2    2/5
   0   1/36   1/30   1/30
   1    1/2    1/3    1/4
```

format

(c) **A(3,:) = −1 ∗ A(1,:) + A(3,:)**

```
A =
   1.0000   0.6667   0.5000   0.4000
        0   0.0278   0.0333   0.0333
        0  -0.1667  -0.1667  -0.1500
```

format rat, A

```
A =
   1    2/3    1/2    2/5
   0   1/36   1/30   1/30
   0   -1/6   -1/6  -3/20
```

format

(d) **temp = A(2,:)**

```
temp =
        0   0.0278   0.0333   0.0333
```

A(2,:) = A(3,:)

```
A =
   1.0000    0.6667    0.5000    0.4000
        0   -0.1667   -0.1667   -0.1500
        0   -0.1667   -0.1667   -0.1500
```

A(3,:) = temp

```
A =
   1.0000    0.6667    0.5000    0.4000
        0   -0.1667   -0.1667   -0.1500
        0    0.0278    0.0333    0.0333
```

format rat, A

```
A =
   1    2/3    1/2     2/5
   0   -1/6   -1/6   -3/20
   0   1/36   1/30    1/30
```

format

ML.3. Enter A into MATLAB, then type **reduce(A)**. Use the menu to select row operations. There are many different sequences of row operations that can be used to obtain the reduced row echelon form. However, the reduced row echelon form is unique and is

```
ans =
      1  0  0
      0  1  0
      0  0  1
      0  0  0
```

ML.4. Enter A into MATLAB, then type **reduce(A)**. Use the menu to select row operations. There are many different sequences of row operations that can be used to obtain the reduced row echelon form. However, the reduced row echelon form is unique and is

```
ans =
      1.0000       0       0    0.0500
           0  1.0000       0   -0.6000
           0       0  1.0000    1.5000
```

format rat, ans

```
ans =
      1  0  0   1/20
      0  1  0  -3/5
      0  0  1   3/2
```

format

ML.5. Enter the augmented matrix **aug** into MATLAB. Then use command **reduce(aug)** to construct row operations to obtain the reduced row echelon form. We obtain

```
ans =
      1  0  0  -1  -2
      0  1  0   0  -2
      0  0  1   2   8
```

We write the equations equivalent to rows of the reduced row echelon form and use back substitution to determine the solution. The last row corresponds to the equation $z - 2w = 8$. Hence we can choose w arbitrarily, $w = r$, r any real number. Then $z = 8 + 2r$. The second row corresponds to the equation $y = -1$. The first row corresponds to the equation $x - w = -2$ hence $x = -2 + w = -2 + r$. Thus the solution is given by

$$\begin{aligned} x &= -2 + r \\ y &= -1 \\ z &= 8 + 2r \\ w &= r. \end{aligned}$$

ML.6. Enter the augmented matrix **aug** into MATLAB. Then use command **reduce(aug)** to construct row operations to obtain the reduced row echelon form. We obtain

```
ans =
      1  0  1  0  0
      0  1  2  0  0
      0  0  0  0  1
```

The last row is equivalent to the equation $0x + 0y + 0z + 0w = 1$, which is clearly impossible. Thus the system is inconsistent.

ML.7. Enter the augmented matrix **aug** into MATLAB. Then use command **reduce(aug)** to construct row operations to obtain the reduced row echelon form. We obtain

```
ans =
        1  0  0  0
        0  1  0  0
        0  0  1  0
        0  0  0  0
```

It follows that this system has only the trivial solution.

ML.8. Enter the augmented matrix **aug** into MATLAB. Then use command **reduce(aug)** to construct row operations to obtain the reduced row echelon form. We obtain

```
ans =
        1  0  −1  0
        0  1   2  0
        0  0   0  0
```

The second row corresponds to the equation $y + 2z = 0$. Hence we can choose z arbitrarily. Set $z = r$, any real real number. Then $y = -2r$. The first row corresponds to the equation $x - z = 0$ which is the same as $x = z = r$. Hence the solution to this system is

$$\begin{aligned} x &= r \\ z &= -2r \\ z &= r \end{aligned}$$

ML.9. After entering A into MATLAB, use command **reduce(5 ∗ eye(size(A)) − A)**. Selecting row operations, we can show that the reduced row echelon form of $5I_2 - A$ is

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$

Thus the solution to the homogeneous system is

$$\mathbf{x} = \begin{bmatrix} .5r \\ r \end{bmatrix}.$$

Hence for any real number r, not zero, we obtain a nontrivial solution.

ML.10. After entering A into MATLAB, use command **reduce(− 4 ∗ eye(size(A)) − A)**. Selecting row operations, we can show that the reduced row echelon form of $-4I_2 - A$ is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus the solution to the homogeneous system is

$$\mathbf{x} = \begin{bmatrix} -r \\ r \end{bmatrix}.$$

Hence for any real number r, not zero, we obtain a nontrivial solution.

ML.11. For a linear system enter the augmented matrix **aug** and use the command **rref**. Then write out the solution

For 27(a):
rref(aug)

```
ans =
        1   0   0  −1
        0   1   0   4
        0   0   1  −3
```

It follows that there is a unique solution $x = -1$, $y = 4$, $z = -3$.

For 27(b):
rref(aug)

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
        0   0   0   1
```

It follows that the only solution is the trivial solution.

For 28(a):
rref(aug)

```
ans =
        1   0  −1   0
        0   1   2   0
        0   0   0   0
```

It follows that $x = r$, $y = -2r$, $z = r$, where r is any real number.

For 28(b):
rref(aug)

```
ans =
        1   0   0   1
        0   1   0   2
        0   0   1   2
        0   0   0   0
        0   0   0   0
```

It follows that there is a unique solution $x = 1$, $y = 2$, $z = 2$.

ML.12. (a) **A = [1 1 1;1 1 0;0 1 1];**
b = [0 3 1]′;
x = A\b

```
x =
      −1
       4
      −3
```

(b) **A = [1 1 1;1 1 − 2;2 1 1];**
b = [1 3 2]′;
x = A\b

```
x =
       1.0000
       0.6667
      −0.0667
```

ML.13. **A = [1 2 3;4 5 6;7 8 9];**
b = [1 0 0]′;

x = A\b

```
x =
    1  0  −1  0
    0  1   2  0
    0  0   0  1
```

This augmented matrix implies that the system is inconsistent. We can also infer that the coefficient matrix is singular.

```
x = A\b
Warning:  Matrix is close to singular or badly scaled.  Results may be inaccurate.
RCOND=2.937385e-018.
x =
    1.0e + 015*
         3.1522
        −6.3044
         3.1522
```

Each element of the solution displayed using \ is huge. This, together with the warning, suggests that errors due to using computer arithmetic were magnified in the solution process. MATLAB uses an LU-factorization procedure when \ is used to solve linear systems (see Section 1.7), while **rref** actually rounds values before displaying them.

ML.16. $\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \begin{array}{rl} x_1\ x_2\ x_3 + x_4 \qquad & = 0 \\ x_2\ x_3 + x_4 + x_5 & = 1 \\ x_3 + x_4 \qquad & = 1 \end{array} \Longrightarrow \begin{array}{l} x_1 = x_4 \\ x_2 = 1 + x_4 + x_5 \\ x_3 = 1 + x_4 \\ x_4 = x_4 \\ x_5 = x_5 \end{array} \Longrightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

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2. Adding twice the first row to the second row produces a row of zeros.

4. Singular.

6. (a) Singular. (b) $\begin{bmatrix} 1 & -1 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & 1 & \frac{1}{5} & \frac{3}{5} \\ \frac{2}{5} & -\frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$.

8. (a) $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} 3 & 2 & -4 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. (c) Singular.

10. (a) $\begin{bmatrix} \frac{3}{5} & -\frac{3}{5} & -\frac{2}{5} \\ \frac{2}{5} & \frac{3}{5} & -\frac{4}{5} \\ -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$. (b) Singular. (c) Singular.

12. (b) and (c).

14. $\begin{bmatrix} -1 & -4 \\ 1 & 3 \end{bmatrix}$.

15. If the jth column A_j of A consists entirely of zeros, then so does the jth column BA_j of BA (Exercise T.9(a), Sec. 1.3), so A is singular. If the ith row A_i of A consists entirely of zeros, then for any B, the ith row A_iB of AB is zero, so again A is singular.

16. $a \neq 0$, $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -\frac{2}{a} & \frac{1}{a} & \frac{1}{a} \end{bmatrix}$.

18. (a) $A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$. (b) $(A^T)^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = (A^{-1})^T$.

19. Yes. $(A^{-1})^T A = (A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n$. By Theorem 1.9, $(A^{-1})^T = A^{-1}$. That is, A^{-1} is symmetric.

20. (a) No. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $(A+B)^{-1}$ exists but A^{-1} and B^{-1} do not. Even supposing they all exist, equality need not hold. For example, let $A = \begin{bmatrix} 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \end{bmatrix}$. Then $(A+B)^{-1} = \begin{bmatrix} \frac{1}{3} \end{bmatrix} \neq \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \end{bmatrix} = A^{-1} + B^{-1}$.

(b) Yes for A nonsingular and $c \neq 0$.

$$(cA)\left(\frac{1}{c}A^{-1}\right) = c\left(\frac{1}{c}\right) A \cdot A^{-1} = 1 \cdot I_n = I_n.$$

22. $A + B$ may be singular: let $A = I_n$ and $B = -I_n$.
$A - B$ may be singular: let $A = B = I_n$.
$-A$ is nonsingular: $(-A)^{-1} = -(A^{-1})$.

24. $\begin{bmatrix} 11 & 19 \\ 7 & 0 \end{bmatrix}$.

26. Singular. Since the given homogeneous system has a nontrivial solution, Theorem 1.12 implies that A is singular.

28. $\left[\begin{array}{rr:rr} 3 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ \hdashline 0 & 0 & -1 & 7 \\ 0 & 0 & 1 & -6 \end{array}\right]$.

30. (a) Singular. (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. (c) Singular.

32. (a) No. (b) Yes.

T.1. B is nonsingular, so B^{-1} exists, and

$$A = AI_n = A(BB^{-1}) = (AB)B^{-1} = OB^{-1} = O.$$

T.2. The case $r = 2$ of Corollary 1.2 is Theorem 1.10(b). In general, if $r > 2$,

$$\begin{aligned}(A_1A_2\cdots A_r)^{-1} &= [(A_1A_2\cdots A_{r-1})A_r]^{-1}\\ &= A_r^{-1}(A_1A_2\cdots A_{r-1})^{-1}\\ &= A_r^{-1}\left[(A_1A_2\cdots A_{r-2})A_{r-1}\right]^{-1}\\ &= A_r^{-1}A_{r-1}^{-1}(A_1A_2\cdots A_{r-2})^{-1}\\ &= \cdots = A_r^{-1}A_{r-1}^{-1}\cdots A_1^{-1}.\end{aligned}$$

T.3. A is row equivalent to a matrix B in reduced row echelon form which, by Theorem 1.11 is not I_n. Thus B has fewer than n nonzero rows, and fewer than n unknowns corresponding to pivotal columns of B. Choose one of the free unknowns — unknowns not corresponding to pivotal columns of B. Assign any nonzero value to that unknown. This leads to a nontrivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$.

T.4. The result follows from Theorem 1.12 and Exercise T.8 of Section 1.5.

T.5. For any angle θ, $\cos\theta$ and $\sin\theta$ are never simultaneously zero. Thus at least one element in column 1 is not zero. Assume $\cos\theta \neq 0$. (If $\cos\theta = 0$, then interchange rows 1 and 2 and proceed in a similar manner to that described below.) To show that the matrix is nonsingular and determine its inverse, we put

$$\left[\begin{array}{cc:cc}\cos\theta & \sin\theta & 1 & 0\\ -\sin\theta & \cos\theta & 0 & 1\end{array}\right]$$

into reduced row echelon form. Apply row operations $\frac{1}{\cos\theta}$ times row 1 and $\sin\theta$ times row 1 added to row 2 to obtain

$$\left[\begin{array}{cc:cc}1 & \dfrac{\sin\theta}{\cos\theta} & \dfrac{1}{\cos\theta} & 0\\ 0 & \dfrac{\sin^2\theta}{\cos\theta} + \cos\theta & \dfrac{\sin\theta}{\cos\theta} & 1\end{array}\right].$$

Since

$$\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta},$$

the $(2,2)$-element is not zero. Applying row operations $\cos\theta$ times row 2 and $\left(-\frac{\sin\theta}{\cos\theta}\right)$ times row 2 added to row 1 we obtain

$$\left[\begin{array}{cc:cc}1 & 0 & \cos\theta & -\sin\theta\\ 0 & 1 & \sin\theta & \cos\theta\end{array}\right].$$

It follows that the matrix is nonsingular and its inverse is

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

T.6. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be a nonsingular upper triangular matrix, where $a_{ij} = 0$ for $i > j$. We seek a matrix $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ such that $AB = I_n$ and $BA = I_n$. Using the equation $BA = I_n$, we find that $\sum_{k=1}^{n} b_{ik}a_{ki} = 1$, and since $a_{ki} = 0$ for $k > i$, this equation reduces to $b_{ii}a_{ii} = 1$. Thus, we must have $a_{ii} \neq 0$ and $b_{ii} = 1/a_{ii}$. The equation $\sum_{k=1}^{n} b_{ik}a_{kj} = 0$ for $i \neq j$ implies that $b_{ij} = 0$ for $i > j$. Hence, $B = A^{-1}$ is upper triangular.

T.7. Let $\mathbf{u}$ be one solution to $A\mathbf{x} = \mathbf{b}$. Since A is singular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{u}_0$. Then for any real number r, $\mathbf{v} = r\mathbf{u}_0$ is also a solution to the homogeneous system. Finally, by Exercise T.13(a), Sec. 1.5, for each of the infinitely many matrices $\mathbf{v}$, the matrix $\mathbf{w} = \mathbf{u} + \mathbf{v}$ is a solution to the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$.

T.8. Let A be nonsingular and symmetric. We have $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, so A^{-1} is symmetric.

T.9. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be a diagonal matrix with nonzero diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$. That is, $a_{ij} \neq 0$ if $i = j$ and 0 otherwise. We seek an $n \times n$ matrix $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ such that $AB = I_n$. The (i, j) entry in AB is $\sum_{k=1}^{n} a_{ik}b_{kj}$, so $\sum_{k=1}^{n} a_{ik}b_{kj} = 1$ if $i = j$ and 0 otherwise. This implies that $b_{ii} = 1/a_{ii}$ and $b_{ij} = 0$ if $i \neq j$. Hence, A is nonsingular and $A^{-1} = B$.

T.10. $B^k = PA^kP^{-1}$.

T.11. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$

T.12. No, because if $AB = O$, then $A^{-1}AB = B = A^{-1}O = O$, which contradicts that B is nonsingular.

T.13. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}.$$

Thus $A^2 = O$ provided

$$\begin{aligned} ab + bd &= b(a + d) = 0 \\ ac + cd &= c(a + d) = 0 \\ a^2 + bc &= 0 \\ d^2 + bc &= 0 \end{aligned}$$

Case $b = 0$. Then $a^2 = 0 \Longrightarrow a = 0$ and $d^2 = 0 \Longrightarrow d = 0$. But $bc = 0$. Hence b could be either 1 or 0. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Case $c = 0$. Similarly $a = d = 0$ and $c = 1$ or 0. So

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Case $a + d = 0 \Longrightarrow a = d = 0$ or $a = d = 1$

(i) $a = d = 0 \Longrightarrow bc = 0$ so we have $c = b = 0$ or $c = 0$, $b = 1$ or $c = 1$, $b = 0$. Thus

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

(ii) $a = d = 1 \Longrightarrow 1 + bc = 0$ thus $bc = 1$ so $b = c = 1$. Then

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Hence $A^2 = O$ provided

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

ML.1. We use the fact that A is nonsingular if **rref(A)** is the identity matrix.

(a) **A = [1 2; − 2 1];**
rref(A)
```
ans =
      1   0
      0   1
```
Thus A is nonsingular.

(b) **A = [1 2 3;4 5 6;7 8 9];**
rref(A)
```
ans =
      1   0  −1
      0   1   2
      0   0   0
```
Thus A is singular.

(c) **A = [1 2 3;4 5 6;7 8 0];**
rref(A)
```
ans =
      1   0   0
      0   1   0
      0   0   1
```
Thus A is nonsingular.

ML.2. We use the fact that A is nonsingular if **rref(A)** is the identity matrix.

(a) **A = [1 2;2 4];**
rref(A)
```
ans =
      1   2
      0   0
```
Thus A is singular.

(b) **A = [1 0 0;0 1 0;1 1 1];**
rref(A)
```
ans =
      1   0   0
      0   1   0
      0   0   1
```
Thus A is nonsingular.

(c) **A = [1 2 1;0 1 2;1 0 0];**
rref(A)
```
ans =
      1   0   0
      0   1   0
      0   0   1
```
Thus A is nonsingular.

ML.3. (a) **A = [1 3;1 2];**
rref([A eye(size(A))])

```
ans =
      1   0   −2    3
      0   1    1   −3
```

(b) **A = [1 1 2;2 1 1;1 2 1];**
rref(A eye(size(A)))

```
ans =
      1.0000        0        0   −0.2500    0.7500  −0.2500
           0   1.0000        0   −0.2500   −0.2500   0.7500
           0        0   1.0000    0.7500   −0.2500  −0.2500
```

format rat, ans

```
ans =
      1   0   0   −1/4    3/4   −1/4
      0   1   0   −1/4   −1/4    3/4
      0   0   1    3/4   −1/4   −1/4
```

format

ML.4. (a) **A = [2 1;2 3];**
rref(A eye(size(A)))

```
ans =
      1.0000        0    0.7500   −0.2500
           0   1.0000   −0.5000    0.5000
```

format rat, ans

```
ans =
      1   0    3/4   −1/4
      0   1   −1/2    1/2
```

format

(b) **A = [1 − 1 2;0 2 1;1 0 0];**
rref(A eye(size(A)))

```
ans =
      1.0000        0        0         0        0    1.0000
           0   1.0000        0   −0.2000   0.4000    0.2000
           0        0   1.0000    0.4000   0.2000   −0.4000
```

format rat, ans

```
ans =
      1   0   0      0     0      1
      0   1   0   −1/5   2/5    1/5
      0   0   1    2/5   1/5   −2/5
```

format

ML.5. We experiment choosing successive values of t then computing the **rref** of **(t ∗ eye(size(A)) − A)**.

(a) **A = [1 3;3 1];**
t = 1; rref(t ∗ eye(size(A)) − A)
(Use the up arrow key to recall and then revise it for use below.)

```
ans =
      1   0
      0   1
```

t = 2;rref(t ∗ eye(size(A)) − A)

```
ans =
      1   0
      0   1
```

```
t = 3;rref(t * eye(size(A)) - A)
ans =
      1   0
      0   1
t = 4;rref(t * eye(size(A)) - A)
ans =
      1  -1
      0   0
```

Thus $t = 4$.

(b)
```
A = [4  1  2;1  4  1;0  0  - 4];
t = 1;rref(t * eye(size(A)) - A)
ans =
      1   0   0
      0   1   0
      0   0   1
t = 2;rref(t * eye(size(A)) - A)
ans =
      1   0   0
      0   1   0
      0   0   1
t = 3;rref(t * eye(size(A)) - A)
ans =
      1   1   0
      0   0   1
      0   0   0
```

Thus $t = 3$.

ML.8. (a) Nonsingular. (b) Singular.

ML.9. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ have inverses, but there are others.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ do not have inverses, but there are others.

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2. $\mathbf{x} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$. 4. $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 5 \end{bmatrix}$.

6. $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -3 & 1 & -2 \\ 0 & 6 & 2 \\ 0 & 0 & -4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$.

8. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ -2 & 3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} -5 & 4 & 0 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ -4 \end{bmatrix}$.

10. $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 1 & 0 & 0 \\ -0.4 & 0.8 & 1 & 0 \\ 2 & -1.2 & -0.4 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 1 & 0.25 & -0.5 \\ 0 & 0.4 & 1.2 & -2.5 \\ 0 & 0 & -0.85 & 2 \\ 0 & 0 & 0 & -2.5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} -1.5 \\ 4.2 \\ 2.6 \\ -2 \end{bmatrix}$.

ML.1. We show the first few steps of the LU-factorization using routine **lupr** and then display the matrices L and U.

[L,U] = lupr(A)

```
++++++++++++++++++++++++++++++++++++++++++++++++++++++++
        ***** Find an LU-FACTORIZATION by Row Reduction *****
L =             U =
    1  0  0         2   8   0
    0  1  0         2   2  -3
    0  0  1         1   2   7
                    OPTIONS
<1> Insert element into L.  <-1> Undo previous operation.  <0> Quit.
ENTER your choice ===> 1
Enter multiplier.  -1
Enter first row number.  1
Enter number of row that changes.  2

++++++++++++++++++++++++++++++++++++++++++++++++++++++++
            Replacement by Linear Combination Complete
L =             U =
    1  0  0         2   8   0
    0  1  0         0  -6  -3
    0  0  1         1   2   7
You just performed operation -1 * Row(1) + Row(2).
                    OPTIONS
<1> Insert element into L.  <-1> Undo previous operation.  <0> Quit.
ENTER your choice ===> 1

++++++++++++++++++++++++++++++++++++++++++++++++++++++++
            Replacement by Linear Combination Complete
L =             U =
    1  0  0         2   8   0
    0  1  0         0  -6  -3
    0  0  1         1   2   7
You just performed operation -1 * Row(1) + Row(2).
Insert a value in L in the position you just eliminated in U. Let the multiplier you just used be called num. It has the value -1.
Enter row number of L to change.  2
Enter column number of L to change.  1
Value of L(2,1) = -num
Correct:  L(2,1)=1
```

++

Continuing the factorization gives

```
L =                       U =
      1       0  0          2   8   0
      1       1  0          0  −6  −3
    0.5  0.3333  1          0   0   8
```

ML.2. We show the first few steps of the LU-factorization using routine **lupr** and then display the matrices L and U.

[L,U] = lupr(A)

++

∗ ∗ ∗ ∗ ∗ Find an LU-FACTORIZATION by Row Reduction ∗ ∗ ∗ ∗ ∗

```
L =              U =
    1  0  0         8  −1   2
    0  1  0         3   7   2
    0  0  1         1   1   5
```

OPTIONS

```
<1> Insert element into L.  <-1> Undo previous operation.  <0> Quit.
ENTER your choice ===> 1

Enter multiplier.  -3/8
Enter first row number.  1
Enter number of row that changes.  2
```

++

Replacement by Linear Combination Complete

```
L =              U =
    1  0  0         8     −1     2
    0  1  0         0  7.375  1.25
    0  0  1         1      1     5
```

You just performed operation $-0.375 * \text{Row}(1) + \text{Row}(2)$

OPTIONS

```
<1> Insert element into L.  <-1> Undo previous operation.  <0> Quit.
ENTER your choice ===> 1
```

++

Replacement by Linear Combination Complete

```
L =              U =
    1  0  0         8     −1     2
    0  1  0         0  7.375  1.25
    0  0  1         1      1     5
```

You just performed operation $-0.375 * \text{Row}(1) + \text{Row}(2)$

Insert a value in L in the position you just eliminated in U. Let the multiplier you just used be called num. It has the value -0.375.

```
Enter row number of L to change.  2
Enter column number of L to change.  1
Value of L(2,1) = -num
Correct:  L(2,1) = 0.375
```

++

Continuing the factorization process we obtain

L =

1	0	0
0.375	1	0
0.125	0.1525	1

U =

8	−1	2
0	7.375	1.25
0	0	4.559

Warning: It is recommended that the row multipliers be written in terms of the entries of matrix U when entries are decimal expressions. For example, $-U(3,2)/U(2,2)$. This assures that the exact numerical values are used rather than the decimal approximations shown on the screen. The preceding display of L and U appears in the routine **lupr**, but the following displays which are shown upon exit from the routine more accurately show the decimal values in the entries.

L =

1.0000	0	0
0.3750	1.0000	0
0.1250	0.1525	1.0000

U =

8.0000	−1.0000	2.0000
0	7.3750	1.2500
0	0	4.5593

ML.3. We first use **lupr** to find an LU-factorization of A. The matrices L and U that we find are different from those stated in Example 2. There can be many LU-factorizations for a matrix. We omit the details from **lupr**. It is assumed that A and **b** have been entered.

L =

1.0000	0	0	0
0.5000	1.0000	0	0
−2.0000	−2.0000	1.0000	0
−1.0000	1.0000	−2.0000	1.0000

U =

6	−2	−4	4
0	−2	−4	−1
0	0	5	−2
0	0	0	8

z = forsub(L,b)

z =

2
−5
2
−32

x = bksub(U,z)

x =

4.5000
6.9000
−1.2000
−4.0000

ML.4. The detailed steps of the solution of Exercises 7 and 8 are omitted. The solution to Exercise 7 is $\begin{bmatrix}2 & -2 & -1\end{bmatrix}^T$ and the solution to Exercise 8 is $\begin{bmatrix}1 & -2 & 5 & -4\end{bmatrix}^T$.

Supplementary Exercises, p. 114

2. $\begin{bmatrix} 8 & -6 \\ -9 & 17 \end{bmatrix}$.

4. (a) When $AB = BA$, since $(A+B)(A-B) = A^2 - AB + BA - B^2$.
   (b) $(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$.

6. $k = 1$.

8. $\begin{bmatrix} 1 & 0 & \frac{11}{5} & 0 \\ 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

10. $x = \frac{7}{3} + \frac{2}{3}r - s$, $y = \frac{2}{3} + \frac{1}{3}r$, $z = r$, $w = s$, r and s any real numbers.

12. $2b_1 + b_2 - b_3 = 0$.

14. $a = 1, 2$.

16. $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

18. $\begin{bmatrix} 3 & 5 & 0 \\ 1 & 3 & -3 \\ 7 & 10 & 4 \end{bmatrix}$.

20. 0, 4.

22. $\frac{1}{c}A^{-1}$.

24. (a) $a = 2, -4$. (b) Any real number a.

26. (a) 3. (b) 6. (c) 10. (d) $\frac{n}{2}(n+1)$.

28. A can be any of the following:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & b \\ 0 & 1 \end{bmatrix}.$$

30. (a) 4. (b) 1. (c) 3.

31. By inspecting the sequence A^2, A^3, A^4, A^5 and higher powers if need be, it appears that

$$A^n = \begin{bmatrix} 1 & \frac{2^n - 1}{2^n} \\ 0 & \frac{1}{2^n} \end{bmatrix}.$$

32. (a) The results must be identical, since an inverse is unique.
    (b) The instructor computes AA_1 and AA_2. If the result is I_{10}, then the answer submitted by the student is correct.

34. $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 2 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

T.1. (a) $\operatorname{Tr}(cA) = \sum_{i=1}^{n} ca_{ii} = c\sum_{i=1}^{n} a_{ii} = c\operatorname{Tr}(A)$.

(b) $\operatorname{Tr}(A+B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \operatorname{Tr}(A) + \operatorname{Tr}(B)$.

(c) Let $AB = C = \begin{bmatrix} c_{ij} \end{bmatrix}$. Then

$$\operatorname{Tr}(AB) = \operatorname{Tr}(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \operatorname{Tr}(BA).$$

(d) $\operatorname{Tr}(A^T) = \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(A)$.

(e) $\operatorname{Tr}(A^TA)$ is the sum of the diagonal entries of A^TA. The ith diagonal entry of A^TA is $\sum_{j=1}^{n} a_{ji}^2$, so

$$\operatorname{Tr}(A^TA) = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} a_{ji}^2\right] = \text{sum of the squares of all entries of } A.$$

Hence, $\operatorname{Tr}(A^TA) \geq 0$.

T.2. From part (e) of Exercise T.1.,

$$\operatorname{Tr}(A^TA) = \sum_{i=1}^{n}\left[\sum_{j=1}^{n} a_{ji}^2\right].$$

Thus, if $\operatorname{Tr}(A^TA) = 0$, then $a_{ji} = 0$ for all i, j, that is, $A = O$.

T.3. If $A\mathbf{x} = B\mathbf{x}$ for all $n \times 1$ matrices $\mathbf{x}$, then $A\mathbf{e}_j = B\mathbf{e}_j$, $j = 1, 2, \ldots, n$, where $\mathbf{e}_j =$ column j of I_n. But then

$$A\mathbf{e}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = B\mathbf{e}_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

Hence column j of $A =$ column j of B for each j and it follows that $A = B$.

T.4. We have $\operatorname{Tr}(AB - BA) = \operatorname{Tr}(AB) - \operatorname{Tr}(BA) = 0$, while

$$\operatorname{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2.$$

T.5. Suppose that A is skew symmetric, so $A^T = -A$. Then $(A^k)^T = (A^T)^k = (-A)^k = -A^k$ if k is a positive odd integer, so A^k is skew symmetric.

T.6. If A is skew symmetric then $A^T = -A$. Note that $\mathbf{x}^T A\mathbf{x}$ is a scalar, thus $(\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A\mathbf{x}$. That is,

$$\mathbf{x}^T A\mathbf{x} = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T\mathbf{x} = -(\mathbf{x}^T A\mathbf{x}).$$

The only scalar equal to its negative is zero. Hence $\mathbf{x}^T A\mathbf{x} = 0$ for all $\mathbf{x}$.

T.7. If A is symmetric and upper (lower) triangular, then $a_{ij} = a_{ji}$ and $a_{ij} = 0$ for $j > i$ ($j < i$). Thus, $a_{ij} = 0$, $i \neq j$, so A is diagonal.

T.8. Assume that A is upper triangular. If A is nonsingular then A is row equivalent to I_n. Since A is upper triangular this can occur only if $a_{ii} \neq 0$ because in the reduction process we must perform the row operations $(1/a_{ii}) \cdot$ (ith row of A). The steps are reversible.

T.9. Suppose that $A \neq O$ but row equivalent to O. Then in the reduction process some row operation must have transformed a nonzero matrix into the zero matrix. However, considering the types of row operations this is impossible. Thus $A = O$. The converse follows immediately.

T.10. Let A and B be row equivalent $n \times n$ matrices. Then there exists a finite number of row operations which when applied to A yield B and vice versa. If B is nonsingular, then B is row equivalent to I_n. Thus A is also row equivalent to I_n, hence A is nonsingular. We repeat the argument with A and B interchanged to prove the converse.

T.11. Assume that B is singular. Then by Theorem 1.13 there exists $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$. Then $(AB)\mathbf{x} = A\mathbf{0} = \mathbf{0}$, which means that the homogeneous system $(AB)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Theorem 1.13 implies that AB is singular, but this is a contradiction. Suppose now that A is singular and B is nonsingular. Then there exists a $\mathbf{y} \neq \mathbf{0}$ such that $A\mathbf{y} = \mathbf{0}$. Since B is nonsingular we can find $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{y} = B\mathbf{x}$ ($\mathbf{x} = B^{-1}\mathbf{y}$). Then $\mathbf{0} = A\mathbf{y} = (AB)\mathbf{x}$, which again implies that AB is singular, a contradiction.

T.12. If A is skew symmetric, $A^T = -A$. Thus $a_{ii} = -a_{ii}$, so $a_{ii} = 0$.

T.13. If $A^T = -A$ and A^{-1} exists, then

$$(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}.$$

Hence A^{-1} is skew symmetric.

T.14. (a) $I_n^2 = I_n$ and $O^2 = O$.

(b) One such matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) If $A^2 = A$ and A^{-1} exists, then $A^{-1}(A^2) = A^{-1}A$. Simplifying gives $A = I_n$.

T.15. (a) Let A be nilpotent. If A were nonsingular, then products of A with itself are also nonsingular. But $A^k = O$, hence A^k is singular. Thus A must be singular.

(b) $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = O$.

(c) $k = 1$, $A = O$; $I_n - A = I_n$; $(I_n - A)^{-1} = I_n$.
$k = 2$, $A^2 = O$; $(I_n - A)(I_n + A) = I_n - A^2 = I_n$; $(I_n - A)^{-1} = I_n + A$.
$k = 3$, $A^3 = O$; $(I_n - A)(I_n + A + A^2) = I_n - A^3 = I_n$; $(I_n - A)^{-1} = I_n + A + A^2$, etc.

T.16. We have that $A^2 = A$ and $B^2 = B$.

(a) $(AB)^2 = ABAB = A(BA)B = A(AB)B \qquad$ (since $AB = BA$)
$= A^2B^2 = AB \qquad$ (since A and B are idempotent)

(b) $(A^T)^2 = A^T A^T = (AA)^T$ (by the properties of the transpose)
$= (A^2)^T = A^T$ (since A is idempotent)

(c) If A and B are $n \times n$ and idempotent, then $A + B$ need not be idempotent., For example, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Both A and B are idempotent and $C = A + B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. However, $C^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \neq C$.

T.17. Let

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}.$$

Then A and B are skew symmetric and

$$AB = \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}$$

which is diagonal. The result is not true for $n > 2$. For example, let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 5 & 6 & -3 \\ 6 & 10 & 2 \\ -3 & 2 & 13 \end{bmatrix}.$$

T.18. Assume that A is nonsingular. Then A^{-1} exists. Hence we can multiply $A\mathbf{x} = \mathbf{b}$ by A^{-1} on the left on both sides obtaining

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \quad \text{or} \quad I_n\mathbf{x} = \mathbf{x} = A^{-1}\mathbf{b}.$$

Thus $A\mathbf{x} = \mathbf{b}$ has a unique solution. Assume that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$. Since $A\mathbf{x} = \mathbf{0}$ has solution $\mathbf{x} = \mathbf{0}$, Theorem 1.13 implies that A is nonsingular.

T.19. (a) $\mathbf{x}\mathbf{y}^T = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$. (b) $\mathbf{x}\mathbf{y}^T = \begin{bmatrix} -1 & 0 & 3 & 5 \\ -2 & 0 & 6 & 10 \\ -1 & 0 & 3 & 5 \\ -2 & 0 & 6 & 10 \end{bmatrix}$.

T.20. It is not true that $\mathbf{x}\mathbf{y}^T$ must be equal to $\mathbf{y}\mathbf{x}^T$. For example, let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Then

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} 4 & 5 \\ 8 & 10 \end{bmatrix} \quad \text{and} \quad \mathbf{y}\mathbf{x}^T = \begin{bmatrix} 4 & 8 \\ 5 & 10 \end{bmatrix}.$$

T.21. $\text{Tr}(\mathbf{x}\mathbf{y}^T) = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \mathbf{x}^T\mathbf{y}$. (See discussion preceding Exercises T.19–T.22.)

T.22. The outer product of $\mathbf{x}$ and $\mathbf{y}$ can be written in the form

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ x_2 \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ \vdots \\ x_n \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \end{bmatrix}$$

If either $\mathbf{x} = \mathbf{0}$ or $\mathbf{y} = \mathbf{0}$, then $\mathbf{x}\mathbf{y}^T = O_n$. Thus assume that there is at least one nonzero component in $\mathbf{x}$, say x_i, and at least one nonzero component in $\mathbf{y}$, say y_j. Then $\frac{1}{x_i}\text{Row}_i(\mathbf{x}\mathbf{y}^T)$ makes the ith row exactly $\mathbf{y}^T$. Since all the other rows are multiples of $\mathbf{y}^T$, row operations of the form $(-x_k)\,R_i + R_p$, for $p \neq i$ can be performed to zero out everything but the ith row. It follows that either $\mathbf{x}\mathbf{y}^T$ is row equivalent to O_n or to a matrix with $n-1$ zero rows.

T.23. (a) $H^T = (I_n - 2\mathbf{w}\mathbf{w}^T)^T = I_n^T - 2(\mathbf{w}\mathbf{w}^T)^T = I_n - 2(\mathbf{w}^T)^T\mathbf{w}^T = I_n - 2\mathbf{w}\mathbf{w}^T = H$.

(b)
$$\begin{aligned} HH^T = HH &= (I_n - 2\mathbf{w}\mathbf{w}^T)(I_n - 2\mathbf{w}\mathbf{w}^T) \\ &= I_n - 4\mathbf{w}\mathbf{w}^T + 4\mathbf{w}\mathbf{w}^T\mathbf{w}\mathbf{w}^T \\ &= I_n - 4\mathbf{w}\mathbf{w}^T + 4\mathbf{w}(\mathbf{w}^T\mathbf{w})\mathbf{w}^T \\ &= I_n - 4\mathbf{w}\mathbf{w}^T + 4\mathbf{w}(I_n)\mathbf{w}^T = I_n \end{aligned}$$

Thus, $H^T = H^{-1}$.

T.24. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 2a & 2b \\ -a+c & -b+d \end{bmatrix} = \begin{bmatrix} 2a-b & b \\ 2c-d & d \end{bmatrix}$$

which yields $a = r$, $b = 0$, $d = s$, $c = d - a = s - r$. Thus, $B = \begin{bmatrix} r & 0 \\ s-r & s \end{bmatrix}$.

Chapter 2

Applications of Linear Equations and Matrices (Optional)

Section 2.1, p. 123

2. (a) No. (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

4. (a) Yes. (b) No such matrix A can be constructed.

6. (a) 2. (b) 3. (c) 2. (d) 1.

8. (a) Odd. (b) Even. (c) Odd. (d) Odd.

10. (a) Yes. (b) No. (c) No. (d) No.

T.1. 1 word with weight zero; 2 words with weight one; 1 word with weight two.

T.2. 1 word with weight zero; 3 words with weight one; 3 words with weight two; 1 word with weight three.

T.3. n words with weight one; $\dfrac{n}{(n-1)}2$ words with weight two.

T.4. (a) There are only two code words 000 and 111. Since $000 + 000 = 000$, $000 + 111 = 111$, and $111 + 111 = 000$, it is linear.

(b) The code words are 000, 011, 101, and 110. 000+any code word = same code word, $011+011 = 000$, $011 + 101 = 110$, $011 + 110 = 101$, $101 + 110 = 011$, so it is linear.

T.5. (a) The code words are 0000, 0101, 1010, 1111.

(b) Yes.

(c) No. If the received word was 11000, no error would be detected, but 1100 is not a code word.

ML.1. (a) $\mathbf{M} = \mathbf{bingen}(\mathbf{0}, \mathbf{15}, \mathbf{4})$

```
A =
      0  0  0  0  0  0  0  0  1  1  1  1  1  1  1  1
      0  0  0  0  1  1  1  1  0  0  0  0  1  1  1  1
      0  0  1  1  0  0  1  1  0  0  1  1  0  0  1  1
      0  1  0  1  0  1  0  1  0  1  0  1  0  1  0  1
```

(b) $\mathbf{s} = \mathbf{sum}(\mathbf{M})$

```
s =
      0  1  1  2  1  2  2  3  1  2  2  3  2  3  3  4
```

(c) $\mathbf{w} = [\mathbf{0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0}]$

```
w =
      0  1  1  0  1  0  0  1  1  0  0  1  0  1  1  0
```

(d) $\mathbf{C} = [\mathbf{M}; \mathbf{w}]$

```
A =
      0  0  0  0  0  0  0  0  1  1  1  1  1  1  1  1
      0  0  0  0  1  1  1  1  0  0  0  0  1  1  1  1
      0  0  1  1  0  0  1  1  0  0  1  1  0  0  1  1
      0  1  0  1  0  1  0  1  0  1  0  1  0  1  0  1
      0  1  1  0  1  0  0  1  1  0  0  1  0  1  1  0
```

Section 2.2, p. 134

2. (a)

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{array}\begin{array}{c} \begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

(b)

$$\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{array}\begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

4. (b).

6. (a) One way: $P_2 \to P_5 \to P_1$.
   (b) Two ways: $P_2 \to P_5 \to P_1 \to P_2$.
   $P_2 \to P_5 \to P_3 \to P_2$.

8. P_2, P_3, and P_4.

10. There is no clique.

12. (a) Strongly connected.
    (b) Not strongly connected.

14. P_1, P_2, or P_3.

T.1. In a dominance digraph, for each i and j, it is not the case that both P_i dominates P_j and P_j dominates P_i.

T.2. Let $r = 2$. For each i and j, $b_{ij}^{(2)}$, the number of ways P_i has two-stage access to P_j, is the number of indices k, $1 \leq k \leq n$, such that P_i has direct access to P_k and P_k has direct access to P_j. This in turn is the number of k such that $a_{ik} = 1$ and $a_{kj} = 1$ where $A(G) = \big[a_{ij}\big]$, which is

$$\sum_{k=1}^{n} a_{ik}a_{kj} = i,\ j \text{ entry of } [A(G)]^2.$$

For $r > 2$, assume that the theorem has been proved for values up to $r - 1$. Then

$$\begin{aligned} b_{ij}^{(r)} &= \text{the number of } k \text{ such that } P_i \text{ has } r-1 \text{ stage access to } P_k \\ &\quad \text{and } P_k \text{ has direct access to } P_j \\ &= \sum_{k=1}^{n} b_{ik}^{(r-1)} \cdot a_{kj} \\ &= \sum_{k=1}^{n} (i,\, k \text{ entry of } [A(G)]^{r-1}) \cdot (k,\, j \text{ entry of } A(G)) \\ &= i,\, j \text{ entry of } [A(G)]^r. \end{aligned}$$

T.3. The implication in one direction is proved in the discussion following the theorem. Next suppose P_i belongs to the clique $\{P_i, P_j, P_k, \ldots, P_m\}$. According to the definition of clique, it contains at least three vertices so we may assume P_i, P_j and P_k all exist in the clique. Then $s_{ij} = s_{ji} = s_{jk} = s_{kj} = s_{ik} = s_{ki} = 1$ and $s_{ii}^{(3)}$ is a sum of nonnegative integer terms including the positive term which represents three stage access from P_i to P_j to P_k to P_i. Thus $s_{ii}^{(3)}$ is positive.

ML.1. **A = [0 0 0 0 0;1 0 1 1 1;0 1 0 1 0;1 1 1 0 0;0 0 1 1 0];**

S = zeros(size(A)); [k,m] = size(A); for i = 1:k, for j = 1:k,

if A(i,j) == 1&A(j,i) == 1&j ~= i,S(i,j) = 1;S(j,i) = 1;end,end,end,S

```
S =
     0     0     0     0     0
     0     0     1     1     0
     0     1     0     1     0
     0     1     1     0     0
     0     0     0     0     0
```

Next we compute S3 as follows.

S^3

```
ans =
     0     0     0     0     0
     0     2     3     3     0
     0     3     2     3     0
     0     3     3     2     0
     0     0     0     0     0
```

It follows that P2, P3, and P4 form a clique.

ML.2. **A = [0 1 1 0 1;1 0 0 1 0;0 1 0 0 1;0 1 1 0 1;1 0 0 1 0];**

Using the up-arrow, recall the following lines.

S = zeros(size(A));[k,m] = size(A); for i = 1:k, for j = 1:k,

if A(i,j) == 1&A(j,i) == 1&j ~= i,S(i,j) = 1; S(j,i) = 1;end,end,end,S

```
S =
     0     1     0     0     1
     1     0     0     1     0
     0     0     0     0     0
     0     1     0     0     1
     1     0     0     1     0
```

Next we compute **S^3**.

S^3

```
ans =
     0   4   0   0   4
     4   0   0   4   0
     0   0   0   0   0
     0   4   0   0   4
     4   0   0   4   0
```

Since no diagonal entry is different than zero, there are no cliques.

ML.3. We use Theorem 2.3.

(a) **A = [0 0 1 1 1;1 0 1 1 0;0 1 0 0 0;0 1 0 0 1;1 1 0 0 0];**
Here $n = 5$, so we form
E = A + A^2 + A^3 + A^4

```
E =
     7   13   10   10   11
    10   11   12   12    9
     5    7    4    4    3
     8   13    7    7    8
    11   11    9    9    6
```

Since E has no zero entries the digraph represented by A is strongly connected.

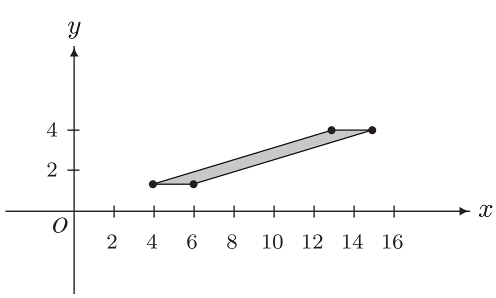
(b) **A = [0 0 0 0 1;0 0 1 1 0;0 1 0 0 1;1 0 0 0 0;0 0 0 1 0];**
Here $n = 5$, so we form
E = A + A^2 + A^3 + A^4

```
E =
     1   0   0   1   2
     3   2   2   4   3
     2   2   2   4   4
     2   0   0   1   1
     1   0   0   2   1
```

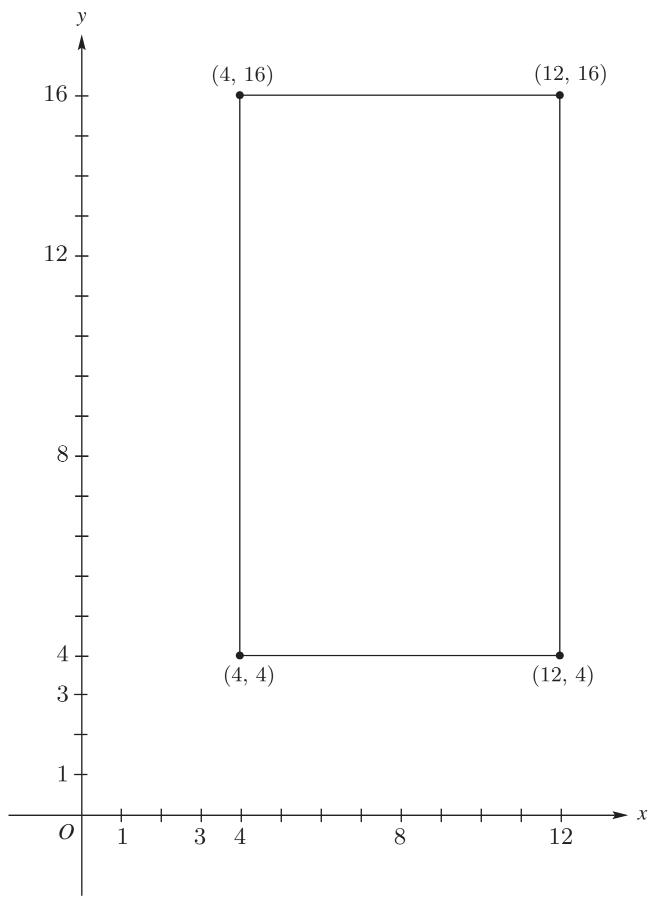
Since E has zero entries the digraph represented by A is not strongly connected.

Section 2.3, p. 141

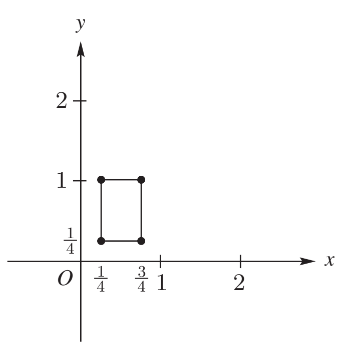
2.



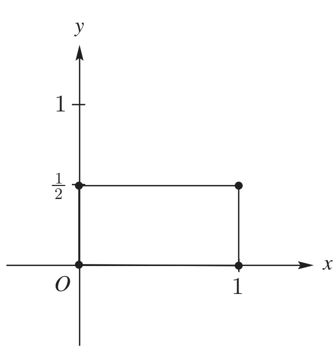
4. (a)



(b)



6.



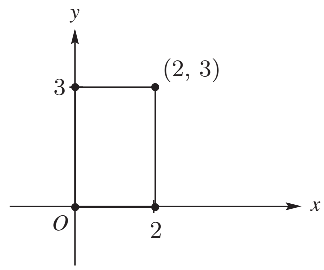
8. $(1,-2)$, $(-3,6)$, $(11,-10)$.

10. We find that

$$(f_1 \circ f_2)(\mathbf{e}_1) = \mathbf{e}_2$$
$$(f_2 \circ f_1)(\mathbf{e}_1) = -\mathbf{e}_2.$$

Therefore $f_1 \circ f_2 \neq f_2 \circ f_1$.

12. Here $f(\mathbf{u}) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u}$. The new vertices are $(0,0)$, $(2,0)$, $(2,3)$, and $(0,3)$.



14. (a) Possible answer: First perform f_1 (45° counterclockwise rotation), then f_2.
    (b) Possible answer: First perform f_3, then f_2.

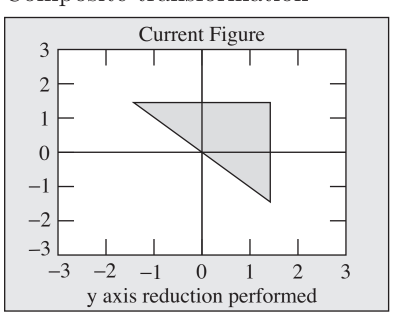
ML.1. (a) Part (a) resulted in an ellipse. Part (b) generated another ellipse within that generated in part (a). The two ellipses are nested.
    (b) Inside the ellip[se generated in part (b).

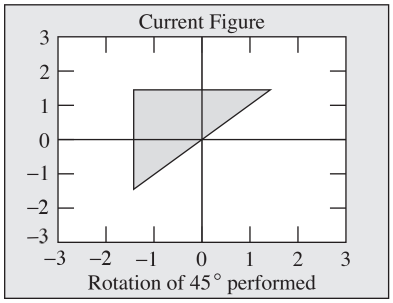
ML.2. (a) The area of the image is 8 square units.
    (b) The area of the composite image is 1 square unit.
    (c) $BA = I_2$, so the composition gave the image of the unit square as the unit square.

ML.3. (a) The area of the house is 5 square units. The area of the image is 5 square units. The areas of the original figure and the image are the same.
    (b) The area of the image is 5 square units. The areas of the original figure and the image are the same.
    (c) The area of the image is 5 square units. The areas of the original figure and the image are the same.

ML.4. (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. (b) $A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.
    (b) The tangent of the angle is 1. Thus the angle is $\pi/4$.

ML.5. (a) Composite transformation



(b) 

$A * B \neq B * A$ because the images of the composite transformations represented by the matrix products are not equal.

ML.6. The parallelogram in Figure 16 may be obtaqined by reflecting the parallelogram about the line $y = -x$.

ML.7. (a) The projection is longer than $\mathbf{w}$ and is in the same direction.

(b) The projection is shorter than $\mathbf{w}$ and is in the opposite direction.

(c) The projection is shorter than $\mathbf{w}$ and is in the same direction.

(d) The projection is shorter than $\mathbf{w}$ and is in the same direction.

Section 2.4, p. 148

2. $I_1 = 1$A from b to a, $I_2 = 5$A from c to b, $I_3 = 4$A from b to f,
$I_4 = 10$A from d to c, $I_5 = 5$A from c to f.

4. $I_1 = 50$A from b to a, $I_2 = 60$A from b to e, $I_3 = 110$A from e to d,
$I_4 = 75$A from b to c, $I_5 = 185$A from d to b.

6. $I_1 = 2$A from a to h, $I_2 = 5$A from c to b, $I_3 = 15$A from f to c,
$E_2 = 20$V, $E_3 = 30$V, $R = 3\Omega$.

8. $I_1 = 21$A from b to a, $I_2 = 40$A from d to c, $I_3 = 10$A from e to f,
$I_4 = 29$A from f to g, $E_1 = 185$V, $E_2 = 370$V.

T.1. We choose the following directions for the currents:

$$\begin{aligned} I &: a \text{ to } b \\ I_1 &: b \text{ to } e \\ I_2 &: b \text{ to } c. \end{aligned}$$

Then we have the following linear equations

$$\begin{aligned} I - \quad I_1 - \quad I_2 &= 0 \quad (\text{node } b) \\ - R_1 I_1 + R_2 I_2 &= 0 \quad (\text{loop } bcdeb) \end{aligned}$$

which leads to the linear system

$$\begin{bmatrix} 1 & 1 \\ R_1 & -R_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

whose solution is

$$I_1 = \left(\frac{R_2}{R_1 + R_2} \right) I = \frac{R}{R_1} I \quad \text{and} \quad I_2 = \left(\frac{R_1}{R_1 + R_2} \right) I = \frac{R}{R_2} I,$$

where

$$R = \frac{R_1 + R_2}{R_1 R_2},$$

so

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

T.2. We choose the following directions for the currents:

$$\begin{aligned} I &: a \text{ to } b \\ I' &: b \text{ to } c \\ I_1 &: b \text{ to } g \\ I_2 &: c \text{ to } f \\ I_3 &: d \text{ to } e \end{aligned}$$

Then we have the following linear equations:

$$\begin{aligned} I - I' - I_1 &= 0 \qquad &&(\text{node } b) \\ I' - I_2 - I_3 &= 0 \qquad &&(\text{node } c) \\ -I_1R_1 + I_2R_2 &= 0 \qquad &&(\text{loop } bgfcb) \\ -I_2R_2 + I_3R_3 &= 0 \qquad &&(\text{loop } cfed) \end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ -R_1 & R_2 & 0 \\ 0 & -R_2 & R_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}$$

whose solution leads to the final result.

Section 2.5, p. 157

2. (b) and (c).

4. $\begin{bmatrix} 0.5 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$.

6. (a) $\mathbf{x}^{(1)} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$, $\mathbf{x}^{(2)} = \begin{bmatrix} 0.06 \\ 0.24 \\ 0.70 \end{bmatrix}$, $\mathbf{x}^{(3)} = \begin{bmatrix} 0.048 \\ 0.282 \\ 0.670 \end{bmatrix}$, $\mathbf{x}^{(4)} = \begin{bmatrix} 0.056 \\ 0.286 \\ 0.658 \end{bmatrix}$.

   (b) Let $*$ stand for any positive matrix entry. Then

$$T^2 = \begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} \cdot \begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix},$$

$$T^3 = T^2 \cdot T = \begin{bmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \cdot \begin{bmatrix} 0 & * & 0 \\ 0 & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} > 0,$$

   hence T is regular; $\mathbf{u} = \begin{bmatrix} 0.057 \\ 0.283 \\ 0.660 \end{bmatrix}$.

8. In general, each matrix is regular, so T^n converges to a state of equilibrium. Specifically, $T^n \to$ a matrix all of whose columns are $\mathbf{u}$, where

$$\text{(a) } \mathbf{u} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \qquad \text{(b) } \mathbf{u} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}, \qquad \text{(c) } \mathbf{u} = \begin{bmatrix} \frac{9}{17} \\ \frac{4}{17} \\ \frac{4}{17} \end{bmatrix}, \qquad \text{(d) } \mathbf{u} = \begin{bmatrix} 0.333 \\ 0.111 \\ 0.555 \end{bmatrix}.$$

10. (a) $\begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{8} \\ \frac{7}{8} \end{bmatrix}$ (c) $\begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{11} \\ \frac{4}{11} \\ \frac{6}{11} \end{bmatrix}$.

12. (a) $T = \begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}$. (b) $T \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.39 \\ 0.61 \end{bmatrix}$; 39% will order a subscription.

14. red, 25%; pink, 50%; white, 25%.

T.1. No. If the sum of the entries of each column is 1, it does not follow that the sum of the entries in each column of A^T will also be 1.

ML.2. Enter the matrix T and initial state vector $\mathbf{x}^{(0)}$ into MATLAB.

T = [.5 .6 .4;.25 .3 .3;.25 .1 .3];

x0 = [.1 .3 .6];

State vector $\mathbf{x}^{(5)}$ is given by

x5 = T^5 ∗ x0

```
x5 =
     0.5055
     0.2747
     0.2198
```

ML.3. The command **sum** operating on a matrix computes the sum of the entries in each column and displays these totals as a row vector. If the output from the **sum** command is a row of ones, then the matrix is a Markov matrix.

(a) **A = [2/3 1/3 1/2;1/3 1/3 1/4;0 1/3 1/4];sum(A)**

```
ans =
      1   1   1
```

Hence A is a Markov matrix.

(b) **A = [.5 .6 .7;.3 .2 .3;.1 .2 0];sum(A)**

```
ans =
      0.9000   1.0000   1.0000
```

A is not a Markov matrix.

(c) **A = [.66 .25 .125;.33 .25 .625;0 .5 .25];sum(A)**

```
ans =
      0.9900   1.0000   1.0000
```

A is not a Markov matrix.

Section 2.6, p. 165

2. $\begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$.   4. $\begin{bmatrix} 15 \\ 3 \\ 14 \end{bmatrix}$.

6. C_1's income is 24; C_2's income is 25; C_3's income is 16.

8. Productive   10. Productive.

12. \$2.805 million of copper, \$2.125 million of transportation, \$4.158 million of electric power.

T.1. We must show that for an exchange matrix A and vector $\mathbf{p}$, $A\mathbf{p} \leq \mathbf{p}$ implies $A\mathbf{p} = \mathbf{p}$. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} p_j \end{bmatrix}$, and $A\mathbf{p} = \mathbf{y} = \begin{bmatrix} y_j \end{bmatrix}$. Then

$$\sum_{j=1}^{n} y_j = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} p_k = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_{jk} \right) p_k = \sum_{k=1}^{n} p_k$$

since the sum of the entries in the kth column of A is 1.

Since $y_j \leq p_j$ for $j = 1, \ldots, n$ and $\sum y_j = \sum p_j$, the respective entries must be equal: $y_j = p_j$ for $j = 1, \ldots, n$. Thus $A\mathbf{p} = \mathbf{p}$.

Section 2.7, p. 178

2. Final average: 14.75;
Detail coefficients: 8.25, 4, -1.5
Compressed data: 14.75, 8.25, 4, 0
Wavelet y-coordinates: 27, 19, 6.5, 8.5.

4. Final average: -0.875
Detail coefficients: 0.625, -2.25, 0.5, 3.5, -3.0, 3.0, 1.0
Compressed data: -0.875, 0, -2.25, 0, 3.5, -3.0, 3.0, 0
Wavelet y-coordinates: 0.375, -6.625, -1.625, 4.375, 2.125, -3.875, -0.875, -0.875.

6. Computing the reduced row echelon for of A_1 and A_2, we find that in each case we obtain I_4.

Supplementary Exercises, p. 179

2. Let $A = \begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix}$. The rectangle has vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(0, 2)$. We must have

$$\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & r & s & t \\ 0 & r & s & t \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix},$$

so $r = 4$ and $4a = 4$. Hence, $a = 1$. Also,

$$\begin{bmatrix} 1 & 2 \\ a & b \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix},$$

so $8 = s$ and $4a + 2b = s = 8$, which implies that $b = 2$.

4. $I_1 = 4\text{A}$ from e to b, $I_2 = 1.5\text{A}$ from d to e, $I_3 = 2.5\text{A}$ from f to e.

6. $p_1 = 12r$, $p_2 = 8r$, $p_3 = 9r$, $r =$ any real number.

T.1. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in R^2. Then

$$\begin{aligned}(f_1 \circ f_2)(\mathbf{u}+\mathbf{v}) &= f_1(f_2(\mathbf{u}+\mathbf{v})) = f_1(f_2(\mathbf{u}) + f_2(\mathbf{v})) \quad \text{since } f_2 \text{ is a linear operator} \\ &= f_1(f_2(\mathbf{u})) + f_1(f_2(\mathbf{v})) \quad \text{since } f_1 \text{ is a linear operator} \\ &= (f_1 \circ f_2)(\mathbf{u}) + (f_1 \circ f_2)(\mathbf{v}).\end{aligned}$$

Moreover, for any scalar c,

$$(f_1 \circ f_2)(c\mathbf{u}) = f_1(f_2(c\mathbf{u})) = f_1(cf_2(\mathbf{u})) = cf_1(f_2(\mathbf{u})) = c(f_1 \circ f_2)(\mathbf{u}).$$

Therefore, $f_1 \circ f_2$ is a linear operator.

Chapter 3

Determinants

Section 3.1, p. 192

2. (a) even. (b) odd. (c) even. (d) odd. (e) even. (f) even.

4. The number of inversions are: (a) 9, 6. (b) 8, 7. (c) 5, 6. (d) 2, 7.

6. (a) 2. (b) 24. (c) -30. (d) 2.

8. $|B| = 4$; $|C| = -8$; $|D| = -4$.

10. $\det(A) = \det(A^T) = 14$.

12. (a) $(\lambda - 1)(\lambda - 2)(\lambda - 3) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$. (b) $\lambda^3 - \lambda$.

14. (a) 1, 2, 3. (b) -1, 0, 1.

16. (a) -144. (b) -168. (c) 72.

18. (a) -120. (b) 29. (c) 9.

20. (a) -1 (b) -120. (c) -22.

22. (a) 16. (b) 256. (c) $-\frac{1}{4}$.

24. (a) 1. (b) 1. (c) 1.

26. (a) 1. (b) 1.

T.1. If j_i and j_{i+1} are interchanged, all inversions between numbers distinct from j_i and j_{i+1} remain unchanged, and all inversions between one of j_i, j_{i+1} and some other number also remain unchanged. If originally $j_i < j_{i+1}$, then after interchange there is one additional inversion due to $j_{i+1}j_i$. If originally $j_i > j_{i+1}$, then after interchange there is one fewer inversion.

Suppose $j_p$and j_q are separated by k intervening numbers. Then k interchanges of adjacent numbers will move j_p next to j_q. One interchange switches j_p and j_q. Finally, k interchanges of adjacent numbers takes j_q back to j_p's original position. The total number of interchanges is the odd number $2k + 1$.

T.2. Parallel to proof for the upper triangular case.

T.3. $cA = \left[ca_{ij}\right]$. By n applications of Theorem 3.5, the result follows.

T.4. If A is nonsingular, then $AA^{-1} = I_n$. Therefore $\det(A)\cdot\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$. Thus $\det(A) \neq 0$ and

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

T.5. $\det(AB) = \det(A)\det(B)$. Thus if $\det(AB) = 0$, then $\det A \cdot \det B = 0$, and either $\det A = 0$ or $\det B = 0$.

T.6. $\det(AB) = \det(A)\cdot\det(B) = \det(B)\cdot\det(A) = \det(BA)$.

T.7. In the summation

$$\det(A) = \sum(\pm)a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

for the definition of $\det(A)$ there is exactly one nonzero term. Thus $\det(A) \neq 0$.

T.8. $\det(A)\det(B) = \det(AB) = \det(I_n) = 1$. Thus $\det(A) \neq 0$ and $\det(B) \neq 0$.

T.9. (a) $[\det(A)]^2 = \det(A)\det(A) = \det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$.

(b) $[\det(A)]^2 = \det(A)\det(A) = \det(A)\det(A^T) = \det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$.

T.10. $\det(A^2) = [\det(A)]^2 = \det(A)$, so $\det(A)$ is a nonzero root of the equation $x^2 - x = 0$.

T.11. $\det(A^TB^T) = \det(A^T)\det(B^T) = \det(A)\det(B^T) = \det(A^T)\det(B)$.

T.12. $$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \begin{vmatrix} a^2 & a & 1 \\ b^2 - a^2 & b - a & 0 \\ c^2 - a^2 & c - a & 0 \end{vmatrix} = \begin{vmatrix} (b-a)(b+a) & b-a \\ (c-a)(c+a) & c-a \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & 1 \\ c+a & 1 \end{vmatrix} = (b-a)(c-a)(b-c).$$

T.13. If A is nonsingular, by Corollary 3.2, $\det(A) \neq 0$ and $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$. Conversely, if $a_{ii} \neq 0$ for $i = 1, \ldots, n$, then clearly A is row equivalent to I_n, and thus is nonsingular.

T.14. $\det(AB) = \det(A)\det(B) = 0\cdot\det(B) = 0$.

T.15. If $\det(A) \neq 0$, then since

$$0 = \det(O) = \det(A^n) = \det(A)\det(A^{n-1}),$$

by Exercise T.5 above, $\det(A^{n-1}) = 0$. Working downward, $\det(A^{n-2}) = 0, \ldots, \det(A^2) = 0$, $\det(A) = 0$, which is a contradiction.

T.16. $\det(A) = \det(A^T) = \det(-A) = (-1)^n\det(A) = \det(A)$, which implies $\det(A) = 0$.

T.17. Follows immediately from Theorem 3.7.

T.18. When all the entries on its main diagonal are nonzero.

T.19. Ten have determinant 0 and six have determinant 1.

ML.1. There are many sequences of row operations that can be used. Here we record the value of the determinant so you may check your result.

(a) $\det(A) = -18$. (b) $\det(A) = 5$.

ML.2. There are many sequences of row operations that can be used. Here we record the value of the determinant so you may check your result.

(a) $\det(A) = -9$. (b) $\det(A) = 5$.

ML.3. (a) **A = [1 − 1 1;1 1 − 1; − 1 1 1];**

```
det(A)
ans =
     4
```

(b) **A = [1 2 3 4;2 3 4 5;3 4 5 6;4 5 6 7];**

```
det(A)
ans =
     0
```

ML.4. (a) **A = [2 3 0;4 1 0;0 0 5];**

```
det(5 * eye(size(A)) − A)
ans =
     0
```

(b) **A = [1 1;5 2];**

```
det(3 * eye(size(A)) − A)^2
ans =
     9
```

(c) **A = [1 1 0;0 1 0;1 0 1];**

```
det(inverse(A) * A)
ans =
     1
```

ML.5. **A = [5 2; − 1 2];**

```
t = 1;
det(t * eye(size(A)) − A)
ans =
     6
t = 2;
det(t * eye(size(A)) − A)
ans =
     2
t = 3;
det(t * eye(size(A)) − A)
ans =
     0
```

Section 3.2, p. 207

2. $A_{21} = 0$, $A_{22} = 0$, $A_{23} = 0$, $A_{24} = 13$, $A_{13} = -9$, $A_{23} = 0$, $A_{33} = 3$, $A_{43} = -2$.

4. (a) 9. (b) 13. (c) -26.

6. (a) -135. (b) -20. (c) -20.

8. (a) $\begin{bmatrix} 2 & -7 & -6 \\ 1 & -7 & -3 \\ -4 & 7 & 5 \end{bmatrix}$. (b) -7.

10. (a) $\begin{bmatrix} \frac{2}{9} & -\frac{1}{9} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{3}{14} & -\frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{5}{7} & -\frac{4}{7} \\ -\frac{1}{14} & \frac{1}{7} & \frac{2}{7} \end{bmatrix}$. (c) Singular.

12. (a) $\begin{bmatrix} 1 & 0 & -1 \\ -2 & \frac{1}{2} & \frac{5}{2} \\ -1 & 0 & 2 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{5}{3} \end{bmatrix}$. (c) $\begin{bmatrix} -\frac{1}{21} & -\frac{2}{21} & \frac{8}{21} \\ \frac{4}{21} & -\frac{5}{42} & -\frac{1}{42} \\ \frac{7}{42} & \frac{7}{84} & -\frac{7}{84} \end{bmatrix}$.

14. (a), (b) and (d) are nonsingular.

16. (a) 0, 5. (b) -1, 0, 1.

18. (a) Has nontrivial solutions. (b) Has only the trivial solution.

20. $x = -2$, $y = 0$, $z = 1$.

22. $x = \frac{22}{5}$, $y = -\frac{26}{5}$, $z = \frac{12}{5}$.

24. (a) is nonsingular.

T.1. Let A be upper triangular. Then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \cdots & \\ 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ & \ddots & \\ 0 & \cdots & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}.$$

T.2. (a) $\det(A) = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

$= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21})$

$= -a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32}$.

T.3. The i, j entry of $\operatorname{adj} A$ is $A_{ji} = (-1)^{j+i}\det(M_{ji})$, where M_{ji} is the submatrix of A obtained by deleting from A the jth row and ith column. Since A is symmetric, that submatrix is the transpose of M_{ij}. Thus

$$A_{ji} = (-1)^{j+i}\det(M_{ji}) = (-1)^{i+j}\det(M_{ij}) = j,\ i \text{ entry of } \operatorname{adj} A.$$

Thus $\operatorname{adj} A$ is symmetric.

T.4. The adjoint matrix is upper triangular if A is upper triangular, since $A_{ij} = 0$ if $i > j$.

T.5. If $\det(A) = ad - bc \neq 0$, then by Corollary 3.3,

$$A^{-1} = \frac{1}{\det(A)}(\operatorname{adj} A) = \frac{1}{ad - bc} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}.$$

T.6. $\dfrac{1}{(b-a)(c-a)(c-b)}\begin{bmatrix} bc(c-b) & ac(a-c) & ab(b-a) \\ b^2-c^2 & c^2-a^2 & a^2-b^2 \\ c-b & a-c & b-a \end{bmatrix}$.

T.7. If $A = O$, then $\operatorname{adj} A = O$ and is singular. Suppose $A \neq O$ but is singular. By Theorem 3.11, $A(\operatorname{adj} A) = \det(A)I_n = O$. Were $\operatorname{adj} A$ nonsingular, it should have an inverse, and

$$A = A(\operatorname{adj} A)(\operatorname{adj} A)^{-1} = O(\operatorname{adj} A)^{-1} = O.$$

Contradiction.

T.8. For the case that A is nonsingular, we have $\operatorname{adj} A = \det(A)A^{-1}$. Hence

$$\det(\operatorname{adj} A) = \det(\det(A)A^{-1}) = [\det(A)]^n \det(A^{-1}) = [\det(A)]^n \frac{1}{\det(A)} = [\det(A)]^{n-1}.$$

If A is singular, then the result is true by Exercise T.7 above.

T.9. $\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc.$

T.10. If $\det(A) \neq 0$, then A is nonsingular, and $B = A^{-1}AB = A^{-1}AC = C$.

T.11. Since the entries of A are integers, the cofactors of entries of A are integers and $\operatorname{adj} A$ is a matrix of integer entries. Since $\det(A) = \pm 1$, A^{-1} is also a matrix of integers.

T.12. $\left(\dfrac{1}{\det(A)}A\right)\operatorname{adj} A = \dfrac{\det(A)}{\det(A)}I_n = I_n$. Thus $\dfrac{1}{\det(A)}A = (\operatorname{adj} A)^{-1}$.

By Corollary 3.3, for any nonsingular matrix B, $\operatorname{adj} B = \det(B)B^{-1}$. Thus for $B = A^{-1}$,

$$\operatorname{adj}(A^{-1}) = \det(A^{-1})(A^{-1})^{-1} = \frac{1}{\det(A)}A.$$

T.13. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$

ML.1. We present a sample of the cofactors.

A = [1 0 − 2;3 1 4;5 2 − 3];

cofactor(1,1,A)	**cofactor(2,3,A)**	**cofactor(3,1,A)**
`ans =`	`ans =`	`ans =`
−11	−2	2

ML.2. **A = [1 5 0;2 − 1 3;3 2 1];**

cofactor(2,1,A)	**cofactor(2,2,A)**	**cofactor(2,3,A)**
`ans =`	`ans =`	`ans =`
−5	1	13

ML.3. **A = [4 0 − 1; − 2 2 − 1;0 4 − 3];**
detA = 4 ∗ cofactor(1,1,A) + (− 1) ∗ cofactor(1,3,A)

```
detA =
      0
```

We can check this using the **det** command.

ML.4. **A = [− 1 2 0 0;2 − 1 2 0; 0 2 − 1 2;0 0 2 − 1];**
(Use expansion about the first column.)
detA = −1 ∗ cofactor(1,1,A) + 2 ∗ cofactor(2,1,A)

```
detA =
      5
```

ML.5. Before using the expression for the inverse in Corollary 3.3, check the value of the determinant to avoid division by zero.

(a) **A = [1 2 − 3; − 4 − 5 2; − 1 1 − 7];**
det(A)

```
ans =
      0
```

The matrix is singular.

(b) **A = [2 3; − 1 2];**
det(A)

```
ans =
      7
```

invA = (1/det(A)) ∗ adjoint(A)

```
invA =
        0.2857   −0.4286
        0.1429    0.2857
```

To see the inverse with rational entries proceed as follows.
format rat, ans

```
ans =
      2/7   −3/7
      1/7    2/7
```

format

(c) **A = [4 0 2;0 3 4;0 1 − 2];**
det(A)

```
ans =
      −40
```

invA = (1/det(A)) ∗ adjoint(A)

```
invA =
        0.2500   −0.0500    0.1500
             0    0.2000    0.4000
             0    0.1000   −0.3000
```

format rat, ans

```
ans =
      1/4   −1/20    3/20
        0     1/5     2/5
        0    1/10   −3/10
```

format

Supplementary Exercises, p. 212

2. (a) $\frac{5}{2}$. (b) 30.

4. (a) 12. (b) 36. (c) 3.

6. -2.

8. $A_{11} = 44$, $A_{12} = -21$, $A_{13} = 8$; $A_{21} = -6$, $A_{22} = 21$, $A_{23} = 11$; $A_{31} = -17$, $A_{32} = -7$, $A_{33} = 9$.

10. (a) $\begin{bmatrix} -7 & 8 & -13 \\ 5 & 4 & -15 \\ -4 & -10 & 12 \end{bmatrix}$. (b) -34.

12. -2, 0.

14. $x = -2$, $y = 1$, $z = -3$.

16. (a) $a \neq 2$ and $a \neq -2$. (b) $a = 2$ or $a = -2$.

18. $a = -1$.

T.1. If rows i and j are proportional with $ta_{ik} = a_{jk}$, $k = 1, 2, \ldots, n$, then

$$\det(A) = \det(A)_{-t\mathbf{r}_i + \mathbf{r}_j \to \mathbf{r}_j} = 0$$

since this row operation makes row j all zeros.

T.2. $\det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = [\det(A)]^2 \geq 0$.

T.3. $$\det(Q - nI_n) = \begin{vmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1-n \end{vmatrix}_{\substack{\mathbf{r}_i + \mathbf{r}_1 \to \mathbf{r}_1 \\ i = 2, 3, \ldots, n}} = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1-n \end{vmatrix} = 0.$$

T.4. $\det(B) = \det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)\dfrac{1}{\det(P)} = \det(A)$.

T.5. From Theorem 3.11, $A(\operatorname{adj} A) = \det(A)I_n$. Since A is singular, $\det(A) = 0$. Therefore $A(\operatorname{adj} A) = O$.

T.6. $\det(AB) = \det(A)\det(B) = 0\det(B) = 0$. Thus AB is singular.

T.7. Compute

$$\begin{vmatrix} A & O \\ O & B \end{vmatrix}$$

by expanding about the first column and expand the resulting $(n-1) \times (n-1)$ determinants about the first column, etc.

T.8. Compute

$$\begin{vmatrix} A & B \\ C & O \end{vmatrix}$$

by expanding about the first column of B and continue expanding the resulting determinants about the first column with zeros toward the bottom, etc.

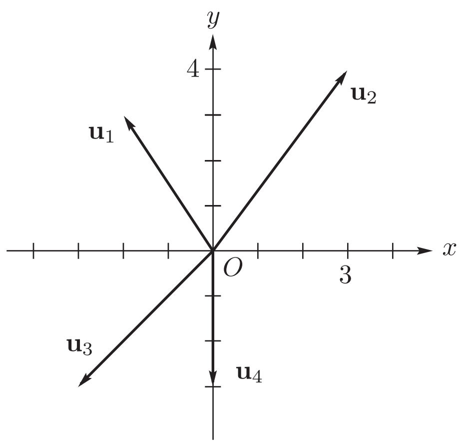
T.9. Since $\det(A) \neq 0$, A is nonsingular. Hence the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$. By Exercise T.11 in Section 3.2 matrix A^{-1} has only integer entries. It follows that the product $A^{-1}\mathbf{b}$ has only integer entries.

Chapter 4

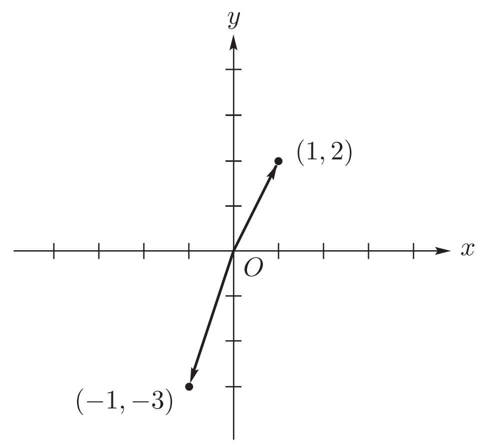
Vectors in R^n

Section 4.1, p. 227

2.



4. $(-1,-3)$



6. (a) $\mathbf{u}+\mathbf{v}=(1,7)$; $\mathbf{u}-\mathbf{v}=(-3,-1)$; $2\mathbf{u}=(-2,6)$; $3\mathbf{u}-2\mathbf{v}=(-7,1)$.
   (b) $\mathbf{u}+\mathbf{v}=(1,-1)$; $\mathbf{u}-\mathbf{v}=(-9,-5)$; $2\mathbf{u}=(-8,-6)$; $3\mathbf{u}-2\mathbf{v}=(-22,-13)$.
   (c) $\mathbf{u}+\mathbf{v}=(1,2)$; $\mathbf{u}-\mathbf{v}=(5,2)$; $2\mathbf{u}=(6,4)$; $3\mathbf{u}-2\mathbf{v}=(13,6)$.

8. (a) $x=-2$, $y=-9$. (b) $x=-6$, $y=8$. (c) $x=5$, $y=-\frac{25}{2}$.

10. (a) $\sqrt{13}$. (b) 3. (c) $\sqrt{41}$. (d) $\sqrt{13}$.

12. (a) 3. (b) $\sqrt{20}$. (c) $\sqrt{18}$. (d) $\sqrt{5}$.

14. Impossible

16. 6.

18. $\frac{41}{2}$.

20. (a) $\left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$. (b) $(0,-1)$. (c) $\left(-\frac{1}{\sqrt{10}},-\frac{3}{\sqrt{10}}\right)$.

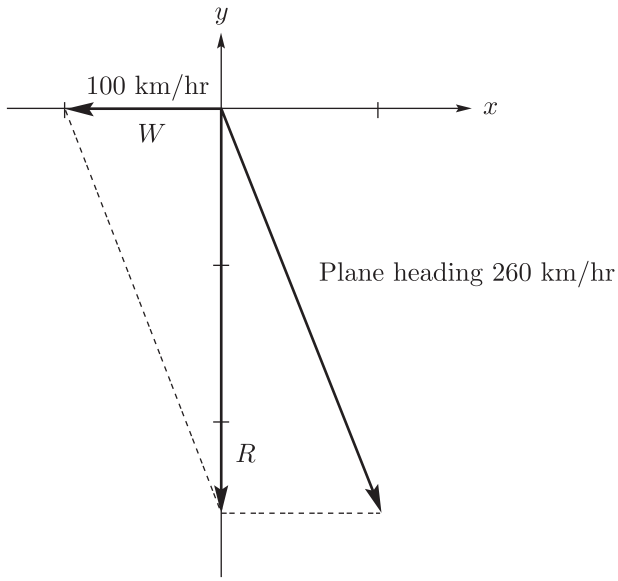
22. (a) 0. (b) $\frac{-1}{\sqrt{2}\sqrt{41}}$. (c) $\frac{-4}{\sqrt{5}\sqrt{13}}$. (d) $-\frac{1}{\sqrt{2}}$.

24. (a) $\mathbf{u}_1$ and $\mathbf{u}_4$, $\mathbf{u}_1$ and $\mathbf{u}_6$, $\mathbf{u}_3$ and $\mathbf{u}_4$, $\mathbf{u}_3$ and $\mathbf{u}_6$, $\mathbf{u}_4$ and $\mathbf{u}_5$, $\mathbf{u}_5$ and $\mathbf{u}_6$.
   (b) $\mathbf{u}_1$ and $\mathbf{u}_5$, $\mathbf{u}_4$ and $\mathbf{u}_6$.
   (c) $\mathbf{u}_1$ and $\mathbf{u}_3$, $\mathbf{u}_3$ and $\mathbf{u}_5$.

26. $a = \pm 2$.

28. (a) $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$. (b) $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. (c) $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

30.



Resultant speed: 240 km/hr.

T.1. Locate the point A on the x-axis which is x units from the origin. Construct a perpendicular to the x-axis through A. Locate B on the y-axis y units from the origin. Construct a perpendicular through B. The intersection of those two perpendiculars is the desired point in the plane.

T.2. $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$.

T.3. $(x, y) + (-1)(x, y) = (x, y) + (-x, -y) = (x - x, y - y) = (0, 0)$.

T.4. $\|c\mathbf{u}\| = \sqrt{(cx)^2 + (cy)^2} = \sqrt{c^2}\,\sqrt{x^2 + y^2} = |c|\,\|\mathbf{u}\|$.

T.5. $\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{x}\|}\,\mathbf{x} \right\| = \frac{1}{\|\mathbf{x}\|}\,\|\mathbf{x}\| = 1$.

T.6. (a) $1\mathbf{u} = 1 \cdot (x, y) = (1 \cdot x, 1 \cdot y) = (x, y) = \mathbf{u}$.

(b) $(rs)\mathbf{u} = (rs)(x, y) = (rsx, rsy) = r(sx, sy) = r(s\mathbf{u})$.

T.7. (a) $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 = x^2 + y^2 \geq 0$; $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $x = 0$ and $y = 0$, that is, $\mathbf{u} = \mathbf{0}$.

(b) $(x_1, y_1) \cdot (x_2, y_2) = x_1x_2 + y_1y_2 = x_2x_1 + y_2y_1 = (x_2, y_2) \cdot (x_1, y_1)$.

(c) $[(x_1, y_1) + (x_2, y_2)] \cdot (x_3, y_3) = (x_1 + x_2)x_3 + (y_1 + y_2)y_3 = x_1x_3 + y_1y_3 + x_2x_3 + y_2y_3 = (x_1, y_1) \cdot (x_3, y_3) + (x_2, y_2) \cdot (x_3, y_3)$.

(d) $(cx_1, cy_1) \cdot (x_2, y_2) = cx_1x_2 + cy_1y_2 = (x_1, y_1) \cdot (cx_2, cy_2) = c(x_1x_2 + y_1y_2) = c[(x_1, y_1) \cdot (x_2, y_2)]$.

T.8. If $\mathbf{w} \cdot \mathbf{u} = 0 = \mathbf{w} \cdot \mathbf{v}$, then $\mathbf{w} \cdot (r\mathbf{u} + s\mathbf{v}) = r(\mathbf{w} \cdot \mathbf{u}) + s(\mathbf{w} \cdot \mathbf{v}) = 0 + 0 = 0$.

T.9. If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then there exists a nonzero scalar k such that $\mathbf{v} = k\mathbf{u}$. Thus

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\mathbf{u} \cdot (k\mathbf{u})}{\|\mathbf{u}\|\,\|k\mathbf{u}\|} = \frac{k(\mathbf{u} \cdot \mathbf{u})}{\|\mathbf{u}\|\,\sqrt{(k\mathbf{u}) \cdot (k\mathbf{u})}} = \frac{k\|\mathbf{u}\|^2}{\|\mathbf{u}\|\,\sqrt{k^2}\,\|\mathbf{u}\|} = \frac{k}{\sqrt{k^2}} = \frac{k}{\pm k} = \pm 1.$$

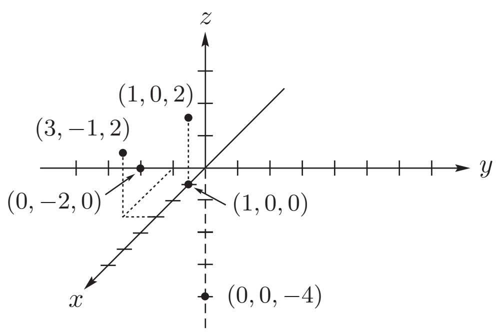
Section 4.2, p. 244

2. (a) $\mathbf{u}+\mathbf{v}=\begin{bmatrix}5\\2\\-3\end{bmatrix}$, $\mathbf{u}-\mathbf{v}=\begin{bmatrix}-1\\-2\\-5\end{bmatrix}$, $2\mathbf{u}=\begin{bmatrix}4\\0\\-8\end{bmatrix}$, $3\mathbf{u}-2\mathbf{v}=\begin{bmatrix}0\\-4\\-14\end{bmatrix}$.

   (b) $\mathbf{u}+\mathbf{v}=\begin{bmatrix}-1\\6\\2\\-2\end{bmatrix}$, $\mathbf{u}-\mathbf{v}=\begin{bmatrix}-5\\4\\-8\\2\end{bmatrix}$, $2\mathbf{u}=\begin{bmatrix}-6\\10\\-6\\0\end{bmatrix}$, $3\mathbf{u}-2\mathbf{v}=\begin{bmatrix}-13\\13\\-19\\4\end{bmatrix}$.

4. (a) $a=12$, $b=9$. (b) $a=2$, $b=-1$, $c=2$, $d=4$. (c) $a=7$, $b=4$.

6.



8. (a) $\begin{bmatrix}-2\\-3\\3\end{bmatrix}$. (b) $\begin{bmatrix}-1\\0\\1\end{bmatrix}$. (c) $\begin{bmatrix}4\\6\\8\end{bmatrix}$. (d) $\begin{bmatrix}-1\\-1\\-2\end{bmatrix}$.

10. (a) $\sqrt{14}$. (b) $\sqrt{30}$. (c) $\sqrt{10}$. (d) 5.

12. (a) $\sqrt{5}$. (b) $\sqrt{6}$. (c) $\sqrt{13}$. (d) $\sqrt{30}$.

14. Impossible.

16. $a=\pm\sqrt{11}$.

18. (a) $\mathbf{u}\cdot\mathbf{u}=1^2+2^2+3^21=1+4+9=14\geq 0$.

    (b) $\mathbf{u}\cdot\mathbf{v}=-7=\mathbf{v}\cdot\mathbf{u}$

    (c) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=(2,4,-1)\cdot(1,0,2)=0$;
    $\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}=(1,2,3)\cdot(1,0,2)+(1,2,-4)\cdot(1,0,2)=7-7=0$.

    (d) $(3,6,9)\cdot(1,2,-4)=(1,2,3)\cdot(3,6,-12)=-21=3(-7)$.

20. (a) $\frac{19}{\sqrt{14}\sqrt{57}}$. (b) $-\frac{2}{7}$. (c) 0. (d) $\frac{-11}{\sqrt{14}\sqrt{39}}$.

24. $\frac{5}{2}$.

26. $\|\mathbf{u}+\mathbf{v}\|=\|(2,2,1,2)\|=\sqrt{4+4+1+4}=\sqrt{13}\leq\sqrt{15}+\sqrt{14}=\|(1,2,3,-1)\|+\|(1,0,-2,3)\|$.

28. (a) $\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$. (b) $(0,0,1,0)$. (c) $\left(-\frac{1}{\sqrt{5}},0,-\frac{2}{\sqrt{5}}\right)$. (d) $\left(0,0,\frac{3}{5},\frac{4}{5}\right)$.

30. (a) $\begin{bmatrix}2\\3\\-4\end{bmatrix}$. (b) $\begin{bmatrix}1\\2\\0\end{bmatrix}$. (c) $\begin{bmatrix}-3\\0\\0\end{bmatrix}$. (d) $\begin{bmatrix}3\\0\\-2\end{bmatrix}$.

34. The value of the inventory of the four types of items.

36.

38.

T.1. (a) $\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ \vdots \\ v_n+u_n \end{bmatrix} = \mathbf{v}+\mathbf{u}.$

(b) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = \begin{bmatrix} u_1+(v_1+w_1) \\ \vdots \\ u_n+(v_n+w_n) \end{bmatrix} = \begin{bmatrix} (u_1+v_1)+w_1 \\ \vdots \\ (u_n+v_n)+w_n \end{bmatrix} = (\mathbf{u}+\mathbf{v})+\mathbf{w}.$

(c) $\mathbf{u}+\mathbf{0} = \begin{bmatrix} u_1+0 \\ \vdots \\ u_n+0 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \mathbf{u}.$

(d) $\mathbf{u}+(-\mathbf{u}) = \begin{bmatrix} u_1+(-u_1) \\ \vdots \\ u_n+(-u_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$

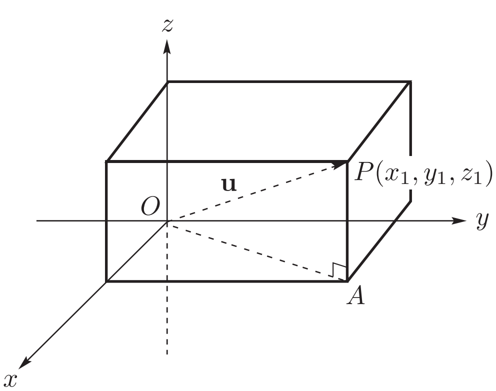
(e) $c(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} c(u_1+v_1) \\ \vdots \\ c(u_n+v_n) \end{bmatrix} = \begin{bmatrix} cu_1+cv_1 \\ \vdots \\ cu_n+cv_n \end{bmatrix} = c\mathbf{u}+c\mathbf{v}.$

(f) $c(d\mathbf{u}) = \begin{bmatrix} c(du_1) \\ \vdots \\ c(du_n) \end{bmatrix} = \begin{bmatrix} (cd)u_1 \\ \vdots \\ (cd)u_n \end{bmatrix} = (cd)\mathbf{u}.$

(g) $1\mathbf{u} = \begin{bmatrix} 1u_1 \\ \vdots \\ 1u_n \end{bmatrix} = \mathbf{u}.$

T.2. $\mathbf{u}+(-1)\mathbf{u}) = (1+(-1))\mathbf{u} = 0\mathbf{u} = \mathbf{0}$. Thus, $(-1)\mathbf{u} = -\mathbf{u}$.

T.3. The origin O and the head of the vector $\mathbf{u} = (x_1, y_1, z_1)$, call it P, are opposite vertices of a parallelepiped with faces parallel to the coordinate planes (see Figure).



The face diagonal OA has length $\sqrt{x_1^2+y_1^2}$ by one application of the Pythagorean Theorem. By a second application, the body diagonal has length

$$\|\mathbf{u}\| = \|OP\| = \sqrt{\left(\sqrt{x_1^2+y_1^2}\right)^2 + z_1^2} = \sqrt{x_1^2+y_1^2+z_1^2}.$$

T.4. (a) $u_1^2 + u_2^2 + \cdots + u_n^2 \geq 0$ and $u_1^2 + u_2^2 + \cdots + u_n^2 = 0$ if and only if all $u_i = 0$.

(b) $\displaystyle\sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} v_i u_i.$

(c) $\displaystyle\sum_{i=1}^{n} (u_i + v_i) w_i = \sum_{i=1}^{n} u_i w_i + \sum_{i=1}^{n} v_i w_i.$

(d) $\displaystyle\sum_{i=1}^{n} (cu_i) v_i = \sum_{i=1}^{n} u_i (cv_i) = c \sum_{i=1}^{n} u_i v_i.$

T.5. See solution to Exercise T.8 of Section 4.1.

T.6. If $\mathbf{u} \cdot \mathbf{v} = 0$ for all $\mathbf{v}$, then in particular, for $\mathbf{v} = \mathbf{u}$, $0 = \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$. By (a) of Theorem 4.3, $\mathbf{u} = \mathbf{0}$.

T.7. $\displaystyle\sum_{i=1}^{n} u_i (v_i + w_i) = \sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} u_i w_i.$

T.8. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ for all $\mathbf{u}$, then $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$ for all $\mathbf{u}$. The result follows by Exercise T.6.

T.9. $\|c\mathbf{u}\| = \left[\displaystyle\sum_{i=1}^{n} (cu_i)^2\right]^{\frac{1}{2}} = \left[c^2 \displaystyle\sum_{i=1}^{n} u_i^2\right]^{\frac{1}{2}} = |c|\,\|\mathbf{u}\|.$

T.10. $$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) &= \mathbf{u} \cdot (\mathbf{u} + \mathbf{v}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u} \cdot \mathbf{v})\end{aligned}$$

Thus $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

T.11. By the remark following Example 10 in Section 1.3, we have

$$(A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T \mathbf{y} = \mathbf{x}^T (A^T \mathbf{y}) = \mathbf{x} \cdot (A^T \mathbf{y}).$$

T.12. (a) and (b) follow from Theorem 4.3(a). For (c): $\|\mathbf{u} - \mathbf{v}\| = \|-(\mathbf{u} - \mathbf{v})\| = \|\mathbf{v} - \mathbf{u}\|$. (d) follows from the Triangle Inequality, Theorem 4.5.

T.13. As in the solution to Exercise T.10 above,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}).$$

Substitute $-\mathbf{v}$ for $\mathbf{v}$ to obtain

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}).$$

Adding these two equations, we find that

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

T.14. $\mathbf{u} = c\mathbf{x}$ for $c = \dfrac{1}{\|\mathbf{x}\|} > 0$. Thus $\mathbf{u}$ is a vector in the direction of $\mathbf{x}$ and

$$\|\mathbf{u}\| = |c|\,\|\mathbf{x}\| = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|} = 1.$$

T.15. As in the solution to Exercise T.13 above,

$$\tfrac{1}{4}\|\mathbf{u} + \mathbf{v}\| - \tfrac{1}{4}\|\mathbf{u} - \mathbf{v}\| = \tfrac{1}{4}\left[\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u} \cdot \mathbf{v})\right] - \tfrac{1}{4}\left[\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u} \cdot \mathbf{v})\right] = \mathbf{u} \cdot \mathbf{v}.$$

T.16. We have $\mathbf{z} \cdot \mathbf{v} = 0$. Moreover, since $|\mathbf{z} \cdot \mathbf{u}| \neq \|\mathbf{z}\|\|\mathbf{u}\|$ and $|\mathbf{z} \cdot \mathbf{w}| \neq \|\mathbf{z}\|\|\mathbf{w}\|$, we conclude that $\mathbf{z}$ is not parallel to $\mathbf{u}$ or $\mathbf{w}$.

T.17. The negative of each vector in B^3 is itself.

T.18 Let $\mathbf{v} = (b_1, b_2, b_3)$. Then we require that

$$\mathbf{v} \cdot \mathbf{v} = b_1 b_1 + b_2 b_2 + b_3 b_3 = 0.$$

This result will be true provided we choose the bits as follows:

$$b_1 = b_2 = b_3 = 0$$

or any two of b_1, b_2 and b_3 as 1.

T.19. The negative of each vector in B^4 is iteself.

T.20. (a) Let $\mathbf{v} = (b_1, b_2, b_3)$. Then we require that $\mathbf{u} \cdot \mathbf{v} = b_1 + b_2 + b_3 = 0$. So we can have $b_1 = b_2 = b_3 = 0$ or that any two of the bits b_1, b_2, b_3 be 1.

$$V_{\mathbf{u}} = \{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\}.$$

(b) $\tilde{V}_{\mathbf{u}}$ is any vector of B^3 not in $V_{\mathbf{u}}$, hence

$$\tilde{V}_{\mathbf{u}} = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}.$$

(c) Yes. Just inspect the sets, they include every vector in B^3.

T.21. (a) Let $\mathbf{v} = (b_1, b_2, b_3, b_4)$. Then we require that $\mathbf{u} \cdot \mathbf{v} = b_1 + b_2 + b_3 + b_4 = 0$. So we can have $b_1 = b_2 = b_3 = b_4 = 0$, $b_1 = b_2 = b_3 = b_4 = 1$, or any two of the bits be ones.

$$V_{\mathbf{u}} = \{(0,0,0,0), (1,1,1,1), (0,0,1,1), (0,1,0,1), (0,1,1,0), (1,0,0,1), (1,0,1,0), (1,1,0,0)\}.$$

(b) $\tilde{V}_{\mathbf{u}}$ is any vector in B^3 with an odd number of 1 bits.

$$\tilde{V}_{\mathbf{u}} = \{(0,0,0,1), (0,0,1,0), (0,1,0,0), (1,0,0,0), (0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0)\}.$$

(c) Yes. Just inspect the sets, they include every vector in B^4.

ML.2. (a) **u = [2 2 − 1]′;norm(u)**
```
ans =
      3
```
(b) **v = [0 4 − 3 0]′;norm(v)**
```
ans =
      5
```
(c) **w = [1 0 1 0 3]′;norm(w)**
```
ans =
      3.3166
```

ML.3. (a) **u = [2 0 3]′;v = [2 − 1 1]′;norm(u − v)**
```
ans =
      2.2361
```
(b) **u = [2 0 0 1];v = [2 5 − 1 3];norm(u − v)**
```
ans =
      5.4772
```

(c) **u = [1 0 4 3];v = [− 1 1 2 2];norm(u − v)**
```
ans =
        3.1623
```

ML.4. Enter A, B, and C as points and construct vectors **v**AB, **v**BC, and **v**CA. Then determine the lengths of the vectors.

A = [1 3 − 2];B = [4 − 1 0];C = [1 1 2];
vAB = B − C
```
vAB =
      3  −2  −2
```
norm(vAB)
```
ans =
      4.1231
```
vBC = C − B
```
vBC =
      −3  2  2
```
norm(vBC)
```
ans =
      4.1231
```
vCA = A − C
```
vCA =
      0  2  −4
```
norm(vCA)
```
ans =
      4.4721
```

ML.5. (a) **u = [5 4 − 4];v = [3 2 1];**
dot(u,v)
```
ans =
        19
```

(b) **u = [3 − 1 0 2];v = [− 1 2 − 5 − 3];**
dot(u,v)
```
ans =
        −11
```

(c) **u = [1 2 3 4 5];**
dot(u, − u)
```
ans =
        −55
```

ML.8. (a) **u = [3 2 4 0];v = [0 2 − 1 0];**
ang = dot(u,v)/((norm(u) ∗ norm(v))
```
ang =
        0
```

(b) **u = [2 2 − 1];v = [2 0 1];**
ang = dot(u,v)/((norm(u) ∗ norm(v))
```
ang =
        0.4472
```
degrees = ang ∗ (180/pi)
```
degrees =
            25.6235
```

(c) **u = [1 0 0 2];v = [0 3 − 4 0];**
ang = dot(u,v)/((norm(u) ∗ norm(v))

```
ang =
     0
```

ML.9. (a) **u = [2 2 − 1]′;**
unit = u/norm(u)

```
unit =
        0.6667
        0.6667
       -0.3333
```

format rat, unit

```
unit =
        2/3
        2/3
       -1/3
```

format

(b) **v = [0 4 − 3 0]′;**
unit = v/norm(v)

```
unit =
             0
        0.8000
       -0.6000
             0
```

format rat, unit

```
unit =
          0
        4/5
       -3/5
          0
```

format

(c) **w = [1 0 1 0 3]′;**
unit = w/norm(w)

```
unit =
       0.3015
            0
       0.3015
            0
       0.9045
```

format

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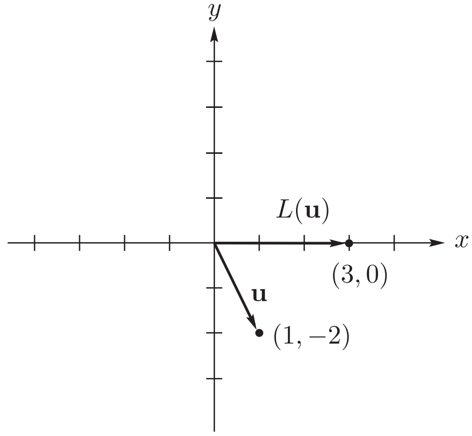
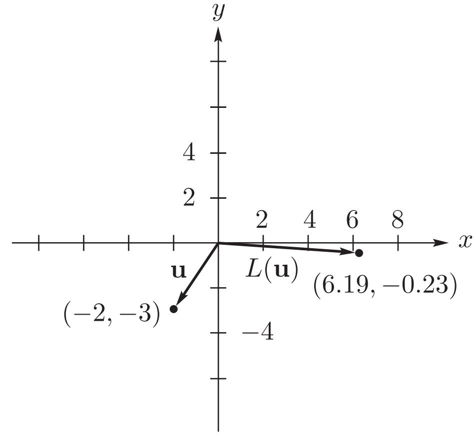
2. (b). 4. (b) and (c).

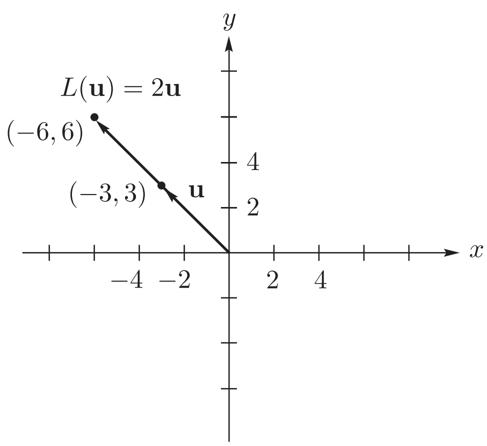
6.



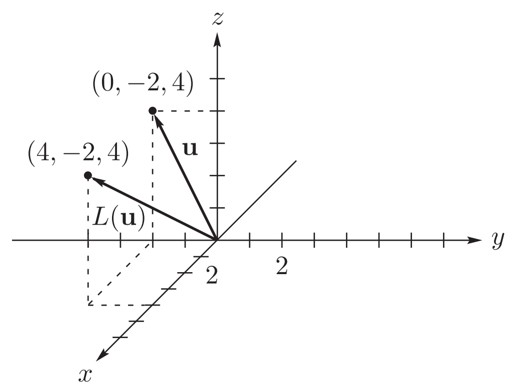
8.



10.



12.



14. (a) No. (b) Yes.

16. $c - b - a = 0$. 18. $\begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$. 20. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

22. (a) Reflection with respect to the line $x = y$.
(b) Reflection about the line $y = -x$.
(c) Dilation by a factor of 2.

24. No. $L(\mathbf{u} + \mathbf{v})$ is not always equal to $L(\mathbf{u}) + L(\mathbf{v})$ and $L(c\mathbf{u})$ is not always equal to $cL(\mathbf{u})$.

26. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. 28. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. 30. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

32. (a) 160 61 123 47 43 17 102 40.
(b) OF COURSE.

T.1. Using properties (a) and (b) in the definition of a linear transformation, we have

$$\begin{aligned} L(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n) &= L((c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{n-1}\mathbf{u}_{n-1}) + c_n\mathbf{u}_n) \\ &= L(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{n-1}\mathbf{u}_{n-1}) + L(c_n\mathbf{u}_n) \quad \text{[by property (a)]} \\ &= L(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{n-1}\mathbf{u}_{n-1}) + c_nL(\mathbf{u}_n) \quad \text{[by property (b)]} \end{aligned}$$

Repeat with $L(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{n-1}\mathbf{u}_{n-1})$.

T.2. (a) We have $\mathbf{0}_{R^n} = \mathbf{0}_{R^n} + \mathbf{0}_{R^n}$ so

$$L(\mathbf{0}_{R^n}) = L(\mathbf{0}_{R^n} + \mathbf{0}_{R^n}) = L(\mathbf{0}_{R^n}) + L(\mathbf{0}_{R^n}).$$

Hence $L(\mathbf{0}_{R^n}) = \mathbf{0}_{R^m}$.

(b) We have

$$L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u} + (-1)\mathbf{v}) = L(\mathbf{u}) + L((-1)\mathbf{v}) = L(\mathbf{u}) + (-1)L(\mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v}).$$

T.3. We have

$$L(\mathbf{u} + \mathbf{v}) = r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v} = L(\mathbf{u}) + L(\mathbf{v})$$

and

$$L(c\mathbf{u}) = r(c\mathbf{u}) = c(r\mathbf{u}) = cL(\mathbf{u}).$$

T.4. We have

$$L(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) + (\mathbf{u}_0) = \mathbf{u} + \mathbf{v} + \mathbf{u}_0$$

and

$$L(\mathbf{u}) + L(\mathbf{v}) = (\mathbf{u} + \mathbf{u}_0) + (\mathbf{v} + \mathbf{u}_0) = \mathbf{u} + \mathbf{v} + 2\mathbf{u}_0.$$

Since $\mathbf{u}_0 \neq \mathbf{0}$, $L(\mathbf{u} + \mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$, so L is not a linear transformation.

T.5. a = any real number, $b = 0$.

T.6. We have

$$O(\mathbf{u} + \mathbf{v}) = \mathbf{0}_W = \mathbf{0}_W + \mathbf{0}_W = O(\mathbf{u}) + O(\mathbf{v})$$

and

$$(c\mathbf{u}) = \mathbf{0}_W = c\mathbf{0}_W = cO(\mathbf{u}).$$

T.7. We have $I(\mathbf{u} + \mathbf{v}) = \mathbf{u} + \mathbf{v} = I(\mathbf{u}) + I(\mathbf{v})$ and $I(c\mathbf{u}) = c\mathbf{u} = cI(\mathbf{u})$.

T.8. We have, by Exercise T.1, $L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v}) = a\mathbf{0} + b\mathbf{0} = \mathbf{0}$.

T.9. (a) Counterclockwise rotation by 60°.

(b) Clockwise rotation by 30°.

(c) $k = 12$.

T.10. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the natural basis for R^n. Then $O(\mathbf{e}_i) = \mathbf{0}$ for $i = 1, \ldots, n$. Hence the standard matrix representing O is the $n \times n$ zero matrix O.

T.11. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the natural basis for R^n. Then $I(\mathbf{e}_i) = \mathbf{0}$ for $i = 1, \ldots, n$. Hence the standard matrix representing I is the $n \times n$ identity matrix I_n.

ML.1. (a) **u = [1 2]';v = [0 3]';**

norm(u + v)

```
ans =
        5.0990
```

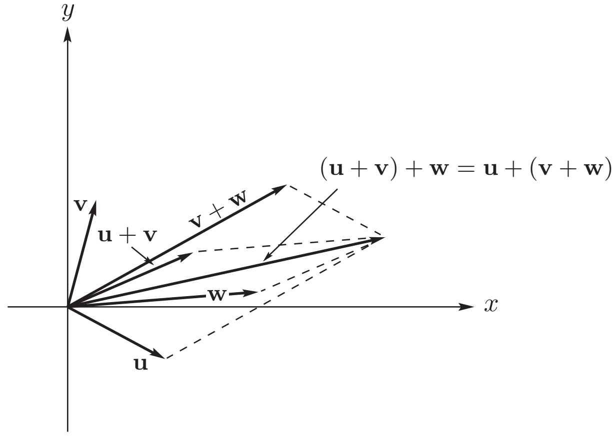
norm(u) + norm(v)

```
ans =
        5.2361
```

(b) **u = [1 2 3]′;v = [6 0 1]′;**
norm(u + v)
```
ans =
        8.3066
```
norm(u) + norm(v)
```
ans =
        9.8244
```

Supplementary Exercises, p. 257

2.



4. $\mathbf{x} = (4, -3, 3)$.

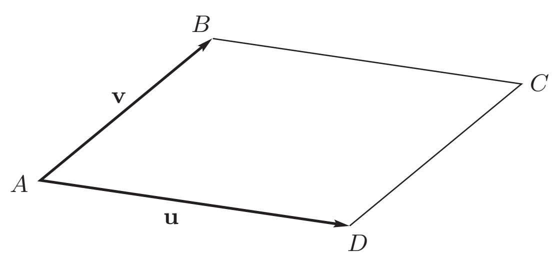
6. $(1, 2) = 3(-2, 3) + 7(1, -1)$.

8. $(x, y) \cdot (-y, x) = -xy + xy = 0$.

10. $a = 1$.

12. $L(\mathbf{u}+\mathbf{v}) = (\mathbf{u}+\mathbf{v})\cdot\mathbf{u}_0 = \mathbf{u}\cdot\mathbf{u}_0 + \mathbf{v}\cdot\mathbf{u}_0 = L(\mathbf{u}) + L(\mathbf{v})$ and $L(c\mathbf{u}) = (c\mathbf{u})\cdot\mathbf{u}_0 = c(\mathbf{u}\cdot\mathbf{u}_0) = cL(\mathbf{u})$.

14. Let the vertices of a parallelogram be denoted by A, B, C, and D, as shown in the figure, and let $\mathbf{u} = \overrightarrow{AD}$ and $\mathbf{v} = \overrightarrow{AB}$.



Suppose that $ABCD$ is a rhombus. Then $|\overrightarrow{AD}| = |\overrightarrow{AB}|$. Therefore $\|\mathbf{u}\| = \|\mathbf{v}\|$ and hence

$$(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2 = 0.$$

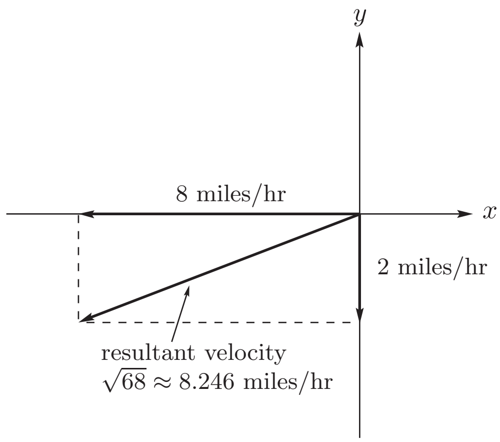
Thus $\mathbf{u}-\mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are perpendicular and it follows that the diagonals $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are orthogonal. Conversely, if $\overrightarrow{BD}$ and $\overrightarrow{AC}$ are orthogonal, then $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = 0$ and hence, since $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$, it follows that $\|\mathbf{u}\| = \|\mathbf{v}\|$. Therefore the sides of $ABCD$ have equal length and hence $ABCD$ is a rhombus.

16. Possible answer: $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

18. $a = 3$, $a = 2$.

20. $\begin{bmatrix} 1 & 2 \end{bmatrix}$.

22.



24. Yes.

T.1. If $\mathbf{u} \cdot \mathbf{v} = 0$, then

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})} = \sqrt{\mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}}$$

and

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})} = \sqrt{\mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}}.$$

Hence $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$. On the other hand, if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u} - \mathbf{v}\|^2.$$

Simplifying, we have $2(\mathbf{u} \cdot \mathbf{v}) = -2(\mathbf{u} \cdot \mathbf{v})$, hence $\mathbf{u} \cdot \mathbf{v} = 0$.

T.2. (a) $(\mathbf{u} + c\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + c(\mathbf{v} \cdot \mathbf{w})$.

(b) $\mathbf{u} \cdot (c\mathbf{v}) = c\mathbf{v} \cdot \mathbf{u} = c(\mathbf{v} \cdot \mathbf{u}) = c(\mathbf{u} \cdot \mathbf{v})$.

(c) $(\mathbf{u} + \mathbf{v}) \cdot c\mathbf{w} = \mathbf{u} \cdot (c\mathbf{w}) + \mathbf{v} \cdot (c\mathbf{w}) = c(\mathbf{u} \cdot \mathbf{w}) + c(\mathbf{v} \cdot \mathbf{w})$.

T.3. Let $\mathbf{v} = (a, b, c)$ be a vector in R^3 that is orthogonal to every vector in R^3. Then $\mathbf{v} \cdot \mathbf{i} = 0$ so $(a, b, c) \cdot (1, 0, 0) = a = 0$. Similarly, $\mathbf{v} \cdot \mathbf{j} = 0$ and $\mathbf{v} \cdot \mathbf{k} = 0$ imply that $b = c = 0$. Therefore $\mathbf{v} = \mathbf{0}$.

T.4. Suppose that $L\colon R^n \to R^m$ is a linear transformation. Then by properties (a) and (b) of the definition of a linear transformation, we have

$$L(a\mathbf{u} + b\mathbf{v}) = L(a\mathbf{u}) + L(b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v}).$$

Conversely, if this property holds, then if $a = b = 1$, we have

$$L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$$

and if $a = 1$ and $b = 0$, then

$$L(a\mathbf{u}) = aL(\mathbf{u}).$$

Hence, L is a linear transformation.

T.5. If $\|\mathbf{u}\| = 0$, then $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = 0$, so $\mathbf{u} \cdot \mathbf{u} = 0$. Part (a) of Theorem 4.3 implies that $\mathbf{u} = \mathbf{0}$.

T.6. Let $\mathbf{u} = (2, 0, -1, 0)$, $\mathbf{v} = (1, -1, 2, 3)$, and $\mathbf{w} = (-5, 2, 2, 1)$.

Chapter 5

Applications of Vectors in R^2 and R^3 (Optional)

Section 5.1, p. 263

2. (a) $-4\mathbf{i}+\mathbf{j}+4\mathbf{k}$. (b) $3\mathbf{i}-8\mathbf{j}-\mathbf{k}$. (c) $0\mathbf{i}+0\mathbf{j}+0\mathbf{k}$. (d) $4\mathbf{i}+4\mathbf{j}+8\mathbf{k}$.

10. $\frac{1}{2}\sqrt{90}$. 12. 1.

T.1. (a) Interchange of the second and third rows of the determinant in (2) changes the sign of the determinant.

(b) $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1+w_1 & v_2+w_2 & v_3+w_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$.

(c) Similar to proof for (b).

(d) Follows from the homogeneity property for determinants: Theorem 3.5.

(e) Follows from Theorem 3.3.

(f) Follows from Theorem 3.4.

(g) First let $\mathbf{u}=\mathbf{i}$ and verify that the result holds. Similarly, let $\mathbf{u}=\mathbf{j}$ and then $\mathbf{u}=\mathbf{k}$. Finally, let $\mathbf{u}=u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$.

(h) First let $\mathbf{w}=\mathbf{i}$. Then

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times\mathbf{i} &= (u_1v_2-u_2v_1)\mathbf{j}-(u_3v_1-u_1v_3)\mathbf{k}\\ &= u_1(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})-v_1(u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k})\\ &= u_1\mathbf{v}-v_1\mathbf{u}=(\mathbf{i}\cdot\mathbf{u})\mathbf{v}-(\mathbf{i}\cdot\mathbf{v})\mathbf{u}.\end{aligned}$$

Thus equality holds when $\mathbf{w}=\mathbf{i}$. Similarly it holds when $\mathbf{w}=\mathbf{j}$, when $\mathbf{w}=\mathbf{k}$, and (adding scalar multiples of the three equations), when $\mathbf{w}=w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k}$.

T.2. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

T.3. We have

$$\mathbf{j}\times\mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{k}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -\mathbf{k}$$

$$\mathbf{k}\times\mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -\mathbf{i}$$

$$\mathbf{i}\times\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -\mathbf{j}.$$

T.4. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad \text{(expand the determinant along the third row).}$$

T.5. If $\mathbf{v} = c\mathbf{u}$ for some c, then $\mathbf{u}\times\mathbf{v} = c(\mathbf{u}\times\mathbf{u}) = \mathbf{0}$. Conversely, if $\mathbf{u}\times\mathbf{v} = \mathbf{0}$, the area of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ is 0, and hence that parallelogram is degenerate: $\mathbf{u}$ and $\mathbf{v}$ are parallel.

T.6. $\|\mathbf{u}\times\mathbf{v}\|^2 + (\mathbf{u}\cdot\mathbf{v})^2 = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2\,(\sin^2\theta + \cos^2\theta) = \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$.

T.7. Using Theorem 5.1(h),

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times\mathbf{w} + (\mathbf{v}\times\mathbf{w})\times\mathbf{u} + (\mathbf{w}\times\mathbf{u})\times\mathbf{v} =& [(\mathbf{w}\cdot\mathbf{u})\mathbf{v} - (\mathbf{w}\cdot\mathbf{v})\mathbf{u}] + [(\mathbf{u}\cdot\mathbf{v})\mathbf{w} - (\mathbf{u}\cdot\mathbf{w})\mathbf{v}] \\ &+ [(\mathbf{v}\cdot\mathbf{w})\mathbf{u} - (\mathbf{v}\cdot\mathbf{u})\mathbf{w}] = \mathbf{0}.\end{aligned}$$

ML.1. (a) **u = [1 − 2 3];v = [1 3 1];cross(u,v)**
```
ans =
         −11   2   5
```
(b) **u = [1 0 3];v = [1 − 1 2];cross(u,v)**
```
ans =
         3   1   −1
```
(c) **u = [1 2 − 3];v = [2 − 1 2];cross(u,v)**
```
ans =
         1   −8   −5
```

ML.2. (a) **u = [2 3 − 1];v = [2 3 1];cross(u,v)**
```
ans =
         6   −4   0
```
(b) **u = [3 − 1 1];v = 2 ∗ u;cross(u,v)**
```
ans =
         0   0   0
```
(c) **u = [1 − 2 1];v = [3 1 − 1];cross(u,v)**
```
ans =
         1   4   7
```

ML.5. Following Example 6 we proceed as follows in Matlab.

u = [3 − 2 1];v = [1 2 3];w = [2 − 1 2];
vol = abs(dot(u,cross(v,w)))
```
vol =
      8
```

ML.6. We find the angle between the perpendicular perxy to plane P1 and the perpendicular pervw to plane P2.

```
x = [2  − 1  2];y = [3  − 2  1];v = [1  3  1];w = [0  2  − 1];
perxy = cross(x,y)
perxy =
        3   4   −1
pervw = cross(v,w)
pervw =
        −5   1   2
angle = dot(perxy,pervw)/(norm(perxy) ∗ norm(pervw))
angle =
        −0.4655
angdeg = (180/pi) ∗ angle
angdeg =
          −26.6697
```

Section 5.2, p. 269

2. (a) $-x+y=0$. (b) $-x+1=0$. (c) $y+4=0$. (d) $-x+y+5=0$.

4. (a), (c).

6. (a) $x=2+2t$, $y=-3+5t$, $z=1+4t$.
   (b) $x=-3+8t$, $y=-2+7t$, $z=-2+6t$.
   (c) $x=-2+4t$, $y=3-6t$, $z=4+t$.
   (d) $x=4t$, $y=5t$, $z=2t$.

8. (a), (d).

10. (a) $x-z+2=0$.
    (b) $3x+y-14z+47=0$.
    (c) $-x-10y+7z=0$.
    (d) $-4x-19y+14z+9=0$.

12. (a) $-4y-z+14=0$ and $4x-3z+2=0$.
    (b) $3y-4z-25=0$ and $3x+2z+2=0$.
    (c) $5x-4y+4=0$ and $x+4z-8=0$.

14. No.

16. (b).

18. $x=3$, $y=-1+t$, $z=-3-5t$.

20. $\left(-\frac{17}{5}, \frac{38}{5}, -6\right)$.

22. $-2x+4y-5z-27=0$.

T.1. Since by hypothesis a, b, and c are not all zero, take $a \neq 0$. Let $P_0 = \left(-\frac{d}{a}, 0, 0\right)$. Then from equations (8) and (9), the equation of the plane through P_0 with normal vector $\mathbf{n} = (a, b, c)$ is

$$a\left(x + \frac{d}{a}\right) + b(y-0) + c(z-0) = 0 \quad \text{or} \quad ax + by + cz + d = 0.$$

T.2. (a) L_1 and L_2 are parallel if and only if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, if and only if $\mathbf{u} = k\mathbf{v}$ for some scalar k.

(b) If L_1 and L_2 are identical, then they are parallel and so $\mathbf{u}$ and $\mathbf{v}$ are parallel. Also, the point $\mathbf{w}_1$ lies on the line L_1 (L_1 and L_2 are the same line), so

$$\mathbf{w}_1 = \mathbf{w}_0 + s\mathbf{u}$$

for some constant s. Thus $\mathbf{w}_1 - \mathbf{w}_0 = s\mathbf{u}$, and $\mathbf{w}_1 - \mathbf{w}_0$ is parallel to $\mathbf{u}$. Conversely, if $\mathbf{w}_1 - \mathbf{w}_0$, $\mathbf{u}$, and $\mathbf{v}$ are mutually parallel, then the point $\mathbf{w}_1$ lies on the line L_1, and so both L_1 and L_2 are lines through $\mathbf{w}_1$ with the same direction and thus are identical lines.

(c) L_1 and L_2 are perpendicular if and only if their direction vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

(d) If L_1 and L_2 intersect in a point $\mathbf{w}_3$, then $\mathbf{w}_3 = \mathbf{w}_0 + s\mathbf{u} = \mathbf{w}_1 + t\mathbf{v}$ for some s and t. Then $\mathbf{w}_1 - \mathbf{w}_0 = s\mathbf{u} - t\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Reversing these steps proves the converse.

T.3. Possible solutions:

$$\begin{aligned} L_1 &: \quad x = s, y = z = 0 \quad \text{(the } x\text{-axis)} \\ L_2 &: \quad x = 0, y = 1, z = t. \end{aligned}$$

T.4. By Exercise T.1, the coefficients of the first degree terms in an equation for a plane give a normal vector for that plane. Thus, if $\mathbf{n}_1 = (a_1, b_1, c_1)$ and $\mathbf{n}_2 = (a_2, b_2, c_2)$ are two normal vectors for the same plane, then $\mathbf{n}_1$ and $\mathbf{n}_2$ must be parallel, and so $\mathbf{n}_2 = a\mathbf{n}_1$ for some nonzero scalar a.

T.5. Expand the determinant about the first row:

$$0 = \begin{vmatrix} x & y & z & 1 \\ a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \end{vmatrix} = xA_{11} + yA_{12} + zA_{13} + 1 \cdot A_{14} \tag{5.1}$$

where A_{1j} is the cofactor of the 1, jth element, and (since it depends upon the second, third and fourth rows of the determinant) is a constant. Thus (5.1) is an equation of the form

$$ax + by + cz + d = 0$$

and so is the equation of some plane. The noncolinearity of the three points insures that the three cofactors A_{11}, A_{12}, A_{13} are not all zero. Next let $(x, y, z) = (a_i, b_i, c_i)$. The determinant has two equal rows, and so has the value zero. Thus the point P_i lies on the plane whose equation is (5.1). Thus (5.1) is an equation for the plane through P_1, P_2, P_3.

Supplementary Exercises, p. 271

2. Possible answer: $\mathbf{u} = (-1, -1, 1)$.

4. $x = -\frac{5}{3} - \frac{5}{3}t$, $y = \frac{2}{3} - \frac{1}{3}t$, $z = t$, $-\infty < t < \infty$.

Chapter 6

Real Vector Spaces

Section 6.1, p. 278

2. Closed under $\oplus$; closed under $\odot$.

4. Closed under $\oplus$; closed under $\odot$.

8. Properties (α), (a), (b), (c), (d) follow as for R^3.

   Regarding (β): (cx, y, z) is a triple of real numbers, so it lies in V.

   Regarding (g): $c \odot (d \odot (x, y, z)) = c \odot (dx, y, z) = (cdx, y, z) = (cd) \odot (x, y, z)$.

   Regarding (h): $1 \odot (x, y, z) = (1 \cdot x, y, z) = (x, y, z)$.

10. P is a vector space. Let $p(t)$ be a polynomial of degree n and $q(t)$ a polynomial of degree m, and $r = \max(n, m)$. Then inside the vector space P_r, addition of $p(t)$ and $q(t)$ is defined and multiplication of $p(t)$ by a scalar is defined, and these operations satisfy the vector space axioms. P_r is contained in P. Thus the additive inverse of $p(t)$, zero polynomial, etc. all lie in P. Hence P is a vector space.

12. Not a vector space; (e), (f), and (h) do not hold.

14. Vector space.

16. Not a vector space; (h) does not hold.

18. No. For example, (a) fails since $2\mathbf{u} - \mathbf{v} \neq 2\mathbf{v} - \mathbf{u}$.

20. (a) Infinitely many.

    (b) The only vector space having a finite number of vectors is $\{\mathbf{0}\}$.

T.1. $c\mathbf{u} = c(\mathbf{u} + \mathbf{0}) = c\mathbf{u} + c\mathbf{0} = c\mathbf{0} + c\mathbf{u}$ by Definition 1(c), (e) and (a). Add the negative of $c\mathbf{u}$ to both sides of this equation to get $\mathbf{0} = c\mathbf{0} + c\mathbf{u} + (-c\mathbf{u}) = c\mathbf{0} + \mathbf{0} = c\mathbf{0}$.

T.2. $-(-\mathbf{u})$ is that unique vector which when added to $-\mathbf{u}$ gives $\mathbf{0}$. But $\mathbf{u}$ added to $-\mathbf{u}$ gives $\mathbf{0}$. Thus $-(-\mathbf{u}) = \mathbf{u}$.

T.3. (cancellation): If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then

$$\begin{aligned}(-\mathbf{u}) + (\mathbf{u} + \mathbf{v}) &= (-\mathbf{u}) + (\mathbf{u} + \mathbf{w})\\ (-\mathbf{u} + \mathbf{u}) + \mathbf{v} &= (-\mathbf{u} + \mathbf{u}) + \mathbf{w}\\ \mathbf{0} + \mathbf{v} &= \mathbf{0} + \mathbf{w}\\ \mathbf{v} &= \mathbf{w}\end{aligned}$$

T.4. If $\mathbf{u} \neq \mathbf{0}$ and $a\mathbf{u} = b\mathbf{u}$, then $(a - b)\mathbf{u} = a\mathbf{u} - b\mathbf{u} = \mathbf{0}$. By Theorem 6.1(c), $a - b = 0$, $a = b$.

T.5. Let $\mathbf{0}_1$ and $\mathbf{0}_2$ be zero vectors. Then $\mathbf{0}_1 \oplus \mathbf{0}_2 = \mathbf{0}_1$ and $\mathbf{0}_1 \oplus \mathbf{0}_2 = \mathbf{0}_2$. So $\mathbf{0}_1 = \mathbf{0}_2$.

T.6. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be negatives of $\mathbf{u}$. Then $\mathbf{u} \oplus \mathbf{u}_1 = \mathbf{0}$ and $\mathbf{u} \oplus \mathbf{u}_2 = \mathbf{0}$. So $\mathbf{u} \oplus \mathbf{u}_1 = \mathbf{u} \oplus \mathbf{u}_2$. Then

$$\begin{aligned}
\mathbf{u}_1 \oplus (\mathbf{u} \oplus \mathbf{u}_1) &= \mathbf{u}_1 \oplus (\mathbf{u} \oplus \mathbf{u}_2) \\
(\mathbf{u}_1 \oplus \mathbf{u}) \oplus \mathbf{u}_1 &= (\mathbf{u}_1 \oplus \mathbf{u}) \oplus \mathbf{u}_2 \\
\mathbf{0} \oplus \mathbf{u}_1 &= \mathbf{0} \oplus \mathbf{u}_2 \\
\mathbf{u}_1 &= \mathbf{u}_2.
\end{aligned}$$

T.7. The sum of any pair of vectors from B^n is, by virtue of entry-by-entry binary addition, a vector in B^n. Thus B^n is closed.

T.8. For $\mathbf{v}$ any vector if B^n, we have $0\mathbf{v} = \mathbf{0}$ and $1\mathbf{v} = \mathbf{v}$. Both $\mathbf{0}$ and $\mathbf{v}$ are in B^n, so B^n is closed under scalar multiplication.

T.9. Let $\mathbf{v} = (b_1, b_2, \ldots, b_n)$ be in B^n. Then $1\mathbf{v} = (1b_1, 1b_2, \ldots, 1b_n) = \mathbf{v}$.

ML.2 **p = [2 5 1 − 2],q = [1 0 3 5]**

```
p =
    2   5   1   −2
q =
    1   0   3   5
```

(a) **p + q**

```
ans =
        3   5   4   3
```

which is $3t^3 + 5t^2 + 4t + 3$.

(b) **5 ∗ p**

```
ans =
        10   25   5   −10
```

which is $10t^3 + 25t^2 + 5t - 10$.

(c) **3 ∗ p − 4 ∗ q**

```
ans =
        2   15   −9   −26
```

which is $2t^3 + 15t^2 - 9t - 26$.

Section 6.2, p. 287

2. Yes.

4. No.

6. (a) and (b).

8. (a).

10. (a) and (b).

12. Since P_n is a subset of P and it is a vector space with respect to the operations in P, it is a subspace of P.

14. Let $\mathbf{w}_1 = a_1\mathbf{u} + b_1\mathbf{v}$ and $\mathbf{w}_2 = a_2\mathbf{u} + b_2\mathbf{v}$ be two vectors in W. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = (a_1\mathbf{u} + b_1\mathbf{v}) + (a_2\mathbf{u} + b_2\mathbf{v}) = (a_1 + a_2)\mathbf{u} + (b_1 + b_2)\mathbf{v}$$

is in W. Also, if c is a scalar, then

$$c\mathbf{w}_1 = c(a_1\mathbf{u} + b_1\mathbf{v}) = (ca_1)\mathbf{u} + (cb_1)\mathbf{v}$$

is in W. Hence, W is a subspace of R^3.

16. (a) and (c). 14. (a) and (c). 16. (b), (c), and (e).

22. Let f_1 and f_2 be solutions to the differential equation $y'' - y' + 2y = 0$, so that $f_1'' - f_1' + 2f_1 = 0$ and $f_2'' - f_2' + 2f_1 = 0$. Then

$$(f_1 + f_2)'' - (f_1 + f_2)' + 2(f_1 + f_2) = (f_1'' - f_1' + 2f_1) + (f_2'' - f_2' + 2f_2) = 0 + 0 = 0.$$

Thus, $f_1 + f_2$ is in V. Also, if c is a scalar, then

$$(cf_1)'' - (cf_1)' + 2cf_1 = c(f_1'' - f_1' + 2f_1) = c(0) = 0$$

so cf_1' is in V. Hence, V is a subspace of the vector space of all real-valued functions defined on R^1.

24. Neither. 26. (a) and (b).

28. No since $0\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, which is not in W.

30. Since the sum of any two vectors in W has first entry zero and for $\mathbf{w}$ in W, $0\mathbf{w}$ and $1\mathbf{w}$ have first entry zero, we have that W is closed under addition and scalar multiplication. So W is a subspace.

32. Yes. Observe that $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$.

T.1. If W is a subspace, then for $\mathbf{u}$, $\mathbf{v} \in W$, $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ lie in W by properties (α) and (β) of Definition 1. Conversely, assume (α) and (β) of Theorem 6.2 hold. We must show that properties (a)–(h) in Definition 1 hold.

Property (a) holds since, if $\mathbf{u}$, $\mathbf{v}$ are in W they are *a fortiori* in V, and therefore $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ by property (a) for V. Similarly for (b). By (β), for $c = 0$, $\mathbf{0} = 0\mathbf{u}$ lies in W. Again by (β), for $c = -1$, $-\mathbf{u} = (-1)\mathbf{u}$ lies in W. Thus (d) holds. Finally, (e), (f), (g), (h) follow for W because those properties hold for any vectors in V and any scalars.

T.2. Let W be a subspace of V, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W, and let a and b be scalars. Then $a\mathbf{u} \in W$, $b\mathbf{v} \in W$, and $a\mathbf{u} + b\mathbf{v} \in W$. Conversely, if $a\mathbf{u} + b\mathbf{v} \in W$ for any $\mathbf{u}$, $\mathbf{v}$ in W and any scalars a, b, then in particular for $a = b = 1$, $\mathbf{u} + \mathbf{v} \in W$ and for $a = c$, $b = 0$, $c\mathbf{u} \in W$. Thus W is a subspace by Theorem 6.2.

T.3. Since $\mathbf{b} \neq \mathbf{0}$, $A \neq O$. If $A\mathbf{x} = \mathbf{b}$ has no solutions, then that empty set of solutions is not a vector space. Otherwise, let $\mathbf{x}_0$ be a solution. Then $A(2\mathbf{x}_0) = 2(A\mathbf{x}_0) = 2\mathbf{b} \neq \mathbf{b}$ since $\mathbf{b} \neq \mathbf{0}$. Thus $\mathbf{x}_0 + \mathbf{x}_0 = 2\mathbf{x}_0$ is not a solution. Hence, the set of all solutions fails to be closed under either vector addition or scalar multiplication.

T.4. We assume S is nonempty. Let

$$\mathbf{v} = \sum_{i=1}^{k} a_i\mathbf{v}_i \quad \text{and} \quad \mathbf{w} = \sum_{i=1}^{k} b_i\mathbf{v}_i$$

be two vectors in span S. Then

$$\mathbf{v} + \mathbf{w} = \sum_{i=1}^{k} (a_i + b_i)\mathbf{v}_i$$

and, for any c,

$$c\mathbf{v} = \sum_{i=1}^{k} (ca_i)\mathbf{v}_i$$

are vectors in span S. Thus span S is a subspace of V.

T.5. W must be closed under vector addition and under multiplication of a vector by an arbitrary scalar. Thus, along with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, W must contain $\sum_{i=1}^{k} a_i\mathbf{v}_i$ for any set of coefficients $a_1, \ldots, a_k$. Thus W contains span S.

T.6. $\{\mathbf{0}\}$. By Theorem 1.13, if A is nonsingular, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

T.7. We have $c\mathbf{x}_0 + d\mathbf{x}_0 = (c + d)\mathbf{x}_0$ is in W, and if r is a scalar then $r(c\mathbf{x}_0) = (rc)\mathbf{x}_0$ is in W.

T.8. No, it is not a subspace. Let $\mathbf{x}$ be in W so $A\mathbf{x} \neq \mathbf{0}$. Letting $\mathbf{y} = -\mathbf{x}$, we have $\mathbf{y}$ is also in W and $A\mathbf{y} \neq \mathbf{0}$. However, $A(\mathbf{x} + \mathbf{y}) = \mathbf{0}$, so $\mathbf{x} + \mathbf{y}$ does not belong to W.

T.9. Let V be a subspace of R^1 which is not the zero subspace and let $\mathbf{v} \neq \mathbf{0}$ be any vector in V. If $\mathbf{u}$ is any nonzero vector in R^1, then $\mathbf{u} = \left[\frac{\mathbf{u}}{\mathbf{v}}\right]\mathbf{v}$, so R^1 is a subset of V. Hence, $V = R^1$.

T.10. Let $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}'_1 + \mathbf{w}'_2$, where $\mathbf{w}_1$ and $\mathbf{w}'_1$ are in W_1 and $\mathbf{w}_2$ and $\mathbf{w}'_2$ are in W_2. Then

$$\mathbf{u} + \mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}'_1 + \mathbf{w}'_2 = (\mathbf{w}_1 + \mathbf{w}'_1) + (\mathbf{w}_2 + \mathbf{w}'_2).$$

Since $\mathbf{w}_1 + \mathbf{w}'_1$ is in W_1 and $\mathbf{w}_2 + \mathbf{w}'_2$ is in W_2, we conclude that $\mathbf{u} + \mathbf{v}$ is in W. Also, if c is a scalar, then $c\mathbf{u} = c\mathbf{w}_1 + c\mathbf{w}_2$, and since $c\mathbf{w}_1$ is in W_1, and $c\mathbf{w}_2$ is in W_2, we conclude that $c\mathbf{u}$ is in W.

T.11. Since $V = W_1 + W_2$, every vector $\mathbf{v}$ in W can be written as $\mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{w}_1$ in W_1 and $\mathbf{w}_2$ in W_2. Suppose now that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$ and $\mathbf{v} = \mathbf{w}'_1 + \mathbf{w}'_2$. Then $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}'_1 + \mathbf{w}'_2$, so

$$\mathbf{w}_1 - \mathbf{w}'_1 = \mathbf{w}'_2 - \mathbf{w}_2. \qquad (6.1)$$

Since $\mathbf{w}_1 - \mathbf{w}'_1$ is in W_1 and $\mathbf{w}'_2 - \mathbf{w}_2$ is in W_2, $\mathbf{w}_1 - \mathbf{w}'_1$ is in $W_1 \cap W_2 = \{\mathbf{0}\}$. Hence $\mathbf{w}_1 = \mathbf{w}'_1$. Similarly, or from (6.1), we conclude that $\mathbf{w}_2 = \mathbf{w}'_2$.

T.12 Let $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_2 = (x_2, y_2, z_2)$ be points on the plane $ax + by + cz = 0$. Then

$$a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = ax_1 + by_1 + cz_1 + ax_2 + by_2 + cz_2 = 0 + 0 = 0$$

and so $\mathbf{x}_1 + \mathbf{x}_2$ lies on the plane. Also, for any scalar r,

$$a(rx_1) + b(ry_1) + c(rz_1) = r(ax_1 + by_1 + cz_1) = r(0) = 0$$

and so $r\mathbf{x}_1$ lies on the plane.

T.13. Let $\mathbf{w}$ be any vector in W. Then $0\mathbf{w} = \mathbf{0}$. But, since $\mathbf{0}$ is not in W, this implies that W is not closed under scalar multiplication, so W cannot be a subspace of V.

T.14. If $\mathbf{w}_1 = \mathbf{0}$, then $\{\mathbf{w}_1\}$ is a subspace; otherwise, no. For example, if $\mathbf{w}_1 = (1, 1, 1)$, $\mathbf{w}_1 + \mathbf{w}_1 = (0, 0, 0)$, which is not in $W = \{\mathbf{w}_1\}$.

T.15. No; one of the vectors in a subspace of B^3 must be $\begin{bmatrix}0\\0\\0\end{bmatrix}$. Then any two different nonzero vectors will have a 1 in different entries, hence their sum will not be one of these two vectors.

T.16. $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$,

$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$,

$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$,

$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

T.17. B^3 itself,

$$\left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}, \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

ML.3. (a) Following Example 1, we construct the augmented matrix that results from the expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$. Note that since the vectors are rows we need to convert them to columns to form this matrix. Next we obtain the reduced row echelon form of the associated linear system.

v1 = [1 0 0 1];v2 = [0 1 1 0];v3 = [1 1 1 1];v = [0 1 1 1];
rref([v1′ v2′ v3′ v′])

```
ans =
        1  0  1  0
        0  1  1  0
        0  0  0  1
        0  0  0  0
```

Since this represents an augmented matrix, the system is inconsistent and hence has no solution. Thus $\mathbf{v}$ is not a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Here the strategy is similar to that in part (a) except that the vectors are already columns. We use the transpose operator to conveniently enter the vectors.

v1 = [1 2 − 1]′;v2 = [2 − 1 0]′;v3 = [− 1 8 − 3]′;v = [0 5 − 2]′;
rref([v1 v2 v3 v])

```
ans =
        1  0   3   2
        0  1  −2  −1
        0  0   0   0
```

Since this matrix represents an augmented matrix, the system is consistent. It follows that $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

ML.4. (a) Apply the procedure in ML.3(a).

v1 = [1 2 1];v2 = [3 0 1];v3 = [1 8 3];v = [− 2 14 4];

rref([v1′ v2′ v3′ v′])

```
ans =
        1   0    4    7
        0   1   −1   −3
        0   0    0    0
```

This system is consistent so $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In the general solution if we set $c_3 = 0$, then $c_1 = 7$ and $c_2 = 3$. Hence $7\mathbf{v}_1 - 3\mathbf{v}_2 = \mathbf{v}$. There are many other linear combinations that work.

(b) After entering the 2×2 matrices into MATLAB we associate a column with each one by 'reshaping' it into a 4×1 matrix. The linear system obtained from the linear combination of reshaped vectors is the same as that obtained using the 2×2 matrices in $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$.

v1 = [1 2;1 0];v2 = [2 − 1;1 2];v3 = [− 3 1;0 1];v = eye(2);
rref([reshape(v1,4,1) reshape(v2,4,1) reshape(v3,4,1) reshape(v,4,1)])

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
        0   0   0   1
```

The system is inconsistent, hence $\mathbf{v}$ is not a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

ML.5. (a) Follow the procedure in ML.3(b).

v1 = [1 2 1 0 1]′;v2 = [0 1 2 − 1 1]′;
v3 = [2 1 0 0 − 1]′;v4 = [− 2 1 1 1 1]′;
v = [0 − 1 1 − 2 1]′;
rref([v1 v2 v3 v4 v])

```
ans =
        1   0   0   0    0
        0   1   0   0    1
        0   0   1   0   −1
        0   0   0   1   −1
        0   0   0   0    0
```

The system is consistent and it follows that $0\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4 = \mathbf{v}$.

(b) Associate a column vector of coefficients with each polynomial, then follow the method in part (a).

v1 = [2 − 1 1]′;v2 = [1 0 − 2]′;v3 = [0 1 − 1]′;v = [4 1 − 5]′;
rref([v1 v2 v3 v])

```
ans =
        1   0   0   1
        0   1   0   2
        0   0   1   2
```

Since the system is consistent, we have that $p_1(t) + 2p_2(t) + 2p_3(t) = p(t)$.

ML.6. Follow the method in ML.4(a).

v1 = [1 1 0 1]; v2 = [1 − 1 0 1]; v3 = [0 1 2 1];

(a) **v = [2 3 2 3];**
rref([v1′ v2′ v3′ v′])

```
ans =
        1   0   0   2
        0   1   0   0
        0   0   1   1
        0   0   0   0
```

Since the system is consistent, $\mathbf{v}$ is in span S. In fact, $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_3$.

(b) **v = [2 − 3 − 2 3];**
rref([v1′ v2′ v3′ v′])

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
        0   0   0   1
```

The system is inconsistent, hence $\mathbf{v}$ is not in span S.

(c) **v = [0 1 2 3];**
rref([v1′ v2′ v3′ v′])

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
        0   0   0   1
```

The system is inconsistent, hence $\mathbf{v}$ is not in span S.

ML.7. Associate a column vector with each polynomial as in ML.5(b).
v1 = [0 1 − 1]′;v2 = [0 1 1]′;v3 = [1 1 1]′;

(a) **v = [1 2 4]′;**
rref([v1′ v2′ v3′ v′])

```
ans =
        1   0   0  −1
        0   1   0   2
        0   0   1   1
```

Since the system is consistent, $p(t)$ is in span S.

(b) **v = [2 1 − 1];**
rref([v1′ v2′ v3′ v′])

```
ans =
        1   0   0   1
        0   1   0  −2
        0   0   1   2
```

Since the system is consistent, $p(t)$ is in span S.

(c) **v = [− 2 0 1];**
rref([v1′ v2′ v3′ v′])

```
ans =
        1.0000        0        0  −0.5000
             0   1.0000        0   2.5000
             0        0   1.0000  −2.0000
```

Since the system is consistent, $p(t)$ is in span S.

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2. (c) and (d).

4. (a) and (c).

6. $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

8. No.

10. (b) $(2, 6, -5) = (4, 2, 1) - 2(1, -2, 3)$.
    (c) $(3, 6, 6) = 2(1, 1, 0) + (0, 2, 3) + (1, 2, 3)$.

12. (c) $3t + 1 = 3(2t^2 + t + 1) - 2(3t^2 + 1)$.
    (d) $5t^2 - 5t - 6 = 2(t^2 - 4) + (3t^2 - 5t + 2)$.

14. Only (d) is linearly dependent: $\cos 2t = \cos^2 t - \sin^2 t$.

16. $\lambda \neq \pm 2$.

18. Yes.

20. Yes.

22. $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_3$.

T.1. If $c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_n\mathbf{e}_n = (c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0) = \mathbf{0}$ in R^n, then $c_1 = c_2 = \cdots = c_n = 0$.

T.2. (a) Since S_1 is linearly dependent, there are vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in S_1 and constants $c_1, c_2, \ldots, c_k$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Those $\mathbf{v}_i$'s also lie in S_2, hence S_2 is linearly dependent.

(b) Suppose S_1 were linearly dependent, then by part (a), S_2 would be linearly dependent. Contradiction.

T.3. Assume that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent. Then there are constants c_i, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Let c_j be a nonzero coefficient. Then, solving the equation for $\mathbf{v}_j$, we find that

$$\mathbf{v}_j = -\frac{c_1}{c_j}\,\mathbf{v}_1 - \frac{c_2}{c_j}\,\mathbf{v}_2 - \cdots - \frac{c_{j-1}}{c_j}\,\mathbf{v}_{j-1} - \frac{c_{j+1}}{c_j}\,\mathbf{v}_{j+1} - \cdots - \frac{c_k}{c_j}\,\mathbf{v}_k.$$

Conversely, if

$$\mathbf{v}_j = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_{j-1}\mathbf{v}_{j-1} + d_{j+1}\mathbf{v}_{j+1} + \cdots + d_k\mathbf{v}_k$$

for some coefficients d_i, then

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + (-1)\mathbf{v}_j + \cdots + d_k\mathbf{v}_k = \mathbf{0}$$

and the set S is linearly dependent.

T.4. Suppose

$$\begin{aligned} c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 &= c_1(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) + c_2(\mathbf{v}_2 + \mathbf{v}_3) + c_3\mathbf{v}_3 \\ &= c_1\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + (c_1 + c_2 + c_3)\mathbf{v}_3 = \mathbf{0}. \end{aligned}$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, $c_1 = 0$, $c_1 + c_2 = 0$ (and hence $c_2 = 0$), and $c_1 + c_2 + c_3 = 0$ (and hence $c_3 = 0$). Thus the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

T.5. Form the linear combination

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

which gives

$$c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_1 + \mathbf{v}_3) + c_3(\mathbf{v}_2 + \mathbf{v}_3) = (c_1 + c_2)\mathbf{v}_1 + (c_1 + c_3)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3 = \mathbf{0}.$$

Since S is linearly independent, we have

$$\begin{aligned} c_1 + c_2 \qquad &= 0 \\ c_1 \qquad + c_3 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

a linear system whose augmented matrix is

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

The reduced row echelon form is

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

thus $c_1 = c_2 = c_3 = 0$ which implies that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

T.6. Form the linear combination

$$c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 = \mathbf{0}$$

which gives

$$c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = (c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

Since S is linearly dependent, this last equation is satisfied with $c_1 + c_2 + c_3$, $c_2 + c_3$, and c_3 not all being zero. This implies that c_1, c_2, and c_3 are not all zero. Hence, $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent.

T.7. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Then one of the $\mathbf{v}_j$'s is a linear combination of the preceding vectors in the list. It must be $\mathbf{v}_3$ since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Thus $\mathbf{v}_3$ belongs to span $\{\mathbf{v}_1, \mathbf{v}_2\}$. Contradiction.

T.8. Let $\mathbf{a}_1, \dots, \mathbf{a}_r$ be the nonzero rows of the reduced row echelon form matrix A, and suppose

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_r\mathbf{a}_r = \mathbf{0}. \tag{6.2}$$

For each j, $1 \le j \le r$, $\mathbf{a}_j$ is the only row with a nonzero entry in the column which holds the leading entry of that row. Thus, in the summation (6.2), c_j must be zero. Hence (6.2) is the trivial dependence relation, and the $\mathbf{a}_i$ are linearly independent.

T.9. Let $\mathbf{v}_i = \sum_{j=1}^{k} a_{ij}\mathbf{u}_j$ for $i = 1, 2, \ldots, m$. Then

$$\mathbf{w} = \sum_{i=1}^{m} b_i\mathbf{v}_i = \sum_{i=1}^{m} b_i \sum_{j=1}^{k} a_{ij}\mathbf{u}_j = \sum_{j=1}^{k}\left(\sum_{i=1}^{m} b_i a_{ij}\right)\mathbf{u}_j$$

is a linear combination of the vectors $\mathbf{u}_j$ in S.

T.10. Form the linear combination

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_nA\mathbf{v}_n = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}.$$

Since A is nonsingular, Theorem 1.13 implies that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent, we have

$$c_1 = c_2 = \cdots = c_n = 0.$$

Hence, $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is linearly independent.

T.11. Let $V = R^2$ and $S_2 = \{(0,0), (1,0), (0,1)\}$. Since S_2 contains the zero vector for R^2 it is linearly dependent. If $S_1 = \{(0,0), (1,0)\}$, it is linearly dependent for the same reason. If $S_1 = \{(1,0), (0,1)\}$, then

$$c_1(1,0) + c_2(0,1) = (0,0)$$

only if $c_1 = c_2 = 0$. Thus in this case S_1 is linearly independent.

T.12. Let $V = R^2$ and $S_1 = \{(1,0)\}$, which is linearly independent. If $S_2 = \{(1,0), (0,1)\}$, then S_2 is linearly independent. If $S_2 = \{(1,0), (0,1), (1,1)\}$, then S_2 is linearly dependent.

T.13. If $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent, then there exist scalars c_1 and c_2, not both zero, such that $c_1\mathbf{u}+c_2\mathbf{v} = \mathbf{0}$. In fact, since neither $\mathbf{u}$ nor $\mathbf{v}$ is the zero vector, it follows that both c_1 and c_2 must be nonzero for $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$. Hence we have $\mathbf{v} = -\frac{c_1}{c_2}\mathbf{u}$.

Alternatively, if $\mathbf{v} = k\mathbf{u}$, then $k \neq 0$ since $\mathbf{v} \neq \mathbf{0}$. Hence we have $\mathbf{v} - k\mathbf{u} = \mathbf{0}$ which implies that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent.

T.14. Form the linear combination

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3(\mathbf{u} \times \mathbf{v}) = \mathbf{0}. \tag{6.3}$$

Take the dot product of both sides with $\mathbf{u} \times \mathbf{v}$, obtaining

$$c_1(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} + c_2(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} + c_3(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

Since $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, this equation becomes

$$c_3(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel, so they are linearly dependent, which contradicts the hypothesis. Hence, $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$, so $c_3 = 0$. Now equation (6.3) becomes

$$c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}.$$

Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, we have $c_1 = c_2 = 0$. Therefore, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ are linearly independent.

T.15. If span $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}\} = W =$ span $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$, then $\mathbf{v}$ must be in W. Hence, $\mathbf{v}$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. Thus, the set of all such vectors $\mathbf{v}$ is W.

ML.1. In each case we form a linear combination of the vectors in A, set it equal to the zero vector, derive the associated linear system and find its reduced row echelon form.

(a) **v1 = [1 0 0 1];v2 = [0 1 1 0];v3 = [1 1 1 1];**
rref([v1′ v2′ v3′ zeros(4,1)])

```
ans =
        1   0   1   0
        0   1   1   0
        0   0   0   0
        0   0   0   0
```

This represents a homogeneous system with 2 equations in 3 unknowns, hence there is a non-trivial solution. Thus S is linearly dependent.

(b) **v1 = [1 2;1 0];v2 = [2 − 1;1 2];v3 = [− 3 1;0 1];**
rref([reshape(v1,4,1) reshape(v2,4,1) reshape(v3,4,1) zeros(4,1)])

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
        0   0   0   0
```

The homogeneous system has only the trivial solution, hence S is linearly independent.

(c) **v1 = [0 1 − 1]′;v2 = [0 1 1]′;v3 = [1 1 1]′;**
rref([v1 v2 v3 zeros(3,1)])

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
```

The homogeneous system has only the trivial solution, hence S is linearly independent.

ML.2. Form the augmented matrix $\left[A \mid \mathbf{0}\right]$ and row reduce it.

A = [1 2 0 1;1 1 1 2;2 − 1 5 7;0 2 − 2 − 2];

rref([A zeros(4,1)])

```
ans =
        1   0   2   3   0
        0   1  -1  -1   0
        0   0   0   0   0
        0   0   0   0   0
```

The general solution is $x_4 = s$, $x_3 = t$, $x_2 = t + s$, $x_1 = -2t - 3s$. Hence

$$\mathbf{x} = \begin{bmatrix} -2t - 3s & t + s & t & s \end{bmatrix}' = t \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix}' + s \begin{bmatrix} -3 & 1 & 0 & 1 \end{bmatrix}'$$

and it follows that $\begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix}'$ and $\begin{bmatrix} -3 & 1 & 0 & 1 \end{bmatrix}'$ span the solution space.

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2. (c).

4. (d).

6. If

$$c_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$\begin{bmatrix} c_1 + c_3 & c_1 + c_4 \\ c_2 + c_4 & c_2 + c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The first three entries imply that $c_3 = -c_1 = c_4 = -c_2$. The fourth entry gives $c_2 - c_2 - c_2 = -c_2 = 0$. Thus $c_i = 0$ for $i = 1, 2, 3, 4$. Hence the set of four matrices is linearly independent. By Theorem 6.9, it is a basis.

8. (b); $(2, 1, 3) = 1(1, 1, 2) + 2(2, 2, 0) - 1(3, 4, -1)$.

10. (a) forms a basis: $5t^2 - 3t + 8 = -3(t^2 + t) + 0t^2 + 8(t^2 + 1)$.

12. Possible answer: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$; $\dim W = 3$.

14. Possible answer: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

16. Possible answer: $\{\cos^2 t, \sin^2 t\}$; $\dim W = 2$.

18. (a) $\{(0, 1, 0), (0, 0, 1)\}$. (b) $\{(1, 0, 1, 0), (0, 1, -1, -1)\}$. (c) $\{(1, 1, 0), (-5, 0, 1)\}$.

20. (a) 3. (b) 2.

22. $\{t^3 + t^2, t + 1\}$.

24. (a) 2. (b) 3. (c) 3. (d) 3.

26. 2.

28. (a) Possible answer: $\{(1, 0, 2), (1, 0, 0), (0, 1, 0)\}$.
    (b) Possible answer: $\{(1, 0, 2), (0, 1, 3), (1, 0, 0)\}$.

30. For $a \neq -1, 0, 1$.

32. $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$.

34. The set of all polynomials of the form $at^3 + bt^2 + (b - a)$, where a and b are any real numbers.

36. Possible answer: $\{(-1, 1, 0), (3, 0, 1)\}$.

38. Yes.

40. No.

T.1. Since the largest number of vectors in any linearly independent set is m, $\dim V = m$. The result follows from Theorem 6.9.

T.2. Let $\dim V = n$. First note that any set of vectors in W that is linearly independent in W is linearly independent in V. If $W = \{\mathbf{0}\}$, then $\dim W = 0$ and we are done. Suppose now that W is a nonzero subspace of V. Then W contains a nonzero vector $\mathbf{v}_1$, so $\{\mathbf{v}_1\}$ is linearly independent in W (and in V). If span $\{\mathbf{v}_1\} = W$, then $\dim W = 1$ and we are done. If span $\{\mathbf{v}_1\} \neq W$, then there exists a vector $\mathbf{v}_2$ in W which is not in span $\{\mathbf{v}_1\}$. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent in W (and in V). Since $\dim V = n$, no linearly independent set in W can have more than n vectors. Continuing the above process we find a basis for W containing at most n vectors. Hence W is finite dimensional and $\dim W \leq \dim V$.

T.3. By Theorem 6.7, any linearly independent set T of vectors in V has at most n elements. Thus a set of $n + 1$ vectors must be linearly dependent.

T.4. Suppose a set S of $n - 1$ vectors in V spans V. By Theorem 6.6, some subset of S would be a basis for V. Thus $\dim V \leq n - 1$. Contradiction.

T.5. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be a linearly independent set of vectors in V and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for V (V is finite-dimensional). Let $S_1 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. S_1 spans V (since the subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ does). If S_1 is linearly independent, then it is a basis for V which contains S. Otherwise some vector in S_1 is a linear combination of the preceding vectors (Theorem 6.4). That vector cannot be one of the $\mathbf{w}_i$'s since S is linearly independent. So it is one of the $\mathbf{v}_j$'s. Delete it to form a new set S_2 with one fewer element than S_1 which also spans V. Either S_2 is a basis or else another $\mathbf{v}_j$ can be deleted. After a finite number of steps we arrive at a set S_p which is a basis for V and which contains the given set S.

T.6. (a) By Theorem 6.8, there is a basis T for V which contains $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Since $\dim V = n$, T cannot have more vectors than S. Thus $T = S$.

(b) By Theorem 6.6, some subset T of S is a basis for V. Since $\dim V = n$, T has n elements. Thus $T = S$.

T.7. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a basis for W. By Theorem 6.8, there is a basis T for V which contains the linearly independent set S. Since $\dim W = m = \dim V$, T must have m elements. Thus $T = S$ and $V = W$.

T.8. Let $V = R^3$. Since every vector space has the subspaces $\{\mathbf{0}\}$ and V, then $\{\mathbf{0}\}$ and R^3 are subspaces of R^3. A line ℓ_0 passing through the origin in R^3 parallel to vector

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is the set of all points $P(x, y, z)$ whose associated vector

$$\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is of the form $\mathbf{u} = t\mathbf{v}$, where t is any real scalar. Let $\mathbf{u}_1 = r\mathbf{v}$ and $\mathbf{u}_2 = s\mathbf{v}$ be the associated vectors for two points on line ℓ_0. Then $\mathbf{u}_1 + \mathbf{u}_2 = (r + s)\mathbf{v}$ and hence is an associated vector for a point on ℓ_0. Similarly, $c\mathbf{u}_1 = (cr)\mathbf{v}$ is an associated vector for a point on ℓ_0. Thus ℓ_0 is a subspace of R^3.

Any plane π in R^3 through the origin has an equation of the form

$$ax + by + cz = 0.$$

Sums and scalar multiples of any point on π will also satisfy this equation, hence π is a subspace of R^3.

To show that $\{\mathbf{0}\}$, V, lines, and planes are the only subspace of R^3, we argue as follows. Let W be any subspace of R^3. Hence W contains the zero vector $\mathbf{0}$. If $W \neq \{\mathbf{0}\}$, then it contains a nonzero vector $\mathbf{u} = \begin{bmatrix} a & b & c \end{bmatrix}^T$, where at least one of a, b, or c is not zero. Since W is a subspace it contains span $\{\mathbf{u}\}$. If $W = \text{span}\ \{\mathbf{u}\}$, then W is a line in R^3 through the origin. Otherwise, there exists a vector $\mathbf{v}$ in W which is not in span $\{\mathbf{u}\}$. Hence $\{\mathbf{v}, \mathbf{u}\}$ is a linearly independent set. But then W contains span $\{\mathbf{v}, \mathbf{u}\}$. If $W = \text{span}\ \{\mathbf{v}, \mathbf{u}\}$, then W is a plane through the origin. Otherwise there is a vector $\mathbf{w}$ in W that is not in span $\{\mathbf{v}, \mathbf{u}\}$. Hence $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ is a linearly independent set in W and W contains span $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$. But $\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ is a linearly independent set in R^3, hence a basis for R^3. It follows in this case that $W = R^3$.

T.9. If $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$$

and hence

$$\mathbf{v} = \frac{d_1}{c}(c\mathbf{v}_1) + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n.$$

Therefore $\mathbf{v}$ is a linear combination of $\{c\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Similarly any vector which is a linear combination of the second set

$$\mathbf{v} = d_1(c\mathbf{v}_1) + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n,$$

is a linear combination of the first set:

$$\mathbf{v} = (d_1c)\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n.$$

Thus the two sets span V. Since the second set has n elements, it is also a basis for V.

T.10. The set $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a set of three vectors in the three-dimensional vector space V. One can solve for the $\mathbf{v}$'s in terms of the $\mathbf{w}$'s:

$$\begin{aligned}
&\mathbf{v}_3 = \mathbf{w}_3\\
&\mathbf{v}_2 = \mathbf{w}_2 - \mathbf{v}_3 = \mathbf{w}_2 - \mathbf{w}_3\\
&\mathbf{v}_1 = \mathbf{w}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{w}_1 - (\mathbf{w}_2 - \mathbf{w}_3) - \mathbf{w}_3 = \mathbf{w}_1 - \mathbf{w}_2.
\end{aligned}$$

Thus $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is contained in span T and so $V = \text{span}\ S$ is contained in span T. Hence T is a basis for V.

T.11. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Since every vector in V can be written as a linear combination of the vectors in S, it follows that S spans V. Suppose now that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

We also have

$$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

From the hypothesis it then follows that $c_1 = 0$, $c_2 = 0, \dots, c_n = 0$. Hence, S is a basis for V.

T.12. If A is nonsingular then the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. Let

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_nA\mathbf{v}_n = \mathbf{0}.$$

Then $A[c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n] = \mathbf{0}$ and by the opening remark it must be that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

However, since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent it follows that $c_1 = c_2 = \cdots = c_n = 0$. Hence $\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$ is linearly independent.

T.13. Since A is singular, Theorem 1.13 implies that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{w}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in R^n, it is a basis for R^n, so

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Observe that $\mathbf{w} \neq \mathbf{0}$, so $c_1, c_2, \ldots, c_n$ are not all zero. Then

$$\mathbf{0} = A\mathbf{w} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_n(A\mathbf{v}_n).$$

Hence, $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is linearly dependent.

T.14. Let $d = \max\{d_1, d_2, \ldots, d_k\}$. The polynomial $t^{d+1} + t^d + \cdots + t + 1$ cannot be written as a linear combination of polynomials of degrees $\leq d$.

T.15. Form the equation

$$a_1t^n + a_2t^{n-1} + \cdots + a_nt + a_{n+1} = 0,$$

where the right side is the zero polynomial in P_n:

$$0t^n + 0t^{n-1} + \cdots + 0t + 0.$$

Since these two polynomials agree for all values of t, it follows that corresponding coefficients of like powers of t must agree. Thus

$$a_1 = a_2 = \cdots = a_n = a_{n+1} = 0.$$

Hence, $\{t^n, t^{n-1}, \ldots, t, 1\}$ is a linearly independent set of vectors in P_n.

T.16. If $\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n = \mathbf{0}$ and any one vector, say $\mathbf{v}_k \neq \mathbf{0}$, then adding $\mathbf{v}_k$ to both sides we have

$$\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n + \mathbf{v}_k = \mathbf{v}_1 + \cdots + \mathbf{v}_{k-1} + \mathbf{v}_{k+1} + \cdots + \mathbf{v}_n = \mathbf{v}_k.$$

Hence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. If $\mathbf{v}_j = \mathbf{0}$ for $j = 1, 2, \ldots, n$, then they cannot be a basis.

T.17. (a) $(1, 0, 0) + (0, 1, 0) + (0, 0, 1) = (1, 1, 1)$.
(b) $(1, 1, 0) + (1, 1, 0) + (1, 1, 1) = (1, 1, 1)$.

ML.1. Follow the procedure in Exercise ML.3 in Section 6.2.

```
v1 = [1 2 1]';v2 = [2 1 1]';v3 = [2 2 1]';
rref([v1 v2 v3 zeros(size(v1))])
ans =
     1  0  0  0
     0  1  0  0
     0  0  1  0
```

It follows that the only solution is the trivial solution so S is linearly independent.

ML.2. Follow the procedure in Exercise ML.5(b) in Section 6.2.

```
v1 = [0 2 -2]';v2 = [1 -3 1]';v3 = [2 -8 4]';
rref([v1 v2 v3 zeros(size(v1))])
ans =
     1  0  -1  0
     0  1   2  0
     0  0   0  0
```

It follows that there is a nontrivial solution so S is linearly dependent and cannot be a basis for V.

ML.3. Proceed as in ML.1.

v1 = [1 1 0 1]′;v2 = [2 1 1 − 1]′;v3 = [0 0 1 1]′;v4 = [1 2 1 2]′;

rref([v1 v2 v3 v4 zeros(size(v1))])

```
ans =
        1   0   0   0   0
        0   1   0   0   0
        0   0   1   0   0
        0   0   0   1   0
```

It follows that S is linearly independent and since $\dim V = 4$, S is a basis for V.

ML.4. Here we do not know dim(span S), but dim(span S) = the number of linearly independent vectors in S. We proceed as we did in ML.1.

v1 = [1 2 1 0]′;v2 = [2 1 3 1]′;v3 = [2 − 2 4 2]′;

rref([v1 v2 v3 zeros(size(v1))])

```
ans =
        1   0  -2   0
        0   1   2   0
        0   0   0   0
        0   0   0   0
```

The leading 1's imply that $\mathbf{v}_1$ and $\mathbf{v}_2$ are a linearly independent subset of S, hence dim(span S) = 2 and S is not a basis for V.

ML.5. Here we do not know dim(span S), but dim(span S) = the number of linearly independent vectors in S. We proceed as we did in ML.1.

v1 = [1 2 1 0]′;v2 = [2 1 3 1]′;v3 = [2 2 1 2]′;

rref([v1 v2 v3 zeros(size(v1))])

```
ans =
        1   0   0   0
        0   1   0   0
        0   0   1   0
        0   0   0   0
```

The leading 1's imply that S is a linearly independent set hence dim(span S) = 3 and S is a basis for V.

ML.6. Any vector in V has the form

$$(a, b, c) = (a, 2a - c, c) = a(1, 2, 0) + c(0, -1, 1).$$

It follows that $T = \{(1, 2, 0), (0, -1, 1)\}$ spans V and since the members of T are not multiples of one another, T is a linearly independent subset of V. Thus $\dim V = 2$. We need only determine if S is a linearly independent subset of V. Let

v1 = [0 1 − 1]′;v2 = [1 1 1]′;

then

rref([v1 v2 zeros(size(v1))])

```
ans =
    1  0  0
    0  1  0
    0  0  0
```

It follows that S is linearly independent and so Theorem 6.9 implies that S is a basis for V.

In Exercises ML.7 through ML.9 we use the technique involving leading 1's as in Example 5.

ML.7. **v1 = [1 1 0 0]′;v2 = [−2 −2 0 0]′;v3 = [1 0 2 1]′;v4 = [2 1 2 1]′;v5 = [0 1 1 1]′;**

rref([v1 v2 v3 v4 v5 zeros(size(v1))])

```
ans =
    1  −2  0  1  0  0
    0   0  1  1  0  0
    0   0  0  0  1  0
    0   0  0  0  0  0
```

The leading 1's point to vectors $\mathbf{v}_1$, $\mathbf{v}_3$ and $\mathbf{v}_5$ and hence these vectors are a linearly independent set which also spans S. Thus $T = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$ is a basis for span S. We have dim(span S) = 3 and span $S \neq R^4$.

ML.8. Associate a column with each 2×2 matrix as in Exercise ML.4(b) in Section 6.2.

v1 = [1 2;1 2]′;v2 = [1 0;1 1]′;v3 = [0 2;0 1]′;v4 = [2 4;2 4]′;v5 = [1 0;0 1]′;

rref([reshape(v1,4,1) reshape(v2,4,1) reshape(v3,4,1) reshape(v4,4,1) reshape(v5,4,1) zeros(4,1)])

```
ans =
    1  0   1  2  0  0
    0  1  −1  0  0  0
    0  0   0  0  1  0
    0  0   0  0  0  0
```

The leading 1's point to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_5$ which are a basis for span S. We have dim(span S) = 3 and span $S \neq M_{22}$.

ML.9. Proceed as in ML.2.

v1 = [0 1 − 2]′;v2 = [0 2 1]′;v3 = [0 4 − 2]′;v4 = [1 − 1 1]′;v5 = [1 2 1]′;

rref([v1 v2 v3 v4 v5 zeros(size(v1))])

```
ans =
    1.0000       0  1.6000       0  0.6000  0
         0  1.0000  1.2000       0  1.2000  0
         0       0       0  1.0000  1.0000  0
```

It follows that $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for span S. We have dim(span S) = 3 and it follows that span $S = P_2$.

ML.10. **v1 = [1 1 0 0]′;v2 = [1 0 1 0]′;**

rref([v1 v2 eye(4) zeros(size(v1))])

```
ans =
      1   0   0    1    0   0   0
      0   1   0    0    1   0   0
      0   0   1   −1   −1   0   0
      0   0   0    0    0   1   0
```

It follows that $\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1 = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}', \mathbf{e}_4 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}'\right\}$ is a basis for V which contains S.

ML.11. **v1 = [1 0 − 1 1]′;v2 = [1 0 0 2]′;**
rref([v1 v2 eye(4) zeros(size(v1))])

```
ans =
      1.0000        0        0        0   −1.0000         0   0
           0   1.0000        0        0    0.5000    0.5000   0
           0        0   1.0000        0    0.5000   −0.5000   0
           0        0        0   1.0000        0         0   0
```

It follows that $\left\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3 = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}', \mathbf{e}_4 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}'\right\}$ is a basis for R^4. Hence, a basis for P_3 is $\{t^3 - t + 1, t^3 + 2, t, 1\}$.

ML.12. Any vector in V has the form $(a, 2d + e, a, d, e)$. It follows that

$$(a, 2d + e, a, d, e) = a(1, 0, 1, 0, 0) + d(0, 2, 0, 1, 0) + e(0, 1, 0, 0, 1)$$

and $T = \{(1, 0, 1, 0, 0), (0, 2, 0, 1, 0), (0, 1, 0, 0, 1)\}$ is a basis for V. Hence let
v1 = [0 3 0 2 − 1]′;w1 = [1 0 1 0 0]′;w2 = [0 2 0 1 0]′;w3 = [0 1 0 0 1]′;
then
rref([v1 w1 w2 w3 eye(4) zeros(size(v1))])

```
ans =
      1   0   0  −1   0
      0   1   0   0   0
      0   0   1   2   0
      0   0   0   0   0
      0   0   0   0   0
```

Thus $\{\mathbf{v}_1, \mathbf{w}_1, \mathbf{w}_2\}$ is a basis for V containing S.

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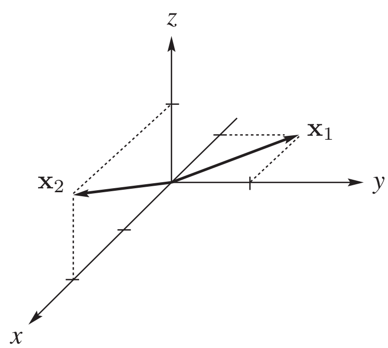
2. (a) $x = -r + 2s$, $y = r$, $z = s$, where r, s are any real numbers.

   (b) Let $\mathbf{x}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}2\\0\\1\end{bmatrix}$. Then

$$\begin{bmatrix}-r + 2s\\r\\s\end{bmatrix} = r\begin{bmatrix}-1\\1\\0\end{bmatrix} + s\begin{bmatrix}2\\0\\1\end{bmatrix} = r\mathbf{x}_1 + s\mathbf{x}_2.$$

(c)



4. $\left\{ \begin{bmatrix} 1\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -4\\0\\6\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\-1\\0\\1 \end{bmatrix} \right\}$; dimension = 3.

6. $\left\{ \begin{bmatrix} 4\\3\\-2\\1\\0 \end{bmatrix}, \begin{bmatrix} 1\\1\\-2\\0\\1 \end{bmatrix} \right\}$; dimension = 2.

8. No basis; dimension = 0.

10. $\left\{ \begin{bmatrix} -2\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 17\\0\\5\\1\\0\\0 \end{bmatrix} \right\}$; dimension = 2.

12. $\left\{ \begin{bmatrix} 0\\3\\1\\1\\0 \end{bmatrix}, \begin{bmatrix} 0\\-6\\-3\\0\\1 \end{bmatrix} \right\}$.

14. $\left\{ \begin{bmatrix} -3\\1 \end{bmatrix} \right\}$.

16. No basis.

18. $\lambda = 3, -2$.

20. $\lambda = 1, 2, -2$.

22. $\mathbf{x}_p = \begin{bmatrix} \frac{32}{23}\\ \frac{13}{23}\\ \frac{2}{23}\\ 0 \end{bmatrix}, \mathbf{x}_h = r \begin{bmatrix} \frac{2}{23}\\ \frac{8}{23}\\ -\frac{20}{23}\\ 1 \end{bmatrix}$.

24. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis. Dimension of solution space = 2.

26. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis. Dimension of solution space = 1.

28. $\mathbf{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_h = b_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

T.1. Since each vector in S is a solution to $A\mathbf{x} = \mathbf{0}$, we have $A\mathbf{x}_i = \mathbf{0}$ for $i = 1, 2, \ldots, n$. The span of S consists of all possible linear combinations of the vectors in S, hence

$$\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

represents an arbitrary member of span S. We have

$$A\mathbf{y} = c_1A\mathbf{x}_1 + c_2A\mathbf{x}_2 + \cdots + c_kA\mathbf{x}_k = c_1\mathbf{0} + c_2\mathbf{0} + \cdots + c_k\mathbf{0} = \mathbf{0}.$$

Thus $\mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{0}$ and it follows that every member of span S is a solution to $A\mathbf{x} = \mathbf{0}$.

T.2. If A has a row or a column of zeros, then $\det(A) = 0$ and it follows that matrix A is singular. Theorem 1.12 then implies that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

T.3. (a) Set $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Since the dimension of the null space of A is 3, the null space of A is R^3. Then the natural basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for the null space of A. Forming $A\mathbf{e}_1 = \mathbf{0}$, $A\mathbf{e}_2 = \mathbf{0}$, $A\mathbf{e}_3 = \mathbf{0}$, we find that all the columns of A must be zero. Hence, $A = O$.

(b) Since $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, the null space of A contains a nonzero vector, so the dimension of the null space of A is not zero. If this dimension is 3, then by part (a), $A = O$, a contradiction. Hence, the dimension is either 1 or 2.

T.4. Since the reduced row echelon forms of matrices A and B are the same it follows that the solutions to the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are the same set of vectors. Hence the null spaces of A and B are the same.

ML.1. Enter A into MATLAB and we find that

rref(A)

```
ans =
     1   0   2   1    2
     0   1   0   1   −1
     0   0   0   0    0
```

Write out the solution to the linear system $A\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = r \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the null space of A consists of the three vectors above. We can compute such a basis directly using the command **homsoln** as shown next.

homsoln(A)

```
ans =
      -2   -1   -2
       0   -1    1
       1    0    0
       0    1    0
       0    0    1
```

ML.2. Enter A into MATLAB and we find that

rref(A)

```
ans =
      1   0   0
      0   1   0
      0   0   1
      0   0   0
      0   0   0
```

The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

ML.3. Enter A into MATLAB and we find that

rref(A)

```
ans =
      1.0000         0   -1.0000   -1.3333
           0    1.0000    2.0000    0.3333
           0         0         0         0
```

format rat, ans

```
ans =
      1   0   -1   -4/3
      0   1    2    1/3
      0   0    0      0
```

format

Write out the solution to the linear system $A\mathbf{x} = \mathbf{0}$ as

$$\mathbf{x} = r \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{4}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}.$$

A basis for the null space of A consists of the two vectors above. We can compute such a basis directly using command **homsoln** as shown next.

homsoln(A)

```
ans =
     1.0000    1.3333
    -2.0000   -0.3333
     1.0000         0
          0    1.0000
```

format rat, ans

```
ans =
     1     4/3
    -2    -1/3
     1       0
     0       1
```

format

ML.4. Form the matrix $3I_2 - A$ in MATLAB as follows.

C = 3 ∗ eye(2) − [1 2;2 1]

```
C =
     2   -2
    -2    2
```

rref(C)

```
ans =
     1   -1
     0    0
```

The solution is $\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix}$, for t any real number. Just choose $t \neq 0$ to obtain a nontrivial solution.

ML.5. Form the matrix $6I_3 - A$ in MATLAB as follows.

C = 6 ∗ eye(3) − [1 2 3;3 2 1;2 1 3]

```
C =
     5   -2   -3
    -3    4   -1
    -2   -1    3
```

rref(C)

```
ans =
     1    0   -1
     0    1   -1
     0    0    0
```

The solution is $\mathbf{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$, for t any nonzero real number. Just choose $t \neq 0$ to obtain a nontrivial solution.

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2. $\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

4. $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

6. (a) $\left\{\left(1,0,0,-\frac{33}{7}\right), \left(0,1,0,\frac{23}{7}\right), \left(0,0,1,-\frac{8}{7}\right)\right\}$. (b) $\{(1,2,-1,3),(3,5,2,0),(0,1,2,1)\}$.

8. (a) $\left\{\begin{bmatrix}1\\0\\4\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-\frac{5}{2}\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$. (b) $\left\{\begin{bmatrix}-2\\-2\\-3\\4\end{bmatrix}, \begin{bmatrix}2\\2\\3\\-2\end{bmatrix}, \begin{bmatrix}3\\4\\2\\1\end{bmatrix}\right\}$.

10. Basis for row space of A: $\left\{\begin{bmatrix}1 & 0 & -2 & 4\end{bmatrix}, \begin{bmatrix}0 & 1 & 1 & -1\end{bmatrix}\right\}$

    Basis for column space of A: $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

    Basis for row space of A^T: $\left\{\begin{bmatrix}1 & 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & 1\end{bmatrix}\right\}$

    Basis for column space of A^T: $\left\{\begin{bmatrix}1\\0\\-2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\\-1\end{bmatrix}\right\}$

    A basis for the column space of A^T consists of the transposes of a corresponding basis for the row space of A. Similarly, a basis for the row space of A^T consists of the transposes of a corresponding basis for the column space of A.

12. 5.

14. rank = 4, nullity = 0.

16. 2.

18. 3.

20. The five rows of A span a row space of dimension rank A, which is at most 3. Thus the five rows are linearly dependent.

22. Linearly independent. Since rank $A = 3$, dim(column space A) = 3. The three column vectors of A span the column space of A and are then a basis for the column space. Hence, they are linearly independent.

24. Nonsingular.

26. No.

28. Yes, linearly independent.

30. Only the trivial solution.

32. Only the trivial solution.

34. Has a solution.

36. Has no solution.

38. 3.

40. 3.

T.1. Rank $A = n$ if and only if A is nonsingular if and only if $\det A \neq 0$.

T.2. Let rank $A = n$. Then Theorem 6.13 implies that A is nonsingular, so $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, then $A\mathbf{x}_1 = A\mathbf{x}_2$ and multiplying both sides on the left by A^{-1}, we have $\mathbf{x}_1 = \mathbf{x}_2$. Thus, $A\mathbf{x} = \mathbf{b}$ has a unique solution. Conversely, suppose that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$. Then the n linear systems $A\mathbf{x} = \mathbf{e}_1, A\mathbf{x} = \mathbf{e}_2, \dots, A\mathbf{x} = \mathbf{e}_n$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are the columns of I_n, have solutions $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Let B be the matrix whose jth column is $\mathbf{x}_j$. Then the n linear systems above can be written as $AB = I_n$. Hence, $B = A^{-1}$, so A is nonsingular and Theorem 6.13 implies that rank $A = n$.

T.3. $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent if and only if A has rank n if and only if $\det(A) \neq 0$.

T.4. If $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, then by Corollary 6.5 rank $A < n$. Hence column rank $A < n$ and it follows that the columns of A are linearly dependent. If the columns of A are linearly dependent, then by Corollary 6.4, $\det(A) = 0$. It follows by Corollary 6.2 that rank $A < n$ and then Corollary 6.5 implies that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

T.5. Let rank $A = n$. Then $n =$ rank $A =$ column rank A implies that the n columns of A are linearly independent. Conversely, suppose the columns of A are linearly independent. Then $n =$ column rank $A =$ rank A.

T.6. If the rows of A are linearly independent, $n =$ rank $A =$ column rank A and the n columns of A span the entire space R^n. Conversely, if the n columns of A span R^n then by Theorem 6.9(n) they are a basis for R^n. Hence the columns of A are linearly independent which implies that $n =$ column rank $A =$ row rank A. Thus the rows of A are linearly independent.

T.7. Let $A\mathbf{x} = \mathbf{b}$ have a solution for every $m \times 1$ matrix $\mathbf{b}$. Then the columns of A span R^m. Thus there is a subset of m columns of A that is a basis for R^m and rank $A = m$. Conversely, if rank $A = m$, then column rank $A = m$. Thus m columns of A are a basis for R^m and hence all the columns of A span R^m. Since $\mathbf{b}$ is in R^m, it is a linear combination of the columns of A; that is, $A\mathbf{x} = \mathbf{b}$ has a solution for every $m \times 1$ matrix $\mathbf{b}$.

T.8. Suppose that the columns of A are linearly independent. Then rank $A = n$, so by Theorem 6.12, nullity $A = 0$. Hence, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Conversely, if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then nullity $A = 0$, so by Theorem 6.12, rank $A = n$. This means that

$$\text{column rank } A = \dim(\text{column space } A) = n.$$

Since A has n columns which span its column space, it follows that they are linearly independent.

T.9. Suppose that the linear system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $m \times 1$ matrix $\mathbf{b}$. Since $A\mathbf{x} = \mathbf{0}$ always has the trivial solution, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Conversely, suppose that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then nullity $A = 0$, so by Theorem 6.12, rank $A = n$. Thus, dim(column space A) $= n$, so the n columns of A, which span its column space, form a basis for the column space. If $\mathbf{b}$ is an $m \times 1$ matrix then $\mathbf{b}$ is a vector in R^m. If $\mathbf{b}$ is in the column space of A, then B can be written as a linear combination of the columns of A in one and only one way. That is, $A\mathbf{x} = \mathbf{b}$ has exactly one solution. If $\mathbf{b}$ is not in the column space of A, then $A\mathbf{x} = \mathbf{b}$ has no solution. Thus, $A\mathbf{x} = \mathbf{b}$ has at most one solution.

T.10. Since the rank of a matrix is the same as its row rank and column rank, the number of linearly independent rows of a matrix is the same as the number of linearly independent columns. It follows that the largest the rank can be is $\min\{m, n\}$. Since $m \neq n$, it must be that either the rows or columns are linearly independent.

T.11. Suppose that $A\mathbf{x} = \mathbf{b}$ is consistent. Assume that there are at least two different solutions $\mathbf{x}_1$ and $\mathbf{x}_2$. Then $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$, so

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

That is, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution so nullity $A > 0$. By Theorem 6.12, rank $A < n$. Conversely, if rank $A < n$, then by Corollary 6.5, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{y}$. Suppose that $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{b}$. Thus, $A\mathbf{y} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$. Then $\mathbf{x}_0 + \mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{b}$, since

$$A(\mathbf{x}_0 + \mathbf{y}) = A\mathbf{x}_0 + A\mathbf{y} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Since $\mathbf{y} \neq \mathbf{0}$, $\mathbf{x}_0 + \mathbf{y} \neq \mathbf{x}_0$, so $A\mathbf{x} = \mathbf{b}$ has more than one solution.

T.12. We must show that the rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ of AA^T are linearly independent. Consider

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m = \mathbf{0}$$

which can be written in matrix form as $\mathbf{x}A = \mathbf{0}$, where $\mathbf{x} = (a_1, a_2, \ldots, a_m)$. Multiplying this equation by A^T we have $\mathbf{x}AA^T = \mathbf{0}$. Since AA^T is nonsingular, Theorem 1.13 implies that $\mathbf{x} = \mathbf{0}$, so $a_1 = a_2 = \cdots = a_m = 0$. Hence rank $A = m$.

ML.2. (a) One basis for the row space of A consists of the nonzero rows of rref(A).

A = [1 3 1;2 5 0;4 11 2;6 9 1];

rref(A)

```
ans =
     1  0  0
     0  1  0
     0  0  1
     0  0  0
```

Another basis is found using the leading 1's of rref(A^T) to point to rows of A that form a basis for the row space of A.

rref(A′)

```
ans =
     1  0  2  0
     0  1  1  0
     0  0  0  1
```

It follows that rows 1, 2, and 4 of A are a basis for the row space of A.

(b) Follow the same procedure as in part (a).

A = [2 1 2 0;0 0 0 0;1 2 2 1;4 5 6 2;3 3 4 1];

```
ans =
     1.0000        0   0.6667  −0.3333
          0   1.0000   0.6667   0.6667
          0        0        0        0
          0        0        0        0
```

format rat, ans

```
ans =
     1  0  2/3  −1/3
     0  1  2/3   2/3
     0  0    0     0
     0  0    0     0
```

format
rref(A′)

```
ans =
        1   0   0   1   1
        0   0   1   2   1
        0   0   0   0   0
        0   0   0   0   0
```

It follows that rows 1 and 2 of A are a basis for the row space of A.

ML.3. (a) The transposes of the nonzero rows of rref(A^T) give us one basis for the column space of A.

A = [1 3 1;2 5 0;4 11 2;6 9 1];
rref(A′)

```
ans =
        1   0   2   0
        0   1   1   0
        0   0   0   1
```

The leading ones of rref(A) point to the columns of A that form a basis for the column space of A.

rref(A)

```
ans =
        1   0   0
        0   1   0
        0   0   1
        0   0   0
```

Thus columns 1, 2, and 3 of A are a basis for the column space of A.

(b) Follow the same procedure as in part (a).

A = [2 1 2 0;0 0 0 0;1 2 2 1;4 5 6 2;3 3 4 1];
rref(A′)

```
ans =
        1   0   0   1   1
        0   0   1   2   1
        0   0   0   0   0
        0   0   0   0   0
```

rref(A)

```
ans =
        1.0000        0   0.6667   −0.3333
             0   1.0000   0.6667    0.6667
             0        0        0         0
             0        0        0         0
```

Thus columns 1 and 2 of A are a basis for the column space of A.

ML.4. (a) **A = [3 2 1;1 2 − 1;2 1 3];**
rank(A)

```
ans =
        3
```

The nullity of A is 0.

(b) **A = [1 2 1 2 1;2 1 0 0 2;1 − 1 − 1 − 2 1;3 0 − 1 − 2 3];**
rank(A)

```
ans =

     2
```

The nullity of $A = 5 - \text{rank}(A) = 3$.

ML.5. Compare the rank of the coefficient matrix with the rank of the augmented matrix as in Theorem 6.14.

(a) **A = [1 2 4 − 1;0 1 2 0;3 1 1 − 2];b = [21 8 16]′;**
rank(A),rank([A b])

```
ans =

     3

ans =

     3
```

The system is consistent.

(b) **A = [1 2 1;1 1 0;2 1 − 1];b = [3 3 3]′;**
rank(A),rank([A b])

```
ans =

     2

ans =

     3
```

The system is inconsistent.

(c) **A = [1 2;2 0;2 1; − 1 2];b = [3 2 3 2]′;**
rank(A),rank([A b])

```
ans =

     2

ans =

     3
```

The system is inconsistent.

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2. $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. 4. $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. 6. $\begin{bmatrix} -1 \\ 2 \\ -2 \\ 4 \end{bmatrix}$. 8. $(3, 1, 3)$. 10. $t^2 - 3t + 2$. 12. $\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

14. (a) $\left[\mathbf{v}\right]_T = \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix}$, $\left[\mathbf{w}\right]_T = \begin{bmatrix} 1 \\ -2 \\ 13 \end{bmatrix}$. (b) $\begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}$.

(c) $\left[\mathbf{v}\right]_S = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\left[\mathbf{w}\right]_S = \begin{bmatrix} -18 \\ -17 \\ 8 \end{bmatrix}$. (d) Same as (c).

(e) $\begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}$. (f) Same as (a).

16. (a) $\left[\mathbf{v}\right]_T = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$, $\left[\mathbf{w}\right]_T = \begin{bmatrix} 0 \\ 8 \\ -6 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$.

(c) $\left[\mathbf{v}\right]_S = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}$, $\left[\mathbf{w}\right]_S = \begin{bmatrix} 8 \\ -4 \\ -2 \end{bmatrix}$. (d) Same as (c).

(e) $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. (f) Same as (a).

18. (a) $\left[\mathbf{v}\right]_T = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 2 \end{bmatrix}$, $\left[\mathbf{w}\right]_T = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$. (b) $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(c) $\left[\mathbf{v}\right]_S = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix}$, $\left[\mathbf{w}\right]_S = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \end{bmatrix}$. (d) Same as (c).

(e) $\begin{bmatrix} -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. (f) Same as (a).

20. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. 22. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. 24. $S = \{t+1, 5t-2\}$. 26. $S = \{-t+5, t-3\}$.

T.1. Let $\mathbf{v} = \mathbf{w}$. The coordinates of a vector relative to basis S are the coefficients used to express the vector in terms of the members of S. A vector has a unique expression in terms of the vectors of a basis, hence it follows that $\left[\mathbf{v}\right]_S$ must equal $\left[\mathbf{w}\right]_S$. Conversely, let

$$\left[\mathbf{v}\right]_S = \left[\mathbf{w}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Then

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Hence $\mathbf{v} = \mathbf{w}$.

T.2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$
$$\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n.$$

Then

$$\left[\mathbf{v}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \left[\mathbf{w}\right]_S = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We also have

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= (a_1+b_1)\mathbf{v}_1+(a_2+b_2)\mathbf{v}_2+\cdots+(a_n+b_n)\mathbf{v}_n\\ c\mathbf{v} &= (ca_1)\mathbf{v}_1+(ca_2)\mathbf{v}_2+\cdots+(ca_n)\mathbf{v}_n,\end{aligned}$$

so

$$\left[\mathbf{v}+\mathbf{w}\right]_S = \begin{bmatrix}a_1+b_1\\ a_2+b_2\\ \vdots\\ a_n+b_n\end{bmatrix} = \begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix} = \left[\mathbf{v}\right]_S + \left[\mathbf{w}\right]_S$$

$$\left[c\mathbf{v}\right] = \begin{bmatrix}ca_1\\ ca_2\\ \vdots\\ ca_n\end{bmatrix} = c\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix} = c\left[\mathbf{v}\right]_S.$$

T.3. Suppose that $\left\{\left[\mathbf{w}_1\right]_S, \left[\mathbf{w}_2\right]_S, \ldots, \left[\mathbf{w}_k\right]_S\right\}$ is linearly dependent. Then there exist scalars a_i, $i = 1, 2, \ldots, k$, that are not all zero, such that

$$a_1\left[\mathbf{w}_1\right]_S + a_2\left[\mathbf{w}_2\right]_S + \cdots + a_n\left[\mathbf{w}_k\right]_S = \left[\mathbf{0}_V\right]_S.$$

Using Exercise T.2 we find that the preceding equation is equivalent to

$$\left[a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k\right]_S = \left[\mathbf{0}_V\right]_S.$$

By Exercise T.1 we have

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_k\mathbf{w}_k = \mathbf{0}_V.$$

Since the $\mathbf{w}$'s are linearly independent, the preceding equation is only true when all $a_i = 0$. Hence we have a contradiction and our assumption that the $\left[\mathbf{w}_i\right]_S$'s are linearly dependent must be false. It follows that $\left\{\left[\mathbf{w}_1\right]_S, \left[\mathbf{w}_2\right]_S, \ldots, \left[\mathbf{w}_k\right]_S\right\}$ is linearly independent.

T.4. From Exercise T.3 we know that $T = \left\{\left[\mathbf{v}_1\right]_S, \left[\mathbf{v}_2\right]_S, \ldots, \left[\mathbf{v}_n\right]_S\right\}$ is a linearly independent set of vectors in R^n. By Theorem 6.9, T spans R^n and is thus a basis for R^n.

T.5. Consider the homogeneous system $M_S\mathbf{x} = \mathbf{0}$, where

$$\mathbf{x} = \begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{bmatrix}.$$

This system can then be written in terms of the columns of M_S as

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0},$$

where $\mathbf{v}_j$ is the jth column of M_S. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, we have $a_1 = a_2 = \cdots = a_n = 0$. Thus, $\mathbf{x} = \mathbf{0}$ is the only solution to $M_S\mathbf{x} = \mathbf{0}$, so by Theorem 1.13 we conclude that M_S is nonsingular.

T.6. Let $\mathbf{v}$ be a vector in V. Then $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$. This last equation can be written in matrix form as

$$\mathbf{v} = M_S\left[\mathbf{v}\right]_S,$$

where M_S is the matrix whose jth column is $\mathbf{v}_j$. Similarly, $\mathbf{v} = M_T\left[\mathbf{v}\right]_T$.

T.7. (a) From Exercise T.6 we have

$$M_S\left[\mathbf{v}\right]_S = M_T\left[\mathbf{v}\right]_T.$$

From Exercise T.5 we know that M_S is nonsingular, so

$$\left[\mathbf{v}\right]_S = M_S^{-1}M_T\left[\mathbf{v}\right]_T.$$

Equation (2) is

$$\left[\mathbf{v}\right]_S = P_{S\leftarrow T}\left[\mathbf{v}\right]_T,$$

so

$$P_{S\leftarrow T} = M_S^{-1}M_T.$$

(b) Since M_S and M_T are nonsingular, M_S^{-1} is nonsingular, so $P_{S\leftarrow T}$, as the product of two nonsingular matrices, is nonsingular.

(c) $M_S = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $M_T = \begin{bmatrix} 6 & 4 & 5 \\ 3 & -1 & 5 \\ 3 & 3 & 2 \end{bmatrix}$, $M_S^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{4}{3} \end{bmatrix}$, $P_{S\leftarrow T} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

ML.1. Since S is a set consisting of three vectors in a 3-dimensional vector space, we can show that S is a basis by verifying that the vectors in S are linearly independent. It follows that if the reduced row echelon form of the three columns is I_3, they are linearly independent.

```
A = [1 2 1;2 1 0;1 0 2];
rref(A)
ans =
      1  0  0
      0  1  0
      0  0  1
```

To find the coordinates of $\mathbf{v}$ we solve the system $A\mathbf{c} = \mathbf{v}$. We can do all three parts simultaneously as follows. Put the three columns whose coordinates we want to find into a matrix B.

```
B = [8 2 4;4 0 3;7 − 3 3];
rref([A B])
ans =
      1  0  0  1  −1  1
      0  1  0  2   2  1
      0  0  1  3  −1  1
```

The coordinates appear in the last three columns of the matrix above.

ML.2. Proceed as in ML.1 by making each of the vectors in S a column in matrix A.

```
A = [1 0 1 1;1 2 1 3;0 2 1 1;0 1 0 0]';
rref(A)
ans =
      1  0  0  0
      0  1  0  0
      0  0  1  0
      0  0  0  1
```

To find the coordinates of **v** we solve a linear system. We can do all three parts simultaneously as follows. Associate with each vector **v** a column. Form a matrix B from these columns.

B = [4 12 8 14;1/2 0 0 0;1 1 1 7/3]′;

rref([A B])

```
ans =
    1.0000         0         0         0    1.0000    0.5000    0.3333
         0    1.0000         0         0    3.0000         0    0.6667
         0         0    1.0000         0    4.0000   -0.5000         0
         0         0         0    1.0000   -2.0000    1.0000   -0.3333
```

The coordinates are the last three columns of the preceding matrix.

ML.3. Associate a column with each matrix and proceed as in ML.2.

A = [1 1 2 2;0 1 2 0;3 − 1 1 0; −1 0 0 0]′;

rref(A)

```
ans =
    1  0  0  0
    0  1  0  0
    0  0  1  0
    0  0  0  1
```

B = [1 0 0 1;2 7/6 10/3 2;1 1 1 1]′;

rref([A B])

```
ans =
    1.0000         0         0         0    0.5000   1.0000    0.5000
         0    1.0000         0         0   -0.5000   0.5000    0.1667
         0         0    1.0000         0         0   0.3333   -0.3333
         0         0         0    1.0000   -0.5000        0   -1.5000
```

The coordinates are the last three columns of the preceding matrix.

ML.4. **A = [1 0 1;1 1 0;0 1 1];**

B = [2 1 1;1 2 1;1 1 2];

rref([A B])

```
ans =
    1  0  0  1  1  0
    0  1  0  0  1  1
    0  0  1  1  0  1
```

The transition matrix from the T-basis to the S-basis is P = **ans(:,4:6)**.

```
P =
  1  1  0
  0  1  1
  1  0  1
```

ML.5. **A = [0 0 1 − 1;0 0 1 1;0 1 1 0;1 0 − 1 0]′;**

B = [0 1 0 0;0 0 − 1 1;0 − 1 0 2;1 1 0 0]′;

rref([A B])

```
ans =
    1.0000        0        0        0  -0.5000  -1.0000  -0.5000        0
         0   1.0000        0        0  -0.5000        0   1.5000        0
         0        0   1.0000        0   1.0000        0  -1.0000   1.0000
         0        0        0   1.0000        0        0        0   1.0000
```

The transition matrix P is found in columns 5 through 8 of the preceding matrix.

ML.6. **A = [1 2 3 0;0 1 2 3;3 0 1 2;2 3 0 1]′;**

B = eye(4);

rref([A B])

```
ans =
    1.0000        0        0        0   0.0417   0.0417   0.2917  -0.2083
         0   1.0000        0        0  -0.2083   0.0417   0.0417   0.2917
         0        0   1.0000        0   0.2917  -0.2083   0.0417   0.0417
         0        0        0   1.0000   0.0417   0.2917  -0.2083   0.0417
```

The transition matrix P is found in columns 5 through 8 of the preceding matrix.

ML.7. We put basis S into matrix A, T into B, and U into C.

A = [1 1 0;1 2 1;1 1 1];
B = [1 1 0;0 1 1;1 0 2];
C = [2 − 1 1;1 2 − 2 1;1 1 1];

(a) The transition matrix from U to T will be the last 3 columns of rref([B C]).

rref([B C])

```
ans =
    1.0000        0        0   1.0000  -1.6667   2.3333
         0   1.0000        0   1.0000   0.6667  -1.3333
         0        0   1.0000        0   1.3333  -0.6667
```

P = ans(:,4:6)

```
P =
    1.0000  -1.6667   2.3333
    1.0000   0.6667  -1.3333
         0   1.3333  -0.6667
```

(b) The transition matrix from T to S will be the last 3 columns of rref([A B]).

rref([A B])

```
ans =
     1     0     0     2     0     1
     0     1     0    -1     1    -1
     0     0     1     0    -1     2
```

Q = ans(:,4:6)

```
Q =
     2     0     1
    -1     1    -1
     0    -1     2
```

(c) The transition matrix from U to S will be the last 3 columns of rref([A C]).

rref([A C])

```
ans =
        1   0   0    2  −2    4
        0   1   0    0   1   −3
        0   0   1   −1   2    0
```

Z = ans(:,4:6)

```
Z =
        2  −2    4
        0   1   −3
       −1   2    0
```

(d) **Q ∗ P** gives Z.

Section 6.8, p. 359

2. (a).

4. $a = b = \frac{1}{2}$ or $a = b = -\frac{1}{2}$.

6. $\left\{\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right), \left(-\frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}, \frac{2}{3\sqrt{5}}\right)\right\}$.

8. $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right), \left(-\frac{2}{\sqrt{33}}, \frac{4}{\sqrt{33}}, \frac{2}{\sqrt{33}}, \frac{3}{\sqrt{33}}\right), \left(\frac{4}{\sqrt{110}}, \frac{3}{\sqrt{110}}, \frac{7}{\sqrt{110}}, -\frac{6}{\sqrt{110}}\right)\right\}$.

10. (a) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$.

    (b) $(2, 3, 1) = \frac{6}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \frac{2}{\sqrt{2}}\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

12. Possible answer: $\left\{(0,0,1), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right\}$.

14. $\left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)\right\}$.

16. $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\right\}$.

18. $\left\{\frac{1}{\sqrt{26}}\begin{bmatrix}-3\\4\\1\end{bmatrix}\right\}$.

20. $\frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = (2, 3)$.

T.1. $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$ and 1 for $i = j$.

T.2. An orthonormal set in R^n is an orthogonal set of nonzero vectors. The result follows from Theorem 6.16.

T.3. Since an orthonormal set of vectors is an orthogonal set, the result follows by Theorem 6.16.

T.4. (a) Let $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ be the expression for $\mathbf{v}$ in terms of the basis S. Then

$$\mathbf{v} \cdot \mathbf{v}_i = \left(\sum_{j=1}^{n} c_j\mathbf{v}_j\right) \cdot \mathbf{v}_i = \sum_{j=1}^{n} c_j(\mathbf{v}_j \cdot \mathbf{v}_i) = c_i$$

for $1 \leq i \leq n$.

(b) If

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n,$$

taking the dot product of both sides of the equation with $\mathbf{u}_i$, $1 \le i \le n$, we have

$$\mathbf{u}_i \cdot \mathbf{v} = c_1(\mathbf{u}_i \cdot \mathbf{u}_1) + c_2(\mathbf{u}_i \cdot \mathbf{u}_2) + \cdots + c_i(\mathbf{u}_i \cdot \mathbf{u}_i) + \cdots + c_n(\mathbf{u}_i \cdot \mathbf{u}_n) = c_i(\mathbf{u}_i \cdot \mathbf{u}_i).$$

Since $\mathbf{u}_i \neq \mathbf{0}$, we conclude that

$$c_i = \frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i}.$$

T.5. If $\mathbf{u}$ is orthogonal to $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $\mathbf{u} \cdot \mathbf{v}_j = 0$ for $j = 1, \ldots, n$. Let $\mathbf{w}$ be in span S. Then $\mathbf{w}$ is a linear combination of the vectors in S:

$$\mathbf{w} = \sum_{j=1}^{n} c_j \mathbf{v}_j.$$

Thus

$$\mathbf{u} \cdot \mathbf{w} = \sum_{j=1}^{n} c_j(\mathbf{u} \cdot \mathbf{v}_j) = \sum_{j=1}^{n} c_j 0 = 0.$$

Hence $\mathbf{u}$ is orthogonal to every vector in span S.

T.6. Let $\mathbf{u}$ be a fixed vector in R^n and let W be the set of all vectors in R^n that are orthogonal to $\mathbf{u}$. We show W is closed under addition of vectors and under scalar multiplication. Let $\mathbf{v}$ and $\mathbf{w}$ be in W. Then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0$. Hence

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = 0 + 0 = 0$$

and it follows that $\mathbf{v} + \mathbf{w}$ is in W. Also, for any real number k,

$$\mathbf{u} \cdot (k\mathbf{v}) = k(\mathbf{u} \cdot \mathbf{v}) = k0 = 0.$$

It follows that $k\mathbf{v}$ is in W. Thus W is a subspace of R^n.

T.7. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $u_1v_1 + u_2v_2 + \cdots + u_nv_n = 0$. We have

$$\mathbf{u} \cdot (c\mathbf{v}) = u_1(cv_1) + u_2(cv_2) + \cdots + u_n(cv_n) = c(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = c(0) = 0.$$

T.8. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal set, by Theorem 6.16 it is linearly independent. Hence, A is nonsingular. Since S is orthonormal,

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This can be written in terms of matrices as

$$\mathbf{v}_i \mathbf{v}_j^T = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

or as $AA^T = I_n$. Then $A^{-1} = A^T$. Examples of such matrices:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{3} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

T.9. Since some of the vectors $\mathbf{v}_j$ can be zero, A can be singular.

T.10. Let $T = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for R^n. Form the set $Q = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. None of the vectors in A is the zero vector. Since A contains more than n vectors, Q is a linearly dependent set. Thus one of the vectors is not orthogonal to the preceding ones. (See Theorem 6.16). It cannot be one of the $\mathbf{u}$'s, so at least one of the $\mathbf{v}$'s is not orthogonal to the $\mathbf{u}$'s. Check $\mathbf{v}_1 \cdot \mathbf{u}_j$, $j = 1, \ldots, k$. If all these dot products are zero, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1\}$ is an orthonormal set, otherwise delete $\mathbf{v}_1$. Proceed in a similar fashion with $\mathbf{v}_i$, $i = 2, \ldots, n$ using the largest subset of A that has been found to be orthogonal so far. What remains will be a set of n orthogonal vectors since A originally contained a basis for V. In fact, the set will be orthonormal since each of the $\mathbf{u}$'s and $\mathbf{v}$'s originally had length 1.

T.11 Let $\mathbf{x}$ be in S. Then we can write $\mathbf{x} = \displaystyle\sum_{j=1}^{k} c_j\mathbf{u}_j$. Similarly if $\mathbf{y}$ is in T, we have $\mathbf{y} = \displaystyle\sum_{i=k+1}^{n} c_i\mathbf{u}_i$. Then

$$\mathbf{x} \cdot \mathbf{y} = \left(\sum_{j=1}^{k} c_j\mathbf{u}_j\right) \cdot \mathbf{y} = \sum_{j=1}^{k} c_j(\mathbf{u}_j \cdot \mathbf{y}) = \sum_{j=1}^{k} c_j\left(\mathbf{u}_j \cdot \sum_{i=k+1}^{n} c_i\mathbf{u}_i\right) = \sum_{j=1}^{k} c_j\left(\sum_{i=k+1}^{n} c_i(\mathbf{u}_j \cdot \mathbf{u}_i)\right).$$

Since $j \neq i$, $\mathbf{u}_j \cdot \mathbf{u}_i = 0$, hence $\mathbf{x} \cdot \mathbf{y} = 0$.

ML.1. Use the following MATLAB commands.

A = [1 1 0;1 0 1;0 0 1];

gschmidt(A)

```
ans =
      0.7071     0.7071          0
      0.7071    -0.7071          0
           0          0     1.0000
```

Write the columns in terms of $\sqrt{2}$. Note that $\frac{\sqrt{2}}{2} \approx 0.7071$.

ML.2. Use the following MATLAB commands.

A = [1 0 1 1;1 2 1 3;0 2 1 1;0 1 0 0]';

gschmidt(A)

```
ans =
      0.5774    -0.2582    -0.1690     0.7559
           0     0.7746     0.5071     0.3780
      0.5774    -0.2582     0.6761    -0.3780
      0.5774     0.5164    -0.5071    -0.3780
```

ML.3. To find the orthonormal basis we proceed as follows in MATLAB.

A = [0 −1 1;0 1 1;1 1 1]';

G = gschmidt(A)

```
G =
           0          0     1.0000
     -0.7071     0.7071          0
      0.7071     0.7071          0
```

To find the coordinates of each vector with respect to the orthonormal basis T which consists of the columns of matrix G we express each vector as a linear combination of the columns of G. It follows that $\left[\mathbf{v}\right]_T$ is the solution to the linear system $G\mathbf{x} = \mathbf{v}$. We find the solution to all three systems at the same time as follows.

coord = rref([G [1 2 0;1 1 1; − 1 0 1]′])

```
coord =
    1.0000         0         0   -1.4142         0    0.7071
         0    1.0000         0    1.4142    1.4142    0.7071
         0         0    1.0000    1.0000    1.0000   -1.0000
```

Columns 4, 5, and 6 are the solutions to parts (a), (b), and (c), respectively.

ML.4. We have that all vectors of the form $(a, 0, a+b, b+c)$ can be expressed as follows:

$$(a, 0, a+b, b+c) = a(1, 0, 1, 0) + b(0, 0, 1, 1) + c(0, 0, 0, 1).$$

By the same type of argument used in Exercises 16–19 we show that

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 0, 1, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}$$

is a basis for the subspace. Apply routine **gschmidt** to the vectors of S.

A = [1 0 1 0;0 0 1 1;0 0 0 1]′;

gschmidt(A,1)

```
ans =
    1.0000   -0.5000    0.3333
         0         0         0
    1.0000    0.5000   -0.3333
         0    1.0000    0.3333
```

The columns are an orthogonal basis for the subspace.

Section 6.9, p. 369

2. $\mathbf{v} = (1, 1, 0, 0 = (1, \frac{1}{2}, \frac{1}{2}, 0) + (0, \frac{1}{2}, -\frac{1}{2}, 0)$, where $\mathbf{w} = (1, \frac{1}{2}, \frac{1}{2}, 0)$ is in W and $\mathbf{u} = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ is in $W^\perp$.

4. (a) $\left\{\left(\frac{7}{5}, -\frac{1}{5}, 1\right)\right\}$. (b) $W^\perp$ is the normal to the plane represented by W.

6. Null space of A has basis $\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

   Basis for row space of A is $\{(1, 0, -2, -3), (0, 1, 1, 2)\}$.

   Null space of A^T has basis $\left\{\begin{bmatrix} -\frac{7}{5} \\ -\frac{13}{10} \\ 1 \end{bmatrix}\right\}$. Basis for column space of A^T is $\left\{\begin{bmatrix} 1 \\ 0 \\ \frac{7}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{13}{10} \end{bmatrix}\right\}$.

8. (a) $(3, 0, -1)$. (b) $(2, 0, 3)$. (c) $(5, 0, 1)$.

10. $\mathbf{w} = \left(-\frac{1}{5}, 2, -\frac{2}{5}\right)$, $\mathbf{u} = \left(\frac{6}{5}, 0, -\frac{3}{5}\right)$.

12. $\frac{3}{5}\sqrt{5}$. 14. $1\sqrt{14}$.

T.1. The zero vector is orthogonal to every vector in W.

T.2. If $\mathbf{v}$ is in $V^\perp$, then $\mathbf{v} \cdot \mathbf{v} = 0$. By Theorem 4.3, $\mathbf{v}$ must be the zero vector. If $W = \{\mathbf{0}\}$, then every vector $\mathbf{v}$ in V is in $W^\perp$ because $\mathbf{v} \cdot \mathbf{0} = 0$. Thus $W^\perp = V$.

T.3. Let $W = \text{span } S$, where $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. If $\mathbf{u}$ is in $W^\perp$, then $\mathbf{u} \cdot \mathbf{w} = 0$ for any $\mathbf{w}$ in W. Hence $\mathbf{u} \cdot \mathbf{v}_i = 0$ for $i = 1, 2, \ldots, m$. Conversely, suppose that $\mathbf{u} \cdot \mathbf{v}_i = 0$ for $i = 1, 2, \ldots, m$. Let

$$\mathbf{w} = \sum_{i=1}^{m} c_i \mathbf{v}_i$$

be any vector in W. Then

$$\mathbf{u} \cdot \mathbf{w} = \sum_{i=1}^{m} c_i (\mathbf{u} \cdot \mathbf{v}_i) = 0.$$

Hence $\mathbf{u}$ is in $W^\perp$.

T.4. Let $\mathbf{v}$ be a vector in R^n. By Theorem 6.22(a), the column space of A^T is the orthogonal complement of the null space of A. This means that

$$R^n = \text{null space of } A \oplus \text{column space of } A^T.$$

Hence, there exist unique vectors $\mathbf{w}$ in the null space of A and $\mathbf{u}$ in the column space of A^T such that $\mathbf{v} = \mathbf{w} + \mathbf{u}$.

T.5. Let W be a subspace of R^n. By Theorem 6.20, we have $R^n = W \oplus W^\perp$. Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a basis for W, so $\dim W = r$, and let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\}$ be a basis for $W^\perp$, so $\dim W^\perp = s$. If $\mathbf{v}$ is in V, then $\mathbf{v} = \mathbf{w} + \mathbf{u}$, where $\mathbf{w}$ is in W and $\mathbf{u}$ is in $W^\perp$. Moreover, $\mathbf{w}$ and $\mathbf{u}$ are unique. Then

$$\mathbf{v} = \sum_{i=1}^{r} a_i \mathbf{w}_i + \sum_{j=1}^{s} b_j \mathbf{u}_j$$

so $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ spans V. We now show that S is linearly independent. Suppose that

$$\sum_{i=1}^{r} a_i \mathbf{w}_i + \sum_{j=1}^{s} b_j \mathbf{u}_j = \mathbf{0}.$$

Then

$$\sum_{i=1}^{r} a_i \mathbf{w}_i = -\sum_{j=1}^{s} b_j \mathbf{u}_j,$$

so $\displaystyle\sum_{i=1}^{r} a_i \mathbf{w}_i$ lies in $W \cap W^\perp = \{\mathbf{0}\}$. Hence

$$\sum_{i=1}^{r} a_i \mathbf{w}_i = \mathbf{0},$$

and since $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ are linearly independent, $a_1 = a_2 = \cdots = a_r = 0$. Similarly, $b_1 = b_2 = \cdots = b_s = 0$. Thus, S is also linearly independent and is then a basis for R^n. This means that

$$n = \dim R^n = r + s = \dim W + \dim W^\perp,$$

and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s$ is a basis for R^n.

T.6. If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is an orthogonal basis for W, then

$$\left\{\frac{1}{\|\mathbf{w}_1\|}\,\mathbf{w}_1, \frac{1}{\|\mathbf{w}_2\|}\,\mathbf{w}_2, \ldots, \frac{1}{\|\mathbf{w}_m\|}\,\mathbf{w}_m\right\}$$

is an orthonormal basis for W, so

$$\begin{aligned}\operatorname{proj}_W \mathbf{v} &= \left[\mathbf{v}\cdot\left(\frac{1}{\|\mathbf{w}_1\|}\,\mathbf{w}_1\right)\right]\frac{1}{\|\mathbf{w}_1\|}\,\mathbf{w}_1 + \left[\mathbf{v}\cdot\left(\frac{1}{\|\mathbf{w}_2\|}\,\mathbf{w}_2\right)\right]\frac{1}{\|\mathbf{w}_2\|}\,\mathbf{w}_2 + \cdots \\ &\quad + \left[\mathbf{v}\cdot\left(\frac{1}{\|\mathbf{w}_m\|}\,\mathbf{w}_m\right)\right]\frac{1}{\|\mathbf{w}_m\|}\,\mathbf{w}_m \\ &= \frac{\mathbf{v}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\,\mathbf{w}_1 + \frac{\mathbf{v}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\,\mathbf{w}_2 + \cdots + \frac{\mathbf{v}\cdot\mathbf{w}_m}{\mathbf{w}_m\cdot\mathbf{w}_m}\,\mathbf{w}_m.\end{aligned}$$

T.7. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ be the new vectors of A. If $A\mathbf{x} = \mathbf{0}$, then $\mathbf{a}_i\mathbf{x} = 0$, so $\mathbf{a}_i^T \cdot \mathbf{x} = 0$. Hence $\mathbf{x}$ is orthgonal to every row vector of A, so $\mathbf{x}$ is orthogonal to every vector in the row space of A. Therefore $\mathbf{x}$ is in $W^\perp$.

ML.1. (a) **v = [1 5 − 1 1]′,w = [0 1 2 1]′**

```
v =
      1
      5
     -1
      2

w =
     0
     1
     2
     1
```

proj = dot(v,w)/norm(w)^2 ∗ w

```
proj =
              0
         0.8333
         1.6667
         0.8333
```

format rat

proj

```
proj =
           0
         5/6
         5/3
         5/6
```

format

(b) **v = [1 − 2 3 0 1]′,w = [1 1 1 1 1]′**

```
v =
    1
   -2
    3
    0
    1

w =
    1
    1
    1
    1
    1
```

proj = dot(v,w)/norm(w)^2 * w

```
proj =
       0.6000
       0.6000
       0.6000
       0.6000
       0.6000
```

format rat

proj

```
proj =
       3/5
       3/5
       3/5
       3/5
       3/5
```

format

ML.2. **w1 = [1 0 1 1]′,w2 = [1 1 − 1 0]′**

```
w1 =
     1
     0
     1
     1

w2 =
      1
      1
     -1
      0
```

(a) We show the dot product of w1 and w2 is zero and since nonzero orthogonal vectors are linearly independent they form a basis for W.

dot(w1,w2)

```
ans =
      0
```

(b) **v = [2 1 2 1]′**

```
v =
    2
    1
    2
    1
```

proj = dot(v,w1)/norm(w1)^2 ∗ w1

```
proj =
        1.6667
             0
        1.6667
        1.6667
```

format rat

proj

```
proj =
        5/3
          0
        5/3
        5/3
```

format

(c) **proj = dot(v,w1)/norm(w1)^2 ∗ w1 + dot(v,w2)/norm(w2)^2 ∗ w2**

```
proj =
        2.0000
        0.3333
        1.3333
        1.6667
```

format rat

proj

```
proj =
          2
        1/3
        4/3
        5/3
```

format

ML.3. **w1 = [1 2 3]′,w2 = [0 − 3 2]′**

```
w1 =
     1
     2
     3
```

```
w2 =
    0
   -3
    2
```

(a) First note that w1 and w2 form an orthogonal basis for plane P.

v = [2 4 8]′

```
v =
    2
    4
    8
```

proj = dot(v,w1)/norm(w1)^2 ∗ w1 + dot(v,w2)/norm(w2)^2 ∗ w2

```
proj =
        2.4286
        3.9341
        7.9011
```

(b) The distance from **v** to P is the length of the vector $-\text{proj} + \mathbf{v}$.

norm(− proj + v)

```
ans =
       0.4447
```

ML.4. Note that the vectors in S are not an orthogonal basis for $W = \text{span } S$. We first use the Gram–Schmidt process to find an orthonormal basis.

x = [[1 1 0 1]′ [2 − 1 0 0]′ [0 1 0 1]′]

```
x =
    1     2   0
    1    -1   1
    0     0   0
    1     0   1
```

b = gschmidt(x)

```
x =
    0.5774    0.7715   -0.2673
    0.5774   -0.6172   -0.5345
         0         0         0
    0.5774   -0.1543    0.8018
```

Name these columns w1, w2, w3, respectively.

w1 = b(:,1);w2 = b(:,2);w3 = b(:,3);

Then w1, w2, w3 is an orthonormal basis for W.

v = [0 0 1 1]′

```
v =
    0
    0
    1
    1
```

(a) **proj = dot(v,w1) ∗ w1 + dot(v,w2) ∗ w2 + dot(v,w3) ∗ w3**

```
proj =
        0.0000
             0
             0
        1.0000
```

(b) The distance from **v** to P is the length of vector $-\text{proj} + \mathbf{v}$.

norm(− proj + v)

```
ans =
      1
```

ML.5. **T = [1 0;0 1;1 1;1 0;1 0]**

```
T =
    1   0
    0   1
    1   1
    1   0
    1   0
```

b = [1 1 1 1 1]′

```
b =
    1
    1
    1
    1
    1
```

(a) **rref([T b])**

```
ans =
      1   0   0
      0   1   0
      0   0   1
      0   0   0
      0   0   0
```

Note that row [0 0 1] implies that the system is inconsistent.

(b) Note that the columns of T are not orthogonal, so we use the Gram–Schmidt process to find an orthonormal basis for the column space.

q = gschmidt(T)

```
q =
    0.5000   -0.1890
         0    0.7559
    0.5000    0.5669
    0.5000   -0.1890
    0.5000   -0.1890
```

Define the columns of q to be w1 and w2 which is an orthonormal basis for the column space.

```
w1 = q(:,1);w2 = q(:,2);
proj = dot(b,w1) * w1 + dot(b,w2) * w2
proj =
        0.8571
        0.5714
        1.4286
        0.8571
        0.8571
```

Supplementary Exercises, p. 372

2. No. 4. Yes. 6. Yes. 8. 2.

10. Possible answer: $\{(1, 2, -1, 2), (0, -1, 3, -6), (1, -1, 0, -2)\}$.

12. $\lambda \neq -1, 0, 1$.

14. $a = 1$ or $a = 2$.

16. $k \neq 1$ and $k \neq -1$.

18. Yes.

20. (a) Yes. (b) W = column space of A.

22. $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ where $\mathbf{e}_1 = (1, 0, 0)$.

24. 5.

25. The solution space is a vector space of dimension d, $2 \leq d \leq 7$.

26. No. If all the nontrivial solutions of the homogeneous system are multiples of each other, then the dimension of the solution space is 1. The rank of the coefficient matrix is ≤ 5. Since nullity $= 7-$rank, nullity $\geq 7 - 5 = 2$.

28. $T = \{7t + 4, t - 3\}$.

30. $\{(2, 1, 0), (-1, 0, 1)\}$

32. $(1, 2, 3) = -\sqrt{2}\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) + 2(0, 1, 0) + 2\sqrt{2}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

34. $\left\{\left(\frac{1}{2}, -\frac{5}{4}, 1, 0\right), \left(-\frac{3}{2}, \frac{13}{4}, 0, 1\right)\right\}$.

36. $\sqrt{8} \approx 2.828$.

T.1. If A is nonsingular then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Thus the dimension of the solution space is zero. Conversely, if $A\mathbf{x} = \mathbf{0}$ has a solution space of dimension zero, then $\mathbf{x} = \mathbf{0}$ is the only solution. Thus A is nonsingular.

T.2. Let $c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_n A\mathbf{v}_n = \mathbf{0}$. Then we have

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}.$$

Since A is nonsingular A^{-1} exists. Multiplying both sides on the left by A^{-1} gives

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

However, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent so $c_1 = c_2 = \cdots = c_n = 0$. It follows that $\{A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n\}$ is linearly independent.

T.3. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{e}_j$, $j = 1, 2, \ldots, n$, be the natural basis for R^n. Then $\mathbf{v}_j \cdot \mathbf{u} = 0$ for $j = 1, 2, \ldots, n$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, there exist scalars $c_1, c_2, \ldots, c_n$ such that $\mathbf{e}_j = \sum_{k=1}^{n} c_k \mathbf{v}_k$. Then

$$\mathbf{u} \cdot \mathbf{e}_j = u_j = \mathbf{u} \cdot \sum_{k=1}^{n} c_k \mathbf{v}_k = \sum_{k=1}^{n} c_k (\mathbf{u} \cdot \mathbf{v}_k) = \sum_{k=1}^{n} c_k(0) = 0$$

for each $j = 1, 2, \ldots, n$. Thus $\mathbf{u} = \mathbf{0}$.

T.4. rank A = row rank A = column rank A^T = rank A^T. (See Theorem 6.11.)

T.5. (a) Theorem 6.10 implies that row space of A = row space of B. Thus rank A = row rank A = row rank B = rank B.

(b) Since A and B are row equivalent they have the same reduced row echelon form. It follows that the solutions of $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are the same. Hence $A\mathbf{x} = \mathbf{0}$ if and only if $B\mathbf{x} = \mathbf{0}$.

T.6. (a) From the definition of a matrix product, the rows of AB are linear combinations of the rows of B. Hence, the row space of AB is a subspace of the row space of B and it follows that $\operatorname{rank}(AB) \leq \operatorname{rank} B$. From Exercise T.4 above, $\operatorname{rank}(AB) \leq \operatorname{rank}((AB)^T) = \operatorname{rank}(B^T A^T)$. A similar argument shows $\operatorname{rank}(AB) \leq \operatorname{rank} A^T = \operatorname{rank} A$. It follows that $\operatorname{rank}(AB) \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$.

(b) One such pair of matrices is $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(c) Since $A = (AB)B^{-1}$, by (a), $\operatorname{rank} A \leq \operatorname{rank}(AB)$. But (a) also implies that $\operatorname{rank}(AB) \leq \operatorname{rank} A$, so $\operatorname{rank}(AB) = \operatorname{rank} A$.

(d) Since $B = A^{-1}(AB)$, by (a), $\operatorname{rank} B \leq \operatorname{rank}(AB)$. But (a) implies that $\operatorname{rank}(AB) \leq \operatorname{rank} B$, thus $\operatorname{rank}(AB) = \operatorname{rank} B$.

(e) $\operatorname{rank}(PAQ) = \operatorname{rank} A$.

T.7. $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for R^n. Hence $\dim V = k$ and

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Let $T = \{a_1\mathbf{v}_1, a_2\mathbf{v}_2, \ldots, a_k\mathbf{v}_k\}$, where $a_j \neq 0$. To show that T is a basis we need only show that it spans R^n and then use Theorem 6.9(b). Let $\mathbf{v}$ belong to R^n. Then there exist scalars c_i, $i = 1, 2, \ldots, k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

Since $a_j \neq 0$, we have

$$\mathbf{v} = \frac{c_1}{a_1} a_1\mathbf{v}_1 + \frac{c_2}{a_2} a_2\mathbf{v}_2 + \cdots + \frac{c_k}{a_k} a_k\mathbf{v}_k$$

so span $T = R^n$. Next we show that the members of T are orthogonal. Since S is orthogonal, we have

$$(a_i\mathbf{v}_i) \cdot (a_j\mathbf{v}_j) = a_i a_j (\mathbf{v}_i \cdot \mathbf{v}_j) = \begin{cases} 0 & \text{if } i \neq j \\ a_i a_j & \text{if } i = j. \end{cases}$$

Hence T is an orthogonal set. In order for T to be an orthonormal set we must have $a_i a_j = 1$ for all i and j. This is only possible if all $a_i = 1$.

T.8. (a) The columns $\mathbf{b}_j$ are in R^m. Since the columns are orthonormal they are linearly independent. There can be at most m linearly independent vectors in R^m. Thus $m \geq n$.

(b) We have

$$\mathbf{b}_i^T \mathbf{b}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

It follows that $B^T B = I_n$, since the (i, j) element of $B^T B$ is computed by taking row i of B^T times column j of B. But row i of B^T is just $\mathbf{b}_i^T$ and column j of B is $\mathbf{b}_j$.

Chapter 7

Applications of Real Vector Spaces (Optional)

Section 7.1, p. 378

2. $Q = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{1}{\sqrt{6}} \\ -\frac{\sqrt{3}}{3} & -\frac{1}{\sqrt{6}} \\ \frac{\sqrt{3}}{3} & -\frac{2}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 0.5774 & 0.4082 \\ -0.5774 & -0.4082 \\ 0.5774 & -0.8165 \end{bmatrix}$, $R = \begin{bmatrix} \sqrt{3} & \frac{5\sqrt{3}}{3} \\ 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 1.7321 & 2.8868 \\ 0 & 0.8165 \end{bmatrix}$.

4. $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 0.8944 & 0.4082 \\ 0.4472 & 0.8165 \\ 0 & 0.4082 \end{bmatrix}$, $R = \begin{bmatrix} \frac{5}{\sqrt{5}} & -\frac{5}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 2.2361 & -2.2361 \\ 0 & 2.4495 \end{bmatrix}$.

6. $Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & -\frac{5}{\sqrt{30}} \end{bmatrix} \approx \begin{bmatrix} 0.8944 & -0.4082 & -0.1826 \\ 0.4472 & 0.8165 & 0.3651 \\ 0 & 0.4082 & -0.9129 \end{bmatrix}$

$$R = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{6} & -\frac{7}{\sqrt{6}} \\ 0 & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix} \approx \begin{bmatrix} 2.2361 & 0 & 0 \\ 0 & 2.4495 & -2.8577 \\ 0 & 0 & -0.9129 \end{bmatrix}.$$

T.1. We have

$$\mathbf{u}_i = \mathbf{v}_i + \frac{\mathbf{u}_i \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\,\mathbf{v}_1 + \frac{\mathbf{u}_i \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\,\mathbf{v}_2 + \cdots + \frac{\mathbf{u}_i \cdot \mathbf{v}_{i-1}}{\mathbf{v}_{i-1} \cdot \mathbf{v}_{i-1}}\,\mathbf{v}_{i-1}.$$

Then

$$r_{ii} = \mathbf{u}_i \cdot \mathbf{w}_i = \mathbf{v}_i \cdot \mathbf{w}_i + \frac{\mathbf{u}_i \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}(\mathbf{v}_1 \cdot \mathbf{w}_i) + \frac{\mathbf{u}_i \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}(\mathbf{v}_2 \cdot \mathbf{w}_i) + \cdots + \frac{\mathbf{u}_i \cdot \mathbf{v}_{i-1}}{\mathbf{v}_{i-1} \cdot \mathbf{v}_{i-1}}(\mathbf{v}_{i-1} \cdot \mathbf{w}_i)$$
$$= \mathbf{v}_i \cdot \mathbf{w}_i$$

because $\mathbf{v}_i \cdot \mathbf{w}_j = 0$ for $i \neq j$. Moreover,

$$\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|}\,\mathbf{v}_i,$$

so $\mathbf{v}_i \cdot \mathbf{w}_i = \dfrac{1}{\|\mathbf{v}_i\|}(\mathbf{v}_i \cdot \mathbf{v}_i) = \|\mathbf{v}_i\|$.

T.2. If A is an $n \times n$ nonsingular matrix, then the columns of A are linearly independent, so by Theorem 7.1, A has a QR-factorization.

Section 7.2, p. 388

2. $\widehat{\mathbf{x}} = \begin{bmatrix} \frac{56}{257} \\ \frac{6}{257} \end{bmatrix} \approx \begin{bmatrix} 0.2179 \\ 0.0233 \end{bmatrix}$.

4. $\widehat{\mathbf{x}} = \begin{bmatrix} \frac{25}{49} \\ \frac{85}{196} \\ \frac{17}{49} \\ -\frac{3}{49} \end{bmatrix} \approx \begin{bmatrix} 0.5102 \\ 0.4337 \\ 0.3469 \\ -0.0612 \end{bmatrix}$.

8. $y = 0.3x + 1.3$

10. $y = 0.321x + 2.786$.

12. $y = -0.3818x^2 + 2.6345x - 2.3600$

14. (a) $y = 0.697x + 1.457$. (b) 7.73.

16. (a) $x = 0.2129t^2 + 2.3962t - 2.1833$. (b) -4.0865 million dollars.

T.1. We have

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}.$$

If at least two x-coordinates are unequal, then $\operatorname{rank} A = 2$. Theorem 7.2 implies that A^TA is nonsingular.

T.2. From Equation (1), the normal system of equations is $A^TA\widehat{\mathbf{x}} = A^T\mathbf{b}$. Since A is nonsingular so is A^T and hence so is A^TA. It follows from matrix algebra that $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$ and multiplying both sides of the preceding equation by $(A^TA)^{-1}$ on the left gives

$$\widehat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = A^{-1}(A^T)^{-1}A^T\mathbf{b} = A^{-1}\mathbf{b}.$$

ML.1. Enter the data into MATLAB.

x = [2 3 4 5 6 7];y = [3 4 3 4 3 4];

c = lsqline(x,y)

We find that the least squares model is:

$$y = 0.08571 * x + 3.114.$$

ML.2. Enter the data into MATLAB.

x = [1 2 3 4 5 6];y = [.8 2.1 2.6 2.0 3.1 3.3];

c = lsqline(x,y)

We find that the least squares model is:

$$y = 0.4257 * x + 0.8267.$$

Using the option to evaluate the model, we find that $x = 7$ gives 3.8067, $x = 8$ gives 4.2324, and $x = 9$ gives 4.6581.

ML.3. Enter the data into MATLAB.

x = [0 2 3 5 9];y = [185 170 166 152 110];

(a) Using command **c = lsqline(x,y)** we find that the least squares model is:

$$y = -8.278 * x + 188.1.$$

(b) Using the option to evaluate the model, we find that $x = 1$ gives 179.7778, $x = 6$ gives 138.3889, and $x = 8$ gives 121.8333.

(c) In the equation for the least squares line set $y = 160$ and solve for x. We find $x = 3.3893$ min.

ML.4. Data for quadratic least squares: (Sample of cos on $[0, 1.5 * \text{pi}]$.)

t	yy
0	1.0000
0.5000	0.8800
1.0000	0.5400
1.5000	0.0700
2.0000	−0.4200
2.5000	−0.8000
3.0000	−0.9900
3.5000	−0.9400
4.0000	−0.6500
4.5000	−0.2100

v = polyfit(t,yy,2)

```
v =
    0.2006   −1.2974   1.3378
```

Thus $y = 0.2006t^2 - 1.2974t + 1.3378$.

ML.5. Data for quadratic least squares:

x	yy
−3.0000	0.5000
−2.5000	0
−2.0000	−1.1250
−1.5000	−1.1875
−1.0000	−1.0000
0	0.9375
0.5000	2.8750
1.0000	4.7500
1.5000	8.2500
2.0000	11.5000

v = polyfit(x,yy,2)

```
v =
    1.0204    3.1238    1.0507
```

Thus $y = 1.0204x^2 + 3.1238x + 1.0507$.

Section 7.3, p. 404

2. The parity $(m, m+1)$ code has code words generated by the function e given by

$$e(\mathbf{b}) = e(b_1, b_2, \dots, b_m) = b_1 b_2 \dots b_m b_{m+1},$$

where

$$b_{m+1} = \begin{cases} 0, & \text{if weight of } \mathbf{b} \text{ is even} \\ 1, & \text{if weight of } \mathbf{b} \text{ is odd.} \end{cases}$$

But the weight of $\mathbf{b}$, $|\mathbf{b}|$, using binary addition is the sum of its bits. If $|\mathbf{b}|$ is even $b_1 + b_2 + \cdots + b_m = 0$ (using binary bits); otherwise, the sum is 1. Hence we have

$$e(\mathbf{b}) = \begin{bmatrix} I_m \\ \mathbf{u} \end{bmatrix} \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ b_{m+1} \end{bmatrix}.$$

4. $GC = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

5. Determine the solution to the linear system $G\mathbf{x} = \mathbf{0}$. The augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

and its reduced row echelon form is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

It follows that

$$\begin{aligned} x_1 &= x_3 + x_5 \\ x_2 &= x_3 + x_4 + x_5 \end{aligned}$$

and that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + x_5 \\ x_3 + x_4 + x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Since x_3, x_4, and x_5 are arbitrary bits, the set

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for the null space of G.

6. The check matrix is $G = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$.

8. $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of G.

10. 2.

12. (a) No. Not all columns are distinct.
    (b) Yes.

14. (a) $G\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so no single error was detected.

    (b) $G\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ so a single error was detected in the 6th bit. The corrected vector is

$$\mathbf{x}_t = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

    (c) $G\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so no single error was detected.

16. $H(4) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$.

18. $C = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

20. (a) $H(5)\mathbf{x}_t = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ so a single error was detected in the 3rd bit. The corrected vector is

$$\mathbf{x}_t = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) $H(5)\mathbf{x}_t = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ so a single error was detected in the 2nd bit. The corrected vector is

$$\mathbf{x}_t = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(c) $H(5)\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so no single error was detected.

22. (a) $H(7)\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so no single error was detected.

(b) $H(7)\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so no single error was detected.

(c) $H(7)\mathbf{x}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so no single error was detected.

T.1. We compute GC:

$$GC = \begin{bmatrix} D & I_{n-m} \end{bmatrix} \begin{bmatrix} I_m \\ D \end{bmatrix} = O + O = O.$$

Thus, the column space of C is in the null space of G.

T.2. The binary representations of the integers from 1 to n are all different and none is all zeros. Hence, G satisfies Theorem 7.5.

T.3. If Q_p is a rearrangement of the columns of Q, then there exists a matrix R, which is a rearrangement of the columns of the identity matrix, so that $Q_p = QR$. The matrix R is an orthogonal matrix, so $R^{-1} = R^T$. It follows that if $\mathbf{y}$ is a column vector then $R^T\mathbf{y}$ is a column vector with its entries rearranged in the same order as the columns of Q when Q_p is formed. Let $\mathbf{x}$ be in the null space of Q so $Q\mathbf{x} = \mathbf{0}$. Then $Q\mathbf{x} = Q_pR^T\mathbf{x} = \mathbf{0}$, hence $R^T\mathbf{x}$ is in the null space of Q_p. Thus the null space of Q_p consists of the vectors in the null space of Q with their entries rearranged in the same manner as the columns of A when Q_p was formed.

T.4. (a) $H(\mathbf{v}, \mathbf{w}) = 2$. (b) $H(\mathbf{v}, \mathbf{w}) = 3$. (c) $H(\mathbf{v}, \mathbf{w}) = 4$.

T.5. If $u_j = v_j$, then $u_j + v_j = u_j - v_j = 0$.

If $u_j \neq v_j$, then $u_j + v_j = u_j - v_j = 1$.

Hence the number of 1s in $\mathbf{u} - \mathbf{v}$ or $\mathbf{u} + \mathbf{v}$ is exactly the number of positions in which $\mathbf{u}$ and $\mathbf{v}$ vary.

T.6. We form the binary representation of the integers from 1 to 15 using four bits and represent these as the columns of the matrix B:

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Next we compute $CB = W$ (using binary arithmetic) to obtain

$$W = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The columns of W are the code words. Adding the entries using base 10 arithmetic we get the weights

$$\mathbf{w} = \begin{bmatrix} 4 & 3 & 3 & 3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 4 & 4 & 3 & 7 \end{bmatrix}.$$

Hence all the code words have weight greater than or equal to 3.

T.7. We form the binary representations of the integers from 1 to 3 using two bits and represent these as the columns of the matrix B:

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Next, we compute $CB = W$ (using binary arithmetic) to obtain

$$W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The columns of W are the code words. Their respective weights are 3, 3, and 4.

T.8. (a) $H(\mathbf{u}, \mathbf{v}) = |\mathbf{u} + \mathbf{v}| = |\mathbf{v} + \mathbf{u}| = H(\mathbf{v}, \mathbf{u})$.

(b) $H(\mathbf{u}, \mathbf{v}) = |\mathbf{u} + \mathbf{v}| \geq 0$.

(c) $H(\mathbf{u}, \mathbf{v}) = |\mathbf{u} + \mathbf{v}| = 0$ if and only if $\mathbf{u} + \mathbf{v} = \mathbf{0}$, that is, if and only if $\mathbf{u} = -\mathbf{v} = \mathbf{v}$.

(d) $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$, since at any position where $\mathbf{u}$ and $\mathbf{v}$ differ one of them must contain a 1. Then

$$H(\mathbf{u}, \mathbf{v}) = |\mathbf{u}+\mathbf{v}| = |\mathbf{u}+\mathbf{w}-\mathbf{w}+\mathbf{v}| = |(\mathbf{u}+\mathbf{w})+(\mathbf{w}+\mathbf{v})| \leq |\mathbf{u}+\mathbf{w}|+|\mathbf{w}+\mathbf{v}| = H(\mathbf{u}, \mathbf{w})+H(\mathbf{w}, \mathbf{v}).$$

T.9. (a) Let $\mathbf{w}_d = \mathbf{col}_j(\mathbf{w})$ in Exercise T.6. We must compute the Hamming distances $H(\mathbf{w}_j, \mathbf{w}_k)$ for $j \neq k$. There are 15 code words, hence there are 105 pairs of vectors. Using the following MATLAB commands we can determine the minimum Hamming distance as the smallest nonzero entry in d. We get the minimum Hamming distance to be 3.

```
d=[]
for jj=1:15
  Wtemp=W;
  for kk=1:15
    Wtemp(:,kk)=binadd(Wtemp(:,kk),W(:,jj));
  end
  dd=sum(Wtemp);
  d=[d  dd];
end
d=sort(d)
```

(b) From Exercise T.7., there are only three code words. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Compute the following Hamming distances.

$$H(\mathbf{u},\mathbf{v}) = 4$$
$$H(\mathbf{u},\mathbf{w}) = 3$$
$$H(\mathbf{v},\mathbf{w}) = 3$$

Thus the minimum distances of the $(5, 2)$ Hamming code is 3.

T.10. Both codes have Hamming distance 3 so each can detect two or fewer errors.

ML.1. (a) Routine **bingen** gives

H8=bingen(1,8,4)

```
H8=
    0   0   0   0   0   0   0   1
    0   0   0   1   1   1   1   0
    0   1   1   0   0   1   1   0
    1   0   1   0   1   0   1   0
```

(b) Using **binreduce** on the homogeneous linear system $H(8)\mathbf{x} = \mathbf{0}$, gives

```
1   0   1   0   1   0   1   0   0
0   1   1   0   0   1   1   0   0
0   0   0   1   1   1   1   0   0
0   0   0   0   0   0   0   1   0
```

The corresponding homogeneous linear system is

$$\begin{aligned} x_1 \quad + x_3 \quad + x_5 \quad\quad + x_7 \quad &= 0 \\ x_2 + x_3 \quad\quad + x_6 + x_7 \quad &= 0 \\ x_4 + x_5 + x_6 + x_7 \quad &= 0 \\ x_8 &= 0 \end{aligned}$$

and we have

$$\begin{aligned} x_1 &= x_3 + x_5 + x_7 \\ x_2 &= x_3 + x_6 + x_7 \\ x_4 &= x_5 + x_6 + x_7 \\ x_8 &= 0 \end{aligned}$$

The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and a code matrix is

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) **binprod(H8,C)**

```
ans=
     0    0    0    0
     0    0    0    0
     0    0    0    0
     0    0    0    0
```

ML.2. Use **bingen** to generate all the binary representations of integers 0 through 8 using 4 bits and then multiply by the code matrix C using **binprod**.

B4=bingen(0,8,4)

```
ans=
     0    0    0    0    0    0    0    0    1
     0    0    0    0    1    1    1    1    0
     0    0    1    1    0    0    1    1    0
     0    1    0    1    0    1    0    1    0
```

binprod(C,B4)

```
B4=
    0    1    0    1    1    0    1    0    1
    0    1    1    0    0    1    1    0    1
    0    0    0    0    0    0    0    0    1
    0    1    1    0    1    0    0    1    0
    0    0    0    0    1    1    1    1    0
    0    0    1    1    0    0    1    1    0
    0    1    0    1    0    1    0    1    0
    0    0    0    0    0    0    0    0    0
```

ML.3. (a) Routine **bingen** gives

H15=bingen(1,15,4)

```
H15=
    Columns 1 through 15
     0    0    0    0    0    0    0    1    1    1    1    1    1    1    1
     0    0    0    1    1    1    1    0    0    0    0    1    1    1    1
     0    1    1    0    0    1    1    0    0    1    1    0    0    1    1
     1    0    1    0    1    0    1    0    1    0    1    0    1    0    1
```

(b) Using **binreduce** on the homogeneous linear system $H(15)\mathbf{x} = \mathbf{0}$, gives

```
1  0  1  0  1  0  1  0  1  0  1  0  1  0  1  0
0  1  1  0  0  1  1  0  0  1  1  0  0  1  1  0
0  0  0  1  1  1  1  0  0  0  0  1  1  1  1  0
0  0  0  0  0  0  0  1  1  1  1  1  1  1  1  0
```

Solving this homogeneous linear system for the unknowns corresponding to leading 1s we get

$$\begin{aligned}
x_1 &= x_3 + x_5 \qquad x_7 + x_9 \qquad x_{11} \qquad + x_{13} \qquad + x_{15} \\
x_2 &= x_3 \qquad + x_6 + x_7 \qquad + x_{10} + x_{11} \qquad + x_{14} + x_{15} \\
x_4 &= \qquad + x_5 + x_6 + x_7 \qquad + x_{12} + x_{13} + x_{14} + x_{15} \\
x_8 &= \qquad + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15}
\end{aligned}$$

and it follows that the general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_9 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_{10} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$+ x_{11} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_{12} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_{13} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_{14} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_{15} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and a code matrix is

```
C=
    1  1  0  1  1  0  1  0  1  0  1
    1  0  1  1  0  1  1  0  0  1  1
    1  0  0  0  0  0  0  0  0  0  0
    0  1  1  1  0  0  0  1  1  1  1
    0  1  0  0  0  0  0  0  0  0  0
    0  0  1  0  0  0  0  0  0  0  0
    0  0  0  1  0  0  0  0  0  0  0
    0  0  0  0  1  1  1  1  1  1  1
    0  0  0  0  1  0  0  0  0  0  0
    0  0  0  0  0  1  0  0  0  0  0
    0  0  0  0  0  0  1  0  0  0  0
    0  0  0  0  0  0  0  1  0  0  0
    0  0  0  0  0  0  0  0  1  0  0
    0  0  0  0  0  0  0  0  0  1  0
    0  0  0  0  0  0  0  0  0  0  1
```

(c) **binprod(H15,C15)**

```
ans=
       0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0
```

ML.4. Use **bingen** to generate all the binary representations of integers 0 through 15 using 11 bits and then multiply by the code matrix C using **binprod**.

B15=bingen(0,15,11)

```
B15=
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
       0  0  0  0  0  0  0  0  1  1  1  1  1  1  1  1
       0  0  0  0  1  1  1  1  0  0  0  0  1  1  1  1
       0  0  1  1  0  0  1  1  0  0  1  1  0  0  1  1
       0  1  0  1  0  1  0  1  0  1  0  1  0  1  0  1
```

binprod(C,B15)

```
ans=
      0  1  0  1  1  0  1  0  0  1  0  1  1  0  1  0
      0  1  1  0  0  1  1  0  0  1  1  0  0  1  1  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  1  1  0  1  0  0  1  1  0  0  1  0  1  1  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  1  1  0  1  0  0  1  1  0  0  1  0  1  1  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
      0  0  0  0  0  0  0  0  1  1  1  1  1  1  1  1
      0  0  0  0  1  1  1  1  0  0  0  0  1  1  1  1
      0  0  1  1  0  0  1  1  0  0  1  1  0  0  1  1
      0  1  0  1  0  1  0  1  0  1  0  1  0  1  0  1
```

Supplementary Exercises, p. 407

2. $$Q = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} \\ -\frac{1}{\sqrt{6}} & \frac{3}{\sqrt{14}} & \frac{2}{\sqrt{21}} \\ \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{14}} & -\frac{1}{\sqrt{21}} \end{bmatrix} \approx \begin{bmatrix} 0.4082 & -0.2673 & 0.8729 \\ -0.4082 & 0.8018 & 0.4364 \\ 0.8165 & 0.5345 & -0.2182 \end{bmatrix}.$$

$$R = \begin{bmatrix} \frac{6}{\sqrt{6}} & \frac{3}{\sqrt{6}} & -\frac{3}{\sqrt{6}} \\ 0 & \frac{7}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ 0 & 0 & -\frac{9}{\sqrt{21}} \end{bmatrix} \approx \begin{bmatrix} 2.4495 & 1.2247 & -1.2247 \\ 0 & 1.8708 & 0.8018 \\ 0 & 0 & 1.9640 \end{bmatrix}.$$

4. $y = -0.2413x^2 + 0.8118x + 2.6627.$

Chapter 8

Eigenvectors, Eigenvalues, and Diagonalization

Section 8.1, p. 420

2. (a) $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$.
   (b) $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$.
   (c) $A\mathbf{x}_3 = \lambda_3\mathbf{x}_3$.

4. $\lambda^2 - 5\lambda + 7$.

6. $p(\lambda) = \lambda^2 - 7\lambda + 6$.

8. $f(\lambda) = \lambda^3$; $\lambda_1 = \lambda_2 = \lambda_3 = 0$; $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$.

10. $f(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$; $\lambda_1 = 0$, $\lambda_2 = 2$; $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

12. $f(\lambda) = \lambda^3 - 7\lambda^2 + 14\lambda - 8$; $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 4$; $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix}$.

14. $f(\lambda) = (\lambda - 2)(\lambda + 1)(\lambda - 3)$; $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 3$; $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

16. (a) $p(\lambda) = \lambda^2 + 1$. The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

   (b) $p(\lambda) = \lambda^3 + 2\lambda^2 + 4\lambda + 8$. The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2i$, and $\lambda_3 = -2i$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -4 \\ 2i \\ 1 \end{bmatrix}, \quad \text{and} \mathbf{x}_3 = \begin{bmatrix} -4 \\ -2i \\ 1 \end{bmatrix}.$$

(c) $p(\lambda) = \lambda^3 + (-2+i)\lambda^2 - 2i\lambda$. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -i$, and $\lambda_3 = 2$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ -i \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

(d) $p(\lambda) = \lambda^2 - 8\lambda + 17$. The eigenvalues are $\lambda_1 = 4 + i$ and $\lambda_2 = 4 - i$. Associated eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1+i \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}.$$

18. Basis for eigenspace associated with $\lambda_1 = \lambda_2 = 2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Basis for eigenspace associated with $\lambda_3 = 1$ is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$.

20. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

22. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

24. (a) $\left\{ \begin{bmatrix} -4 \\ 2i \\ 1 \end{bmatrix} \right\}$. (b) $\left\{ \begin{bmatrix} -4 \\ -2i \\ 1 \end{bmatrix} \right\}$.

26. The eigenvalues of A with associated eigenvectors are

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \lambda_2 = 4, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvalues and associated eigenvectors of

$$A^2 = \begin{bmatrix} 11 & -5 \\ -10 & 6 \end{bmatrix}$$

are

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \lambda_2 = 16, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

28. $\begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix}$.

T.1. Let $\mathbf{u}$ and $\mathbf{v}$ be in S, so that $A\mathbf{u} = \lambda_j \mathbf{u}$ and $A\mathbf{v} = \lambda_j \mathbf{v}$. Then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \lambda_j \mathbf{u} + \lambda_j \mathbf{v} = \lambda_j (\mathbf{u} + \mathbf{v}),$$

so $\mathbf{u} + \mathbf{v}$ is in S. Moreover, if c is any real number, then

$$A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda_j \mathbf{u}) = \lambda_j (c\mathbf{u}),$$

so $c\mathbf{u}$ is in S.

T.2. An eigenvector must be a nonzero vector, so the zero vector must be included in S.

T.3. $(\lambda I_n - A)$ is a triangular matrix whose determinant is the product of its diagonal elements, thus the characteristic polynomial of A is

$$f(\lambda) = (\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}).$$

It follows that the eigenvalues of A are the diagonal elements of A.

T.4. $\left|\lambda I_n - A^T\right| = \left|(\lambda I_n - A)^T\right| = |\lambda I_n - A|$. Associated eigenvectors need not be the same. (But the dimensions of the eigenspace associated with λ, for A and A^T, are equal.)

T.5. $A^k\mathbf{x} = A^{k-1}(A\mathbf{x}) = A^{k-1}(\lambda\mathbf{x}) = \lambda A^{k-1}\mathbf{x} = \cdots = \lambda^k\mathbf{x}$.

T.6. If A is nilpotent and $A^k = O$, and if λ is an eigenvalue for A with associated eigenvector $\mathbf{x}$, then $\mathbf{0} = A^k\mathbf{x} = \lambda^k\mathbf{x}$ implies $\lambda^k = 0$ (since $\mathbf{x} \neq \mathbf{0}$), so $\lambda = 0$.

T.7. (a) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the roots of the characteristic polynomial of A. Then

$$f(\lambda) = \det(\lambda I_n - A) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

Hence

$$f(0) = \det(-A) = (-\lambda_1)(-\lambda_2) \cdots (-\lambda_n) = (-1)^n \lambda_1\lambda_2 \cdots \lambda_n.$$

Since $\det(-A) = (-1)^n \det(A)$ we have $\det(A) = \lambda_1 \cdots \lambda_n$.

(b) A is singular if and only if for some nonzero vector $\mathbf{x}$, $A\mathbf{x} = \mathbf{0}$, if and only if 0 is an eigenvalue of A. Alternatively, A is singular if and only if $\det(A) = 0$, if and only if [by (a)] 0 is a real root of the characteristic polynomial of A.

T.8. If $A\mathbf{x} = \lambda\mathbf{x}$, $\lambda \neq 0$, then $\lambda^{-1}\mathbf{x} = \lambda^{-1}A^{-1}A\mathbf{x} = \lambda^{-1}A^{-1}(\lambda\mathbf{x}) = \lambda^{-1}\lambda A^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, and thus λ^{-1} is an eigenvalue of A^{-1} with associated eigenvector $\mathbf{x}$.

T.9. (a) The characteristic polynomial of A is

$$\det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{12} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n\,n-1} & \lambda - a_{nn} \end{vmatrix}.$$

Any product in $\det(\lambda I_n - A)$, other than the product of the diagonal entries, can contain at most $n-2$ of the diagonal entries of $\lambda I_n - A$. This follows because at least two of the column indices must be out of natural order in every other product appearing in $\det(\lambda I_n - A)$. This implies that the coefficient of λ^{n-1} is formed by the expansion of the product of the diagonal entries. The coefficient of λ^{n-1} is the sum of the coefficients of λ^{n-1} from each of the products

$$-a_{ii}(\lambda - a_{11}) \cdots (\lambda - a_{i-1\,i-1})(\lambda - a_{i+1\,i+1}) \cdots (\lambda - a_{nn})$$

$i = 1, 2, \ldots, n$. The coefficient of λ^{n-1} in each such term is $-a_{ii}$, so the coefficient of λ^{n-1} in the characteristic polynomial is

$$-a_{11} - a_{22} - \cdots - a_{nn} = -\operatorname{Tr}(A).$$

(b) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A then $\lambda - \lambda_i$, $i = 1, 2, \ldots, n$ are factors of the characteristic polynomial $\det(\lambda I_n - A)$. It follows that

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Proceeding as in (a), the coefficient of λ^{n-1} is the sum of the coefficients of λ^{n-1} from each of the products

$$-\lambda_i(\lambda - \lambda_1)\cdots(\lambda - \lambda_{i-1})(\lambda - \lambda_{i+1})\cdots(\lambda - \lambda_n)$$

for $i = 1, 2, \ldots, n$. The coefficient of λ^{n-1} in each such term is $-\lambda_i$, so the coefficient of λ^{n-1} in the characteristic polynomial is $-\lambda_1 - \lambda_2 - \cdots - \lambda_n = -\operatorname{Tr}(A)$ by (a). Thus, $\operatorname{Tr}(A)$ is the sum of the eigenvalues of A.

(c) We have

$$\det(\lambda I_n - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

so the constant term is $\pm\lambda_1\lambda_2\cdots\lambda_n$.

(d) If $f(\lambda) = \det(\lambda I_n - A)$ is the characteristic polynomial of A, then $f(0) = \det(-A) = (-1)^n \det(A)$. Since $f(0) = a_n$, the constant term of $f(\lambda)$, $a_n = (-1)^n \det(A)$. The result follows from part (c).

T.10. Suppose there is a vector $\mathbf{x} \neq \mathbf{0}$ in both S_1 and S_2. Then $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$. So $(\lambda_2 - \lambda_1)\mathbf{x} = \mathbf{0}$. Hence $\lambda_1 = \lambda_2$ since $\mathbf{x} \neq \mathbf{0}$, a contradiction. Thus the zero vector is the only vector in both S_1 and S_2.

T.11. If $A\mathbf{x} = \lambda\mathbf{x}$, then, for any scalar r,

$$(A + rI_n)\mathbf{x} = A\mathbf{x} + r\mathbf{x} = \lambda\mathbf{x} + r\mathbf{x} = (\lambda + r)\mathbf{x}.$$

Thus $\lambda + r$ is an eigenvalue of $A + rI_n$ with associated eigenvector $\mathbf{x}$.

T.12. (a) Since $A\mathbf{u} = \mathbf{0} = 0\mathbf{0}$, it follows that 0 is an eigenvalue of A with associated eigenvector $\mathbf{u}$.

(b) Since $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, it follows that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, namely $\mathbf{x} = \mathbf{v}$.

T.13. We have

(a) $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x} = (\lambda + \mu)\mathbf{x}$.

(b) $(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mu\mathbf{x}) = \mu(A\mathbf{x}) = (\mu\lambda)\mathbf{x} = (\lambda\mu)\mathbf{x}$.

T.14. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

The product

$$A^T\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{x} = 1\mathbf{x}$$

so $\lambda = 1$ is an eigenvalue of A^T. By Exercise T.4, $\lambda = 1$ is also an eigenvalue of A.

T.15. Let W be the eigenspace of A with associated eigenvalue λ. Let $\mathbf{w}$ be in W. Then $L(\mathbf{w}) = A\mathbf{w} = \lambda\mathbf{w}$. Therefore $L(\mathbf{w})$ is in W since W is closed under scalar multiplication.

ML.1. Enter each matrix A into MATLAB and use command **poly(A)**.

(a) $\mathbf{A = [1 \quad 2;2 \quad -1];}$
$\mathbf{v = poly(A)}$

```
v =
    1.0000   0   −5.0000
```

The characteristic polynomial is $\lambda^2 - 5$.

(b) $\mathbf{A = [2 \quad 4 \quad 0;1 \quad 2 \quad 1;0 \quad 4 \quad 2];}$
$\mathbf{v = poly(A)}$

```
v =
    1.0000   −6.0000   4.0000   8.0000
```

The characteristic polynomial is $\lambda^3 - 6\lambda^2 + 4\lambda + 8$.

(c) $\mathbf{A = [1 \quad 0 \quad 0 \quad 0;2 \quad -2 \quad 0 \quad 0;0 \quad 0 \quad 2 \quad -1;0 \quad 0 \quad -1 \quad 2];}$
$\mathbf{v = poly(A)}$

```
v =
    1   −3   −3   11   −6
```

The characteristic polynomial is $\lambda^4 - 3\lambda^3 - 3\lambda^2 + 11\lambda - 6$.

ML.2. The eigenvalues of matrix A will be computed using MATLAB command **roots(poly(A))**.

(a) $\mathbf{A = [1 \quad -3;3 \quad -5];}$
$\mathbf{r = roots(poly(A))}$

```
r =
    −2
    −2
```

(b) $\mathbf{A = [3 \quad -1 \quad 4; \quad -1 \quad 0 \quad 1;4 \quad 1 \quad 2];}$
$\mathbf{r = roots(poly(A))}$

```
r =
     6.5324
    −2.3715
     0.8392
```

(c) $\mathbf{A = [2 \quad -2 \quad 0;1 \quad -1 \quad 0;1 \quad -1 \quad 0];}$
$\mathbf{r = roots(poly(A))}$

```
r =
    0
    0
    1
```

(d) $\mathbf{A = [2 \quad 4;3 \quad 6];}$
$\mathbf{r = roots(poly(A))}$

```
r =
    0
    8
```

ML.3. We solve the homogeneous system $(\lambda I_2 - A)\mathbf{x} = \mathbf{0}$ by finding the reduced row echelon form of the corresponding augmented matrix and then writing out the general solution.

(a) **A = [1 2; − 1 4];**
M = (3 ∗ eye(size(A)) − A)
rref([M [0 0]′])

```
ans =
      1   −1   0
      0    0   0
```

The general solution is $x_1 = x_2$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & 1\end{bmatrix}'$ is an eigenvector.

(b) **A = [4 0 0;1 3 0;2 1 − 1];**
M = (− 1 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

```
ans =
      1   0   0   0
      0   1   0   0
      0   0   0   0
```

The general solution is $x_3 = r$, $x_2 = 0$, $x_1 = 0$. Let $r = 1$ and we have that $\begin{bmatrix}0 & 0 & 1\end{bmatrix}'$ is an eigenvector.

(c) **A = [2 1 2;2 2 − 2;3 1 1];**
M = (2 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

```
ans =
      1   0   −1   0
      0   1    2   0
      0   0    0   0
```

The general solution is $x_3 = r$, $x_2 = -2x_3 = -2r$, $x_1 = x_3 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & -2 & 1\end{bmatrix}'$ is an eigenvector.

ML.4. Approximately $\begin{bmatrix}1.0536 \\ -0.47 \\ -0.37\end{bmatrix}$.

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2. Not diagonalizable. The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$. Associated eigenvectors are $\mathbf{x}_1 = \mathbf{x}_2 = r\begin{bmatrix}0 \\ 1\end{bmatrix}$, where r is any nonzero real number.

4. Diagonalizable. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_3 = 2$. The result follows by Theorem 8.5.

6. Diagonalizable. The eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = 3$. Associated eigenvectors are, respectively,
$$\mathbf{x}_1 = \begin{bmatrix}-1 \\ 1\end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix}2 \\ 5\end{bmatrix}.$$

8. Not diagonalizable. The eigenvalues of A are $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 5$. An eigenvector associated with λ_1 is
$$\mathbf{x}_1 = \begin{bmatrix}1 \\ -1 \\ 0 \\ 0\end{bmatrix}.$$

eigenvectors associated with $\lambda_2 = \lambda_3$ are

$$r \begin{bmatrix} -2 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Since we cannot find two linearly independent eigenvectors associated with $\lambda_2 = \lambda_3$ we conclude that A is not diagonalizable.

10. $\begin{bmatrix} 3 & 5 & -5 \\ 5 & 3 & -5 \\ 5 & 5 & -7 \end{bmatrix}$.

12. $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Associated eigenvectors are the columns of P. (P is not unique.)

14. $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$. Associated eigenvectors are the columns of P. (P is not unique.)

16. Not possible.

18. $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 6$. Associated eigenvectors are the columns of P. (P is not unique.)

20. Not possible.

22. Not possible.

24. $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

26. $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

28. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Other answers are possible.

30. No.

32. No.

34. The eigenvalues of the given matrix are $\lambda_1 = 0$ and $\lambda_2 = 7$. By Theorem 8.5, the given matrix is diagonalizable. $D = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ is similar to the given matrix.

36. The eigenvalues of the given matrix are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$. Associated eigenvectors are, respectively,

$$\mathbf{x}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

By Theorem 8.4, the given matrix is diagonalizable. $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is similar to the given matrix.

38. A is upper triangular with multiple eigenvalue $\lambda_1 = \lambda_2 = 2$ and associated eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

40. A has the multiple eigenvalue $\lambda_1 = \lambda_2 = 2$ with associated eigenvector $\begin{bmatrix} -3 \\ -7 \\ 8 \\ 0 \end{bmatrix}$.

42. Not defective.

44. Not defective.

46. $\begin{bmatrix} 768 & -1280 \\ 256 & -768 \end{bmatrix}$.

T.1. (a) $A = P^{-1}AP$ for $P = I_n$.

(b) If $B = P^{-1}AP$, then $A = PBP^{-1}$ and so A is similar to B.

(c) If $B = P^{-1}AP$ and $C = Q^{-1}BQ$ then $C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ)$ with PQ nonsingular.

T.2. If A is diagonalizable, then there is a nonsingular matrix P so that $P^{-1}AP = D$, a diagonal matrix. Then $A^{-1} = PD^{-1}P^{-1} = (P^{-1})^{-1}D^{-1}P^{-1}$. Since D^{-1} is a diagonal matrix, we conclude that A^{-1} is diagonalizable.

T.3. Necessary and sufficient conditions are: $(a-d)^2 + 4bc > 0$ for $b = c = 0$.

For the characteristic polynomial of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad f(\lambda) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 + \lambda(-a-d) + ad - bc.$$

Then $f(\lambda)$ has real roots if and only if $(a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc \geq 0$. If $(a-d)^2 + 4bc > 0$, then the eigenvalues are distinct and we can diagonalize. On the other hand, if $(a-d)^2 + 4bc = 0$, then the two eigenvalues λ_1 and λ_2 are equal and we have $\lambda_1 = \lambda_2 = \frac{a+d}{2}$. To find associated eigenvectors we solve the homogeneous system

$$\begin{bmatrix} \frac{d-a}{2} & -b \\ -c & \frac{a-d}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this case A is diagonalizable if and only if the solution space has dimension $= 2$; that is, if and only if the rank of the coefficient matrix $= 0$, thus, if and only if $b = c = 0$ so that A is already diagonal.

T.4. We show that the characteristic polynomials of AB^{-1} and $B^{-1}A$ are the same. The characteristic polynomial of AB^{-1} is

$$\begin{aligned} f(\lambda) = |\lambda I_n - AB^{-1}| &= |\lambda BB^{-1} - AB^{-1}| \\ &= |(\lambda B - A)B^{-1}| = |\lambda B - A|\,|B^{-1}| = |B^{-1}|\,|\lambda B - A| \\ &= |B^{-1}(\lambda B - A)| = |\lambda B^{-1}B - B^{-1}A| = |\lambda I_n - B^{-1}A|, \end{aligned}$$

which is the charcteristic polynomial of $B^{-1}A$.

T.5. $A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1$, but all the eigenvectors are of the form $r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Clearly A has only one linearly independent eigenvector and is not diagonalizable. However, $\det(A) \neq 0$, so A is nonsingular. (See also Example 6 in Section 8.2.)

T.6. We have $BA = A^{-1}(AB)A$, so AB and BA are similar. By Theorem 8.3, AB and BA have the same eigenvalues.

T.7. (a) If $P^{-1}AP = D$, a diagonal matrix, then $P^T A^T (P^{-1})^T = (P^{-1}AP)^T = D^T$ is diagonal, and $P^T = ((P^{-1})^T)^{-1}$, so A is similar to a diagonal matrix.

(b) $P^{-1}A^kP = (P^{-1}AP)^k = D^k$ is diagonal.

T.8. Suppose that A and B are similar, so that $B = P^{-1}AP$. Then it follows that $B^k = P^{-1}A^kP$, for any nonnegative integer k. Hence, A^k and B^k are similar.

T.9. Suppose that A and B are similar, so that $B = P^{-1}AP$. Then

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det(A).$$

T.10. We have $B = P^{-1}AP$ and $A\mathbf{x} = \lambda\mathbf{x}$. Therefore $BP^{-1} = P^{-1}APP^{-1} = P^{-1}A$ and hence

$$B(P^{-1}\mathbf{x}) = (BP^{-1})\mathbf{x} = P^{-1}A\mathbf{x} = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

which shows that $P^{-1}\mathbf{x}$ is an eigenvector of B associated with the eigenvalue λ.

T.11. The proof proceeds as in the proof of Theorem 8.5, with $k = n$.

T.12. The result follows at once from Theorems 8.2 and 8.3.

ML.1. (a) **A = [0 2; − 1 3];**
r = roots(poly(A))

```
r =
    2
    1
```

The eigenvalues are distinct so A is diagonalizable. We find the corresponding eigenvectors.
M = (2 ∗ eye(size(A)) − A)
rref([M [0 0]′])

```
ans =
      1   −1   0
      0    0   0
```

The general solution is $x_1 = x_2$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & 1\end{bmatrix}'$ is an eigenvector.
M = (1 ∗ eye(size(A)) − A)
rref([M [0 0]′])

```
ans =
      1   −2   0
      0    0   0
```

The general solution is $x_1 = 2x_2$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}2 & 1\end{bmatrix}'$ is an eigenvector.
P = [1 1;2 1]′

```
P =
    1  2
    1  1
```

invert(P) ∗ A ∗ P

```
ans =
      2  0
      0  1
```

(b) **A = [1 − 3;3 − 5];**
r = roots(poly(A))

```
r =
   −2
   −2
```

Next we determine eigenvectors corresponding to the eigenvalue −2.
M = (− 2 ∗ eye(size(A)) − A)
rref([M [0 0]′])

```
ans =
    1   −1   0
    0    0   0
```

The general solution is $x_1 = x_2$, $x_2 = r$. Let $r = 1$ and it follows that $\begin{bmatrix}1 & 1\end{bmatrix}'$ is an eigenvector, but there is only one linearly independent eigenvector. Hence A is not diagonalizable.

(c) **A = [0 0 4;5 3 6;6 0 5];**
r = roots(poly(A))

```
r =
    8.0000
    3.0000
   −3.0000
```

The eigenvalues are distinct, thus A is diagonalizable. We find corresponding eigenvectors.
M = (8 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

```
ans =
    1.0000        0   −0.5000   0
         0   1.0000   −1.7000   0
         0        0         0   0
```

The general solution is $x_1 = 0.5x_3$, $x_2 = 1.7x_3$, $x_3 = r$. Let $r = 10$ and we have that $\begin{bmatrix}2 & 17 & 10\end{bmatrix}'$ is an eigenvector.
M = (3 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

```
ans =
    1   0   0   0
    0   0   1   0
    0   0   0   0
```

The general solution is $x_1 = 0$, $x_3 = 0$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}0 & 1 & 0\end{bmatrix}'$ is an eigenvector.
M = (− 3 ∗ eye(size(A)) − A)
rref([M [0 0 0]′])

```
ans =
    1.0000        0    1.3333   0
         0   1.0000   −0.1111   0
         0        0         0   0
```

The general solution is $x_1 = -\frac{4}{3}x_3$, $x_2 = \frac{1}{9}x_3$, $x_3 = r$. Let $r = 9$ and we have that $\begin{bmatrix}-12 & 1 & 9\end{bmatrix}'$ is an eigenvector. Thus P is

$$P = \begin{bmatrix} 2 & 0 & -12 \\ 17 & 1 & 1 \\ 10 & 0 & 9 \end{bmatrix}.$$

invert(P) ∗ A ∗ P

```
ans =
      8  0   0
      0  3   0
      0  0  −3
```

ML.2. We find the eigenvalues and corresponding eigenvectors.

A = [− 1 1 − 1; − 2 2 − 1; − 2 2 − 1];

r = roots(poly(A))

```
r =
          0
    −1.0000
     1.0000
```

The eigenvalues are distinct, hence A is diagonalizable.

M = (0 ∗ eye(size(A)) − A)

rref([M [0 0 0]′])

```
ans =
      1  −1  0  0
      0   0  1  0
      0   0  0  0
```

The general solution is $x_1 = x_2$, $x_3 = 0$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & 1 & 0\end{bmatrix}'$ is an eigenvector.

M = (− 1 ∗ eye(size(A)) − A)

rref([M [0 0 0]′])

```
ans =
      1  0  −1  0
      0  1  −1  0
      0  0   0  0
```

The general solution is $x_1 = x_2$, $x_2 = x_3$, $x_3 = r$. Let $r = 1$ and we have that $\begin{bmatrix}1 & 1 & 1\end{bmatrix}'$ is an eigenvector.

M = (− 1 ∗ eye(size(A)) − A)

rref([M [0 0 0]′])

```
ans =
      1  0   0  0
      0  1  −1  0
      0  0   0  0
```

The general solution is $x_1 = 0$, $x_2 = x_3$, $x_3 = r$. Let $r = 1$ and we have that $\begin{bmatrix}0 & 1 & 1\end{bmatrix}'$ is an eigenvector.

P = [1 1 0;1 1 1;0 1 1]′

```
P =
    1  1  0
    1  1  1
    0  1  1
```

A30 = P * diag([0 − 1 1]) * invert(P)

```
A30 =
      1  −1  1
      0   0  1
      0   0  1
```

ML.3. **A = [− 1 1.5 − 1.5;− 2 2.5 − 1.5; − 2 2 − 1]′**

r = roots(poly(A))

```
r =
     1.0000
    −1.0000
     0.5000
```

The eigenvalues are distinct, hence A is diagonalizable.

M = (1 * eye(size(A)) − A)

rref([M [0 0 0]′)

```
ans =
      1  0   0  0
      0  1  −1  0
      0  0   0  0
```

The general solution is $x_1 = 0$, $x_2 = x_3$, $x_3 = r$. Let $r = 1$ and we have that $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}'$ is an eigenvector.

M = (− 1 * eye(size(A)) − A)

rref([M [0 0 0]′)

```
ans =
      1  0  −1  0
      0  1  −1  0
      0  0   0  0
```

The general solution is $x_1 = x_3$, $x_2 = x_3$, $x_3 = r$. Let $r = 1$ and we have that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}'$ is an eigenvector.

M = (.5 * eye(size(A)) − A)

rref([M [0 0 0]′)

```
ans =
      1  −1  0  0
      0   0  1  0
      0   0  0  0
```

The general solution is $x_1 = x_3$, $x_3 = 0$, $x_2 = r$. Let $r = 1$ and we have that $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}'$ is an eigenvector. Hence let

P = [0 1 1;1 1 1;1 1 0]′

```
P =
     0   1   1
     1   1   1
     1   1   0
```

then we have

A30 = P ∗ (diag([1 − 1 .5])^30 ∗ invert(P)

```
A30 =
        1.0000   −1.0000   1.0000
             0    0.0000   1.0000
             0         0   1.0000
```

Since all the entries are not displayed as integers we set the format to long and redisplay the matrix to view its contents for more detail.

format long

A30

```
A30 =
        1.0000000000000   −0.99999999906868   0.99999999906868
                      0    0.00000000093132   0.99999999906868
                      0                   0   1.00000000000000
```

ML.4. **A = [− 1 1 − 1; − 2 2 − 1; − 2 2 − 1];**

A,A^3,A^5

```
A =
     −1   1   −1
     −2   2   −1
     −2   2   −1

ans =
        −1   1   −1
        −2   2   −1
        −2   2   −1

ans =
        −1   1   −1
        −2   2   −1
        −2   2   −1
```

Further computation shows that A raised to an odd power gives A, hence sequence $A, A\hat{}3, A\hat{}5, \ldots$ converges to A.

A^2,A^4,A^6

```
A =
     1   −1   1
     0    0   1
     0    0   1
```

```
ans =
        1   −1   1
        0    0   1
        0    0   1

ans =
        1   −1   1
        0    0   1
        0    0   1
```

Further investigation shows that A raised to an even power gives the same matrix as displayed above. Hence the sequence $A\hat{}2, A\hat{}4, A\hat{}6 \ldots$ converges to this matrix.

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2. (a) $A^{-1} = A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$. (b) $B^{-1} = B$.

6. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$; $P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

8. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$; $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$.

10. $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$; $P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

12. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. 14. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

16. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. 18. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$.

T.1. In Exercise T.14 in Section 1.3, we showed that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$. We now have

$$(A\mathbf{x}) \cdot \mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{x}^T A^T \mathbf{y} = \mathbf{x} \cdot (A^T\mathbf{y}).$$

T.2. By Exercise T.1, we have

$$(A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot (A^T A\mathbf{y}) = \mathbf{x} \cdot (I_n\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$$

since $A^T A = I_n$ for the orthogonal matrix A.

T.3. The i, j entry of the matrix product $A^T A$ represents the ith row of A^T times the jth column of A. That is, the dot product of the ith and jth columns of A. If $A^T A = I_n$, then the dot product of the ith and jth columns of A is 1 if $i = j$ and 0 if $i \neq j$. Thus the columns of A form an orthonormal set in R^n. The converse is proved by reversing the steps in this argument.

T.4. If $A^T A = I_n$,

$$[\det(A)]^2 = \det(A^T) \cdot \det(A) = \det(A^T A) = \det(I_n) = 1.$$

Thus $\det(A) = \pm 1$.

T.5. Let

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

be a 2×2 symmetric matrix. Then its characteristic polynomial is $\lambda^2 - (a+d)\lambda + (ad - b^2)$. The roots of this polynomial are

$$\lambda = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - b^2)}}{2} = \frac{a + d \pm \sqrt{(a-d)^2 + 4b^2}}{2}.$$

If $b = 0$, A is already diagonal. If $b \neq 0$, the discriminant $(a-d)^2 + 4b^2$ is positive and there are two distinct real eigenvalues. Thus A is diagonalizable. By Theorem 8.4, there is a diagonalizing matrix P whose columns are linearly independent eigenvectors of A. We may assume further that those columns are unit vectors in R^2. By Theorem 8.7, the two columns are orthogonal. Thus P is an orthogonal matrix.

T.6. $(AB)^T(AB) = B^T A^T AB = B^T I_n B = I_n$.

T.7. $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = I_n^{-1} = I_n$.

T.8. (a) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = I_2$.

(b) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be orthogonal. Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

is orthogonal and its first column is a unit vector: $a^2 + b^2 = 1$. Let $a = \cos\theta$ and $b = -\sin\theta$ for some θ. Then $0 = ac + bd = c\cos\theta + d\sin\theta$ implies $c = \mu\sin\theta$, $d = \mu\cos\theta$ for some real number μ. But $1 = c^2 + d^2 = \mu^2(\sin^2\theta + \cos^2\theta) = \mu^2$ implies $\mu = \pm 1$.

T.9. For an $n \times n$ matrix A, if $A^T A\mathbf{y} = \mathbf{y} = I_n\mathbf{y}$ for all $\mathbf{y}$ in R^n, then $(A^T A - I_n)\mathbf{y} = \mathbf{0}$ for all $\mathbf{y}$ in R^n. Thus $A^T A - I_n$ is the zero matrix, so $A^T A = I_n$.

T.10. If A is nonsingular and diagonalizable, then there is an orthogonal matrix P so that $P^{-1}AP = D$, a diagonal matrix. We now have $A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$ (as in Exercise T.2 in Section 8.2). Since D^{-1} is diagonal and P^{-1} is orthogonal by Exercise T.7, we conclude that A^{-1} is orthogonally diagonalizable.

ML.1. (a) **A = [6 6;6 6];**

[V,D] = eig(A)

```
V =
    0.7071   0.7071
   -0.7071   0.7071
```

```
D =
   0    0
   0   12
```

Let $P = V$, then

P = V;P′ ∗ A ∗ P

```
ans =
        0          0
        0    12.0000
```

(b) **A = [1 2 2;2 1 2;2 2 1];**

[V,D] = eig(A)

```
V =
    0.7743   -0.2590   0.5774
   -0.6115   -0.5411   0.5774
   -0.1629    0.8001   0.5774
```

```
D =
   -1.0000         0        0
         0   -1.0000        0
         0         0   5.0000
```

Let $P = V$, then

P = V;P′ ∗ A ∗ P

```
ans =
        -1.0000    0.0000    0.0000
         0.0000   -1.0000   -0.0000
         0.0000   -0.0000    5.0000
```

(c) **A = [4 1 0;1 4 1;0 1 4];**
[V,D] = eig(A)

```
V =
    0.5000   -0.7071   -0.5000
    0.7071   -0.0000    0.7071
    0.5000    0.7071   -0.5000

D =
    5.4142         0         0
         0    4.0000         0
         0         0    2.5858
```

Let $P = V$, then
P = V;P′ ∗ A ∗ P

```
ans =
         5.4142   -0.0000   -0.0000
        -0.0000    4.0000    0.0000
        -0.0000    0.0000    2.5858
```

ML.2. (a) **A = [1 2; − 1 4];**
[V,D] = eig(A)

```
V =
   -0.8944   -0.7071
   -0.4472   -0.7071

D =
    2    0
    0    3
```

V′ ∗ V

```
ans =
        1.0000    0.9487
        0.9487    1.0000
```

Hence V is not orthogonal. However, since the eigenvalues are distinct A is diagonalizable, so V can be replaced by an orthogonal matrix.

(b) **A = [2 1 2;2 2 − 2;3 1 1];**
[V,D] = eig(A)

```
V =
   -0.5482    0.7071    0.4082
    0.6852   -0.0000   -0.8165
    0.4796    0.7071    0.4082

D =
   -1.0000         0         0
         0    4.0000         0
         0         0    2.0000
```

V′ ∗ V

```
ans =
     1.0000   -0.0485   -0.5874
    -0.0485    1.0000    0.5774
    -0.5874    0.5774    1.0000
```

Hence V is not orthogonal. However, since the eigenvalues are distinct A is diagonalizable, so V can be replaced by an orthogonal matrix.

(c) **A = [1 − 3;3 − 5];**
[V,D] = eig(A)

```
V =
    0.7071   0.7071
    0.7071   0.7071

D =
    -2    0
     0   -2
```

Inspecting V, we see that there is only one linearly independent eigenvector, so A is not diagonalizable.

(d) **A = [1 0 0;0 1 1;0 1 1];**
[V,D] = eig(A)

```
V =
    1.0000        0         0
         0   0.7071    0.7071
         0   0.7071   -0.7071

D =
    1.0000        0        0
         0   2.0000        0
         0        0   0.0000
```

V′ ∗ V

```
ans =
    1.0000        0        0
         0   1.0000        0
         0        0   1.0000
```

Hence V is orthogonal. We should have expected this since A is symmetric.

Supplementary Exercises, p. 445

2. Not diagonalizable.

4. $P = \begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 6 \\ 2 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

6. Not diagonalizable.

8. No.

10. $P = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$, $D = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

12. (a) $p_1(\lambda)p_2(\lambda)$. (b) $p_1(\lambda)p_2(\lambda)$.

T.1. Let P be a nonsingular matrix such that $P^{-1}AP = D$. Then

$$\mathrm{Tr}(D) = \mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}(P^{-1}(AP)) = \mathrm{Tr}((AP)P^{-1}) = \mathrm{Tr}(APP^{-1}) = \mathrm{Tr}(AI_n) = \mathrm{Tr}(A).$$

T.2. In Exercise T.1, the diagonal entries of D are the eigenvalues of A; thus $\mathrm{Tr}(A) = \mathrm{Tr}(D) =$ sum of the eigenvalues of A.

T.3. Let P be such that $P^{-1}AP = B$.

(a) $B^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T = P^T A^T (P^T)^{-1}$; hence A^T and B^T are similar.

(b) $\mathrm{rank}(B) = \mathrm{rank}(P^{-1}AP) = \mathrm{rank}(P^{-1}A)$ (See Exercise T.6(c) in the Supplementary Exercises to Chapter 6.) $= \mathrm{rank}(A)$ (See Exercise T.6(d) in the Supplementary Exercises to Chapter 6.)

(c) $\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = (1/\det(P))\det(A)\det(P) = \det(A)$. Thus $\det(B) \neq 0$ if and only if $\det(A) \neq 0$.

(d) Since A and B are nonsingular and $B = P^{-1}AP$, $B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}P$. That is, A^{-1} and B^{-1} are similar.

(e) $\mathrm{Tr}(B) = \mathrm{Tr}(P^{-1}AP) = \mathrm{Tr}((P^{-1}A)P) = \mathrm{Tr}(P(P^{-1}A)) = \mathrm{Tr}(A)$. (See Supplementary Exercise T.1 in Chapter 1.)

T.4. If A is orthogonal, then $A^T = A^{-1}$. Since

$$(A^T)^T = (A^{-1})^T = (A^T)^{-1},$$

we have that A^T is orthogonal.

T.5. $(cA)^T = (cA)^{-1}$ if and only if $cA^T = \frac{1}{c}A^{-1} = \frac{1}{c}A^T$. That is, $c = \frac{1}{c}$. Hence $c = \pm 1$.

T.6. The characteristic polynomial of A is

$$f(\lambda) = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) - bc = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

T.7. (a) The characteristic polynomial is

$$f(\lambda) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 5 \\ 3 & 2 & 1-\lambda \end{vmatrix} = -\lambda^3 + \lambda^2 + 24\lambda + 36.$$

(b) The characteristic polynomial is $f(\lambda) = (1 - \lambda)(2 - \lambda)(-3 - \lambda)$.

(c) The characteristic polynomial is

$$f(\lambda) = \begin{vmatrix} 3-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 6.$$

T.8. From the Cayley–Hamilton Theorem we have

$$A^n + a_1 A^{n-1} + \cdots + a_{n-2}A^2 + a_{n-1}A + a_n I_n = O.$$

Since A is nonsingular, $\det(A) \neq 0$, so $a_n \neq 0$. Multiplying the last equation by A^{-1} we have

$$A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-2}A + a_{n-1}I_n + a_n A^{-1} = O.$$

Solving for A^{-1} we obtain the desired result.

Chapter 9

Applications of Eigenvectors and Eigenvalues (Optional)

Section 9.1, p. 450

2. $A^k = (PBP^{-1})(PBP^{-1})\cdots(PBP^{-1}) = PB(P^{-1}P)B(P^{-1}P)\cdots(P^{-1}P)BP^{-1} = PB^kP^{-1}$.

4. (a) $u_0 = u_1 = 1$, $u_n = u_{n-1} + 2u_{n-2}$ for $n \geq 2$.

   (b) $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = A^{n-1}\mathbf{u}_0$, A is similar to the diagonal matrix

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

   $u_n = \frac{1}{3}\left[2^{n+1} + (-1)^n\right]$.

T.1. Let us define u_{-1} to be 0. Then for $n = 0$,

$$A^1 = A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u_1 & u_0 \\ u_0 & u_{-1} \end{bmatrix}$$

and, for $n = 1$,

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} u_2 & u_1 \\ u_1 & u_0 \end{bmatrix}.$$

Suppose that the formula

$$A^{n+1} = \begin{bmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{bmatrix} \tag{9.1}$$

holds for values up to and including n, $n \geq 1$. Then

$$\begin{aligned} A^{n+2} = A \cdot A^{n+1} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} u_{n+1} + u_n & u_n + u_{n-1} \\ u_{n+1} & u_n \end{bmatrix} = \begin{bmatrix} u_{n+2} & u_{n+1} \\ u_{n+1} & u_n \end{bmatrix}. \end{aligned}$$

Thus the formula (9.1) also holds for $n + 1$, so it holds for all natural numbers n. Using (9.1), we see that

$$u_{n+1}u_{n-1} - u_n^2 = \begin{vmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{vmatrix} = \det\left(A^{n+1}\right) = (\det(A))^{n+1} = (-1)^{n+1}.$$

Section 9.2, p. 460

2. (a) $\mathbf{x}(t) = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + b_3 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} e^{3t}$.

   (b) $\mathbf{x}(t) = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + 4 \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} e^{3t}$.

4. $\mathbf{x}(t) = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + b_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + b_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} e^{t}$.

6. $\mathbf{x}(t) = b_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$.

8. $\mathbf{x}(t) = b_1 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^{t} + b_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + b_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{3t}$.

10. The system of differential equations is

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} & \frac{2}{30} \\ \frac{1}{10} & -\frac{2}{30} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The characteristic polynomial of the coefficient matrix is $p(\lambda) = \lambda^2 + \frac{1}{6}\lambda$. Eigenvalues and associated eigenvectors are:

$$\lambda_1 = 0, \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \qquad \lambda_2 = -\tfrac{1}{6}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Hence the general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = b_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-\frac{1}{6}t} + b_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}.$$

Using the initial conditions $x(0) = 10$ and $y(0) = 40$, we find that $b_1 = 10$ and $b_2 = 30$. Thus, the particular solution, which gives the amount of salt in each tank at time t, is

$$x(t) = -10e^{-\frac{1}{6}t} + 20$$
$$y(t) = 10e^{-\frac{1}{6}t} + 30.$$

T.1. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the equation $\mathbf{x}' = A\mathbf{x}$, and let a and b be scalars. Then

$$\frac{d}{dt}(a\mathbf{x}_1 + b\mathbf{x}_2) = a\mathbf{x}_1' + b\mathbf{x}_2' = aA\mathbf{x}_1 + bA\mathbf{x}_2 = A(a\mathbf{x}_1 + b\mathbf{x}_2).$$

Thus $a\mathbf{x}_1 + b\mathbf{x}_2$ is also a solution to the given equation.

ML.1. **A = [1 1 0;0 0 1;8 − 14 7];**

[v,d] = eig(A)

```
v =
    −0.5774   0.2182   0.0605
    −0.5774   0.4364   0.2421
    −0.5774   0.8729   0.9684
```

```
d =
    1.0000       0       0
         0  2.0000       0
         0       0  4.0000
```

The general solution is given by

$$\mathbf{x}(t) = b_1 \begin{bmatrix} -0.5774 \\ -0.5774 \\ -0.5774 \end{bmatrix} e^{t} + b_2 \begin{bmatrix} 0.2182 \\ 0.4364 \\ 0.8729 \end{bmatrix} e^{2t} + b_3 \begin{bmatrix} 0.0605 \\ 0.2421 \\ 0.9684 \end{bmatrix} e^{4t}.$$

ML.2. **A = [1 0 0;0 3 − 2;0 − 2 3];**
[v,d] = eig(A)

```
v =
    1.0000        0        0
         0  −0.7071  −0.7071
         0  −0.7071   0.7071

d =
    1  0  0
    0  1  0
    0  0  5
```

The general solution is given by

$$\mathbf{x}(t) = b_1 \begin{bmatrix} 1.0000 \\ 0 \\ 0 \end{bmatrix} e^{t} + b_2 \begin{bmatrix} 0 \\ -0.7071 \\ -0.7071 \end{bmatrix} e^{2t} + b_3 \begin{bmatrix} 0 \\ -0.7071 \\ 0.7071 \end{bmatrix} e^{5t}.$$

ML.3. **A = [1 2 3;0 1 0;2 1 2];**
[v,d] = eig(A)

```
v =
    −0.8321  −0.7071  −0.1374
          0        0   0.8242
     0.5547  −0.7071  −0.5494

d =
    −1  0  0
     0  4  0
     0  0  1
```

The general solution is given by

$$\mathbf{x}(t) = b_1 \begin{bmatrix} -0.8321 \\ 0 \\ 0.5547 \end{bmatrix} e^{-t} + b_2 \begin{bmatrix} -0.7071 \\ 0 \\ -0.7071 \end{bmatrix} e^{4t} + b_3 \begin{bmatrix} -0.1374 \\ 0.8242 \\ -0.5494 \end{bmatrix} e^{t}.$$

Section 9.3, p. 474

2. The eigenvalues of the coefficient matrix are $\lambda_1 = 2$ and $\lambda_2 = 1$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus the origin is an unstable equilibrium. The phase portrait shows all trajectories tending away from the origin.

4. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = -2$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus the origin is a saddle point. The phase portrait shows trajectories not in the direction of an eigenvector heading toward the origin, but bending away as $t \to \infty$.

6. The eigenvalues of the coefficient matrix are $\lambda_1 = -1+i$ and $\lambda_2 = -1-i$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} -1 \\ i \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Since the real part of the eigenvalues is negative the origin is a stable equilibrium with trajectories spiraling in toward it.

8. The eigenvalues of the coefficient matrix are $\lambda_1 = -2+i$ and $\lambda_2 = -2-i$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Since the real part of the eigenvalues is negative the origin is a stable equilibrium with trajectories spiraling in toward it.

10. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = 5$ with associated eigenvectors $\mathbf{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the origin is an unstable equilibrium. The phase portrait shows all trajectories tending away from the origin.

T.1. (a) Just compute $Ax(t)$ and note that $x_1'(t) = x_2(t)$ and $x_2'(t) = -\frac{k}{m}x_1(t) - 2rx_2(t)$. Replace $x_2(t)$ in the second expression by $x_1'(t)$ and we obtain

$$x_1''(t) = -\frac{k}{m}x_1(t) - 2rx_1'(t)$$

which is equivalent to Equation (12).

(b) (i) $\lambda = -\frac{1}{2}, -\frac{3}{2}$. All trajectories tend toward the equilibrium point at the origin, which is an attractor.

(ii) $\lambda = -1, -1$. All trajectories tend to a stable equilibrium point at the origin.

(iii) $\lambda = -1 \pm i$. The trajectories spiral inward to the origin which is a stable equilibrium point.

(iv) $\lambda = -1 \pm 3i$. The trajectories spiral inward to the origin which is a stable equilibrium point.

(c) $A = \begin{bmatrix} 0 & 1 \\ -k & -2 \end{bmatrix}$. Its characteristic polynomial is $\lambda^2 + 2\lambda + k$. The eigenvalues are given by $\lambda = -1 \pm \sqrt{1-k}$ and are complex for $k > 1$.

(d) (i) $\lambda = \pm i$. Trajectories are elliptical.

(ii) $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. The trajectories spiral inward to the origin which is a stable equilibrium point.

(iii) $\lambda = -1, -1$. Trajectories tend to a stable equilibrium point at the origin.

(iv) $\lambda = -\sqrt{2} \pm 1$. Trajectories tend toward the origin which is an attractor.

(e) The rider would experience an oscillatory up and down motion.

(f) $A = \begin{bmatrix} 0 & 1 \\ -1 & -2r \end{bmatrix}$. The characteristic polynomial is $\lambda^2 + 2r\lambda + 1$ and the eigenvalues are given by $\lambda = -r \pm \sqrt{r^2 - 1}$. The eigenvalues will be real provided $r \geq 1$.

Section 9.4, p. 483

2. (a) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & -3 & 3 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(b) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

(c) $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

6. $y_1^2 + 2y_2^2$. 8. $4y_3^2$. 10. $5y_1^2 - 5y_2^2$. 12. $y_1^2 + y_2^2$.

14. $y_1^2 + y_2^2 + y_3^2$. 16. $y_1^2 + y_2^2 + y_3^2$. 18. $y_1^2 - y_2^2 - y_3^2$; rank $= 3$; signature $= -1$.

20. $y_1^2 = 1$, which represents the two lines $y_1 = 1$ and $y_1 = -1$. The equation $-y_2^2 = 1$ represents no conic at all.

22. g_1, g_2, and g_4 are equivalent. The eigenvalues of the matrices associated with the quadratic forms are: for g_1: 1, 1, −1; for g_2: 9, 3, −1; for g_3: 2, −1, −1; for g_4: 5, 5, −5. The rank r and signature s of g_1, g_2 and g_4 are $r = 3$ and $s = 2p - r = 1$.

24. The eigenvalues of the matrices are:

(a) 1, −1. (b) 0, 2. (c) 0, 3, −1. (d) 3, 3, 15. (e) 2, 1, 3, −3.

T.1. $(P^TAP)^T = P^TA^TP = P^TAP$ since $A^T = A$.

T.2. (a) $A = P^TAP$ for $P = I_n$.

(b) If $B = P^TAP$ with nonsingular P, then $A = (P^{-1})^TBP^{-1}$ and B is congruent to A.

(c) If $B = P^TAP$ and $C = Q^TBA$ with P, Q nonsingular, then $C = Q^TP^TAPQ = (PQ)^TA(PQ)$ with PQ nonsingular.

T.3. By Theorem 8.9, for the symmetric matrix A, there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal. Since P is orthogonal, $P^{-1} = P^T$. Thus A is congruent to D.

T.4. Let

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

and let the eigenvalues of A be λ_1 and λ_2. The characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (a + d)\lambda + ad - b^2.$$

If A is positive definite then both λ_1 and λ_2 are > 0, so $\lambda_1\lambda_2 = \det(A) > 0$. Also,

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a > 0.$$

Conversely, let $\det(A) > 0$ and $a > 0$. Then $\lambda_1\lambda_2 = \det(A) > 0$ so λ_1 and λ_2 are of the same sign. If λ_1 and λ_2 are both < 0 then $\lambda_1 + \lambda_2 = a + d < 0$, so $d < -a$. Since $a > 0$, we have $d < 0$ and $ad < 0$. Now $\det(A) = ad - b^2 > 0$, which means that $ad > b^2$ so $ad > 0$, a contradiction. Hence, λ_1 and λ_2 are both positive.

T.5. Let A be positive definite and $Q(\mathbf{x}) = \mathbf{x}^TA\mathbf{x}$. By Theorem 9.3, $Q(\mathbf{x})$ is a quadratic form which is equivalent to

$$Q'(\mathbf{y}) = y_1^2 + y_2^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_r^2.$$

If Q and Q' are equivalent then $Q'(\mathbf{y}) > 0$ for each $\mathbf{y} \neq \mathbf{0}$. However, this can happen if and only if all terms in $Q'(\mathbf{y})$ are positive; that is, if and only if A is congruent to I_n, or if and only if $A = P^TI_nP = P^TP$.

ML.1. (a) **A = [−1 0 0;0 1 1;0 1 1];**
eig(A)

```
ans =
    -1.0000
     2.0000
     0.0000
```

If we set the format to long e, then the eigenvalues are displayed as

```
ans =
    -1.000000000000000e + 000
     2.000000000000000e + 000
     2.220446049250313e - 016
```

Since the last value is extremely small, we will consider it zero. The **eig** command approximates the eigenvalues, hence errors due to using machine arithmetic can occur. Thus it follows that $\text{rank}(A) = 2$ and the signature of the quadratic form is 0.

(b) **A = ones(3);**
eig(A)

```
ans =
     0.0000
    -0.0000
     3.0000
```

If we set the format to long e, then the eigenvalues are displayed as

```
ans =
     2.343881062810587e - 017
    -7.011704839834072e - 016
     2.999999999999999e + 000
```

We will consider the first two eigenvalues zero. Hence $\text{rank}(A) = 1$ and the signature of the quadratic form is 1.

(c) **A = [2 1 0 −2;1 −1 1 3;0 1 2 −1;−2 3 −1 0];**
eig(A)

```
ans =
     2.2896
     1.6599
     3.5596
    -4.5091
```

It follows that $\text{rank}(A) = 4$ and the signature of the quadratic form is 2.

(d) **A = [2 −1 0 0;−1 2 −1 0;0 −1 2 −1;0 0 −1 2];**
eig(A)

```
ans =
    1.3820
    0.3820
    2.6180
    3.6180
```

It follows that $\text{rank}(A) = 4$ and the signature of the quadratic form is 4.

ML.2. By Theorem 9.4, only the matrix in part (d) is positive definite.

Section 9.5, p. 491

2. Parabola 4. Two parallel lines. 6. Straight line 8. Hyperbola.

10. None. 12. Hyperbola; $\dfrac{x'^2}{4} - \dfrac{y'^2}{4} = 1.$ 14. Parabola; $x'^2 + 4y'^2 = 0.$

16. Ellipse; $4x'^2 + 5y'^2 = 20.$

18. None; $2x'^2 + y'^2 = -2.$

20. Possible answer: hyperbola; $\dfrac{x'^2}{2} - \dfrac{y'^2}{2} = 1.$

22. Possible answer: parabola; $x'^2 = 4y'$

24. Possible answer: ellipse; $\dfrac{x'^2}{\frac{1}{2}} + y'^2 = 1.$

26. Possible answer: ellipse; $\dfrac{x''^2}{4} + y''^2 = 1.$

28. Possible answer: ellipse; $x''^2 + \dfrac{y''^2}{\frac{1}{2}} = 1.$

30. Possible answer: parabola; $y''^2 = -\frac{1}{8}x''.$

Section 9.6, p. 499

2. Ellipsoid. 4. Elliptic paraboloid. 6. Hyperbolic paraboloid.

8. Hyperboloid of one sheet. 10. Hyperbolic paraboloid. 12. Hyperboloid of one sheet.

14. Ellipsoid.

16. HYperboloid of one sheet; $\dfrac{x''^2}{8} + \dfrac{y''^2}{4} - \dfrac{z''^2}{8} = 1.$

18. Ellipsoid; $\dfrac{x''^2}{9} + \dfrac{y''^2}{9} + \dfrac{z''^2}{\frac{9}{5}} = 1.$

20. Hyperboloid of two sheets; $x''^2 - y''^2 - z''^2 = 1.$

22. Ellipsoid; $\dfrac{x''^2}{-\frac{25}{2}} + \dfrac{y''^2}{\frac{25}{4}} + \dfrac{z''^2}{\frac{25}{10}} = 1.$

24. Hyperbolic paraboloid; $\dfrac{x''^2}{\frac{1}{2}} - \dfrac{y''^2}{\frac{1}{2}} = z''.$

26. Ellipsoid; $\dfrac{x''^2}{\frac{1}{2}} + \dfrac{y''^2}{\frac{1}{2}} + \dfrac{z''^2}{\frac{1}{4}} = 1.$

28. Hyperboloid of one sheet; $\dfrac{x''^2}{4} + \dfrac{y''^2}{2} - \dfrac{z''^2}{1} = 1.$

Supplementary Exercises, p. 500

2. (a) $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (b) $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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6. $y_1^2 + y_2^2$.

8. (a) One answer is $k = 8$.
   (b) One answer is $k = -2$.

T.1. Proceed by showing corresponding entries of the matrices involved are equal.

(a)

$$\begin{aligned}\left[\frac{d}{dt}[c_1A(t) + c_2B(t)]\right]_{ij} &= \frac{d}{dt}[c_1A(t) + c_2B(t)]_{ij} \\ &= \frac{d}{dt}[c_1a_{ij}(t) + c_2b_{ij}(t)] = c_1\frac{d}{dt}a_{ij} + c_2\frac{d}{dt}b_{ij}(t) \\ &= c_1\left[\frac{d}{dt}A(t)\right]_{ij} + c_2\left[\frac{d}{dt}B(t)\right]_{ij} = \left[c_1\frac{d}{dt}A(t) + c_2\frac{d}{dt}B(t)\right]_{ij}.\end{aligned}$$

(b)

$$\begin{aligned}\left[\int_a^t (c_1A(s) + c_2B(s))\,ds\right]_{ij} &= \int_a^t [c_1A(s) + c_2B(s)]_{ij}\,ds \\ &= \int_a^t [c_1a_{ij}(s) + c_2b_{ij}(s)]\,ds = c_1\int_a^t a_{ij}\,ds + c_2\int_a^t b_{ij}\,ds \\ &= c_1\left[\int_a^t A(s)\,ds\right]_{ij} + c_2\left[\int_a^t B(s)\,ds\right]_{ij} \\ &= \left[c_1\int_a^t A(s)\,ds + c_2\int_a^t B(s)\,ds\right]_{ij}.\end{aligned}$$

(c)

$$\begin{aligned}\left[\frac{d}{dt}[A(t)B(t)]\right]_{ij} &= \frac{d}{dt}[A(t)B(t)]_{ij} = \frac{d}{dt}\left(\sum_{k=1}^n a_{ik}(t)b_{kj}(t)\right) \\ &= \sum_{k=1}^n \frac{d}{dt}[a_{ik}(t)b_{kj}(t)] = \sum_{k=1}^n \left(\frac{d}{dt}[a_{ik}(t)]b_{kj}(t) + a_{ik}(t)\frac{d}{dt}[b_{kj}(t)]\right) \\ &= \sum_{k=1}^n \left(\frac{d}{dt}[a_{ik}(t)]b_{kj}(t)\right) + \sum_{k=1}^n \left(a_{ik}(t)\frac{d}{dt}[b_{kj}(t)]\right) \\ &= \left[\frac{d}{dt}[A(t)B(t)\right]_{ij} + \left[A(t)\frac{d}{dt}[B(t)]\right]_{ij} \\ &= \left[\frac{d}{dt}[A(t)B(t) + A(t)\frac{d}{dt}[B(t)]\right]_{ij}.\end{aligned}$$

T.2. (a) Assuming that the differentiation of the series can be carried out term by term,

$$\begin{aligned}\frac{d}{dt}[e^{At}] &= \frac{d}{dt}\left[I_n + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots\right] \\ &= A + A^2t + A^3\frac{t^2}{2!} + \cdots \\ &= A[I_n + At + A^2\frac{t^2}{2!} + \cdots] \\ &= Ae^{At}\end{aligned}$$

(b) Note that $A^k = B^k = O$ for $k = 2, 3, \ldots$, with $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so the infinite series terminates. Now

$$e^A e^B = [I_2 + A][I_2 + B] = I_2 + A + B + AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$e^{A+B} = \left[I_2 + (A+B) + (A+B)^2 \frac{1}{2!} \right] = I_2 + A + B + \frac{1}{2!}(AB + BA) = \begin{bmatrix} \frac{3}{2} & 1 \\ 1 & \frac{3}{2} \end{bmatrix}.$$

(c)
$$\begin{aligned} e^{iA} &= I_n + iA + (iA)^2 \frac{1}{3!} + (iA)^4 \frac{1}{4!} + (iA)^5 \frac{1}{5!} + \cdots \\ &= I_n + iA - A^2 \frac{1}{2!} - iA^3 \frac{1}{3!} - A^4 \frac{1}{4!} + iA^5 \frac{1}{5!} + \cdots \\ &= \left(I_n - A^2 \frac{1}{2!} + A^4 \frac{1}{4!} - \cdots \right) + i \left(A - A^3 \frac{1}{3!} + A^5 \frac{1}{5!} - \cdots \right) \\ &= \cos A + i \sin A \end{aligned}$$

T.3. If $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$, $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$, and in general for $k = 2, 3, \ldots$

$$(A+B)^k = A^k + \binom{k}{1} A^{k-1}B + \binom{k}{2} A^{k-2}B^2 + \cdots + \binom{k}{k-1} AB^{k-1} + B^k.$$

Applying these results to the product of the series in the following, we see that

$$\begin{aligned} e^A e^B &= \left[I_n + A + A^2 \frac{1}{2!} + A^3 \frac{1}{3!} + \cdots \right] \left[I_n + B + B^2 \frac{1}{2!} + B^3 \frac{1}{3!} + \cdots \right] \\ &= I_n + (A+B) + \frac{1}{2!}(A^2 + 2AB + B^2) + \frac{1}{3!}(A^3 + 3A^2B + 3AB^2 + B^3) + \cdots \\ &= e^{A+B} \end{aligned}$$

T.4. Note that e^{Dt} is a sum of diagonal matrices since D is diagonal. We proceed by showing that $\left[B(t)\right]_{ii} = \left[e^{Dt}\right]_{ii}$:

$$\begin{aligned} \left[e^{tD}\right]_{ii} &= \left[I_n + Dt + D^2 \frac{t^2}{2!} + D^3 \frac{t^3}{3!} + \cdots \right] \\ &= 1 + [Dt]_{ii} + \left[D^2 \frac{t^2}{2!} \right]_{ii} + \left[D^3 \frac{t^3}{3!} \right]_{ii} + \cdots \\ &= 1 + d_{ii}t + \frac{(\lambda_{ii}t)^2}{2!} + \frac{(\lambda_{ii}t)^3}{3!} + \cdots \\ &= e^{\lambda_{ii}t} = \left[B(t)\right]_{ii}. \end{aligned}$$

T.5. Recall that if C is $n \times n$ and R is $n \times 1$ then $CR = r_1 \text{col}_1(C) + r_2 \text{col}_2(C) + \cdots + r_n \text{col}_n(C)$. It follows that

$$\mathbf{x}(t) = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix} \begin{bmatrix} e^{c_1 t} b_1 \\ e^{c_2 t} b_2 \\ \vdots \\ e^{c_n t} b_n \end{bmatrix} = P \begin{bmatrix} e^{c_1 t} & 0 & \cdots & 0 \\ 0 & e^{c_2 t} & \cdots & \vdots \\ 0 & 0 & \cdot & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{c_n t} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = P e^{Dt} B.$$

T.6. From Exercise T.5, $\mathbf{x}(t) = Pe^{Dt}B$ is the general solution to the system of differential equations. Applying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ we have

$$\mathbf{x}_0 = \mathbf{x}(0) = Pe^{D\cdot 0}B = PI_nB = PB.$$

Solving for B we have $B = P^{-1}\mathbf{x}_0$, so the solution to the system of differential equations that satisfies the initial condition is

$$\begin{aligned}
\mathbf{x}(t) &= Pe^{Dt}P^{-1}\mathbf{x}_0 \\
&= P\left[I_n + Dt + D^2\frac{t^2}{2!} + D^3\frac{t^3}{3!} + \cdots\right]P^{-1}\mathbf{x}_0 \\
&= \left[I_n + PDP^{-1}t + PD^2P^{-1}\frac{t^2}{2!} + PD^3P^{-1}\frac{t^3}{3!} + \cdots\right]\mathbf{x}_0 \\
&= \left[I_n + PDP^{-1}t + PDP^{-1}PDP^{-1}\frac{t^2}{2!} + PDP^{-1}PDP^{-1}PDP^{-1}\frac{t^3}{3!} + \cdots\right]\mathbf{x}_0 \\
&= \left[I_n + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots\right]\mathbf{x}_0 = e^{At}\mathbf{x}_0,
\end{aligned}$$

where $A = PDP^{-1}$.

Chapter 10

Linear Transformations and Matrices

Section 10.1, p. 507

2. (a).

4. (a) No.

$$\begin{aligned} L((a_1t^2 + b_1t + c_1) + (a_2t^2 + b_2t + c_2)) &= L((a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)) \\ &= (a_1 + a_2)t + (b_1 + b_2) + 1 \end{aligned}$$

while

$$\begin{aligned} L(a_1t^2 + b_1t + c_1) + L(a_2t^2 + b_2t + c_2) &= a_1t + b_1 + 1 + a_2t + b_2 + 1 \\ &= (a_1 + a_2)t + (b_1 + b_2) + 2. \end{aligned}$$

(b) Yes.

$$\begin{aligned} L((a_1t^2 + b_1t + c_1) + (a_2t^2 + b_2t + c_2)) &= L((a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)) \\ &= 2(a_1 + a_2) - (b_1 + b_2) \\ &= 2a_1 - b_1 + 2a_2 - b_2 \\ &= L(a_1t^2 + b_1t + c_1) + L(a_2t^2 + b_2t + c_2). \end{aligned}$$

Also, if k is a scalar, then

$$L(k(at^2 + bt + c)) = L(kat^2 + kbt + kc) = 2ka - kb = k(2a - b) = kL(at^2 + bt + c).$$

(c) No.

$$\begin{aligned} L((a_1t^2 + b_1t + c_1) + (a_2t^2 + b_2t + c_2)) &= L((a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)) \\ &= (a_1 + a_2 + 2)t + (b_1 + b_2) - (a_1 + a_2) \end{aligned}$$

while

$$\begin{aligned} L(a_1t^2 + b_1t + c_1) + L(a_2t^2 + b_2t + c_2) &= (a_1 + 2)t + (b_1 - a_1) + (a_2 + 2)t + (b_2 - a_2) \\ &= (a_1 + a_2 + 4)t + (b_1 + b_2) - (a_1 + a_2). \end{aligned}$$

6. Let A and B be elements of M_{nn}. Then

$$L(A + B) = C(A + B) = CA + CB = L(A) + L(B).$$

Also, if k is a scalar, then

$$L(kA) = C(kA) = kCA = kL(A).$$

8. No. 10. No. 12. No.

14. $L(f+g) = \int_0^1 (f(x)+g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = L(f) + L(g)$,
$L(cf) = \int_0^1 (cf(x))\,dx = c\int_0^1 f(x)\,dx = cL(f)$.

16. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n. Then

$$L(\mathbf{u}+\mathbf{v}) = P_{S\leftarrow T}(\mathbf{u}+\mathbf{v}) = P_{S\leftarrow T}\mathbf{u} + P_{S\leftarrow T}\mathbf{v} = L(\mathbf{u}) + L(\mathbf{v}).$$

Also, if c is a scalar, then

$$L(c\mathbf{u}) = P_{S\leftarrow T}(c\mathbf{u}) = c(P_{S\leftarrow T}\mathbf{u}) = cL(\mathbf{u}).$$

18. (a) $2t^3 - 5t^2 + 2t + 3$. (b) $at^3 + bt^2 + at + c$.

T.1.
$$\begin{aligned} L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) &= L(c_1\mathbf{v}_1 + +L(c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= c_1L(\mathbf{v}_1) + L(c_2\mathbf{v}_2) + L(c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k) \\ &= \cdots = c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + \cdots + c_kL(\mathbf{v}_k). \end{aligned}$$

T.2. $L(\mathbf{u}-\mathbf{v}) = L(\mathbf{u}+(-1)\mathbf{v}) = L(\mathbf{u}) + (-1)L(\mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$.

T.3. See the proof of Corollary 4.1 in Section 4.3.

T.4. If L is a linear transformation, then

$$L(a\mathbf{u} + b\mathbf{v}) = aL(\mathbf{u}) + bL(\mathbf{v}) \tag{10.1}$$

by Theorem 10.1. Conversely, if Equation (10.1) holds for any a and b and vectors $\mathbf{u}$ and $\mathbf{v}$, then in particular it holds for $a = b = 1$, which gives (a) of the definition of linear transformation, and for $a = c$, $b = 0$, which gives (b) of that definition.

T.5. Let A and B be in M_{nn}. Then

$$\mathrm{Tr}(A+B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

and

$$\mathrm{Tr}(cA) = \sum_{i=1}^{n}(ca_{ii}) = c\sum_{i=1}^{n} a_{ii} = c\,\mathrm{Tr}(A).$$

Hence Tr is a linear transformation.

T.6. No. Let $n = 2$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$L(A+B) = L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq \mathbf{0} + \mathbf{0} = L(A) + L(B).$$

T.7. We have $L(\mathbf{u}+\mathbf{v}) = \mathbf{0}_W = \mathbf{0}_W + \mathbf{0}_W = L(\mathbf{u}) + L(\mathbf{v})$ and $L(c\mathbf{u}) = \mathbf{0}_W = c\mathbf{0}_W = cL(\mathbf{u})$.

T.8. We have $I(\mathbf{u}+\mathbf{v}) = \mathbf{u}+\mathbf{v} = I(\mathbf{u}) + I(\mathbf{v})$ and $I(c\mathbf{u}) = c\mathbf{u} = cI(\mathbf{u})$.

T.9. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in $L(V_1)$ and let c be a scalar. Then $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V_1. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2) \quad \text{and} \quad c\mathbf{w}_1 = cL(\mathbf{v}_1) = L(c\mathbf{v}_1).$$

Since $\mathbf{v}_1 + \mathbf{v}_2$ and $c\mathbf{v}_1$ are in V_1, we conclude that $\mathbf{w}_1 + \mathbf{w}_2$ and $c\mathbf{w}_1$ lie in $L(V_1)$. Hence $L(V_1)$ is a subspace of V.

T.10. Let $\mathbf{v}$ be any vector in V. Then

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

We now have

$$\begin{aligned} L_1(\mathbf{v}) &= L_1(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) \\ &= c_1L_1(\mathbf{v}_1) + c_2L_1(\mathbf{v}_2) + \cdots + c_nL_1(\mathbf{v}_n) \\ &= c_1L_2(\mathbf{v}_1) + c_2L_2(\mathbf{v}_2) + \cdots + c_nL_2(\mathbf{v}_n) \\ &= L_2(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = L_2(\mathbf{v}). \end{aligned}$$

T.11. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in $L^{-1}(W_1)$ and let c be a scalar. Then $L(\mathbf{v}_1 + \mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2)$ is in W_1 since $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ are in W_1 and W_1 is a subspace of V. Hence $\mathbf{v}_1 + \mathbf{v}_2$ is in $L^{-1}(W_1)$. Similarly, $L(c\mathbf{v}_1) = cL(\mathbf{v}_1)$ is in W_1 so $c\mathbf{v}_1$ is in $L^{-1}(W_1)$. Hence, $L^{-1}(W_1)$ is a subspace of V.

T.12. T is not a linear transformation because $T(\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{b}$ but $T(\mathbf{v}_1) + T(\mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{b}) + (\mathbf{v}_2 + \mathbf{b}) = (\mathbf{v}_1 + \mathbf{v}_2) + 2\mathbf{b}$.

T.13. Let U be the eigenspace associated with λ. Then we have already shown in Exercise T.1 in Section 8.1 that U is a subspace of R^n. Now let $\mathbf{x}$ be a vector in U, so that $A\mathbf{x} = \lambda\mathbf{x}$. Then $L(\mathbf{x}) = A\mathbf{x} = \lambda\mathbf{x}$, which is in U, since U is a subspace of R^n and $\lambda\mathbf{x}$ is a scalar multiple of $\mathbf{x}$. So $L(\mathbf{x})$ is also an eigenvector of A associated with λ. Hence, U is an invariant subspace of R^n.

ML.1. (a) **A = [1 0;0 0];B = [0 0;0 1];**
det(A + B)

```
ans =
        1
```

det(A) + det(B)

```
ans =
        0
```

(b) **A = eye(3);B = −ones(3);**
det(A + B)

```
ans =
       -2
```

det(A) + det(B)

```
ans =
        1
```

ML.2. (a) **A = eye(2);B = eye(2);**
rank(A + B)

```
ans =
        2
```

rank(A) + rank(B)

```
ans =
        4
```

(b) Repeat the preceding with A = eye(3) and B = eye(3).

Section 10.2, p. 519

2. (a) No. (b) Yes. (c) Yes. (d) No. (e) $\left\{\begin{bmatrix}-2r\\ r\end{bmatrix}\right\}$, $r =$ any real number.

   (f) Possible answer: $\left\{\begin{bmatrix}1\\ 2\end{bmatrix}\right\}$.

4. (a) Possible answer: $\{(1,-1,-1,1)\}$.
   (b) Possible answer: $\{(1,0,0),(0,1,0),(0,0,1)\}$.
   (c) $\dim(\ker L) + \dim(\text{range } L) = 1 + 3 = 4 = \dim V$.

6. (a) No. (b) 2.

8. (a) $\ker L = \{0\}$; it has no basis.
   (b) Possible answer: $\{(1,1,0),(-1,2,0),(0,0,1)\}$.

10. (a) Possible answer: $\left\{\begin{bmatrix}1\\ -\frac{8}{3}\\ \frac{4}{3}\\ 1\end{bmatrix}\right\}$. (b) Possible answer: $\left\{\begin{bmatrix}1\\ 2\\ 1\\ 4\end{bmatrix}, \begin{bmatrix}2\\ 1\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}1\\ -1\\ 0\\ -1\end{bmatrix}\right\}$.

12. (a) No. (b) No. (c) Yes. (d) No. (e) Possible answer: $\{t+1, t^3+t^2\}$.
    (f) Possible answer: $\{t^3, t\}$.

14. (a) $\{1\}$. (b) $\left\{\begin{bmatrix}1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}\right\}$.

16. (a) $\ker L = \left\{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right\}$; it has no basis.

    (b) $\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$.

18. (a) Possible answer: $\left\{t^2 - \frac{1}{3}, t - \frac{1}{2}\right\}$. (b) Possible answer: $\{1\}$.

20. (a) 7. (b) 5.

T.1. Let $\mathbf{x}$ and $\mathbf{y}$ be solutions to $A\mathbf{x} = \mathbf{b}$, so $L(\mathbf{x}) = \mathbf{b}$ and $L(\mathbf{y}) = \mathbf{b}$. Then

$$L(\mathbf{x} - \mathbf{y}) = L(\mathbf{x}) - L(\mathbf{y}) = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence $\mathbf{x} - \mathbf{y} = \mathbf{z}$ is in $\ker L$.

T.2. By Theorem 10.7 and the assumption that $\dim V = \dim W$,

$$\dim(\ker L) + \dim(\text{range } L) = \dim W.$$

(a) If L is one-to-one, $\dim(\text{range } L) = \dim W$, so L is onto.
(b) If L is onto, $\dim(\ker L) = 0$, so L is one-to-one.

T.3. If $\mathbf{w}$ lies in the range of L, then $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v}$ in R^n, and $\mathbf{w}$ is a linear combination of columns of A. Thus $\mathbf{w}$ lies in the column space of A. Conversely, if $\mathbf{w}$ is in the column space of A, then $\mathbf{w}$ is a linear combination of the columns of A. Hence $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v}$ in R^n, which implies that $\mathbf{w}$ is in the range of L.

T.4. L is one-to-one if and only if $\ker L = \{\mathbf{0}\}$ if and only if $\dim(\text{range } L) = n$ if and only if $\dim(\text{column space of } A) = n$ if and only if $\text{rank } A = n$ if and only if $\det(A) \neq 0$.

T.5. Let $\mathbf{w}$ be any vector in range L. Then there exists a vector $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{w}$. Next, there exist scalars $c_1, \ldots, c_k$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. Thus

$$\mathbf{w} = L(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1L(\mathbf{v}_1) + \cdots + c_kL(\mathbf{v}_k).$$

Hence $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$ spans range L.

T.6. (a) $\dim(\ker L) + \dim(\text{range } L) = \dim V$, $\dim(\ker L) \geq 0$; thus $\dim(\text{range } L) \leq \dim V$.

(b) If L is onto, then range $L = W$ and the result follows from part (a).

T.7. Suppose that S is linearly dependent. Then there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Then

$$L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = L(\mathbf{0}) = \mathbf{0}$$

or

$$c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + \cdots + c_nL(\mathbf{v}_n) = \mathbf{0}$$

which implies that T is linearly dependent, a contradiction.

T.8. L is one-to-one if and only if $\dim(\ker L) = 0$ if and only if $\dim(\text{range } L) = \dim V$.

T.9. Let L be one-to-one and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in V. Suppose that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$ is linearly dependent. Then there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + \cdots + c_kL(\mathbf{v}_k) = \mathbf{0} \quad \text{or} \quad L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = \mathbf{0} = L(\mathbf{0}).$$

Since L is one-to-one, we have $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, which implies that S is linearly dependent, a contradiction.

T.10. The "only if" portion follows from Exercise T.9. If the image of a basis for V is a basis for W, then range L has dimension $= \dim W = \dim V$, and hence $\ker L$ has dimension 0, so L is one-to-one.

T.11. (a) Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in V. From Equations (2) and (3) in Section 6.7, we have

$$L(\mathbf{v} + \mathbf{w}) = \left[\mathbf{v} + \mathbf{w}\right]_S = \left[\mathbf{v}\right]_S + \left[\mathbf{w}\right]_S = L(\mathbf{v}) + L(\mathbf{w})$$

and

$$L(c\mathbf{v}) = \left[c\mathbf{v}\right]_S = c\left[\mathbf{v}\right]_S = cL(\mathbf{v}).$$

(b) Let

$$L(\mathbf{v}) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \left[\mathbf{v}\right]_S \quad \text{and} \quad L(\mathbf{w}) = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left[\mathbf{w}\right]_S$$

and assume that $L(\mathbf{v}) = L(\mathbf{w})$. Then $a_i = b_i$, $1 \leq i \leq n$. This implies that

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n = \mathbf{w}.$$

Hence, L is one-to-one.

(c) Let

$$\mathbf{w} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

be an arbitrary vector in R^n. If we let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

then $\mathbf{v}$ is in V and $L(\mathbf{v}) = \left[\mathbf{v}\right]_S = \mathbf{w}$. Hence, L is onto.

ML.1. **A = [1 2 5 5; − 2 − 3 − 8 − 7];**

rref(A)

```
ans =
     1   0   1  −1
     0   1   2   3
```

It follows that the general solution to $A\mathbf{x} = \mathbf{0}$ is obtained from

$$\begin{aligned} x_1 \quad + \ x_3 - \ x_4 &= 0 \\ x_2 + 2x_3 + 3x_4 &= 0. \end{aligned}$$

Let $x_3 = r$ and $x_4 = s$, then $x_2 = -2r - 3s$ and $x_1 = -r + s$. Thus

$$\mathbf{x} = \begin{bmatrix} -r+s \\ -2r-3s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\ker L$.

To find a basis for range L proceed as follows.

rref(A′)′

```
ans =
     1   0   0   0
     0   1   0   0
```

Then $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for range L.

ML.2. **A = [− 3 2 − 7; 2 − 1 4; 2 − 2 6];**

rref(A)

```
ans =
     1   0   1
     0   1  −2
     0   0   0
```

It follows that the general solution to $A\mathbf{x} = \mathbf{0}$ is obtained from

$$\begin{aligned} x_1 \quad + \quad x_3 &= 0 \\ x_2 - 2x_3 &= 0. \end{aligned}$$

Let $x_3 = r$, then $x_2 = 2r$ and $x_1 = -r$. Thus

$$\mathbf{x} = \begin{bmatrix} -r \\ 2r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

and $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for $\ker L$. To find a basis for range L proceed as follows.

rref(A′)′

```
ans =
        1    0   0
        0    1   0
       -2   -2   0
```

Then $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis for range L.

ML.3. **A = [3 3 − 3 1 11; − 4 − 4 7 − 2 − 19;2 2 − 3 1 9];**

rref(A)

```
ans =
       1  1  0  0   2
       0  0  1  0  -1
       0  0  0  1   2
```

It follows that the general solution to $A\mathbf{x} = \mathbf{0}$ is obtained from

$$\begin{aligned} x_1 + x_2 \qquad\qquad + 2x_5 &= 0 \\ x_3 \qquad - \quad x_5 &= 0 \\ x_4 + 2x_5 &= 0. \end{aligned}$$

Let $x_5 = r$ and $x_2 = s$, then $x_4 = -2r$ and $x_3 = r$, $x_1 = -s - 2r$. Thus

$$\mathbf{x} = \begin{bmatrix} -2r - s \\ s \\ r \\ -2r \\ r \end{bmatrix} = r \begin{bmatrix} -2 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $\ker L$. To find a basis for range L proceed as follows.

rref(A′)′

```
ans =
    1  0  0  0  0
    0  1  0  0  0
    0  0  1  0  0
```

Thus the columns of I_3 are a basis for range L.

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2. (a) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{3}{4} \end{bmatrix}$. (c) $\begin{bmatrix} 3 & 2 \\ -4 & 4 \end{bmatrix}$. (d) $\begin{bmatrix} -2 & 2 \\ \frac{1}{2} & 2 \end{bmatrix}$. (e) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$. (d) $\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$.
   (e) $(1, 1, 0)$.

6. (a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & \frac{2}{3} \\ -1 & \frac{4}{3} \end{bmatrix}$. (c) $\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$. (c) $-3t^2 + 3t + 3$.

10. (a) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 \end{bmatrix}$. (b) $\begin{bmatrix} 3 & -2 & 5 & -3 \\ -3 & 3 & -5 & 5 \\ 2 & 0 & 3 & 0 \\ -2 & 2 & -3 & 3 \end{bmatrix}$. (c) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & 3 & 2 \\ 2 & 2 & 3 & 3 \end{bmatrix}$.
    (d) $\begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 3 & 0 & 2 \\ 2 & 0 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 0 & 3 & -2 & 0 \\ 2 & 3 & 0 & -2 \\ -3 & 0 & -3 & 3 \\ 0 & -3 & 2 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 3 & 6 & -4 & 0 \\ 2 & -3 & 2 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} -3 & 6 & -7 & 3 \\ 5 & -3 & 7 & -5 \\ -3 & 0 & -3 & 3 \\ 3 & -3 & 5 & -3 \end{bmatrix}$.
    (d) $\begin{bmatrix} 3 & 3 & -2 & 0 \\ 5 & 3 & -2 & 0 \\ -3 & 0 & 0 & 0 \\ -3 & -3 & 2 & 0 \end{bmatrix}$.

14. (a) $[L(\mathbf{v}_1)]_T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $[L(\mathbf{v}_2]_T = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $[L(\mathbf{v}_3)]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
    (b) $L(\mathbf{v}_1) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $L(\mathbf{v}_2) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $L(\mathbf{v}_3) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
    (c) $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$ (d) $\begin{bmatrix} a + b + 2c \\ 2a + 5b - 2c \end{bmatrix}$.

16. (a) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 8 \\ 1 \\ 5 \end{bmatrix}$. (c) $\begin{bmatrix} a+2b+c \\ a \\ b+c \end{bmatrix}$.

18. $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$.

20. (a) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} 2 & -2 & 1 \\ 1 & 1 & 0 \\ -2 & 5 & -2 \end{bmatrix}$. (c) Same as (b).

22. (a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (c) $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. (d) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

24. Let $\mathbf{x} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_2 \\ -\frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_2 \end{bmatrix} \quad \text{and} \quad A\mathbf{y} = \begin{bmatrix} \frac{1}{\sqrt{2}}\,b_1 - \frac{1}{\sqrt{2}}\,b_2 \\ -\frac{1}{\sqrt{2}}\,b_1 - \frac{1}{\sqrt{2}}\,b_2 \end{bmatrix}.$$

We have

$$\begin{aligned}(A\mathbf{x})\cdot(A\mathbf{y}) &= \left(\frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_2\right)\left(\frac{1}{\sqrt{2}}\,b_1 - \frac{1}{\sqrt{2}}\,b_2\right) + \left(-\frac{1}{\sqrt{2}}\,a_1 - \frac{1}{\sqrt{2}}\,a_2\right)\left(-\frac{1}{\sqrt{2}}\,b_1 - \frac{1}{\sqrt{2}}\,b_2\right) \\ &= a_1b_1 + a_2b_2 = \mathbf{x}\cdot\mathbf{y}.\end{aligned}$$

25. It follows from Example 10 in Section 4.3 that the matrix of L with respect to the natural basis for R^2 is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

which is orthogonal.

26. Verify that $L(\mathbf{x})\cdot L(\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$.

T.1. If $\mathbf{x} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ is a vector in V, then

$$L(\mathbf{x}) = L\left(\sum_{j=1}^{n} a_j\mathbf{v}_j\right) = \sum_{j=1}^{n} a_j L(\mathbf{v}_j).$$

Then

$$\begin{aligned}\left[L(\mathbf{x})\right]_T &= \left[L\left(\sum_{j=1}^{n} a_j\mathbf{v}_j\right)\right]_T = \sum_{j=1}^{n} a_j\left[L(\mathbf{v}_j)\right]_T \\ &= \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} a_1 + \begin{bmatrix} c_{12} \\ c_{22} \\ \vdots \\ c_{m2} \end{bmatrix} a_2 + \cdots + \begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{mn} \end{bmatrix} a_n \\ &= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = A\left[\mathbf{v}\right]_S.\end{aligned}$$

We now show that A is unique. Assume that $A^* = \left[c^*_{ij}\right]$ is another matrix having the same properties as A does, with $A \neq A^*$. Since all the elements of A and A^* are not equal, say the kth columns of these matrices are unequal. Now $\left[\mathbf{v}_k\right]_S = \mathbf{e}_k$. Then

$$\left[L(\mathbf{v}_k)\right]_T = A\left[\mathbf{v}_k\right]_S = A\mathbf{e}_k = k\text{th column of } A,$$

and

$$\left[L(\mathbf{v}_k)\right]_T = A^*\left[\mathbf{v}_k\right]_S = A^*\mathbf{e}_k = k\text{th column of } A^*.$$

Thus, $L(\mathbf{v}_k)$ has two different coordinate vectors with respect to T, which is impossible. Hence A is unique.

T.2. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. For each j, $1 \leq j \leq n$,

$$I(\mathbf{v}_j) = \mathbf{v}_j = 0 \cdot \mathbf{v}_1 + \cdots + 1 \cdot \mathbf{v}_j + \cdots + 0 \cdot \mathbf{v}_n.$$

Thus the jth column of the matrix representing the identity transformation is $\mathbf{e}_j$. Hence the entire matrix is I_n.

T.3. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V, $T = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ a basis for W. Then $O(\mathbf{v}_j) = \mathbf{0}_W = 0 \cdot \mathbf{w}_1 + \cdots + 0 \cdot \mathbf{w}_m$. If A is the matrix of the zero transformation with respect to these bases, then the jth column of A is $\mathbf{0}$. Thus A is the $m \times n$ zero matrix.

T.4. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V. Then $L(\mathbf{v}_j) = c\mathbf{v}_j = 0 \cdot \mathbf{v}_1 + \cdots + c\mathbf{v}_j + \cdots + 0 \cdot \mathbf{v}_n$. If A is the matrix of L with respect to A, then the jth column of A is $c\mathbf{e}_j$, where $\mathbf{e}_j$ is the jth natural basis vector. Thus $A = cI_n$.

T.5. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$. The matrix of I with respect to S and T is the matrix whose jth column is $\left[I(\mathbf{v}_j)\right]_T = \left[\mathbf{v}_j\right]_T$. This is precisely the transition matrix $P_{T \leftarrow S}$ from the S-basis to the T-basis. (See Section 6.7).

T.6. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a basis for U and $T = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n\}$ a basis for V (Theorem 6.8). Now $L(\mathbf{v}_j)$ for $j = 1, 2, \dots, m$ is a vector in U, so $L(\mathbf{v}_j)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. Thus

$$L(\mathbf{v}_j) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + \cdots + 0\mathbf{v}_n.$$

Hence,

$$\left[L(\mathbf{v}_j)\right]_T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

T.7. Suppose that L is one-to-one and onto. Then $\dim(\ker L) = 0$. Since $\ker L$ is the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$, this homogeneous system has only the trivial solution. Theorem 1.13 implies that A is nonsingular. Conversely, if A is nonsingular, then Theorem 1.13 implies that the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial one.

T.8. Equation (10) shows that L preserves the dot product: $L(\mathbf{u}) \cdot L(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$. We now have

$$\|L(\mathbf{u})\| = \sqrt{L(\mathbf{u}) \cdot L(\mathbf{u})} = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \|\mathbf{u}\|.$$

For nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, the cosine of the angle between $L(\mathbf{u})$ and $L(\mathbf{v})$ is

$$\frac{L(\mathbf{u})\cdot L(\mathbf{v})}{\|L(\mathbf{u})\|\,\|L(\mathbf{v})\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \cos\theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

T.9. Suppose that L is an isometry. Then $L(\mathbf{v}_i)\cdot L(\mathbf{v}_j) = \mathbf{v}_i\cdot\mathbf{v}_j$ so $L(\mathbf{v}_i)\cdot L(\mathbf{v}_j) = 1$ if $i = j$ and 0 if $i \neq j$. Hence, $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is an orthonormal basis for R^n. Conversely, suppose that T is an orthonormal basis for R^n. Then $L(\mathbf{v}_i)\cdot L(\mathbf{v}_j) = 1$ if $i = j$ and 0 if $i \neq j$. Thus, $L(\mathbf{v}_i)\cdot L(\mathbf{v}_j) = \mathbf{v}_i\cdot\mathbf{v}_j$, so L is an isometry.

T.10. Since

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + 2(\mathbf{x}\cdot\mathbf{y}) + \mathbf{y}\cdot\mathbf{y} = \|\mathbf{x}\|^2 + 2(\mathbf{x}\cdot\mathbf{y}) + \|\mathbf{y}\|^2$$

it follows that

$$\mathbf{x}\cdot\mathbf{y} = \tfrac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right).$$

Then if L preserves length,

$$\begin{aligned} L(\mathbf{x})\cdot L(\mathbf{y}) &= \tfrac{1}{2}\left(\|L(\mathbf{x})+L(\mathbf{y})\|^2 - \|L(\mathbf{x})\|^2 - \|L(\mathbf{y})\|^2\right) \\ &= \tfrac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right) = \mathbf{x}\cdot\mathbf{y}. \end{aligned}$$

ML.1. From the definition of L, note that we can compute images under L using matrix multiplication: $L(\mathbf{v}) = C\mathbf{v}$, where

C = [2 − 1 0;1 1 − 3]

```
C =
     2   -1    0
     1    1   -3
```

This observation makes it easy to compute $L(\mathbf{v}_i)$ in MATLAB. Entering the vectors in set S and computing their images, we have

v1 = [1 1 1]′; v2 = [1 2 1]′; v3 = [0 1 − 1]′;

Denote the images as **Lvi**:

Lv1 = C ∗ v1

```
Lv1 =
         1
        -1
```

Lv2 = C ∗ v2

```
Lv2 =
        0
        0
```

Lv3 = C ∗ v3

```
Lv3 =
        -1
         4
```

To find the coordinates of **Lvi** with respect to the T basis we solve the three systems involved all at once using the **rref** command.

rref([[1 2;2 1] Lv1 Lv2 Lv3])

```
ans =
       1   0   −1   0    3
       0   1    1   0   −2
```

The last 3 columns give the matrix A representing L with respect to bases S and T.

A = ans(:,3:5)

```
A =
    −1   0    3
     1   0   −2
```

ML.2. Enter C and the vectors from the S and T bases into MATLAB. Then compute the images of $\mathbf{v}_i$ as $L(\mathbf{v}_i) = C * \mathbf{v}_i$.

C = [1 2 0;2 1 − 1;3 1 0; − 1 0 2]

```
C =
     1   2    0
     2   1   −1
     3   1    0
    −1   0    2
```

v1 = [1 0 1]′; v2 = [2 0 1]′; v3 = [0 1 2]′;

w1 = [1 1 1 2]′; w2 = [1 1 1 0]′; w3 = [0 1 1 − 1]′; w4 = [0 0 1 0]′;

Lv1 = C ∗ v1; Lv2 = C ∗ v2; Lv3 = C ∗ v3;

rref([w1 w2 w3 w4 Lv1 Lv2 Lv3])

```
ans =
       1.0000        0        0        0   0.5000   0.5000    0.5000
            0   1.0000        0        0   0.5000   1.5000    1.5000
            0        0   1.0000        0        0   1.0000   −3.0000
            0        0        0   1.0000   2.0000   3.0000    2.0000
```

It follows that A consists of the last 3 columns of ans.

A = ans(:,5:7)

```
A =
    0.5000   0.5000    0.5000
    0.5000   1.5000    1.5000
         0   1.0000   −3.0000
    2.0000   3.0000    2.0000
```

ML.3. Note that images under L can be computed by matrix multiplication using the matrix

C = [− 1 2;3 − 1]

```
C =
    −1    2
     3   −1
```

Enter each of the basis vectors into MATLAB.

v1 = [1 2′; v2 = [− 1 1]′;

w2 = [− 2 1]′; w2 = [1 1]′;

(a) Compute the images of $\mathbf{v}_i$ under L.

Lv1 = C * v1

```
Lv1 =
     3
     1
```

Lv2 = C * v2

```
Lv2 =
      3
     -4
```

To compute the matrix representing L with respect to S, compute

rref([v1 v2 Lv1 Lv2])

```
ans =
     1.0000         0    1.3333   -0.3333
          0    1.0000   -1.6667   -3.3333
```

A = ans(:,3:4)

```
A =
     1.3333   -0.3333
    -1.6667   -3.3333
```

(b) Compute the images of $\mathbf{w}_i$ under L.

Lw1 = C * w1

```
Lw1 =
      4
     -7
```

Lw2 = C * w2

```
Lw2 =
     1
     2
```

To compute the matrix representing L with respect to T, compute

rref([w1 w2 Lw1 Lw2])

```
ans =
     1.0000         0   -3.6667    0.3333
          0    1.0000   -3.3333    1.6667
```

B = ans(:,3:4)

```
B =
    -3.6667    0.3333
    -3.3333    1.6667
```

(c) To find the transition matrix from T to S we find the coordinates of the T-basis vectors in terms of S. This is done by solving two linear systems with coefficient matrix consisting of the S-basis vectors and right hand sides the T-basis vectors. We use

rref([v1 v2 w1 w2])

```
ans =
     1.0000         0   -0.3333    0.6667
          0    1.0000    1.6667   -0.3333
```

Then the transition matrix P is

P = ans(:,3:4)

```
P =
    -0.3333    0.6667
     1.6667   -0.3333
```

(d) We have

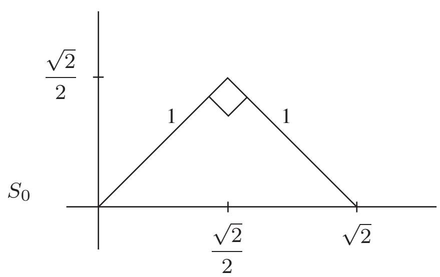
invert(P) ∗ A ∗ P

```
ans =
    -3.6667   0.3333
    -3.3333   1.6667
```

which is indeed B.

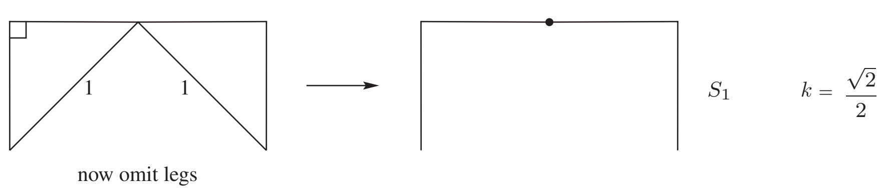
Section 10.4, p. 547

2. (a)

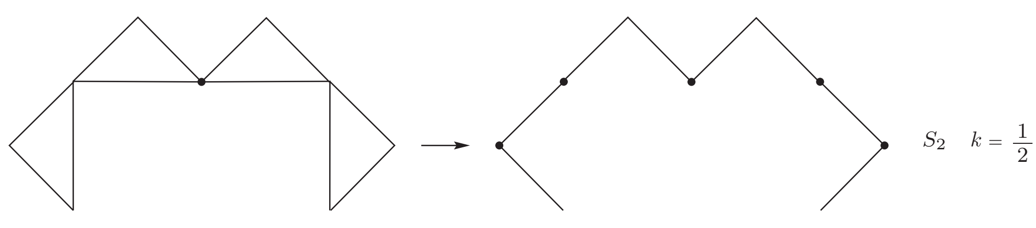


(b) It is half of the square with side = 1 that is obtained when the isosceles triangle is reflected about the x-axis and both the original triangle and its reflection are retained.

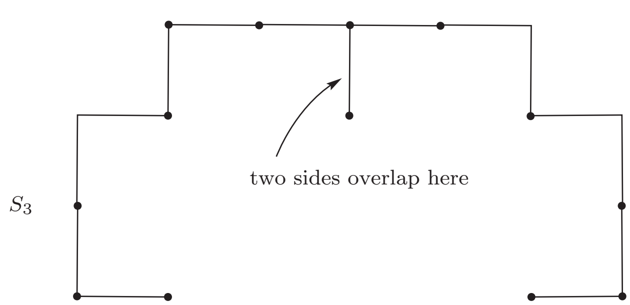
(c)



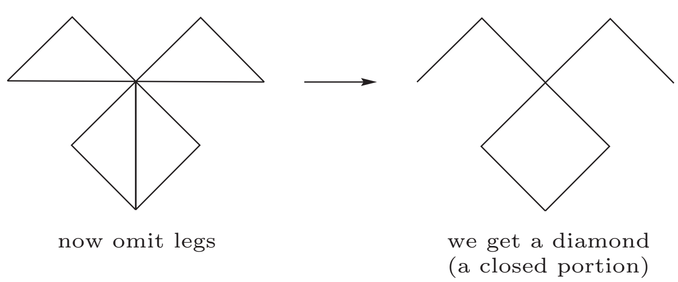
(d)



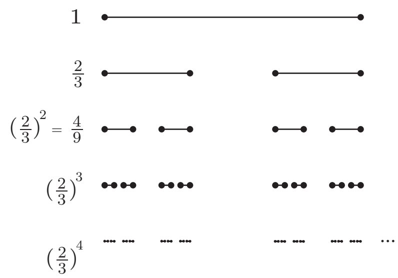
(e)



(f)



4. (a)



(b) $\left(\frac{2}{3}\right)^3$ and $\left(\frac{2}{3}\right)^4$ respectively.

(c) 16 and 32 respectively.

(d) 2^{10}.

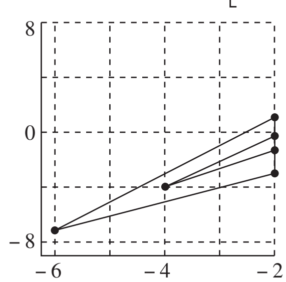
6. $T(T(\mathbf{v})) = T(A\mathbf{v}+\mathbf{b}) = A^2\mathbf{v} + A\mathbf{b} + \mathbf{b}$
$T(T(T(\mathbf{v}))) = T(A^2\mathbf{v} + A\mathbf{b} + \mathbf{b}) = A^3\mathbf{v} + A^2\mathbf{b} + A\mathbf{b} + \mathbf{b}$
$T^k(\mathbf{v}) = A^k\mathbf{v} + (A^{k-1} + A^{k-2} + \cdots + A)\mathbf{b} + \mathbf{b}$

8. $A = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

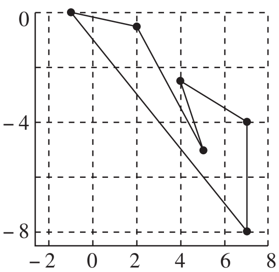
10. $A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$.

12. $S = \begin{bmatrix} 0 & 1.5 & 3 & 2.5 & 4 & 4 & 0 \\ 0 & 1.5 & 1 & 2.5 & 4 & 0 & 0 \end{bmatrix}$. Recall from Example 1 that to compute $T(S)$ we compute AS, then add the vector $\mathbf{b}$ to each column of AS.

(a) $$T(S) = AS + \mathbf{b} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1.5 & 3 & 2.5 & 4 & 4 & 0 \\ 0 & 1.5 & 1 & 2.5 & 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -2 & -2 & -4 & -2 & -2 & -6 & -2 \\ 1 & -\frac{1}{2} & -4 & -\frac{3}{2} & -3 & -7 & 1 \end{bmatrix}$$

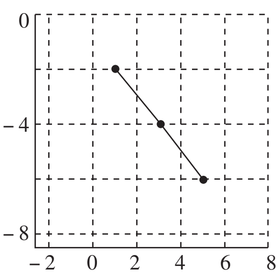


(b) $$T(S) = AS + \mathbf{b} = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1.5 & 3 & 2.5 & 4 & 4 & 0 \\ 0 & 1.5 & 1 & 2.5 & 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 & 5 & 4 & 7 & 7 & -1 \\ 0 & -\frac{3}{2} & -5 & -\frac{5}{2} & -4 & -8 & 0 \end{bmatrix}$$



(c) $T(S) = AS + \mathbf{b} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1.5 & 3 & 2.5 & 4 & 4 & 0 \\ 0 & 1.5 & 1 & 2.5 & 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -2 & -2 & -2 & -2 & -2 & -2 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 5 & 1 \\ -2 & -2 & -4 & -2 & -2 & -6 & -2 \end{bmatrix}$



14. $\mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

16. We have the following information:

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} p & r \\ s & t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} p + 2r + b_1 \\ s + 2t + b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} p & r \\ s & t \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} p + 4r + b_1 \\ s + 4t + b_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} p & r \\ s & t \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2p + 3r + b_1 \\ 2s + 3t + b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

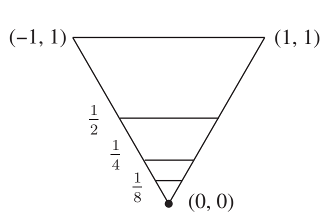
Equating corresponding entries in the relations above leads to two linear systems of equations with three equations in each system. We get the systems

$$\begin{aligned} p + 2r + b_1 &= 1 \\ p + 4r + b_1 &= -1 \\ 2p + 3r + b_1 &= 1 \end{aligned} \qquad \begin{aligned} p + 2r + b_1 &= 0 \\ p + 4r + b_1 &= 0 \\ 2p + 3r + b_1 &= 1. \end{aligned}$$

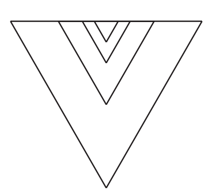
We form the corresponding augmented matrices and compute the RREF to solve the systems. (Note that the coefficient matrices are identical.) We obtain $p = 1$, $r = -1$, $b_1 = 2$ and $s = 1$, $t = 0$, $b_2 = -1$. Thus

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

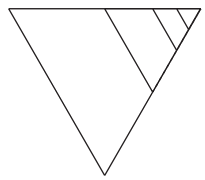
18. (a)



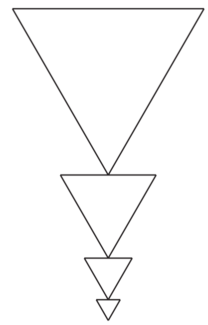
(b)



(c)



(d)



T.1. (a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

T.2. (a) $R_\phi = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$. (b) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

T.3. (a) $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. (b) $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{b}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b}_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_5 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

T.4. (a) We determine when the matrix $A - I_2$ is nonsingular because that will guarantee that the linear system $(A - I_2)\mathbf{v} = -\mathbf{b}$ has a unique solution. Computing the determinant of $A - I_2$ we get $(p-1)(t-1) - rs$. Hence when this quantity is not zero we have a unique fixed point.

(b) We compute the solution(s) to $(A - I_2)\mathbf{v} = -\mathbf{b}$ in each case.

$$\textbf{rref}\left(\left[\begin{array}{cc:c} -\frac{5}{6} & \frac{\sqrt{3}}{6} & -\frac{1}{2} \\ -\frac{\sqrt{3}}{6} & -\frac{5}{6} & -\frac{\sqrt{3}}{6} \end{array}\right]\right) = \left[\begin{array}{cc:c} 1 & 0 & 0.6429 \\ 0 & 1 & 0.1237 \end{array}\right]$$

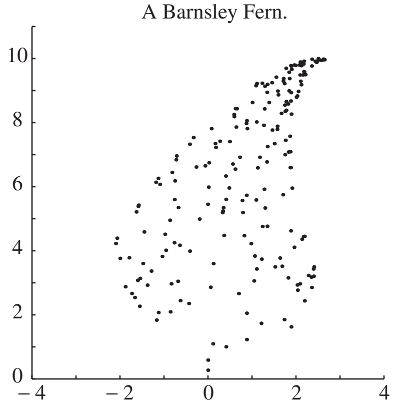
hence

$$\mathbf{v} = \begin{bmatrix} \frac{9}{14} \\ \frac{\sqrt{3}}{14} \end{bmatrix} \approx \begin{bmatrix} 0.6429 \\ 0.1237 \end{bmatrix}.$$

T.5. Here we consider solutions $\mathbf{v}$ of the matrix equation $A\mathbf{v} = \mathbf{v}$. The equation $A\mathbf{v} = \mathbf{v}$ implies that $\lambda = 1$ is an eigenvalue of A. Using this observation we can state the following. T has a fixed point when 1 is an eigenvalue of A. In this case there will be infinitely many fixed points since any nonzero multiple of an eigenvector is another eigenvector.

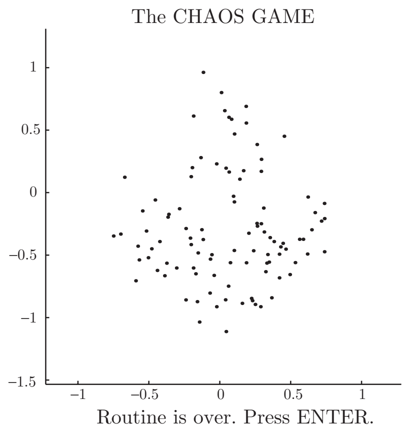
ML.1. Command **fernifs([0 0.2],30000)** produces the following figure.

A Barnsley Fern.

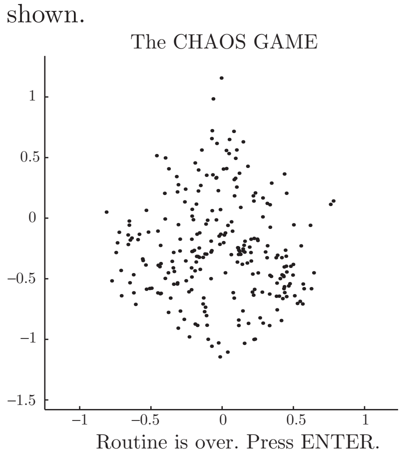


Routine is over. Press ENTER twice.

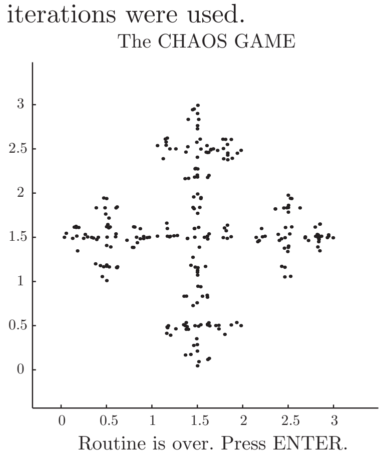
ML.2. (a) After 3000 iterations, the attractor looks like a leaf. In the simulation, the starting vector $[0.5, 0.5]$ was used.



More iterations give a more well-defined leaf. After 10,000 iterations, the resulting figure is shown.



(b) The five affine transformations from Exercise T.3 produced the following figure when 25,000 iterations were used.



Supplementary Exercises, p. 552

2. No.

4. (a) Possible answer: $\{2t^2 + t + 1\}$. (b) No.

6. 2.

8. (a) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$. (b) $4t^2 - 4t + 1$.

10. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

12. No. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A + B = I_2$. However, $L(A + B) = L(I_2) = 1$, but $L(A) + L(B) = 0$.

14. Yes. We have

$$L(B_1 + B_2) = A(B_1 + B_2) - (B_1 + B_2)A = (AB_1 - B_1A) + (AB_2 - B_2A) = L(B_1) + L(B_2)$$

and

$$L(cB) = A(cB) - (cB)A = c(AB - BA) = cL(B).$$

15. (a) We have

$$\begin{aligned} L(f + g) &= (f + g)(0) = f(0) + g(0) = L(f) + L(g) \\ L(cf) &= (cf)(0) = cf(0) = cL(f) \end{aligned}$$

(b) The kernel of L consists of any continuous function f such that $L(f) = f(0) = 0$. That is, f is in $\ker L$ provided the value of f at $x = 0$ is zero. The following functions are in $\ker L$:

$$x, \quad x^2, \quad x\cos x, \quad \sin x, \quad \frac{x}{x^2+1}, \quad xe^x.$$

(c) Yes. In this case

$$\begin{aligned} L(f + g) &= (f + g)\left(\tfrac{1}{2}\right) = f\left(\tfrac{1}{2}\right) + g\left(\tfrac{1}{2}\right) = L(f) + L(g) \\ L(cf) &= (cf)\left(\tfrac{1}{2}\right) = cf\left(\tfrac{1}{2}\right) = cL(f). \end{aligned}$$

16. (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. (b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

T.1. Since $\left[\mathbf{v}_j\right]_S = \mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ jth position , $\left\{\left[\mathbf{v}_1\right]_S, \ldots, \left[\mathbf{v}_n\right]_S\right\}$ is the standard basis for R^n.

T.2. Let

$$[\mathbf{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [\mathbf{w}]_S = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}.$$

Then

$$\begin{aligned}[\mathbf{v}+\mathbf{w}]_S = \left[\sum_{i=1}^{n} c_i\mathbf{v}_i + \sum_{i=1}^{n} d_i\mathbf{v}_i\right]_S &= \left[\sum_{i=1}^{n}(c_i+d_i)\mathbf{v}_i\right]_S \\ &= \begin{bmatrix} c_1+d_1 \\ c_2+d_2 \\ \vdots \\ c_n+d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{v}]_S + [\mathbf{w}]_S.\end{aligned}$$

and, for any scalar k,

$$[k\mathbf{v}]_S = \left[k\sum_{i=1}^{n} c_i\mathbf{v}_i\right]_S = \left[\sum_{i=1}^{n} kc_i\mathbf{v}_i\right]_S = \begin{bmatrix} kc_1 \\ kc_2 \\ \vdots \\ kc_n \end{bmatrix} = k\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = k\,[\mathbf{v}]_S.$$

T.3. (a) $(L_1 \boxplus L_2)(\mathbf{u}+\mathbf{v}) = L_1(\mathbf{u}+\mathbf{v}) + L_2(\mathbf{u}+\mathbf{v})$
$= L_1(\mathbf{u}) + L_1(\mathbf{v}) + L_2(\mathbf{u}) + L_2(\mathbf{v}) = (L_1 \boxplus L_2)(\mathbf{u}) + (L_1 \boxplus L_2)(\mathbf{v})$

$(L_1 \boxplus L_2)(k\mathbf{u}) = L_1(k\mathbf{u}) + L_2(k\mathbf{u})$
$= kL_1(\mathbf{u}) + kL_2(\mathbf{u}) = k(L_1(\mathbf{u}) + L_2(\mathbf{u})) = k(L_1 \boxplus L_2)(\mathbf{u})$

(b) $(c \boxdot L)(\mathbf{u}+\mathbf{v}) = cL(\mathbf{u}+\mathbf{v}) = cL(\mathbf{u}) + cL(\mathbf{v}) = (c \boxdot L)(\mathbf{u}) + (c \boxdot L)(\mathbf{v})$
$(c \boxdot L)(k\mathbf{u}) = cL(k\mathbf{u}) = ckL(\mathbf{u}) = kcL(\mathbf{u}) = k(c \boxdot L)(\mathbf{u})$

(c) $(L_1 \boxplus L_2)(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v}) = (v_1+v_2, v_2+v_3) + (v_1+v_3, v_2) = (2v_1+v_2+v_3, 2v_2+v_3)$.
$(-2 \boxdot L_1)(\mathbf{v}) = -2L(\mathbf{v}) = -2(v_1+v_2, v_2+v_3) = (-2v_1-2v_2, -2v_2-2v_3)$.

T.4. We verify Definition 1 in Section 6.1. Property (α) follows from Exercise T.3(a) and (β) follows from T.3(b).

(a) $(L_1 \boxplus L_2)(\mathbf{v}) = L_1(\mathbf{v}) + L_2(\mathbf{v})$
$= L_2(\mathbf{v}) + L_1(\mathbf{v})$ (since W is a vector space)
$= (L_2 \boxplus L_1)(\mathbf{v})$

(b) $(L_1 \boxplus (L_2 \boxplus L_3))(\mathbf{v}) = L_1(\mathbf{v}) + (L_2 \boxplus L_3)(\mathbf{v})$
$= L_1(\mathbf{v}) + (L_2(\mathbf{v}) + L_3(\mathbf{v}))$
$= (L_1(\mathbf{v}) + L_2(\mathbf{v})) + L_3(\mathbf{v})$ (since W is a vector space)
$= (L_1 \boxplus L_2)(\mathbf{v}) + L_3(\mathbf{v})$
$= ((L_1 \boxplus L_2) \boxplus L_3)(\mathbf{v})$

(c) Let $O\colon V \to W$ be the zero linear transformation defined by $O(\mathbf{v}) = \mathbf{0}_W$, for any $\mathbf{v}$ in V. (See Exercise T.6 in Section 10.1.) Then

$$(L_1 \boxplus O)(\mathbf{v}) = L_1(\mathbf{v}) + O(\mathbf{v}) = L_1(\mathbf{v}) + \mathbf{0}_W = L_1(\mathbf{v})$$

and

$$(O \boxplus L_1)(\mathbf{v}) = O(\mathbf{v}) + L_1(\mathbf{v}) = \mathbf{0}_W + L_1(\mathbf{v}) = L_1(\mathbf{v}).$$

Hence

$$L_1 \boxplus O = O \boxplus L_1 = L_1.$$

(d) Let $-L_1$ be the linear transformation from V to W defined by $(-L_1)(\mathbf{v}) = -L_1(\mathbf{v})$ for any $\mathbf{v}$ in V. Then

$$(L_1 \boxplus -L_1)(\mathbf{v}) = L_1(\mathbf{v}) + (-L_1(\mathbf{v})) = L_1(\mathbf{v}) - L_1(\mathbf{v}) = \mathbf{0}_W.$$

Thus for any $\mathbf{v}$ in V, $L \boxplus -L_1$ gives the zero element in W. From (c), $L_1 \boxplus -L_1 = O$.

(e) $$\begin{aligned}(c \boxdot (L_1 \boxplus L_2))(\mathbf{v}) &= c(L_1 \boxplus L_2(\mathbf{v})) \\ &= c(L_1(\mathbf{v}) + L_2(\mathbf{v})) \\ &= cL_1(\mathbf{v}) + cL_2(\mathbf{v}) \\ &= (c \boxdot L_1)(\mathbf{v}) + (c \boxdot L_2)(\mathbf{v}) \\ &= ((c \boxdot L_1) \boxplus (c \boxdot L_2))(\mathbf{v})\end{aligned}$$

(f) $$\begin{aligned}((c+d) \boxdot L_1)(\mathbf{v}) &= (c+d)L_1(\mathbf{v}) \\ &= cL_1(\mathbf{v}) + dL_1(\mathbf{v}) = (c \boxdot L_1)(\mathbf{v}) + (d \boxdot L_1)(\mathbf{v}) = ((c \boxdot L_1) \boxplus (d \boxdot L_1))(\mathbf{v})\end{aligned}$$

(g) $(c \boxdot (d \boxdot L_1))(\mathbf{v}) = c(d \boxdot L_1)(\mathbf{v}) = c(dL_1(\mathbf{v})) = cdL_1(\mathbf{v}) = (cd \boxdot L_1)(\mathbf{v})$

(h) $(1 \boxdot L_1)(\mathbf{v}) = 1L_1(\mathbf{v}) = L_1(\mathbf{v})$ since W is a vector space.

T.5. (a) By Theorem 10.5, $L(\mathbf{x}) = A\mathbf{x}$ is one-to-one if and only if $\ker(L)$, which is the solution space of $A\mathbf{x} = \mathbf{0}$, is equal to $\{\mathbf{0}_V\}$. The solution space of $A\mathbf{x} = \mathbf{0}$ is $\{\mathbf{0}_V\}$ if and only if the only linear combination of the columns of A that gives the zero vector is the one in which all the coefficients are zero. This is the case if and only if the columns of A are linearly independent, which is equivalent to $\operatorname{rank} A = n$.

(b) $L(A) = A\mathbf{x}$ is onto if and only if $\operatorname{range} L$ is R^m. But $\operatorname{range} L$ is equal to the column space of A and

$$m = \dim(\operatorname{range} L) = \dim(\text{column space of } A) = \text{column rank of } A = \operatorname{rank} A.$$

T.6. For $\mathbf{v} = (a_1, a_2, \ldots, a_n)$, $L(\mathbf{v}) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$.

(a) Let $\mathbf{w} = (b_1, b_2, \ldots, b_n)$. Then

$$\begin{aligned}L(\mathbf{v}+\mathbf{w}) &= (a_1+b_1)\mathbf{v}_1 + (a_2+b_2)\mathbf{v}_2 + \cdots + (a_n+b_n)\mathbf{v}_n \\ &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) + (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n) \\ &= L(\mathbf{v}) + L(\mathbf{w})\end{aligned}$$

and, for any scalar k,

$$\begin{aligned}L(k\mathbf{v}) &= (ka_1)\mathbf{v}_1 + (ka_2)\mathbf{v}_2 + \cdots + (ka_n)\mathbf{v}_n \\ &= k(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= kL(\mathbf{v}).\end{aligned}$$

Thus L is a linear transformation.

(b) From Theorem 10.5, L is one-to-one provided $\ker L = \{\mathbf{0}\}$. Assume that $L(\mathbf{v}) = \mathbf{0}$. Then we have

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}.$$

Since S is a basis, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are linearly independent so $a_1 = a_2 = \cdots = a_n = 0$ and it follows that $\mathbf{v} = \mathbf{0}$. Hence $\ker L = \{\mathbf{0}\}$ and L is one-to-one.

(c) That L is onto follows from part (b) and Corollary 10.2(a).

Chapter 11

Linear Programming (Optional)

Section 11.1, p. 572

2. Maximize $z = 0.08x + 0.10y$

   subject to

$$\begin{aligned} x + y &\leq 6000 \\ x &\geq 1500 \\ y &\leq 4000 \\ x \geq 0, y &\geq 0 \end{aligned}$$

4. Maximize $z = 30x + 60y$

   subject to

$$\begin{aligned} 6x + 12y &\leq 18{,}000 \\ 3x + y &\geq 1800 \\ x \geq 0, y &\geq 0 \end{aligned}$$

6. Maximize $z = 60x + 50y$

   subject to

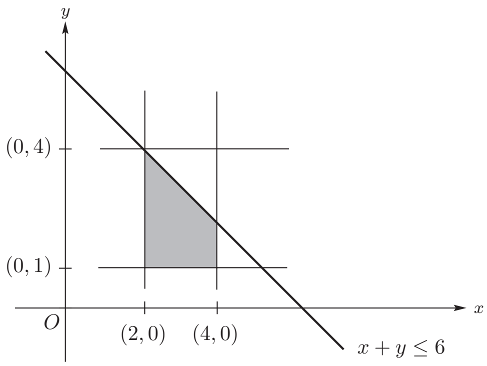
$$\begin{aligned} 3x + 5y &\leq 15 \\ 4x + 4y &\geq 16 \\ x \geq 0, y &\geq 0 \end{aligned}$$

8. Maximize $z = 10{,}000{,}000x + 15{,}000{,}000y$

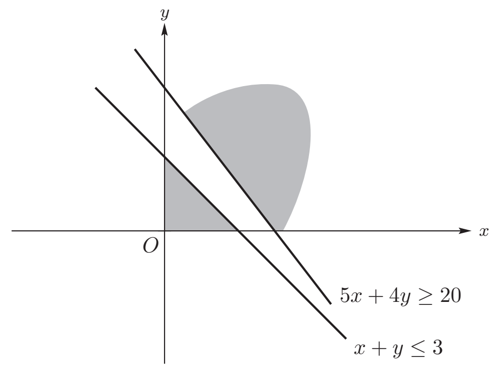
   subject to

$$\begin{aligned} 40x + 60y &\geq 300 \\ 2x + 3y &\leq 12 \\ x \geq 0, y &\geq 0 \end{aligned}$$

10.



12.



14. $x = \frac{18}{5}$, $y = \frac{2}{5}$, optimal value of z is $\frac{58}{5}$.

16. $\frac{3}{2}$ tons of regular steel and $\frac{5}{2}$ tons of special steel; maximum profit is \$430.

18. Invest \$4000 in bond A and \$2000 in bond B; maximum return is \$520.

20. Use 2 minutes of advertising and 28 minutes of programming; maximum number of viewer-minutes is 1,340,000.

22. Use 4 units of A and 3 units of B; maximum amount of protein is 34 units.

24. (b).

26. Maximize $z = 2x_1 - 3x_2 - 2x_3$

subject to

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\le 12 \\ x_1 + x_2 - 3x_3 &\le 8 \\ x_1 \ge 0, x_2 \ge 0, x_3 &\ge 0 \end{aligned}$$

28. Maximize $z = 2x + 8y$

subject to

$$\begin{aligned} 2x + 3y + u \qquad &= 18 \\ 3x - 2y \qquad + v &= 6 \\ x \ge 0, y \ge 0, u \ge 0, v &\ge 0 \end{aligned}$$

Section 11.2, p. 589

2.

	x_1	x_2	x_3	x_4	x_5	z	
x_4	3	−2	1	1	0	0	4
x_5	2	4	5	0	1	0	6
	−2	−3	4	0	0	1	0

4.

	x_1	x_2	x_3	x_4	x_5	x_6	z	
x_4	1	−2	4	1	0	0	0	5
x_5	2	2	4	0	1	0	0	5
x_6	3	1	−1	0	0	1	0	7
	−2	3	−1	0	0	0	1	0

6. $x = 0$, $y = \frac{6}{7}$, optimal z value is $\frac{30}{7}$.

8. No finite optimal solution.

10. $x_1 = 0$, $x_2 = \frac{5}{2}$, $x_3 = 0$, optimal z value is $z = 10$.

12. $\frac{3}{2}$ tons of regular steel and $\frac{5}{2}$ tons of special steel; maximum profit is \$430.

14. 4 units of A and 3 units of B; maximum amount of protein is 34 units.

T.1. We must show that if $\mathbf{x}$ and $\mathbf{y}$ are any two feasible solutions, then for any $0 \leq r \leq 1$, the vector $r\mathbf{x} + (1 - r)\mathbf{y}$ is also a feasible solution. First, since $r \geq 0$ and $(1 - r) \geq 0$, and $A\mathbf{x} \leq \mathbf{b}$, $A\mathbf{y} \leq \mathbf{b}$,

$$A[r\mathbf{x} + (1 - r)\mathbf{y})] = rA\mathbf{x} + (1 - r)A\mathbf{y} \leq r\mathbf{b} + (1 - r)\mathbf{b} = \mathbf{b}.$$

Also, since $\mathbf{x} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$,

$$r\mathbf{x} + (1 - r)\mathbf{y} \geq r \cdot \mathbf{0} + (1 - r) \cdot \mathbf{0} = \mathbf{0}.$$

Thus $r\mathbf{x} + (1 - r)\mathbf{y}$ is a feasible solution.

T.2. Suppose that in a certain step of the simplex method, the minimum positive θ-ratio is not chosen. After a reindexing of the variables, if necessary, we may assume that all the nonbasic variables occur first followed by all the basic variables. That is, we may assume that we start with the situation given by Tableau 2 which precedes Equation (16). Let us further assume that x_1 is the entering variable and x_{n+1} the departing variable and that the θ-ratio b_2/a_{21} associated with the second row is positive and smaller than b_1/a_{11} associated with the first row — the row of the incorrectly chosen departing variable x_{n+1}. Thus

$$0 < \frac{b_2}{a_{21}} < \frac{b_1}{a_{11}}$$
$$a_{11}b_2 < a_{21}b_1,$$
$$a_{11}b_2 - a_{21}b_1 < 0.$$

The new set of nonbasic variables is $\{x_2, \ldots, x_n, x_{n+1}\}$. Set these nonbasic variables equal to zero and solve the first equation for x_1:

$$a_{11}x_1 + a_{12} \cdot 0 + \cdots + a_{1n} \cdot 0 + 0 = b_1$$
$$x_1 = \frac{b_1}{a_{11}}.$$

Next substitute this value for x_1 into the second equation and solve for the basic variable x_{n+2}:

$$a_{21}\left(\frac{b_1}{a_{11}}\right) + a_{22} \cdot 0 + \cdots + a_{2n} \cdot 0 + x_{n+2} = b_2,$$
$$x_{n+2} = b_2 - \frac{a_{21}b_1}{a_{11}} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}} = \frac{\text{neg}}{\text{pos}} = \text{neg},$$

a contradiction to the fact that the coordinate x_{n+2} of any feasible solution must be ≥ 0.

ML.5. As a check, the solution is $x_4 = \frac{2}{3}$, $x_3 = 0$, $x_2 = \frac{1}{3}$, $x_1 = 1$, all other variables zero and the optimal value of z is 11. The final tableau is as follows:

0	0	1.6667	1	0.0000	−0.1667	0	0.6667
0	1	3.3333	0	1.0000	−0.3333	0	0.3333
1	0	−3.0000	0	−1.0000	0.5000	0	1.0000
0	0	1.0000	0	1.0000	1.0000	1	11.0000

ML.6. As a check, the solution is $x_3 = \frac{4}{3}$, $x_2 = 4$, $x_7 = \frac{22}{3}$, all other variables zero, and the optimal value of z is $\frac{28}{3}$. The final tableau is as follows:

0.4444	0	1	−0.1111	0.3333	−0.1111	0	0	1.3333
0.6667	1	0	1.3333	0	0.3333	0	0	4.0000
0.7778	0	0	−2.4444	−0.6667	−0.4444	1	0	7.3333
0.7778	0	0	1.5556	0.3333	0.5556	0	1	9.3333

Section 11.3, p. 598

2. Maximize $z' = 5w_1 + 6w_2$
   subject to

$$\begin{aligned} w_1 + 2w_2 &\geq 10 \\ 3w_1 - 4w_2 &\geq 12 \\ 4w_1 - 5w_2 &\geq 15 \\ w_1 \geq 0, w_2 &\geq 0 \end{aligned}$$

4. Maximize $z' = 9w_1 + 12w_2$
   subject to

$$\begin{aligned} 3w_1 + 5w_2 &\leq 14 \\ 5w_1 + 2w_2 &\leq 12 \\ -4w_1 + 7w_2 &\leq 18 \\ w_1 \geq 0, w_2 &\geq 0 \end{aligned}$$

6. $w_1 = \frac{2}{3}$, $w_2 = 0$, optimal value is 4.
8. $w_1 = \frac{1}{10}$, $w_2 = \frac{7}{10}$, optimal value is $\frac{69}{10}$.
10. Use 6 oz. of dates and no nuts or raisins. Total cost is 90 cents.

Section 11.4, p. 612

2.

$$\begin{array}{ccc} & & \text{C} \\ & & \begin{array}{ccc} \text{stone} & \text{scissors} & \text{paper} \end{array} \\ \text{R} & \begin{array}{c} \text{stone} \\ \text{scissors} \\ \text{paper} \end{array} & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

4.

		C's guess	
		Nickel	Dime
R's Choice	Nickel	-5	5
	Dime	10	10

6. (a) $\mathbf{p} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = 3$. (b) $\mathbf{p} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v = -1$.

(c) $\mathbf{p} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v = -2$ or $\mathbf{p} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v = -2$.

8. (a) $\frac{5}{3}$. (b) $\frac{1}{12}$.

10. $p_1 = \frac{2}{3}$, $p_2 = \frac{1}{3}$, $q_1 = \frac{5}{6}$, $q_2 = \frac{1}{6}$, $v = \frac{14}{3}$. 12. $p_1 = 0$, $p_2 = \frac{3}{4}$, $p_3 = \frac{1}{4}$, $q_1 = \frac{3}{4}$, $q_2 = \frac{1}{4}$, $v = \frac{29}{4}$.

14. $\mathbf{p} = \begin{bmatrix} \frac{3}{7} & 0 & \frac{4}{7} \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} \frac{2}{7} \\ \frac{5}{7} \\ 0 \end{bmatrix}$, $v = -\frac{1}{7}$. 16. $\mathbf{p} = \begin{bmatrix} \frac{11}{20} & \frac{9}{20} \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} \frac{11}{20} \\ \frac{9}{20} \end{bmatrix}$, $v = \frac{1}{20}$.

18. $\mathbf{p} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v = 50$.

20. For labor: $p_1 = 0$, $p_2 = \frac{3}{4}$, $p_3 = \frac{1}{4}$; for management: $q_1 = \frac{3}{4}$, $q_2 = \frac{1}{4}$, $v = \frac{11}{4}$.

T.1. The expected payoff to R is the sum of terms of the form (Probability that R plays row i and C plays column j) $\times$ (Payoff to R when R plays i and C plays j) $= p_i q_j a_{ij}$. Summing over all $1 \leq i \leq m$ and $1 \leq j \leq n$, we get

$$\text{Expected payoff to } R = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = \mathbf{p} A \mathbf{q}.$$

T.2. Let $\mathbf{p}_0$ be an optimal strategy for R for the original game with payoff matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$. Then for any strategy $\mathbf{q}$ for C and any strategy $\mathbf{p}$ for R,

$$E(\mathbf{p}_0, \mathbf{q}) = \mathbf{p}_0 A \mathbf{q} \geq \mathbf{p} A \mathbf{q} = E(\mathbf{p}, \mathbf{q})$$

or

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_i^{(0)} a_{ij} a_j \geq \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j,$$

where $\mathbf{p}_0 = \begin{bmatrix} p_i^{(0)} \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} p_i \end{bmatrix}$, and $\mathbf{q} = \begin{bmatrix} q_j \end{bmatrix}$.

Let $A' = \begin{bmatrix} a_{ij} + r \end{bmatrix}$ be the payoff matrix for the new game in which each payoff to R has been increased by the constant r. Let E' be the expected payoff to R for the new game. Then

$$\begin{aligned} E'(\mathbf{p},\mathbf{q}) &= \sum_{i=1}^{m}\sum_{j=1}^{n} p_i(a_{ij}+r)q_j \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} p_i a_{ij} q_j + \sum_{i=1}^{m}\sum_{j=1}^{n} p_i r q_j \\ &= E(\mathbf{p},\mathbf{q}) + r\left(\sum_{i=1}^{m} p_i\right)\left(\sum_{j=1}^{n} q_j\right) \\ &= E(\mathbf{p},\mathbf{q}) + r. \end{aligned}$$

Likewise, $E'(\mathbf{p}_0,\mathbf{q}) = E(\mathbf{p}_0,\mathbf{q}) + r$. Then

$$E'(\mathbf{p}_0,\mathbf{q}) = E(\mathbf{p}_0,\mathbf{q}) + r \geq E(\mathbf{p},\mathbf{q}) + r = E'(\mathbf{p},\mathbf{q}),$$

so $\mathbf{p}_0$ is also an optimal strategy for R for the new game.

Similarly, the optimal strategy $\mathbf{q}_0$ for C is the same for both the original game and the new game. Finally, the value v' of the new game is

$$E'(\mathbf{p}_0,\mathbf{q}_0) = E(\mathbf{p}_0,\mathbf{q}_0) + r = v + r,$$

where v is the value of the original game.

Supplementary Exercises, p. 614

2. Manufacture 100 units of model A and 500 units of model B daily. The maximum profit is \$58,000.

4. Maximize $z' = 6y_1 + 10y_2$

   subject to

$$\begin{aligned} 2y_1 + 5y_2 &\leq 6 \\ 3y_1 + 2y_2 &\leq 5 \\ y_1 \geq 0, y_2 &\geq 0 \end{aligned}$$

6. $\mathbf{p} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$, $v = \frac{8}{3}$.

Appendix A

Complex Numbers

Appendix A.1, p. A7

2. (a) $-\frac{1}{5} + \frac{2}{5}i$. (b) $\frac{9}{10} - \frac{7}{10}i$. (c) $4 - 3i$. (d) $\frac{1}{26} - \frac{5}{26}i$.

4. (a) $\sqrt{20}$. (b) $\sqrt{10}$. (c) $\sqrt{13}$. (d) $\sqrt{17}$.

6. (a) $\begin{bmatrix} 2+4i & 5i \\ -2 & 4-2i \end{bmatrix}$. (b) $\begin{bmatrix} 4-3i \\ -2-i \end{bmatrix}$. (c) $\begin{bmatrix} -4+4i & -2+16i \\ -4i & -8i \end{bmatrix}$. $\begin{bmatrix} 3i \\ -1-3i \end{bmatrix}$.

   (e) $\begin{bmatrix} 2i & -1+3i \\ -2 & -1-i \end{bmatrix}$. (f) $\begin{bmatrix} -2i & 1-2i \\ 0 & 3+i \end{bmatrix}$. (g) $\begin{bmatrix} 3+i \\ -3+3i \end{bmatrix}$. (h) $\begin{bmatrix} 3-6i \\ -2-6i \end{bmatrix}$.

8. (a) Hermitian, normal. (b) None. (c) Unitary, normal. (d) Normal.

   (e) Hermitian, normal. (f) None. (g) Normal. (h) Unitary, normal.

10. (a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. (b) $\begin{bmatrix} 4 & 18 \\ 0 & 4 \end{bmatrix}$. (c) $\begin{bmatrix} -5 & 5i \\ 5i & -5 \end{bmatrix}$. (d) $\begin{bmatrix} 4 & 7i \\ 0 & -3 \end{bmatrix}$.

12. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

T.1. (a) $\text{Re}\,(c_1 + c_2) = \text{Re}\,((a_1 + a_2) + (b_1 + b_2)i) = a_1 + a_2 = \text{Re}\,(c_1) + \text{Re}\,(c_2)$.
$\text{Im}\,(c_1 + c_2) = \text{Im}\,((a_1 + a_2) + (b_1 + b_2)i) = b_1 + b_2 = \text{Im}\,(c_1 + \text{Im}\,(c_2)$.

(b) $\text{Re}\,(kc) = \text{Re}\,(ka + kbi) = ka = \text{Re}\,(c)$.
$\text{Im}\,(kc) = \text{Im}\,(ka + kbi) = kb = k\,\text{Im}\,(c)$.

(c) No.

(d) $\text{Re}\,(c_1c_2) = \text{Re}\,((a_1 + b_1i)(a_2 + b_2i)) = \text{Re}\,((a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i) = a_1a_2 - b_1b_2 \neq \text{Re}\,(c_1)\,\text{Re}\,(c_2)$.

T.2. (a) $\left(\overline{A+B}\right)_{ij} = \overline{a_{ij} + b_{ij}} = \overline{a_{ij}} + \overline{b_{ij}} = \left(\overline{A}\right)_{ij} + \left(\overline{B}\right)_{ij}$

(b) $\left(\overline{kA}\right)_{ij} = \overline{ka_{ij}} = k\,\overline{a_{ij}} = k\left(\overline{A}\right)_{ij}$.

(c) $\overline{CC^{-1}} = \overline{C^{-1}C} = \overline{I_n} = I_n$; thus $(\overline{C})^{-1} = \overline{C^{-1}}$.

T.3. (a) $\overline{a_{ii}} = a_{ii}$, hence a_{ii} is real. (See Property 4 in Section B1.)

(b) First, $\overline{A^T} = A$ implies that $A^T = \overline{A}$. Let $B = \dfrac{A + \overline{A}}{2}$. Then

$$\overline{B} = \overline{\left(\frac{A+\overline{A}}{2}\right)} = \frac{\overline{A} + \overline{\overline{A}}}{2} = \frac{\overline{A} + A}{2} = \frac{A + \overline{A}}{2} = B,$$

so B is a real matrix. Also,

$$B^T = \left(\frac{A+\overline{A}}{2}\right)^T = \frac{A^T + \overline{A}^T}{2} = \frac{A^T + \overline{A^T}}{2} = \frac{\overline{A} + A}{2} = \frac{A+\overline{A}}{2} = B$$

so B is symmetric.

Next, let $C = \dfrac{A-\overline{A}}{2i}$. Then

$$\overline{C} = \overline{\left(\frac{A-\overline{A}}{2i}\right)} = \frac{\overline{A} - \overline{\overline{A}}}{-2i} = \frac{A-\overline{A}}{2i} = C$$

so C is a real matrix. Also,

$$C^T = \left(\frac{A-\overline{A}}{2i}\right)^T = \frac{A^T - \overline{A}^T}{2i} = \frac{A^T - \overline{A^T}}{2i} = \frac{\overline{A} - A}{2i} = -\frac{A-\overline{A}}{2i} = -C$$

so C is also skew symmetric. Moreover, $A = B + iC$.

(c) If $A = A^T$ and $A = \overline{A}$, then $\overline{A^T} = \overline{A} = A$. Hence, A is Hermitian.

T.4. (a) If A is real and orthogonal, then $A^{-1} = A^T$ or $AA^T = I_n$. Hence A is unitary.

(b) $\overline{(A^T)^T} A^T = \overline{(A^T)}^T A^T = (A\overline{A^T})^T = I_n^T = I_n$. Note: $\overline{(A^T)^T} = (\overline{A^T})^T$.
Similarly, $A^T\overline{(A^T)^T} = I_n$.

(c) $\overline{(A^{-1})^T} A^{-1} = \overline{(A^T)^{-1}} A^{-1} = (\overline{A^T})^{-1} A^{-1} = (A\overline{A^T})^{-1} = I_n^{-1} = I_n$.
Note: $\overline{(A^{-1})^T} = \overline{(A^T)^{-1}}$ and $\overline{(A^T)^{-1}} = (\overline{A^T})^{-1}$. Similarly, $A^{-1}\overline{(A^{-1})^T} = I_n$.

T.5. (a) Let

$$B = \frac{A + \overline{A^T}}{2} \quad \text{and} \quad C = \frac{A - \overline{A^T}}{2i}.$$

Then

$$\overline{B^T} = \overline{\left(\frac{A+\overline{A^T}}{2}\right)^T} = \frac{\overline{A^T} + \overline{(\overline{A^T})^T}}{2} = \frac{\overline{A^T} + A}{2} = \frac{A + \overline{A^T}}{2} = B$$

so B is Hermitian. Also,

$$\overline{C^T} = \overline{\left(\frac{A-\overline{A^T}}{2i}\right)^T} = \frac{\overline{A^T} - \overline{(\overline{A^T})^T}}{-2i} = \frac{A - \overline{A^T}}{2i} = C$$

so C is Hermitian. Moreover, $A = B + iC$.

(b) We have

$$\begin{aligned}
\overline{A^T}A = \overline{(B^T + iC^T)}(B+iC) &= (\overline{B^T} + \overline{iC^T})(B+iC) \\
&= (B - iC)(B+iC) \\
&= B^2 - iCB + iBC - i^2C^2 \\
&= (B^2 + C^2) + i(BC - CB).
\end{aligned}$$

Similarly,

$$\begin{aligned}
A\overline{A^T} = (B+iC)\overline{(B^T + iC)^T} &= (B+iC)(\overline{B^T} + \overline{iC^T}) \\
&= (B+iC)(B-iC) \\
&= B^2 - iBC + iCB - i^2C^2 \\
&= (B^2 + C^2) + i(CB - BC).
\end{aligned}$$

Since $\overline{A^T}A = A\overline{A^T}$, we equate imaginary parts obtaining $BC - CB = CB - BC$, which implies that $BC = CB$. The steps are reversible, establishing the converse.

T.6. (a) If $\overline{A^T} = A$, then $\overline{A^T}A = A^2 = A\overline{A^T}$, so A is normal.

(b) If $\overline{A^T} = A^{-1}$, then $\overline{A^T}A = A^{-1}A = AA^{-1} = A\overline{A^T}$, so A is normal.

(c) One example is $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$. Note that this matrix is not symmetric since it is not a real matrix.

T.7. Let $A = B + iC$ be skew Hermitian. Then $\overline{A^T} = -A$ so $B^T - iC^T = -B - iC$. Then $B^T = -B$ and $C^T = C$. Thus, B is skew symmetric and C is symmetric. Conversely, if B is skew symmetric and C is symmetric, then $B^T = -B$ and $C^T = C$ so $B^T - iC^T = -B - iC$ or $\overline{A^T} = -A$. Hence, A is skew Hermitian.

Appendix A.2, p. A17

2. (a) $x = -\frac{7}{30} - \frac{4}{30}i$, $y = -\frac{11}{15}(1 + 2i)$, $z = \frac{3}{5} - \frac{4}{5}i$.
   (b) $x = -1 + 4i$, $y = \frac{1}{2} + \frac{3}{2}i$, $z = 2 - i$.

4. (a) $4i$. (b) 0. (c) $-9 - 8i$. (d) -10.

6. (a) Yes. (b) No. (c) Yes.

8. (a) No. (b) No.

10. (a) $-7 - 6i$. (b) $10 + 19i$.

12. (a) 4. (b) 4. (c) $\sqrt{3}$. (d) $\sqrt{19}$.

14. (a) $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. (b) $P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.

(c) $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ i & 0 & -i \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ i & -i & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix}$.

T.1. (a) Let A and B be Hermitian and let k be a complex scalar. Then

$$(\overline{A + B})^T = (\overline{A} + \overline{B})^T = \overline{A}^T + \overline{B}^T = A + B,$$

so the sum of Hermitian matrices is again Hermitian. Next,

$$(\overline{kA})^T = \overline{k}\,\overline{A}^T = \overline{k}A \neq kA,$$

so the set of Hermitian matrices is not closed under scalar multiplication and hence is not a complex subspace of C_{nn}.

(b) From (a), we have closure of addition and since the scalars are real here, $\overline{k} = k$, hence $(\overline{kA})^T = kA$. Thus, W is a real subspace of the real vector space of $n \times n$ complex matrices.

T.2. The zero vector $\mathbf{0}_n$ is not unitary, so W cannot be a subspace.

T.3. (a) Let A be Hermitian and suppose that $A\mathbf{x} = \lambda\mathbf{x}$, $\lambda \neq 0$. We show that $\lambda = \overline{\lambda}$. We have

$$(\overline{A\mathbf{x}})^T = \left(\overline{A}\,\overline{\mathbf{x}}\right)^T = \overline{\mathbf{x}}^T\,\overline{A} = \overline{\mathbf{x}}^T A.$$

Also, $(\overline{\lambda}\,\overline{\mathbf{x}})^T = \overline{\lambda}\,\overline{\mathbf{x}}^T$, so $\overline{\mathbf{x}}^T A = \overline{\lambda}\,\overline{\mathbf{x}}^T$. Multiplying both sides by $\mathbf{x}$ on the right, we obtain $\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x}$. However, $\overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T\lambda\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}$. Thus, $\overline{\lambda}\overline{\mathbf{x}}^T\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x}$. Then $(\lambda - \overline{\lambda})\overline{\mathbf{x}}^T\mathbf{x} = 0$ and since $\overline{\mathbf{x}^T}\mathbf{x} > 0$, we have $\lambda = \overline{\lambda}$.

(b) $A^T = \begin{bmatrix} 2 & 0 & \overline{0} \\ 0 & 2 & \overline{-i} \\ 0 & \overline{i} & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{bmatrix} = A.$

(c) No, see 11(b). An eigenvector $\mathbf{x}$ associated with a real eigenvalue λ of a complex matrix A is in general complex, because $A\mathbf{x}$ is in general complex. Thus $\lambda\mathbf{x}$ must also be complex.

T.4. If A is unitary, then $\overline{A}^T = A^{-1}$. Let $A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$. Since

$$I_n = A\overline{A^T} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \overline{\mathbf{u}_1^T} \\ \overline{\mathbf{u}_2^T} \\ \vdots \\ \overline{\mathbf{u}_n^T} \end{bmatrix},$$

then

$$\mathbf{u}_k\overline{\mathbf{u}_j^T} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

It follows that the columns $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ form an orthonormal set. The steps are reversible establishing the converse.

T.5. Let A be a skew symmetric matrix, so that $\overline{A^T} = -A$, and let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. We show that $\overline{\lambda} = -\lambda$. We have $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides of this equation by $\overline{\mathbf{x}^T}$ on the left we have $\overline{\mathbf{x}^T}A\mathbf{x} = \overline{\mathbf{x}^T}\lambda\mathbf{x}$. Taking the conjugate transpose of both sides yields

$$\overline{\mathbf{x}^T A^T \mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}^T}\mathbf{x}.$$

Therefore $-\overline{\mathbf{x}^T}A\mathbf{x} = \overline{\lambda}\,\overline{\mathbf{x}^T}\mathbf{x}$, or $-\lambda\overline{\mathbf{x}^T}\,\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}^T}\mathbf{x}$, so $(\lambda + \overline{\lambda})(\overline{\mathbf{x}^T}\mathbf{x}) = 0$. Since $\mathbf{x} \neq \mathbf{0}$, $\overline{\mathbf{x}^T}\mathbf{x} \neq 0$, so $\overline{\lambda} = -\lambda$. Hence, the real part of λ is zero.

T.6. (a) Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Then

$$\mathbf{u}\cdot\mathbf{u} = u_1\overline{u_1} + u_2\overline{u_2} + \cdots + u_n\overline{u_n}.$$

Let $u_j = a_j + b_j i$, $j = 1, 2, \ldots, n$. Then $u_j\overline{u_j} = a_j^2 + b_j^2$, so $\mathbf{u}\cdot\mathbf{u} > 0$. Moreover, $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $a_j = b_j = 0$ for $j = 1, 2, \ldots, n$. Thus $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(b) We have $\mathbf{u}\cdot\mathbf{v} = \overline{\mathbf{u}}^T\mathbf{v}$ and

$$\overline{\mathbf{v}\cdot\mathbf{u}} = \overline{\mathbf{v}^T\overline{\mathbf{u}}} = \mathbf{u}\overline{\mathbf{v}^T} = \overline{\mathbf{v}^T\overline{\mathbf{u}}} = \overline{\mathbf{v}\cdot\mathbf{u}}.$$

Thus, $\mathbf{u}\cdot\mathbf{v} = \overline{\mathbf{v}\cdot\mathbf{u}}$.

(c) We have

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}+\mathbf{v})^T\overline{\mathbf{w}} = \left(\mathbf{u}^T + \mathbf{v}^T\right)\overline{\mathbf{w}} = \mathbf{u}^T\overline{\mathbf{w}} + \mathbf{v}^T\overline{\mathbf{w}} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}.$$

(d) We have

$$(c\mathbf{u})\cdot\mathbf{v} = (c\mathbf{u})^T\overline{\mathbf{v}} = \left(c\,\mathbf{u}^T\right)\overline{\mathbf{v}} = c\left(\mathbf{u}^T\overline{\mathbf{v}}\right) = c(\mathbf{u}\cdot\mathbf{v}).$$

Appendix B

Further Directions

Appendix B.1, p. A28

1. (b) $(\mathbf{v}, \mathbf{u}) = a_1 b_1 - a_2 b_1 - a_1 b_2 - 3a_2 b_2 = (\mathbf{u}, \mathbf{v})$.

   (c) $$\begin{aligned}(\mathbf{u}+\mathbf{v}, \mathbf{w}) &= (a_1+b_1)c_1 - (a_2+b_2)c_1 - (a_1+b_1)c_2 + 3(a_2+b_2)c_2 \\ &= (a_1c_1 - a_2c_1 - a_1c_2 + 3a_2c_2) + (b_1c_1 - b_2c_1 - b_1c_2 + 3b_2c_2) \\ &= (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w}).\end{aligned}$$

   (d) $(c\mathbf{u}, \mathbf{v}) = (ca_1)b_1 - (ca_2)b_1 - (ca_1)b_2 + 3(ca_2)b_2 = c(a_1b_1 - a_2b_1 - a_1b_2 + 3a_2b_2) = c(\mathbf{u}, \mathbf{v})$.

2. Almost the same as the verification given in Example 4.

3. We have

   (a) $(\mathbf{u}, \mathbf{u}) = u_1^2 + 5u_2^2 \geq 0$ and $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $u_1 = u_2 = 0$, that is, if and only if $\mathbf{u} = \mathbf{0}$.

   (b) $(\mathbf{u}, \mathbf{v}) = u_1v_1 + 5u_2v_2$ and $(\mathbf{v}, \mathbf{u}) = v_1u_1 + 5v_2u_2$, so $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$.

   (c) $(\mathbf{u}+\mathbf{v}, \mathbf{w}) = (u_1+v_1)w_1 + 5(u_2+v_2)w_2 = u_1w_1 + 5u_2w_2 + v_1w_1 + 5v_2w_2 = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.

   (d) $(c\mathbf{u}, \mathbf{v}) = (cu_1)v_1 + 5(cu_2)v_2 = c(u_1v_1 + 5u_2v_2) = c(\mathbf{u}, \mathbf{v})$.

4. (a) $(A, A) = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$, and $(A, A) = 0$ if and only if $a_{11} = a_{12} = a_{21} = a_{22} = 0$, that is, if and only if $A = O$.

   (b) $(A, B) = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$ and $(B, A) = b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22}$, so $(A, B) = (B, A)$.

   (c) $$\begin{aligned}(A+B, C) &= (a_{11}+b_{11})c_{11} + (a_{12}+b_{12})c_{12} + (a_{21}+b_{21})c_{21} + (a_{22}+b_{22})c_{22} \\ &= a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + a_{22}c_{22} + b_{11}c_{11} + b_{12}c_{12} + b_{21}c_{21} + b_{22}c_{22} \\ &= (A, C) + (B, C).\end{aligned}$$

   (d) $$\begin{aligned}(cA, B) &= (ca_{11})b_{11} + (ca_{12})b_{12} + (ca_{21})b_{21} + (ca_{22})b_{22} \\ &= c(a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}) \\ &= c(A, B).\end{aligned}$$

5. (a) If $A = \left[a_{ij}\right]$, then

   $$(A, A) = \text{Tr}(A^TA) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}^2 \geq 0.$$

   Also, $(A, A) = 0$ if and only if $a_{ij} = 0$, that is, if and only if $A = O$.

(b) If $B = [b_{ij}]$, then $(A, B) = \mathrm{Tr}(B^T A)$ and $(B, A) = \mathrm{Tr}(A^T B)$. Now,

$$\mathrm{Tr}(B^T A) = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}^T a_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ki} a_{ki}$$

and

$$\mathrm{Tr}(A^T B) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^T b_{ki} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ki} b_{ki},$$

so $(A, B) = (B, A)$.

(c) If $C = [c_{ij}]$, then

$$(A + B, C) = \mathrm{Tr}\left[C^T(A + B)\right] = \mathrm{Tr}\left[C^T A + C^T B\right] = \mathrm{Tr}(C^T A) + \mathrm{Tr}(C^T B) = (A, C) + B, C).$$

(d) $(cA, B) = \mathrm{Tr}(B^T(cA)) = c\,\mathrm{Tr}(B^T A) = c(A, B)$.

6. Almost the same as the verification given in Example 4.

8. 17. 10. $\frac{1}{2}\sin^2 1$. 12. 2.

14. (a) $\sqrt{\frac{1}{5}}$. (b) $\sqrt{\frac{1}{2}\left(e^2 - 1\right)}$.

16. (a) $\sqrt{21}$. (b) $\sqrt{96}$.

18. (a) $\frac{16}{3\sqrt{29}}$. (b) $-\frac{17}{7\sqrt{6}}$.

20. $3a = -5b$.

22. (a) $5a = -3b$. (b) $b = \dfrac{2a(\cos 1 - 1)}{e(\sin 1 - \cos 1 + 1)}$.

24. Cauchy–Schwarz inequality:

$$\left|\int_0^1 p(t)q(t)\,dt\right| \le \sqrt{\int_0^1 (p(t))^2\,dt}\,\sqrt{\int_0^1 (q(t))^2\,dt}.$$

Triangle inequality:

$$\sqrt{\int_0^1 p(t)q(t)\,dt} \le \sqrt{\int_0^1 (p(t))^2\,dt} + \sqrt{\int_0^1 (q(t))^2\,dt}.$$

26. (a) $\left\{\sqrt{\frac{3}{7}}\,(t+1), \frac{1}{\sqrt{7}}\,(9t-5)\right\}$. (b) $\begin{bmatrix} \frac{11}{2}\sqrt{\frac{3}{7}} \\ \frac{1}{2\sqrt{7}} \end{bmatrix}$.

28. $\left\{\sqrt{3}\,t, \dfrac{e^t - 3t}{\sqrt{\frac{e^2}{2} - \frac{7}{2}}}\right\}$.

30. $\left\{\frac{5}{2}t^4 - \frac{10}{3}t^3 + t^2, 10t^4 - 10t^3 + t, 45t^4 - 40t^3 + 1\right\}$.

32. (c) $\left(\dfrac{e^{\pi} - e^{-\pi}}{2\pi}\right) + \left(\dfrac{e^{-\pi} - e^{\pi}}{2\pi}\right)\cos t + \left(\dfrac{e^{\pi} - e^{-\pi}}{2\pi}\right)\sin t$.

34. $\mathbf{w} = \dfrac{2}{3}\dfrac{\pi^3}{\sqrt{2\pi}}\left(\dfrac{1}{\sqrt{2\pi}}\right) - \dfrac{4\pi}{\sqrt{\pi}}\left(\dfrac{1}{\sqrt{\pi}}\cos t\right) + 0\left(\dfrac{1}{\sqrt{\pi}}\sin t\right) = \frac{1}{3}\pi^2 - 4\cos t.$
$\mathbf{u} = t^2 - \frac{1}{3}\pi^2 + 4\cos t.$

36. 0.

38. $$\text{proj}_W e^t = \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) + \frac{1}{\pi}\left(-\frac{1}{2}e^{\pi} + \frac{1}{2}e^{-\pi}\right)\cos t$$
$$+ \frac{1}{\pi}\left(\frac{1}{2}e^{\pi} - \frac{1}{2}e^{-\pi}\right)\sin t + \frac{1}{\pi}\left(\frac{1}{5}e^{\pi} - \frac{1}{5}e^{-\pi}\right)\cos 2t$$
$$+ \frac{1}{\pi}\left(-\frac{2}{5}e^{\pi} + \frac{2}{5}e^{-\pi}\right)\sin 2t.$$

T.1. (a) $\mathbf{0} + \mathbf{0} = \mathbf{0}$ so $(\mathbf{0}, \mathbf{0}) = (\mathbf{0}, \mathbf{0} + \mathbf{0}) = (\mathbf{0}, \mathbf{0}) + (\mathbf{0}, \mathbf{0})$, and then $(\mathbf{0}, \mathbf{0}) = 0$. Hence $\|\mathbf{0}\| = \sqrt{(\mathbf{0}, \mathbf{0})} = \sqrt{0} = 0$.

(b) $(\mathbf{u}, \mathbf{0}) = (\mathbf{u}, \mathbf{0} + \mathbf{0}) = (\mathbf{u}, \mathbf{0}) + (\mathbf{u}, \mathbf{0})$ so $(\mathbf{u}, \mathbf{0}) = 0$.

(c) If $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{v}$ in V, then $(\mathbf{u}, \mathbf{u}) = 0$ so $\mathbf{u} = \mathbf{0}$.

(d) If $(\mathbf{u}, \mathbf{w}) = (\mathbf{v}, \mathbf{w})$ for all $\mathbf{w}$ in V, then $(\mathbf{u} - \mathbf{v}, \mathbf{w}) = 0$ and so $\mathbf{u} = \mathbf{v}$.

(e) If $(\mathbf{w}, \mathbf{u}) = (\mathbf{w}, \mathbf{v})$ for all $\mathbf{w}$ in V, then $(\mathbf{w}, \mathbf{u} - \mathbf{v}) = 0$ or $(\mathbf{u} - \mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w}$ in V. Then $\mathbf{u} = \mathbf{v}$.

T.2. (a) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \geq 0$.

(b) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = (\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = 0$ if and only if $\mathbf{u} - \mathbf{v} = \mathbf{0}$.

(c) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\| = d(\mathbf{v}, \mathbf{u})$.

(d) We have $\mathbf{u}-\mathbf{v} = (\mathbf{u}-\mathbf{w}) + (\mathbf{w}-\mathbf{v})$ and $\|\mathbf{u}-\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\|$ so $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$.

T.3. Let $T = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for an inner product space V. If

$$[\mathbf{v}]_T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

then $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$. Since $(\mathbf{u}_i, \mathbf{u}_j) = 0$ if $i \neq j$ and 1 if $i = j$, we conclude that

$$\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}.$$

T.4. Let $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$ and $\mathbf{w} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$. By Exercise T.2, $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$. Then

$$d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\| = \sqrt{(\mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w})}$$
$$= \sqrt{((a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n, (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2 + \cdots + (a_n - b_n)\mathbf{v}_n)}$$
$$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$$

since $(\mathbf{v}_i, \mathbf{v}_j) = 0$ if $i \neq j$ and 1 if $i = j$.

T.5. $\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}) = (\mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}, \mathbf{v}) + \|\mathbf{v}\|^2$, and $\|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u}, \mathbf{u}) - 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = \|\mathbf{u}\|^2 - 2(\mathbf{u}, \mathbf{v}) + \|\mathbf{v}\|^2$. Hence

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

T.6. $\|c\mathbf{u}\| = \sqrt{(c\mathbf{u}, c\mathbf{u})} = \sqrt{c^2(\mathbf{u}, \mathbf{u})} = \sqrt{c^2}\,\sqrt{(\mathbf{u}, \mathbf{u})} = |c|\,\|\mathbf{u}\|$.

T.7. $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}) = (\mathbf{u}, \mathbf{u}) + 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = \|\mathbf{u}\|^2 + 2(\mathbf{u}, \mathbf{v}) + \|\mathbf{v}\|^2$. Thus $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $(\mathbf{u}, \mathbf{v}) = 0$.

T.8. 3.

T.9. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$. If $\mathbf{u}$ is in span S, then

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k.$$

Let $\mathbf{v}$ be orthogonal to $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. Then

$$\begin{aligned}(\mathbf{v}, \mathbf{u}) = (\mathbf{v}, c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_k\mathbf{w}_k) &= c_1(\mathbf{v}, \mathbf{w}_1) + c_2(\mathbf{v}, \mathbf{w}_2) + \cdots + c_k(\mathbf{v}, \mathbf{w}_k)\\ &= c_1(0) + c_2(0) + \cdots + c_k(0) = 0.\end{aligned}$$

Appendix B.2, p. A35

2. (a) $\begin{bmatrix} 14 \\ -2 \\ 6 \end{bmatrix}$. (b) $\begin{bmatrix} 5x - 4y \\ 2y \\ 2x - 2y \end{bmatrix}$.

4. (a) -3. (b) 2. (c) b. (d) a.

6. (a) $B = \begin{bmatrix} -1 & -4 \\ 0 & 2 \\ 2 & 0 \end{bmatrix}$. (b) $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ -4 & 3 \end{bmatrix}$. (c) $\begin{bmatrix} 9 & -8 \\ -3 & 3 \end{bmatrix}$. (d) $\begin{bmatrix} 11 & -6 \\ -5 & 3 \end{bmatrix}$.

10. (a) $L^{-1}\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

12. Invertible. $L^{-1}(b_1, b_2) = \left(\frac{3}{4}b_1 - \frac{1}{4}b_2, \frac{1}{4}b_1 + \frac{3}{4}b_2\right)$.

14. Not invertible. 16. Invertible. $L^{-1}(ct + d) = dt - c$.

18. Not invertible. 20. Not invertible. 22. Invertible.

24. (b) $\begin{bmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$. 26. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

T.1. Let $\mathbf{u}$ and $\mathbf{v}$ be in V_1 and let c be a scalar. We have

$$\begin{aligned}(L_2 \circ L_1)(\mathbf{u} + \mathbf{v}) = L_2(L_1(\mathbf{u} + \mathbf{v})) &= L_2(L_1(\mathbf{u}) + L_1(\mathbf{v}))\\ &= L_2(L_1(\mathbf{u})) + L_2(L_1(\mathbf{v}))\\ &= (L_2 \circ L_1)(\mathbf{u}) + (L_2 \circ L_1)(\mathbf{v})\end{aligned}$$

and

$$(L_2 \circ L_1)(c\mathbf{u}) = L_2(L_1(c\mathbf{u})) = L_2(cL_1(\mathbf{u})) = cL_2(L_1(\mathbf{u})) = c(L_2 \circ L_1)(\mathbf{u}).$$

T.2. We have

$$(L \circ I_V)(\mathbf{v}) = L(I_V(\mathbf{v})) = L(\mathbf{v}) \quad \text{and} \quad (I_W \circ L)(\mathbf{w}) = I_W(L(\mathbf{w})) = L(\mathbf{w}).$$

T.3. We have

$$(L \circ O_V)(\mathbf{v}) = L(O_V(\mathbf{v})) = L(\mathbf{0}) = \mathbf{0} = O_V(\mathbf{v})$$

and

$$(O_V \circ L)(\mathbf{v}) = O_V(L(\mathbf{v})) = \mathbf{0} = O_V(\mathbf{v}).$$

T.4. From Theorem B.2, it follows directly that A^2 represents $L^2 = L \circ L$. Now Theorem B.2 implies that A^3 represents $L^3 = L \circ L^2$. We continue this argument as long as necessary. A more formal proof can be given using induction.

T.5. Suppose that L_1 and L_2 are one-to-one and onto. We first show that $L_2 \circ L_1$ is also one-to-one and onto. First, one-to-one. Suppose that $(L_2 \circ L_1)(\mathbf{v}_1) = (L_2 \circ L_1)(\mathbf{v}_2)$. Then $L_2(L_1(\mathbf{v}_1)) = L_2(L_1(\mathbf{v}_2))$ so $L_1(\mathbf{v}_1) = L_1(\mathbf{v}_2)$, since L_2 is one-to-one. Hence $\mathbf{v}_1 = \mathbf{v}_2$ since L_1 is one-to-one. Next, let L_1 and L_2 be onto, and let $\mathbf{w}$ be any vector in V. Since L_2 is onto, there exists a vector $\mathbf{v}_1$ in V such that $L_2(\mathbf{v}_1) = \mathbf{w}$. Since L_1 is onto, there exists a vector $\mathbf{v}_2$ in V such that $L_1(\mathbf{v}_2) = \mathbf{v}_1$. Then we have

$$(L_2 \circ L_1)(\mathbf{v}_2) = L_2(L_1(\mathbf{v}_2)) = L_2(\mathbf{v}_1) = \mathbf{w},$$

and therefore $L_2 \circ L_1$ is onto. Hence $L_2 \circ L_1$ is invertible. Since

$$(L_2 \circ L_1) \circ (L_1^{-1} \circ L_2^{-1}) = I_V \quad \text{and} \quad (L_1^{-1} \circ L_2^{-1}) \circ (L_2 \circ L_1) = I_V,$$

we conclude that $(L_2 \circ L_1)^{-1} = L_1^{-1} \circ L_2^{-1}$.

T.6. If L is one-to-one and onto, so is cL, $c \neq 0$. Moreover,

$$(cL) \circ \left(\frac{1}{c}L^{-1}\right) = \left(c \cdot \frac{1}{c}\right) L \circ L^{-1} = I_V \quad \text{and} \quad \left(\frac{1}{c}L^{-1}\right) \circ (cL) = \left(\frac{1}{c}c\right)(L^{-1} \circ L) = I_V.$$

T.7. Since L is one-to-one and onto, it is invertible. First, L is one-to-one. To show this, let $L(A) = L(B)$. Then $A^T = B^T$ so $(A^T)^T = (B^T)^T$ which implies that $A = B$. Also, if B is any element in M_{22}, then $L(B^T) = (B^T)^T = B$, so L is onto. We have $L^{-1}(A) = A^T$.

T.8. Since L is one-to-one and onto, it is invertible. First, L is one-to-one. To show this, let $L(A_1) = L(A_2)$. Then $BA_1 = BA_2$. Since B is nonsingular, it follows that $A_1 = A_2$. Also, if C is any element in M_{22}, then $B^{-1}C$ is in M_{22} and $L(B^{-1}C) = C$, so L is onto. We have

$$L^{-1}(C) = B^{-1}C = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} C.$$

T.9. We show that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a).

(a) $\Longrightarrow$ (b): Suppose that L is invertible. Then L is one-to-one and onto, so $\dim(\operatorname{range} L) = n = \operatorname{rank} L$.

(b) $\Longrightarrow$ (c): If $\operatorname{rank} L = n$, then $\dim(\ker L) = 0$ so $\operatorname{nullity} L = 0$.

(c) $\Longrightarrow$ (a): If $\operatorname{nullity} L = 0$, then $\operatorname{rank} L = n$, which means that $\dim(\operatorname{range} L) = n$. Hence L is one-to-one and onto and is then invertible.

T.10. Assume that $(L_1 + L_2)^2 = L_1^2 + 2(L_1 \circ L_2) + L_2^2$. Then

$$L_1^2 + L_1 \circ L_2 + L_2 \circ L_1 + L_2^2 = L_1^2 + 2(L_1 \circ L_2) + L_2^2,$$

and simplifying gives $L_1 \circ L_2 = L_2 \circ L_1$. The steps are reversible.

T.11 We have

$$L(\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = (\mathbf{v}_1, \mathbf{w}) + (\mathbf{v}_2, \mathbf{w}) = L(\mathbf{v}_1) + L(\mathbf{v}_2).$$

Also, $L(c\mathbf{v}) = (c\mathbf{v}, \mathbf{w}) = c(\mathbf{v}, \mathbf{w}) = cL(\mathbf{v})$.