Ex # 1.6 orthogonal trajectories.

Note:- As logarithm & exponential are inverse of each other.

Q4:- $y = x^2 + c$.

Sol:- To find OTs.

As $y = x^2 + c$.

$$\frac{dy}{dx} = 2x.$$

$$y' = 2x \longrightarrow \text{ODE}.$$

or

$$\frac{dy}{dx} = 2x$$

So OTs to given ODE is;

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2x}.$$

$$dy = -\frac{1}{2x}\,dx.$$

$$\int dy = -\frac{1}{2}\int \frac{1}{x}\,dx$$

$$y = -\frac{1}{2}\ln x + c.$$

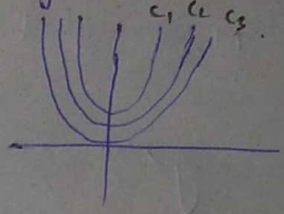
or. $y = -\ln x^{1/2} + c$

$$\boxed{y = -\ln\sqrt{x} + c}$$ Required OTs.

Sketch

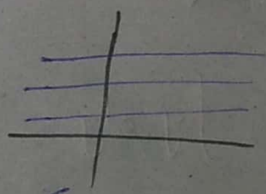
Graph of

$y = x^2 + c$



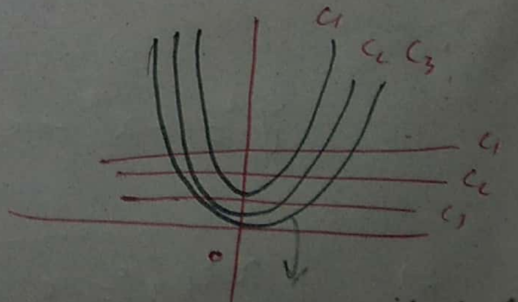
Graph of

$y = x + c$.



So

Combined graph will be.



Note:- Here it should not touch the origin.

(5) $y = cx$.

Sol:- $y = cx \Rightarrow c = y/x \longrightarrow ①$

$y' = c \longrightarrow ②$ (ODE)

Put eq ① in ②.

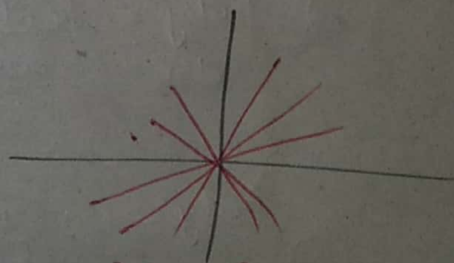
$y' = y/x$ (Required ODE)

Now to find OTS to ODE.

$y' = -x/y$. $\qquad m_1 = -1/m_2$.

$y y' = -x$.

Graph of $y = cx$.



Straight line passing through origin.

$\int y\, dy = -\int x\, dx$.

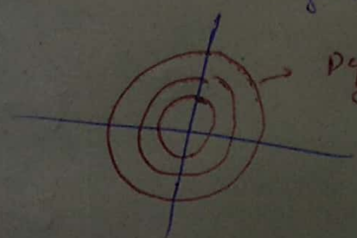
$y^2/2 = -\frac{x^2}{2} + c/2$.

$y^2 = -x^2 + c$.

$\Rightarrow \boxed{x^2 + y^2 = c}$

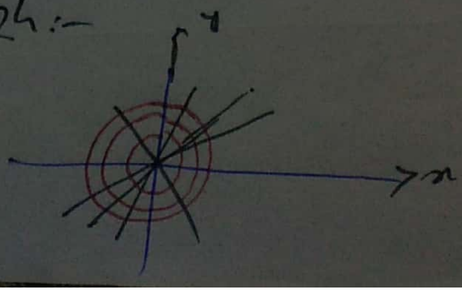
Eqn of Circle

Graph of $x^2 + y^2 = c$.
Circle eqn with Centre is origin



Depend on c value

Combine graph:-



(6) $xy = c$.

Sol:- $xy = c$

$\Rightarrow c = xy$

$y = c/x \longrightarrow (1)$

$y' = -\dfrac{c}{x^2}$ (ODE)

$\dfrac{dy}{dx} = -c/x^2 \longrightarrow$ (ODE)

So ODE of OTS is;

$\dfrac{dy}{dx} = \dfrac{x^2}{c}$

Put c value

$\dfrac{dy}{dx} = \dfrac{x^2}{xy}$

$\dfrac{dy}{dx} = x/y$.

$y\,dy = x\,dx$

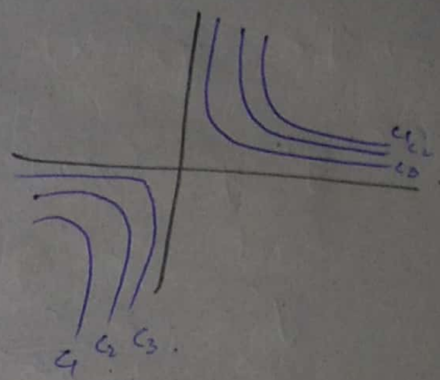
$\int y\,dy = \int x\,dx$

$\dfrac{y^2}{2} = \dfrac{x^2}{2} + \dfrac{c}{2}$

$\boxed{y^2 - x^2 = c}$

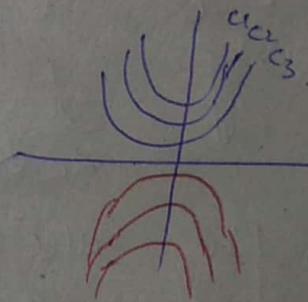
Sketch

Graph of $y = c/x$



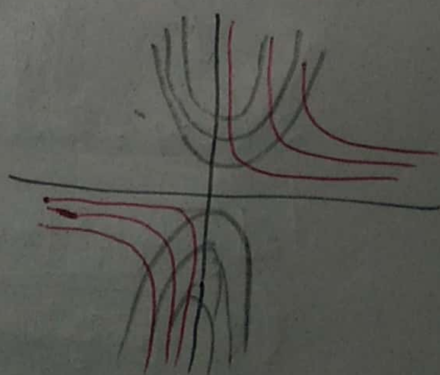
Graph of Parabola

$y^2 - x^2 = c$ or $y^2 = x^2 + c$



Depend upon value of c

Combined graph



Note:- Both graphs should not touch the origin.

Q7:- $y = c/x^2$.

Sol:- $y = c/x^2 \longrightarrow (1)$

$\Rightarrow c = yx^2$.

$\&(1) \Rightarrow \quad y = c/x^2$.

$y' = -2\frac{c}{x^3}$.

$y' = \frac{-2c}{x^3}$. (ODE)

ODE of OTs is;

$y' = \frac{x^3}{2c}$.

Put value of c.

$y' = \frac{x^{\cancel{3}\,1}}{2y\cancel{x^2}}$

$y' = x/2y$.

$2y\,dy = x\,dx$.

$2\int y\,dy = \int x\,dx$

$$\boxed{y^2 = \frac{x^2}{2} + c.}$$

Required OTC.

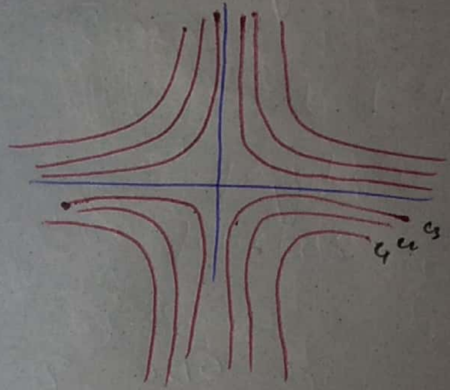
Note:- Both graphs should not touch origin.

Sketches:-

Graph of $y = c/x^2$.

As here $\frac{1}{x^2}$ so even if we give -ve value we will get +ve value for y

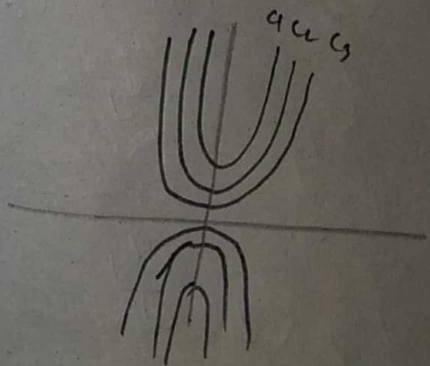
so;



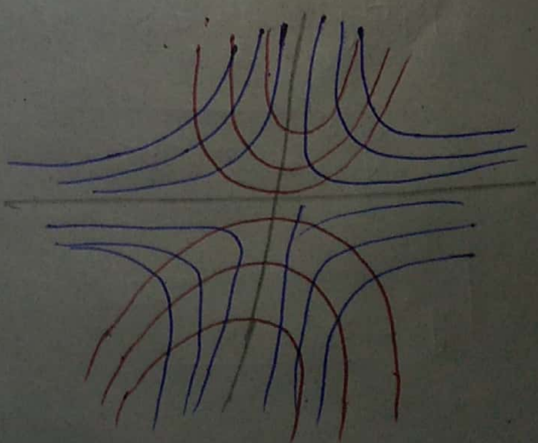
Graph of

$y = x^2/2 + c$.

As parabola so;



Combined graph:-



Q8:- $y = \sqrt{x+c}$.

Sol:- $y = \sqrt{x+c} \longrightarrow (1)$

$$y' = \frac{1}{2}(x+c)^{-1/2}$$

$$y' = \frac{1}{2(x+c)^{1/2}}$$

$$y' = \frac{1}{2\sqrt{x+c}} \longrightarrow (2)$$

Put of (1) in (2).

$$y' = \frac{1}{2y} \quad \text{(ODE)}.$$

So ODE of OTS is;

$$y' = -2y.$$

$$\frac{1}{y}\,dy = -2\,dx.$$

$$\int \frac{1}{y}\,dy = -2\int dx$$

$$\ln y = -2x + c.$$

$$\boxed{y = e^{-2x} + c}$$

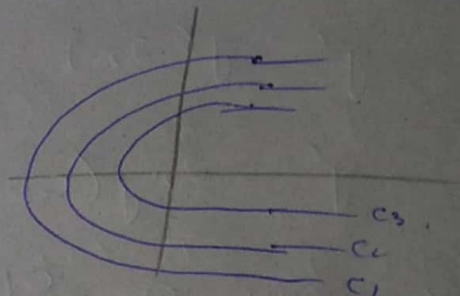
Required OTS.

Graph of $y = \sqrt{x+c}$.

$y = \sqrt{x+c}$

$y^2 = x + c$

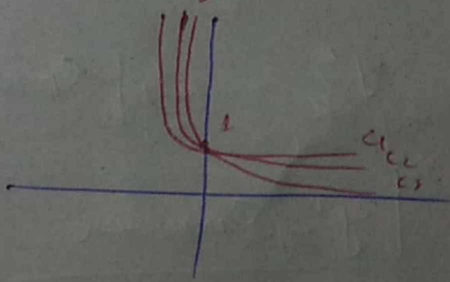
(Parabola with respect to x axis) as $a = 1$ so open to right.



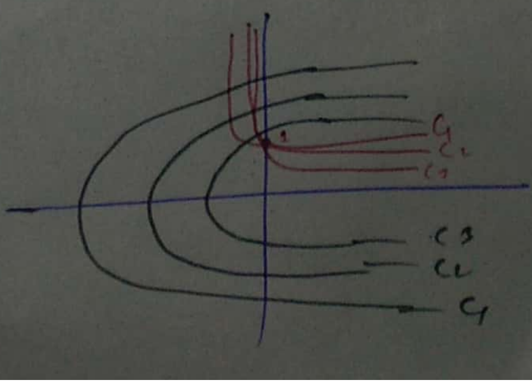
Not move to right after origin.

Graph of $y = e^{-2x} + c$

As for exponential function the graph will always passes through 1 so.



Combined graphs



Q9:- $y = ce^{-x^2}$.

Sol:- $y = ce^{-x^2} \longrightarrow ①$

$y' = ce^{-x^2}(-2x)$

$y' = ce^{-x^2}(-2x) \longrightarrow ②$

But Put eq ① in ②.

$y' = y(-2x)$

$y' = -2xy$ (ODE)

So ODE of OTS is;

$$\frac{dy}{dx} = \frac{1}{2xy}$$

$$y\,dy = \frac{1}{2x}\,dx$$

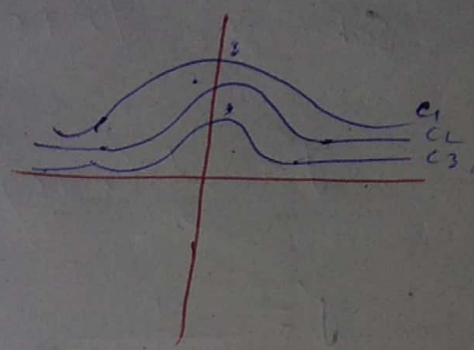
$$\int y\,dy = \frac{1}{2}\int \frac{1}{x}\,dx$$

$$\frac{y^2}{\cancel{2}} = \frac{1}{\cancel{2}}\left[\ln x + c\right]$$

$$\boxed{y^2 = \ln x + c}$$

(Required OTS)

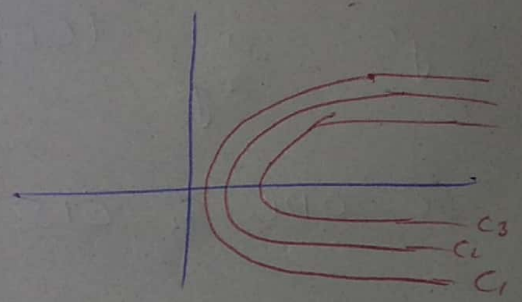
Graph of ce^{-x^2}



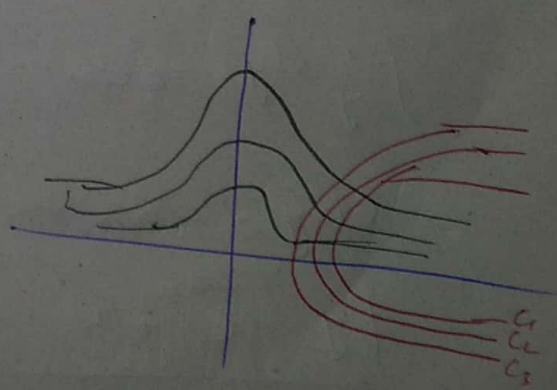
Graph of $y^2 = \ln x + c$

As parabolic function & $a = 1$.

So;



Combined graph



Q10:- $x^2+(y-c)^2=c^2$.

Sol:- $x^2+y^2+\cancel{c^2}-2yc=\cancel{c^2}$

$x^2+y^2-2yc=0$.

or

$\underbrace{x^2+y^2=2yc}_{①} \Rightarrow c=\dfrac{x^2+y^2}{2y}$.

Diff eq ①.

$2x+2yy'=2y'c$.

$x+yy'=y'c. \Rightarrow y'c-yy'=x$.

$y'(c-y)=x \Rightarrow y'=\dfrac{x}{c-y} \longrightarrow ②$.

Put value of c in eq ②.

$y'=\dfrac{x}{\dfrac{x^2+y^2}{2y}-y} \Rightarrow \dfrac{x}{\dfrac{x^2+y^2-2y^2}{2y}} \Rightarrow \dfrac{2xy}{x^2-y^2}$.

So the ODE is;

$$y'=\frac{2xy}{x^2-y^2} \longrightarrow ③$$

So OTS of ODE is;

$$\frac{dy}{dx} = \frac{-(x^2-y^2)}{2xy}.$$

$$2xy\,dy = -(x^2-y^2)\,dx.$$

$$(x^2-y^2)\,dx + 2xy\,dy = 0 \longrightarrow (4)$$

$P = M = (x^2-y^2) \qquad Q = N = 2xy.$

$P_y = M_y = -2y \qquad Q_x = N_x = 2y.$

As $N_x \neq M_y$ so not exact.

So find its I. Factor:

$$I.F = e^{\int \frac{P_y - Q_x}{Q} dx}.$$

$$= e^{\int \frac{-2y-2y}{2xy} dx} \Rightarrow e^{\int \frac{-4y}{2xy} dx} = e^{-2\int \frac{1}{x} dx}$$

$$= e^{-2\ln x} \Rightarrow e^{\ln x^{-2}} \Rightarrow \boxed{I.F = x^{-2}}$$

Multiply with eq (4).

$$(x^2-y^2)x^{-2}\,dx + 2xy\,x^{-2}\,dy = 0.$$

$$(1-x^{-2}y^2)\,dx + 2x^{-1}y\,dy = 0 \longrightarrow (5).$$

Now; $M = 1 - y^2/x^2. \qquad N = 2y/x.$

$M_y = -2y/x^2. \qquad N_x = -\frac{2y}{x^2}.$

As $\boxed{M_y = N_x}$ so exact.

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To find solution;

$$\frac{\partial u}{\partial x} = M.$$

$$\partial u = \left(1 - \frac{y^2}{x^2}\right)\partial x.$$

$$\int \partial u = \int 1\, \partial x - y^2 \int \frac{1}{x^2}\, \partial x.$$

$$u = x + \frac{y^2}{x} + C(y). \quad \text{--- (6)}$$

To find $C(y) = ?$

$$\frac{\partial u}{\partial y} = + \frac{2y}{x} + C'(y). \qquad \frac{\partial u}{\partial y} = N$$

$$N = \frac{2y}{x} + C'(y). \qquad N = \frac{2y}{x}$$

$$\cancel{\frac{2y}{x}} = \cancel{\frac{2y}{x}} + C'(y).$$

$$C'(y) = 0.$$

$$\int C'(y) = \int 0 \implies \boxed{C(y) = C}$$

eq (6) →

$$u = x + \frac{y^2}{x} + C$$

But $u = c_1$.

$$c_1 - c = x + \frac{y^2}{x}$$

$$x + \frac{y^2}{x} = c.$$

$$\frac{y^2}{x} = c - x.$$

$$y^2 = cx - x^2.$$

$$\boxed{y^2 + x^2 = cx}$$

Graph:-