Statistics
lecture # 12

Joint Distribution:-

"When we have two variables of random experiment are observed, the distribution of such variable is called Joint distribution."

- Distribution involving two, three or many r.variables simultaneously called bivariate, trivariate or multivariate.

Joint Probability Distribution:-

"As probability distribution is tabular form of random variables & its (their) corresponding probabilities. But as here two variables (Random) are included i.e x & y. So Joint Pd can be written as;

X \ Y	y_1	y_2	y_3	...	y_n	Marginal Pd of X
x_1	$f(x_1,y_1)$	$f(x_1,y_2)$	$f(x_1,y_3)$	---	$f(x_1,y_n)$	$\sum_{j=1}^{n}(x_1,y_j) = g(x_1)$
x_2	$f(x_2,y_1)$	$f(x_2,y_2)$	$f(x_2,y_3)$	---	$f(x_2,y_n)$	$\sum_{j=1}^{n}(x_2,y_j) = g(x_2)$
x_3	$f(x_3,y_1)$	$f(x_3,y_2)$	$f(x_3,y_3)$	---	$f(x_3,y_n)$	$\sum_{j=1}^{n}(x_3,y_j) = g(x_3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	$f(x_m,y_1)$	$f(x_m,y_2)$	$f(x_m,y_3)$		$f(x_m,y_n)$	$\sum_{j=1}^{n}(x_m,y_j) = g(x_m)$
	$\sum_{i=1}^{m}(x_i,y_1) = h(y_1)$	$\sum_{i=1}^{m}(x_i,y_2) = h(y_2)$	$\sum_{i=1}^{m}(x_i,y_3) = h(y_3)$	...	$\sum_{i=1}^{m}(x_i,y_n)$ $h(y_m)$	$\underline{1}$

Marginal Pd of y.

• In Marginal Pd of x, The term x is constant & value of y get varry & same for marginal Pd of y, value of y is constant & x varry in column.

• It should be noted that.

$$\boxed{\sum_{i=1}^{n} h(y_i) = 1.}$$

$$\boxed{\sum_{i=1}^{m} g(x_i) = 1.}$$

Condition for Pd:-

A function $f(x_i, y_j)$ will be Pd. If;

(i) $f(x_i, y_j) \geq 0$.

(ii) $\sum_i \sum_j f(x_i, y_j) = 1$

∴ Here one summation is for adding all terms row wise & than second summation is to add all those along Column.

Marginal P.d of x.

$$g(x_i) = \sum_{j=1}^{n} f(x_i, y_j).$$

∴ Here x constant y varies.

Marginal P.d of y:-

$$h(y_j) = \sum_{i=1}^{m} f(x_i, y_j).$$

∴ Here y constant x varies.

Conditional Probability Distribution:-

$$f(x_i/y_j) = \frac{f(x_i, y_j)}{h(y_j)} \quad \because P(A/B) = \frac{P(A \cap B)}{P(B)},$$

$$f(y_j/x_i) = \frac{f(x_i, y_j)}{g(x_i)}$$

$f(x_i/y_j)$ means $P(X = x_i / Y = y_j)$.

- If X & Y are Independant then $f(x, y) = g(x)\, h(y)$.

- Two r.v. will be Independant if multiplication of marginal Pd of X & Marginal Pd of Y equal to their corresponding Prob. distribution.

Distribution function:-

$$F(X, Y) = P(X \le x, Y \le y).$$

$$\boxed{F(X, Y) = \sum_i \sum_j f(x_i, y_j)}$$

For e.g $F(1, 1)$ $\because$ It start from $(0, 0)$.

so, $F(1, 1) = f(0,0) + f(0,1) + f(1,0) + f(1,1)$

Properties:-

(1) $F(-\infty, y) = F(x, -\infty) = 0$.

$-\infty$ = Lower limit (behind).

(2) $F(\infty, \infty) = 1$.

Both x & y if have upper limit

(*) . $F(\infty, 0)$ or $F(0, \infty) \neq 1$.

If not 1 mean we are leaving some data. As from marginal Pd of X & Y we found that sum = 1.

Example:-

$X = 0, 1, 2.$

$Y = 0, 1.$

$F(1, 2) = ?$

$$F(1, 2) = f(0,0) + f(1,0) + f(2,0) + f(0,1) + f(0,2), f(1,1)$$

$$\underset{\substack{\infty \\ \text{extreme value}}}{1}, \underset{\substack{\infty \\ \text{extreme value}}}{2}$$

so;

$$\boxed{F(1, 2) = 1}$$

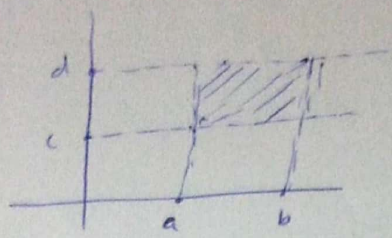
(3) The distribution function is non-decreasing.

(4) The distribution function is continous from right.

$P(a < x \leq b, c \leq y \leq d) = ?$

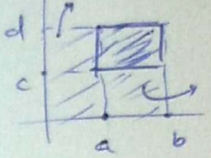
$P(a < x \leq b, c < y \leq d) = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

$\therefore$ As in process we remove (a,c) portion two times so we add it once at last.



Explaination:-

We have to find the shaded region. So total is $F(b,d)$ i.e less from b & less from d on x & y-axis respectively.



remove this by $F(a,d)$.

remove this by $F(b,c)$.

Example # 7.6 An urn contain 3 Black, 2 red and 3 green balls and 2 balls are selected at random from it. If X is no. of black balls & Y is no. of red balls selected. Then find.

i) The Joint Probability function.

Solution:-

3B
2R
3G

If X is no. of Black balls.

$X: B = 0, 1, 2.$

If Y is no. of red balls.

$Y: R = 0, 1, 2.$

$X \backslash Y$	0	1	2	
0	3/28	6/28	1/28	10/28
1	9/28	6/28	0	15/28
2	3/28	0	0	3/28.
	15/28	12/28	1/28	1 = Σ

$\because n(S) = {}^8C_2$

$= \frac{8!}{2!(8-2)!}$

$= 28.$

$f(0,0) = P(X=0, Y=0)$

$$= \frac{\binom{3}{0}\binom{2}{0}\binom{3}{2}}{28} = \frac{3}{28}.$$

$f(0,1) = P(X=0, Y=1)$

$$= \frac{\binom{3}{0}\binom{2}{1}\binom{3}{1}}{28} = \frac{6}{28}.$$

$f(0,2) = P(X=0, Y=2)$

$$= \frac{\binom{3}{0}\binom{2}{2}\binom{3}{0}}{28} = \frac{1}{28}.$$

$f(1,0) = P(X=\cancel{0}\,1, Y=0)$

$$= \frac{\binom{3}{1}\binom{2}{0}\binom{3}{1}}{28} = 9/28.$$

$f(1,1) = P(X=1, Y=1)$

$$= \binom{3}{1}\binom{2}{1}\binom{3}{0} = \frac{6}{28}$$

$f(1,2) = P(X=1, Y=2)$

$= 0$. As Impossible since we have to select just two balls but here $2+1=3$. (Not possible)

$f(2,0) = P(X=2, Y=0)$

$$= \frac{\binom{3}{2}\binom{2}{0}\binom{3}{0}}{28} = 3/28$$

$P(2,1) = 0$. (Impossible event)

$P(2,2) = 0$.

(ii) $P(X+Y \leq 1)$.

$P(X+Y \leq 1) = f(0,0) + f(0,1) + f(1,0)$.

From table.

$= 3/28 + 6/28 + 9/28$

$$\boxed{P(X+Y \leq 1) = \frac{18}{28}}$$

(iii) The marginal Pd $g(x)$ & $h(y)$.

x	0	1	2
$g(x)$	$10/28$	$15/28$	$3/28$

y	0	1	2
$h(y)$	$15/28$	$\frac{12}{28}$	$1/28$

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iv) The Conditional Probability dist $f(x|1)$

As $f(x|1) = P(X = x / Y = 1)$.

$$f(x|1) = \frac{P(x, 1)}{P(Y = 1)} = \frac{f(x, 1)}{h(1)}.$$

As from table $h(1) = \frac{12}{28}$.

where $x = 0, 1, 2$.

$$\boxed{\text{So, } f(x|1) = \frac{P(x, 1)}{12/28}}$$

(v) $P(X = 0 / Y = 1)$.

so; $P(X = 0 / Y = 1) = \frac{f(0, 1)}{h(1)}$

$$= \frac{\cancel{6}^{1}/\cancel{28}}{_{2}\cancel{12}/\cancel{28}}$$ ∵ From table

$$\boxed{P(X = 0 / Y = 1) = 1/2}$$

(vi) Are X & Y Independant.

X & Y are not Independant as;

$f(0,0) \neq g(0) . h(0)$.

$\frac{3}{28} \neq \frac{15}{28} \times \frac{10}{28}$.

So not Independant. Ans 2

Example # 7.7

The joint Pd of two discrete r.v's X & Y is given by.

$$f(x,y) = \frac{xy^2}{30} \quad \text{for} \quad x = 1, 2, 3 \ \& \ y = 1, 2.$$

Are X & Y Independant?

Solution:- For X & Y to be Independant this condition must be satisfied.

$$f(x,y) = g(x) \cdot h(y) \longrightarrow ①$$

Marginal Pd of $x = g(x)$.

$$g(x) = f(x,1) + f(x,2)$$

$$= \frac{x}{30} + \frac{4x}{30} = \frac{x}{6}.$$

$$h(y) = f(1,y) + f(2,y) + f(3,y)$$

$$= \frac{y^2}{30} + \frac{2y^2}{30} + \frac{3y^2}{30} = \frac{6y^2}{30}.$$

$$= y^2/5.$$

Eq ① $\Rightarrow$ $f(x,y) = \frac{x}{6} \cdot \frac{y^2}{5}$

$$= \frac{xy^2}{30} \quad \text{Satisfy.}$$

So X & Y are Independant.

Joint Continous:-

Joint Pdf:-

Let X & Y be continous random variable than $f(x,y)$ is said to be joint Pdf if

① $f(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$

Note:-
Probability at $x=a$ & $y=b$ is zero.

For Interval Case:-

$$P(a \leq x \leq b,\ c \leq y \leq d) = \int_{a}^{b}\int_{c}^{d} f(x,y)\,dy\,dx. \quad \text{(A)}$$

This equation is used to find probability at any Intervals when Pd is given.

To find distribution function.

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx.$$

To find P.d when distribution is given.

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\,\partial y}$$

· Joint density function & D.F are same

Marginal Pd of x

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

Here x is constant & y varry as dy.

Marginal Pd of y.

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

Here y is constant & x varries.

Example #7.8 Given the following joint Pdf

$$f(x,y) = \frac{1}{8}(6-x-y), \quad 0 \le x \le 2; \; 2 \le y \le 4.$$

$$= 0 \quad \text{else where}$$

(a) Verify that $f(x,y)$ is a joint density function.

(b) Calculate (i) $P\left(X \le 3/2, Y \le 5/2\right)$ (ii) $P(X+Y<3)$.

(c) Find the marginal Pdf $g(x)$ & $h(y)$.

(d) Find the Conditional Pdf $f(x/y)$ & $f(y/x)$

(a) $f(x,y)$ will be joint density function i.e Pdf if.

i) $f(x,y) \geq 0$, & (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

As condition(i) is satisfy since for both upper limit is 2 & 4 if we put we get zero & if we put In b/w then so not get any negative number.

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \frac{1}{8}\int_{0}^{2}\int_{2}^{4}(6-x-y)\,dy\,dx$

$= \frac{1}{8}\int_{0}^{2}\left(6y - xy - y^2/2\right)dx\Big|_{2}^{4}$

$= \frac{1}{8}\int_{0}^{2}(6-2x)\,dx$

$= \frac{1}{8}\left(6x - x^2\right)\Big|_{0}^{2}$

$= 1/8\,(12-4) = 1/8 \times 8$

$= 1.$

Thus $f(x,y)$ holds properties. So Pdf.

(b) (i) $P(X \le 3/2,\ Y \le 5/2)$.

$P(X \le 3/2,\ Y \le 5/2) = \frac{1}{8}\int_{x=0}^{3/2}\int_{2}^{5/2}(6-x-y)\,dy\,dx$

$= \frac{1}{8}\int_{0}^{3/2}\left(6y - xy - y^2/2\right)dx\Big|_{2}^{5/2}$

$= 1/8\int_{0}^{3/2}\left(\frac{15}{8} - x/2\right)dx$

$= \frac{1}{8}\left(\frac{15}{8}x - \frac{x^2}{4}\right)\Big|_{0}^{3/2}$

$= 9/32.$

(ii) $P(X+Y<3)$.

We can write

$x+y<3$

$y = 3-x$.

So;

$$P(X+Y<3) = \frac{1}{8}\int_0^1 \int_2^{3-x} (6-x-y)\,dy\,dx.$$

$$= \frac{1}{8}\int_0^1 \left(\left(6y - xy - \frac{y^2}{2}\right)dx\right)\Big|_2^{3-x}$$

$$= \frac{1}{8}\int_0^1 \left(\frac{x^2}{2} - 4x + \frac{7}{2}\right)dx.$$

$$= \frac{1}{8}\left(\frac{x^3}{6} - 2x^2 + \frac{7x}{2}\right)\Big|_0^1.$$

$$= \frac{1}{8}\left(\frac{10}{6}\right) = \frac{5}{24}$$

(c) Marginal Pd of X:-

Since $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$

$$= \frac{1}{8}\int_2^4 (6-x-y)\,dy$$

$$g(x) = \frac{1}{8}\left(6y - xy - \frac{y^2}{2}\right)\Big|_2^4.$$

$$g(x) = \frac{1}{4}(3-x)$$ $\because$ $x<0$ or $x \geq 2$

$g(x) = 0$.

Marginal Pdf of y :-

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

$$= \frac{1}{8}\int_0^2 (6-x-y)\,dx$$

$$= \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)\Big|_0^2$$

$$h(y) = \frac{1}{4}(5-y).$$ $h(y)=0$ else where

(d) Conditional Pdf of x given y.

$$f(x/y) = \frac{f(x,y)}{h(y)}, \quad h(y)>0.$$

By putting value.

$$f(x/y) = \frac{\frac{1}{\cancel{8}_2}(6-x-y)}{\frac{1}{\cancel{4}}(5-y)} = \frac{\frac{1}{2}(6-x-y)}{(5-y)}$$

Conditional Pdf of y given x is;

$$f(y/x) = \frac{f(x,y)}{g(x)} = \frac{\frac{1}{\cancel{8}_2}(6-x-y)}{\frac{1}{\cancel{4}}(3-x)}$$

$$\boxed{f(y/x) = \frac{1}{2}\frac{(6-x-y)}{(3-x)}}$$ Ans 2.

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Example # 7.9 :-

Let the bivariate continous random variable (X,Y) have the joint density Probability function given by.

$$f(x,y) = x^2 + \frac{xy}{3}, \quad 0 \le x \le 1, \quad 0 \le y \le 2.$$

$$= 0 \quad \text{else where}$$

(a) Check that $f(x,y)$ is a Pdf.

(b) Find the marginal Pdf.

(c) Find the Conditional Pdf & verify that $f(x/y)$ is a Pdf.

Sol:- $f(x,y)$ is Pdf if.

(i) $f(x,y) \ge 0$. (ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

As condition 1 is satisfy since both upper & lower limit we have not any -ve no. So always give +ve no.

(ii) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \left(x^2 + \frac{xy}{3}\right)dx\,dy$$

$$= \int_0^2 \left(\frac{x^3}{3} + \frac{x^2 y}{2\times 3}\right)\Big|_0^1 dy.$$

$$= \int_0^2 \left(\frac{1}{3} + \frac{y}{6}\right)dy.$$

$$= \left(\frac{1}{3}y + \frac{y^2}{12}\right)\Big|_0^2$$

$$= 2/3 + 4/12 = \frac{8+4}{12} = \frac{12}{12} = 1$$

Since both Conditions Satisfied so it is Pdf

(b) Marginal Pdf :-

of x.

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

$$= \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy.$$

$$= \left(x^2 y + \frac{xy^2}{6}\right)\Big|_0^2.$$

$$= 2x^2 + \tfrac{2}{3}x \Rightarrow$$

Marginal Pdf of y.

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$$

$$= \int_0^1 \left(x^2 + xy/3\right) dx.$$

$$= \left(\frac{x^3}{3} + \frac{x^2 y}{6}\right)\Big|_0^1$$

$$= \left(\tfrac{1}{3} + y/6\right).$$

(c) The Conditional Pdf of Y given X.

$$f(y/x) = \frac{f(x,y)}{g(x)} = \frac{\left(x^2 + \frac{xy}{3}\right)}{2x^2 + 2/3\,x}.$$

$$= \frac{\cancel{x}(x + y/3)}{\cancel{x}(2x + 2/3)} = \frac{x + y/3}{2x + 2/3}.$$

$$= \frac{\frac{3x+y}{3}}{\frac{3\times 2x + 2}{3}} = \boxed{\frac{3x+y}{6x+2}}$$

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Conditional Pdf of X given Y.

$$f(x/y) = \frac{f(x,y)}{h(y)}$$

$$= \frac{\left(x^2 + \frac{xy}{3}\right)}{\left(\frac{1}{3} + \frac{y}{6}\right)}$$

$$= \frac{\frac{3x^2 + xy}{3}}{\frac{2+y}{6}} \Rightarrow \frac{3x^2 + xy}{3} \times \frac{6}{2+y}$$

$$= \frac{2(3x^2 + xy)}{2+y} = \frac{6x^2 + 2xy}{2+y}.$$

$$\boxed{f(x/y) = \frac{6x^2 + 2xy}{2+y}}$$ Ans

Mathematical Expectation of Random Variable :-

Mathematical expectation represent mean of random variable.

Let X is discrete random variable than.

$$E(x) = \mu = \sum_{i=1}^{n} x_i (f(x_i)) \longrightarrow ①$$

symbollically

or For Infinite series.

$$E(x) = \mu = \sum_{i=1}^{\infty} x_i (f(x_i)) \longrightarrow ②$$

For this case the series should be convergent.

It simply means multiply each value of R.V with their corresponding probability & than add them.

- For Infinite series should be convergent.

If x is Continous R.V.

$$E(x) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx \rightarrow (3).$$

- Here series should be convergent or absolute convergent.

Eq (1) is also called weighted mean.

If values are equally likely than

$$\boxed{E(x) = \frac{1}{n} \sum x_i}.$$

Example # 7.10 :-

(a) What is the mathematical expectation of the number of heads when 3 fair coins are tossed?

Note (b) Not study yet.

Sol:- In order to find mathematical Exp first Find P.d. As three coins are tossed.

x	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

So, $E(x) = 0 \times 1/8 + 1 \times 3/8 + 2 \times \frac{3}{8} + 3 \times 1/8.$

$$\boxed{E(x) = 1.5}$$

It means in 3 exp. the average of heads is 1.5.

Example # 7.11

(a) If it rains, an umberalla salemman can earn $30 per day. If it is fair, he can lose $ 6 per day. What is his expectation if the probability of rain is 0.3?

Solution:- let x represents the no. of dollar he earn.

Than

x	30 $	-6 $
f(x)	0.3	0.7

This is used when we have to do multiple buisness. We find expected value for each buisness & whose expected value is more then consider that job.

So the expected value is:

$$E(x) = 30 * 0.3 + (-6)(0.7)$$

$$\boxed{E(x) = 4.8}$$

(b). A man draws 2 balls from bag containing 3 white and 5 black balls. If he recives Rs. 70 for every white ball he draws and Rs. 7 for every black ball, Find his expectation.

Solution:- As total balls are 8, the man can choose any two so use combination.

$$n(s) = {}^8C_2 = 28$$

there are 3 possibilities he can choose 2 balls.

(1) Both are white.

Than $P_1 = \dfrac{\binom{3}{2}}{28} = 3/28$

(2) One white & one black.

$P_2 = \binom{3}{1}\binom{5}{1}/28 = 15/28.$

(3) Both are Black

$$P_3 = \frac{\binom{5}{2}}{28} = \frac{10}{28}$$

Let X denote the amount he can receive

If both are white

$$X_1 = 2 * Rs\ 70 = 140\ R/s$$

If one white & one black.

$$x_2 = 1 \times 70 + 1 \times 7 = 77\ R/s.$$

If both are black.

$$X_3 = 1 \times 7 + 1 \times 7 = 14\ R/s.$$

X	140	77	14
$f(x)$	3/28	15/28	10/28

So;

$$E(X) = 140 \times \frac{3}{28} + 77 \times \frac{15}{28} + 14 * \frac{10}{28}$$

$$\boxed{E(X) = 61.25}$$ Ans.

If we have to play same as discuss in example if investment is 50 R/s. Since expected value is more (61.25) than Investment so go for this game. If Investment is 100 than 61.25 < 100 So not go for game.

Expectation of function of a Random Variable:-

Let $H(x)$ is a function of r.v X than.

$$E(H(X)) = H(x_1)f(x_1) + H(x_2)f(x_2) + \cdots + H(x_n)f(x_n).$$

$$= \sum_{i=1}^{n} H(x_i) f(x_i). \longrightarrow (1)$$

- Provided the series is convergent.
- eq (1) is for discrete case.

For Continous Case:-

$$E(H(X)) = \int_{-\infty}^{\infty} H(x). f(x_i)\, dx.$$

For expected value for x^2 in both cases can be written as;

$$E(x^2) = \sum x_i^2 f(x_i) \quad \text{or} \quad E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

Variance can be find as;

$$E(X - \mu^2) = \sigma^2 \text{ (variance).}$$

Short method

$$\sigma^2 = E(x^2) - (E(x))^2.$$

If we take square root we get Standard deviation.

To get other (higher) consider this;
To find higher moments

$$E(X-\mu)^r = \mu_r$$

If

$$E(X-\mu)^r = \mu_r.$$

If $\mu = a$.

$$E(X-a)^r = \mu_r'.$$

Remember:-

$$E(X-a) = \mu_r'.$$

About arbitary constant let $a=0$.

Then expected value is;

$$E(X) = \mu = \mu_1'.$$

$$E(X^2) = \mu_2'.$$

$$E(X^3) = \mu_3' \quad \text{\& so on.}$$

Example # 7.13:- Let X have the following probability distribution

x_i	1	2	3	4	5
$f(x_i)$	0.2	0.3	0.2	0.2	0.1

Find the probability function of $3X-1$, X^2 and X^2+2 and find $E(3X-1)$, $E(X^2)$

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and $E(X^2+2)$.

Solution:-

To find Probability ~~function~~ of $(3X-1)$.

x_i	1	2	3	4	5
$f(x_i)$	0.2	0.3	0.2	0.2	0.1
$3x-1$	2	5	8	11	14

The expected value of $3X-1$ will be;

$$E(3X-1) = 2\times 0.2 + 5\times 0.3 + 8\times 0.2 + 0.2\times 11 + 14\times 0.1$$

$$\boxed{E(3X-1) = 7.1}$$

The probability ~~function~~ distribution of X^2 is;

X	1	2	3	4	5
$f(X)$	0.2	0.3	0.2	0.2	0.1
X^2	1	4	9	16	25

The expected value of X^2 is;

$$E(X^2) = 1\times 0.2 + 4\times 0.3 + 9\times 0.2 + 16\times 0.2 + 25\times 0.1$$

$$\boxed{E(X^2) = 8.9}$$

The probability distribution of X^2+1.

x	1	2	3	4	5
$f(x)$	0.2	0.3	0.2	0.2	0.1
X^2+1	2	5	10	17	26

$E(X^2+1) = 2\times 0.2 + 0.3\times 5 + 0.2\times 10 + 0.2\times 17 + 0.1\times 26 = 10.9$ Ans

Example 7.14 let X be a r.v with pdf

$$f(x) = 2(x-1) \qquad 1 < x < 2.$$

$$= 0 \qquad \text{else where}$$

Find the expected values of $H(x) = 2X-1$ & $H(X) = X^2$.

Solution:-

To find expected value of $H(x) = 2x-1$,

$$E(x) = \int_{-\infty}^{\infty} H(x) f(x)\, dx \qquad -\alpha = 1 \quad \alpha = 2.$$

$$E(2X-1) = \int_1^2 2(x-1)(2x-1)\, dx.$$

$$= 2\int_1^2 (x-1)(2x-1)\, dx$$

$$\boxed{E(2X-1) = 2.33}$$

& $H(X) = X^2$.

$$E(X) = \int_{-\infty}^{\infty} H(x) f(x)\, dx.$$

$$E(x^2) = \int_1^2 x^2 (2(x-1))\, dx.$$

$$= 2\int_1^2 x^2(x-1)\, dx.$$

$$\boxed{E(x^2) = 2.833}$$ Ans

Properties of Expected Values:-

① Expected value of constant is also constant as mean.

$E(a) = a.$

② $E(ax+b) = aE(x) + b.$

where a & b are constant.

③ $E(X+Y) = E(X) + E(Y).$

For Independant events.

If X & Y are Independant events.

$E(XY) = E(X)\,E(Y).$

∵ Leave examples from page # 257 → 265.

Note:- For finding these we have to check whether the given function is pdf or not i.e satisfying the two conditions or not?

Median & Mode of Continous Random Variable:-

Median:- In case of Continous random variable.

$$\text{Median} = \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}.$$

Solve for "m" where m is our required median.

"$1/2$" is for median.

If "$1/4$" First Quartile.

if "$3/4$" 3rd Quartile.

∵ $-\infty$ is for lower limit

For Discrete case:-

$$\sum_{-\infty}^{m} f(x)dx = 1/2.$$

Mode:- For mode we find that value at which first derivative of function is zero & second derivative of function on that value we get negative number.

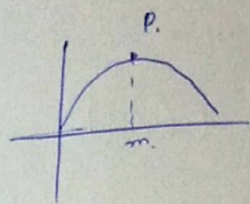
Mathematically:

$$f'(m) = 0 \quad \& \quad f''(m) < 0.$$

Then m will consider mode.

So mode is the max. repeated value in data.

Since at point P first derivative is zero & second derivative is -ve so P is mode.



For Maximum value.

① $f'(x) = 0.$
& $f''(x) < 0$ > Second derivative test.

Harmonic Mean:-

It is reciprocal of expected value of "$1/x$". of random experiment.

$$H = \left(E\left(1/x\right)\right)^{-1}.$$

Geometric Mean:-

$$G = \text{Antilog}\left(E(\log x)\right).$$

It is antilog of expected value of log of x. of random experiment.

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Example 7.15

If the continous r.v X has pdf

$$f(x) = \frac{3}{4}(3-x)(x-5) \qquad 3 \le x \le 5.$$

$$= 0 \qquad \text{else where.}$$

Calculate arithematic mean, variance & standard deviation of x.

Solution:- As for arithematic mean.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

$$= \int_3^5 \tfrac{3}{4} x(3-x)(x-5) dx.$$

$$= \tfrac{3}{4} \int_3^5 x(3-x)(x-5) dx.$$

$$\boxed{E(X) = 4}$$

As variance is

$$\sigma^2 = E(X^2) - [E(X)]^2 \longrightarrow ①.$$

Now find $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx.$

$$= \int_3^5 x^2 (\tfrac{3}{4})(3-x)(x-5) dx.$$

$$= \tfrac{3}{4} \int_3^5 x^2 (3-x)(x-5) dx.$$

$E(x^2) = 16.2$

$\therefore$ variance $= E(x^2) - [E(x)]^2$.

$= 16.2 - (4)^2$

$\boxed{\text{Variance} = 0.2}$

$\therefore$ standard deviation is;

$S.D = \sqrt{\text{variance}}$

$S.D = \sqrt{0.2}$.

$\boxed{S.D = 0.44}$

<u>Example # 7.16</u>

The continous r.v X has pdf $f(x)$ where $f(x) = \frac{3}{4}(\cancel{x}\,1 + x^2)$ for $0 \le x \le 1$ If $E(x) = \mu$ & $Var(x) = \sigma^2$, Find $P(|X - \mu| < \sigma)$.

<u>Solution:-</u>

As; $P(|X - \mu| < \sigma)$.

$P(-\sigma < X - \mu < \sigma)$

$P(-\sigma + \mu < X < \sigma + \mu)$. $\longrightarrow$ ①

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So first find value of δ & $\mu = ?$

As $\mu = E(x) = \int_0^1 3/4\, x(1+x^2)$

$= 3/4 \int_0^1 x(1+x^2)$

$= 0.5625$

& $\delta = E(x^2) - [E(x)]^2 \rightarrow$ ②

To find $E(x^2) = ?$

$E(x^2) = 3/4 \int_0^1 x^2(1+x^2)\,dx = 0.4$

$V = 0.4 - (0.5625)^2$

$= V(x) = 0.0836$

So, $\delta = \sqrt{V(x)} = 0.289$

& ① $\Rightarrow$ $P(-0.289 + 0.5625 < x < 0.289 + 0.5625)$

$\Rightarrow P(0.2735 < x < 0.8515)$

So, $P(0.2735 < x < 0.8515) = 3/4 \int_{0.2735}^{0.8515} (1+x^2)\,dx$

$= 0.8527$ Ans

Example #7.17

A con. r.v X has the Pdf.

$$f(x) = \tfrac{3}{4}x(2-x) \qquad 0 \le x \le 2.$$

$$= 0 \qquad \text{else where}$$

Find the first four moments about mean & Co-efficient of Skewness.

Sol:- As first four moments about ~~mean~~ origin is easily find i-e

$\mu_1' = E(x)$, $\mu_2' = E(x^2)$, $\mu_3' = E(x^3)$ & $\mu_4' = E(x^4)$

From that we find moment about mean

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_3' - 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6(\mu_1')^2\mu_2' - 3(\mu_1')^4$$

By putting values we get the Correspond-

To find Co-efficient of Skewness.

Since we know

$$C.S = \frac{\mu_3}{\sigma^3} = 0/\sigma^3$$

$$= 0 \quad \text{Ans.}$$

Example # 7.18

Let X & Y be two discrete r.v with the following joint Pd.

$y \backslash x$	2	4
1	0.1	0.15
3	0.2	0.3
5	0.1	0.15

Find $E(X)$, $E(Y)$, $E(X+Y)$, $E(2X-3Y)$ & $E(XY)$.

Sol:- As $E(x) = \sum x_i g(x_i)$

$E(Y) = \sum y_i h(y_i)$

so first find marginal Pd of x & y.

$y \backslash x$	2	4	$h(y)$
1	0.1	0.15	0.25
3	0.2	0.3	0.5
5	0.1	0.15	0.25
$g(x)$	0.4	0.6	1

$E(X) = \sum x_i g(x_i)$

$= 2(0.4) + 4(0.6)$

$E(x) = 3.2$.

$E(Y) = \sum y_i h(y_i)$.

$= 1\times 0.25 + 3\times 0.5 + 5\times 0.25$.

$E(Y) = 3$.

$E(X+Y) = \sum_i \sum_j (x_i + y_j)(f(x_i, y_j))$

? $E(X) + E(Y)$.

$= 3.2 + 3 = 6.2$.

$E(2X - 3Y) = 2E(X) - 3E(X)$.

$= 2(3.2) - 3(3)$

$= -2.6$.

$E(XY) = ?$

As X & Y are Independant.

$E(XY) = E(X)E(Y)$

$= 3.2(3) = 9.6$ Ans

Example #7.21

let X & Y be Independant r.v's with joint Pdf

$$f(x,y) = \frac{x(1+3y^2)}{4} \quad 0 \le x < 2, \; 0 < y < 1.$$

$$0 \quad \text{else where.}$$

Find $E(X)$, $E(Y)$, $E(X+Y)$ & $E(XY)$.

Sol:- As

$$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx.$$

so first find marginal pd of X & Y.

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~~h(x)~~ $g(x) = \int_0^1 \frac{x(1+3y^2)}{4} dy$.

$= x/2$.

$h(y) = \int_0^2 \frac{x(1+3y^2)}{4} dx$.

$= 1/2 (1+3y^2)$.

Thus $E(x) = \int_{-\infty}^{\infty} x g(x) dx$.

$= \int_0^2 x \left(\overset{(x/2)}{\cancel{\frac{x(1+3y^2)}{4}}} \right) dx$.

$= 4/3$.

$E(y) = \int_{-\infty}^{\infty} y h(y) dy$.

$= \frac{1}{2} \int_0^1 y(1+3y^2) dy$.

$\boxed{E(y) = 5/8}$

$E(X+Y) = ?$

as X & Y are Independant

$E(X+Y) = E(X) + E(Y)$.

$= 4/3 + 5/8$.

$\boxed{E(X+Y) = 47/24}$

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$E(XY) = ?$

As Independant;

So, $E(XY) = E(X)\,E(Y)$.

$= (4/3)(5/8) = 5/6$ Ans

To find Variance for Joint Pd :- For Discrete case

As; $V(X) = E(X^2) - [E(X)]^2$,

so; $E(X) = \sum x\, g(x)$.

$E(X^2) = \sum x^2 g(x)$

$E(Y) = \sum y\, h(y)$.

$E(Y^2) = \sum y^2 h(y)$.

$Cov(X,Y) = E(XY) - E(X)E(Y)$.

For Continous Case:-

$E(X) = ?$

$V(X) = E(X^2) - [E(X)]^2$.

$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx$ $\qquad g(x) = \int_{-\infty}^{\infty} (f(x,y))\, dy$.

$E(X^2) = \int_{-\infty}^{\infty} x^2 g(x)\, dx$ $\qquad h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$

EXP#7.24

let X be a r.v with pdf.

$$f(x) = k(x - x^2) \quad 0 \le x \le 1.$$

$$= 0 \quad \text{else where}$$

where k is constant, Find.

Sol:- First find value of k.

It will be pdf if.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\int_0^1 k(x - x^2)\,dx = 1.$$

$$k \int_0^1 (x - x^2)\,dx = 1.$$

$$\boxed{k = 6}$$

So

$$f(x) = 6(x - x^2) \quad 0 \le x \le 1.$$

$$0 \quad \text{else where}$$

To find mean.

As $\text{mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= 6\int_0^1 x(x - x^2)\,dx$$

$$= \tfrac{1}{2}.$$

Median:-

As $\int_{-\infty}^{m} f(x)dx = 1/2$

$6\int_{0}^{m}(x-x^2)dx = 1/2$

$4m^3 - 6m^2 + 1 = 0.$

we get

$\boxed{m_1 = -0.366}$ discard as -ve & not come in range

$\boxed{m_2 = 1.366}$ Discard,

$\boxed{m_3 = 1/2}$ Select as come in range

So median is $1/2$.

Mode:-

As $f(x) = 6(x-x^2)$

$f'(x) = 6(x-2x)$.

$6(1-2x) = 0.$

$1 = 2x$

$\boxed{x = 1/2}$

$f''(x) = (-2)6 \Rightarrow -12$

Thus $x = 1/2$ is mode

Harmonic mean = ?

As $H.M = \left(E(1/x)\right)^{-1}$

$$E\left(\frac{1}{x}\right) = \int_{-\infty}^{\infty} 1/x \, f(x).$$

$$= 6\int_{0}^{1} \frac{1}{x}(x - x^2)\,dx.$$

$$= 3.$$

$$H.M = (3)^{-1} = 1/3. \text{ Ans}$$

To find S.D = ?

As $S.D = \sqrt{V(X)}$.

$$V(X) = E(x^2) - [E(x)]^2.$$

Frist find

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

$$= 6\int_{0}^{1} x^2(x - x^2)\,dx = 3/10.$$

$$[E(x)]^2 = (1/2)^2$$

so; $V(X) = 3/10 - 1/4 = 1/20$.

$$S.D = \sqrt{V(X)} = \sqrt{1/20}$$

$$\boxed{S.D = 0.2234}$$

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