Conversion to Systems. (Ex # 4.1)

Find the general solution of given ODE (a) by first converting it to a system (b) as given. show the detail of your work.

(10) $y'' + 4y' + 3y = 0. \longrightarrow ①$

Sol- Let $y = y_1$

$y_1' = y_2 \longrightarrow ③$

$y_1'' = y_2'$

eq ① ⇒ $y_2' + 4y_2 + 3y_1 = 0.$

$y_2' = -3y_1 - 4y_2 \longrightarrow ②$

From eq ① & eq ② we can write as;

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$y' = AY \longrightarrow ④$

its linear in y so,

so; Solution is;

$y = \underline{x} e^{\lambda t}$

$y' = \lambda \underline{x} e^{\lambda t} \longrightarrow ⑤$

Put eq ④ in ⑤

$AY = \underline{x} \lambda e^{\lambda t}$

$A\underline{x} e^{\lambda t} = \underline{x} \lambda e^{\lambda t}$

$A\underline{x} = \underline{x} \lambda$

where λ are
eigen vector.

$A\underline{X} - \underline{X}\lambda = 0$

$(A - \lambda I)\underline{X} = 0 \quad \text{—} \textcircled{A}$

for $\lambda \neq 0$

for A^{-1} not exist or solution not zero

det $= 0$.

To find value of λ.

So, $\det(A - \lambda I)\lambda = 0$.

$$\det\left(\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0.$$

$$\det\left(\begin{pmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{pmatrix}\right) = 0.$$

$-\lambda(-4-\lambda) + 3 = 0.$

$\lambda^2 + 4\lambda + 3 = 0$

$\lambda^2 + 3\lambda + \lambda + 3 = 0.$

$\lambda(\lambda+3) + 1(\lambda+3) = 0.$

$(\lambda+3)(\lambda+1) = 0.$

$\lambda = -3$ or $\lambda = -1$

eq $\textcircled{A} \Rightarrow$ $(A - \lambda I)\underline{x} = 0$.

$$\left(\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -\lambda & 1 \\ -3 & -4-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$-\lambda x_1 + x_2 = 0 \longrightarrow (B)$

$-3x_1 - (4+\lambda)x_2 = 0 \longrightarrow (C)$

Put $\lambda = -3$

$3x_1 + \cancel{x_2} = 0$

$-3x_1 + \cancel{x_2} = 0$

$\boxed{x_2 = 0}$

$x_1 = r$ when $r \in \mathbb{R}$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$x^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ when $\boxed{\lambda_1 = -3}$

Now eq (B) & (C) Put $\lambda = -1$.

multiply & add

$x_1 + x_2 = 0$

$-3x_1 - 2x_2 = 0$

$2x_1 + \cancel{2x_2} = 0$

$\boxed{x_1 = 0}$

$\boxed{x_2 = r}$ when $r \in \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} = r\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$x^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ when $\lambda = -1$

$Y_1 = x^{(1)} e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t}$ & $Y_2 = x^{(2)} e^{\lambda t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t}$

$\boxed{Y = c_1 e^{-3t} + c_2 e^{-t}}$ Ans

Q13:- $y'' + y' - 12y = 0$.

Sol:- $y'' + y' - 12 = 0$.

Let $y = y_1 \Rightarrow y_1'' + y_1' - 12y = 0 \longrightarrow (A)$

$y_1' = y_2 \longrightarrow (1)$

$y_2' = y_1''$.

eq (A) $\Rightarrow$ $y_2' + y_2 - 12y_1 = 0$.

$y_2' = 12y_1 - y_2 \longrightarrow (2)$

From eq (1) & (2) we can write as;

$y_1' = y_2$

$y_2' = 12y_1 - y_2$.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

$Y' = AY \longrightarrow (3)$.

As linear in Y so solution is;

$y = \underline{x} e^{\lambda t}$.

$y' = \underline{x} \lambda e^{\lambda t} \longrightarrow (4)$.

Put eq (3) in (4).

$Ay = \underline{x} \lambda e^{\lambda t}$ $\quad \because y = \underline{x} e^{\lambda t}$

$A \underline{x} \cancel{e^{\lambda t}} = \underline{x} \lambda \cancel{e^{\lambda t}}$

$A \underline{x} = \underline{x} \lambda$

$A \underline{x} - \underline{x} \lambda = 0$.

Ex # 4.1

$$(A - \lambda I)\underline{x} = 0. \rightarrow (A)$$

To find $\lambda = ?$

Consider.

$$\det(A - \lambda I) = 0.$$

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 12 & -1-\lambda \end{bmatrix}\right) = 0.$$

$$-\lambda(-1-\lambda) - 12 = 0.$$

$$\lambda^2 + \lambda - 12 = 0.$$

$$\lambda^2 - 3\lambda + 4\lambda - 12 = 0 \Rightarrow \lambda(\lambda - 3) + 4(\lambda - 3) = 0$$

$$\boxed{\lambda = 3} \quad \text{or} \quad \boxed{\lambda = -4}$$

So, ξ (A) $\Rightarrow$. $(A - \lambda I)\underline{x} = 0$.

To find eigen vector.

$$\left(\begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\begin{bmatrix} -\lambda & 1 \\ 12 & -1-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\cancel{-\lambda x_1 + x_2 = 0 \Rightarrow x_1 = 0}$$

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$$-\lambda x_1 + x_2 = 0 \longrightarrow (A)$$

$$12x_1 + (-1-\lambda)x_2 = 0$$

For ~~xq~~ $\lambda = 3$.

$$-3x_1 + x_2 = 0$$

$$\boxed{x_1 = \frac{1}{3} x_2}$$

let $x_2 = r$ when $r \in \mathbb{R}$.

$$x^{(1)} = \begin{bmatrix} \frac{1}{3} r \\ r \end{bmatrix} = r \cancel{\frac{1}{3}} \begin{bmatrix} \frac{1}{3} \\ \cancel{\frac{1}{3}}\ 1 \end{bmatrix}.$$

For $\lambda = -4$.

ξ (A) $\Rightarrow$

$$4x_1 + x_2 = 0.$$

$$x_1 = -\frac{1}{4} x_2.$$

let $x_2 = r$ where $r \in \mathbb{R}$.

$$x^{(2)} = \begin{bmatrix} -\frac{1}{4} r \\ r \end{bmatrix} = r \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix}.$$

So, the general solution is.

$$y = c_1 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -\frac{1}{4} \\ 1 \end{bmatrix} e^{-4t}$$

Ans

Z.

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$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & (3/2-\lambda) \end{bmatrix}\right) = 0.$$

$$-\lambda(3/2-\lambda) - 1 = 0.$$

$$\lambda^2 - 3/2\,\lambda - 1 = 0.$$

$$2\lambda^2 - 3\lambda - 2 = 0.$$

$$2\lambda^2 - 4\lambda + \lambda - 2 = 0$$

$$2\lambda(\lambda-2) + (\lambda-2) = 0$$

$$(\lambda-2)(2\lambda+1) = 0.$$

$$\boxed{\lambda = 2} \quad \& \quad \boxed{\lambda = -1/2}$$

Consider eq C.

$$(A - \lambda I)\underline{x} = 0.$$

$$\left(\begin{bmatrix} -\lambda & 1 \\ 1 & (3/2-\lambda) \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$-\lambda x_1 + x_2 = 0. \longrightarrow D$$

$$x_1 + (3/2-\lambda)x_2 = 0. \longrightarrow E$$

For $\boxed{\lambda = 2}$

Consider D $\Rightarrow$.

$$-2x_1 + x_2 = 0 \Rightarrow \boxed{x_1 = \frac{1}{2}x_2}$$

let $x_2 = r$ than

where $r \in \mathbb{R}$.

$$x^{(1)} = \begin{bmatrix} 1/2\, r \\ r \end{bmatrix}.$$

$$x^{(1)} = r\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}.$$

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Q11:- $2y'' - 3y' - 2y = 0$

Sol:- $y'' - 3/2\, y' - \frac{2}{2} y = 0 \longrightarrow (1).$

let

$y = y_1$

$\& \quad y_1' = y_2 \longrightarrow (2).$

than $y_1'' = y_2'$.

$\&$ (1) $\Rightarrow$

$y_2' - 3/2\, y_2 - y_1 = 0$

$y_2' = y_1 + 3/2\, y_2 \longrightarrow (3)$

From eq (2) $\&$ (3) we can write;

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So $y' = Ay \longrightarrow (A)$

Linear in y so sol. is

$y = \underline{x}\, e^{\lambda t}$.

$y' = \underline{x}\, \lambda\, e^{\lambda t} \longrightarrow (B)$

By comparing (A) $\&$ (B) we get.

$(A - \lambda I)\underline{x} = 0. \longrightarrow (C).$

To find $\lambda = ?$

Consider $\det(A - \lambda I) = 0$

Put $\lambda = -1/2$.

$\frac{1}{2} x_1 + x_2 = 0.$ let $x_2 = r$

$x_1 = -2x_2$ where $r \in \mathbb{R}$.

$$x^{(2)} = \begin{bmatrix} -2r \\ r \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So general eqn is;

$$\boxed{y = c_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-1/2 t}.}$$

Ans

Q_{12}:-

$$y''' - 2y'' - y' + 2y = 0 \longrightarrow ①.$$

Sol:- let $y = y_1$.

$y_1' = y_2$.

$y_2' = y_1'' \longrightarrow ①$

$y_2'' = y_1'''$.

$$y_2'' - 2y_2' - y_2 + 2y_1 = 0. \longrightarrow ②.$$