Lecture#02

Points:-

Operator del:-

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

Gradient of ϕ.

$$\text{Grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}.$$

Directional derivatives:-

Directional derivative of function $\phi(x,y,z)$ in direction of vector $\vec{u}$ is defined at Point P

as;

$$D.D = \vec{\nabla}\phi(P) \cdot \hat{u}.$$

EXP#09 Find the D.D of $\phi = x^2y^2z^2$ at point $(1,1,-1)$ in direction of tangent to the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.

Solution:- As;

$$D.D = \vec{\nabla}\phi\,(\text{Point}) \cdot \hat{u} \longrightarrow (1)$$

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(x^2y^2z^2)\hat{i} + \frac{\partial}{\partial y}(x^2y^2z^2)\hat{j} + \frac{\partial}{\partial z}(x^2y^2z^2)\hat{k}$$

Scanned with CamScanner

$= 4xy^2z^2\hat{i} + 2x^2yz\hat{j} + 2xy^2z\hat{k}$.

$\nabla\phi$

D.D (Point) $= \nabla\phi(1,1,-1) = 2\hat{i} + 2\hat{j} - 2\hat{k}$

Now as positional vector is;

$r = \hat{i}x + \hat{j}y + \hat{k}z$

By putting values of x, y & z from given.

$r = e^t\hat{i} + (\sin 2t + 2)\hat{j} + (1 - \cos 2t)\hat{k}$.

$$\vec{U} = \frac{d\vec{r}}{dt} = e^t\hat{i} + 2\cos 2t\hat{j} + 2\sin 2t\hat{k}$$

$\vec{U} = e^t\hat{i} + 2\cos 2t\hat{j} + \sin 2t\hat{k}$.

At $t = 0$

$\vec{U} = 1\hat{i} + 2\hat{j} + 0\hat{k}$

Unit vector is;

$$\hat{U} = \frac{\vec{U}}{|\vec{U}|} = \frac{1\hat{i} + 2\hat{j} + 0\hat{k}}{\sqrt{5}}$$

Put values in eq ①

$$D.D = (2\hat{i} + 2\hat{j} - 2\hat{k}) \cdot \frac{(1\hat{i} + 2\hat{j} + 0\hat{k})}{\sqrt{5}} \Rightarrow \frac{2+4}{\sqrt{5}}$$

$6/\sqrt{5}$ Ans

Example #10

Find The unit vector normal to surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

Sol:- As we know that;

normal vector is $\vec{\nabla}\phi$ to surface $\phi(x,y,z)$.

So

normal vector $= \vec{\nabla}\phi = ?$

$$\phi = xy^3z^2 - 4$$

$$\nabla\phi = \frac{\partial}{\partial x}(xy^3z^2-4)\hat{i} + \frac{\partial}{\partial y}(xy^3z^2-4)\hat{j} + \frac{\partial}{\partial z}(xy^3z^2-4)\hat{k}$$

$$= \hat{i}y^3z^2 + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

Normal vector at $(-1,-1,2)$ is;

$$= 4\hat{i} - 12\hat{j} + 4\hat{k}$$

Unit vector normal to surface;

$$= \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{4\hat{i}-12\hat{j}+4\hat{k}}{\sqrt{16+144+16}}$$

$$= \frac{4\hat{i}-12\hat{j}+4\hat{k}}{\sqrt{176}} = \frac{4\hat{i}-12\hat{j}+4\hat{k}}{\sqrt{16}.\sqrt{11}}$$

$$= \frac{4\hat{i}-12\hat{j}+4\hat{k}}{4\sqrt{11}} \Rightarrow \frac{\hat{i}-3\hat{j}+\hat{k}}{\sqrt{11}}$$ Ans

Example # 11

Note:- Rate of change = Directional derivative.

Find the rate of change of $\phi = xyz$ in the direction normal to surface $x^2y + y^2x + yz^2 = 3$ at $(1,1,1)$.

Sol:- As given that;

$$\phi = xyz$$

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (xyz)$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}$$

$$\vec{\nabla}\phi \text{ (Point)} = \hat{i} + \hat{j} + \hat{k}$$

ψ = Normal to surface $= x^2y + y^2x + yz^2 - 3$.

$$\vec{\nabla}\psi = \nabla \text{ N.T.S} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2y + y^2x + yz^2 - 3\right)$$

$$\vec{\nabla}\psi = (2xy + y^2)\hat{i} + (x^2 + 2xy + z^2)\hat{j} + 2yz\hat{k}$$

Normal to surface at $(1,1,1)$

$$\vec{\nabla}\psi \text{ (Points)} = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\text{unit normal} = \frac{\vec{\nabla}\psi \text{ (Point)}}{|\vec{\nabla}\psi|}$$

$$= \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{3^2 + 4^2 + 2^2}} \Rightarrow \frac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{29}}$$

So,

$$\text{rate of change} = \vec{\nabla}\phi \cdot \frac{\overrightarrow{\vec{\nabla}\psi \text{ (Point)}}}{|\vec{\nabla}\psi|}$$

By putting values

$$= (\hat{i}+\hat{j}+\hat{k}) \cdot \frac{(3\hat{i}+4\hat{j}+2\hat{k})}{\sqrt{29}}$$

$$= \frac{3+4+2}{\sqrt{29}} = \frac{9}{\sqrt{29}} \text{ Ans}$$

Example #12

Find the constant m & n such that; surface $mx^2 - 2nyz = (m+4)x$ will be orthogonal to surface $4x^2y + z^3 = 4$ at point $(1, -1, 2)$.

Solution:- To find $n = ?$

As the point lies on both surfaces. So put in first one;

$$mx^2 - 2nyz = (m+4)x$$

$$m(1)^2 - 2(n)(-1)(2) = (m+4)(1)$$

$$\cancel{m} + 4n = \cancel{m} + 4$$

$$4n = 4 \Rightarrow \boxed{n = 1}$$

To find $m = ?$

Let $\phi_1 = mx^2 - 2nyz - (m+4)x$ & $\phi_2 = 4x^2y + z^3 - 4 = 0$

Normal to $\phi_1 = mx^2 - 2nyz - (m+4)x$ is:

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(mx^2 - 2nyz - (m+4)x\right)$$

Scanned with CamScanner

$= [2mx - (m+4)]\hat{i} - 2z\hat{j} - 2y\hat{k}$

$= (2mx - m - 4)\hat{i} - 2z\hat{j} - 2y\hat{k}$

Normal to ϕ_1 at $(1,-1,2)$.

$= (2m - m - 4)\hat{i} - 2(2)\hat{j} - 2(-1)\hat{k}$

$= (m-4)\hat{i} - 4\hat{j} + 2\hat{k} \longrightarrow (x)$

Normal to ϕ_2 is;

$\vec{\nabla}\phi_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2y + z^3 - 4)$

~~$3x\hat{i} + 2y\hat{j}$~~

$= 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

Normal to ϕ_2 at $(1,-1,2)$ is;

$= -8\hat{i} + 4\hat{j} + 12\hat{k}$

Since ~~normal~~ ϕ_1 & ϕ_2 are perpendicular to each other so their normal also perpendicular

$\boxed{\text{As for orthogonal vector dot product} = 0}$

$\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 = 0$

$((m-4)\hat{i} - 4\hat{j} + 2\hat{k}) \cdot (-8\hat{i} + 4\hat{j} + 12\hat{k}) = 0$

$-8(m-4) - 4(4) + 2(12) = 0$

$-8m + 32 - 16 + 24 = 0$

$8m = 40 \Rightarrow \boxed{m = 5}$

Hence $n = 1$ & $\boxed{m = 5}$ Ans.

Example # 13

Find the values of constants λ & μ so that the surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$, $4x^2y + z^3 = 4$ intersect orthogonally at point $(1, -1, 2)$.

Sol:- As given that;

$$\lambda x^2 - \mu yz = (\lambda+2)x.$$

The point $(1, -1, 2)$ satisfy as it lies on it

$$\cancel{\lambda} + 2\mu = \cancel{\lambda} + 2$$

$$2\mu = 2 \Rightarrow \boxed{\mu = 1}$$

To find $\lambda = ?$

Let $\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x.$

& $\phi_2 = 4x^2y + z^3 - 4.$

Normal to $\phi_1 = \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\lambda x^2 - \mu yz - (\lambda+2)x\right)$

$$= (2\lambda x - \lambda - 2)\hat{i} - \mu z\hat{j} - \mu y\hat{k}.$$

Normal at $(1, -1, 2)$

$$= (2\lambda - \lambda - 2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}.$$

$$= (\lambda - 2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}. \longrightarrow (1).$$

Normal to $\phi_2 = \vec{\nabla}\phi_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(4x^2y + z^3 - 4\right)$

$$= 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}.$$

Normal to ϕ_2 at $(1,-1,2)$.

$$= -8\hat{i} + 4\hat{j} + 12\hat{k} \longrightarrow ②$$

As, ϕ_1 & ϕ_2 are orthogonal so their normals are also orthogonal.

[Normal to ϕ_1]. [Normal to ϕ_2] = 0.

$$[(\lambda-2)\hat{i} - 2\mu\hat{j} + \mu\hat{k}].[-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0$$

$$-8(\lambda-2) - 8\mu + 12\mu = 0.$$

$$-8\lambda + 16 - 8\mu + 12\mu = 0.$$

$$-8\lambda + 16 + 4\mu = 0.$$

$$\because \mu = 1$$

$$-8\lambda = -20(1)$$

$$\lambda = \frac{20}{8}$$

$$\boxed{\lambda = \frac{5}{2}}$$ Ans

Scanned with CamScanner

Example # 14

Find the angle between surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at point $(2,-1,2)$.

Sol:- Since

let $\phi_1 = x^2+y^2+z^2-9$.

& $\phi_2 = x^2+y^2-z-3$

As; $\phi_1 \cdot \phi_2 = \phi_1 \phi_2 \cos\theta$

$$\cos\theta = \frac{|\phi_1||\phi_2|}{\phi_1 \phi_2} \longrightarrow (i)$$

Normal vector to $\phi_1 = \nabla\phi_1$.

$$\nabla\phi_1 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Normal to ϕ at $(2,-1,2)$

$$= 4\hat{i} - 2\hat{j} + 4\hat{k} \longrightarrow (ii)$$

Normal to $\phi_2 = \nabla\phi_2$

$$\nabla\phi_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$$= 2x\hat{i} + 2y\hat{j} - 1\hat{k}$$

Normal at $(2,-1,2)$.

$$= 4\hat{i} - 2\hat{j} - \hat{k}.$$

$|\phi_1| = \sqrt{(4)^2+(-2)^2+(4)^2} \Rightarrow \phi_1 = \sqrt{36} = \phi_1 = 6$

$|\phi_2| = \sqrt{(4)^2+(-2)^2+(-1)^2} = \sqrt{21}$

By putting values in eq ①

$$\cos\theta = \frac{(4\hat{i}-2\hat{j}+4\hat{k}).(4\hat{i}-2\hat{j}-\hat{k})}{6\sqrt{21}}$$

$$= \frac{16+(+4)-4}{6\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right) \quad \underline{\underline{Ans}}$$

Example # 15

Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$, where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Sol:- As Directional derivative is;

$$D.D = \nabla\phi . \hat{r} \quad \longrightarrow ①$$

Here $\phi = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$

so, $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left((x^2+y^2+z^2)^{-1/2}\right)$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{i}\,\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}(x^2+y^2+z^2) + \hat{j}\left(\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial y}(x^2+y^2+z^2)\right)$$

$$+ \hat{k}\left(\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial z}(x^2+y^2+z^2)\right)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)\hat{i} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y)\hat{j} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z)$$

$$= \frac{-x\hat{i}}{(x^2+y^2+z^2)^{3/2}} - \frac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} \quad \frac{-\hat{k}z}{(x^2+y^2+z^2)^{3/2}}$$

$$-\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} \longrightarrow (1)$$

As, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\hat{r} = \frac{\vec{r}}{|r|} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{1/2}} \longrightarrow (2).$$

By putting values.

$3/2 + 1/2$

$\frac{3+1}{2} = \frac{4}{2} = 2$

$$\nabla\phi\cdot\hat{r} = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}}\cdot\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{1/2}}$$

$$= -\frac{\cancel{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^{\cancel{2}}} = -\frac{1}{(x^2+y^2+z^2)}$$

$x^2+y^2+z^2 = r^2$

$\boxed{D.D = -1/r^2}$ Ans.

Example # 16

Find the direction in which the directional derivative of $\phi(x,y) = \frac{x^2+y^2}{xy}$ at $(1,1)$ is zero and hence find out components of velocity of vector $\vec{r} = (t^3+1)\hat{i} + t^2\hat{j}$ in same direction at $t=1$.

Sol:- As given That;

$$\phi = \frac{x^2+y^2}{xy}$$

To find direction of directional derivative:?

Directional derivative of $\phi = \vec{\nabla}\phi$.

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x^2+y^2}{xy}\right).$$

$$= \left[\frac{xy(2x) - (x^2+y^2)(y)}{(xy)^2}\right]\hat{i} + \hat{j}\left[\frac{xy(2y) - (y^2+x^2)x}{(xy)^2}\right].$$

$$= \left(\frac{2x^2y - x^2y - y^3}{x^2y^2}\right)\hat{i} + \left(\frac{2xy^2 - xy^2 - x^3}{x^2y^2}\right).$$

$$= \left(\frac{x^2y - y^3}{x^2y^2}\right)\hat{i} + \left(\frac{xy^2 - x^3}{x^2y^2}\right)\hat{j}.$$

Direction derivative at $(1,1)$ Put $x=1, y=1$

$$= \frac{1-1}{1}\hat{i} + \frac{1-1}{1}\hat{j} = 0\hat{i} + 0\hat{j} = 0$$

Since $(\nabla\phi)_{(1,1)} = 0$ the directional derivative

$\therefore$ d at ~~any~~ (1, 1) is zero at any direction.

Now; Since; Given That;

$$\vec{r} = (t^3+1)\hat{i} + t^2\hat{j}$$

To find velocity at $t = 1$.

$$\frac{d\vec{r}}{dt} = \vec{v} = \hat{i}\,3t^2 + 2t\hat{j} \qquad \text{at } t = 1.$$

$$\vec{v} = 3\hat{i} + 2\hat{j}$$

Component of velocity in same direction;

$$\text{Component} = \vec{v} \cdot \frac{\vec{v}}{|\vec{v}|} \rightarrow ①$$

$$|\vec{v}| = \sqrt{9+4} = \sqrt{13}$$

By putting values

$$= (3\hat{i}+2\hat{j}) \cdot \frac{(3\hat{i}+2\hat{j})}{\sqrt{13}}$$

$$= \frac{9+4}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \frac{\sqrt{13}\cdot\cancel{\sqrt{13}}}{\cancel{\sqrt{13}}}$$

$$= \sqrt{13} \quad \text{Ans}$$

Example # 17

Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,1)$ in direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find greatest rate of increase of ϕ.

$\therefore$ Greatest rate of Increase $= |\text{D.Derivative}|$.

Solution:- Since;

$$\phi = x^2yz + 4xz^2$$

~~Directional~~ del $\phi = \nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2yz + 4xz^2)$

$$= \frac{\partial}{\partial x}(x^2yz + 4xz^2)\hat{i} + \frac{\partial}{\partial y}(x^2yz + 4xz^2)\hat{j} + \frac{\partial}{\partial z}(x^2yz + 4xz^2)\hat{k}.$$

$$= (2xyz + 4z^2)\hat{i} + \hat{j}x^2z + (x^2y + 8xz)\hat{k}.$$

$\nabla\phi$ at $(1,-2,1)$

$\nabla\phi(1,-2,1) = (-4+4)\hat{i} + \hat{j} + 6\hat{k}$.

$= 0\hat{i} + \hat{j} + 6\hat{k}$.

$= \hat{j} + 6\hat{k} \longrightarrow (1)$

$\text{D.D} = \nabla\phi(P) \cdot \hat{u} \longrightarrow (A)$

$\vec{u} = 2\hat{i} - \hat{j} - 2\hat{k}$.

$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-1)^2}} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}}$

$\hat{u} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3} \rightarrow ②$

By putting values in eq ①

$D.D = \hat{j} + 6\hat{k} \cdot \left(\frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}\right)$

$= \frac{-1-12}{3} = \frac{-13}{3}$

$$\boxed{D.D = \frac{-13}{3}}$$

Greatest rate of Increase $= |D.D|_{(Point)}$

$(D.D)(P) = \hat{j} + 6\hat{k}$

$= \sqrt{(1)^2 + (6)^2}$

$= \sqrt{1+36}$

$= \sqrt{37}$ Ans.

Example #18

Find the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at Point $P(1,2,3)$ in direction of line PQ where Q is $(5,0,4)$

Sol: As $D \cdot D = \nabla\phi(\text{Point}) \cdot \hat{u} \rightarrow ①$

$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (x^2 - y^2 + 2z^2)$

Scanned with CamScanner

$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$\Rightarrow \nabla\phi \text{ (Point)} = \nabla\phi(1,2,3)$

$\nabla\phi(\text{Point}) = 2\hat{i} - 4\hat{j} + 12\hat{k} \longrightarrow \text{(ii)}$

Since; Given that;

$P = (1,2,3)$ & $Q = (5,0,4)$

so, $\vec{PQ} = (4,-2,1)$

so, $\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

$$\hat{PQ} = \hat{u} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{(4)^2 + (-2)^2 + (1)^2}}$$

$$= \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$$\hat{PQ} = \hat{u} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}} \longrightarrow \text{(iii)}$$

By putting values in eq (i)

$$D.D = (2\hat{i} - 4\hat{j} + 12\hat{k}) \cdot \frac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}}$$

$$\boxed{D.D = \frac{28}{\sqrt{21}}} \text{ Ans}$$

Scanned with CamScanner

Example # 19

Find directional derivative of $\phi = 4e^{2x-y+z}$ at point $(1,1,-1)$ in directional towards the point $(-3,5,6)$.

Sol:- Since D.D $= \nabla\phi(\text{Point})\,\hat{u}$ $\longrightarrow$ ①.

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4e^{2x-y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(4e^{2x-y+z}\right)\hat{i} + \frac{\partial}{\partial y}\left(4e^{2x-y+z}\right)\hat{j} + \frac{\partial}{\partial z}\left(4e^{2x-y+z}\right)\hat{k}$$

$$= 4\left[e^{2x-y+z}\cdot 2\hat{i} + e^{2x-y+z}(-1)\hat{j} + e^{2x-y+z}(1)\hat{k}\right]$$

$$= 4\left[2e^{2x-y+z}\hat{i} - e^{2x-y+z}\hat{j} + e^{2x-y+z}\hat{k}\right]$$

$\nabla\phi(1,1,-1)$

$$\nabla\phi(1,1,-1) = 4\left[2e^{0}\hat{i} - e^{0}\hat{j} + e^{0}\hat{k}\right)$$

$$= 4\left(2\hat{i} - \hat{j} + \hat{k}\right) = 8\hat{i} - 4\hat{j} + 4\hat{k} \rightarrow ②$$

As $P_1(1,1,-1)$ & $P_2(-3,5,6)$

$$\overrightarrow{P_1P_2} = \vec{U} = (-4, 4, 7)$$

$$\overrightarrow{P_1P_2} = \vec{U} = -4\hat{i} + 4\hat{j} + 7\hat{k}$$

Scanned with CamScanner

unit vector $= \hat{U} = \dfrac{\vec{U}}{|\vec{U}|}$.

$$= \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{\sqrt{(-4)^2+(4)^2+(7)^2}}$$

$$= \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{\sqrt{81}} \longrightarrow (2)$$

By putting values in eq (1).

$$D.D = \left| \frac{(8\hat{i}-4\hat{j}+4\hat{k}).(-4\hat{i}+4\hat{j}+7\hat{k})}{\sqrt{81}} \right|$$

$$= \left| \frac{-32-16+28}{9} \right|$$

$$D.D = \left| \frac{-20}{9} \right|$$

$$= \frac{20}{9} \text{ Ans}$$

Note:-

∴ Direction derivative should always be +ve.

Example # 20

For the function $\phi(x,y) = \frac{x}{x^2+y^2}$, find the magnitude of D.D along a line making an 30° with +ve x-axis at (0, 2).

Sol:- Since $\phi = \frac{x}{x^2+y^2}$

Required Ans

$$D.D = \nabla\phi(\text{Point})\ \hat{u} \longrightarrow ①$$

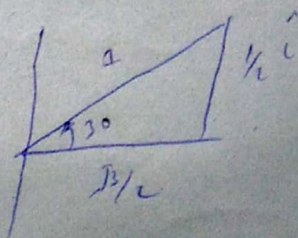
$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{x^2+y^2}\right)$$

$$= \left(\frac{(x^2+y^2) - 2x \cdot x}{(x^2+y^2)^2}\right)\hat{i} + \hat{j}\left(\frac{0 - x(2y)}{(x^2+y^2)^2}\right)$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\hat{i} - \frac{2xy\,\hat{j}}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}\hat{i} - \frac{2xy\,\hat{j}}{(x^2+y^2)^2} \longrightarrow ⑪$$

$$\underset{(0,2)}{\nabla\phi(\text{Point})} = \frac{4\hat{i}}{16} - 0 = \frac{1}{4}\hat{i} \longrightarrow ③$$



As it is in Two Components x & y.

So, $x = \cos\theta$.

$x = \cos 30° = \sqrt{3}/2\ \hat{i}$, $y = \sin\theta = \sin 30° = 1/2\ \hat{j}$

$\vec{U} = x\hat{i} + y\hat{j}$ (vector in direction of $\hat{i}$ & $\hat{j}$)

Putting values.

$$\vec{U} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$\hat{U} = \frac{\vec{U}}{|\vec{U}|} = \frac{\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}}{\sqrt{(\sqrt{3}/2)^2 + (1/2)^2}}$$

$$\hat{U} = \frac{\frac{\sqrt{3}\hat{i} + 1\hat{j}}{2}}{\sqrt{3/4 + 1/4}} \longrightarrow ②$$

By putting values in eq ①

$$\nabla\phi(\text{Point})\,\hat{U} = \frac{\frac{1}{4}\hat{i}\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{\hat{j}}{2}\right)}{\sqrt{\frac{3+1}{4}}}$$

$$= \frac{\sqrt{3}}{8} + 0 / \sqrt{1}$$

$$= \frac{\sqrt{3}}{8} \text{ Ans}$$

Scanned with CamScanner

Example #21

Find the directional derivative of $\vec{V}^2$, where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at point $(2,0,3)$ in direction of outward normal to sphere $x^2+y^2+z^2 = 14$ at point $(3,2,1)$.

Sol:- As; $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$

$$\vec{V}.\vec{V} = (xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}).(xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k})$$

$$\vec{V}^2 = x^2y^4 + z^2y^4 + x^2z^4$$

$$D.D = \nabla\vec{V}^2(\text{Point}).\hat{u} \longrightarrow (A)$$

$$\nabla\vec{V}^2 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4z^2y^3)\hat{j} + (2zy^4 + 4x^2z^3)\hat{k}$$

$\nabla\vec{V}^2(\text{Point})$ at $(2,0,3)$

$$= 0 + 324\hat{i} + (0+0)\hat{j} + (0+432)\hat{k}$$

$$= 324\hat{i} + 432\hat{k} \longrightarrow (1)$$

Find normal to $u = x^2+y^2+z^2 = 14$

$$\nabla\vec{u} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2+z^2 = 14)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Normal vector at $(3,2,1)$ is:

$\vec{u}(3,2,1) = 6\hat{i} + 4\hat{j} + 2\hat{k}$.

$\dfrac{\vec{U}}{|\vec{U}|} = \hat{U}$

$\hat{U} = \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{6^2+4^2+2^2}}$

$= \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}} = \dfrac{6\hat{i}+4\hat{j}+2\hat{k}}{2\sqrt{14}}$

$= \dfrac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$

Put values in eq ①.

$D.D = (324\hat{i}+432\hat{k}) \cdot \dfrac{(3\hat{i}+2\hat{j}+\hat{k})}{\sqrt{14}}$

$= \dfrac{972 + \overset{432}{\cancel{864}}}{\sqrt{14}}$

$D.D. = \dfrac{1404}{\sqrt{14}}$ Ans

Example # 22

Find the directional derivative of $\nabla(\nabla f)$ at point $(1,-2,1)$ in direction of normal to surface $xy^2z = 3x+z^2$, where $f = 2x^3y^2z^4$.

Sol:- As given that;

$$f = 2x^3y^2z^4$$

we know that;

$$D.D \text{ of } \nabla(\nabla f) = \nabla(\nabla(\nabla f))(\text{Point})\,\hat{n} \longrightarrow ①$$

$$\vec{\nabla} f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\hat{k}\right)(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\hat{i} + \hat{j}\,4x^3yz^4 + 8x^3y^2z^3\hat{k}$$

$$\vec{\nabla}(\vec{\nabla} f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}\right)$$

$$= \cancel{(12xy^2z^4 + 12x^2yz^4)\hat{i} + (}$$

$$= \frac{\partial}{\partial x}\hat{i}(6x^2y^2z^4)\hat{i} + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Now to find direction derivative of $\nabla(\nabla f)$

$$\nabla(\nabla(\nabla f)) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2\right)$$

$$= (12xy^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz^4 + 48x^3yz^2)\hat{j}$$

$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

Scanned with CamScanner

Let $\nabla\phi = \nabla(\nabla(\nabla f))$

$\nabla\phi$ at point $(1,-2,1)$ is,

$= (48+12+288)\hat{i} + (-48-96)\hat{j} + (192+16+192)\hat{k}$

$= 348\hat{i} - 144\hat{j} + 400\hat{k} \longrightarrow$ (ii)

Normal to surface $\psi = xy^2z - 3x - z^2$ is,

$= \nabla(\psi)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$.

$= (y^2z-3)\hat{i} + (2xzy)\hat{j} + (xy^2 - 2z)$.

Normal to surface at $(1,-2,1)$.

$= \hat{i} - 4\hat{j} + 2\hat{k}$

Unit normal vector $= \hat{U} = \frac{\text{Normal vector}}{|\text{Normal vector}|}$

$\hat{U} = \frac{1 - 4\hat{j} + 2\hat{k}}{\sqrt{21}}$

By putting values.

$D.D(\nabla(\nabla f)) = 348\hat{i} - 144\hat{j} + 400\hat{k} \cdot \frac{(1 - 4\hat{j} + 2\hat{k})}{\sqrt{21}}$

$= \frac{348 + 576 + 800}{\sqrt{21}}$

$= \frac{1724}{\sqrt{21}}$ Ans

Example # 23

If directional derivative of $\phi = ax^2y + by^2z + cz^2x$ at point $(1,1,1)$ has max. magnitude 15 in direction parallel to line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$.

Find values of a, b & c.

Sol:- $D.D = \nabla\phi(\text{point})\,\hat{u} \longrightarrow ①$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(ax^2y + by^2z + cz^2x)$$

$$= (2axy + cz^2)\hat{i} + (ax^2 + 2byz)\hat{j} + (by^2 + 2czx)$$

$\nabla\phi$ at point $(1,1,1)$

$$= (2a+c)\hat{i} + (a+2b)\hat{j} + (b+2c)\hat{k} \hookrightarrow Ⓐ$$

We know $\max|\nabla\phi| = 15$. (Given)

$$|\nabla\phi| = (2a+c)^2 + (2b+a)^2 + (b+2c)^2 = (15)^2 \hookrightarrow ②$$

D.D given to parallel line b;

$$\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$$

So, parallel line b; $2\hat{i} - 2\hat{j} + \hat{k} = 0$ $\hookrightarrow Ⓑ$

On comparing co-efficient of Ⓐ & Ⓑ

$$\frac{2a+c}{2} = \frac{2b+a}{-2} = \frac{2c+b}{1}$$

Scanned with CamScanner

Comparing each two to get two eqns.

$$\frac{2a+c}{\cancel{2}} = \frac{2b+a}{-\cancel{2}}$$

$$2a+c = -2b-a$$

$$2a+2b+a+c=0$$

$$3a+2b+c=0 \longrightarrow 3$$

Also;

2 $$\frac{2b+a}{-2} = \frac{2c+b}{1}$$

$$2b+a = -4c-2b$$

$$a+2b+4c+2b=0$$

$$a+\overset{4}{\cancel{3}}b+4c=0 \longrightarrow 4$$

Rewriting eq 3 & 4.

Rough work
Writing coefficient
of eq 3 & 4

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{matrix} 2 & 1 & 3 & 2 \\ 4 & 4 & 1 & 4 \end{matrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\frac{a}{8-4} = \frac{b}{1-12} = \frac{c}{12-2} = k$$

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$$

$$a=4k, \quad b=-11k, \quad c=10k \longrightarrow C$$

Put these values in eq 1.

$$\text{eq } 1 \Rightarrow (2a+c)^2+(2b+a)^2+(2c+b)^2=(15)^2$$

$$(5k+10k)^2+(-22k+4k)^2+(20k-11k)^2=(15)^2$$

$$(15k)^2+(-18k)^2+(9k)^2=(15)^2$$

$$729k^2=225$$

$$k^2=\frac{225}{729} \Rightarrow k=\pm\frac{15}{27}$$

$$\boxed{k=\frac{\pm 5}{9}}$$

& ©

$\Rightarrow$

$$a=4(\pm 5/9) \Rightarrow \boxed{\frac{\pm 20}{9}=a}$$

$$b=-11(\pm 5/9)= \pm 55/9=b$$

$$c=10k=10(\pm 5/9) \Rightarrow \boxed{\pm 50/9=c}$$ Ans

Example #24

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, show that

(i) grad $r=\frac{\vec{r}}{r}$ (ii) Grad$\left(\frac{1}{r}\right)=-\frac{\vec{r}}{r^3}$.

Sol:- ① grad $r=\frac{\vec{r}}{r}$.

grad $r=\nabla r=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{j}\right).r$

$$\text{grad } r = \left(i\frac{\partial}{\partial x} r + \frac{\partial}{\partial y} r\hat{j} + \frac{\partial}{\partial z} r\hat{k} \right) \rightarrow ①$$

Since,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|r| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r^2}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2) = 2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \rightarrow ②$$

$$\frac{\partial r^2}{\partial y} = 2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \rightarrow ③$$

$$\frac{\partial r^2}{\partial z} = \frac{2r\,\partial r}{\partial z} = 2z = \frac{\partial r}{\partial z} = \frac{z}{r} \rightarrow ④$$

Put eq ②, ③, ④ in ①.

$$\Rightarrow \left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k} \right) \Rightarrow \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{r}$$

$$\boxed{\text{grad } r = \vec{r}/r}$$ Proved

(ii) $\text{grad}\left(\frac{1}{r}\right) = -\vec{r}/r^3$

$$\Rightarrow \text{grad}\left(1/r\right) = \nabla\left(1/r\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \left(\frac{\partial}{\partial x}\frac{1}{r}\hat{i}\right) + \left(\frac{\partial}{\partial y}\frac{1}{r}\hat{j}\right) + \left(\frac{\partial}{\partial z}\frac{1}{r}\hat{k}\right)$$

$$= \left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)\hat{i} + \left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)\hat{j} + \left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)\hat{k}$$

By putting values.

$$= -\frac{1}{r^2}\left(\frac{x}{r}\right)\hat{i} - \frac{1}{r^2}\left(\frac{y}{r}\right)\hat{j} - \frac{1}{r^2}\left(\frac{z}{r}\right)\hat{k}$$

$$= -\frac{x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}$$

$$= \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{r^3} = \frac{-\vec{r}}{r^3}$$ $\boxed{\text{proved}}$

Example # 25

Prove that;

$$\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$$

Sol:- Take L.H.S.

$$L.H.S = \nabla^2 f(r) \longrightarrow (1)$$

$$\nabla \cdot f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \hat{i}\frac{\partial}{\partial x}f(r) + \hat{j}\frac{\partial}{\partial y}f(r) + \hat{k}\frac{\partial}{\partial z}f(r) \longrightarrow (2)$$

$$r^2 = x^2 + y^2 + z^2$$

$$\partial r^2/\partial x = 2x \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \longrightarrow (3)$$

Scanned with CamScanner

$$\frac{\partial}{\partial y} r^2 = \frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$\frac{\cancel{2}r\,\partial r}{\partial y} = \cancel{2}y \Rightarrow \frac{\partial r}{\partial y} = y/r \rightarrow (4)$$

$$\frac{\partial r^2}{\partial z} = 2z \Rightarrow \frac{\cancel{2}r\,\partial r}{\partial z} = \cancel{2}z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r} \rightarrow (5)$$

~~By putting values~~.

& (1) $\Rightarrow$ $\vec{\nabla} f(r) = \hat{i}\frac{\partial}{\partial x} f(r) + \hat{j}\frac{\partial}{\partial y} f(r) + \hat{k}\frac{\partial}{\partial z} f(r)$

$$= f'(r)\frac{\partial r}{\partial x} + f'(r)\frac{\partial r}{\partial y} + f'(r)\frac{\partial r}{\partial z} \rightarrow (6)$$

Now put values from eq (6), (3), & (4) & (5)

$$= f'(r)\left(\frac{x}{r}\right)\hat{i} + f'(r)\frac{y}{r}\hat{j} + f'(r)\left(\frac{z}{r}\right)\hat{k}.$$

$$= f'(r)\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) = f'(r)\left[\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right]$$

Now;

$$\nabla(\nabla f(r)) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(f'(r)\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right)\right)$$

$$= \frac{\partial}{\partial x}\left[f'(r)\left(\frac{x}{r}\right)\right] + \frac{\partial}{\partial y}\left[f'(r)\left(\frac{y}{r}\right)\right]\hat{j} + \frac{\partial}{\partial z}\left(f'(r)\frac{z}{r}\right)\hat{k}.$$

: use product rule { for second in division form use quotient r

$$= \left(\frac{x}{r} f''(r)\cdot\frac{\partial r}{\partial x}\right) + f'(r)\frac{\left(r(1) - x\cdot\frac{\partial r}{\partial x}\right)}{r^2} + \left(f''(r)\cdot\frac{\partial r}{\partial y}\cdot\frac{y}{r}\right)$$

$$+ f'(r)\frac{\left(r(1) - y\frac{\partial r}{\partial y}\right)}{r^2} + f''(r)\frac{\partial r}{\partial z}\left(\frac{z}{r}\right) + f'(r)\left(\frac{r(1) - z\frac{\partial r}{\partial z}}{r^2}\right.$$

$$= f''(r)\left(\frac{x}{r}\right)\left(\frac{x}{r}\right) + f'(r)\left(\frac{r-\frac{x^2}{r}}{r^2}\right) + f''(r)\left(\frac{y}{r}\right)\left(\frac{y}{r}\right) + f'(r)\,\frac{r-\frac{y^2}{r}}{r^2}$$

$$+ \left(f''(r)\cdot\frac{z}{r}\right)\left(\frac{z}{r}\right) + f'(r)\,\frac{r-z^2/r}{r^2}.$$

The above is obtain by putting values

of $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$ & $\frac{\partial r}{\partial z}$.

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{r^2-x^2}{r^3}\right) + f''(r)\frac{y^2}{r^2} + f'(r)\frac{r^2-y^2}{r^3}$$

$$+ f''(r)\cdot\frac{z^2}{r^2} + f'(r)\frac{r^2-z^2}{r^3}.$$

$$= f''(r)\frac{x^2}{r^2} + f''(r)\frac{y^2}{r^2} + f''(r)\frac{z^2}{r^2} + f'(r)\left(\frac{r^2-x^2}{r^3}\right) + f'(r)\left(\frac{r^2-y^2}{r^3}\right) + f'(r)\left(\frac{r^2-z^2}{r^3}\right)$$

$$\boxed{= f''(r)\frac{x^2}{r^2} + f''(r)\frac{y^2}{r^2} + f''(r)\frac{z^2}{r^2} + f'(r)\frac{y^2+z^2}{r^3} + f'(r)\left(\frac{x^2+z^2}{r^3}\right) + f'(r)\left(\frac{x^2+y^2}{r^3}\right)}$$

$$= f''(r)\left[\frac{x^2+y^2+z^2}{r^2}\right] + f'(r)\left(\frac{+x^2+y^2+x^2+y^2+z^2+z^2}{r^3}\right)$$

$r^2 = x^2+y^2+z^2$

$\because x^2+y^2+z^2 = r^2$

$$= f''(r)\left(\frac{r^2}{r^2}\right) + f'(r)\,\frac{2\left(x^2+y^2+z^2\right)}{r^3}$$

$$= f''(r)(1) + f'(r)\cdot 2\frac{r^2}{r^3}$$

$$= f''(r) + \frac{2}{r}f'(r)$$

$\because x^2+y^2+z^2 = r^2$

$\boxed{\text{Proved}}$ L.H.S = R.H.S.

Scanned with CamScanner

Divergence of $\vec{F}$.

Div $= \vec{\nabla}.F$.

Example # 28

For what value of n, the vector $r^n\vec{r}$ is solinoidal, when $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Sol:- As given that;

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}.$$

$$r = \sqrt{x^2+y^2+z^2} \quad \text{or} \quad r = (x^2+y^2+z^2)^{1/2}.$$

$$r^n\vec{r} = (x^2+y^2+z^2)^{m/2}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= x(x^2+y^2+z^2)^{m/2}\hat{i} + y(x^2+y^2+z^2)^{m/2}\hat{j} + z(x^2+y^2+z^2)^{m/2}\hat{k}.$$

Since; div $= \nabla.(r^n\vec{r}) = 0$ (As for solinoidal div $= 0$)

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(x(x^2+y^2+z^2)^{m/2}\hat{i}+y(x^2+y^2+z^2)^{m/2}\hat{j}+z(x^2+y^2+z^2)^{m/2}\hat{k}\right) = 0$$

$$= \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{m/2}x\right] + \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{m/2}y\right] + \frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{m/2}z\right) = 0$$

$$= x\left(\frac{m}{2}\right)(x^2+y^2+z^2)^{m/2-1}(2x) + (x^2+y^2+z^2)^{m/2} + \left(y\left(\frac{m}{2}\right)(x^2+y^2+z^2)^{m/2-1}(2y) + (x^2+y^2+z^2)^{m/2}\right)$$

$$+ \left(z\left(\frac{m}{2}\right)(x^2+y^2+z^2)^{m/2-1}(2z) + (x^2+y^2+z^2)^{m/2}\right) = 0$$

$= x^2 n(x^2+y^2+z^2)^{n/2-1} + 3(x^2+y^2+z^2)^{n/2} + y^2 n(x^2+y^2+z^2)^{n/2-1} + z^2 n(x^2+y^2+z^2)^{n/2-1} = 0$

$= n(x^2+y^2+z^2)^{n/2-1}\left[x^2+y^2+z^2\right] + 3(x^2+y^2+z^2)^{n/2} = 0$

$= n(x^2+y^2+z^2)^{n/2-1+1} + 3(x^2+y^2+z^2)^{n/2} = 0$

$= n(x^2+y^2+z^2)^{n/2} + 3(x^2+y^2+z^2)^{n/2} = 0$

$(x^2+y^2+z^2)^{n/2}\left[n+3\right] = 0.$

So.

either

$(x^2+y^2+z^2)^{n/2} = 0$

Discard as position vecter cannot be negative.

or

$n+3=0$

$\boxed{n=-3}$

Ans.

Example #29

show that, $\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right) = \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$

Sol:- Take L.H.S.

$L.H.S = \nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right)$ —(1)

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

So, $\frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{r^n}$

$$\phi = \frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{a_1x + a_2y + a_3z}{r^n}$$

$$\frac{\partial}{\partial x}\phi = \frac{\partial}{\partial x}\left(\frac{a_1x + a_2y + a_3z}{r^n}\right).$$

Using Quotient rule;

$$= \frac{r^n \frac{\partial}{\partial x}(a_1x + a_2y + a_3z) - a_1x + a_2y + a_3z \frac{\partial}{\partial x} r^n}{(r^n)^2}$$

$$= \frac{r^n a_1 - (a_1x + a_2y + a_3z)\, n r^{n-1}\left(\frac{\partial r}{\partial x}\right)}{r^{2n}} \longrightarrow ①$$

Since; $r^2 = x^2 + y^2 + z^2$

$$2r \cdot \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = y/r$$

$$\frac{\partial r}{\partial z} = z/r$$

By putting values in eq ①.

$$= \frac{r^n a_1 - (a_1x + a_2y + a_3z)\, n r^{n-1}(x/r)}{r^{2n}}$$

$$= \frac{a_1 r^n - (a_1x + a_2y + a_3z)\, n x\, r^{n-2}}{r^{2n}}$$

Scanned with CamScanner

$$= \frac{a_1 r^n}{r^{2n}} - \frac{(a_1x+a_2y+a_3z)\,nx}{r^{n+2}\cdot r^{2n}}$$

$$= \frac{a_1}{r^n} - \frac{(a_1x+a_2y+a_3z)\,nx}{r^{n+2}}$$

$$= \frac{a_1}{r^n} - \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k})}{r^{n+2}}$$

$\because$ $a_1\hat{i}+a_2\hat{j}+a_3\hat{k} = \vec{a}$

$x\hat{i}+y\hat{j}+z\hat{k} = \vec{r}.$

By putting values

$$= \frac{a_1}{r^n} - \frac{(\vec{a}\,\vec{r})(\vec{r}.)}{r^{n+2}}$$

$\boxed{\text{Proved}}$ Ans

Example #30

Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, $r = |\vec{r}|$ and $\vec{a}$ is a constant vector. Find the value of

$$\text{div}\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)$$

Sol:- $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$.

By putting values in below eqn.

$$\frac{\vec{a}\times\vec{r}}{r^n} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(x\hat{i}+y\hat{j}+z\hat{k})}{r^n} \rightarrow ①$$

Scanned with CamScanner

$$\vec{a} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}.$$

$$= \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x).$$

$$\because r^n = (x^2 + y^2 + z^2)^{n/2}$$

Put in eq ①.

$$\frac{\vec{a} \times \vec{r}}{r^n} = \frac{\hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)}{(x^2 + y^2 + z^2)^{n/2}}$$

Now;

$$\operatorname{div}\left(\frac{\vec{a} \times \vec{r}}{r^n}\right) = \nabla \cdot \left(\frac{\vec{a} \times \vec{r}}{r^n}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)}{(x^2 + y^2 + z^2)^{n/2}}\right)$$

$$= \frac{\partial}{\partial x}\frac{(a_2z - a_3y)}{(x^2 + y^2 + z^2)^{n/2}} - \frac{\partial}{\partial y}\frac{(a_1z - a_3x)}{(x^2 + y^2 + z^2)^{n/2}} + \frac{\partial}{\partial z}\left(\frac{a_1y - a_2x}{(x^2 + y^2 + z^2)^{n/2}}\right).$$

Apply Quotient rule;

$$= \frac{\frac{\partial}{\partial x}(a_2z - a_3y)(x^2 + y^2 + z^2)^{n/2} - \frac{\partial}{\partial x}(x^2 + y^2 + z^2)^{n/2}(a_2z - a_3y)}{\left((x^2 + y^2 + z^2)^{n/2}\right)^2} - \frac{0(x^2 + y^2 + z^2)^{n/2} + (a_1z - a_3x)\frac{\partial}{\partial y}(x^2 + y^2 + z^2)^{n/2}}{(x^2 + y^2 + z^2)^{n/2}}$$

$$+ 0(x^2 + y^2 + z^2)^{n/2} - (a_1y - a_2x)\frac{\partial}{\partial z}(x^2 + y^2 + z^2)^{n/2}$$

Scanned with CamScanner

$$0-\frac{\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x)(a_2z-a_3y)}{(x^2+y^2+z^2)^n}+\frac{(a_1z-a_3x)\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2y)}{(x^2+y^2+z^2)^n}$$

$$-\frac{(a_1y-a_2x)\frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2z)}{(x^2+y^2+z^2)^n}.$$

$$=-\frac{n(x^2+y^2+z^2)^{\frac{n-2}{2}}(a_2z-a_3y)x}{(x^2+y^2+z^2)^n}+\frac{n(a_1z-a_3x)(x^2+y^2+z^2)^{\frac{n-2}{2}}y}{(x^2+y^2+z^2)^n}-\frac{n(x^2+y^2+z^2)^{\frac{n-2}{2}}(a_1y-a_2x)}{(x^2+y^2+z^2)^n}$$

$$=\frac{-n(x^2+y^2+z^2)^{n-2/2}}{(x^2+y^2+z^2)^n}\left[x(a_2z-a_3y)-(a_1z+a_3x)y-(a_1y-a_2x)z\right].$$

$$\frac{-n(x^2+y^2+z^2)^{\frac{n-2}{2}}}{(x^2+y^2+z^2)^n}\left[\cancel{a_2xz}-\cancel{a_3xy}-\cancel{a_1zy}+\cancel{a_3xy}+\cancel{a_1yz}-\cancel{a_2xz}\right]$$

$$=\frac{-n(x^2+y^2+z^2)^{\frac{n+2}{2}}}{(x^2+y^2+z^2)^n}(0).$$

$$=0$$ Ans

Example # 31 Find D.D of $div(\vec{u})$ at point $(1,2,2)$ in direction of outer normal of sphere $x^2+y^2+z^2=9$ for $\vec{u}=x^4\hat{i}+y^4\hat{j}+z^4\hat{k}$.

Sol:- Find D.D of $Div(\vec{u})$ at $(1,2,2)$.

$$Div\ \vec{u} = \nabla\cdot\vec{u}.$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^4\hat{i}+y^4\hat{j}+z^4\hat{k}\right)$$

$$= 4x^3+4y^3+4z^3$$

Now;

$$D.D\ of\ div(\vec{u}) = \nabla.(\nabla\vec{u})\ (Point)\ \hat{v}. \longrightarrow (A)$$

$$\nabla(\nabla\vec{u}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(4x^3+4y^3+4z^3\right)$$

$$= 12x^2\hat{i}+12y^2\hat{j}+12z^2\hat{k}.$$

at point $(1,2,2)$

$$= 12\hat{i}+48\hat{j}+48\hat{k}. \longrightarrow (1)$$

Outer normal of sphere $= \nabla(x^2+y^2+z^2-9)$

$$= \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\left(x^2+y^2+z^2-9\right)$$

$$= 2x+2y+2z.$$

At point $(1,2,2)$

$\vec{U}' = 2\hat{i} + 4\hat{j} + 4\hat{k}$.

Unit vector is,

$$\hat{U}' = \frac{\vec{U}'}{|\vec{U}'|}.$$

$$= \frac{2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{4+16+16}} \Rightarrow \frac{2\hat{i}+4\hat{j}+4\hat{k}}{6}$$

Put values in eq ①.

$$D.D = \nabla(\vec{U}\vec{U})(\text{Point})\hat{U}' = (12\hat{i}+48\hat{j}+48\hat{k}) \cdot \left(\frac{2\hat{i}+4\hat{j}+4\hat{k}}{6}\right)$$

$$= \frac{24+192+192}{6} = 68 \text{ Ans}$$

Example #32

Show That $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$, where $r = \sqrt{x^2+y^2+z^2}$.

Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$.

Sol:- Since $r^n = (x^2+y^2+z^2)^{n/2}$.

$$\operatorname{grad}(r^n) = \nabla r^n$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) r^n$$

$= \frac{\partial}{\partial x} r^n \hat{i} + \frac{\partial}{\partial y} r^n \hat{j} + \frac{\partial}{\partial z} r^n \hat{k}$

$= n r^{n-1} \cdot \frac{\partial r}{\partial x} \hat{i} + n r^{n-1} \frac{\partial r}{\partial y} \hat{j} + n r^{n-1} \frac{\partial r}{\partial z} \hat{k}.$ $\longrightarrow$ (A).

Since;

$x^2 + y^2 + z^2 = r^2.$

$\frac{\partial r^2}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)$

$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly; $\frac{\partial r}{\partial y} = y/r.$, $\frac{\partial r}{\partial z} = \frac{z}{r}.$

By putting values.

$= n r^{n-1} \cdot \frac{x}{r} \hat{i} + n r^{n-1} \left(\frac{y}{r}\right) \hat{j} + n r^{n-1} \left(\frac{z}{r}\right) \hat{k}.$

$= n r^{n-2} x \hat{i} + n r^{n-2} y \hat{j} + n r^{n-2} z \hat{k}.$

$\text{grad}(r^n) = n r^{n-2} x \hat{i} + n r^{n-2} y \hat{j} + n r^{n-2} z \hat{k} \longrightarrow$ (B).

Now;

$\text{Div}(\text{grad } r^n) = \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) \cdot \left(n r^{n-2} x \hat{i} + n r^{n-2} y \hat{j} + n r^{n-2} z \hat{k}\right)$

$= \frac{\partial}{\partial x}(n r^{n-2} x) + \frac{\partial}{\partial y}(n r^{n-2} y) + \frac{\partial}{\partial z}(n r^{n-2} z)$

Apply product rule

$$= \left(x\, n(n-2) r^{n-3} \frac{\partial r}{\partial x} + n r^{n-2}\right) + \left(y\, n(n-2) r^{n-3} \frac{\partial r}{\partial y} + n r^{n-2}\right)$$

$$+ z\, n(n-2) r^{n-3} \cdot \frac{\partial r}{\partial z} + n r^{n-2}\Big)$$

By putting values of $\partial r/\partial x$.

$$= \left(x\, n(n-2) r^{n-3} \left(\frac{x}{r}\right) + n r^{n-2}\right) + \left(y\,(n(n-2))\, r^{n-3} \left(\frac{y}{r}\right) + n r^{n-2}\right)$$

$$+ z\, n(n-2) r^{n-3} \frac{z}{r} + n r^{n-2}\Big)$$

$$= 3n r^{n-2} + x^2 n(n-2) r^{n-4} + y^2 n(n-2) r^{n-4} + z^2 n(n-2) r^{n-4}$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} (x^2 + y^2 + z^2)$$

$$= 3n r^{n-2} + n(n-2) r^{n-4} \cdot (r^2)$$

$$= 3n r^{n-2} + n(n-2) r^{n-2}$$

$$= r^{n-2} (3n + n^2 - 2n) \Rightarrow r^{n-2} (n^2 + n)$$

$$= r^{n-2} (n(n+1))$$

or $n(n+1) r^{n-2}$ ① Proved

for $\nabla^2 \left(\frac{1}{r}\right) = 0$.

eq. ③ $\Rightarrow$ $n(n+1) r^{n-1}$ = div (grad r).

If we put $n = -1$.

div (grad r^{-1}) $= -1(-1+1) r^{-1-2} \Rightarrow 0$ Ans Proved

Example # 34

If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ find value of Curl $\vec{V}$.

Sol:- Curl V = $\vec{\nabla} \times \vec{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}.$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right).$$

$$= \hat{i}\left(z\left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}y) + \frac{1}{\cancel{2}}(y)(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}z)\right) - \hat{j}\left(z\left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}x)\right.$$

$$\left. + x\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}z)\right] + \hat{k}\left(y\left(-\frac{1}{\cancel{2}}\right)(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}x)\right.$$

$$\left. + \frac{1}{\cancel{2}}x(x^2+y^2+z^2)^{-\frac{1}{2}-1}(\cancel{2}y)\right]$$

$$= \left(\cancel{-yz(x^2+y^2+z^2)^{-3/2}} + \cancel{yz(x^2+y^2+z^2)^{-3/2}}\right)\hat{i} - \hat{j}\left(\cancel{-xz(x^2+y^2+z^2)} + \cancel{\frac{xz}{(x^2+y^2+z^2)^{3/2}}}\right)$$

$$+ \hat{k}\left(\cancel{-yx(x^2+y^2+z^2)^{-3/2}} + \cancel{yx(x^2+y^2+z^2)^{-3/2}}\right).$$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= 0$ Ans

Example #35

Prove that $(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solinoidal & irrotational.

Sol:-

For solinoidal:-

$$\vec{\nabla} \cdot \vec{F} = 0.$$

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[(y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}\right].$$

$$= \cancel{-2} + \cancel{2x} - \cancel{2x} + \cancel{2} = 0.$$

Hence F is solinoidal.

For Irrotational;

$$\vec{\nabla} \times \vec{F} = 0.$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}(\cancel{3x} - \cancel{3x}) - \hat{j}(\cancel{3y} - \cancel{2z} - \cancel{3y} + \cancel{2z}) + \hat{k}(\cancel{3z} + \cancel{2y} - \cancel{3z} - \cancel{2y})$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= 0$$

Hence F is irrotational also

Exp # 36

Determine the constant a, & b such that the curl of vector $A = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero.

Sol:- $\text{Curl } A = \vec{\nabla} \times \vec{A}$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy + 3yz & x^2 + axz - 4z^2 & -3xy - byz \end{vmatrix}$$

$$= \hat{i}(-3x - bz - ax + 8z) - \hat{j}(-3y - 3y) + \hat{k}(\cancel{2x} + az - \cancel{2x} - 3z)$$

$$= \hat{i}(-3x - ax - bz + 8z) + 6y\hat{j} + \hat{k}(az - 3z)$$

$$= \hat{i}(-x(3+a) - z(b-8))\hat{i} + 6y\hat{j} + \hat{k}(z(a-3))$$

Since Curl $A = 0$.

$$0 = [-x(3+a) - z(b-8)]\hat{i} + 6y\hat{j} + \hat{k}(z(a-3))$$

By comparing coefficients of $\hat{i}, \hat{j}$ & $\hat{k}$

$3 + a = 0 \Rightarrow a = -3, 3$

$-a + 3 = 0$

$a = +3$

$b - 8 = 0 \Rightarrow b = 8$

Ans

Example # 37

If a vector field is given by;

$F = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. Is this field irrotational

If so. find its scalar potential

Sol:- For irrotational field.

$\vec{\nabla} \times \vec{F} = 0$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 + x & -2xy - y & +0 \end{vmatrix}$$

$$= \hat{i}(0 - 0) - \hat{j}(0 - 0) + \hat{k}(-2y + 2y) = 0.$$

Hence field is irrotational.

Finding Scalar potential.

Let $\vec{F} = \nabla\phi$ —, ①.

$$d\phi = \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz).$$

$$= \nabla\phi \cdot d\vec{r} \rightarrow ②.$$

Put eq ① in ②.

$d\phi = F \cdot d\vec{r}$

By putting values;

$$d\phi = \left((x^2 - y^2 + x)\hat{i} - (2x+y)\hat{j}\right)\cdot\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$d\phi = (x^2 - y^2 + x)dx - (2x+y)dy.$$

Now $\int$ on b/s.

$$\int d\phi = \int (x^2 - y^2 + x)\,dx - \int (2xy + y)\,dy$$

$$\cancel{= \frac{x^3}{3} - \frac{y^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} + c}$$

$$\phi = \frac{x^3}{3} + \frac{x^2}{2} - xy^2 - \frac{y^2}{2} + c \quad \underline{\text{Ans}}$$

<u>Exp #39</u> A vector field is given by $\vec{A} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$. show that field is irrotational & find its scalar potential.

Sol:- Given that; $A = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$

For irrotational curl $\vec{A} = 0$.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + xy^2 & y^2 + x^2y & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(2xy - 2xy)$$

$= 0$ Hence vector field irrotational

Scanned with CamScanner

To find Scalar potential:-

Let $\vec{F} = \nabla\phi \rightarrow (1)$.

As

$$d\phi = \left(\frac{\partial}{\partial x}\phi\, dx + \frac{\partial}{\partial y}\phi\, dy + \frac{\partial\phi}{\partial z} dz\right)$$

$$d\phi = \left(\frac{\partial}{\partial x}\phi\,\hat{i} + \frac{\partial}{\partial y}\phi\,\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right).\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right).$$

$$d\phi = \nabla\phi\ d\vec{r} \rightarrow (2).$$

Put eq (1) in (2).

$$d\phi = F.d\vec{r}$$

By putting values..

$$d\phi = \left((x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}\right).\left(\hat{i}dx + \hat{j}dy + dz\,\hat{k}\right).$$

$$= x^2 + xy^2\, dx + y^2 + x^2y\, dy.$$

Integrating on b/s

$$\int d\phi = \int (x^2 + xy^2)dx + \int (y^2 + x^2y)\, dy.$$

$$= \frac{x^3}{3} + \frac{x^2 y^2}{2} + \frac{y^3}{3} + c \text{ Ans}$$

Example # 40

show that $\vec{v}(x,y,z) = 2xyz\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\,\hat{k}$

is irrotational & find scalar pot function

$u(x,y,z)$ such that $\vec{v} = grad(u)$.

Sol:- As; $\vec{v}(x,y,z) = 2xyz\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\,\hat{k}$

For irrotational

Curl $\vec{v} = 0$.

$$\text{curl}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz & x^2z + 2y & x^2y \end{vmatrix}$$

$$= \hat{i}(\cancel{x^2} - \cancel{x^2}) - \hat{j}(\cancel{2xy} - \cancel{2xy}) + \hat{k}(\cancel{2xz} - \cancel{2xz}).$$

$= 0$ Hence $\vec{v}$ is irrotational.

to find $\phi = ?$

let $\vec{v} = grad(\phi^u)$

$\vec{v} = \nabla\phi^u \longrightarrow (1)$

$$du = \frac{\partial}{\partial x}u\,dx + \frac{\partial}{\partial y}u\,dy + \frac{\partial}{\partial z}u\,dz$$

$$du = \left(\frac{\partial}{\partial x}u\,\hat{i} + \frac{\partial}{\partial y}u\,\hat{j} + \frac{\partial}{z}u\,\hat{k}\right)(\hat{i}dx + \hat{j}dy + dz\,\hat{k})$$

Scanned with CamScanner

$du = \vec{\nabla} u \cdot d\vec{r}$ — ①

Put eq ② in eq ①

$du = \vec{V} \cdot d\vec{r}$

By putting values

$du = (2xyz\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\,\hat{k})(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$= 2xyz\,dx + (x^2z + 2y)j + x^2y\,dz$

By Integrating -

$\int du = \int 2xyz\,dx + (x^2z + 2y)j + x^2y\,dz$

$u = x^2yz + \cancel{2}\frac{y^2}{\cancel{2}} + c$

$\boxed{u = x^2yz + y^2 + c}$ Ans

Exp #41 A fluid motion is given by $\vec{v} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. Show that motion is irrotational & hence find velocity potential.

Sol :- For irrotational;

Curl $\vec{v} = 0$

Curl $\vec{v} = \nabla \times \vec{V}$.

Scanned with CamScanner

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \hat{i}\,(1-1) - \hat{j}\,(1-1) + \hat{k}\,(1-1)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$= 0$ Hence irrotational.

To find velocity potential.

Let $\vec{v} = \nabla\phi \rightarrow (1)$

Since $d\phi = \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \nabla\phi \cdot d\vec{r}$$

Put eq (1) in above eqn.

$d\phi = \vec{v}\cdot d\vec{r}$.

By putting values

$$= \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= (y+z)dx + (z+x)dy + (x+y)dz$$

Integrating

$$\int d\phi = \int (y+z)dx + \int (z+x)dy + \int (x+y)dz$$

Scanned with CamScanner

$= yx + zx + zy + c$

Velocity potential $= xy + xz + yz + c$ Ans

Example #42

A fluid motion is given by

$\vec{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$

is motion irrotational? If so find velocity potential

Sol:- For irrotational;

curl $\vec{V} = 0$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$= \hat{i}(x\cos z + 2y - x\cos z - 2y) - \hat{j}(y\cos z - y\cos z)$

$+ \hat{k}(\sin z - \sin z)$.

$= 0 + 0 + 0 = 0$ field irrotational.

To find velocity potential?

Let $\vec{V} = \vec{\nabla}\phi$ where ϕ is velocity potential
$\hookrightarrow (1)$.

As, $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$d\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz).$$

$$d\phi = \nabla\phi \cdot d\vec{r} \rightarrow ②$$

Put eq ① in eq ②.

$$d\phi = \vec{V}\cdot d\vec{r}$$

By putting values.

$$= \left[(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}\right].$$
$$\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right].$$

$$= (y\sin z - \sin x)\,dx + (x\sin z + 2yz)\,dy + (xy\cos z + y^2)\,dz$$

$\int$ on b/s.

$$\int d\phi = \int (y\sin z - \sin x)\,dx + \int (x\sin z + 2yz)\,dy + \int (xy\cos z + y^2)\,dz$$

$$= \cos x + y^2 z + xy\sin z + c.$$

<u>Example#43</u> Prove that $\vec{F} = r^2\vec{r}$ is conservative and find the scalar potential ϕ such that $F = \nabla\phi$.

Sol:- As given that; $\vec{F} = r^2\vec{r}$.

$$\because r = (\hat{i}x + \hat{j}y + \hat{k}z).$$

$$= r^2(x\hat{i} + y\hat{j} + z\hat{k}).$$

Scanned with CamScanner

$= r^2 x\hat{i} + r^2 y\hat{j} + r^2 z\hat{k}$.

For irrotational; Curl $\vec{F} = 0$

$$\text{Curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r^2 x & r^2 y & r^2 z \end{vmatrix}$$

$$= \hat{i}\left(\partial/\partial y\,(r^2 z) - \partial/\partial z\,(r^2 y)\right) - \hat{j}\left(\partial/\partial x\,(r^2 z) - \partial/\partial z\,(r^2 x)\right)$$
$$+ \hat{k}\left(\partial/\partial x\,(r^2 y) - \partial/\partial y\,(r^2 x)\right)]$$

$$= \left(2rz\frac{\partial r}{\partial y} - 2ry\frac{\partial r}{\partial z}\right) - \hat{j}\left(2rz\frac{\partial r}{\partial x} - 2rx\frac{\partial r}{\partial z}\right) + \hat{k}\left(2ry\frac{\partial r}{\partial x} - 2rx\frac{\partial r}{\partial y}\right) \hookrightarrow (1)$$

Since; $x^2 + y^2 + z^2 = r^2$

$\partial/\partial x\,(r^2) = \partial/\partial x\,(x^2 + y^2 + z^2)$

$2r\,\partial r/\partial x = 2x \Rightarrow \frac{\partial r}{\partial x} = x/r \longrightarrow (2)$

Similarly, $\frac{\partial r}{\partial y} = y/r \longrightarrow (3)$

& $\frac{\partial r}{\partial z} = z/r$.

By putting values in eq (1)

$$= \left(\cancel{2rz\frac{y}{r}} - \cancel{2ry\frac{z}{r}}\right)\hat{i} - \hat{j}\left(\cancel{2rz\frac{x}{r}} - \cancel{2rx\frac{z}{r}}\right) + \hat{k}\left(\cancel{2ry\frac{x}{r}} - \cancel{2rx\frac{y}{r}}\right)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

Curl $\vec{F}$. $= 0$ ~~Ans~~ Hence irrotational.

To find scalar potential;

let $F = \nabla\phi \rightarrow ①$. ϕ is scalar potential

Since; $d\phi = \hat{i}\left(\frac{\partial}{\partial x}\phi\, dx + \frac{\partial}{\partial y}\phi\, dy + \frac{\partial}{\partial z}\phi\, dz\right)$.

$= \left(\frac{\partial}{\partial x}\phi\hat{i} + \frac{\partial}{\partial y}\phi\hat{j} + \frac{\partial}{\partial z}\phi\hat{k}\right).\left(\frac{\partial}{\partial}\hat{i}dx + \hat{j}dy + \hat{k}dz\right)$

$d\theta = \nabla\phi . d\hat{r} \longrightarrow ②$

By putting eq ① in eq ②.

$d\phi = \vec{F}.d\vec{r}$ By putting values

$d\phi = r^2\hat{r}(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$\because r^2 = x^2+y^2+z^2$ $\hat{r} = (x\hat{i}+y\hat{j}+z\hat{k})$

By putting values

$= (x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}dx+\hat{j}dy+\hat{k})$

$= (x^2+y^2+z^2)(xdx+ydy+zdz)$

$= x^3dx + x^2ydy + x^2zdz + xy^2dx + y^3dy + y^2zdz + xz^2dx + z^2ydy + z^3dz$

Take $\int$ on b/s

$\int d\phi = \int\left(x^3dx + x^2ydy + x^2zdz + xy^2dx + y^3dy + y^2zdz + xz^2dx + z^2ydy + z^3dz\right)$

$= \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^2z^2}{2} + \frac{y^4}{4} + \frac{y^2z^2}{2} + \frac{z^4}{4} + c$

Ans

Scanned with CamScanner

Example # 44 Show that the vector field $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ is irrotational as well as solinoidal. Find the scalar potential.

Sol:- As given that;

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$$

$\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$|\vec{r}|^3 = (x^2+y^2+z^2)^{3/2}$

$$\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

For solinoidal Curl $\vec{F} = 0$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{(x^2+y^2+z^2)^{3/2}} & \dfrac{y}{(x^2+y^2+z^2)^{3/2}} & \dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}$$

$$= \hat{i}\left(z(-\tfrac{3}{2})(x^2+y^2+z^2)^{-3/2-1}(2y) - y(-\tfrac{3}{2})(x^2+y^2+z^2)^{-3/2-1}(2z)\right) - \hat{j}\left(z(-\tfrac{3}{2})(x^2+y^2+z^2)^{-3/2}(2x) - (x(-\tfrac{3}{2})(x^2+y^2+z^2)^{-3/2-1}(2z)\right] + \hat{k}\left(y(-\tfrac{3}{2})(x^2+y^2+z^2)^{-3/2-1}(2x) - x(-\tfrac{3}{2})(x^2+y^2+z^2)^{-3/2-1}(2y)\right)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

$= 0$ Hence field is irrotational

For solinoidal;

$\nabla . F = 0.$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{3/2-1}(\cancel{2}x)(x)}{\left((x^2+y^2+z^2)^{3/2}\right)^2} + \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{3/2-1}(\cancel{2}y)(y)}{\left((x^2+y^2+z^2)^{3/2}\right)^2}$$

$$+ \frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{3/2-1}(\cancel{2}z)(z)}{\left((x^2+y^2+z^2)^{3/2}\right)^2}$$

$$= \frac{(x^2+y^2+z^2)^{3/2}\left(3(x^2+y^2+z^2)\right)^{1/2}}{(x^2+y^2+z^2)^3}\left[-x^2-y^2-z^2+1+1+1\right].$$

$$= \cancel{\frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}\left[1+1+1-x^2-y^2-z^2\right]}$$

$$= \frac{1}{x^2+y^2+z^2}\left(1+1+1-(x^2+y^2+z^2)\right).$$

=

To find scalar potential;

Let $\vec{F} = \nabla\phi \rightarrow (1)$

$d\phi = \left(\frac{\partial}{\partial x}\phi dx + \frac{\partial}{\partial y}\phi dy + \frac{\partial}{\partial y}\phi dz\right)$

$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz\right)$

$d\phi = \nabla\phi\, d\vec{r}$

Put val of (1).

$d\phi = \vec{F} \cdot d\vec{r}$ By putting values.

$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} \left(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz\right)$

$= \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$

Taking $\int$ on b/s.

$\int d\phi = \int \frac{x\,dx + 2y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$

$= \frac{1}{2}\int \frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{2}\int (2x\,dx + 2y\,dy + 2z\,dz)(x^2+y^2+z^2)^{-3/2}$

~~Using power rule.~~ As function & its derivative are there

$= \frac{1}{2}\int (2x\,dx + 2y\,dy + 2z\,dz)(x^2+y^2+z^2)^{-3/2}$

$= \frac{1}{2} \frac{(x^2+y^2+z^2)^{-3/2+1}}{-3/2+1} \Rightarrow \frac{1}{2} \frac{(x^2+y^2+z^2)^{-1/2}}{-1/2} \Rightarrow \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2)$

Scanned with CamScanner

$$= \frac{1}{(x^2+y^2+z^2)^{1/2}} = \frac{1}{|\vec{r}|} \quad \underline{\text{Ans}}$$

Example #45

Given vector field $\vec{v} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$. Find curl v. Show that the vector given by curl v at $P_0(1, 4, -3)$ & $P_1(2, 3, 12)$ are orthogonal.

Sol:- Find curl:-

$$\text{Curl} = \nabla \times \vec{F} \text{ or } \nabla \times \vec{v}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$$

$$= \hat{i}(0 - x - y) - \hat{j}(2x - 2x) + \hat{k}(z - y + 2y)$$

$$= (-x - y)\hat{i} + \hat{k}(y + z)$$

So curl at point $(1, 4, -3)$
Putting values;

$$= -3\hat{i} + \hat{k}$$

Curl at point $(2, 3, 12)$

$$= (-5\hat{i}) + 15\hat{k}$$

To show vector given by curl are orthogonal;

$$(-3\hat{i}-1\hat{k}).(-5\hat{i}+15\hat{k})=0$$

$$15-15=0.$$

Hence proved that they are orthogonal.

Example #46

Find the constant a, b, c so that

$$\vec{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$$

is irrotational & hence find function ϕ such that $F=\vec{\nabla}\phi$.

Sol:- As; To find value of a, b & c

$\text{Curl } \vec{F}=?$

$$\text{Curl } \vec{F}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}.$$

$$\text{Curl } \vec{F}=\hat{i}(c+1)-\hat{j}(4-a)+\hat{k}(b-2).$$

As for such value field is irrotational so.

$$\text{Curl } \vec{F}=0$$

$$0=\hat{i}(c+1)-\hat{j}(4-a)+\hat{k}(b-2).$$

By comparing coefficient of $\hat{i}, \hat{j}$ & $\hat{k}$

$0 = c + 1 = 0 \Rightarrow c = -1$

$4 - a = 0 \Rightarrow a = 4$

$b - 2 = 0 \Rightarrow b = 2$

Now find scalar potential ϕ.

Let $\vec{F} = \nabla \phi \rightarrow ①$.

Since; $d\phi = \frac{\partial}{\partial x}\phi\, dx + \frac{\partial}{\partial y}\phi\, dy + \frac{\partial}{\partial z}\phi\, dz$.

$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial}{\partial z}\phi\,\hat{k}\right) \cdot \left(\hat{i}dx + \hat{j}dy + \hat{z}dz\right)$

$= \nabla \phi . d\vec{r} \rightarrow ②$.

Put eq ① in ②.

$= \vec{F} . d\vec{r}$

By putting values.

$= \left[(x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2x)\hat{k}\right] . \left[\hat{i}dx + \hat{j}dy + \hat{k}dz\right]$.

$= (x + 2y + 4z)dx + (2x - 3y - z)dy + (4x - y + 2z)dz$.

Taking $\int$ on b/s.

$\int d\phi = \int \left[(x + 2y + 4z)dx + (2x - 3y - z)dy + (4x - y + 2z)dz\right]$

$= \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + \int(2y\, dx + 2x\, dy) + \int 4z\, dx + 4x\, dz - \int(z\, dy + y\, dz)$

$= \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - zy. + C$ Ans

Example # 47

Let $\vec{V}(x,y,t)$ be a differentiable vector function & $\phi(x,y,t)$ be a scalar function. Derive an expression for $div(\phi\vec{V})$ in term of $\phi.\vec{V}$, $div\vec{V}$ & $\nabla\phi$.

Sol. - $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$.

Let ϕ be scalar function than;

$$V\phi = \phi V_1\hat{i} + \phi_2 V_2\hat{j} + \phi V_3\hat{k}.$$

$$div(\phi\vec{V}) = \nabla.(\phi V) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\phi V_1\hat{i} + \phi V_2\hat{j} + \phi V_3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(\phi V_1) + \frac{\partial}{\partial y}(\phi V_2) + \frac{\partial}{\partial z}(\phi V_3)$$

$$= \left(\phi\frac{\partial}{\partial x}V_1 + V_1\frac{\partial\phi}{\partial x}\right) + \left(\phi\frac{\partial}{\partial y}V_2 + V_2\frac{\partial\phi}{\partial y}\right) + \left(\phi\frac{\partial}{\partial z}V_3 + V_3\frac{\partial\phi}{\partial z}\right).$$

$$= \phi\left(\frac{\partial}{\partial x}V_1 + \frac{\partial}{\partial y}V_2 + \frac{\partial}{\partial z}V_3\right) + \phi\left(\frac{\partial V_1}{\partial x} + \frac{\partial}{\partial y}V_2 + \frac{\partial V_3}{\partial z}\right).$$

$$= \phi(\nabla.\cancel{\phi}^{V}) + \left(\frac{\partial\phi}{\partial x}V_1 + \frac{\partial\phi V_2}{\partial y} + \frac{\partial\phi V_3}{\partial z}\right).$$

$$= \phi(\nabla.V) + (\nabla\phi.V).$$

$$= \phi\, div(V) + (grad\,\phi).\vec{V}.$$

Ans

Example # 48

If A is a constant vector & $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$ Then prove that;

$$\text{Curl}\left[(\vec{A}\cdot\vec{R})\vec{R}\right] = \vec{A}\times\vec{R}.$$

Sol:- Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\vec{A}\cdot\vec{R} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (A_1x + A_2y + A_3z)$$

$$(\vec{A}\cdot\vec{R})\vec{R} = (A_1x + A_2y + A_3z)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \hat{i}A_1x^2 + A_1xy\hat{j} + A_1xz\hat{k} + A_2xy\hat{i} + A_2y^2\hat{j} + A_2yz\hat{k} + A_3xz\hat{i} + A_3zy\hat{j} + A_3z^2\hat{k}$$

$$= (A_1x^2 + A_2xy + A_3xz)\hat{i} + (A_1xy + A_2y^2 + A_3yz)\hat{j} + (A_1xz + A_2yz + A_3z^2)$$

Now Curl of $(\vec{A}\cdot\vec{R})\vec{R} = ?$

$$\text{Curl}(\vec{A}\cdot\vec{R})R = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1x^2 + A_2xy + A_3xz & A_1xy + A_2y^2 + A_3yz & A_1xz + A_2yz + A_3z^2 \end{vmatrix}$$

$$= \hat{i}(A_2z - A_3y) - \hat{j}(A_1z - A_3x) + \hat{k}(A_1y - A_2x)$$

$\hookrightarrow$ (1)

L.H.S. $\vec{A} \times \vec{R}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(A_2 z - A_3 y) - \hat{j}(A_1 z - A_3 x) + \hat{k}(A_1 y - A_2 x) \quad \to (2)$$

From eq (1) & (2)

L.H.S = R.H.S Proved

Example # 49:-

Suppose $\vec{U}, \vec{V}$ & f are continously differentiable fields, then proved that $\text{div}(\vec{U} \times \vec{V}) = \vec{V}.\text{curl}\vec{U} - U.\text{curl}V$

Sol:- Let $\vec{U} = U_1 x + U_2 y + U_3 z$

$\vec{V} = V_1 x + V_2 y + V_3 z$

$$\vec{U} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i(U_2 V_3 - U_3 V_2) - j(U_1 V_3 - U_3 V_1) + \hat{k}(U_1 V_2 - U_2 V_1)$$

Now;

$$\text{div}(\vec{U} \times \vec{V}) = \nabla.(\vec{U} \times \vec{V})$$

Scanned with CamScanner

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left[(u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k} \right]$

$= \frac{\partial}{\partial x}(u_2v_3 - v_2u_3) - \frac{\partial}{\partial y}(u_1v_3 - u_3v_1) + \frac{\partial}{\partial z}(u_1v_2 - u_2v_1)$

→ Using product rule.

$= \left(u_2\frac{\partial v_3}{\partial x} + v_3\frac{\partial u_2}{\partial x} - u_3\frac{\partial v_2}{\partial x} - v_2\frac{\partial u_3}{\partial x} \right) + \left(\frac{v_3\partial u_1}{\partial y} + u_1\frac{\partial v_3}{\partial y} - v_1\frac{\partial u_3}{\partial y} - u_3\frac{\partial}{\partial y}v_1 \right) + \left(u_1\frac{\partial v_2}{\partial z} + v_2\frac{\partial u_1}{\partial z} - v_1\frac{\partial u_2}{\partial z} - u_2\frac{\partial v_1}{\partial z} \right)$

$= v_1\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) + v_2\left(-\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z} \right) + v_3\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right)$

$+ u_1\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + u_2\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z} \right) + u_3\left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} \right)$

$= (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \cdot \left[\hat{i}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) + \hat{j}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) + \hat{k}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \right]$

$- (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \left[\hat{i}\left(-\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y} \right) + \hat{j}\left(-\frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} \right) + \hat{k}\left(-\frac{\partial v_2}{\partial x} + \frac{\partial v_1}{\partial y} \right) \right]$

$\therefore \vec{\nabla} \times \vec{U} =$

$= \vec{V} \cdot \text{curl}\,\vec{U} - \vec{U} \cdot \text{curl}\,\vec{V}$

So; $\vec{V} \cdot (\vec{U} \times \vec{\nabla}) - \vec{U} \cdot (\nabla \vec{V})$

Ans proved

Exp # 52

If A is constant vector, show that;

$$\vec{a} \times (\vec{\nabla} \times \vec{r}) = \vec{\nabla}(\vec{a} \cdot \vec{r}) - (\vec{a} \cdot \vec{\nabla})\vec{r}.$$

Sol: let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

$\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$.

$$L.H.S = \vec{a} \times (\vec{\nabla} \times \vec{r})$$

$$\vec{\nabla} \times \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(r_1\hat{i} + r_2\hat{j} + r_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ r_1 & r_2 & r_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}r_3 - \frac{\partial}{\partial z}r_2\right) - \hat{j}\left(\frac{\partial}{\partial x}r_3 - \frac{\partial}{\partial z}r_1\right) + \hat{k}\left(\frac{\partial}{\partial x}r_2 - \frac{\partial}{\partial y}r_1\right).$$

Now;

$$\vec{a} \times (\vec{\nabla} \times \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ \left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right) & -\left(\frac{\partial r_3}{\partial x} - \frac{\partial r_1}{\partial z}\right) & \frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y} \end{vmatrix}$$

$$= \hat{i}\left[a_2\left(\frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}\right) - a_3\left(-\frac{\partial}{\partial x}r_3 + \frac{\partial r_1}{\partial z}\right)\right] - \hat{j}\left[a_1\left(\frac{\partial r_2}{\partial x} - \frac{\partial r_1}{\partial y}\right) - a_3\left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right)\right] + \hat{k}\left[a_1\left(-\frac{\partial r_3}{\partial x} + \frac{\partial}{\partial z}r_1\right) - a_2\left(\frac{\partial r_3}{\partial y} - \frac{\partial r_2}{\partial z}\right)\right]$$

$$= \hat{i}\left(a_2\frac{\partial r_2}{\partial x} - \frac{a_2\partial r_1}{\partial y} + a_3\frac{\partial r_3}{\partial x} - a_3\frac{\partial r_1}{\partial z}\right) - \hat{j}\left(a_1\frac{\partial r_2}{\partial x} - a_1\frac{\partial r_1}{\partial y} + a_3\frac{\partial r_2}{\partial z} - \frac{a_3\partial r_3}{\partial y}\right.$$

Exp#31 Prove that for every field $\vec{v}$;

div curl $\vec{v} = 0$.

Sol:- Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$.

Since Curl = $\vec{\nabla} \times$ function (vector)

Divergerence = $\vec{\nabla}$ (function).

$$\text{Curl} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & + \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\partial/\partial y\, v_3 - \partial/\partial z\, v_2\right)\hat{i} - \hat{j}\left(\partial/\partial x\, v_3 - \partial/\partial z\, v_1\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

Now,

$$\text{div. curl } \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \text{ curl } \vec{v}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left[\left(\frac{\partial}{\partial y}v_3 - \frac{\partial}{\partial z}v_2\right)\hat{i} - \hat{j}\left(\frac{\partial}{\partial x}v_3 - \frac{\partial}{\partial z}v_1\right) + \hat{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial}{\partial y}v_1\right)\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}v_3 - \frac{\partial}{\partial z}v_2\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}v_3 - \frac{\partial}{\partial z}v_1\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial}{\partial y}v_1\right)$$

$$= \cancel{\frac{\partial^2 v_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 v_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 v_3}{\partial x \partial y}} + \cancel{\frac{\partial^2 v_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 v_2}{\partial z \partial x}} - \cancel{\frac{\partial^2 v_1}{\partial z \partial y}}$$

$= 0$

Proved

Ans

Exp # 55

If f & g are two scalar point function

Prove that $\operatorname{div}(f \nabla g) = f \nabla^2 g + \nabla f \nabla g$.

Sol:- Take L.H.S.

$$\operatorname{div}(f \nabla g) = ?$$

$$\nabla g = \frac{\partial}{\partial x} g\hat{i} + \frac{\partial}{\partial y} g\hat{j} + \frac{\partial}{\partial z} g\hat{k}.$$

$$f. \nabla g = (f_1\hat{i} + f_2\hat{j} + f_3\hat{k}).\left(\frac{\partial}{\partial x} g\hat{i} + \frac{\partial}{\partial y} g\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).$$

$$= \cancel{\frac{\partial f}{\partial}}\; f_1 \frac{\partial g}{\partial x} + f_2 \frac{\partial g}{\partial y} + f_3 \frac{\partial g}{\partial z}$$

Now;

$$\operatorname{div} f(\nabla g) = \frac{\partial}{\partial x}\left(f_1 \frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f_2 \frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f_3 \frac{\partial g}{\partial z}\right).$$

$\because$ Product rule;

$$= \left(\frac{\partial^2 g}{\partial x^2} f_1 + f \frac{\partial^2 g}{\partial y^2} + f \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right).$$

$$= fg\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right).\left(\frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k}\right)$$

$$= fg\,\nabla^2 + \nabla f . \nabla g.$$

$\boxed{\text{Proved}}$

Ans

$$= \hat{i}\left(\frac{\partial^2}{\partial x \partial y} f_2 - \frac{\partial^2}{\partial y^2} f_1 - \frac{\partial^2}{\partial z^2} f_1 + \frac{\partial^2}{\partial z \partial x} f_3\right) - \hat{j}\left(\frac{\partial^2}{\partial x^2} f_2 - \frac{\partial^2}{\partial x \partial y} f_1 - \frac{\partial^2}{\partial y \partial z} f_3 + \frac{\partial^2}{\partial z^2} f_2\right)$$

$$+ \hat{k}\left(\frac{\partial^2}{\partial x \partial z} f_1 - \frac{\partial^2}{\partial x^2} f_3 - \frac{\partial^2}{\partial y^2} f_3 + \frac{\partial^2}{\partial y \partial z} f_2\right)$$

$$= \hat{i}\left(\frac{\partial^2}{\partial x \partial y} f_2 + \frac{\partial^2}{\partial x^2} f_1 + \frac{\partial^2 f_3}{\partial z \partial x} - \left(\frac{\partial^2}{\partial x^2} f_1 + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\right)\right)$$

$$- \hat{j}\left(-\frac{\partial^2 f_1}{\partial x \partial y} - \frac{\partial^2 f_3}{\partial y \partial z} - \frac{\partial^2}{\partial y^2} f_2 + \left(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2}\right)\right)$$

$$+ \hat{k}\left(\frac{\partial^2 f_1}{\partial x \partial z} + \frac{\partial^2 f_2}{\partial y \partial z} + \frac{\partial^2 f_3}{\partial z^2} - \left(\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\right)\right)$$

$$= \hat{i}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial z} f_3\right) - f_1\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)$$

$$+ \hat{j}\left(\frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial}{\partial z} f_3\right) - f_2\left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} f_1 + \frac{\partial}{\partial y} f_2 + \frac{\partial}{\partial z} f_3\right) - f_4\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)$$

Exp #156:- Establish the reaction;

$$\text{curl curl } \vec{f} = \vec{\nabla} \operatorname{div} \vec{f} - \vec{\nabla}^2 \vec{f}.$$

Sol:- Let $\vec{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$.

Taking L.H.S

$$\text{curl curl } \vec{f} = ?$$

$$\text{curl } \vec{f} = \vec{\nabla} \times \vec{f} \qquad \because \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \hat{i}\left(\partial/\partial y \, f_3 - \frac{\partial}{\partial z} f_2\right) - \hat{j}\left(\partial/\partial x \, f_3 - \partial/\partial z \, f_1\right) + \hat{k}\left(\frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1\right).$$

Now;

$$\text{curl} \times \text{curl } \vec{f} = ?$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \left(\frac{\partial f_3}{\partial y} - \frac{\partial}{\partial z} f_2\right) & -\frac{\partial}{\partial x} f_3 + \frac{\partial}{\partial z} f_1 & \frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1 \end{vmatrix}.$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial}{\partial y} f_1\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} f_1 - \frac{\partial}{\partial x} f_3\right)\right] - \hat{j}\left[\partial/\partial x\left(\frac{\partial}{\partial x} f_2 - \frac{\partial}{\partial y} f_1\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_3}{\partial y} - \frac{\partial}{\partial z} f_2\right)\right].$$

$$+ \hat{k}\left[\partial/\partial x\left(-\partial/\partial x \, f_3 + \frac{\partial}{\partial z} f_1\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y} f_3 - \frac{\partial}{\partial z} f_2\right)\right].$$

Exp #1 57

For solinoid vector $\vec{F}$

Show that curl curl curl curl $\vec{F} = \nabla^4 \vec{F}$.

Sol:- As solinoidal vector so;

$$\text{div}. \vec{f} = 0 \longrightarrow (1).$$

Remember curl curl $\vec{F}$ = grad div $\vec{F} - \nabla^2 \vec{F}$.

Since

$$\text{Curl Curl } \vec{F} = \text{grad div } \vec{F} - \nabla^2 \vec{F} \longrightarrow (2).$$

Put eq (1) in eq (2).

i.e div $\vec{F} = 0$.

$$\text{Curl Curl } \vec{F} = -\nabla^2 \vec{F} \longrightarrow (3).$$

Now;

$$\text{Curl Curl Curl Curl } \vec{F} = -\underset{\text{div}}{\text{grad}}\ \nabla^2 \vec{F} + \nabla^2(\nabla^2 \vec{F}).$$

$$= -\text{grad div } \nabla^2 \vec{F} + \nabla^4 \vec{F}.$$

$$= -\text{grad}\,(\nabla . \nabla^2 \vec{F}) + (\nabla^4 \vec{F}). \qquad \because \text{div} = \nabla .$$

$$= -\text{grad}\,(\nabla^2 . \nabla \vec{F}) + (\nabla^4 \vec{F}).$$

$$= 0 + \nabla^4 \vec{F} \qquad \because \text{using eq (1)}$$

$$= \nabla^4 \vec{F} \quad \boxed{\text{Proved}} \quad \text{Ans}$$

$$\therefore \left(a_1\frac{\partial r_1}{\partial x} + a_2\frac{\partial r_2}{\partial x} + a_3\frac{\partial r_3}{\partial x}\right)\hat{i} + \hat{j}\left(a_1\frac{\partial r_1}{\partial y} + a_2\frac{\partial r_2}{\partial y} + a_3\frac{\partial r_3}{\partial y}\right)$$

$$+\hat{k}\left(a_1\frac{\partial r_1}{\partial z} + a_2\frac{\partial r_2}{\partial z} + a_3\frac{\partial r_3}{\partial z}\right) - \left(a_1\frac{\partial r_1}{\partial x}\hat{i} + a_1\frac{\partial r_2}{\partial y}\hat{j} + a_1\frac{\partial r_3}{\partial z}\hat{k}\right)$$

$$+\left(a_2\frac{\partial r_1}{\partial y}\hat{i} + a_2\frac{\partial r_2}{\partial y}\hat{j} + a_2\frac{\partial r_3}{\partial z}\hat{k}\right) + \left(a_3\hat{i}\frac{\partial r_1}{\partial z} + a_3\frac{\partial r_2}{\partial z}\hat{j} + a_3\frac{\partial r_3}{\partial z}\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(a_1r_1 + a_2r_2 + a_3r_3\right) - \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)$$
$$\left(r_1\hat{i} + r_2\hat{j} + r_3\hat{k}\right)$$

Since;

$\partial/\partial x\,\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \nabla.$

where $a_1r_1 + a_2r_2 + a_3r_3 = \vec{a}.\vec{r}$

& $\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}.$

By putting values.

$$\nabla(\vec{a}\cdot\vec{r}) - (\vec{\nabla}.\vec{a})\vec{r}.$$

Proved

L.H.S = R.H.S