Equation of tangent plane & Normal line to surface:-

$$(x-x_0)f_x(P)+(y-y_0)f_y(P)+(z-z_0)f_z(P)=0$$

(Eqn of tangent plane)

$$\frac{x-x_0}{f_x(P)}=\frac{y-y_0}{f_y(P)}=\frac{z-z_0}{f_z(P)}.$$

Exp #126 Find eqn of tangent plane & normal line to surface $2x^2+y^2+2z=3$ at $(2,1,-3)$.

Sol:- Since;

$$(x-x_0)f_x(P)+(y-y_0)f_y(P)+(z-z_0)f_z(P)=0$$

$$f=2x^2+y^2+2z-3.$$

so,

$$f_x=4x \Rightarrow f_x(P)=8.$$

$$f_y=2y \Rightarrow f_y(P)=2.$$

$$f_z=2 \Rightarrow f_z=(2).$$

By putting values;

$$(x-2)(8)+(y-1)2+(z+3)2=0.$$

$$8x-16+2y-2+2z+6=0$$

$$8x+2y+2z-12=0.$$

or

$$\boxed{4x+y+z-6=0}$$ eqn of tangent plane

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Eqn of normal:-

As;

$$\frac{x-x_0}{f_x(P)} = \frac{y-y_0}{f_y(P)} = \frac{z-z_0}{f_z(P)}$$

$$\frac{x-2}{8} = \frac{y-1}{2} = \frac{z+3}{2}$$

Multiply 2 on all sides

$$\boxed{\frac{x-2}{4} = \frac{y-1}{1} = \frac{z+3}{1}}$$ Ans

Example # 27 :-

Show that the surface $x^2 - 2yz + y^3 = 4$ is perpendicular to any number of the family of surfaces $x^2 + 1 = (2-4a)y^2 + az^2$ at the point of intersection $(1,-1,2)$.

Sol:- $f(x,y,z) = x^2 - 2yz + y^3 - 4$.

$F(x,y,z) = x^2 + 1 - (2-4a)y^2 - az^2$.

$f_x'(x,y,z) = 2x$

$f_y' = -2z + 3y^2$.

$f_z = -2y$.

Direction ratios are;

$2x, -2z+3y^2, -2y$.

Direction ratios at $(1,-1,2)$ are.

$2(1), -2(2)+3(-1)^2, -2(2)$.

$2, -1, 2$.
$l_1 \quad m_1 \quad n_1$

Now;

$F(x,y,z) = x^2+1-(2-4a)y^2 - az^2$.

Differentiate w.r.t to x, y & z partially.

$F_x = 2x$

$F_y = -2y(2-4a)$

$F_z = -2az$

Direction ratios are;

$2x, -2y(2-4a), -2az$.

Direction ratios at $(1,-1,2)$ are,

$2, 2(2-4a), -4a$.

$2, 4-8a, -4a$.
$l_2 \quad m_2 \quad n_2$

For perpendicularity;
$l_1 . l_2 + m_1 . m_2 + n_1 . n_2 = 0$.

$\Rightarrow 2(2) + (4-8a)(-1) + 2(-4a)$

$\cancel{4} - \cancel{4} + \cancel{8a} - \cancel{8a}$.

$= 0$.

Hence the given surfaces are perpendicular to each other.

$\underline{\underline{Ex \Rightarrow 2\text{-}3}}$

Q1 - $xyz = 6$ at $(1,2,3)$.

Sol:- $f_x = yz$

$f_x(P) = 6$.

$f_y = xz \Rightarrow f_y(P) = 3$.

$f_z = xy \Rightarrow f_z(P) = 2$.

Now tangent plane eqn;

$(x-1)(6) + (y-2)3 + (z-3)2 = 0$.

$6x - 6 + 3y - 6 + 2z - 6 = 0$.

$6x + 3y + 2z - 18 = 0$

<u>Eqn of normal:-</u>

$$\boxed{\frac{x-1}{6} = \frac{y-2}{3} = \frac{z-3}{2}}$$ Ans.

Q2 - $z^2 = 4(1+x^2+y^2)$ at $(2,2,6)$

Sol:- $f = 4x^2 + 4y^2 + 4 - z^2$.

$f_x = 8x \Rightarrow f_x(P) = 16$.

$f_y = 8y \Rightarrow f_y(P) = 16$.

$f_z = -2z \Rightarrow f_z(P) = -12$.

Tangent eqn

$$(x-x_0)f_x(P) + (y-y_0)f_y(P) + (z-z_0)f_z(P) = 0.$$

By putting values.

$$(x-2)(16) + (y-2)16 + (z-6)(12) = 0$$

$$16x - 32 + 16y - 32 - 12z + 72 = 0$$

$$\boxed{4x + 4y - 3z + 2 = 0}$$

Equation of normal:-

$$\frac{x-x_0}{f_x(P)} = \frac{y-y_0}{f_y(P)} = \frac{z-z_0}{f_z(P)}$$

$$\frac{x-2}{16} = \frac{y-2}{16} = \frac{z-6}{-12}$$

Multiply 4 on all sides;

$$\frac{x-2}{4} = \frac{y-2}{4} = \frac{z-6}{-3}$$ Ans 2.

Q3.- Eqn of tangent & normal?

$$\frac{x^2}{2} - \frac{y^2}{3} = z \quad \text{at } (2, 3, -1)$$

Sol:- $3x^2 - 2y^2 - 6z = 0.$

$$f = 3x^2 - 2y^2 - 6z$$

$f_x = 6x$ $\quad f_x(P) = 12$ $\quad f_z = -6$

$f_y = -4y$ $\quad f_y(P) = -12$ $\quad f_z(P)$

Eqn of tangent;

$$(x-x_0)f_x(P) + (y-y_0)f_y(P) + (z-z_0)f_z(P) = 0.$$

$$(x-2)12 + (y-3)(-12) + (z+1)(-6) = 0.$$

Divide by "6".

$$(x-2)2 + (y-3)(-2) + (z+1)(-1) = 0.$$

$$2x - 4 - 2y + 6 + (-z) - 1 = 0.$$

$$\boxed{2x - 2y - z + 1 = 0.}$$ Ans

Eqn of normal;

$$\frac{x-x_0}{f_x(P)} = \frac{y-y_0}{f_y(P)} = \frac{z-z_0}{f_z(P)}$$

$$\frac{x-2}{12} = \frac{y-3}{-12} = \frac{z+1}{-6}.$$ Multiply "6."

$$\boxed{\frac{x-2}{2} = \frac{y-3}{-2} = \frac{z+1}{-1}}$$ Ans

Q#4:- show that plane $3x+12y-6z-17=0$ touch the conicoid $3x^2-6y^2+9z^2+17=0$. find also point of contact.

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Exp# 3 Show that minimum value of

$u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$.

Sol:- $f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$

$f_x = y - \frac{a^3}{x^2}$, $f_y = x - \frac{a^3}{y^2}$

Put $f_x = 0$ Put $f_y = 0$

$y = \frac{a^3}{x^2} \longrightarrow (1)$ $x = \frac{a^3}{y^2} \longrightarrow (2)$

or $a^3 = yx^2$.

Put eq (1) in eq (2)

$x = \frac{a^3}{(a^3/x^2)^2} \Rightarrow \frac{a^3}{a^6/x^4}$

$x = \frac{x^{\cancel{4}\,3}}{a^3} \Rightarrow x^3 = a^2 \Rightarrow \boxed{x = a}$

Put in eq (1).

eq (1) $\Rightarrow$ $y = \frac{a^3}{x^2} \Rightarrow \frac{a^3}{a^2} \Rightarrow \boxed{y = a}$

Use $y = a$. Put in eq (1).

$a = \frac{a^3}{x^2} \Rightarrow x^2 = \frac{a^3}{a} = x^2 = a^2$.

$\boxed{x = \pm a}$

Thus stationary points are (a,a) & $(a,+a)$

Now;

~~For (a, a)~~

$$\cancel{f_{xx} = \frac{2(1)}{2}\frac{a^3}{(a)x^3} = f_{xx}(P) = 2\left(\frac{1}{2}\right)\frac{a^3}{(a)^3}}$$

$$\cancel{f_{xx} = \frac{2(1)}{2} = 1.}$$

$$\cancel{f_{yy} = \frac{a^3}{y^3}.}$$

For (a, a)

$$f_x = y - a^3/x^2$$

$$f_{xx} = \frac{2a^3}{x^3}.$$

$$f_{xx}(P) = \frac{2a^3}{a^3} = 2.$$

$$f_y = x - a^3/y^2$$

$$f_{yy} = 2a^3/y^3.$$

$$f_{yy}(P) = \frac{2(a^3)}{a^3} = 2.$$

$$f_{xy} = 1 + a^3(y+x).$$

$$f_{xy}(P) = 1 + a^3(a+a) \Rightarrow 1 + 2a^4.$$

$$f = xy + a^3\left(\frac{y+x}{xy}\right)$$

$f_{xy} = ?$

$$f_{xy} = 1 + \frac{(-a^3(y+x))}{xy^{+2}}.$$

$$= 1 - \frac{a^3(2a)}{a^3}$$

$$= 1 - 2a.$$

Since;

$$\Rightarrow f_{xx}.f_{yy} - (f_{xy})^2.$$

$$\Rightarrow 2(2) - (1-2a)^2 \Rightarrow 4 - 1 + 4a^2 + 4a > 0 \text{ (+ve)}$$

As $f_{xx} >$ +ve. So minimum value occur at (a, a)

Put in function;

$$f = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) \Rightarrow a^2 + a^3\left(\frac{a+a}{a^2}\right) = a^2 + 2a^2$$

$= 3a^2$ Ans

Example #4

Examine $f(x,y) = x^3 + y^3 - 3axy$ for max & min value

Sol:- As; $f = x^3 + y^3 - 3axy$.

$$f_x = 3x^2 - 3ay \rightarrow ①$$

$$f_y = 3y^2 - 3ax \rightarrow ②$$

Put $f_x = 0$.

$$0 = 3x^2 - 3ay \Rightarrow x^2 = ay \rightarrow ③$$

Put $f_y = 0$.

$$3y^2 - 3ax = 0$$

$$3ax = 3y^2$$

$$y^2 = ax \Rightarrow x = y^2/a \rightarrow ④$$

Put eq ④ in ③.

$$\left(y^2/a\right)^2 = ay$$

$$\frac{y^{\cancel{4}\,3}}{a^2} = \cancel{y}a \Rightarrow y^3 = a^3 \Rightarrow \boxed{y = a}$$

or

$$\frac{y^4}{a^2} = ya \Rightarrow y^4 = ya^3 \Rightarrow y^4 - ya^3 = 0.$$

$$y(y^3 - a^3) = 0.$$

$$\boxed{y = 0} \text{ or } \boxed{y = a}$$

Put in eq ④.

$$\boxed{x = 0} \quad \text{or} \quad \boxed{x = a}$$

For $(x, y) = (a, a)$.

$f_x = 3x^2 - 3ay$.

$f_{xx} = 6x$

$f_y = 3y^2 - 3ax$.

$f_{yy} = 6y$

$f_{xx}(P) = 6a > 0$.

$f_{yy}(P) = 6a$.

$f = x^3 + y^3 - 3axy$.

$f_{xy} = -3a$.

$f_{xy} = -3a$.

Since;

$z = f_{xx} f_{yy} - (f_{xy})^2$. Putting values

$= (6a)(6a) - (-3a)^2$.

$= 36a^2 - 9a^2$.

$= 27a^2 > 0 \, (+ve)$.

Thus minimum value occur at (a, a).

Minimum value Put (a, a) in f.

$f(x, y) = a^3 + a^3 - 3a(a^2) \Rightarrow -a^3$.

For other point $(0, 0)$.

$f_{xx} = 6x$ at $(0, 0)$

$f_{xx} = 0$.

$f_{yy} = 6y$. at $(0, 0)$.

$= 0$.

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$f_{xy} = -3a.$

$f_{xx} f_{yy} - (f_{xy})^2$

$0(0) - (-3a)^2 \Rightarrow \quad -9a^2 < 0.$

Thus this test fail

& Then no extrema at $(0,0)$ <u>Ans</u>

<u>Example #7</u>

Show that function

$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$ is maximum

at $(7,-7)$ & minimum at $(3,3)$

<u>Sol:-</u> $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy.$

$f_x = 3x^2 - 63 + 12y. \longrightarrow (1)$

$f_y = 3y^2 - 63 + 12x. \longrightarrow (2)$

Put f_x & $f_y = 0$

$0 = 3x^2 - 63 + 12y \longrightarrow (3)$

$0 = 3y^2 - 63 + 12x. \longrightarrow (4)$

Subtract eq (3) from (4)

$0 = 3x^2 - 63 + 12y$

$0 = \underset{-}{+}3y^2 \underset{+}{-} 63 \underset{-}{+} 12x$

$3x^2 - 3y^2 + 12y - 12x = 0.$

$3(x^2 - y^2) - 12(x-y) = 0$ .. Divide by 3

$(x^2 - y^2) - 4(x-y) = 0.$

$\Rightarrow (x+y)(x-y) - 4(x-y) = 0$

$(x-y)(x+y-4)=0$

$x-y=0$ or $x+y=4$

$x=y$ or $x+y=4$

For $x=y$.

& ③ ⇒

$0 = 3x^2 - 63 + 12x$

Divide by 3.

$0 = x^2 - 21 + 4x \rightarrow$ ⑤

$0 = x^2 + 7x - 3x - 21$

$= x(x+7) - 3(x+7)$

$(x+7)(x-3) = 0$

$x = -7$ or $x = 3$

~~Put~~ since; $x = y$.

So two stationary points are;

$(-7,-7)$ & $(3,+3)$.

~~For; $x+y=4$ or $x = 4-y$~~ $\rightarrow$ ⑥.

~~Put in~~ ~~eq ⑤~~

$\cancel{3(4-y)^2 - 63 + 12(4-y)} = 0$.

$\cancel{3(16+y^2-8y) - 63}$

$\cancel{(4-y)^2 + 4(4-y) - 21 = 0}$

$\cancel{16+y^2-8y+16-4y-21} = 0$.

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$y = 4 - x \rightarrow (A)$

Put it in eq (3).

$3x^2 - 63 + 12y = 0$

$x^2 - 21 + 4y = 0$

$x^2 - 21 + 4(4 - x) = 0$

$x^2 - 4x - 21 + 16 = 0$

$x^2 - 4x - 5 = 0$

$x^2 - 5x + x - 5 = 0$

$x(x-5) + 1(x-5) = 0$

$(x-5)(x+1) = 0$

$x = 5$ or $x = -1$

eq (A) $\Rightarrow$

$y = 4 - x$

$y = -1$

or $y = 4 - x$

$y = 4 - (-1)$

$y = 5$

So other stationary points are; $(5,-1)$ & $(-1,5)$

Now for $(-7,-7)$

$f_{xx} = 6x \Rightarrow f_{xx}(P) = -42 < 0$

$f_{yy} = 6y \Rightarrow f_{yy}(P) = -42$

$f_{xy} = 12$

$f_{xx} \cdot f_{yy} - (f_{xy})^2 \Rightarrow (-42)(-42) - (12)^2$

$1620 > 0$. Thus max. at $(-7,-7)$.

at $(3,3)$

$f_{xx} = 6x = 18 > 0$

$f_{yy} = 6y = 18$

$f_{xy} = 12$

$= (18)(18) - (12)^2 \Rightarrow 180 > 0$

Thus minimum at $(x,y) = (3,3)$.

At $(-1,5)$

$f_{xx} = 6(-1) = -6 < 0$.

$f_{yy} = 6y = 6(5) = 30$.

$f_{xy} = 12$.

$f_{xx} f_{yy} - (f_{xy})^2$

$-6(30) - (12)^2 = -ve$ Hence saddle point

At $(5,-1)$

$f_{xx} = 6(5) = 30 > 0$.

$f_{xy} = 12$.

$f_{yy} = 6(-1) = -6$.

$(f_{xx})(f_{yy}) - (f_{xy})^2 = 30(-6) - (12)^2$.

$= -ve$

Thus this is also saddle point Ans

So max at $(-7,-7)$ & minimum at $(3,3)$ Ans

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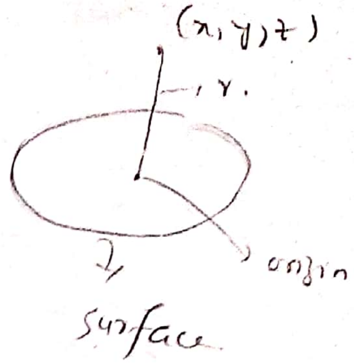
Example #06

: Find the point on surface $z^2 = xy+1$ nearest to origin?

Sol:- Let (x,y,z) be the point nearest to origin.



So; the distance from origin to that point be

$$r = \sqrt{x^2+y^2+z^2}$$

$$\cdot\ u = r^2 = x^2+y^2+z^2 \longrightarrow (A)$$

To find point i.e value of x, y & z for which u is minimum.

Put as; $z^2 = xy+1 \longrightarrow (B)$

Put in eq (A)

$$u = x^2+y^2+xy+1.$$

$$\frac{\partial u}{\partial x} = 2x+y \longrightarrow (1) \quad \& \quad \frac{\partial u}{\partial y} = 2y+x \longrightarrow (2)$$

Put these eqns equal to zero

$$0 = 2y+x \longrightarrow (3)$$

$$2x+y = 0$$

$$2x = -y$$

$$x = -y/2 \longrightarrow (4)$$

Put (4) in eq (3).

eq ④ $\Rightarrow$ $2y + x = 0.$

$2y - y/2 = 0$

$\frac{4y - y}{2} = 0.$

$\frac{3y}{2} = 0 \Rightarrow y = 0$

eq ④ $\Rightarrow$ $x = 0$

Put in eq Ⓑ.

$z^2 = 0 + 1$

$z^2 = 1 \Rightarrow z = \pm 1$

As;

$f_x = 2x + y$, $f_y = 2y + x$

$f_{xx} = 2 > 0.$ $f_{yy} = 2 \ldots$

$f_{xy} = 1.$

So; $f_{xx} f_{yy} - (f_{xy})^2 \Rightarrow 2(2) - (1)^2.$

$= 3 > 0.$ (minimum) occur at $(0, 0, \pm 1)$

Thus

U is minimum for

$(0, 0, \pm 1)$ Ans.

Vector Integration:-

Compute $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = \frac{\hat{i}y - \hat{j}x}{x^2+y^2}$, and C is the circle $x^2+y^2=1$ traversed c. clockwise

Sol:- As given that;

$$\vec{F} = \frac{\hat{i}y - \hat{j}x}{x^2+y^2}$$

while $d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$.

Since;

$$\int_C \vec{F} \cdot d\vec{r} = \int \frac{\hat{i}y - \hat{j}x}{x^2+y^2} \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int_C \frac{y\,dx - x\,dy}{x^2+y^2} \qquad x^2+y^2=1$$

$$= \int_C y\,dx - x\,dy \longrightarrow (1)$$

Since; unit circle.

$x = (1)\cos\theta$

$x = \cos\theta \longrightarrow (2)$

$\frac{dx}{d\theta} = -\sin\theta$

$dx = -\sin\theta\,d\theta \longrightarrow (3)$

while

$y = (1)\sin\theta$

$y = \sin\theta \longrightarrow (4)$

$\frac{dy}{d\theta} = \cos\theta$

$dy = \sin\theta \longrightarrow (5)$

Put values in eq (1)

$$\int_C (\sin\theta(-\sin\theta) - \cos\theta\cos\theta)\,d\theta$$

$$\int_C (-\sin^2\theta - \cos^2\theta)\,d\theta$$

$$= -\int(\sin^2\theta + \cos^2\theta)\,d\theta = -\int(1)\,d\theta$$

$$= -\int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} \Rightarrow -2\pi$$

Ans

Example #10S

The acceleration of particle at time t is given by; $\vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$.

if velocity $\vec{v}$ & displacement $\vec{r}$ be zero at $t=0$ at any point t.

Find $\vec{v}$ & $\vec{r}$

Sol:- As we know that;

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = 18\cos 3t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}.$$

Integrate it w.r.t t.

$$\int \frac{d^2\vec{r}}{dt^2} = 18\,\frac{\sin 3t}{3}\,\hat{i} + 8\,\frac{\cos 2t}{2}\,\hat{j} + 3t^2 + c_1$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2 + c \quad \text{--- (1)}$$

Put $\frac{d\vec{r}}{dt} = v = 0$ at $t=0$.

$$= 6\sin(0)\hat{i} + 4(\cos 0)\hat{j} + c.$$

$$c = -4\hat{j}.$$

Put in eq (1).

$$\vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\hat{k} - 4\hat{j}.$$

$$= 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} - 4\hat{j} + 3t^2\hat{k}.$$

$$= 6\sin 3t\,\hat{i} + 4(\cos t - 4)\hat{j} + 3t^2\hat{k}.$$

Again $\int$.

$$\int \frac{d\vec{r}}{dt} = \int \left(6\sin 3t\,\hat{i} + 4(\cos t - 4)\hat{j} + 3t^2\hat{k}\right)dt$$

$$\vec{r} = \frac{-6\cos 3t}{3}\,\hat{i} + 4\sin t\hat{j} - 4t\hat{j} + t^3\hat{k} + c_2.$$

$$\vec{r} = -2\cos 3t\,\hat{i} + 4\sin t\hat{j} - 4t\hat{j} + t^3\hat{k} + c_2 \quad \text{--- (2)}$$

As at $t=0$, $\vec{r}=0$

$$0 = -2\cos(0)\hat{i} + 0 + 0 + c_2$$

$$\boxed{c_2 = 2\hat{i}}.$$

Put in eq ②

$$\vec{r} = -2\cos 3t\,\hat{i} + 2\hat{i} + (4\sin t - 4t)\hat{j} + t^3\hat{k} \cancel{+c_2}$$

$$\vec{r} = (-2\cos 3t + 2)\hat{i} + (4\sin t - 4t)\hat{j} + t^3\hat{k} \quad \text{Ans 2.}$$

Example#06.

If $A = (3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\oint \vec{A}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve C. $x=t$, $y=t^2$, $z=t^3$

Solution:-

Required;

$$\int_c \vec{A}\cdot d\vec{r}. \qquad d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

By putting values.

$$\oint_c \vec{A}\cdot d\vec{r} = \int_c [(3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}].[\hat{i}dx + \hat{j}dy + \hat{k}dz]$$

$$= \int_c [(3x^2+6y)dx - 14yz\,dy + 20xz^2 dz].$$

If $x=t$, $y=t^2$, $z=t^3$ & The point $(0,0,0)$ to and $(1,1,1)$ corresponds to $t=0$ & $t=1$.

so,
$$\int_0^1 \vec{A}\cdot d\vec{r} = \int_0^1 [(3t^2+6t^2)d(t) - 14t^5 d(t^2) + 20t(t^3)^2 d(t^3)]$$

$$= \int_0^1 9t^2 dt - 14t^5 \cdot 2t\,dt + 20t^7(3t^2)dt]$$

$$= \int_0^1 (9t^2 dt - 28t^6 dt + 60t^9 dt)$$

Apply Integration;

$$\left(\frac{9t^3}{3} - \frac{28}{7} t^7 + \frac{60t^{10}}{10} \right) \Big|_0^1$$

$$= \left(3t^3 - 4t^7 + 6t^{10} \right) \Big|_0^1$$

$$= \left(3(1)^3 - 4(1)^7 + 6(1)^{10} \right) - 0.$$

$$= 3 - 4 + 6 = 5 \text{ Ans}$$

Example # 18:-

show that the vector field is conservative i.e $\vec{F} = 2x(y^2 + z^3)\hat{i} + 2x^2y\,\hat{j} + 3x^2z^2\,\hat{k}$. Find its scalar potential and work done in moving a particle from $(-1,2,1)$ to $(2,3,4)$.

Solution:- As given that,

$$\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}.$$

For field to be conservative;

$$\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = 0$$

$$\vec{\nabla} = \hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z$$

$$\text{Curl} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x(y^2+z^3) & 2x^2y & 3x^2z^2 \end{vmatrix}$$

$$= \hat{i}[0-0] - \hat{j}(6xz^2 - 6xz^2) + \hat{k}(4xy - 4xy)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$= 0$ Hence vector field is irrotational.

To find scalar potential.

Let $\vec{F} = \vec{\nabla}\phi \rightarrow ①$

Since $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz.$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi\,(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$d\phi = \vec{\nabla}\cdot\phi\cdot d\vec{r} \rightarrow ②$

Put eq ① in ②

$d\phi = \vec{F}\cdot d\vec{r}$

By putting values

$= \left[2x(y^2+z^3)dx\,\hat{i} + 2x^2y\,dy\,\hat{j} + 3x^2z^2\hat{k}\right].\left[\hat{i}dx + \hat{j}dy + \hat{k}dz\right]$

$d\phi = \left[2x(y^2+z^3)dx + 2x^2y\,dy + 3x^2z^2dz\right].$

Integrating;

$\int d\phi = \int\left[2x(y^2+z^3)dx + 2x^2y\,dy + 3x^2z^2dz\right].$

$\phi = x^2y^2 + x^2z^3 + c.$

Hence the scalar potential is;

$\phi = x^2y^2 + x^2z^3 + c$

Now for conservative field work done is;

$W = \int_{(-1,2,1)}^{(2,3,4)} \vec{F}.\vec{dr} = \int_{(-1,2,1)}^{(2,3,4)} d\phi = \left(x^2y^2 + x^2z^3\right)\Big|_{(-1,2,1)}^{(2,3,4)}$

$= \left((2)^2(3)^2 + (2)^2(4)^3\right) - \left((-1)^2(2)^2 + (-1)^2(1)^3\right).$

$= (36 + 256) - (4+1).$

$\boxed{W = 287}$ Ans

Example #109. A vector field is given by
$F = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$ Evaluate The line integral over a circular path $x^2+y^2=a^2$, $z=0$.

Sol:- As we have;

work done $= \int_C \vec{F}.d\vec{r}$ —→ ①

$\vec{F} = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$.

$d\vec{r} = \hat{i}dx + \hat{j}dy$. $\quad dz = 0$ as $z=0$.

By putting values.

$$= \int_C [(\sin y)\hat{i} + x(1+\cos y)\hat{j}].[\hat{i}dx + \hat{j}dy]$$

$$= \int_C [\sin y \, dx + x(1+\cos y)dy].$$

$$= \int_C (\sin y \, dx + x\,dy + x\cos y \, dy)$$

$$= \int_C (\sin y \, dx + x\cos y \, dy + x\,dy)$$

$$= \int_C d(x\sin y) + \int_C x\,dy \longrightarrow ①$$

Since; we are given that;

$x^2+y^2=a^2$ for this case

$x = a\cos\theta$

$y = a\sin\theta$

By putting values in eq ①

Take limit from 0 to 2π as circle;

$x = a\cos\theta$ — ①

$\frac{dx}{d\theta} = -a\sin\theta$.

$dx = -a\sin\theta d\theta$.

$y = a\sin\theta$

$\frac{dy}{d\theta} = a\cos\theta$.

$dy = a\cos\theta d\theta$.

$$= \int_0^{2\pi} [d(a\cos\theta \sin(a\sin\theta))] + \int_0^{2\pi} a\cos\theta . a\cos\theta d\theta$$

$$= (a\cos\theta . \sin(a\sin\theta))\Big|_0^{2\pi} + \int_0^{2\pi} a^2\cos^2\theta d\theta$$

$$= (a\cos(2\pi) . \cancel{\sin(a\sin 2\pi)}^{0}) - (a\cos(0) . \cancel{\sin(a\sin 0)}^{0}) + a^2\int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right)$$

$$= 0 + a^2\int_0^{2\pi} \frac{1}{2} d\theta + \int_0^{2\pi} \frac{\cos 2\theta}{2}$$

$$= \frac{a^2}{2}\left[\theta\Big|_0^{2\pi} + \frac{\sin 2\theta}{2}\Big|_0^{2\pi}\right.$$

$$= \frac{a^2}{2}\left[2\pi + 0 + \frac{\sin(2)(2\pi)}{2}\right]$$

$$= \frac{a^2}{2}[2\pi + 0 + 0]$$

$$= a^2\pi \quad \text{Ans}$$

Example #10

Determine whether the line integral

$$\int (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$

is independant of path of Integration? If so, then evaluate it from $(1, 0, 1)$ to $(0, \pi/2, 1)$.

Sol.- As given that;

$$\int_C (2xyz^2)\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz.$$

$$= \int_C (2xyz^2)\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k}\cdot(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \int \vec{F}\cdot d\vec{r}$$

for Independant of path;

$$\vec{\nabla}\times\vec{F} = 0.$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= \hat{i}\left(\cancel{2x^2z} + \cancel{\cos yz} - \cancel{y^2 \sin yz}\cdot z - \cancel{2x^2z} + \cancel{zy \sin yz} - \cancel{\cos yz}\right)$$
$$- \hat{j}\left(\cancel{4xyz} - \cancel{4xyz}\right) + \hat{k}\left(\cancel{2xz^2} - \cancel{2xz^2}\right)$$

$$= 0$$

Hence independant of path.

let $\vec{F} = \vec{\nabla}\phi \rightarrow (1)$.

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} dz \cdot$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz).$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} \rightarrow (2)$$

Put eq (1) in (2).

$$d\phi = \vec{F} \cdot d\vec{r}$$

Taking Integration.

$$\int d\phi = \int \vec{F} \cdot d\vec{r}$$

By putting values

$$\phi = \int (2xyz^2)\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k} \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz).$$

$$= \int 2xyz^2 dx + (x^2z^2 + z\cos yz)dy + (2x^2yz + y\cos yz)dz.$$

$$\phi = x^2yz^2 + \sin yz$$

Now; $[\phi]_A^B = \phi(B) - \phi(A)$

$$(x^2yz^2 + \sin yz) = \Big|_{(1,0,1)}^{(0,\pi/2,1)}$$

$$= \left[0 + \sin \pi/2\right] - \left[0 - 0\right]$$

$$= 1 - 0$$

$$= \underline{1} \text{ Ans}$$