Statistics
lecture # 11

Random Variables:-

A numerical quantity whose value is determined by outcome of random experiment is called random variable.

or Random variable is real valued function defined on sample space.

Explaination & Concept:-

1. Tossing a coin:-

When coin is tossed, the possible outcomes are head & tail. let we interested in head.

so let x represent number of head

so the possible values for x will be

$X = 0, 1$

$0 =$ No head (mean tail)

$1 =$ One head ()

2. If two coin tossed together
Then X represent no.s of head

$X = 0, 1, 2$

3. Tossing Coin 3 times:-

X = No's of head

X = 0, 1, 2, 3.

∴ For Tossing "n" times

X = 1, 2, 3, ... n.

4 Throwing 2 dices:-

Let X represent sum of dots

∴ Sum on face of dots that appear after throwing.

so;

X = 2, 3, 4, ..., 12.

. A function has been named a random variable.

Random variable also called chance variable, stochastic variable or simply variable & is abbreviated as r.v.

The random variables are usually denoted by the capital letters such as X, Y, Z while the value taken by them are represented by small letters.

Random Variable

Discrete Random Variable

All the experiment we discussed i.e 1, 2, 3 & 4 are discrete random variable

→ If there is counting.

→ We have a fixed value for X.

→ It is found in considering counting.

Continous Random Variable.

Consider we have a box & there are 100 of different lengths whose lengths are from 3 to 6 feet. or 3 to 6". It X represent the rod selected random from that box. So the value of X ranges from 3 to 6. As value of X is not fixed. Here we have given interval for value of X

→ If there is measurement

→ When we defined R.V than we shift too to function.

→ functions are define differently in discrete & in continous.

→ **For Discrete Case:-**

(1) Probability distribution- (Pd)

As X ~~xxx~~ is random variable & whose

corresponding values can be obtained from random experiment & that have (values) corresponding probabilities.

Consider Coin tossing.
let x represent no's of head.
$X = 0, 1$

As 0 & 1 are obtain from random experiment so its 0 mean no head i.e tail whose probability is $1/2$.
Similarly for 1, probability is $1/2$.

So we can write;

x	0	1
$f(x)$	$1/2$	$1/2$

∴ Probability Distribution is a function that use values of random variable & generate probability of corresponding values.

For tossing two times.

x	0	1	2
$f(x)$	$1/4$	$1/2$	$1/4$

So **Def:-** "The tabular form of random experiment values "X" with their corresponding probability is called probability distribution."

X	x_1	x_2	x_3				x_n
$f(X)$							

For Third case (Experiment).

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8.

(4) For throwing a dice two times. & x represent sum of dots.

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

From ~~freq~~. Probability distribution we can easily find probability of specific no.

for Probability of 8 is 5/36

$P(8)$ or $P(11) = 5/36 + 2/36 = 7/36$.

Pb(between 5 & 8)

$= 4/36 + 5/36 + 6/36 + 5/36 = \frac{20}{36}$

Probability of 6 & 9 is zero as both are mutually exclusive

or Probability of prime & even number

so $\boxed{P = 1/36}$

If table is given,

whether probability distribution or not?

If we given a table as shown below.

x	-2	-1	0	1	2	3	4
$f(x)$	0	0.1	0.2	0.1	0.3	0.2	0.1

As we not know what X represent but we given its random values

As random experiment (what) is performed?

Now to check whether Pd?

Two condition:-

Function Pd if satisfies.

① $F(x) \geq 0$.

As from table, the probability value is not -ve & satisfy the condition.

② $\sum_x f(x) = 1$.

The sum of probability should be 1.

So the given function is Pd.

Explicit representation:-

The functions that discuss at beginning can be expressed explicitly as it is in tabular form so we can write them in function like.

Experiment #1

$f(x) = 1/2$

As for 0 it is $1/2$ & for 1 is also $1/2$.

Experiment #02

As we not take care of order can be select TH or HT.

So used combination.

$$f(x) = \frac{{}^2C_x}{4}, \quad x = 0, 1, 2$$

For $x = 0$ $\quad f(0) = \frac{{}^2C_0}{4} = \frac{1}{4}$ $\quad$ for $x = 1$ $\quad f(1) = \frac{{}^2C_1}{4} = 1/2$

As two coins are tossed so we have total heads of 2. $\leftarrow$ ${}^{2}C_x$

Experiment #03 :-

$$f(x) = \frac{\left({}^3C_x\right)}{8}, \quad x = 0, 1, 2, \ldots$$

For tossing n times $\quad \boxed{f(x) = \frac{{}^nC_x}{2^n}}$

Experiment # 04

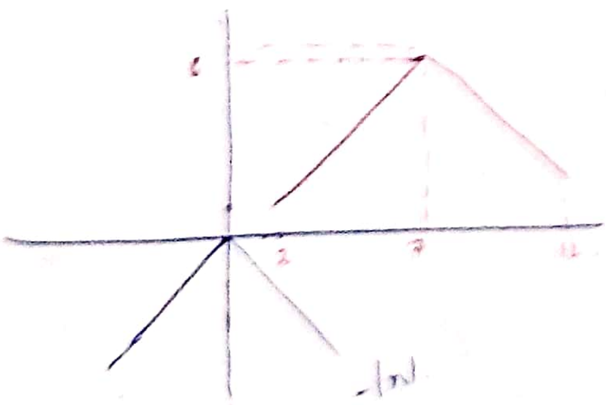
For experiment # 4 let consider it again

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

So,

$$\boxed{f(x) = \frac{6 - |x-7|}{36}}$$

This can be understand from graph



We get graph of -ve absolute. We will add 6 to move up through distance of 6.

$6 - |x|$.

Now to move to right of distance of 7 units so subtract 7 from x in modulus.

So,

$$f(x) = 6 - |x-7|$$

2. Distribution Function:- (DF)

Notation $F(x)$ Also called commulative Prob Fn

$F(x)$ is probability of $(X \leq x)$

$$F(x) = P(X \leq x)$$

$$= \sum_{y \leq x} (x = y)$$

let consider example 3

X	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

So the probability of $(X \leq 2)$

or $F(2) = ?$

$P(2) = P(X=0) + P(X=1) + P(X=2)$

$$= 1/8 + 3/8 + 3/8$$

$$\boxed{P(2) = 7/8}$$

Similarly for $P(X \leq 3)$ is again 7/8 Ans

- Distribution function is the sum of all probability including that mentioned number that consider other less than that value of probability

Properties:-

1. $F(-\infty) = 0$.

F of $(-\infty)$ mean F of anything behind lower limit is zero

Let consider example # 03

x	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

So, $F(-1) = 0$.

Or consider example # 04

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

So, In this case $F(1) = 0$
→ lower limit is 2. So 1 is behind the lower limit.

2 → $F(\infty) = 1$.

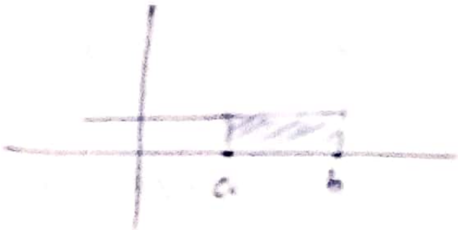
Where ∞ is for upper higher limit

So, In example # 4
$P(12) = 1$
→ we have 12 higher limit
So also $P(100) = 1$

2) $P(a < x \le b)$.

$$P(a < x \le b) = P(x \le b) - P(x \le a)$$

$$= P(b) - P(a)$$



~~If $P(a \le x \le b)$~~

~~Then~~

~~$P(a \le x \le b) = P(b) + P(a)$~~

Property #03

$F(x)$ is non decreasing

non decreasing mean steadily Increasing

At some point slope can be zero & mostly slope Increasing.

- Strickly Increasing. (Related to Ravi's topic).

Property #4 :-

$F(x)$ is at least continuous from right Side.

Example # 1

Find the probability distribution and distribution function for the number of heads when 3 balanced coins are tossed. Construct the probability histogram and graph of distribution.

Solution: When 3 coins are tossed then sample space is

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Let X represent no. of Heads

$X = 0, 1, 2, 3$

So the values of X are $0, 1, 2$ & 3, their corresponding probabilities are

$f(0) = P(X=0) = P(\{TTT\}) = 1/8$

$f(1) = P(X=1) = P(\{HTT, THT, TTH\}) = 3/8$

$f(2) = P(X=2) = P(\{HHT, HTH, THH\}) = 3/8$

$f(3) = P(X=3) = P(\{HHH\}) = 1/8$

As probability distribution in tabular form

x_i	0	1	2	3
$f(x_i)$	1/8	3/8	3/8	1/8

This is probability distribution.

In this case to find distribution function we will apply a class condition on x

i.e

① If $x < 0$ Then $P(X < x) = 0$
(As from properties)

② If $0 \leq x < 1$

$P(X < x) = F(x) = 1/8$.

③ If $1 \leq x < 2$

$P(X < x) = P(X = 1) + P(X = 0)$

$= 1/8 + 3/8 = 1/2$.

④ If $2 \leq x < 3$.

$P(X < x) = P(X=0) + P(X = 1) + P(X = 2)$

$= 1/8 + 3/8 + 3/8$

$= 7/8$

⑤ If $x \geq 3$

$P(X < x) = \sum_{i=1}^{3} P(X = i)$

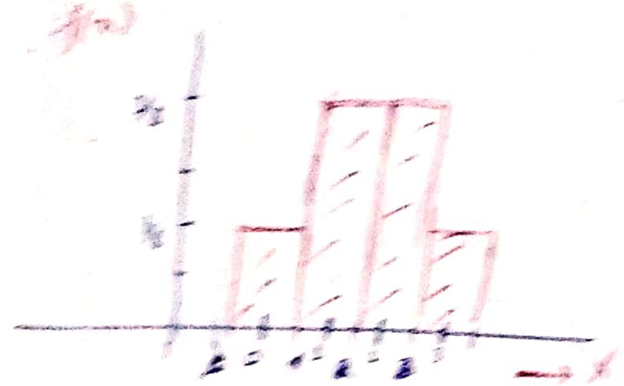
$= 1/8 + 3/8 + 3/8 + 1/8$.

$= 1$

So the distribution function can be written as;

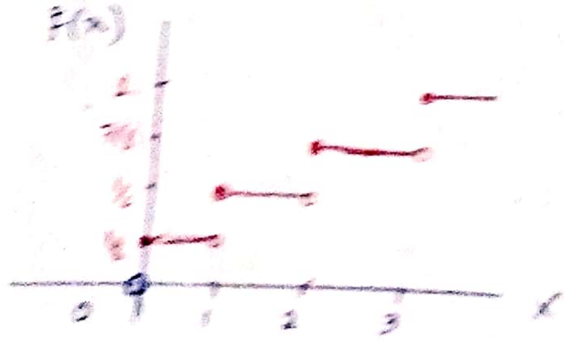
$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ 1/8 & \text{for } 0 \le x < 1 \\ 1/2 & \text{for } 1 \le x < 2 \\ 7/8 & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \ge 3 \end{cases}$$

How to make a histogram:-



- Histogram is for probability

Graph:



Example #7.2

(a) Find the probability distribution of sum of dots when two fair dices are thrown

(b) Use the probability to find the probability of obtaining (i) sum of 5 or 11, (ii) a sum that is greater than 8 (iii) A sum that is

. greater than 5 but less than or equal to 10.

Solution ⓐ

As probability distribution is tabular form of values of random experiment & their corresponding probabilities.

When two dice are thrown
let X represent sum of dots.

$X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.$

$f(2) = P(X=2) = P(\{1,1\}) = 1/36$

$P(3) = P(X=3) = P(\{(1,2),(2,1)\}) = 2/36.$

|

$P(12) = P(X=12) = P(\{(6,6)\}) = 1/36.$

So probability distribution is,

x_i	2	3	4	5	6	7	8	9	10	11	12
$f(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

It can also be written (Probability $= f(x)$)

$$f(x) = \frac{6 - |x-7|}{36}.$$

ⓑ Probability of sum of 8 or 11.

$P(\text{a sum of } 8 \text{ or } 11) = P[(X=8) \text{ or } (X=11)]$

$= P(X=8) + P(X=11)$

$= f(6) + f(11)$ From table

$= \frac{5}{36} + \frac{2}{36}$

$= \frac{7}{36}$

(ii) Probability sum is greater than 8
$= P(X > 8)$

$P(\text{sum is greater than } 8) = f(9) + f(10) + f(11) + f(12)$

$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$

$$\boxed{P(X > 8) = \frac{10}{36}}$$

(iii) Probability of sum is greater than 5 but less than or equal to 10.

$P(5 < x \leq 10)$

$= P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= f(6) + f(7) + f(8) + f(9) + f(10)$

$= \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36}$

$= \frac{5+6+5+4+3}{36}$

$\boxed{f(3 < x \leq 10) = \frac{23}{36}}$ Ans

Continous Random Variable:-

When there is measurement in random experiment and the variable that represent length, hight i.e measurement of that experiment is called Continous random variable.

→ The probability distribution as in discrete case is replaced by probability density (mass) function abbreviated as Pdf

Conditions for Pdf:-

A function $f(x)$ is said to be Pdf if

① $f(x) \geq 0$. for all interval in x

② $\int_{-\infty}^{\infty} f(x)dx = 1$

Here Integration is used Instead of Summation

Distribution function:-

$F(x) = P(X < x)$

$F(x) = \int_{-\infty}^{x} f(x)dx$

Note- In continous random variable probability of fixed number is zero

$P(X=a)=0.$

∴ As the exact value for Continous variable case cannot be find so $P(a)=0.$

- Probability distribution can be find as;

$f(x)=?$

$f(x)=F'(x)$

∴ Here used fundamental theoram of Calculas.

Probability for Interval:-

As in Continous r.v interval is given.

So; For Probability density function (Pdf)

$$P(a \le x \le b) = \int_a^b f(x)\,dx.$$

For Distribution function:-

$$P(a \le x \le b) = F(b) - F(a).$$

Example #7-3

(a) Find the value of K so that the function $f(x)$ defined as follows, may be density function

$f(x) = Kx \quad 0 \le x \le 2.$

$= 0 \quad$ else where.

(b). Find also the probability that both of two sample values will exceed 1.

(c) Compute the distribution function. $F(x)$.

Solution:- $f(x) = \begin{cases} Kx & 0 \le x \le 2 \\ 0 & \text{else where} \end{cases}$

To find value of K & to check whether it satisfies ~~&~~ condition for Pdf or not.

As for Pdf.

$f(x) \ge 0$

This condition will satisfy if $K > \cancel{=} 0$

~~50~~ (ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

As for given function lower limit $= 0$ & upper limit is 2.

So; $\int_0^2 Kx = 1 \Rightarrow K\left(\frac{x^2}{2}\right)\Big|_0^2 = 1.$

$k(4/2) = 1$.

$\boxed{k = 1/2}$

As $k > 0$ so the given function is pdf.

So;

$$f(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le 2 \\ 0 & \text{else where} \end{cases}$$

(b) Find probability both sample exceed 1.
If one sample exceed 1.

$$P(X > 1) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{1}{2}x\,dx = \left(\frac{x^2}{4}\right)\Big|_1^2$$

$$= 4/4 - 1/4 = \frac{4-1}{4} = 3/4$$

So when both samples exceed 1

$$P(\text{two samples exceed } 1) = 3/4 * 3/4$$

$$= 9/16.$$

(c) Distribution function:-

Apply conditions

(i) If $x < 0$ $F(x) = 0$.

(ii) If $0 < x < 2$ $F(x) = \int_{-\infty}^{0} f(x)dx + \int_0^x f(x)dx$

$$= 0 + \frac{x^2}{4}$$

$$F(x) = \frac{x^2}{4}$$

③ If $x \geq 2$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= 0 + \left(\frac{x^2}{4}\right)\Big|_0^2 + 0$$

$$= \frac{4}{4} \Rightarrow 1$$

$$\boxed{F(x) = 1}$$

So distribution function can be written as

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases}$$

Ans

Example # 7.4

A random variable X is of Continous type with Pdf

$$f(x) = 2x \quad 0 < x < 1$$
$$= 0 \quad \text{else where.}$$

Find (i) $P(X = 1/2)$ (ii) $P(X \leq 1/2)$ (iii) $P(X > \frac{1}{4})$

iv) $P(\frac{1}{4} \leq X \leq 1/2)$ & (v) $P(X \leq 1/2 \mid 1/3 \leq x \leq 2/3)$.

Solution:-

As for Continous random variable.

The probability of fixed number is zero.

So.

① $P(X = 1/2) = 0$.

(ii) $P(X \leq 1/2) = ?$ As there is function for such limits.

As; $P(x \leq 1/2) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/2} 2x\,dx$

$$= 0 + 2\left(\frac{x^2}{2}\right)\Big|_0^{1/2}$$

$$= (1/4) - 0 \Rightarrow \boxed{P(X \leq 1/2) = 1/4}$$

Ans

(iii) $P(X > 1/4) = \int_{1/4}^{1} 2x\,dx + \int_{1}^{\infty} 0\,dx$

$$(x^2)\Big|_{1/4}^{1} + 0$$

$$= \frac{1}{4}\left((1)^2 - \left(\tfrac{1}{4}\right)^2\right)$$

$$= \left(1 - \tfrac{1}{16}\right) \Rightarrow \left(\frac{16-1}{16}\right) = \tfrac{15}{16}.$$

$$\boxed{P(x > \tfrac{1}{4}) = \tfrac{15}{16}}$$ Ans

iv) $P\left(\tfrac{1}{4} \leq x < \tfrac{1}{2}\right).$

$$P\left(\tfrac{1}{4} \leq x < \tfrac{1}{2}\right) = \int_{1/4}^{1/2} 2x\,dx.$$

$$= (x^2)\Big|_{1/4}^{1/2}$$

$$= \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{4}\right)^2 = \tfrac{1}{4} - \tfrac{1}{16}.$$

$$= \frac{4-1}{16} = \tfrac{3}{16}$$ Ans

(v). $P\left(x \leq \tfrac{1}{2} \,\middle|\, \tfrac{1}{3} \leq x \leq \tfrac{2}{3}\right).$

As; $P\left(x \leq \tfrac{1}{2} \,\middle|\, \tfrac{1}{3} \leq x \leq \tfrac{2}{3}\right) = \dfrac{P\left(\tfrac{1}{3} \leq X \leq \tfrac{1}{2}\right)}{P\left(\tfrac{1}{3} \leq X < \tfrac{2}{3}\right)}$

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$$= \frac{\int_{1/3}^{1/2} 2x\,dx}{\int_{1/3}^{2/3} 2x\,dx}$$

$$= \frac{(x^2)\Big|_{1/3}^{1/2}}{(x^2)\Big|_{1/3}^{2/3}}$$

$$= \frac{(1/2)^2 - (1/3)^2}{(2/3)^2 - (1/3)^2}$$

$$= \frac{1/4 - 1/9}{4/9 - 1/9} \Rightarrow \frac{\frac{9-4}{36}}{\frac{4-1}{9}}$$

$$= \frac{5}{36} \div 3/9$$

$$= 5/36 \times 9/3$$

$$=$$

$$\boxed{P(x \le 1/2 \mid 1/3 \le x \le 2/3) = 5/12}$$ Ans

Example # 7.5

A continuous r.v X has the c.d.f $F(x)$ as follows

$F(x) = 0$ for $x < 0$

$= \frac{2x^2}{5}$ for $0 < x \le 1$

$= -\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right)$ for $1 \le x \le 2$

$= 1$ elsewhere for $x \ge 2$

Find p.d.f & $P(|X| < 1.5)$

Solution:

To find pdf

As for continuous,

$f(x) = F'(x)$

$f(x) = \frac{4x}{5}$ $\quad 0 < x \le 1$

$= \frac{2}{5}(3 - x)$ $\quad 1 < x \le 2$

$= 0$ elsewhere

Now $P(|x| < 1.5)$ $\quad$ as $|x| < 1.5$ ⇒ $-1.5 < x < 1.5$

$P(|x| < 1.5) = P(-1.5 < x < 1.5)$

$= \int_{-\infty}^{-1.5} 0\,dx + \int_{-1.5}^{0} 0\,dx + \int_{0}^{1} \frac{4x}{5} + \int_{1}^{1.5} \frac{2}{5}(3 - x)\,dx$

$$= 0 + 0 + \left[\frac{2x^5}{5}\right]_0^1 + \left[\frac{2}{5}\left(3x - \frac{x^2}{2}\right)\right]_1^{1.5}$$

$$= \frac{2}{5} + \frac{2}{5}\left[\left(3(1.5) - \frac{(1.5)^2}{2}\right) - \left(3(1) - \frac{1}{2}\right)\right]$$

$$= 2/5 + \frac{2}{5}\left[(4.5 - 1.125) - (3 - 1/2)\right]$$

$$= 2/5 + 2/5($$

$$= 2/5 + 0.35$$

$$\boxed{P(|x| < 1.5) = 0.75} \quad \text{Ans.}$$