Example #10 | Higher Eng. Maths
Lecture # 10
15/05/2019

Find The unit vector to normal

the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

Sol:- Here we have to find gradient i.e perpendicular to surface. (normal)

Let $\phi = xy^3z^2 - 4$.

As gradient is;

$$\vec{\nabla}\phi = \hat{i}\frac{\partial}{\partial x}\phi + \hat{j}\frac{\partial}{\partial y}\phi + \hat{k}\frac{\partial}{\partial z}\phi.$$

$$\vec{\nabla}\phi = \frac{\partial}{\partial x}(xy^3z^2-4)\hat{i} + \frac{\partial}{\partial y}(xy^3z^2-4)\hat{j} + \frac{\partial}{\partial z}(xy^3z^2-4)\hat{k}.$$

$$= \hat{i}\, y^3z^2 + 3y^2xz^2\hat{j} + (2xy^3z)\hat{k}.$$

$$\vec{\nabla}\phi(-1,-1,2) = (-1)^3(2)^2\hat{i} + 3(-1)^2(-1)(2)^2\hat{j} + 2(-1)(-1)^3(2)\hat{k}$$

$$\vec{\nabla}\phi = -4\hat{i} - 12\hat{j} + 4\hat{k}$$

As unit vector $= \hat{\nabla}\phi = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} \rightarrow (1)$.

$$|\vec{\nabla}\phi| = \sqrt{(-4)^2+(-12)^2+(4)^2} \Rightarrow +\sqrt{11\times16} \Rightarrow 4\sqrt{11}.$$

& (1) $\Rightarrow$

$$\hat{\vec{\nabla}}\phi = \frac{-4\hat{i}-12\hat{j}+4\hat{k}}{4\sqrt{11}} \Rightarrow \frac{1}{\sqrt{11}}(-\hat{i}-3\hat{j}+\hat{k})$$ Ans

Unit vector ⊥ to surface $xy^3z^2 = 4$

Scanned with CamScanner

Exp#11 Find rate of change of $\phi = xyz$ in the direction normal to surface $x^2y + y^2x + yz^2 = 3$ at point $(1,1,1)$.

Sol:- As rate of change is

$$= \nabla\phi.$$

$\because \nabla$ in 3-D.

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$

as here 3-D i.e xyz.

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(xyz).$$

$$= \hat{i}\frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz)\hat{j} + \frac{\partial}{\partial z}(xyz)\hat{k}.$$

$$= yz\hat{i} + xz\hat{j} + xy\hat{k}.$$

rate of change at $(1,1,1)$

$$= \hat{i} + \hat{j} + \hat{k}.$$

Normal to surface = gradient.

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2y + y^2x + yz^2 - 3).$$

$$\cancel{\vec{\nabla} = 2xy\hat{i} + 2yx\hat{j} + 2zx\hat{k}}.$$

$$\cancel{\nabla(1,1,1) = 2\hat{i} + 2\hat{j} + 2\hat{k}}.$$

$$\vec{\nabla} = (2xy + y^2)\hat{i} + (x^2 + 2yx + z^2) + 2yz\hat{k}.$$

$$\vec{\nabla}(1,1,1) = 3\hat{i} + 4\hat{j} + 2\hat{k}.$$

Now unit vector is;

Scanned with CamScanner

$$\hat{v} = \frac{3\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{(3)^2+(4)^2+(2)^2}} = \frac{3\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{9+16+4}}$$

$$= \frac{3\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{29}}$$

So now rate of change is;

$$= \nabla\phi \cdot \hat{v}$$

$$= (\hat{i}+\hat{j}+\hat{k}) \cdot \left(\frac{3\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{29}}\right)$$

$$= \frac{3+4+2}{\sqrt{29}} \Rightarrow \frac{9}{\sqrt{29}} \quad \underline{\underline{\text{Ans}}}$$

<u>Exp #12</u>

Find the constants m & n such that the surface $mx^2 - 2nyz = (m+4)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at pt $(1,-1,2)$.

<u>Sol:-</u> Here we have two vectors are given & we will find normal i.e gradient for both.

let $\phi_1 = mx^2 - 2xyz = (m+4)x$

or

$\phi_1 = mx^2 - 2xyz - (m+4)x$

& $\phi_2 = 4x^2y + z^3 - 4$.

So normal or gradient to ϕ_1 is:

$$\nabla\phi_1 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(mx^2 - 2yz - (m+4)x\right)$$

$$= (2mx - m - 4)\hat{i} \mp (2z)\hat{j} + \hat{k}(2y)$$

Now at pt $(1,-1,2)$

$$\nabla\phi_1\Big|_{(1,-1,2)} = (2m - m - 4)\hat{i} \mp \overset{(2)}{2}\hat{j} + 2\hat{k}$$

$$= (m-4)\hat{i} \mp 4\hat{j} + 2\hat{k} \longrightarrow (1)$$

Now normal to ϕ_2.

$$\vec{\nabla}\phi_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4x^2y + z^3 - 4\right)$$

$$= 8xy\hat{i} + 4x^2\hat{j} + 3z^2 k$$

At point $(1,-1,2)$

$$= -8\hat{i} + 4\hat{j} + 12\hat{k}$$

As ϕ_1 & ϕ_2 are orthogonal so the their normals are $\perp$ to each other

Thus $\vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 = 0$.

$$\left((m-4)\hat{i} + 4\hat{j} + 2\hat{k}\right)\cdot\left(-8\hat{i} \mp 4\hat{j} + 12\hat{k}\right) = 0$$

$$-8(m-4) \mp 16 + 24 = 0$$

$$m - 4 = 8/8 \Rightarrow m = 4+1$$

$$m = 5$$

Now to find $n = ?$

As; $mx^2 - 2nyz = (m+4)x$.

Put $m = 5$ & Point i.e $(1, -1, 2)$ $x\ y\ z$.

$$5(1)^2 - 2(n)(-1)(2) = 9(1)$$

$$4n = 4$$

$$n = 1$$

Hence $m = 5$ & $n = 1$. Ans

Example # 13.

Find the values of constant λ & μ so that the surface $\lambda x^2 - \mu xyz = (\lambda + 2)x$, $4x^2y + z^3 = 4$ intersect orthogonally at point $(1, -1, 2)$.

Sol :- Let

$$\phi_1 = \lambda x^2 - \mu yz - (\lambda + 2)x$$

$$\phi_2 = 4x^2y + z^3 - 4$$

We have to find normal vector to these. As gradient is normal vector to surface

So find gradient

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\lambda x^2 - \mu yz - (\lambda + 2)x\right)$$

$$= (2x\lambda - (\lambda + 2))\hat{i} - \mu z\hat{j} - \mu y\hat{k}$$

At pt $(1,-1,2)$.

$$\vec{\nabla}\phi_1(1,-1,2) = \hat{i}(2\lambda-\lambda-2) - \hat{j}(-2\mu) + \hat{k}\mu.$$

$$= \hat{i}(\lambda-2) + 2\mu\hat{j} + \hat{k}\mu.$$

Now gradient to ϕ_2

$$\vec{\nabla}\phi_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4x^2y + z^3 = 4\right)$$

$$= 8xy\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}.$$

at pt $(1,-1,2)$

$$= -8\hat{i} + 4\hat{j} + 12\hat{k}.$$

Since normal vectors are orthogonal. So.

$$\left[\hat{i}(\lambda-2) + 2\mu\hat{j} + \hat{k}\mu\right].\left[-8\hat{i} + 4\hat{j} + 12\hat{k}\right] = 0.$$

$$-8(\lambda-2) + 8\mu + 12\mu = 0.$$

$$-8\lambda + 16 + 8\mu + 12\mu = 0.$$

$$-8\lambda + 20\mu + 16 = 0 \quad \rightarrow ①$$

As, $\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x = 0$.

it will satisfy $(1,-1,2)$

$$= \lambda - \mu 2 - \lambda + 2 = 0$$

$$2\mu = 2 \Rightarrow \mu = 1$$

eq ① $\Rightarrow$ $8\lambda + 20 + 16 \Rightarrow 8\lambda = 36 \Rightarrow \boxed{\lambda = 36/8}$ Ans

Exp #14 Find angle b/w the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at pt $(2,-1,2)$.

Sol:- As we know

$$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta \quad \text{— (A)}$$

Let θ be the angle.

Step # 1

Find the normal vector to both surface i.e gradient.

Let $\phi_1 = x^2+y^2+z^2-9$

$\phi_2 = x^2+y^2-3-z$

So;

$$\nabla\phi_1 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2-9)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

at pt $(2,-1,2)$

$$= 4\hat{i}-2\hat{j}+4\hat{k}$$

Now;

$$\nabla\phi_2 = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2-3-z)$$

$$= 2x\hat{i}+2y\hat{j}-1\hat{k}$$

at $(2,-1,2)$

$$= 4\hat{i}-2\hat{j}-\hat{k}$$

Now eq (A) $\Rightarrow$

$$(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k}) = \sqrt{4^2+(-2)^2+4^2}\,\sqrt{4^2+(-2)^2+(-1)^2}\cos\theta$$

$$16+4-4 = \sqrt{16+4+16}\,\sqrt{16+4+1}\cos\theta$$

$$16 = 6\sqrt{21}\cos\theta$$
$$8 \quad 3$$

$$\cos\theta = \frac{8}{3\sqrt{21}} \Rightarrow \boxed{\theta = \cos^{-1} 8/3\sqrt{21}}$$

Ans

Exp # 15.

Find the directional derivative of $1/r$ in the direction of $\vec{r}$ when $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol:- As given that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Since

$$D.D = \vec{\nabla}\vec{r} \cdot \hat{r} \; \text{---}(i)$$

First find $\vec{\nabla}\vec{r}$.

Since we can write.

$$1/\vec{r} = \frac{1}{x\hat{i} + y\hat{j} + z\hat{k}} \Rightarrow$$

$$\& \; 1/r = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2} \quad \because r \text{ mean magnitude}$$

Now.

$$\vec{\nabla}r = \left(\frac{\partial}{\partial x}\hat{i} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2x\hat{i} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2y\hat{j} - \frac{1}{2}(x^2+y^2+z^2)^{-3/2} 2z\hat{k}$$

$$= -(x\hat{i} + y\hat{j} + z\hat{k})(x^2+y^2+z^2)^{-3/2}$$

$$= -\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}} \; \text{---}(ii) \quad \because \text{Take } (x^2+y^2+z^2)^{-3/2} \text{ common.}$$

Scanned with CamScanner

$$\hat{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$\{(1) \Rightarrow$

$$\nabla\phi \cdot \hat{r} = \frac{-(x\hat{i}+y\hat{j}+z\hat{k}) \cdot (x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2} \cdot (x^2+y^2+z^2)^{1/2}}$$

$$= \frac{-\cancel{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^{2}} \Rightarrow \frac{1}{-(x^2+y^2+z^2)}$$

$$\boxed{\nabla\phi \cdot \hat{r} = \frac{-1}{r^2}}$$ Ans

As $r = x+y+z$

$\frac{1}{2}+\frac{1}{2}$

Exp # 16

Find the direction in which the directional derivative of $\phi(x,y) = \frac{x^2+y^2}{xy}$ at $(1,1)$ is zero & hence find out component of velocity of vector $\vec{r} = (t^3+1)\hat{i} + t^2\hat{j}$ in same direction at $t=1$.

Sol:- (a) Find directional derivative at $(1,1) = 0$.
(b) Component of velocity.

(a) As $D.D = \vec{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x^2+y^2}{xy}\right)$$

$$= \left(\frac{2x(xy) - y(x^2+y^2)}{(xy)^2}\right)\hat{i} + \left(\frac{2y(xy) - x(x^2+y^2)}{(xy)^2}\right)$$

$$= \left(\frac{2x^2y - y(x^2+y^2)}{x^2y^2}\right)\hat{i} + \left(\frac{2xy^2 - x^3 - xy^2}{x^2y^2}\right)$$

Scanned with CamScanner

$$= \frac{(2x^2y - xy^2 - y^3)}{x^2y^2}\hat{i} + \frac{(xy^2 - y^3)}{x^2y^2}.$$

At $(1,1)$.

$$\vec{\nabla}\phi_{(1,1)} = 0\hat{i} + 0\hat{j} = 0. \quad \text{Proved}$$

$(\nabla\phi)_{(1,1)} = 0$ · is at any direction.

(b) Since $\vec{r} = (t^3+1)\hat{i} + t^2\hat{j}$.

$$\text{Velocity} = \frac{d\vec{r}}{dt} = 3t^2\hat{i} + 2t\hat{j}.$$

$$(\text{Velocity})_{(1,1)} = 3\hat{i} + 2\hat{j}.$$

Component of velocity at same direction.

So; $|\text{velocity}_{(1,1)}| = \sqrt{3^2+2^2} \Rightarrow \sqrt{13}$.

So;

$$\widehat{(\text{Velocity})} = \frac{3\hat{i}+2\hat{j}}{\sqrt{13}}.$$

Component of Velocity in same direction $= \dfrac{(3\hat{i}+2\hat{j}).(3\hat{i}+2\hat{j})}{\sqrt{13}} = \dfrac{9+4}{\sqrt{13}} = \dfrac{13}{\sqrt{13}}$.

$$= \frac{\sqrt{13}.\sqrt{13}}{\sqrt{13}} \Rightarrow \sqrt{13} \quad \text{Ans}$$

Exp #17 Find the directional derivative of $\phi(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find greatest rate of increase of ϕ.

Sol:- Req.

(a) Directional derivative.

(b) Greatest rate of increase

(a) As $D.D = \nabla\phi \cdot \text{unit vector} \rightarrow (1)$

First find $\vec{\nabla}\phi$.

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2yz + 4xz^2\right)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$\vec{\nabla}\phi$ at $(1,-2,1)$

$$(\vec{\nabla}\phi)_{(1,-2,1)} = \left(\underbrace{2(1)(-2)(1) + 4(1)^2}_{(4-4) \rightarrow 0}\right)\hat{i} + ((1)^2(1))\hat{j} + ((1)^2(-2) + 8(1)(1))\hat{k}$$

$$= 0 + \hat{j} + 6\hat{k} \Rightarrow \hat{j} + 6\hat{k}$$

Now let $\hat{u}$ be unit vector.

$$\hat{u} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + (-1)^2 + (-2)^2}} \Rightarrow \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}} =$$

$$= \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}$$

eq (1) $\Rightarrow$ $\hat{j} + 6\hat{k} \cdot \left(\frac{2\hat{i} - \hat{j} - 2\hat{k}}{3}\right)$

$\frac{1}{3}(-1,-12) = \frac{-13}{3}$

(b) Greatest rate of Increase.

$= |\nabla\phi|$

$= \sqrt{(1)^2+(6)^2} = \sqrt{37}$ Ans

Exp #18 Find D.D of function $\phi = x^2 - y^2 + 2z^2$ at point P (1, 2, 3) in the direction of line PQ where Q is the point (5, 0, 4).

Sol:- As D.D = $\vec{\nabla}\phi \cdot$ (unit vector)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) \rightarrow (1)$$

$$= 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$$

At point (1, 2, 3).

$$\vec{\nabla}\phi_{(1,2,3)} = 2\hat{i} - 4\hat{j} + 12\hat{k}$$

Now As P(1, 2, 3) & Q(5, 0, 4)

So, $\vec{PQ} = (4, -2, 1)$

$\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

So unit vector is

$$\hat{PQ} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} \Rightarrow \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$$

$$D.D = \frac{(2\hat{i}-4\hat{j}+12\hat{k})\cdot(4\hat{i}-2\hat{j}+\hat{k})}{\sqrt{21}}$$

$$= \frac{8+8+12}{\sqrt{21}} \Rightarrow 28/\sqrt{21} \quad \underline{\underline{Ans}}$$

<u>Exp # 19</u> Find D.D of $\phi = 4e^{2x-y+z}$ at $(1,1,-1)$ in direction towards point $(-3,5,6)$.

<u>Sol:-</u> As $D.D = \vec{\nabla}\phi_{(1,1,-1)} \cdot (\text{unit vector}) \longrightarrow (A)$.

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(4e^{2x-y+z}\right)$$

$$= \left(4e^{2x-y+z}(2)\right)\hat{i} + \left(4e^{2x-y+z}(-1)\right)\hat{j} + \left(4e^{(2x-y+z)}(1)\right)$$

$$= 8e^{2x-y+z}\hat{i} - 4e^{2x-y+z}\hat{j} + 4e^{2x-y+z}\hat{k}$$

$$= e^{2x-y+z}\left(8\hat{i}-4\hat{j}+4\hat{k}\right)$$

$$\nabla\phi_{(1,1,-1)} = e^{0}(8\hat{i}-4\hat{j}+4\hat{k}) = 8\hat{i}-4\hat{j}+4\hat{k}.$$

<u>Unit vector = ?</u>

Let $\vec{U}$ is vector. $\vec{U} = (-3,5,6)-(1,1,-1)$

$= (-4,+4,7) \Rightarrow -4\hat{i}+4\hat{j}+7\hat{k}$

$$\hat{U} = \frac{\vec{U}}{|U|} \Rightarrow \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{\sqrt{(-4)^2+(4)^2+(7)^2}}$$

$$= \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{\sqrt{81}} \Rightarrow \frac{-4\hat{i}+4\hat{j}+7\hat{k}}{9}.$$

$\text{Eq } ① \Rightarrow$

$$D.D = (8\hat{i}-4\hat{j}+4\hat{k}) \cdot \frac{(-4\hat{i}+4\hat{j}+7\hat{k})}{9}.$$

$$= \frac{-32-16+28}{9} = \left|\frac{-20}{9}\right| = \frac{20}{9} \text{ Ans}$$

<u>Exp#20.</u>
For the function $\phi(x,y) = \frac{x}{x^2+y^2}$, find the mag of directional der along a line mak an angle $30°$ with +ve x-axis at $(0,2)$

<u>Sol</u>:- As $D.D = \vec{\nabla}\phi \cdot (\text{Unit vector}) \rightarrow ①$
or
(Along a line)

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{x^2+y^2}\right).$$

$$= \left(\frac{1(x^2+y^2) - 2x \cdot x}{(x^2+y^2)^2}\right)\hat{i} + \left(\frac{2y \cdot x}{(x^2+y^2)^2}\right)\hat{j}.$$

$$= \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right)\hat{i} + \left(2xy/(x^2+y^2)^2\right)\hat{j}$$

$$= \left(\frac{y^2 - x^2}{(x^2+y^2)^2}\right)\hat{i} + \frac{2xy}{(x^2+y^2)^2}\hat{j}.$$

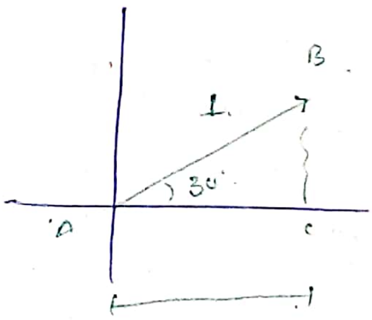
At pt $(0, 2)$.

$$\vec{\nabla}\phi_{(0,2)} = \frac{4}{16}\hat{i} + 0.$$

$$= \frac{1}{4}\hat{i} \rightarrow (2).$$

Now to find a line. i.e 30° incline.

$$OC = \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right).$$



(Head to tail)

From law of triangle.

$$\vec{AB} = \vec{OC} + \vec{CB}. \rightarrow (1).$$

$$OC = (1)\cos 30\,\hat{i}.$$

$$= \frac{\sqrt{3}}{2}\hat{i}.$$

$$\& \; CB = (1)\sin 30\,\hat{j}.$$

$$= \frac{1}{2}\hat{j}$$

& (A).

$$D.D = \frac{1}{4}\hat{i} \cdot \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$\boxed{D.D = \frac{\sqrt{3}}{2\times4}}$$

$$\Rightarrow \boxed{D.D = \frac{\sqrt{3}}{8}}$$ Ans

Exp #21 Find the D.D of $\vec{V}^2$, where $\vec{V} = x^2y\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at $(2,0,3)$ in direction outward normal to sphere $x^2+y^2+z^2 = 14$ at pt $(3,2,1)$.

Sol:- As given that;

$$\vec{V} = x^2y\hat{i} + zy^2\hat{j} + xz^2\hat{k}$$

$$\vec{V}^2 = (x^2y\hat{i} + zy^2\hat{j} + xz^2\hat{k}).(x^2y\hat{i} + zy^2\hat{j} + xz^2\hat{k})$$

$$= x^2y^4 + z^2y^4 + x^2z^4$$

Now;

the DD$(\vec{V}^2)$

$$D.D = \vec{\nabla}\phi.(\text{Unit vector}) \longrightarrow (1).$$

$$\nabla\vec{V}^2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^4 + z^2y^4 + x^2z^4)$$

$$= (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4z^2y^3)\hat{j} + (4x^2z^3)\hat{k}.$$

$$(\nabla\vec{V}^2)_{(2,0,3)} = (0 + 2(2)(3)^4)\hat{i} + (0+0)\hat{j} + (4(2)^2(3)^3)\hat{k}$$

$$= 4\times81\hat{i} + 0\hat{j} + 16\times27\hat{k}$$

$$= 324\hat{i} + 0\hat{j} + 432\hat{k}$$

Now;

Normal i.e ~~unit vector~~ gradient to

$$\phi = x^2+y^2+z^2-14 \text{ is;}$$

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-14).$$

$\vec{\nabla}\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$.

At $(3,2,1)$

$\nabla\phi_{(3,2,1)} = 6\hat{i} + 4\hat{j} + 2\hat{k}$.

Now unit vector is;

$$\hat{U} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{36+16+4}} = \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}}$$

& ⓐ $\Rightarrow$

$$D.D = (324\hat{i} + 0\hat{j} + 432\hat{k}) \cdot \frac{(6\hat{i}+4\hat{j}+2\hat{k})}{\sqrt{56}}$$

$$= \frac{1944 + 864}{\sqrt{56}} \Rightarrow \frac{2808}{\sqrt{56}} \Rightarrow \frac{2808}{\sqrt{4}\sqrt{14}}$$

$$= \frac{2808}{2\sqrt{14}}$$

$$\boxed{D.D = \frac{1404}{\sqrt{14}}}$$ Ans

Exp # 22

Find the D.D of $\nabla(\nabla f)$ at $(1,-2,1)$ in direction of normal to surface $xy^2z = 3x+z^2$ where $f = 2x^3y^2z^4$

Sol:- As given that,

$f = 2x^3y^2z^4$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\nabla(\nabla f) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k})$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

Now D.D of $\nabla(\nabla f)$

$$D.D = \nabla(\nabla(\nabla f))_{P(1,-2,1)} \cdot \text{Unitvector} \rightarrow \hat{e}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2)$$

$$= (12y^2z^4 + 12x^2z^4 + 72x^2y^2z^2)\hat{i} + (24xyz^4 + 48x^3yz^2)\hat{j}$$

At $(1,-2,1)$.

$$=\left(12(-2)^2(1)^4 + 12(1)^2(1)^4 + 72(1)^2(-2)^2(1)^2\right)\hat{i} + \left(24(1)(-2)(1)^4 + 48(1)(-2)(1)^{2}\right)\hat{j}$$

$$+\left(48(1)(-2)^2(1)^3 + 16(1)^3(1)^3 + 48(1)^3(-2)^2(1)\right)\hat{k}$$

$$= (48+12+288)\hat{i} + (-48-96)\hat{j} + (192+16+192)\hat{k}$$

$$= 348\hat{i} + (-144\hat{j}) + 400\hat{k}$$

Now normal to surface.

$$=\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(xy^2z - 3x - z^2\right)$$

$$= (y^2z-3)\hat{i} + (2xzy)\hat{j} + (xy^2-2z)\hat{k}$$

At point $(1,-2,1)$.

$$= 1\hat{i} + (-4\hat{j}) + 2\hat{k}$$

Unit vector to normal.

$$\hat{u} = \frac{1\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1^2 + (4)^2 + 2^2}}$$

$$\hat{u} = \frac{1\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{21}}$$

D.D $= (348\hat{i} - 144\hat{j} + 400\hat{k}) \cdot (\hat{i} - 4\hat{j} + 2\hat{k}) /$

$= \frac{1}{\sqrt{21}} (348 + 576 + 800)$

$$\boxed{D.D = \frac{1724}{\sqrt{21}}}$$ Ans.

Exp # 23 If the D.D of $\phi = ax^2y + by^2z + cz^2x$ at $(1,1,1)$ has max mag is in the direction parallel to line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$. Find values of a, b & c.

Exp # 24 if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that;

(i) $grad\ r = \frac{\vec{r}}{r}$ (ii) $grad\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$.

Sol:- As we know

$\vec{r}$ is positional vector i.e

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ — (i).

$\because r$ is magnitude

$r = \sqrt{x^2 + y^2 + z^2}$

$r^2 = x^2 + y^2 + z^2$ — (ii)

As;

$$\text{grad } r = \vec{\nabla} r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r$$

$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} \longrightarrow \text{(iii)}$$

Eq (2) $\Rightarrow$

$$r^2 = x^2 + y^2 + z^2.$$

$$\frac{\partial r^2}{\partial x} = 2x \Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\frac{\partial r}{\partial x} = x/r \longrightarrow \text{(iv)}$$

Eq (2) $\Rightarrow$

$$\frac{\partial r^2}{\partial y} = 2y \Rightarrow \cancel{2}r\frac{\partial r}{\partial y} = \cancel{2}y$$

$$\frac{\partial r}{\partial y} = y/r \longrightarrow \text{(v)}$$

Eq (ii) $\Rightarrow$

$$\frac{\partial r^2}{\partial} = 2z \Rightarrow \cancel{2}r\frac{\partial r}{\partial z} = \cancel{2}z$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \longrightarrow \text{(vi)}$$

Put values in eq (iii).

$$\text{grad } r = x/r\,\hat{i} + y/r\,\hat{j} + z/r\,\hat{k}.$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \longrightarrow \text{(vii)}.$$

Put eq (1) in eq (vii).

$$\boxed{\text{grad } r = \vec{r}/r} \text{ (proved)}$$

Scanned with CamScanner

(ii) $\text{grad}(1/r) = -\vec{r}/r^3$

Sol:-

- $\text{grad}\left(\frac{1}{r}\right) = \nabla \frac{1}{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(1/r\right)$

$$= \hat{i}\frac{\partial}{\partial x}\frac{1}{r} + \hat{j}\frac{\partial}{\partial y}\frac{1}{r} + \hat{k}\frac{1}{\partial z}\frac{1}{r}.$$

$$= -\hat{i}\frac{1}{r^2}\cdot\frac{\partial r}{\partial x} - \hat{j}\frac{1}{r^2}\frac{\partial r}{\partial y} - \hat{k}\frac{1}{r^2}\frac{\partial r}{\partial z}.$$

$\hookrightarrow$ (A)

Put eq (iv), (v) & (vi) in eq (A).

$$= -\hat{i}\frac{1}{r^2}\left(x/r\right) - \hat{j}\frac{1}{r^2}\left(y/r\right) - \hat{k}\frac{1}{r^2}\left(z/r\right)$$

$$= \frac{-(x\hat{i} + y\hat{j} + z\hat{k})}{r^3}$$

$\because x\hat{i} + y\hat{j} + \hat{k}z = \vec{r}$

$$\boxed{\text{grad}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}}$$

Proved

Example # 26

$$\vec{V} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \qquad \text{Find } \operatorname{div}\vec{V}.$$

Sol:-

As $\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V}$.

$$\therefore \vec{\nabla} = \left(\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\,\partial/\partial x + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\partial/\partial y + \frac{\partial}{\partial z}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{(x^2+y^2+z^2)^{1/2} - \frac{x}{2}(x^2+y^2+z^2)^{-1/2}(2x)}{x^2+y^2+z^2} + \frac{(\sqrt{x^2+y^2+z^2}) - \frac{y}{2}(x^2+y^2+z^2)^{-1/2}(2y)}{x^2+y^2+z^2}$$

$$+ \frac{\sqrt{x^2+y^2+z^2} - \frac{z}{2}(x^2+y^2+z^2)^{-1/2}(2z)}{x^2+y^2+z^2}.$$

$$\frac{(x^2+y^2+z^2) - x^2}{(x^2+y^2+z^2)^{1/2}\cdot(x^2+y^2+z^2)} + \frac{(x^2+y^2+z^2) - y^2}{(x^2+y^2+z^2)(x^2+y^2+z^2)^{1/2}}$$

$$+ \frac{x^2+y^2+z^2 - z^2}{(x^2+y^2+z^2)(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2+z^2-y^2}{(x^2+y^2+z^2)^{3/2}} + \frac{(x^2+y^2+z^2)-z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2}{(x^2+y^2+z^2)^{-1}(x^2+y^2+z^2)^{3/2}}$$

$$\frac{3}{2}-1$$

$$\frac{3-2}{2} = \frac{1}{2}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}} \quad \underline{\text{Ans}}$$

Exp #1 27 & 28 in copy →

Exp #130. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ & $\vec{a}$ is a constant vector. Find the value of $\operatorname{div}\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)$

Sol:- As $\vec{a}$ is vector so,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

Expand by R_1.

$$= \hat{i}(a_2 z - a_3 y) - (a_1 z - a_3 x)\hat{j} + (a_1 y - a_2 x)\hat{k}$$

$$\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n} = \frac{(a_2z-a_3y)\hat{i}-(a_1z-a_3x)\hat{j}+(a_1y-a_2x)\hat{k}}{(x^2+y^2+z^2)^{n/2}}$$

$\because \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$|\vec{r}|^n = \left((x^2+y^2+z^2)^{1/2}\right)^n$

= Now apply div.

$$\text{div}\left(\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}\right) = \vec{\nabla}\left(\frac{\vec{a}\times\vec{r}}{|\vec{r}|^n}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\quad\right)$$

constant

$\frac{n}{2}-1 = \frac{n-2}{2}$

$$= \frac{\partial}{\partial x}\frac{a_2z-a_3y}{(x^2+y^2+z^2)^{n/2}} - \frac{\partial}{\partial y}\frac{(a_1z-a_3x)}{(x^2+y^2+z^2)^{n/2}} + \frac{\partial}{\partial z}\frac{(a_1y-a_2x)}{(x^2+y^2+z^2)^{n/2}}$$

$$= (a_2z-a_3y)\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-n/2} - (a_1z-a_3x)\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-n/2} + (a_1y-a_2x)\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-n/2}$$

$$= (a_2z-a_3y)\frac{-n}{\cancel{2}}(x^2+y^2+z^2)^{-n/2-1}(\cancel{2}x) - (a_1z-a_3x)\left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-n/2-1}(2y) + (a_1y-a_2x)\left(-\frac{n}{2}\right)(x^2+y^2+z^2)^{-n/2-1}2z$$

$$= -n(a_2z-a_3y)(x^2+y^2+z^2)^{\frac{-n-2}{2}}x + n(a_1z-a_3x)(x^2+y^2+z^2)^{\frac{-n-2}{2}}y - n(a_1y-a_2x)(x^2+y^2+z^2)^{\frac{-n-2}{2}}z$$

$$= \frac{-nx(a_2z-a_3y)}{(x^2+y^2+z^2)^{\frac{n+2}{2}}} + \frac{ny(a_1z-a_3x)}{(x^2+y^2+z^2)^{n+2/2}} - \frac{nz(a_1y-a_2x)}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}$$

$$= \frac{n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}\left[(a_2z-a_3y)x + y(a_1z-a_3x) + z(a_1y-a_2x)\right]$$

$\Rightarrow (0)$

$$= \frac{-n}{(x^2+y^2+z^2)^{\frac{n+2}{2}}}\left[\cancel{a_2zx} - \cancel{a_3yx} - \cancel{a_1zy} + \cancel{a_3xy} + \cancel{a_1zy} - \cancel{a_2zx}\right] = 0$$

Ans

Example # 32

Show that;

$$\operatorname{div}(\operatorname{grad} r^n) = n(n+1)\, r^{n-2} \quad \text{where } r = \sqrt{x^2+y^2+z^2}$$

Hence show that $\nabla^2(1/r) = 0$.

Sol:- $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)\, r^{n-2} \longrightarrow ①$.

$L.H.S = \operatorname{div}(\operatorname{grad} r^n) \longrightarrow ②$.

$\operatorname{grad} r^n = ?$

$\because \nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$

As $\operatorname{grad} r^n = \nabla r^n$.

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(r^n)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} r^n + \hat{j}\frac{\partial}{\partial y} r^n + \hat{k}\frac{\partial}{\partial z} r^n\right)$$

$$= n r^{n-1}\frac{\partial r}{\partial x}\hat{i} + n r^{n-1}\frac{\partial r}{\partial y}\hat{j} + n r^{n-1}\frac{\partial r}{\partial x}\hat{k}$$

$$= n r^{n-1}\left[\frac{\partial r}{\partial x}\hat{i} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right] \longrightarrow (A)$$

As $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$\frac{\partial r^2}{\partial x} = 2x$

$2r\frac{\partial r}{\partial x} \Rightarrow 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = x/r}$

Similarly

$\frac{\partial r}{\partial y} = y/r$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

Scanned with CamScanner

use in eq (4)

$$= n r^{n-1}\left[\left(\frac{x}{r}\right)\hat{i} + \left(\frac{y}{r}\right)\hat{j} + \left(\frac{z}{r}\right)\hat{k}\right].$$

$$= n r^{n-1}\left(\frac{1}{r}\right)\left[x\hat{i} + y\hat{j} + z\hat{k}\right].$$

$$= n\, r^{n-1}\, r^{-1}\left[\vec{r}\right]$$

$$= n\, r^{n-2}\left[\vec{r}\right]. \longrightarrow (3).$$

Put in eq (2).

$$\operatorname{div}(\operatorname{grad} r^n) = \operatorname{div}\left[n\, r^{n-2}\vec{r}\right]$$

$$= \operatorname{div}\left(n\, r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})\right)$$

$\because \operatorname{div} = \vec{\nabla}.$ grad.

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(n\, r^{n-2}x\,\hat{i} + n\, r^{n-2}y\,\hat{j} + n\, r^{n-2}z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(n\, r^{n-2}x\right) + \frac{\partial}{\partial y}\left(n\, r^{n-2}y\right) + \frac{\partial}{\partial z}\left(n\, r^{n-2}z\right)$$

product rule

$$= \left(n r^{n-2} + x n(n-2)\, r^{n-2-1}\cdot\frac{\partial r}{\partial x}\right) + \left(n r^{n-2} + y n(n-2)\, r^{n-2-1}\cdot\frac{\partial r}{\partial y}\right) + \left(n r^{n-2} + z n(n-2)\, r^{n-2-1}\frac{\partial z}{\partial r}\right)$$

$$= \left(n r^{n-2} + x n(n-2) r^{n-3}\frac{\partial r}{\partial x}\right) + \left(n r^{n-2} + y(n)(n-2) r^{n-3}\frac{\partial r}{\partial y}\right) + \left(n r^{n-2} + z n(n-2) r^{n-3}\frac{\partial z}{\partial r}\right)$$

$$= 3 n r^{n-2} + n(n-2) r^{n-3}\left(x\,\frac{\partial r}{\partial x} + y\,\frac{\partial r}{\partial y} + z\,\frac{\partial r}{\partial z}\right)$$

45

$$= 3nr^{n-2} + n(n-2)r^{n-3}\left[x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) + z\left(\frac{z}{r}\right)\right].$$

$\frac{\partial x}{\partial r} = \frac{x}{r}$, $\frac{\partial y}{\partial r} = \frac{y}{r}$, $\frac{\partial z}{\partial r} = \frac{z}{r}$

$$= 3nr^{n-2} + n(n-2)r^{n-3}\left[\frac{x^2+y^2+z^2}{r}\right].$$

$$= 3nr^{n-2} + n(n-2)r^{n-3}\cdot r^{-1}(x^2+y^2+z^2).$$

$$= 3nr^{n-2} + n(n-2)r^{n-4}\cdot r^2.$$

$r^2 = x^2+y^2+z^2$

$$= 3nr^{n-2} + n(n-2)r^{n-2}.$$

$$= r^{n-2}[3n + n(n-2)].$$

$$= r^{n-2}[3n + n^2 - 2n] \Rightarrow r^{n-2}(n^2+n).$$

$$\boxed{\operatorname{div}(\;) = r^{n-2}n(n+1)} \Rightarrow \nabla^2(r) = r^{n-2}n(n+1).$$

Proved

Put $n = 0$

As for r = Power = 1.

For $1/r$ = Power = −1.

$$\nabla^2(1/r) = r^{-1-2}(-1)(-1+1)$$

$$= r^{-3}(0)$$

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = 0}$$

Proved.

Exp # 34 if $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$. find value of Curl $\vec{V}$.

Sol:- As Curl $= \vec{\nabla} \times \vec{V}$.

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right).$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x/\sqrt{x^2+y^2+z^2} & \dfrac{y}{(x^2+y^2+z^2)^{1/2}} & \dfrac{z}{(x^2+y^2+z^2)^{1/2}} \end{vmatrix}$$

Expand by R_1.

$$= \cancel{\hat{i}(0-0) + \hat{j}(0-0)}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2+z^2)^{1/2}}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\left(\frac{z}{(x^2+y^2+z^2)^{1/2}}\right) - \right.$$

$$\left.\frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right)\right) + \hat{k}\left(\frac{\partial}{\partial x}\frac{y}{(x^2+y^2+z^2)^{1/2}} - \frac{\partial}{\partial y}\frac{x}{(x^2+y^2+z^2)^{1/2}}\right)$$

$$= \hat{i}\left(z(-1/2)\frac{1 \quad 2y}{(x^2+y^2+z^2)^{3/2}} - \frac{yz(-1/2)(2)}{(x^2+y^2+z^2)^{3/2}}\right) + \hat{j}\left(\frac{z(1/2)(2x)}{(x^2+y^2+z^2)^{3/2}} + \frac{x\,y\,\frac{1}{2}(2)}{(x^2+y^2+z^2)^{3/2}}\right)$$

Scanned with CamScanner

$$+\hat{k}\left(\frac{y\, 2x\,(-\tfrac{1}{2})}{(x^2+y^2+z^2)^{3/2}} + \frac{(\tfrac{1}{2})\, x\, 2y}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$=\hat{i}\left(\cancel{\frac{-yz}{(x^2+y^2+z^2)^{3/2}}}+\cancel{\frac{yz}{(x^2+y^2+z^2)^{3/2}}}\right)-\hat{j}\left(\cancel{\frac{-xz}{(x^2+y^2+z^2)^{3/2}}}+\cancel{\frac{xz}{(x^2+y^2+z^2)^{3/2}}}\right)+\hat{k}\left(\cancel{\frac{-xy}{(x^2+y^2+z^2)^{3/2}}}+\cancel{\frac{xy}{(x^2+y^2+z^2)^{3/2}}}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0 \quad \text{Ans.}$$

Example #36

Determine the constant a & b such that the curl of vector $A = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k}$ is zero

Sol.- As we know that;

Curl $= \vec{\nabla} \times A$

Since Curl $= 0$

So,

$$\vec{\nabla} \times A = 0.$$

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$0 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(2xy+3yz)\hat{i} + (x^2+axz-4z^2)\hat{j} - (3xy+byz)\hat{k}\right]$$

$$0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+3yz & x^2+axz-4z^2 & -(3xy+byz) \end{vmatrix}$$

~~Expand by R_1.~~

Expand by R_1.

$$0 = \hat{i}\left(\frac{\partial}{\partial y}\left(-(3xy+byz)\right) - \frac{\partial}{\partial z}\left(x^2+axz-4z^2\right)\right) - \hat{j}\left[\frac{\partial}{\partial x}\left(-(3xy+byz)\right) - \frac{\partial}{\partial z}\left(2xy+3yz\right)\right]$$

$$+\hat{k}\left(\frac{\partial}{\partial x}\left(x^2+axz-4z^2\right) - \frac{\partial}{\partial y}\left(2xy+3yz\right)\right].$$

$$0 = \hat{i}\left(-(3x+bz) - (ax-8z)\right) - \hat{j}\left(-3y+0-3y\right) + \hat{k}\left(2x+az-2x-3z\right)$$

$$= \hat{i}\left(-3x-bz-ax+8z\right) - \hat{j}\left(-6y\right) + \hat{k}\left(az-3z\right)$$

$$= \hat{i}\left(-x(3+a) - z(b-8)\right) + 6y\hat{j} + z(a-3)\hat{k}$$

$$0\hat{i}+0\hat{j}+0\hat{k} = \left(-x(3+a) - z(b-8)\right)\hat{i} + 6y\hat{j} + z(a-3)\hat{k}.$$

By Comparing co-efficient of $\hat{k}$.

$a-3=0$ & $a+3=0$

$a=3$ $\quad$ $a=-3$

Now ~~$3+a$~~ $b-8=0$

$b-8=0$

$b=8$

Ans

Scanned with CamScanner

Sample #40 Show that $\vec{v}(x,y,z) = 2xyz\,\hat{i} + (x^2z + 2y)\,\hat{j} + x^2y\,\hat{k}$ is irrotational

& find a scalar function $u(x,y,z)$ such that $\vec{v} = \text{grad}(u)$.

ⓐ Show $\vec{v}$ is irrotational.

ⓑ & Find scalar function $u(x,y,z)$.

Sol:- ⓐ As for irrotation

$$\vec{\nabla} \times \vec{v} = 0$$

So, $L.H.S = \vec{\nabla} \times \vec{v}$.

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(2xyz\,\hat{i} + (x^2z + 2y)\hat{j} + x^2y\,\hat{k}\right).$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz & x^2z + 2y & x^2y \end{vmatrix}$$

Exp by R_1

~~$= \hat{i}\left(\partial/\partial x\right.$~~

$$= \hat{i}\left(\partial/\partial y\,(x^2y) - \partial/\partial z\,(x^2z + 2y)\right) - \hat{j}\left(\partial/\partial x\,(x^2y) - \frac{\partial}{\partial z}(2xyz)\right) + \hat{k}\left(\partial/\partial x\,(x^2z + 2y) - \partial/\partial y\,(2xyz)\right).$$

$$= \hat{i}\left(\cancel{x^2} - \cancel{x^2}\right) - \hat{j}\left(\cancel{2xy} - \cancel{2xy}\right) + \hat{k}\left(\cancel{2xz} - \cancel{2xz}\right)$$

$$= 0$$

Hence $\vec{v}$ is irrotational.

(b) To find $u = ?$

Let $\vec{A} = \vec{\nabla} \phi (u)$, $\phi = ?$ $\Rightarrow$ $\vec{A} = \vec{\nabla} u$.

To find ϕu.

$du = \text{grad}(u)$.

$= \vec{\nabla}(u)$.

$= \left(\hat{i} \frac{\partial u}{\partial x} dx + \hat{j} \frac{\partial u}{\partial y} dy + \hat{k} \frac{\partial u}{\partial z} dz \right)$

$= \left(\hat{i} \frac{\partial u}{\partial x} + \hat{j} \frac{\partial u}{\partial y} + \hat{k} \frac{\partial u}{\partial z} \right) \cdot \left(\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz \right)$.

$du = \vec{\nabla} u \cdot d\vec{r}$.

$du = \vec{V} \cdot d\vec{r}$ $\quad \because \vec{\nabla} U = \vec{V}$.

~~Integrating b/s.~~

$du = \left(\left(2xyz\hat{i} + (x^2z + 2y)\hat{j} + x^2y\hat{k} \right) \cdot (\hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz) \right)$

$= \left(2xyz\, dx + (x^2z + 2y)\, dy + x^2y\, dz \right)$

Integrating b/s :

$\int du = \int 2xyz\, dx + \int (x^2z + 2y)\, dy + \int x^2y\, dz$.

$= x^2yz + \cancel{x^2yz} + y^2 + \cancel{x^2yz}\, 0$

$= x^2yz + y^2$ Ans.

As we take single from multiple

Exp # 42

A fluid motion is given by $\vec{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$

is motion irrotational? If so, find velocity potential.

Sol:- As for Irrotational

$$\vec{\nabla} \times \vec{V} = 0.$$

So;

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left((y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}.$$

$$= \hat{i}\left(\partial/\partial y\,(xy\cos z + y^2) - \partial/\partial z\,(x\sin z + 2yz)\right) - \hat{j}\left[\partial/\partial x\,(xy\cos z + y^2) - \partial/\partial z\,(y\sin z - \sin x)\right] + \hat{k}\left[\left(\partial/\partial x\,(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right)\right]$$

$$= \hat{i}\left(\cancel{x\cos z} + \cancel{2y} - (\cancel{x\sin z} + \cancel{2y})\right)\Big] - \hat{j}\left(\cancel{y\cos z} - \cancel{y\cos z}\right) + \hat{k}\left(\cancel{\cos z} - \cancel{\cos z}\right)$$

$= 0$ As $\vec{\nabla} \times \vec{V} = 0$ So irrotational.

Now; velocity Potential = ?

let $\vec{V} = \vec{\nabla} \times V$ → (B) where V is velocity potential.

To find v:

$$dv = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz$$

It can be written as

$$= \left(\frac{\partial}{\partial x}\hat{i} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$dv = \bar{\nabla} v . d\vec{r} \longrightarrow (A)$$

Put (B) in (A)

$$dv = \bar{A} . d\vec{r}$$

$\because r = x + y + z$

$dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$dv = \left((y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}\right).$$

$$\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

Now Integ b/s

$$\int dv = \int (y\sin z - \sin x)dx + \int (x\sin z + 2yz)dy + \int (xy\cos z + y^2)dz$$

$$\boxed{v = xy\sin z + \cos x + y^2 z + C \quad \text{Ans.}}$$

Ans

Exp # 44 Show that the vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ is irrotational as well as solinoidal.

Find Scalar potential.

Sol:-

As given that;

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

For irrotational.

$$\vec{\nabla} \times \vec{F} = 0.$$

$$\vec{\nabla} \times \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right).$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix}.$$

$\frac{3}{2} - 1 = \frac{3-2}{2} = \frac{1}{2}.$

$$= \hat{i}\left(\frac{\partial}{\partial y}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\right)$$

$$+ \hat{k}\left[\left(\frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)\right)\right.$$

$$= \hat{i}\left(\frac{\cancel{z(-3/2)2y}}{\cancel{(x^2+y^2+z^2)^{5/2}}} - \frac{\cancel{y(-3/2)(2z)}}{(x^2+y^2+z^2)^{5/2}}\right) - \hat{j}\left(\frac{\cancel{z(-3/2)(2x)}}{\cancel{(x^2+y^2+z^2)^{5/2}}} - \frac{\cancel{x(-3/2)(2z)}}{(x^2+y^2+z^2)^{5/2}}\right)$$

$$+\hat{k}\left(\frac{\cancel{y(-3/2)2x}}{\cancel{(x^2+y^2+z^2)^{5/2}}} - \frac{\cancel{x(-3/2)(2y)}}{\cancel{(x^2+y^2+z^2)^{5/2}}}\right)$$

$$= 0.$$

Thus $\vec{F}$ is irrotational.

<u>Now for Solinoid:-</u>

As we know for solinoidal.

$$Div\ \vec{F} = \vec{\nabla}.\vec{F} = 0.$$

$$\vec{\nabla}.\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= \left(\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\left(\frac{xz}{(x^2+y^2+z^2)^{3/2}}\right)\right)$$

$$= \left[\frac{(x^2+y^2+z^2)^{+3/2} - 3/2\,(x^2+y^2+z^2)^{+3/2-1}(2x)}{(x^2+y^2+z^2)^3}\right] + \left[\frac{(x^2+y^2+z^2)^{3/2} - \frac{3}{2}(x^2+y^2+z^2)^{-3/2-1}(2y)}{(x^2+y^2+z^2)^3}\right]$$

$$+ \left[\frac{(x^2+y^2+z^2)^{3/2} - 3/2\,(x^2+y^2+z^2)^{-3/2-1}(2z)}{(x^2+y^2+z^2)^3}\right].$$

$$= \left(\frac{(x^2+y^2+z^2)^{3/2} - 3x(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right) + \left(\frac{(x^2+y^2+z^2)^{3/2} - 3y(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right) + \left(\frac{(x^2+y^2+z^2)^{3/2} - 3z(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\right)$$

$$\frac{(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^{3/2}}\left[\cancel{(x^2+y^2+z^2-3x^2+x^2+y^2+z^2-3y^2+x^2+y^2+z^2-3z^2)}\right]$$
$$=0$$

$$=\frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}}(0)=0$$

Thus it is solinoidal.

(c) Scalar potential = ?

let $\vec{F}=\vec{\nabla}.\phi$. ; let ϕ = Scala potential.

$\hookrightarrow$ (1).

As; $d\phi=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz\right)$.

It can be write as;

$$d\theta=\underbrace{\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)}_{\vec{\nabla}\phi}\underbrace{(\hat{i}dx+\hat{j}dy+\hat{k}dz)}_{dr}.$$

$$d\theta=\vec{\nabla}\phi.dr.\longrightarrow(2)$$

$d\phi=$ Put $\vec{F}$ (2).

$d\phi=\vec{F}.dr.$

By putting value of $\vec{F}$.

$$d\phi = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)$$

$$= \frac{x\,dx}{(x^2+y^2+z^2)^{3/2}} + \frac{y\,dy}{(x^2+y^2+z^2)^{3/2}} + \frac{z\,dz}{(x^2+y^2+z^2)^{3/2}}.$$

Integrating b/s.

$$\int d\phi = \int \frac{\begin{matrix} dx & dy & dz. \\ x & +y & +z. \end{matrix}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \int \frac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{2}\int \frac{2x\,dx+2y\,dy+2z\,dz}{(x^2+y^2+z^2)^{3/2}}.$$

$$= \frac{1}{2}\int (2x\,dx+2y\,dy+2z\,dz)(x^2+y^2+z^2)^{-3/2}.$$

$$= \frac{1}{2}\int (x^2+y^2+z^2)^{-3/2}(2x\,dx+2y\,dy+2z\,dz)$$

$$= \frac{1}{2}\frac{(x^2+y^2+z^2)^{-3/2+1}}{-3/2+1}+c. \qquad \because \int f\,f' = \frac{f^{n+1}}{n+1}+c$$

$$= \frac{1}{\cancel{2}}\frac{(x^2+y^2+z^2)^{-1/2}}{-1/\cancel{2}} \Rightarrow -1(x^2+y^2+z^2)^{-1/2} = \frac{-1}{(x^2+y^2+z^2)^{1/2}}.$$

$$= -\frac{1}{|r|} \text{ Ans} \qquad \because |r| = \sqrt{x^2+y^2+z^2}$$

Example # 46

Find the constants a, b, c so that $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational. & hence find function ϕ such that $\vec{F} = \nabla\phi$.

Sol:- As given $\vec{F}$ is irrotational so,

$$\vec{\nabla}\times\vec{F} = 0.$$

$$0 = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left((x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}\right)$$

$$0 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

Expand by R_1.

$$= \hat{i}\left(\partial/\partial y(4x+cy+2z) - \partial/\partial z(bx-3y-z)\right) - \hat{j}\left(\partial/\partial x(4x+cy+2z) - \partial/\partial z(x+2y+az)\right) + \hat{k}\left(\partial/\partial x(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right)$$

$$0 = \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0\hat{i} + 0\hat{j} + 0\hat{k}.$$

By comparing co-efficient.

$c+1=0$, $\quad 4-a=0$, $\quad b-2=0$.

$C=-1$, $\quad a=4$, $\quad b=2$.

Put values in eq of $\vec{F}$.

$$\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z);$$

Scanned with CamScanner

Now as;

$\vec{F} = \vec{\nabla}\phi \longrightarrow ①$

First we will find ϕ as required.

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

We can write it as;

$$d\phi = \underbrace{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)}_{\vec{\nabla}\phi} \underbrace{\left(\hat{i}dx + \hat{j}dy + \hat{k}dz\right)}_{d\vec{r}}.$$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} \longrightarrow ②$$

Put eq ① in eq ②.

$$d\phi = \vec{F}\cdot d\vec{r}$$

By putting values:

$$d\phi = \left[(x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}\right]\cdot(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$d\phi = (x+2y+4z)dx + (2x-3y-z)dy + (4x-y+2z)dz$$

Integrate b/s

$$\int d\phi = \int (x+2y+4z)dx + \int (2x-3y-z)dy + \int (4x-y+2z)dz$$

$$= \frac{x^2}{2} + 2xy + 4xz + x^2 - \frac{3y^2}{2} - zy + z^2 + c.$$

$$= \frac{x^2}{2} + 2xy + 4xz + x^2 - \frac{3y^2}{2} - zy + z^2 + c$$

Ans.

Exp # 48

If A is constant vector & $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$. than prove that $\text{curl}\left[(\vec{A}.\vec{R})\vec{R}\right] = \vec{A} \times \vec{R}$.

Sol:- Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$.

$$\vec{A}.\vec{R} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{A}.\vec{R} = A_1x + A_2y + A_3z.$$

$$(\vec{A}.\vec{R})R = (A_1x + A_2y + A_3z)(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= (A_1x^2 + A_2xy + A_3xz)\hat{i} + (A_1xy + A_2y^2 + A_3yz)\hat{j} + (A_1xz + A_2yz + A_3z^2)\hat{k}.$$

Now As $\text{Curl} = \vec{\nabla} \times [(\vec{A}.\vec{R})R]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left((A_1x^2 + A_2xy + A_3xz)\hat{i} + \hat{j}(A_1xy + A_2y^2 + A_3yz) + (A_1xz + A_2yz + A_3z^2)\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1x^2 + A_2xy + A_3xz & A_1xy + A_2y^2 + A_3yz & A_1xz + A_2yz + A_3z^2 \end{vmatrix}.$$

Exp. by R_1.

$$= \hat{i}\left(\partial/\partial y(A_1xz + A_2yz + A_3z^2) - \partial/\partial z(A_1xy + A_2y^2 + A_3yz)\right] - \hat{j}\left[\partial/\partial x(A_1xz + A_2yz + A_3z^2) - \partial/\partial z(A_1x^2 + A_2xy + A_3xz)\right] + \hat{k}\left(\partial/\partial x(A_1xy + A_2y^2 + A_3yz) - \partial/\partial y(A_1x^2 + A_2xy + A_3xz)\right.$$

$= \hat{i}(A_2 z - A_3 y) - \hat{j}(A_1 z - A_3 x) + \hat{k}(A_1 y - A_2 x)$

$\hookrightarrow$ (A)

Now,

~~$\vec{A} \times \vec{R} = (A_1\hat{i} + A_2$~~

$$\vec{A} \times \vec{R} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix}$$

Expand by R_1

$$= \hat{i}(\cancel{A_2 y}\ A_2 z - A_3 y) - \hat{j}(A_1 z - A_3 x) + \hat{k}(A_1 y - A_2 x)$$

$\hookrightarrow$ (B)

From (A) & (B).

L.H.S = R.H.S.

Thus

$\text{curl}[(\vec{A} \cdot \vec{R}) \cdot \vec{R}] = \vec{A} \times \vec{R}$

Proved