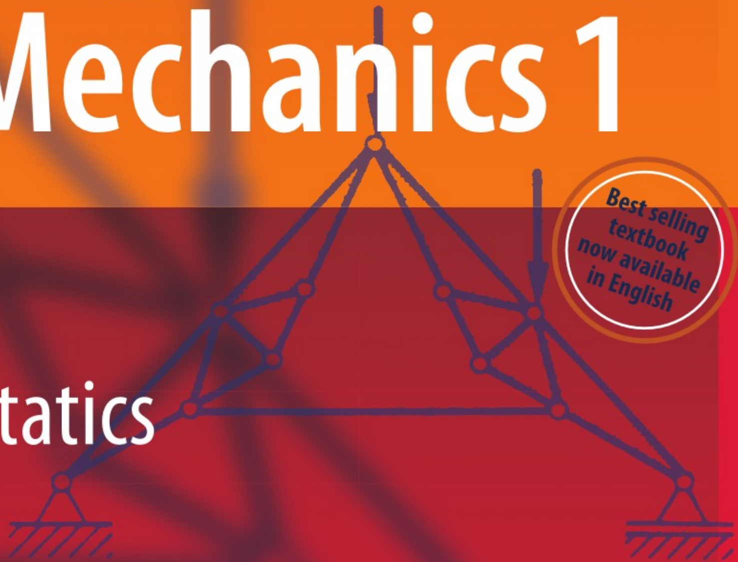
Dietmar Gross · Werner Hauger
Jörg Schröder · Wolfgang A. Wall
Nimal Rajapakse

Engineering Mechanics 1



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Statics

Second Edition

Springer

Springer Textbook

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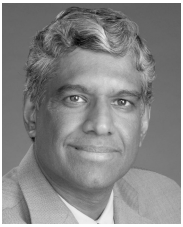

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Preface

Statics is the first volume of a three-volume textbook on Engineering Mechanics. Volume 2 deals with *Mechanics of Materials*; Volume 3 contains *Particle Dynamics and Rigid Body Dynamics.* The original German version of this series is the bestselling textbook on mechanics for nearly three decades and its 11th edition has already been published.

It is our intention to present to engineering students the basic concepts and principles of mechanics in the clearest and simplest form possible. A major objective of this book is to help the students to develop problem solving skills in a systematic manner.

The book developed out of many years of teaching experience gained by the authors while giving courses on engineering mechanics to students of mechanical, civil and electrical engineering. The contents of the book correspond to the topics normally covered in courses on basic engineering mechanics at universities and colleges. The theory is presented in as simple a form as the subject allows without being imprecise. This approach makes the text accessible to students from different disciplines and allows for their different educational backgrounds. Another aim of the book is to provide students as well as practising engineers with a solid foundation to help them bridge the gaps between undergraduate studies, advanced courses on mechanics and practical engineering problems.

A thorough understanding of the theory cannot be acquired by merely studying textbooks. The application of the seemingly simple theory to actual engineering problems can be mastered only if the student takes an active part in solving the numerous examples in this book. It is recommended that the reader tries to solve the problems independently without resorting to the given solutions. To demonstrate the principal way of how to apply the theory we deliberately placed no emphasis on numerical solutions and numerical results.

As a special feature the textbook offers the TM-Tools. Students may solve various problems of mechanics using these tools. They can be found at the web address <www.springer.com/engineering/grundlagen/tm-tools>.

In the second edition the text was revised and part of the notation was changed to make it compatible with the usual notation in English speaking countries. To provide the students with more material to develop their skills in solving problems, additional Supplementary Examples are supplied.

We gratefully acknowledge the support and the cooperation of the staff of Springer who were responsive to our wishes and helped to create the present layout of the books.

Darmstadt, Essen, Munich and Vancouver,
Summer 2012

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Introduction

Mechanics is the oldest and the most highly developed branch of physics. As important foundation of engineering, its relevance continues to increase as its range of application grows.

The tasks of mechanics include the description and determination of the motion of bodies, as well as the investigation of the forces associated with the motion. Technical examples of such motions are the rolling wheel of a vehicle, the flow of a fluid in a duct, the flight of an airplane and the orbit of a satellite. “Motion” in a generalized sense includes the deflection of a bridge or the deformation of a structural element under the influence of a load. An important special case is the state of rest; a building, dam or television tower should be constructed in such a way that it does *not* move or collapse.

Mechanics is based on only a few laws of nature which have an *axiomatic* character. These are statements based on numerous observations and regarded as being known from experience. The conclusions drawn from these laws are also confirmed by experience. Mechanical quantities such as velocity, mass, force, momentum or energy describing the mechanical properties of a system are connected within these axioms and within the resulting theorems.

Real bodies or real technical systems with their multifaceted properties are neither considered in the basic principles nor in their applications to technical problems. Instead, models are investigated that possess the essential mechanical characteristics of the real bodies or systems. Examples of these idealisations are a *rigid body* or a *mass point*. Of course, a real body or a structural element is always deformable to a certain extent. However, they may be considered as being *rigid bodies* if the deformation does not play an essential role in the behaviour of the mechanical system. To investigate the path of a stone thrown by hand or the orbit of a planet in the solar system, it is usually sufficient to view these bodies as being *mass points*, since their dimensions are very small compared with the distances covered.

In mechanics we use mathematics as an exact language. Only mathematics enables precise formulation without reference to a certain place or a certain time and allows to describe and comprehend mechanical processes. If an engineer wants to solve a technical problem with the aid of mechanics he or she has to replace the real technical system with a model that can be analysed mathematically by applying the basic mechanical laws. Finally, the mathematical solution has to be interpreted mechanically and evaluated technically.

Since it is essential to learn and understand the basic principles and their correct application from the beginning, the question of modelling will be mostly omitted in this text, since it requires a high degree of competence and experience. The mechanical analysis of an idealised system in which the real technical system may not always be easily recognised is, however, not simply an unrealistic game. It will familiarise students with the principles of mechanics and thus enable them to solve practical engineering problems independently.

Mechanics may be classified according to various criteria. Depending on the state of the material under consideration, one speaks of the *mechanics of solids*, *hydrodynamics* or *gasdynamics*. In this text we will consider solid bodies only, which can be classified as *rigid*, *elastic* or *plastic* bodies. In the case of a liquid one distinguishes between a *frictionless* and a *viscous* liquid. Again, the characteristics *rigid*, *elastic* or *viscous* are idealisations that make the essential properties of the real material accessible to mathematical treatment.

According to the main task of mechanics, namely, the investigation of the state of rest or motion under the action of forces, mechanics may be divided into *statics* and *dynamics*. Statics (Latin: status = standing) deals with the equilibrium of bodies subjected to forces. Dynamics (Greek: dynamis = force) is subdivided into *kinematics* and *kinetics*. Kinematics (Greek: kinesis = movement) investigates the motion of bodies without referring to forces as a cause or result of the motion. This means that it deals with the geometry of the motion in time and space, whereas kinetics relates the forces involved and the motion.

Alternatively, mechanics may be divided into *analytical mechanics* and *engineering mechanics*. In analytical mechanics, the analytical methods of mathematics are applied with the aim of gaining principal insight into the laws of mechanics. Here, details of the problems are of no particular interest. Engineering mechanics concentrates on the needs of the practising engineer. The engineer has to analyse bridges, cranes, buildings, machines, vehicles or components of microsystems to determine whether they are able to sustain certain loads or perform certain movements.

The historical origin of mechanics can be traced to ancient Greece, although of course mechanical insight derived from experience had been applied to tools and devices much earlier. Several cornerstones on statics were laid by the works of Archimedes (287–212): lever and fulcrum, block and tackle, center of gravity and buoyancy. Nothing more of great importance was discovered until the time of the Renaissance. Further progress was then made by Leonardo da Vinci (1452–1519) with his observations of the equilibrium on an inclined plane, and by Simon Stevin (1548–1620) with his discovery of the law of the composition of forces. The first investigations on dynamics can be traced back to Galileo Galilei (1564–1642) who discovered the law of gravitation. The laws of planetary motion by Johannes Kepler (1571–1630) and the numerous works of Christian Huygens (1629–1695) finally led to the formulation of the laws of motion by Isaac Newton (1643–1727). At this point, tremendous advancement was initiated, which went hand in hand with the development of analysis and is associated with the Bernoulli family (17th and 18th century), Leonhard Euler (1707–1783), Jean le Rond d'Alembert (1717–1783) and Joseph Louis Lagrange (1736–1813). As a result of the progress made in analytical and numerical methods – the latter especially boosted by computer technology – mechanics today continues to enlarge its field of application and makes more complex problems accessible to exact analysis. Mechanics also has its place in branches of sciences such as medicine, biology and the social sciences through the application of modelling and mathematical analysis.

Chapter 1

Basic Concepts

1

1 Basic Concepts

Objectives: *Statics* is the study of forces acting on bodies that are in equilibrium. To investigate statics problems, it is necessary to be familiar with some basic terms, formulas, and work principles. Of particular importance are the *method of sections*, the *law of action and reaction*, and the *free-body diagram*, as they are used to solve nearly all problems in statics.

1.1 Force

1.1

The concept of *force* can be taken from our daily experience. Although forces cannot be seen or directly observed, we are familiar with their effects. For example, a helical spring stretches when a weight is hung on it or when it is pulled. Our muscle tension conveys a qualitative feeling of the force in the spring. Similarly, a stone is accelerated by gravitational force during free fall, or by muscle force when it is thrown. Also, we feel the pressure of a body on our hand when we lift it. Assuming that gravity and its effects are known to us from experience, we can characterize a force as a quantity that is comparable to gravity.

In statics, bodies at rest are investigated. From experience we know that a body subject *only* to the effect of gravity, falls. To prevent a stone from falling, to keep it in equilibrium, we need to exert a force on it, for example our muscle force. In other words:

> A force is a physical quantity that can be brought into equilibrium with gravity.

1.2 Characteristics and Representation of a Force

1.2

A single force is characterized by three properties: magnitude, direction, and point of application.

The quantitative effect of a force is given by its *magnitude*. A qualitative feeling for the magnitude is conveyed by different muscle tensions when we lift different bodies or when we press against a wall with varying intensities. The magnitude F of a force can be measured by comparing it with gravity, i.e., with calibrated or standardized weights. If the body of weight W in Fig. 1.1 is in equilibrium, then $F = W$. The "Newton", abbreviated N (cf. Section 1.6), is used as the unit of force.

From experience we also know that force has a *direction*. While gravity always has an effect downwards (towards the earth's center), we can press against a tabletop in a perpendicular or in an inclined manner. The box on the smooth surface in Fig. 1.2 will

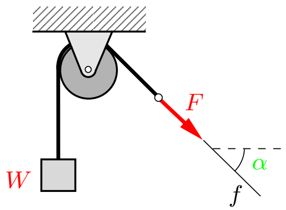


Fig. 1.1

move in different directions, depending on the direction of the force exerted upon it. The direction of the force can be described by its *line of action* and its sense of direction (orientation). In Fig. 1.1, the line of action f of the force F is inclined under the angle α to the horizontal. The sense of direction is indicated by the arrow.

Finally, a single force acts at a certain *point of application*. Depending on the location of point A in Fig. 1.2, the force will cause different movements of the box.

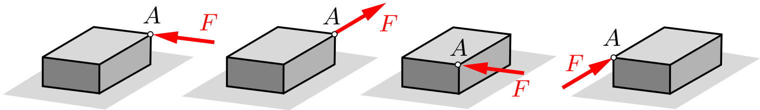


Fig. 1.2

A quantity determined by magnitude and direction is called a vector. In contrast to a free vector, which can be moved arbitrarily in space provided it maintains its direction, a force is tied to its line of action and has a point of application. Therefore, we conclude:

The force is a bound vector.

According to standard vector notation, a force is denoted by a boldfaced letter, for example by $\boldsymbol{F}$, and its magnitude by $|\boldsymbol{F}|$ or simply by F. In figures, a force is represented by an arrow, as shown in Figs. 1.1 and 1.2. Since the vector character usually is uniquely determined through the arrow, it is usually sufficient to write only the magnitude F of the force next to the arrow.

In Cartesian coordinates (see Fig. 1.3 and Appendix A.1), the force vector can be represented using the unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$

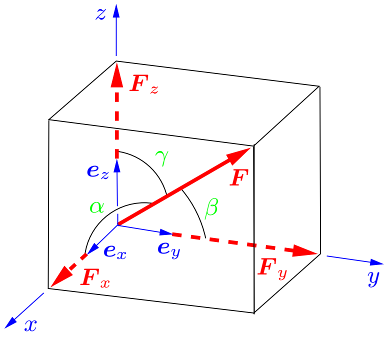


Fig. 1.3

by

$$F = F_x + F_y + F_z = F_x\, e_x + F_y\, e_y + F_z\, e_z\,. \tag{1.1}$$

Applying Pythagoras' theorem in space, the force vector's magnitude F is given by

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}\,. \tag{1.2}$$

The direction angles and therefore the direction of the force follow from

$$\cos\alpha = \frac{F_x}{F}\,, \quad \cos\beta = \frac{F_y}{F}\,, \quad \cos\gamma = \frac{F_z}{F}\,. \tag{1.3}$$

1.3 The Rigid Body

1.3

A body is called a *rigid body* if it does not deform under the influence of forces; the distances between different points of the body remain constant. This is, of course, an idealization of a real body composed of a reasonably stiff material which in many cases is fulfilled in a good approximation. From experience with such bodies it is known that a single force may be applied at any point on the line of action without changing its effect on the body as a whole (*principle of transmissibility*).

This principle is illustrated in Fig. 1.4. In the case of a deformable sphere, the effect of the force depends on the point of

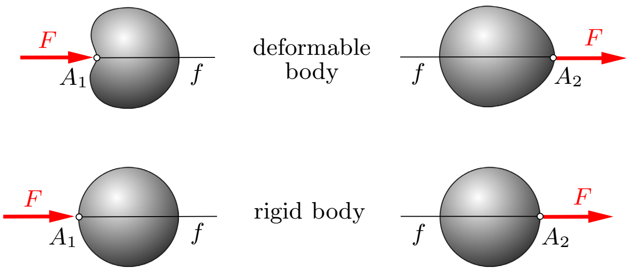


Fig. 1.4

application. In contrast, for a rigid sphere the effect of the force F on the entire body is the same, regardless of whether the body is pulled or pushed. In other words:

> The effect of a force on a rigid body is independent of the location of the point of application on the line of action. The forces acting on rigid bodies are "sliding vectors": they can arbitrarily be moved along their action lines.

A parallel displacement of forces changes their effect considerably. As experience shows, a body with weight W can be held in equilibrium if it is supported appropriately (underneath the center of gravity) by the force F, where $F = W$ (Fig. 1.5a). Displacing force F in a parallel manner causes the body to rotate (Fig. 1.5b).

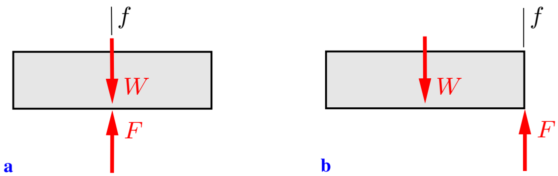


Fig. 1.5

1.4 Classification of Forces, Free-Body Diagram

A single force with a line of action and a point of application, called a *concentrated force*, is an idealization that in reality does not exist. It is almost realized when a body is loaded over a thin wire or a needlepoint. In nature, only two kinds of forces exist: volume forces and surface or area forces.

A *volume force* is a force that is distributed over the volume of a body or a portion thereof. Weight is an example of a volume force. Every small particle (infinitesimal volume element $\mathrm{d}V$) of the entire volume has a certain small (infinitesimal) weight $\mathrm{d}W$ (Fig. 1.6a). The sum of the force elements $\mathrm{d}W$, which are continuously distributed within the volume yields the total weight W. Other examples of volume forces include magnetic and electrical forces.

Area forces occur in the regions where two bodies are in contact. Examples of forces distributed over an area include the water pressure p at a dam (Fig. 1.6b), the snow load on a roof or the pressure of a body on a hand.

A further idealization used in mechanics is the *line force*, which comprises forces that are continuously distributed along a line. If a blade is pressed against an object and the finite thickness of the blade is disregarded, the line force q will act along the line of contact (Fig. 1.6c).

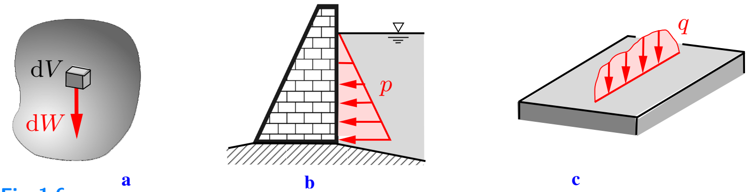


Fig. 1.6

Forces can also be classified according to other criteria. *Active forces* refer to the physically prescribed forces in a mechanical system, as for example the weight, the pressure of the wind or the snow load on a roof.

Reaction forces are generated if the freedom of movement of a body is constrained. For example, a falling stone is subjected only to an active force due to gravity, i.e., its weight. However, when the stone is held in the hand, its freedom of movement is constrained; the hand exerts a reaction force on the stone.

Reaction forces can be visualized only if the body is separated from its geometrical constraints. This procedure is called *freeing* or *cutting free* or *isolating* the body. In Fig. 1.7a, a beam is loaded by an active force W. Supports A and B prevent the beam from moving: they act on it through reaction forces that, for simplicity, are also denoted by A and B. These reaction forces are made visible in the so-called *free-body diagram* (Fig. 1.7b). It shows the forces acting on the body instead of the geometrical constraints through the supports. By this "freeing", the relevant forces become accessible to analysis (cf. Chapter 5). This procedure is still valid when a mechanical system becomes movable (dynamic) due to freeing. In this case, the system is regarded as being frozen when the reaction forces are determined. This is known as the *principle of solidification* (cf. Section 5.3).

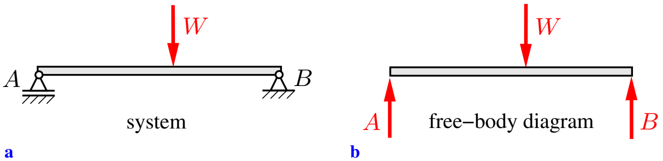


Fig. 1.7

A further classification is introduced by distinguishing between *external forces* and *internal forces*. An *external force* acts from the outside on a mechanical system. Active forces as well as reaction forces are external forces. *Internal forces* act between the parts of a system. They also can be visualized only by imaginary cutting or *sectioning* of the body. If the body in Fig. 1.8a is sectioned by an imaginary cut, the internal area forces p distributed over the cross-section must be included in the free-body diagram; they replace the initially perfect bonding between the two exposed surfaces (Fig. 1.8b). This procedure is based on the hypothesis, which is confirmed by experience, that the laws of mechanics are equally valid for parts of the system. Accordingly, the system initially consists of the complete body at rest. After the cut, the system consists of two parts that act on each other through area forces in such a way that each part is in equilibrium. This procedure, which enables calculation of the internal forces, is called the *method of sections*. It is valid for systems in equilibrium as well as for systems in motion.

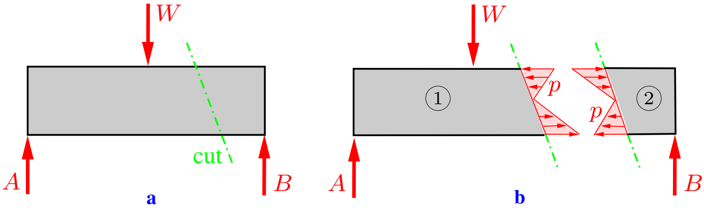


Fig. 1.8

Whether a force is an external or an internal force depends on the system to be investigated. If the entire body in Fig. 1.8a is considered to be the system, then the forces that become exposed by the cut are internal forces: they act between the parts of the system. On the other hand, if only part ① or only part ② of the body in Fig. 1.8b is considered to be the system, the corresponding forces are external forces.

As stated in Section 1.3, a force acting on a rigid body can be displaced along its line of action without changing its effect on the body. Consequently, the principle of transmissibility can be used in the analysis of the external forces. However, this principle can generally not be applied to internal forces. In this case, the body is sectioned by imaginary cuts, therefore it matters whether an external force acts on one or the other part.

The importance of internal forces in engineering sciences is derived from the fact that their magnitude is a measure of the stress in the material.

1.5 Law of Action and Reaction

1.5

A universally accepted law, based on everyday experience, is the *law of action and reaction*. This axiom states that a force always has a counteracting force of the same magnitude but of opposite direction. Therefore, a force can never exist alone. If a hand is pressed against a wall, the hand exerts a force F on the wall (Fig. 1.9a). An opposite force of the same magnitude acts from the wall on the hand. These forces can be made visible if the two bodies are separated at the area of contact. Note that the forces act upon two different bodies. Analogously, a body on earth has a certain weight W due to gravity. However, the body acts upon the earth

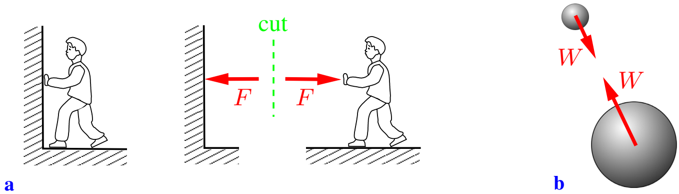


Fig. 1.9

with a force of equal magnitude: they attract each other (Fig. 1.9b). In short:

The forces that two bodies exert upon each other are of the same magnitude but of opposite directions and they lie on the same line of action.

This principle, which Newton succinctly expressed in Latin as

actio = reactio

is the third of Newton's axioms (cf. Volume 3). It is valid for long-range forces as well as for short-range forces, and it is independent of whether the bodies are at rest or in motion.

1.6

1.6 Dimensions and Units

In mechanics the three basic physical quantities length, time and mass are considered. Force is another important element that is considered; however, from a physical point of view, force is a derived quantity. All other mechanical quantities, such as velocity, momentum or energy can be expressed by these four quantities. The geometrical space where mechanical processes take place is three-dimensional. However, as a simplification the discussion is limited sometimes to two-dimensional or, in some cases, one-dimensional problems.

Associated with length, time, mass and force are their dimensions $[l]$, $[t]$, $[M]$ and $[F]$. According to the international SI unit system (Système International d'Unités), they are expressed using the base units meter (m), second (s) and kilogram (kg) and the derived unit newton (N). A force of 1 N gives a mass of 1 kg the acceleration of $1\,\mathrm{m/s^2}$: $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$. Volume forces have the dimension force per volume $[F/l^3]$ and are measured, for example, as a multiple of the unit $\mathrm{N/m^3}$. Similarly, area and line forces have the dimensions $[F/l^2]$ and $[F/l]$ and the units $\mathrm{N/m^2}$ and N/m, respectively.

The magnitude of a physical quantity is completely expressed by a number and the unit. The notations $F = 17\,\mathrm{N}$ or $l = 3\,\mathrm{m}$ represent a force of 17 newtons or a distance of 3 meters, respectively. In numerical calculations units are treated in the same way as numbers. For example, using the above quantities, $F \cdot l = 17\,\mathrm{N} \cdot 3\,\mathrm{m} = 17 \cdot 3\,\mathrm{Nm} = 51\,\mathrm{Nm}$. In physical equations, each side and each additive term must have the same dimension. This should always be kept in mind when equations are formulated or checked.

Very large or very small quantities are generally expressed by attaching prefixes to the units meter, second, newton, and so forth: k (kilo $= 10^3$), M (mega $= 10^6$), G (giga $= 10^9$) and m (milli $= 10^{-3}$), μ (micro $= 10^{-6}$), n (nano $= 10^{-9}$), respectively; for example: $1\,\mathrm{kN} = 10^3\,\mathrm{N}$.

Table 1.1

	U.S. Customary Unit		SI Equivalent
Length	1 ft		0.3048 m
	1 in	(12 in = 1 ft)	25.4 mm
	1 yd	(1 yd = 3 ft)	0.9144 m
	1 mi		1.609344 km
Force	1 lb		4.4482 N
Mass	1 slug		14.5939 kg

In the U.S. and some other English speaking countries the U.S. Customary system of units is still frequently used although the SI system is recommended. In this system length, time, force and mass are expressed using the base units foot (ft), second (s), pound (lb) and the derived mass unit, called a *slug*: 1 slug $= 1\ \mathrm{lb\,s^2/ft}$. As division and multiples of length the inch (in), yard (yd) and mile (mi) are used. In Table 1.1 common conversion factors are listed.

1.7

1.7 Solution of Statics Problems, Accuracy

To solve engineering problems in the field of mechanics a careful procedure is required that depends to a certain extent on the type of the problem. In any case, it is important that engineers express themselves clearly and in a way that can be readily understood since they have to present the formulation as well as the solution of a problem to other engineers and to people with no engineering background. This clarity is equally important for one's own process of understanding, since clear and precise formulations are the key to a correct solution. Although, as already mentioned, there is no fixed scheme for handling mechanical problems, the following steps are usually necessary:

1. Formulation of the engineering problem.
2. Establishing a mechanical model that maps all of the essential characteristics of the real system. Considerations regarding the quality of the mapping.
3. Solution of the mechanical problem using the established model. This includes:
   – Identification of the given and the unknown quantities. This is usually done with the aid of a sketch of the mechanical system. Symbols must be assigned to the unknown quantities.
   – Drawing of the free-body diagram with all the forces acting on the system.
   – Formulation of the mechanical equations, e.g. the equilibrium conditions.

- Formulation of the geometrical relationships (if needed).
- Solving the equations for the unknowns. It should be ensured in advance that the number of equations is equal to the number of unknowns.
- Display of the results.

4. Discussion and interpretation of the solution.

In the examples given in this textbook, usually the mechanical model is provided and Step 3 is concentrated upon, namely the solution of mechanical problems on the basis of models. Nevertheless, it should be kept in mind that these models are mappings of real bodies or systems whose behavior can sometimes be judged from daily experience. Therefore, it is always useful to compare the results of a calculation with expectations based on experience.

Regarding the accuracy of the results, it is necessary to distinguish between the numerical accuracy of calculations and the accuracy of the model. A numerical result depends on the precision of the input data and on the precision of our calculation. Therefore, the results can never be more precise than the input data. Consequently, results should never be expressed in a manner that suggests a non-existent accuracy (e.g., by many digits after the decimal point).

The accuracy of the result concerning the behavior of the real system depends on the quality of the model. For example, the trajectory of a stone that has been thrown can be determined by taking air resistance into account or by disregarding it. The results in each case will, of course, be different. It is the task of the engineer to develop a model in such a way that it has the potential to deliver the accuracy required for the concrete problem.

1.8

1.8 Summary

- Statics deals with bodies that are in equilibrium.
- A force acting on a rigid body can be represented by a vector that can be displaced arbitrarily along its line of action.
- An active force is prescribed by a law of physics. Example: the weight of a body due to earth's gravitational field.
- A reaction force is induced by the constrained freedom of movement of a body.
- Method of sections: reaction forces and internal forces can be made visible by virtual cuts and thus become accessible to an analysis.
- Free-body diagram: representation of all active forces and reaction forces which act on an isolated body. Note: mobile parts of the body can be regarded as being "frozen" (principle of solidification).
- Law of action and reaction: actio = reactio.
- Basic physical quantities are length, mass and time. The force is a derived quantity: $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$.
- In mechanics idealized models are investigated which have the essential characteristics of the real bodies or systems. Examples of such idealizations: rigid body, concentrated force.

Chapter 2

2

Forces with a Common Point of Application

2 Forces with a Common Point of Application

Objectives: In this chapter, systems of concentrated forces that have a common point of application are investigated. Such forces are called *concurrent forces*. Note that forces always act on a body; there are no forces without action on a body. In the case of a rigid body, the forces acting on it do not have to have the same point of application; it is sufficient that their lines of action intersect at a common point. Since in this case the force vectors are sliding vectors, they may be applied at any point along their lines of action without changing their effect on the body (principle of transmissibility). If all the forces acting on a body act in a plane, they are called *coplanar forces*.

Students will learn in this chapter how to determine the resultant of a system of concurrent forces and how to resolve force vectors into given directions. They will also learn how to correctly isolate the body under consideration and draw a free-body diagram, in order to be able to formulate the conditions of equilibrium.

2.1 Addition of Forces in a Plane

Consider a body that is subjected to two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$, whose lines of action intersect at point A (Fig. 2.1a). It is postulated that the two forces can be replaced by a statically equivalent force $\boldsymbol{R}$. This postulate is an axiom; it is known as the *parallelogram law of forces*. The force $\boldsymbol{R}$ is called the *resultant* of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. It is the diagonal of the parallelogram for which $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are adjacent sides. The axiom may be expressed in the following way:

> The effect of two nonparallel forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting at a point A of a body is the same as the effect of the single force $\boldsymbol{R}$ acting at the same point and obtained as the diagonal of the parallelogram formed by $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$.

The construction of the parallelogram is the geometrical representation of the summation of the vectors (see Appendix A.1):

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2\,. \tag{2.1}$$

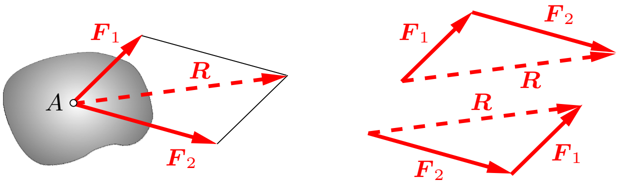
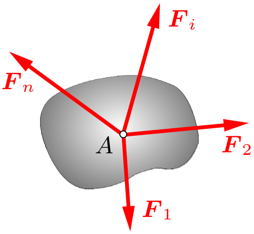


Fig. 2.1 a b

Now consider a system of n forces that all lie in a plane and whose lines of action intersect at point A (Fig. 2.2a). Such a system is called a *coplanar* system of *concurrent forces*. The resultant can be obtained through successive application of the parallelogram law of forces. Mathematically, the summation may be written in the form of the following vector equation:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \ldots + \boldsymbol{F}_n = \sum \boldsymbol{F}_i\,. \tag{2.2}$$

Since the system of forces is reduced to a *single* force, this process is called *reduction*. Note that the forces that act on a rigid body



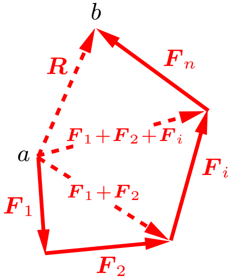


Fig. 2.2

are sliding vectors. Therefore, they do not have to act at point A; only their lines of action have to intersect at this point.

It is not necessary to draw the complete parallelogram to graphically determine the resultant; it is sufficient to draw a *force triangle*, as shown in Fig. 2.1b. This procedure has the disadvantage that the lines of action cannot be seen to intersect at one point. This disadvantage, however, is more than compensated for by the fact that the construction can easily be extended to an arbitrary number of n forces, which are added head-to-tail as shown in Fig. 2.2b. The sequence of the addition is arbitrary; in particular, it is immaterial which vector is chosen to be the first one. The resultant $\boldsymbol{R}$ is the vector that points from the initial point a to the endpoint b of the *force polygon*.

It is appropriate to use a *layout plan* (also called layout diagram) and a *force plan* (also denoted a vector diagram) to solve a problem graphically. The layout plan represents the geometrical specifications of the problem; in general, it has to be drawn to scale (e.g., $1\,\text{cm} \mathrel{\widehat{=}} 1\,\text{m}$). In the case of a system of concurrent forces, it contains only the lines of action of the forces. The force polygon is constructed in a force plan. In the case of a graphical solution, it must be drawn using a scale (e.g., $1\,\text{cm} \mathrel{\widehat{=}} 10\,\text{N}$).

Sometimes problems are solved with just the aid of a sketch of the force plan. The solution is obtained from the force plan, for example, by trigonometry. It is then not necessary to draw the plan to scale. The corresponding method is partly graphical and partly analytical and can be called a “graphic-analytical” method. This procedure is applied, for example, in the Examples 2.1 and 2.4.

Example 2.1 A hook carries two forces F_1 and F_2, which define the angle α (Fig. 2.3a). **E2.1**

Determine the magnitude and direction of the resultant.

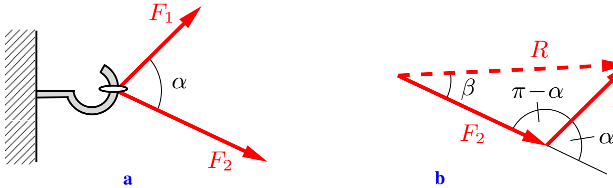


Fig. 2.3

Solution Since the problem will be solved by trigonometry (and since the magnitudes of the forces are not given numerically), a sketch of the force triangle is drawn, but not to scale (Fig. 2.3b).

We assume that the magnitudes of the forces F_1 and F_2 and the angle α are known quantities in this force plan. Then the magnitude of the resultant follows from the law of cosines:

$$R^2 = F_1^2 + F_2^2 - 2\,F_1\,F_2 \cos(\pi - \alpha)$$

or

$$\underline{\underline{R = \sqrt{F_1^2 + F_2^2 + 2\,F_1\,F_2 \cos\alpha}}}\,.$$

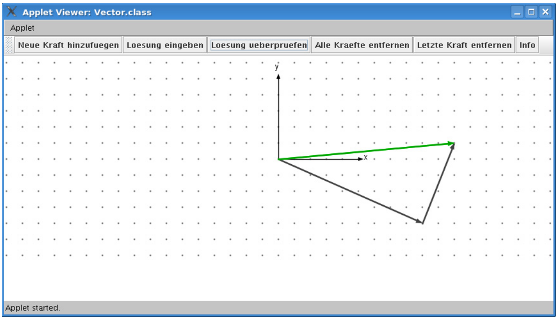
The angle β gives the direction of the resultant R with respect to the force F_2 (Fig. 2.3b). The law of sines yields

$$\frac{\sin\beta}{\sin(\pi - \alpha)} = \frac{F_1}{R}\,.$$

Introducing the result for R and using the trigonometrical relation $\sin(\pi - \alpha) = \sin\alpha$ we obtain

$$\underline{\underline{\sin\beta = \frac{F_1 \sin\alpha}{\sqrt{F_1^2 + F_2^2 + 2\,F_1\,F_2 \cos\alpha}}}}\,.$$

Students may solve this problem and many others concerning the addition of coplanar forces with the aid of the TM-Tool "Resultant of Systems of Coplanar Forces" (see screenshot). This and other TM-Tools can be found at the web address given in the Preface.



E2.2

Example 2.2 An eyebolt is subjected to four forces ($F_1 = 12\,\text{kN}$, $F_2 = 8\,\text{kN}$, $F_3 = 18\,\text{kN}$, $F_4 = 4\,\text{kN}$) that act under given angles ($\alpha_1 = 45°$, $\alpha_2 = 100°$, $\alpha_3 = 205°$, $\alpha_4 = 270°$) with respect to the horizontal (Fig. 2.4a).

Determine the magnitude and direction of the resultant.

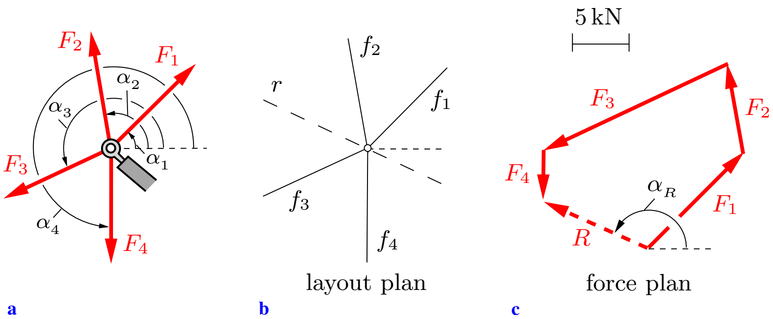


Fig. 2.4

Solution The problem can be solved graphically. First, the layout plan is drawn, showing the lines of action $f_1, \ldots, f_4$ of the forces $F_1, \ldots, F_4$ with their given directions $\alpha_1, \ldots, \alpha_4$ (Fig. 2.4b). Then the force plan is drawn to a chosen scale by adding the given vectors head-to-tail (see Fig. 2.4c and compare Fig. 2.2b). Within the limits of the accuracy of the drawing, the result

$$\underline{\underline{R = 10.5\,\mathrm{kN}}}\,, \qquad \underline{\underline{\alpha_R = 155^\circ}}$$

is obtained. Finally, the action line r of the resultant R is drawn into the layout plan.

There are various possible ways to draw the force polygon. Depending on the choice of the first vector and the sequence of the others, different polygons are obtained. They all yield the same resultant R.

2.2

2.2 Decomposition of Forces in a Plane, Representation in Cartesian Coordinates

Instead of adding forces to obtain their resultant, it is often desired to replace a force $\boldsymbol{R}$ by two forces that act in the directions of given lines of action f_1 and f_2 (Fig. 2.5a). In this case, the force triangle is constructed by drawing straight lines in the directions of f_1 and f_2 through the initial point and the terminal point of $\boldsymbol{R}$, respectively. Thus, two different force triangles are obtained that unambiguously yield the two unknown force vectors (Fig. 2.5b).

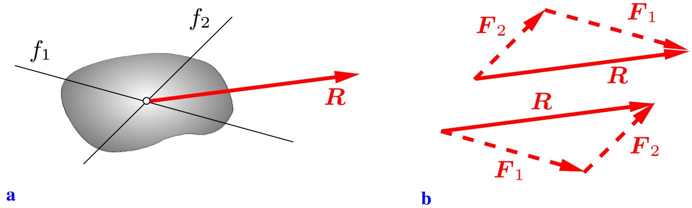


Fig. 2.5

The forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are called the *components* of $\boldsymbol{R}$ in the directions f_1 and f_2, respectively. In coplanar problems, the decomposition of a force into two different directions is unambiguously possible. Note that the resolution into more than two directions cannot be done uniquely: there are an infinite number of ways to resolve the force.

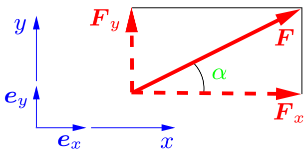


Fig. 2.6

It is usually convenient to resolve forces into two components that are perpendicular to each other. The directions of the components may then be given by the axes x and y of a Cartesian coordinate system (Fig. 2.6). With the unit vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$, the components of $\boldsymbol{F}$ are then written as (compare Appendix A.1)

$$\boldsymbol{F}_x = F_x\,\boldsymbol{e}_x\,, \quad \boldsymbol{F}_y = F_y\,\boldsymbol{e}_y \tag{2.3}$$

and the force $\boldsymbol{F}$ is represented by

$$\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y = F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y\,. \tag{2.4}$$

The quantities F_x and F_y are called the *coordinates* of the vector $\boldsymbol{F}$. Note that they are also often called the components of $\boldsymbol{F}$, even though, strictly speaking, the components of $\boldsymbol{F}$ are the vectors $\boldsymbol{F}_x$ and $\boldsymbol{F}_y$. As mentioned in Section 1.2, a vector will often be referred to by writing simply F (instead of $\boldsymbol{F}$) or F_x (instead of $\boldsymbol{F}_x$), especially when this notation cannot lead to confusion (see, for example, Figs. 1.1 and 1.2).

From Fig. 2.6, it can be found that

$$\begin{aligned} F_x &= F\cos\alpha\,, & F_y &= F\sin\alpha\,, \\ F &= \sqrt{F_x^2+F_y^2}\,, & \tan\alpha &= \frac{F_y}{F_x}\,. \end{aligned} \tag{2.5}$$

In the following, it will be shown that the coordinates of the resultant of a system of concurrent forces can be obtained by simply adding the respective coordinates of the forces. This procedure is demonstrated in Fig. 2.7 with the aid of the example of two forces. The x- and y-components, respectively, of the force $\boldsymbol{F}_i$ are designated with $\boldsymbol{F}_{ix} = F_{ix}\,\boldsymbol{e}_x$ and $\boldsymbol{F}_{iy} = F_{iy}\,\boldsymbol{e}_y$. The resultant then

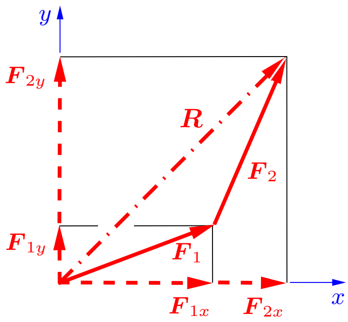


Fig. 2.7

can be written as

$$\boldsymbol{R} = R_x\,\boldsymbol{e}_x + R_y\,\boldsymbol{e}_y = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{F}_{1x} + \boldsymbol{F}_{1y} + \boldsymbol{F}_{2x} + \boldsymbol{F}_{2y}$$
$$= F_{1x}\,\boldsymbol{e}_x + F_{1y}\,\boldsymbol{e}_y + F_{2x}\,\boldsymbol{e}_x + F_{2y}\,\boldsymbol{e}_y = (F_{1x} + F_{2x})\,\boldsymbol{e}_x + (F_{1y} + F_{2y})\boldsymbol{e}_y\,.$$

Hence, the coordinates of the resultant are obtained as

$$R_x = F_{1x} + F_{2x}\,, \quad R_y = F_{1y} + F_{2y}\,.$$

In the case of a system of n forces, the resultant is given by

$$\boldsymbol{R} = R_x\,\boldsymbol{e}_x + R_y\,\boldsymbol{e}_y = \sum \boldsymbol{F}_i = \sum (F_{ix}\,\boldsymbol{e}_x + F_{iy}\,\boldsymbol{e}_y)$$
$$= \left(\sum F_{ix}\right)\boldsymbol{e}_x + \left(\sum F_{iy}\right)\boldsymbol{e}_y \tag{2.6}$$

and the coordinates of the resultant $\boldsymbol{R}$ follow from the summation of the coordinates of the forces:

$$R_x = \sum F_{ix}\,, \quad R_y = \sum F_{iy}\,. \tag{2.7}$$

The magnitude and direction of the resultant are given by (compare (2.5))

$$R = \sqrt{R_x^2 + R_y^2}\,, \quad \tan\alpha_{_R} = \frac{R_y}{R_x}\,. \tag{2.8}$$

In the case of a coplanar force group, the two scalar equations (2.7) are equivalent to the vector equation (2.2).

E2.3

Example 2.3 Solve Example 2.2 with the aid of the representation of the vectors in Cartesian coordinates.

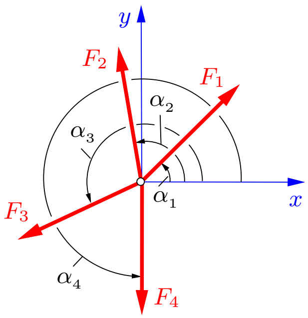


Fig. 2.8

Solution We choose the coordinate system shown in Fig. 2.8, such that the x-axis coincides with the horizontal. The angles are measured from this axis. Then, according to (2.7), the coordinates

$$\begin{aligned} R_x &= F_{1x} + F_{2x} + F_{3x} + F_{4x} \\ &= F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + F_3 \cos\alpha_3 + F_4 \cos\alpha_4 \\ &= 12\cos 45^\circ + 8\cos 100^\circ + 18\cos 205^\circ + 4\cos 270^\circ \\ &= -\,9.22\,\text{kN} \end{aligned}$$

and

$$\begin{aligned} R_y &= F_{1y} + F_{2y} + F_{3y} + F_{4y} \\ &= F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + F_3 \sin\alpha_3 + F_4 \sin\alpha_4 = 4.76\,\text{kN} \end{aligned}$$

are obtained. The magnitude and direction of the resultant follow from (2.8):

$$\underline{\underline{R}} = \sqrt{R_x^2 + R_y^2} = \sqrt{9.22^2 + 4.76^2} = \underline{\underline{10.4\,\text{kN}}}\,,$$

$$\tan\alpha_R = \frac{R_y}{R_x} = -\frac{4.76}{9.22} = -0.52 \quad \to \quad \underline{\underline{\alpha_R = 152.5^\circ}}\,.$$

2.3 Equilibrium in a Plane

We now investigate the conditions under which a body is in *equilibrium* when subjected to the action of forces. It is known from experience that a body that was originally at rest stays at rest if two forces of equal magnitude are applied that have the same line of action and are oppositely directed (Fig. 2.9). In other words:

Two forces are in equilibrium if they are oppositely directed on the same line of action and have the same magnitude.

This means that the sum of the two forces, i.e., their resultant, has to be the zero vector:

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 = \boldsymbol{0}\,. \qquad (2.9)$$

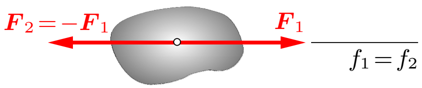


Fig. 2.9

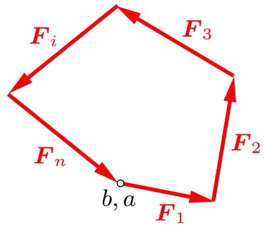


Fig. 2.10

It is also known from Section 2.1 that a system of n concurrent forces $\boldsymbol{F}_i$ can always unambiguously be replaced by its resultant

$$\boldsymbol{R} = \sum \boldsymbol{F}_i\,.$$

Therefore, the *equilibrium condition* (2.9) can immediately be extended to an arbitrary number of forces. A system of concurrent forces is in equilibrium if the resultant is zero:

$$\boldsymbol{R} = \sum \boldsymbol{F}_i = \boldsymbol{0}\,. \qquad (2.10)$$

The geometrical interpretation of (2.10) is that of a closed force polygon, i.e., the initial point a and the terminal point b have to coincide (Fig. 2.10).

The resultant force is zero if its components are zero. Therefore, in the case of a coplanar system of forces, the two scalar equilibrium conditions

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0 \tag{2.11}$$

are equivalent to the vector condition (2.10), (compare (2.7)). Thus, a coplanar system of concurrent forces is in equilibrium if the sums of the respective coordinates of the force vectors (here the x- and y-coordinates) vanish.

Consider a problem where the magnitudes and/or the directions of forces need to be determined. Since we have two equilibrium conditions (2.11), only two unknowns can be calculated. Problems that can be solved by applying only the equilibrium conditions are called *statically determinate*. If there are more than two unknowns, the problem is called *statically indeterminate*. Statically indeterminate systems cannot be solved with the aid of the equilibrium conditions alone.

Before the equilibrium conditions for a given problem are written down, a free-body diagram must be constructed. Therefore, the body in consideration must be isolated by imaginary cuts, and all of the forces acting *on* this body (known and unknown forces) must be drawn into the diagram. Only these forces should appear in the equilibrium conditions. Note that the forces exerted *by* the body to the surroundings are not drawn into the free-body diagram.

To solve a given problem analytically, it is generally necessary to introduce a coordinate system. In principle, the directions of the coordinate axes may be chosen arbitrarily. However, an appropriate choice of the axes may save computational work. To apply the equilibrium conditions (2.11), it suffices to determine the coordinates of the forces; in coplanar problems, the force vectors need not be written down explicitly (compare, e.g., Example 2.4).

2.4 Examples of Coplanar Systems of Forces

To be able to apply the above theory to specific problems, a few idealisations of simple structural elements must be introduced. A structural element whose length is large compared to its cross-sectional dimensions and that can sustain only tensile forces in the direction of its axis, is called a *cable* or a *rope* (Fig. 2.11a). Usually the weight of the cable may be neglected in comparison to the force acting in the cable.

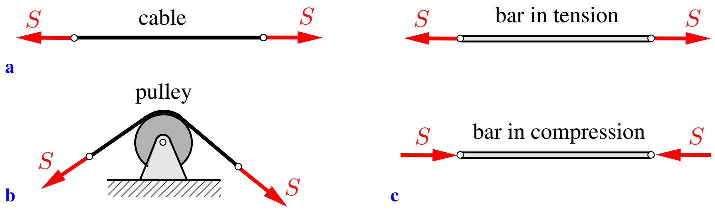


Fig. 2.11

Often a cable is guided over a pulley (Fig. 2.11b). If the bearing friction of the pulley is negligible (ideal pulley), the forces at both ends of the cable are equal in magnitude (see Examples 2.6, 3.3).

A straight structural member with a length much larger than its cross-sectional dimensions that can transfer compressive as well as tensile forces in the direction of its axis is called a *bar* or a *rod* (Fig. 2.11c) (compare Section 5.1.1).

As explained in Section 1.5, the force acting at the point of contact between two bodies can be made visible by separating the bodies (Fig. 2.12a, b). According to Newton's third law (actio = reactio) the contact force K acts with the same magnitude and in an opposite direction on the respective bodies (Fig. 2.12b). It may be resolved into two components, namely, the *normal force* N and the *tangential force* T, respectively. The normal force is perpendicular to the plane of contact, whereas the tangential force lies in this plane. If the two bodies are merely touching each other (i.e., if no connecting elements exist) they can only be pressed against each other (pulling is not possible). Hence, the normal force is oriented towards the interior of the respective body. The tangential force is due to an existing roughness of the surfaces of

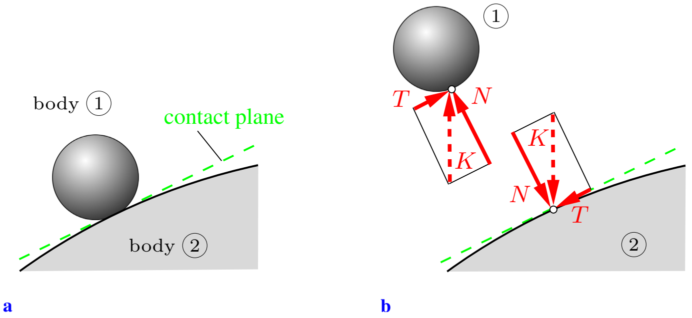


Fig. 2.12

the bodies. In the case of a completely smooth surface of one of the bodies (= idealisation), the tangential force T vanishes. The contact force then coincides with the normal force N.

E2.4

Example 2.4 Two cables are attached to an eye (Fig. 2.13a). The directions of the forces F_1 and F_2 in the cables are given by the angles α and β.

Determine the magnitude of the force $\boldsymbol{H}$ exerted from the wall onto the eye.

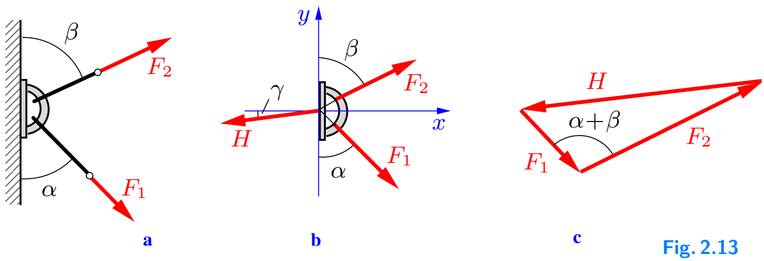


Fig. 2.13

Solution The free-body diagram (Fig. 2.13b) is drawn as the first step. To this end, the eye is separated from the wall by an imaginary cut. Then all of the forces acting on the eye are drawn into the figure: the two given forces F_1 and F_2 and the force H. These three forces are in equilibrium. The free-body diagram contains two unknown quantities, namely, the magnitude of the force H and the angle γ.

The equilibrium conditions are formulated and solved in the second step. We will first present a "graphic-analytical" solution, i.e., a solution that is partly graphical and partly analytical. To this end, the geometrical condition of equilibrium is sketched: the closed force triangle (Fig. 2.13c). Since trigonometry will now be applied to the force plan, it need not be drawn to scale. The law of cosine yields

$$\underline{\underline{H = \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos(\alpha+\beta)}}}\,.$$

The problem may also be solved analytically by applying the scalar equilibrium conditions (2.11). Then we choose a coordinate system (see Fig. 2.13b), and the coordinates of the force vectors are determined and inserted into (2.11):

$$\begin{aligned}
\sum F_{ix} = 0: \quad & F_1 \sin\alpha + F_2 \sin\beta - H\cos\gamma = 0 \\
& \to \quad H\cos\gamma = F_1\sin\alpha + F_2\sin\beta\,, \\
\sum F_{iy} = 0: \quad & -F_1\cos\alpha + F_2\cos\beta - H\sin\gamma = 0 \\
& \to \quad H\sin\gamma = -F_1\cos\alpha + F_2\cos\beta\,.
\end{aligned}$$

These are *two* equations for the *two* unknowns H and γ. To obtain H, the two equations are squared and added. Using the trigonometrical relation

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

yields

$$H^2 = F_1^2 + F_2^2 - 2\,F_1\,F_2\cos(\alpha+\beta)\,.$$

This result, of course, coincides with the result obtained above.

Example 2.5 A wheel with weight W is held on a *smooth* inclined plane by a cable (Fig. 2.14a). **E2.5**

Determine the force in the cable and the contact force between the plane and the wheel.

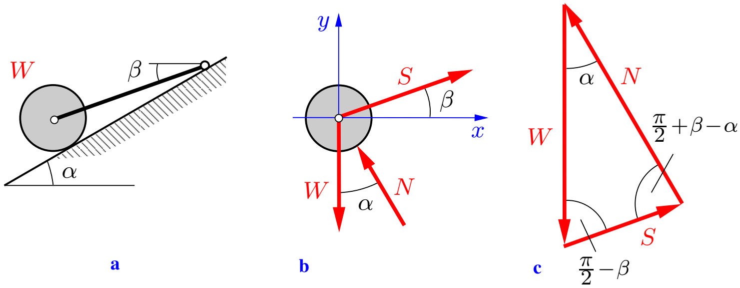


Fig. 2.14

Solution The forces acting at the wheel must satisfy the equilibrium condition (2.10). To make these forces visible, the cable is cut and the wheel is separated from the inclined plane. The free-body diagram (Fig. 2.14b) shows the weight W, the force S in the cable (acting in the direction of the cable) and the contact force N (acting perpendicularly to the inclined plane: smooth surface, $T = 0$!). The three forces S, N and W are concurrent forces; the unknowns are S and N.

First, we solve the problem by graphic-analytical means by sketching (not to scale) the geometrical equilibrium condition, namely, a closed force triangle (Fig. 2.14c). The law of sines yields

$$\underline{\underline{S}} = W\,\frac{\sin\alpha}{\sin(\frac{\pi}{2}+\beta-\alpha)} = \underline{\underline{W\,\frac{\sin\alpha}{\cos(\alpha-\beta)}}}\,,$$

$$\underline{\underline{N}} = W\,\frac{\sin(\frac{\pi}{2}-\beta)}{\sin(\frac{\pi}{2}+\beta-\alpha)} = \underline{\underline{W\,\frac{\cos\beta}{\cos(\alpha-\beta)}}}\,.$$

To solve the problem analytically with the aid of the equilibrium conditions (2.11), we choose a coordinate system (see Fig. 2.14b). Inserting the coordinates of the forces into (2.11) leads to two equations for the two unknowns:

$$\sum F_{ix} = 0 \ : \qquad S\cos\beta - N\sin\alpha = 0\,,$$

$$\sum F_{iy} = 0 \ : \qquad S\sin\beta + N\cos\alpha - W = 0\,.$$

Their solution coincides with the solution given above.

Example 2.6 Three boxes (weights W_1, W_2 and W_3) are attached to two cables as shown in Fig. 2.15a. The pulleys are frictionless. **E2.6**

Calculate the angles α_1 and α_2 in the equilibrium configuration.

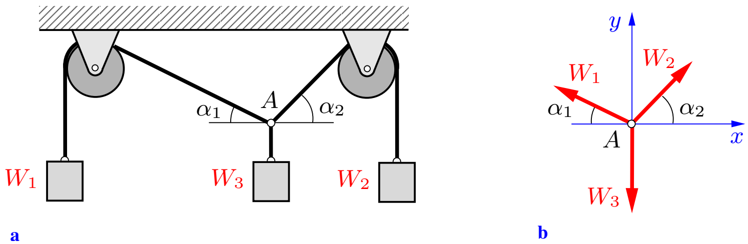


Fig. 2.15

Solution First, point A is isolated by passing imaginary cuts adjacent to this point. The free-body diagram (Fig. 2.15b) shows the forces acting at A; the angles α_1 and α_2 are unknown. Then the coordinate system shown in Fig. 2.15b is chosen and the equilibrium conditions are written down. In plane problems, we shall from now on adopt the following notation: instead of $\sum F_{ix} = 0$ and $\sum F_{iy} = 0$, the symbols $\rightarrow:$ and $\uparrow:$, respectively, will be used (sum of all force components in the directions of the arrows equal to zero). Thus,

$$\rightarrow: \qquad -W_1 \cos\alpha_1 + W_2 \cos\alpha_2 = 0\,,$$

$$\uparrow: \quad W_1 \sin\alpha_1 + W_2 \sin\alpha_2 - W_3 = 0\,.$$

To compute α_1, the angle α_2 is eliminated by rewriting the equations:

$$W_1 \cos\alpha_1 = W_2 \cos\alpha_2\,,$$

$$W_1 \sin\alpha_1 - W_3 = -W_2 \sin\alpha_2\,.$$

Squaring these equations and then adding them yields

$$\underline{\underline{\sin\alpha_1 = \frac{W_3^2 + W_1^2 - W_2^2}{2\,W_1\,W_3}}}\,.$$

Similarly, we obtain

$$\underline{\underline{\sin\alpha_2 = \frac{W_3^2 + W_2^2 - W_1^2}{2\,W_2\,W_3}}}\,.$$

A physically meaningful solution (i.e., an equilibrium configuration) exists only for angles α_1 and α_2 satisfying the conditions $0 < \alpha_1,\ \alpha_2 < \pi/2$. Thus, the weights of the three boxes must be chosen in such a way that both of the numerators are positive and smaller than the denominators.

E2.7

Example 2.7 Two bars 1 and 2 are attached at A and B to a wall by smooth pins. They are pin-connected at C and subjected to a weight W (Fig. 2.16a).

Calculate the forces in the bars.

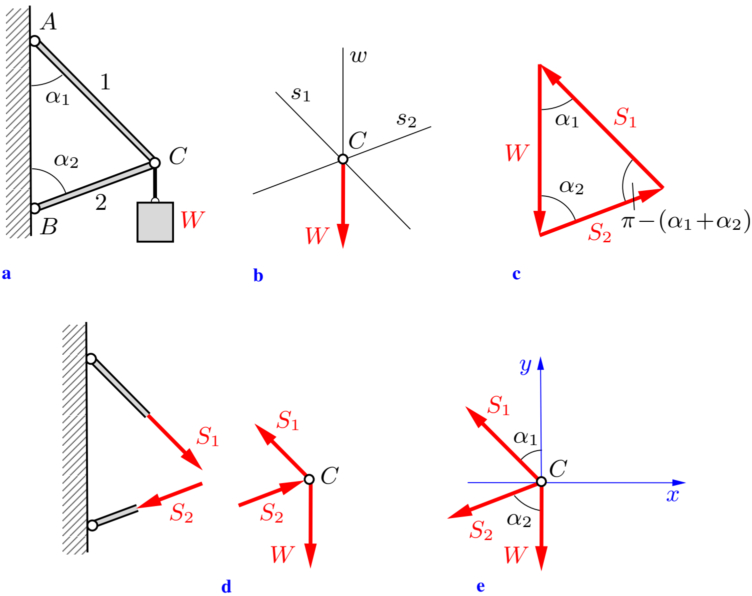


Fig. 2.16

Solution Pin C is isolated by passing cuts through the bars. The forces that act at C are shown in Fig. 2.16d.

First, the graphical solution is indicated. The layout plan is presented in Fig. 2.16b. It contains the given lines of action w, s_1 and s_2 (given by the angles α_1 and α_2) of the forces W, S_1 and S_2, which enables us to draw the closed force triangle (equilibrium condition!) in Fig. 2.16c. To obtain the graphical solution it would be necessary to draw the force plan to scale; in the case of a graphic-analytical solution, no scale is necessary. The law of sines then yields

$$\underline{\underline{S_1 = W\,\frac{\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}}}\,, \quad \underline{\underline{S_2 = W\,\frac{\sin\alpha_1}{\sin(\alpha_1+\alpha_2)}}}\,.$$

The orientation of the forces S_1 and S_2 can be seen in the force plan. This plan shows the forces that are exerted from the bars onto pin C. The forces exerted from the pin onto the bars have the same magnitude; however, according to Newton's third law they are reversed in direction (Fig. 2.16d). It can be seen that bar 1 is subject to tension and that bar 2 is under compression.

The problem will now be solved analytically with the aid of the equilibrium conditions (2.11). The free-body diagram is shown in Fig. 2.16e. The lines of action of the forces S_1 and S_2 are given. The orientations of the forces along their action lines may, in principle, be chosen arbitrarily in the free-body diagram. It is, however, common practice to assume that the forces in bars are tensile forces, as shown in Fig. 2.16e (see also Sections 5.1.3 and 6.3.1). If the analysis yields a negative value for the force in a bar, this bar is in reality subjected to compression.

The equilibrium conditions in the horizontal and vertical directions

$$\begin{aligned} \rightarrow: \quad & -S_1\sin\alpha_1 - S_2\sin\alpha_2 = 0\,, \\ \uparrow: \quad & S_1\cos\alpha_1 - S_2\cos\alpha_2 - W = 0 \end{aligned}$$

lead to

$$S_1 = W\,\frac{\sin\alpha_2}{\sin(\alpha_1+\alpha_2)}\,, \quad S_2 = -\,W\,\frac{\sin\alpha_1}{\sin(\alpha_1+\alpha_2)}\,.$$

Since S_2 is negative, the orientation of the vector $\boldsymbol{S}_2$ along its action line is opposite to the orientation chosen in the free-body diagram. Therefore, in reality bar 2 is subjected to compression.

2.5

2.5 Concurrent Systems of Forces in Space

It was shown in Section 2.2 that a force can unambiguously be resolved into two components in a plane. Analogously, a force can be resolved uniquely into three components in space. As indicated

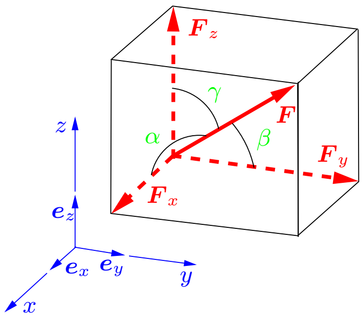


Fig. 2.17

in Section 1.2, a force $\boldsymbol{F}$ may be represented by

$$\boldsymbol{F} = \boldsymbol{F}_x + \boldsymbol{F}_y + \boldsymbol{F}_z = F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y + F_z\,\boldsymbol{e}_z \tag{2.12}$$

in a Cartesian coordinate system x, y, z (Fig. 2.17). The magnitude and the direction of $\boldsymbol{F}$ are given by

$$\begin{gathered} F = \sqrt{F_x^2 + F_y^2 + F_z^2}\,, \\ \cos\alpha = \frac{F_x}{F}\,, \quad \cos\beta = \frac{F_y}{F}\,, \quad \cos\gamma = \frac{F_z}{F}\,. \end{gathered} \tag{2.13}$$

The angles α, β and γ are not independent of each other. If the first equation in (2.13) is squared and F_x, F_y and F_z are inserted according to the second equation the following relation is obtained:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,. \tag{2.14}$$

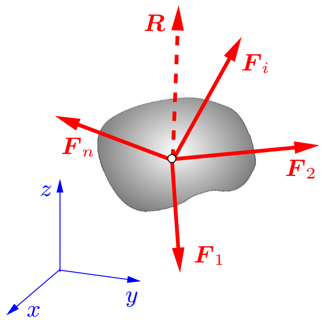


Fig. 2.18

The resultant $\boldsymbol{R}$ of two forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ is obtained by constructing the parallelogram of the forces (see Section 2.1) which is expressed mathematically by the vector equation

$$\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2\,. \tag{2.15}$$

In the case of a spatial system of n concurrent forces (Fig. 2.18), the resultant is found through a successive application of the parallelogram law of forces in space. As in the case of a system of coplanar forces, the resultant is the sum of the force vectors. Mathematically, this is written as (compare (2.2))

$$\boldsymbol{R} = \sum \boldsymbol{F}_i\,. \tag{2.16}$$

If the forces $\boldsymbol{F}_i$ are represented by their components $\boldsymbol{F}_{ix}$, $\boldsymbol{F}_{iy}$ and $\boldsymbol{F}_{iz}$ according to (2.12), we obtain

$$\begin{aligned}\boldsymbol{R} = R_x\,\boldsymbol{e}_x + R_y\,\boldsymbol{e}_y + R_z\,\boldsymbol{e}_z &= \sum(\boldsymbol{F}_{ix} + \boldsymbol{F}_{iy} + \boldsymbol{F}_{iz})\\ &= \sum(F_{ix}\,\boldsymbol{e}_x + F_{iy}\,\boldsymbol{e}_y + F_{iz}\,\boldsymbol{e}_z)\\ &= \Big(\sum F_{ix}\Big)\,\boldsymbol{e}_x + \Big(\sum F_{iy}\Big)\,\boldsymbol{e}_y + \Big(\sum F_{iz}\Big)\,\boldsymbol{e}_z\,.\end{aligned}$$

The coordinates of the resultant in space are thus given by

$$R_x = \sum F_{ix}\,, \quad R_y = \sum F_{iy}\,, \quad R_z = \sum F_{iz}\,. \tag{2.17}$$

The magnitude and direction of $\boldsymbol{R}$ follow as (compare (2.13))

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}\,, \qquad \cos\alpha_R = \frac{R_x}{R}\,, \quad \cos\beta_R = \frac{R_y}{R}\,, \quad \cos\gamma_R = \frac{R_z}{R}\,. \tag{2.18}$$

A spatial system of concurrent forces is in *equilibrium* if the resultant is the zero vector (compare (2.10)):

$$\boldsymbol{R} = \sum \boldsymbol{F}_i = \boldsymbol{0}\,. \tag{2.19}$$

This vector equation is equivalent to the *three* scalar *equilibrium conditions*

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0\,, \quad \sum F_{iz} = 0\,, \tag{2.20}$$

which represent a system of three equations for three unknowns.

E2.8 **Example 2.8** A structure consists of two bars 1 and 2 and a rope 3 (weights negligible). It is loaded in A by a box of weight W (Fig. 2.19a).

Determine the forces in the bars and in the rope.

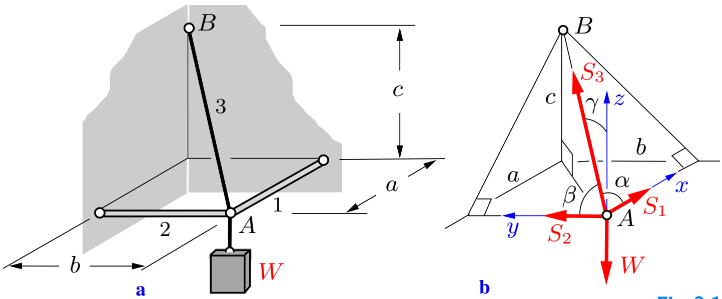


Fig. 2.19

Solution We isolate pin A by passing imaginary sections through the bars and the rope. The internal forces are made visible in the free-body diagram (Fig. 2.19b); they are assumed to be tensile forces. The equilibrium conditions are

$$\begin{aligned} \sum F_{ix} = 0: &\quad S_1 + S_3 \cos\alpha = 0\,, \\ \sum F_{iy} = 0: &\quad S_2 + S_3 \cos\beta = 0\,, \\ \sum F_{iz} = 0: &\quad S_3 \cos\gamma - W = 0\,. \end{aligned} \tag{a}$$

With the diagonal $\overline{AB} = \sqrt{a^2+b^2+c^2}$, the angles α, β and γ can be taken from Fig. 2.19b:

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}\,, \quad \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}\,,$$
$$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}\,.$$

This yields

$$\underline{\underline{S_3}} = \frac{W}{\cos\gamma} = \underline{\underline{W\,\frac{\sqrt{a^2+b^2+c^2}}{c}}}\,,$$
$$\underline{\underline{S_1}} = -S_3\cos\alpha = -W\,\frac{\cos\alpha}{\cos\gamma} = \underline{\underline{-W\,\frac{a}{c}}}\,,$$
$$\underline{\underline{S_2}} = -S_3\cos\beta = -W\,\frac{\cos\beta}{\cos\gamma} = \underline{\underline{-W\,\frac{b}{c}}}\,.$$

The negative signs for the forces in the bars indicate that the bars are actually in a state of compression; the rope is subjected to tension. This can easily be verified by inspection.

As can be seen, the geometry of this problem is very simple. Therefore, it was possible to apply the equilibrium conditions without resorting to the vector formalism. In the case of a complicated geometry it is, however, recommended that the forces be written down in vector form. This more formal and therefore safer way to solve the problem is now presented.

The force vectors

$$\boldsymbol{S}_1 = S_1\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \boldsymbol{S}_2 = S_2\begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \boldsymbol{W} = W\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

(represented as column vectors, see Appendix A.1) can easily be written down. To obtain the vector $\boldsymbol{S}_3$, we first represent the vector $\boldsymbol{r}_{AB}$, which is directed from A to B:

$$\boldsymbol{r}_{AB} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

If this vector is divided by its magnitude $r_{AB} = \sqrt{a^2+b^2+c^2}$, the unit vector

$$\boldsymbol{e}_{AB} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

in the direction from A to B is obtained. The force vector $\boldsymbol{S}_3$ has the same direction; it is therefore given by

$$\boldsymbol{S}_3 = S_3\, \boldsymbol{e}_{AB} = \frac{S_3}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} .$$

The equilibrium condition $\sum \boldsymbol{F}_i = \boldsymbol{0}$, i.e.,

$$\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \boldsymbol{W} = \boldsymbol{0}$$

reads

$$S_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + S_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{S_3}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} + W \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

Evaluation yields

$$\begin{aligned}
\sum F_{ix} = 0: &\qquad S_1 + \frac{aS_3}{\sqrt{a^2+b^2+c^2}} = 0\,, \\
\sum F_{iy} = 0: &\qquad S_2 + \frac{bS_3}{\sqrt{a^2+b^2+c^2}} = 0\,, \\
\sum F_{iz} = 0: &\quad -W + \frac{cS_3}{\sqrt{a^2+b^2+c^2}} = 0\,.
\end{aligned}$$

These equations coincide with equations (a).

Note that the quantities S_j are the forces in the bars and in the rope, respectively, which were assumed to be tensile forces. They are *not* the magnitudes of the vectors $\boldsymbol{S}_j$. The determination of the forces leads to $S_1 < 0$ and $S_2 < 0$ (bars in compression), whereas magnitudes of vectors are always non-negative quantities.

Example 2.9 A vertical mast M is supported by two ropes 1 and 2. The force F in rope 3 is given (Fig. 2.20a). **E2.9**

Determine the forces in ropes 1 and 2 and in the mast.

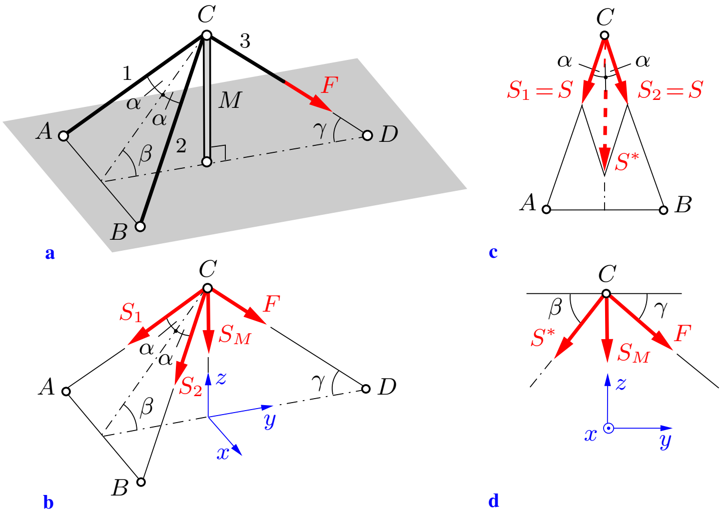


Fig. 2.20

Solution We isolate point C by passing imaginary cuts through the ropes and the mast. The internal forces are assumed to be tensile forces, and are shown in the free-body diagram (Fig. 2.20b). Since the y, z-plane is a plane of symmetry, the forces S_1 and S_2 have to be equal: $S_1 = S_2 = S$ (this may be confirmed by applying the equilibrium condition in the x-direction). The forces S_1 and S_2 are added to obtain their resultant (Fig. 2.20c)

$$S^* = 2\,S\cos\alpha\,.$$

The forces S^*, S_M and F act in the y,z-plane (Fig. 2.20d). The equilibrium conditions

$$\sum F_{iy} = 0 : \quad -S^* \cos\beta + F\cos\gamma = 0\,,$$
$$\sum F_{iz} = 0 : \quad -S^* \sin\beta - S_M - F\sin\gamma = 0$$

yield, after inserting the relation for S^*,

$$\underline{\underline{S = F\,\frac{\cos\gamma}{2\cos\alpha\cos\beta}}}\,, \qquad \underline{\underline{S_M = -F\,\frac{\sin(\beta+\gamma)}{\cos\beta}}}\,.$$

As could be expected, the ropes are subjected to tension ($S > 0$), whereas the mast is under compression ($S_M < 0$).

The special case $\gamma = \pi/2$ is used as a simple check. Then force F acts in the direction of the mast, and $S = 0$ and $S_M = -F$ are obtained with $\cos(\pi/2) = 0$ and $\sin(\beta + \pi/2) = \cos\beta$.

2.6

2.6 Supplementary Problems

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011, or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E2.10

Example 2.10 A hook is subjected to three forces ($F_1 = 180$ N, $\alpha_1 = 45°$, $F_2 = 50$ N, $\alpha_2 = 60°$, $F_3 = 30$ N) as shown in Fig. 2.21.

Determine the magnitude and direction of the resultant.

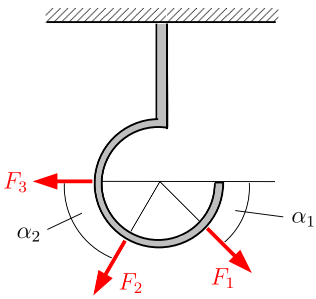
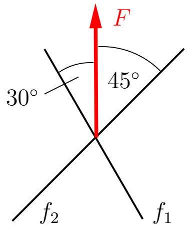


Fig. 2.21

Results: $R = 185$ N, $\alpha_R = 67°$.

E2.11

Example 2.11 Determine the magnitudes F_1 and F_2 of the components of force $\boldsymbol{F}$ with magnitude $F = 5\,\mathrm{kN}$ in the directions f_1 and f_2 (Fig. 2.22).



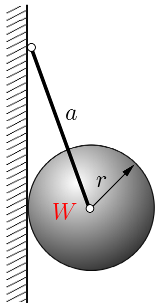
Results: $F_1 = 3.7\,\mathrm{kN}\,, \quad F_2 = 2.6\,\mathrm{kN}\,.$

Fig. 2.22

E2.12

Example 2.12 A smooth sphere (weight $W = 20\,\mathrm{N}$, radius $r = 20\,\mathrm{cm}$) is suspended by a wire (length $a = 60\,\mathrm{cm}$) as shown in Fig. 2.23.

Determine the magnitude of force S in the wire.



Result: see (**A**) $\quad S = 21.2\,\mathrm{N}\,.$

Fig. 2.23

E2.13

Example 2.13 Fig. 2.24 shows a freight elevator. The cable of the winch passes over a smooth pin K. A crate (weight W) is suspended at the end of the cable.

Determine the magnitude of forces S_1 and S_2 in bars 1 and 2.

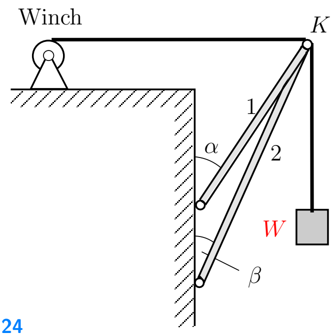


Fig. 2.24

Results: see (**B**) $\quad S_1 = \dfrac{\sin\beta - \cos\beta}{\sin(\alpha-\beta)}\,W\,, \quad S_2 = \dfrac{\cos\alpha - \sin\alpha}{\sin(\alpha-\beta)}\,W\,.$

E2.14

Example 2.14 A smooth circular cylinder (weight W, radius r) touches an obstacle (height h) as shown in Fig. 2.25.

Find the magnitude of force F necessary to roll the cylinder over the obstacle.

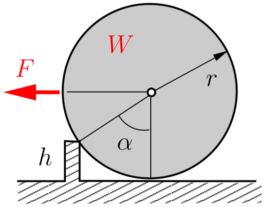


Fig. 2.25

Result: see (**A**) $F = W \tan\alpha$.

E2.15

Example 2.15 A large cylinder (weight $4W$, radius $2r$) lies on top of two small cylinders (each having weight W and radius r) as shown in Fig. 2.26. The small cylinders are connected by a wire S (length $3r$). All surfaces are smooth.

Determine all contact forces and the magnitude of force S in the wire.

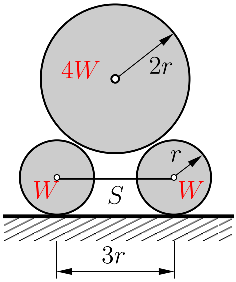


Fig. 2.26

Results: see (**A**) $N_1 = \frac{4}{\sqrt{3}} W, \quad N_2 = 3\,W, \quad S = \frac{2}{\sqrt{3}} W$.

E2.16

Example 2.16 A cable (length l, weight negligible) is attached to two walls at A and B (Fig. 2.27). A cube (weight W) on a frictionless pulley (radius negligible) is suspended by the cable.

Find the distance d of the cube from the left side in the equilibrium position and calculate the force S in the cable.

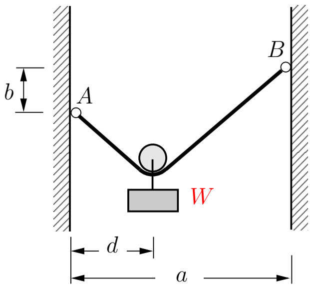


Fig. 2.27

Results: see (**B**) $S = \frac{W\,l}{2\sqrt{l^2 - a^2}}, \quad d = \frac{a}{2}\left(1 - \frac{b}{\sqrt{l^2 - a^2}}\right)$.

Example 2.17 A smooth circular cylinder (weight $W = 500$ N) rests on two fixed supports as shown in Fig. 2.28. It is subjected to a force $F = 200$ N. **E2.17**

a) Calculate the contact forces.

b) Determine the allowable magnitude of F in order to avoid the cylinder from lifting off.

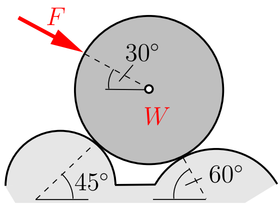


Fig. 2.28

Results: a) $N_l = 155$ N, $N_r = 566$ N, b) $F_{\text{allow}} \leq 500$ N.

Example 2.18 Two cylinders (weights W_1 and W_2) are pin-connected by a bar (weight negligible). They rest on two smooth inclined planes as shown in Fig. 2.29. Given: $W_1 = 200$ N, $W_2 = 300$ N, $\alpha = 60°$. **E2.18**

Calculate the angle φ in the equilibrium position and the corresponding force S in the bar.

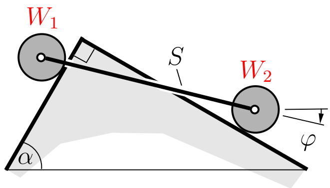


Fig. 2.29

Results: $\varphi = -19°$, $S = 229$ N.

Example 2.19 A spatial system of concurrent forces consists of the three forces **E2.19**

$$\boldsymbol{F}_1 = a_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \boldsymbol{F}_2 = a_2 \begin{pmatrix} -9 \\ 6 \\ 9 \end{pmatrix}, \quad \boldsymbol{F}_3 = a_3 \begin{pmatrix} 8 \\ -7 \\ 1 \end{pmatrix}.$$

Their resultant is given by $\boldsymbol{R} = \begin{pmatrix} 30 \\ -28 \\ 44 \end{pmatrix}$ kN.

Determine the unknown constants a_1, a_2, a_3, the magnitudes of the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3$ and the angle α between $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$.

Results: (selected values) $a_1 = 2$ kN, $a_2 = 4$ kN, $\alpha = 134.3°$.

E2.20

Example 2.20 A smooth sphere (weight W, radius R) rests on three points A, B and C. These three points form an equilateral triangle in a horizontal plane. The height of the triangle is $3a = \frac{3}{4}\sqrt{3}\,R$ (see Fig. 2.28). The action line of the applied force F passes through the center of the sphere.

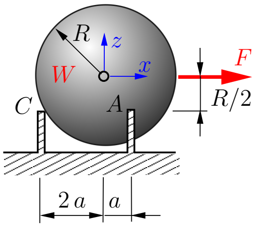


Fig. 2.30

Determine the contact forces at A, B and C. Find the force F required to lift the sphere off at C.

Results: see (**A**) $A = B = \frac{2}{3}\Big(W + \frac{1}{\sqrt{3}}F\Big)$, $C = \frac{2}{3}\Big(W - \frac{2}{\sqrt{3}}F\Big)$,

$$\boldsymbol{A} = \frac{A}{4}\begin{pmatrix} -\sqrt{3} \\ 3 \\ 2 \end{pmatrix}, \quad \boldsymbol{B} = \frac{B}{4}\begin{pmatrix} -\sqrt{3} \\ -3 \\ 2 \end{pmatrix}, \quad \boldsymbol{C} = \frac{C}{4}\begin{pmatrix} 2\sqrt{3} \\ 0 \\ 2 \end{pmatrix},$$

$$F = \frac{\sqrt{3}}{2}W\,.$$

E2.21

Example 2.21 The construction shown in Fig. 2.31 consists of three bars that are pin-connected at K. A rope attached to a wall is guided without friction through an eye at K. The free end of the rope is loaded with a crate (weight W).

Calculate the forces in the bars.

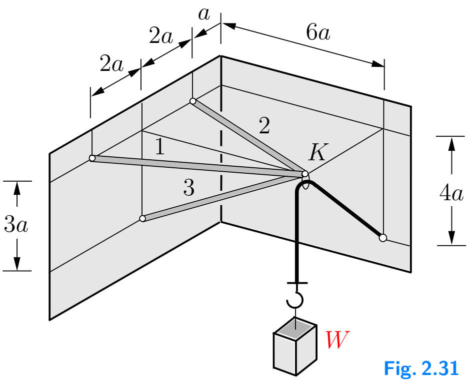


Fig. 2.31

Results: see (**B**) $S_1 = \frac{9}{\sqrt{10}}W\,,\quad S_2 = \frac{3}{\sqrt{10}}W\,,\quad S_3 = -\frac{9}{\sqrt{5}}W\,.$

2.7 Summary

- The lines of action of a system of concurrent forces intersect at a point.
- The resultant of a system of concurrent forces is given by the vector $\boldsymbol{R} = \sum \boldsymbol{F}_i$. In coordinates,

$$R_x = \sum F_{ix}\,, \qquad R_y = \sum F_{iy}\,, \qquad R_z = \sum F_{iz}\,.$$

  In the case of a coplanar system, the z-components vanish. Note: the coordinate system may be chosen arbitrarily; an appropriate choice may save computational work.
- The equilibrium condition for a system of concurrent forces is $\sum \boldsymbol{F}_i = \boldsymbol{0}$. In coordinates,

$$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,, \qquad \sum F_{iz} = 0\,.$$

  In the case of a coplanar problem, the z-components vanish.
- In order to solve force problems the following steps are usually necessary:
  - ◇ Isolate the body (point).
  - ◇ Sketch the free-body diagram: introduce all of the forces exerted *on* the body; assume the internal forces in bars to be tensile forces.
  - ◇ Choose a coordinate system.
  - ◇ Formulate the equilibrium conditions (3 equations in spatial problems, 2 equations in coplanar problems).
  - ◇ Solve the equilibrium conditions.
- The force acting at the point of contact between two bodies can be made visible by separating the bodies. In the case of smooth surfaces, it is perpendicular to the plane of contact.

Chapter 3

General Systems of Forces, Equilibrium of a Rigid Body

3

3 General Systems of Forces, Equilibrium of a Rigid Body

Objectives: In this chapter general systems of forces are considered, i.e., forces whose lines of action do not intersect at a point. For the analysis, the notion *moment* has to be introduced. Students should learn how coplanar or spatial systems of forces can be reduced and under which conditions they are in equilibrium. They should also learn how to apply the method of sections to obtain a free-body diagram. A correct free-body diagram and an appropriate application of the equilibrium conditions are the key to the solution of a coplanar or a spatial problem.

3.1 General Systems of Forces in a Plane

3.1.1 Couple and Moment of a Couple

In Section 2.1 it was shown that a system of *concurrent forces* can always be reduced to a resultant force. In the following, it

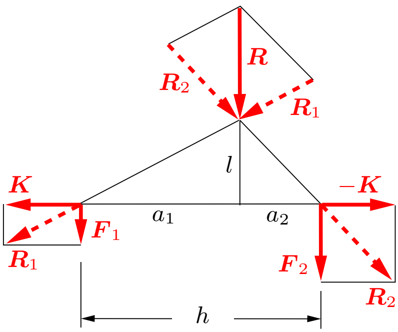


Fig. 3.1

is demonstrated how the resultant $\boldsymbol{R}$ of two *parallel forces* $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ can be found (Fig. 3.1). As a first step, the two forces $\boldsymbol{K}$ and $-\boldsymbol{K}$, which have the same line of action, are added. Since they are in equilibrium they have no effect on a rigid body. Then two parallelograms (here rectangles) are drawn to obtain the forces $\boldsymbol{R}_1 = \boldsymbol{F}_1 + \boldsymbol{K}$ and $\boldsymbol{R}_2 = \boldsymbol{F}_2 + (-\boldsymbol{K})$. These forces, which are statically equivalent to the given system of parallel forces, represent a system of concurrent forces. They may be moved along their respective lines of action (sliding vectors!) to their point of intersection. There another parallelogram of forces is constructed that yields the resultant

$$\boldsymbol{R} = \boldsymbol{R}_1 + \boldsymbol{R}_2 = \boldsymbol{F}_1 + \boldsymbol{F}_2 \,. \tag{3.1}$$

This graphical solution yields the magnitude R of the resultant as well as the location of its action line. From Fig. 3.1 it can be found that

$$\begin{aligned} &R = F_1 + F_2 \,, \\ &h = a_1 + a_2 \,, \quad \frac{a_1}{l} = \frac{K}{F_1} \,, \quad \frac{a_2}{l} = \frac{K}{F_2} \,. \end{aligned} \tag{3.2}$$

Thus, the magnitude of the resultant $\boldsymbol{R}$ of the parallel forces is simply the algebraic sum of the magnitudes of the forces. Equation (3.2) also yields the *principle of the lever* by Archimedes

$$a_1 F_1 = a_2 F_2 \tag{3.3}$$

and the distances

$$a_1 = \frac{F_2}{F_1 + F_2}\,h = \frac{F_2}{R}\,h\,, \quad a_2 = \frac{F_1}{F_1 + F_2}\,h = \frac{F_1}{R}\,h\,. \tag{3.4}$$

Hence, the method described above always gives the resultant force and the location of its action line unless the denominator in (3.4) is zero. In this case the two forces are said to form a *couple*.

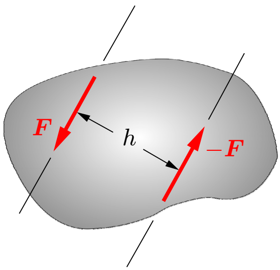


Fig. 3.2

A *couple* consists of two forces having equal magnitude, parallel action lines and opposite directions (Fig. 3.2). In this case the method to find the resultant fails to work. With $F_2 = -F_1$ the equations (3.2) and (3.4) lead to $R = 0$ and $a_1, a_2 \to \pm\infty$. Hence, a couple can *not* be reduced to a resultant single force.

Although the resultant force of a couple is zero the couple has an effect on the body on which it acts: it tends to rotate the body. Fig. 3.3 shows three examples of couples: a) a wheel which is to be turned, b) a screw driver acting on the head of a screw and c) a "clamped" beam whose free end is twisted. As can be seen, a couple has a *sense of rotation*: either clockwise or counterclockwise. Similar to the notion of a "concentrated" force the couple is an idealization which replaces the action of the area forces.

We now investigate the quantities which define a couple and its properties. The effect of a couple on a rigid body is unambiguously determined by its *moment*. The moment incorporates two

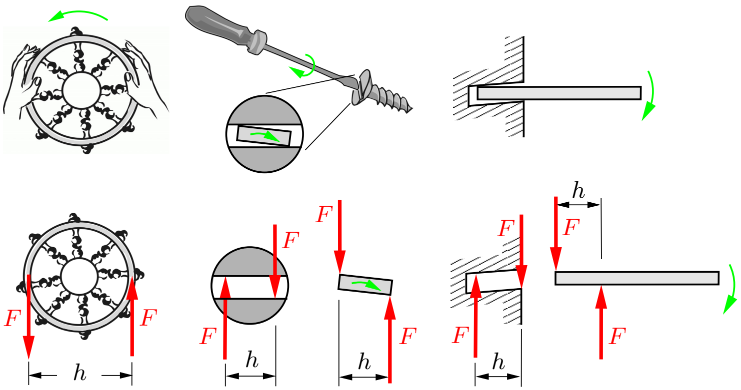


Fig. 3.3 **a** **b** **c**

quantities: first, its magnitude M which is given by the product of the perpendicular distance h of the action lines (Fig. 3.2) and the magnitude F of the forces

$$M = h\,F\,, \tag{3.5}$$

and, secondly, its sense of rotation. In the figures, the sense of rotation is represented by a curved arrow ($\curvearrowleft$ or $\curvearrowright$). The quantities *magnitude* M and *sense of rotation* $\curvearrowleft$ indicate that a couple moment is a vector in three-dimensional space. The moment has the dimension length times force $[l\,F]$, and is expressed, for example, in the unit Nm. In order to avoid confusion with the unit mN $\widehat{=}$ Milli-Newton, the sequence of the units of length and force is exchanged: Nm $\widehat{=}$ Newton-Meter.

Fig. 3.4 shows that a couple with a given moment can be produced by arbitrarily many different pairs of forces. If the forces K and $-K$ are added to the given couple (forces F, perpendicular distance h) a statically equivalent couple is obtained (forces F', perpendicular distance h'). The couple moment, i.e., the sense of rotation and the magnitude of the moment

$$M = h'\,F' = (h\sin\alpha)\left(\frac{F}{\sin\alpha}\right) = h\,F$$

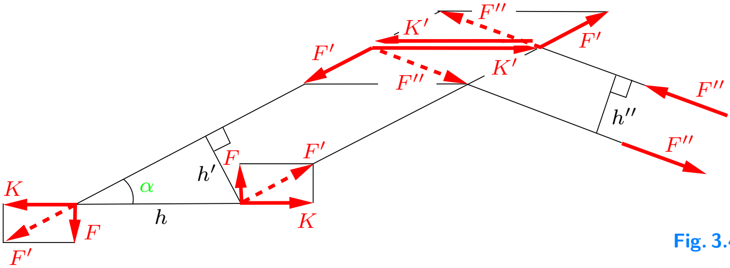


Fig. 3.4

remain unchanged. By successive application of this procedure, a couple may be moved arbitrarily in the plane without any change of its moment. Hence, in contrast to a force, a couple is *not* bound to a line of action. Therefore, it may be applied at arbitrary points of a rigid body: it always has the same turning effect.

A couple is uniquely described by its moment. Hence, the two forces F and $-F$ in the following are replaced by the couple moment. In particular, in the figures we shall replace the two forces by a curved arrow, i.e., $\curvearrowleft M$, as shown in Fig. 3.5. This notation incorporates the magnitude M of the couple moment and the sense of rotation (curved arrow); it is analogous to the notation $\nearrow F$ (arrow and magnitude of the force).

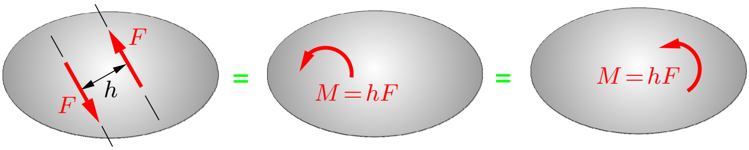


Fig. 3.5

The law of action and reaction (Section 1.5) states that every force has a counteracting force of the same magnitude but opposite direction. By analogy, every couple moment has a counteracting couple moment of equal magnitude but with an opposite sense of rotation. For example, the screwdriver in Fig. 3.3b exerts a moment $M = h\,F$ on the screw which acts clockwise, whereas the screw exerts a moment of equal magnitude in the counterclockwise direction on the screwdriver.

If several couples act on a rigid body they may be appropriately moved and rotated and then added to yield a resultant moment

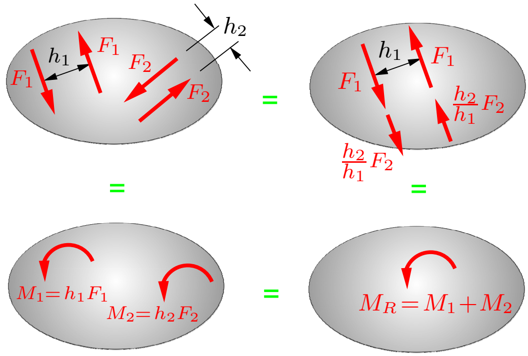


Fig. 3.6

M_R (Fig. 3.6). The couple moments are added algebraically taking into account their algebraic signs (given by their respective senses of rotation):

$$M_R = \sum M_i \,. \tag{3.6}$$

If the sum of the moments is zero, the resultant couple moment, and therefore the tendency to rotate the body, vanish. Thus, the equilibrium condition for a system of couple moments is

$$M_R = \sum M_i = 0 \,. \tag{3.7}$$

3.1.2 Moment of a Force

A force acting on a rigid body is a sliding vector: it may be moved along its line of action without changing the effect on the body. With the aid of the notion of the couple moment, we now will investigate how a force may be moved to a parallel line of action. Consider in Fig. 3.7 a force F whose line of action f is assumed to be moved to the line f', which is parallel to f and passes through point 0. The perpendicular distance of the two lines is given by h. As a first step, the forces F and $-F$ are introduced on the line f'. These two forces are in equilibrium. One of the forces and the originally given force (action line f) represent a couple. The couple moment is given by its magnitude $M^{(0)} = h\,F$ and the sense of rotation. The system consisting of force F with action line f' and couple moment $M^{(0)} = h\,F$ is statically equivalent to force F with action line f. The quantity $M^{(0)} = hF$ is called the *moment of the force* F about (with respect to) point 0. The superscript (0) indicates the *reference point*. The perpendicular distance of point 0 from the action line f is called the *lever arm* of force F with respect to 0. The *sense of rotation* of the moment is given by the sense of rotation of force F about 0.

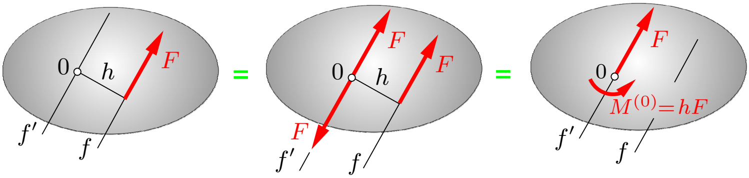


Fig. 3.7

It should be noted that a couple moment is independent of the point of reference, whereas the magnitude and sense of rotation of the moment of a force depend on this point.

Often it is advantageous to replace a force $\boldsymbol{F}$ by its Cartesian components $\boldsymbol{F}_x = F_x\,\boldsymbol{e}_x$ and $\boldsymbol{F}_y = F_y\,\boldsymbol{e}_y$ (Fig. 3.8). Adopting the commonly used sign convention that a moment is *positive* if it tends to rotate the body *counterclockwise* when viewed from above ($\curvearrowleft$), the moment of the force F about point 0 in Fig. 3.8 is given by $M^{(0)} = h\,F$. Using the relations

$$h = x \sin\alpha - y \cos\alpha$$

and

$$\sin\alpha = F_y/F\,, \qquad \cos\alpha = F_x/F$$

the moment can also be represented as

$$M^{(0)} = hF = \left(x\,\frac{F_y}{F} - y\,\frac{F_x}{F} \right) F = x\,F_y - y\,F_x\,. \tag{3.8}$$

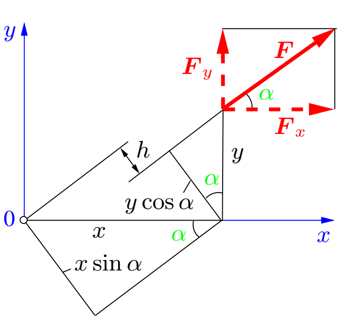


Fig. 3.8

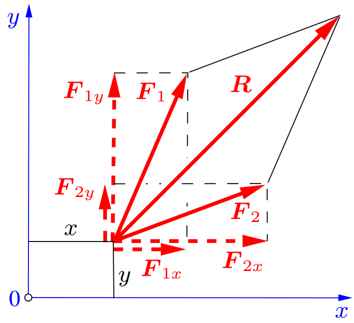


Fig. 3.9

Hence, the moment is equal to the sum of the moments of the force components about 0. Note the senses of rotation of the respective components: they determine the algebraic signs in the summation.

Consider now two forces F_1 and F_2 and their resultant R (Fig. 3.9). The moments of the two forces with respect to point 0 are

$$M_1^{(0)} = x\,F_{1y} - y\,F_{1x}\,, \quad M_2^{(0)} = x\,F_{2y} - y\,F_{2x}\,,$$

and their sum is given by

$$M_1^{(0)} + M_2^{(0)} = x\,(F_{1y} + F_{2y}) - y\,(F_{1x} + F_{2x}) = x\,R_y - y\,R_x = M_R^{(0)}\,.$$

Therefore, it is immaterial whether the forces are added first and then the moment is determined or if the sum of the individual moments is calculated. This property holds for an arbitrary number of forces:

The sum of the moments of single forces is equal to the moment of their resultant.

3.1.3 Resultant of Systems of Coplanar Forces

Consider a rigid body that is subjected to a general system of coplanar forces (Fig. 3.10). To investigate how this system can be reduced to a simpler system, a reference point A is chosen and the action lines of the forces are moved without changing their

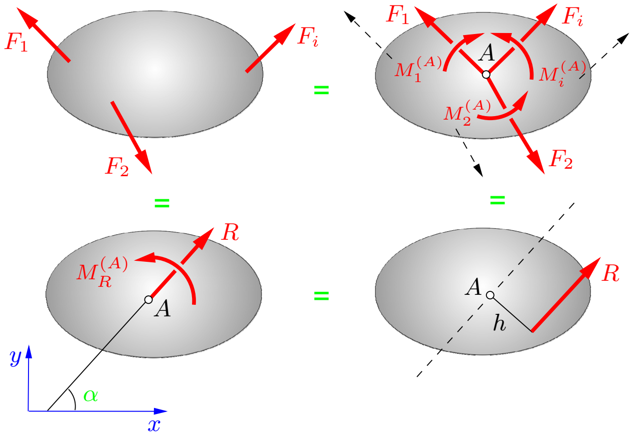


Fig. 3.10

directions until they pass through A. To avoid changing the effect of the forces on the body, the respective moments of the forces about A must be introduced. Hence, the given general system of forces is replaced by a system of concurrent forces and a system of moments. These two systems can be reduced to a resultant force R with the components R_x and R_y and a resultant moment $M_R^{(A)}$. According to (2.7) and (3.6), they are given by

$$R_x = \sum F_{ix}\,, \quad R_y = \sum F_{iy}\,, \quad M_R^{(A)} = \sum M_i^{(A)}\,. \qquad (3.9)$$

The magnitude and direction of the resultant force can be calculated from

$$R = \sqrt{R_x^2 + R_y^2}\,, \quad \tan\alpha = \frac{R_y}{R_x}\,. \qquad (3.10)$$

The system of the resultant R (action line through A) and the moment $M_R^{(A)}$ may be further simplified. It is equivalent to the single force R alone if the action line is moved appropriately. The perpendicular distance h (Fig. 3.10) must be chosen in such a way that the moment $M_R^{(A)}$ equals hR, i.e., $hR = M_R^{(A)}$, which yields

$$h = \frac{M_R^{(A)}}{R}\,. \qquad (3.11)$$

If $M_R^{(A)} = 0$ and $R \neq 0$, Equation (3.11) gives $h = 0$. In this case the action line of the resultant of the general system of forces passes through A. On the other hand, if $R = 0$ and $M_R^{(A)} \neq 0$, a further reduction is not possible: the system of forces is reduced to only a moment (i.e., a couple), which is independent of the choice of the reference point.

Equations (3.9) to (3.11) can be used to calculate the magnitude and direction of the resultant as well as the location of its action line.

Example 3.1 A disc is subjected to four forces as shown in Fig. 3.11a. The forces have the given magnitudes F or $2\,F$, respectively. **E3.1**

Determine the magnitude and direction of the resultant and the location of its line of action.

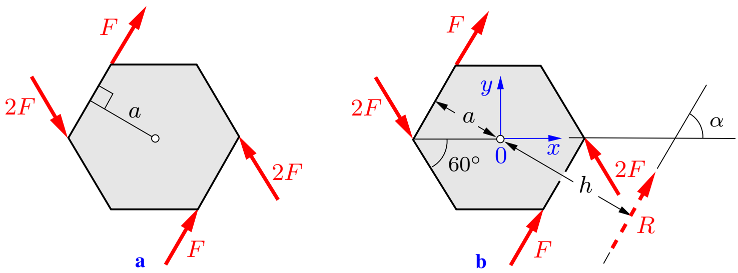


Fig. 3.11

Solution We choose a coordinate system x, y (Fig. 3.11b), and its origin 0 is taken as the point of reference. According to the sign convention, positive moments tend to rotate the disk counterclockwise ($\curvearrowleft$). Thus, from (3.9) we obtain

$$
\begin{aligned}
R_x &= \sum F_{ix} = 2\,F\cos 60^\circ + F\cos 60^\circ \\
&\qquad\qquad + F\cos 60^\circ - 2\,F\cos 60^\circ = F\,, \\
R_y &= \sum F_{iy} = -\,2\,F\sin 60^\circ + F\sin 60^\circ \\
&\qquad\qquad + F\sin 60^\circ + 2\,F\sin 60^\circ = \sqrt{3}\,F\,, \\
M_R^{(0)} &= \sum M_i^{(0)} = 2\,a\,F + a\,F + 2\,a\,F - a\,F = 4\,a\,F\,,
\end{aligned}
$$

which yield (see (3.10))

$$\underline{\underline{R}} = \sqrt{R_x^2 + R_y^2} = \underline{\underline{2\,F}}\,, \quad \tan\alpha = \frac{R_y}{R_x} = \sqrt{3} \quad \rightarrow \quad \underline{\underline{\alpha = 60^\circ}}\,.$$

The perpendicular distance of the resultant from point 0 follows from (3.11):

$$\underline{\underline{h}} = \frac{M_R^{(0)}}{R} = \frac{4\,a\,F}{2\,F} = \underline{\underline{2\,a}}\,.$$

3.1.4 Equilibrium Conditions

As shown in Section 3.1.3 a general system of coplanar forces can be reduced to a system of concurrent forces and a system of moments with respect to an arbitrary reference point A. The system of moments consists of the moments of the forces and of possible couple moments. The systems will be subjected to the conditions of equilibrium (2.11) and (3.7), respectively. Hence, a rigid body under the action of a general system of coplanar forces is in equilibrium if the following equilibrium conditions are satisfied:

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0\,, \quad \sum M_i^{(A)} = 0\,. \tag{3.12}$$

The number of equilibrium conditions (three) equals the number of the possible motions (three) of a body in a coplanar problem: translations in the x- and y-directions, respectively, and a rotation about an axis that is perpendicular to the x,y-plane. The body is said to have three *degrees of freedom*.

It is now shown that the point of reference in the moment equation of (3.12) can be chosen arbitrarily. In order to do this, we formulate the moment equation with respect to point A (see Fig. 3.12):

$$\begin{aligned}\sum M_i^{(A)} &= \sum\{(x_i - x_A)F_{iy} - (y_i - y_A)F_{ix}\} \\ &= \sum(x_i F_{iy} - y_i F_{ix}) - x_A \sum F_{iy} + y_A \sum F_{ix} \qquad (3.13) \\ &= \sum M_i^{(B)} - x_A \sum F_{iy} + y_A \sum F_{ix}\,.\end{aligned}$$

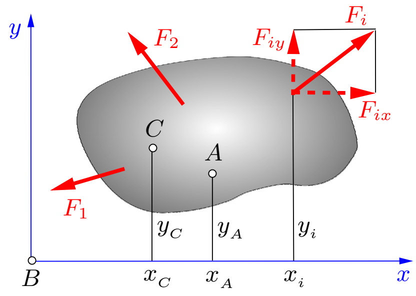


Fig. 3.12

If the equilibrium conditions (3.12) are satisfied, Equation (3.13) immediately yields $\sum M_i^{(B)} = 0$. On the other hand, if $\sum F_{ix} = 0$, $\sum F_{iy} = 0$ and $\sum M_i^{(B)} = 0$, then $\sum M_i^{(A)} = 0$ also has to be satisfied. Therefore it is immaterial which point is chosen as the point of reference.

Instead of using *two* force conditions and *one* moment condition, *one* force condition and *two* moment conditions may be applied. Introducing the conditions

$$\sum F_{ix} = 0\,, \quad \sum M_i^{(A)} = 0\,, \quad \sum M_i^{(B)} = 0 \tag{3.14}$$

into (3.13), $\sum F_{iy} = 0$ is also satisfied if $x_A \neq 0$. Hence, the equilibrium conditions (3.14) are equivalent to the conditions (3.12) if the two points A and B are not lying on a straight line (here the y-axis) that is perpendicular to the direction of the force equilibrium (here the x-direction). Similarly, the conditions

$$\sum F_{iy} = 0\,, \quad \sum M_i^{(A)} = 0\,, \quad \sum M_i^{(B)} = 0 \tag{3.15}$$

also lead to $\sum F_{ix} = 0$ if $y_A \neq 0$.

Three points A, B and C may also be chosen and only moment equations of equilibrium used, as follows:

$$\sum M_i^{(A)} = 0\,, \quad \sum M_i^{(B)} = 0\,, \quad \sum M_i^{(C)} = 0\,. \tag{3.16}$$

These equations are equivalent to (3.12) if the points A, B and C are not lying on a straight line. In order to prove this statement we use (3.13) and the corresponding relation for an arbitrary point C:

$$\begin{aligned}\sum M_i^{(A)} &= \sum M_i^{(B)} - x_A \sum F_{iy} + y_A \sum F_{ix}\,,\\ \sum M_i^{(C)} &= \sum M_i^{(B)} - x_C \sum F_{iy} + y_C \sum F_{ix}\,.\end{aligned} \tag{3.17}$$

Introducing (3.16) yields

$$-x_A \sum F_{iy} + y_A \sum F_{ix} = 0\,, \quad -x_C \sum F_{iy} + y_C \sum F_{ix} = 0\,,$$

and eliminating $\sum F_{iy}$ and $\sum F_{ix}$, respectively, leads to

$$\Big(-x_C\,\frac{y_A}{x_A} + y_C\Big) \sum F_{ix} = 0\,, \quad \Big(-x_C + \frac{x_A}{y_A}\,y_C\Big) \sum F_{iy} = 0\,.$$

Therefore, $\sum F_{ix} = 0$ and $\sum F_{iy} = 0$ is ensured if the terms in the parentheses are nonzero, i.e., if $y_A/x_A \neq y_C/x_C$. This means that the points A and C, respectively, must not lie on the same straight line passing through the origin B of the coordinate system.

In principle, it is irrelevant whether one applies the equilibrium conditions (3.12), (3.14) or (3.16) to solve a given problem. In practice, however, it may be advantageous to use one form or the other.

To apply a moment equilibrium condition (e.g., $\sum M_i^{(A)} = 0$), it is necessary to specify a reference point and a positive sense of rotation (e.g., counterclockwise). In the following, the symbol $\overset{\curvearrowleft}{A}$: is used to signify that the sum of all moments about point A must be equal to zero and that moments in the direction of the curved arrow are taken to be positive. This notation is analogous to the notation for the equilibrium of forces (e.g. $\rightarrow$:).

Consider again a general system of coplanar forces. According to (3.12) and to the results of Section 3.1.3, one can always reduce the system to one of the following four cases:

1. *Resultant* does not pass through the reference point A (Fig. 3.13a):

$$\boldsymbol{R} \neq \boldsymbol{0}\,, \quad M^{(A)} \neq 0\,.$$

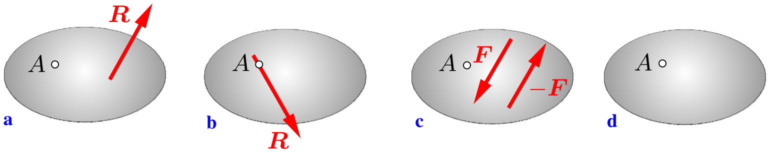


Fig. 3.13

2. *Resultant* passes through the reference point A (Fig. 3.13b):

$$\boldsymbol{R} \neq \boldsymbol{0}\,, \quad M^{(A)} = 0\,.$$

3. *Couple* (independent of the reference point A)(Fig. 3.13c):

$$\boldsymbol{R} = \boldsymbol{0}\,, \quad M^{(A)} = M \neq 0\,.$$

4. *Equilibrium* (Fig. 3.13d):

$$\boldsymbol{R} = \boldsymbol{0}\,, \quad M^{(A)} = 0\,.$$

Example 3.2 The beam shown in Fig. 3.14a can rotate about its support (see Chapter 5). It is loaded by two forces F_1 and F_2. Its weight may be neglected. E3.2

Determine the location of the support so that the beam is in equilibrium. Find the force A exerted on the beam from the support.

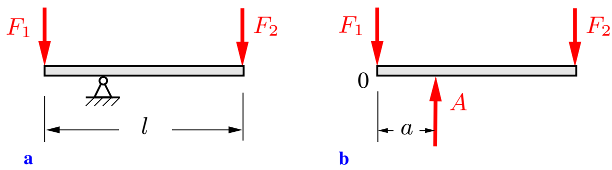


Fig. 3.14

Solution The required distance of the support from point 0 is denoted by a (Fig. 3.14b). The beam is isolated in the free-body diagram and the force A (= support reaction, see Chapter 5) is introduced. Since the forces F_1 and F_2 act in the vertical direction (the horizontal components are zero), force A also has to be vertical. This follows from the equilibrium condition in the horizontal direction.

The reference point for the equilibrium condition of moments may be chosen arbitrarily. It is, however, practical to choose a point on the action line of one of the forces. Then the lever arm of this force is zero and the force does not appear in the moment equation. If point 0 is chosen, the equilibrium conditions are (the force equilibrium in the horizontal direction is identically satisfied)

$$\uparrow: \quad A - F_1 - F_2 = 0\,, \qquad \overset{\curvearrowleft}{0}: \quad a\,A - l\,F_2 = 0\,.$$

This yields

$$\underline{\underline{A = F_1 + F_2}}\,, \qquad \underline{\underline{a = \frac{F_2}{F_1 + F_2}\,l}}\,.$$

As a check, point A is chosen as the reference point. Then the moment equilibrium is given by

$$\overset{\curvearrowleft}{A}: \quad a\,F_1 - (l - a)\,F_2 = 0\,,$$

which leads to the same result as found above.

E3.3 **Example 3.3** A cable is guided over an ideal pulley and subjected to forces S_1 and S_2, which act under the given angles α and β (Fig. 3.15a). The two forces are in equilibrium.

If force S_1 is given, determine the required force S_2 and the force exerted at 0 from the support on the pulley.

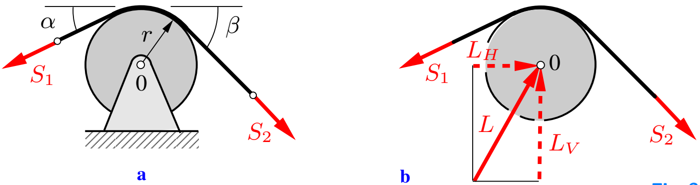


Fig. 3.15

Solution In order to solve the problem, the pulley is isolated (Fig. 3.15b). The solution of the first part of the problem is found with the aid of the equilibrium of moments about point 0. The force L acting at 0 on the pulley has no lever arm, and in an ideal pulley there exists no moment induced by friction. Therefore, the moment equilibrium condition yields

$$\overset{\curvearrowleft}{0}: \quad r\,S_1 - r\,S_2 = 0 \quad \rightarrow \quad \underline{\underline{S_2 = S_1}}\,.$$

This result is already known from Section 2.4, Fig. 2.11b.

Since the direction of the force L is unknown, it is resolved into its components, L_H and L_V, in the horizontal and the vertical directions, respectively. The equilibrium conditions

$$\uparrow: \quad L_V - S_1 \sin\alpha - S_2 \sin\beta = 0\,,$$

$$\rightarrow: \quad L_H - S_1 \cos\alpha + S_2 \cos\beta = 0$$

and $S_2 = S_1$ lead to the result

$$\underline{\underline{L_V = S_1(\sin\alpha + \sin\beta)}}\,, \qquad \underline{\underline{L_H = S_1(\cos\alpha - \cos\beta)}}\,.$$

In the special case of $\alpha = \beta$, we get $L_V = 2\,S_1 \sin\alpha$, $L_H = 0$. If $\alpha = \beta = \pi/2$, then $L_V = 2\,S_1$.

Example 3.4 A homogeneous beam (length $4\,a$, weight W) is suspended at C by a rope. The beam touches the smooth vertical walls at A and B (Fig. 3.16a). E3.4

Find the force in the rope and the contact forces at A and B.

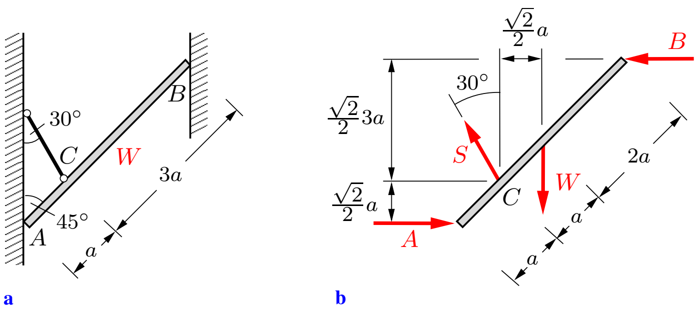


Fig. 3.16

Solution We isolate the beam by cutting the rope and removing the two walls. The free-body diagram (Fig. 3.16b) shows the contact forces A and B acting perpendicularly to the planes of contact (smooth walls), the internal force S in the rope and the weight W (acting at the center of the beam, compare Chapter 4). To formulate the sum of the moments, point C is chosen as reference point; the lever arms of the forces follow from simple geometrical relations. The equilibrium conditions then are given by

$$\begin{aligned}
\uparrow: &\quad S\cos 30^\circ - W = 0\,,\\
\rightarrow: &\quad A - B - S\sin 30^\circ = 0\,,\\
\overset{\curvearrowleft}{C}: &\quad \frac{\sqrt{2}}{2}\,a\,A - \frac{\sqrt{2}}{2}\,a\,W + \frac{\sqrt{2}}{2}\,3\,a\,B = 0\,.
\end{aligned}$$

With $\cos 30^\circ = \sqrt{3}/2$, $\sin 30^\circ = 1/2$ the three unknown forces are obtained as

$$\underline{\underline{S = \frac{2\sqrt{3}}{3}\,W}}\,,\qquad \underline{\underline{A = \frac{1+\sqrt{3}}{4}\,W}}\,,\qquad \underline{\underline{B = \frac{3-\sqrt{3}}{12}\,W}}\,.$$

E3.5 **Example 3.5** A beam (length $l = \sqrt{2}\,r$, weight negligible) lies inside a smooth spherical shell with radius r (Fig. 3.17a).

If a weight W is attached to the beam, determine the distance x from the left end point of the beam required to keep the beam in equilibrium with the angle $\alpha = 15^\circ$. Calculate the contact forces at A and B.

Solution The free-body diagram (Fig. 3.17b) shows the forces acting on the isolated beam. The contact forces are orthogonal to the respective contact planes (smooth surfaces). Therefore, they are directed towards the center 0 of the sphere. The isosceles triangle $0\,A\,B$ displays a right angle at 0 because of the given lengths r and $l = \sqrt{2}\,r$. Forces A and B are inclined with angles 60° and 30°, respectively. If forces A and B are resolved into their components perpendicular and parallel to the beam, the following equilibrium conditions are obtained:

$$\begin{aligned}
\uparrow: &\quad A\sin 60^\circ + B\sin 30^\circ - W = 0\,,\\
\rightarrow: &\quad A\cos 60^\circ - B\cos 30^\circ = 0\,,
\end{aligned}$$

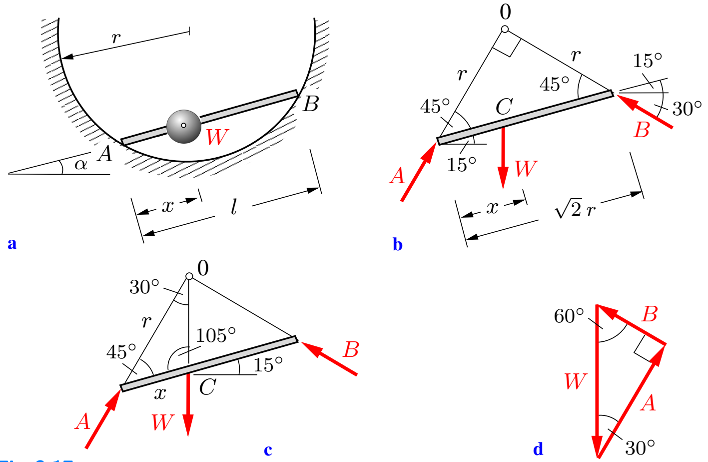


Fig. 3.17

$$\overset{\curvearrowleft}{C}: \qquad -x(A\sin 45°) + (l-x)(B\sin 45°) = 0\,.$$

These are three equations for the three unknowns A, B and x. The force equations lead to the contact forces (note: $\sin 30° = \cos 60° = 1/2$, $\sin 60° = \cos 30° = \sqrt{3}/2$):

$$\underline{\underline{A = \frac{\sqrt{3}}{2}\,W}}\,, \qquad \underline{\underline{B = \frac{1}{2}\,W}}\,.$$

If these results and $\sin 45° = \sqrt{2}/2$ are introduced into the moment equation, the required distance is obtained:

$$\underline{\underline{x}} = l\,\frac{B}{A+B} = \underline{\underline{\frac{l}{\sqrt{3}+1}}}\,.$$

The problem may also be solved graphic-analytically. If three forces are in equilibrium, they must be concurrent forces. Since the action lines of A and B intersect at 0, the action line of W also must pass through this point (Fig. 3.17c). The law of sines is now applied to the triangle $0\,A\,C$. With $\sin 105° = \sin(45°+60°) = \sin 45°\cos 60° + \cos 45°\sin 60° = (\sqrt{2}/4)(1+\sqrt{3})$ and $r = l/\sqrt{2}$ it leads again to

$$\underline{\underline{x}} = r\,\frac{\sin 30°}{\sin 105°} = \frac{l}{\sqrt{2}}\,\frac{1/2}{\frac{\sqrt{2}}{4}(1+\sqrt{3})} = \underline{\underline{\frac{l}{1+\sqrt{3}}}}\,.$$

The contact forces A and B are determined from a sketch of the force plan (not to scale, see Fig. 3.17d):

$$\underline{\underline{A}} = W\cos 30° = \underline{\underline{\frac{\sqrt{3}}{2}W}}\,, \quad \underline{\underline{B}} = W\sin 30° = \underline{\underline{\frac{W}{2}}}\,.$$

E3.6 **Example 3.6** A lever (length l) that is subjected to a vertical force F (Fig. 3.18a) exerts a contact force on a circular cylinder (radius r, weight W). The weight of the lever may be neglected. All surfaces are smooth.

Determine the contact force between the cylinder and the floor if the height h of the step is equal to the radius r of the cylinder.

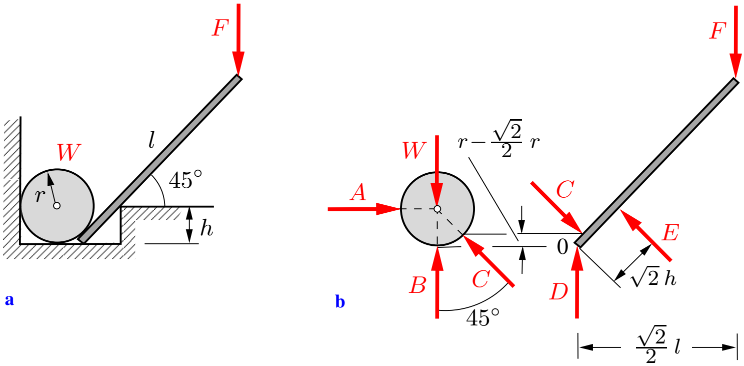


Fig. 3.18

Solution The cylinder and the lever are isolated and the contact forces A to E, which are perpendicular to the planes of contact at the respective points of contact, are introduced (Fig. 3.18b). Note that the floor and the lever represent the planes of contact at D and E, respectively. The equilibrium conditions for the lever are given by

$$\rightarrow: \quad \frac{\sqrt{2}}{2}C - \frac{\sqrt{2}}{2}E = 0\,,$$

$$\uparrow: \quad D - \frac{\sqrt{2}}{2}C + \frac{\sqrt{2}}{2}E - F = 0\,,$$

$$\overset{\curvearrowleft}{0}: \quad \sqrt{2}\,r\Big(1 - \frac{\sqrt{2}}{2}\Big)C - \sqrt{2}\,h\,E + \frac{\sqrt{2}}{2}\,l\,F = 0$$

and equilibrium at the cylinder (concurrent forces) requires

$$\rightarrow: \quad A - \frac{\sqrt{2}}{2}C = 0\,,$$

$$\uparrow: \quad B + \frac{\sqrt{2}}{2}C - W = 0\,.$$

These are five equations for the five unknown forces A to E. With $h = r$, they yield the contact force at point B:

$$B = W - \frac{l}{2\,r}F\,.$$

In the case of $F = (2\,r/l)W$, contact force B vanishes. For larger values of F there is no equilibrium: the cylinder will be lifted.

3.2 General Systems of Forces in Space

3.2.1 The Moment Vector

In order to investigate general systems of forces in space, the *moment vector* is now introduced. To this end, the coplanar problem that was already treated in Section 3.1.2 (see Fig. 3.8) is reconsidered in Fig. 3.19. The force $\boldsymbol{F}$, which acts in the x, y-plane, has a moment $M^{(0)}$ about point 0. With $\boldsymbol{F}_x = F_x\,\boldsymbol{e}_x$, etc., it is given by

$$M_z^{(0)} = h\,F = x\,F_y - y\,F_x \qquad (3.18)$$

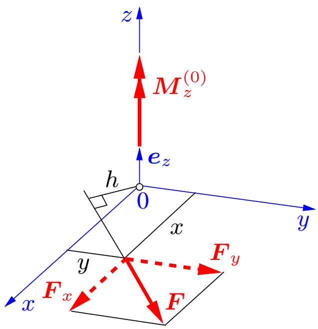
(compare (3.8)). The algebraic sign (positive sense of rotation) is chosen as in Section 3.1.2. The subscript z indicates that $M_z^{(0)}$ exerts a moment about the z-axis.

The two quantities (magnitude and sense of rotation) by which a couple is defined in coplanar problems may be expressed mathematically by the moment vector

$$\boldsymbol{M}_z^{(0)} = M_z^{(0)} \boldsymbol{e}_z \, . \tag{3.19}$$

The vector $\boldsymbol{M}_z^{(0)}$ points in the direction of the z-axis. It incorporates the magnitude $M_z^{(0)}$ and the positive sense of rotation. The positive sense of rotation is determined by the *right-hand rule* (*corkscrew rule*): if we look in the direction of the positive z-axis, a positive moment tends to rotate the body clockwise.

In order to distinguish between force vectors and moment vectors in the figures, a moment vector is represented with a double head, as shown in Fig. 3.19. Note: force vectors and moment vectors have different dimensions; therefore they can never be added.



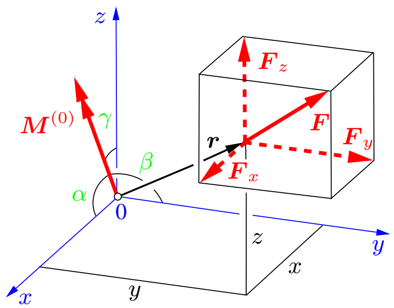


Fig. 3.19 **Fig. 3.20**

In coplanar problems, the body can only be rotated about the z-axis. Therefore, the moment vector has only the component $M_z^{(0)}$. In spatial systems, there are three possibilities of rotation (about the three axes x, y and z). Hence, the moment vector has the three components $M_x^{(0)}$, $M_y^{(0)}$ and $M_z^{(0)}$:

$$\boldsymbol{M}^{(0)} = M_x^{(0)} \, \boldsymbol{e}_x + M_y^{(0)} \, \boldsymbol{e}_y + M_z^{(0)} \, \boldsymbol{e}_z \, . \tag{3.20}$$

Fig. 3.20 shows that the components, i.e., the moments about the coordinate axes, are obtained as follows:

$$M_x^{(0)} = yF_z - zF_y,\ M_y^{(0)} = zF_x - xF_z,\ M_z^{(0)} = xF_y - yF_x. \quad (3.21)$$

The magnitude and direction of the moment vector are given by

$$\begin{aligned} &|\boldsymbol{M}^{(0)}| = M^{(0)} = \sqrt{[M_x^{(0)}]^2 + [M_y^{(0)}]^2 + [M_z^{(0)}]^2}\,, \\ &\cos\alpha = \frac{M_x^{(0)}}{M^{(0)}}\,, \quad \cos\beta = \frac{M_y^{(0)}}{M^{(0)}}\,, \quad \cos\gamma = \frac{M_z^{(0)}}{M^{(0)}}\,. \end{aligned} \quad (3.22)$$

Formally, the moment vector $\boldsymbol{M}^{(0)}$ may be represented by the *vector product*

$$\boldsymbol{M}^{(0)} = \boldsymbol{r} \times \boldsymbol{F}\,, \quad (3.23)$$

where vector $\boldsymbol{r}$ is the *position vector* pointing from the reference point 0 to the point of application of force $\boldsymbol{F}$, i.e., to an arbitrary point on the action line of $\boldsymbol{F}$. With

$$\boldsymbol{r} = x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z\,, \quad \boldsymbol{F} = F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y + F_z\,\boldsymbol{e}_z$$

and the following cross-products (compare Appendix A.1),

$$\begin{aligned} &\boldsymbol{e}_x \times \boldsymbol{e}_x = \boldsymbol{0}\,, && \boldsymbol{e}_x \times \boldsymbol{e}_y = \boldsymbol{e}_z\,, && \boldsymbol{e}_x \times \boldsymbol{e}_z = -\boldsymbol{e}_y\,, \\ &\boldsymbol{e}_y \times \boldsymbol{e}_x = -\boldsymbol{e}_z\,, && \boldsymbol{e}_y \times \boldsymbol{e}_y = \boldsymbol{0}\,, && \boldsymbol{e}_y \times \boldsymbol{e}_z = \boldsymbol{e}_x\,, \\ &\boldsymbol{e}_z \times \boldsymbol{e}_x = \boldsymbol{e}_y\,, && \boldsymbol{e}_z \times \boldsymbol{e}_y = -\boldsymbol{e}_x\,, && \boldsymbol{e}_z \times \boldsymbol{e}_z = \boldsymbol{0} \end{aligned}$$

equation (3.23) yields

$$\begin{aligned} \boldsymbol{M}^{(0)} &= (x\,\boldsymbol{e}_x + y\,\boldsymbol{e}_y + z\,\boldsymbol{e}_z) \times (F_x\,\boldsymbol{e}_x + F_y\,\boldsymbol{e}_y + F_z\,\boldsymbol{e}_z) \\ &= (y\,F_z - z\,F_y)\,\boldsymbol{e}_x + (z\,F_x - x\,F_z)\,\boldsymbol{e}_y + (x\,F_y - y\,F_x)\,\boldsymbol{e}_z \\ &= M_x^{(0)}\,\boldsymbol{e}_x + M_y^{(0)}\,\boldsymbol{e}_y + M_z^{(0)}\,\boldsymbol{e}_z\,. \end{aligned} \quad (3.24)$$

According to the properties of a cross-product, the moment vector $\boldsymbol{M}^{(0)}$ is perpendicular to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$ (Fig. 3.21). Its magnitude is numerically equal to the area of the parallelogram formed by $\boldsymbol{r}$ and $\boldsymbol{F}$:

$$M^{(0)} = r\,F\sin\varphi = h\,F\,. \quad (3.25)$$

Hence, the moment is equal to the product of the lever arm h and the force F.

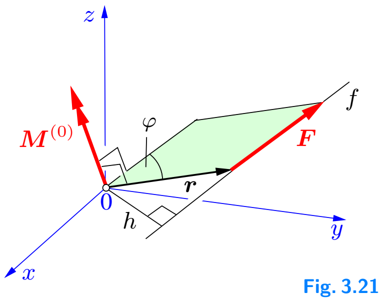


Fig. 3.21

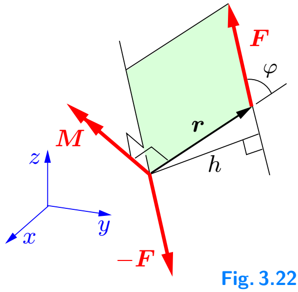


Fig. 3.22

The moment of a couple in space (Fig. 3.22) may be represented by the same formalism. Here,

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} . \tag{3.26}$$

The vector $\boldsymbol{r}$ points from an arbitrary point on the action line of $-\boldsymbol{F}$ to an arbitrary point on the action line of $\boldsymbol{F}$. As before, the moment vector $\boldsymbol{M}$ is orthogonal to the plane determined by $\boldsymbol{r}$ and $\boldsymbol{F}$. Its sense of rotation follows from the rule of the right-handed screw and its magnitude is numerically equal to the area of the parallelogram formed by $\boldsymbol{r}$ and $\boldsymbol{F}$ (lever arm times force):

$$M = h\,F . \tag{3.27}$$

The properties of *couple moments* and of *moments of a force* in space correspond to their properties in coplanar problems. In a plane, couple moments may be moved without changing the effect on a rigid body. In space, the vector of a couple moment can be moved parallel to its line of action and along this line without changing the effect. Whereas the force vector is bound to its action line (sliding vector) the vector of a couple moment is a *free vector*.

If a body in space is subjected to several couple moments $\boldsymbol{M}_i$, the resultant moment $\boldsymbol{M}_R$ is obtained as the vector sum

$$\boldsymbol{M}_R = \sum \boldsymbol{M}_i , \tag{3.28}$$

which reads in components

$$M_{Rx} = \sum M_{ix}\,, \quad M_{Ry} = \sum M_{iy}\,, \quad M_{Rz} = \sum M_{iz}\,. \tag{3.29}$$

If the sum of the moments is zero, the resulting moment $\boldsymbol{M}_R$ and hence the rotational effect on the body vanishes. Then the moment equilibrium condition

$$\boldsymbol{M}_R = \sum \boldsymbol{M}_i = \boldsymbol{0} \tag{3.30}$$

is satisfied. In components,

$$\sum M_{ix} = 0\,, \quad \sum M_{iy} = 0\,, \quad \sum M_{iz} = 0\,. \tag{3.31}$$

Example 3.7 A rope passes over an ideal pulley as shown in Fig. 3.23a. It carries a crate with weight W and is held at point C. The radius of the pulley may be neglected. E3.7

Determine the resultant moment of the forces in the rope about point A.

Solution The internal forces S_1 and S_2 in the rope are made visible by cuts through the rope. Their action on point B of the structure is shown in Fig. 3.23b. Since the bearing friction of the pulley is negligible (ideal pulley) both forces are equal, and equilibrium at the crate yields $S_1 = S_2 = W$.

To represent the moments of the forces, a coordinate system is introduced. Moment $\boldsymbol{M}_1^{(A)}$ of force $\boldsymbol{S}_1$ about A (represented as a column vector) has a component only in the x-direction:

$$\boldsymbol{M}_1^{(A)} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} aW\,. \tag{a}$$

Moment $\boldsymbol{M}_2^{(A)}$ of force $\boldsymbol{S}_2$ may be obtained with the aid of the

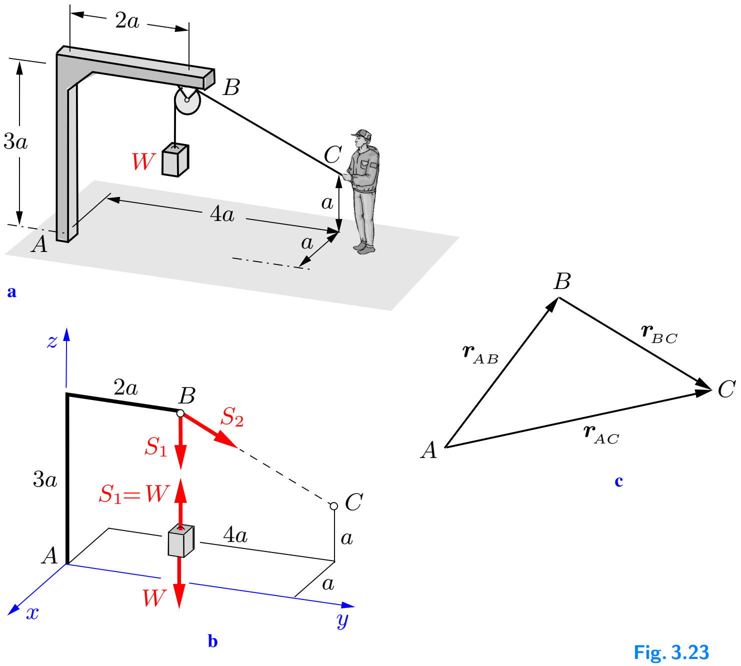


Fig. 3.23

cross-product (see (3.23))

$$\boldsymbol{M}_2^{(A)} = \boldsymbol{r}_{AB} \times \boldsymbol{S}_2 \,. \tag{b}$$

The vector $\boldsymbol{r}_{AB}$ from reference point A to the point of application B of force $\boldsymbol{S}_2$ is given by

$$\boldsymbol{r}_{AB} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} a \,. \tag{c}$$

The force $\boldsymbol{S}_2$ may be represented by its magnitude $S_2 = W$ and the unit vector $\boldsymbol{e}_2$, which points in the direction of $\boldsymbol{S}_2$, i.e., $\boldsymbol{S}_2 = S_2\,\boldsymbol{e}_2$. To obtain vector $\boldsymbol{e}_2$, we first give vector $\boldsymbol{r}_{BC}$, which points from B to C (see Fig. 3.23c):

$$\boldsymbol{r}_{BC} = \boldsymbol{r}_{AC} - \boldsymbol{r}_{AB} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} a - \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} a = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} a \,.$$

If this vector is divided by its magnitude $|\boldsymbol{r}_{BC}|$, the unit vector $\boldsymbol{e}_2$ is obtained:

$$\boldsymbol{e}_2 = \frac{\boldsymbol{r}_{BC}}{|\boldsymbol{r}_{BC}|} = \frac{1}{a\sqrt{1+4+4}} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} a = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} .$$

Therefore,

$$\boldsymbol{S}_2 = S_2\, \boldsymbol{e}_2 = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} W . \tag{d}$$

With (c) and (d) the vector product (b) yields (compare Appendix (A.31))

$$\begin{aligned} \boldsymbol{M}_2^{(A)} &= \boldsymbol{r}_{AB} \times \boldsymbol{S}_2 = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ 0 & 2a & 3a \\ -W/3 & 2W/3 & -2W/3 \end{vmatrix} \\ &= (-4/3 - 2)aW\, \boldsymbol{e}_x - aW\, \boldsymbol{e}_y + (2/3)aW\, \boldsymbol{e}_z . \end{aligned}$$

This vector may be written as a column vector:

$$\boldsymbol{M}_2^{(A)} = \frac{1}{3} \begin{pmatrix} -10 \\ -3 \\ 2 \end{pmatrix} aW . \tag{e}$$

The resultant moment $\boldsymbol{M}_R^{(A)}$ of the forces $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ about point A is the sum of the moments (a) and (e), (compare (3.32)):

$$\underline{\underline{\boldsymbol{M}_R^{(A)}}} = \boldsymbol{M}_1^{(A)} + \boldsymbol{M}_2^{(A)} = \underline{\underline{\frac{1}{3} \begin{pmatrix} -16 \\ -3 \\ 2 \end{pmatrix} aW}} .$$

3.2.2 Equilibrium Conditions

Consider a general system of forces in space (Fig. 3.24). This system can be reduced to a statically equivalent system that consists of a resultant force and a resultant moment. Similar to the procedure used in a coplanar problem (compare Section 3.1.3), an

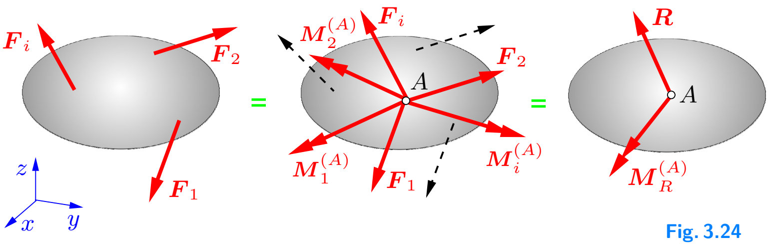


Fig. 3.24

arbitrary reference point A is chosen in space. The forces $\boldsymbol{F}_i$ are then moved to parallel lines of action that pass through this point. Since the effect of the forces on the body must not be changed, the corresponding moments $\boldsymbol{M}_i^{(A)}$ of the forces have to be introduced. Now the system of concurrent forces and the system of moments may be represented by the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$, respectively:

$$\boldsymbol{R} = \sum \boldsymbol{F}_i\,, \quad \boldsymbol{M}_R^{(A)} = \sum \boldsymbol{M}_i^{(A)}\,. \tag{3.32}$$

The resultant force $\boldsymbol{R}$ is independent of the choice of point A; the resultant moment $\boldsymbol{M}_R^{(A)}$, however, depends on this choice. Hence, there are many possible ways to reduce a given general system of forces to a resultant force and a resultant moment.

A general system of forces is in equilibrium if the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ about an arbitrary point A vanish:

$$\sum \boldsymbol{F}_i = \boldsymbol{0}\,, \quad \sum \boldsymbol{M}_i^{(A)} = \boldsymbol{0}\,. \tag{3.33}$$

In components,

$$\begin{aligned} \sum F_{ix} &= 0\,, & \sum M_{ix}^{(A)} &= 0\,, \\ \sum F_{iy} &= 0\,, & \sum M_{iy}^{(A)} &= 0\,, \\ \sum F_{iz} &= 0\,, & \sum M_{iz}^{(A)} &= 0\,. \end{aligned} \tag{3.34}$$

The fact that there are six scalar equilibrium conditions corresponds to the six degrees of freedom of a rigid body in space: translations in the x-, y- and z-directions and rotations about the corresponding coordinate axes. It can be shown that the reference point A may be chosen arbitrarily, as in a coplanar problem.

Consider now the special case of a system of parallel forces. Let, for example, all the forces act in the z-direction. Then $F_{ix} = 0$ and $F_{iy} = 0$, and the equilibrium conditions (3.34) reduce to

$$\sum F_{iz} = 0\,, \quad \sum M_{ix}^{(A)} = 0\,, \quad \sum M_{iy}^{(A)} = 0\,. \tag{3.35}$$

In this case, the equilibrium conditions in the x- and y-directions for the forces and the moment equation about the axis through A which is parallel to the z-axis are identically satisfied.

Example 3.8 A rectangular block (lengths a, b and c) is subjected to six forces, F_1 to F_6 (Fig. 3.25a). **E3.8**

Calculate the resultant $\boldsymbol{R}$, the resultant moments $\boldsymbol{M}_R^{(A)}$ and $\boldsymbol{M}_R^{(B)}$ with respect to points A and B and their magnitudes. Assume $F_1 = F_2 = F$, $F_3 = F_4 = 2\,F$, $F_5 = F_6 = 3\,F$, $b = a$, $c = 2\,a$.

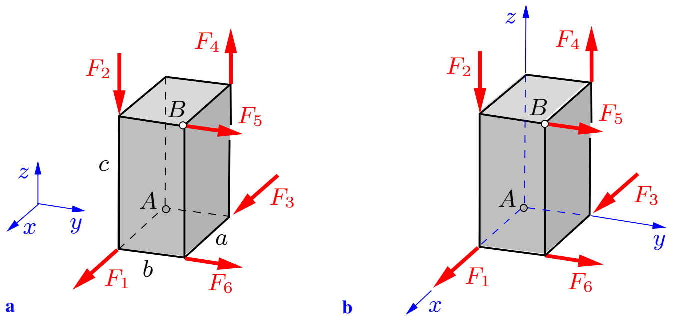


Fig. 3.25

Solution The components of the resultant force vector are obtained as the sum of the given force components:

$$R_x = F_1 + F_3 = 3F, \quad R_y = F_5 + F_6 = 6F, \quad R_z = -F_2 + F_4 = F.$$

Hence, $\boldsymbol{R}$ can be written as the column vector (see Appendix A.1)

$$\underline{\underline{\boldsymbol{R} = \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} F}}\,, \qquad \underline{\underline{R}} = \sqrt{3^2+6^2+1^2}\,F = \underline{\underline{\sqrt{46}\,F}}\,.$$

To determine the resultant moment about point A, the coordinate system is chosen such that its origin is at A (Fig. 3.25b). The components of the moment then are obtained as

$$M_{Rx}^{(A)} = \sum M_{ix}^{(A)} = b\,F_4 - c\,F_5 = -\,4\,a\,F\,,$$

$$M_{Ry}^{(A)} = \sum M_{iy}^{(A)} = a\,F_2 = a\,F\,,$$

$$M_{Rz}^{(A)} = \sum M_{iz}^{(A)} = a\,F_5 + a\,F_6 - b\,F_3 = 4\,a\,F\,,$$

and vector $\boldsymbol{M}_R^{(A)}$ can be written as

$$\underline{\underline{\boldsymbol{M}_R^{(A)} = \begin{pmatrix} M_{Rx}^{(A)} \\ M_{Ry}^{(A)} \\ M_{Rz}^{(A)} \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix} a\,F}}\,,$$

$$\underline{\underline{M_R^{(A)}}} = \sqrt{4^2+1^2+4^2}\,a\,F = \underline{\underline{\sqrt{33}\,a\,F}}\,.$$

Similarly, for point B

$$M_{Rx}^{(B)} = b\,F_2 + c\,F_6 = 7\,a\,F\,,$$

$$M_{Ry}^{(B)} = -\,c\,F_1 - c\,F_3 + a\,F_4 = -\,4\,a\,F\,,$$

$$M_{Rz}^{(B)} = b\,F_1 = a\,F$$

and

$$\underline{\underline{\boldsymbol{M}_R^{(B)} = \begin{pmatrix} 7 \\ -4 \\ 1 \end{pmatrix} a\,F}}\,, \qquad \underline{\underline{M_R^{(B)}}} = \sqrt{7^2+4^2+1^2}\,a\,F = \underline{\underline{\sqrt{66}\,a\,F}}$$

are obtained. The resultant moments about A and B, respectively, are different!

Example 3.9 A homogeneous plate with weight W is supported by six bars and loaded by a force F (Fig. 3.26a). **E3.9**

Calculate the forces in the bars.

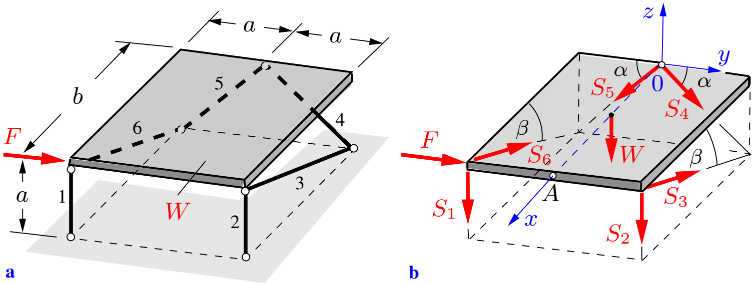


Fig. 3.26

Solution First, the free-body diagram is sketched (Fig. 3.26b). It displays the weight W (acting at the center of the plate, compare Chapter 4), the load F and the internal forces S_1 to S_6 in the bars (these are assumed to be tensile forces). In addition, the auxiliary angles α and β are introduced. Choosing the coordinate system such that as many moments as possible about its origin are zero, the following equilibrium conditions are obtained:

$$
\begin{aligned}
\sum F_{ix} &= 0: \quad -S_3\cos\beta - S_6\cos\beta = 0\,,\\
\sum F_{iy} &= 0: \quad S_4\cos\alpha - S_5\cos\alpha + F = 0\,,\\
\sum F_{iz} &= 0: \quad -S_1 - S_2 - S_3\sin\beta - S_6\sin\beta - S_4\sin\alpha\\
&\qquad\quad - S_5\sin\alpha - W = 0\,,\\
\sum M_{ix}^{(0)} &= 0: \quad a\,S_1 - a\,S_2 + a\,S_6\sin\beta - a\,S_3\sin\beta = 0\,,\\
\sum M_{iy}^{(0)} &= 0: \quad \frac{b}{2}W + b\,S_1 + b\,S_2 + b\,S_6\sin\beta + b\,S_3\sin\beta = 0\,,\\
\sum M_{iz}^{(0)} &= 0: \quad b\,F + a\,S_3\cos\beta - a\,S_6\cos\beta = 0\,.
\end{aligned}
$$

Using the trigonometrical relations

$$\cos\alpha = \sin\alpha = \frac{a}{\sqrt{2\,a^2}} = \frac{\sqrt{2}}{2}\,,$$

$$\cos\beta = \frac{b}{\sqrt{a^2+b^2}}\,, \qquad \sin\beta = \frac{a}{\sqrt{a^2+b^2}}$$

the first and the sixth equilibrium condition lead to

$$\underline{\underline{S_3 = -\,S_6 = -\frac{\sqrt{a^2+b^2}}{2\,a}\,F}}\,.$$

Then the fourth and fifth equations yield

$$\underline{\underline{S_1 = -\frac{W}{4} - \frac{F}{2}}}\,, \qquad \underline{\underline{S_2 = -\frac{W}{4} + \frac{F}{2}}}\,.$$

Finally, from the second and third equations

$$\underline{\underline{S_4 = -\frac{1}{\sqrt{2}}\left(\frac{W}{2}+F\right)}}\,, \qquad \underline{\underline{S_5 = -\frac{1}{\sqrt{2}}\left(\frac{W}{2}-F\right)}}$$

are obtained. As a check, it is verified that the equilibrium of moments is satisfied if an axis is chosen that is parallel to the y-axis and passes through point A:

$$\begin{aligned}\sum M_{iy}^{(A)} &= -\frac{b}{2}\,W - b\,S_4\sin\alpha - b\,S_5\sin\alpha \\ &= -b\left[\frac{W}{2} - \frac{1}{\sqrt{2}}\left(\frac{W}{2}+F\right)\frac{\sqrt{2}}{2} - \frac{1}{\sqrt{2}}\left(\frac{W}{2}-F\right)\frac{\sqrt{2}}{2}\right] = 0.\end{aligned}$$

E3.10 **Example 3.10** An angled member is in equilibrium under the action of four forces (Fig. 3.27a). The forces are perpendicular to the plane determined by the member; the weight of the member is negligible.

If the force F is given, calculate the forces A, B and C.

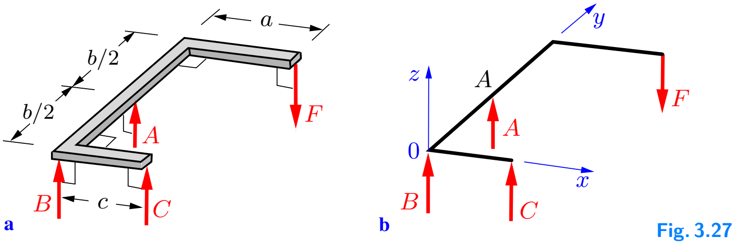


Fig. 3.27

Solution We draw the free-body diagram and introduce a coordinate system (Fig. 3.27b). If the origin 0 is chosen as the reference point, the equilibrium conditions (3.35) are

$$\sum F_{iz} = 0 : \quad A + B + C - F = 0 ,$$
$$\sum M_{ix}^{(0)} = 0 : \quad \frac{b}{2} A - b F = 0 ,$$
$$\sum M_{iy}^{(0)} = 0 : \quad -c\, C + a\, F = 0 .$$

They have the solution

$$\underline{\underline{A = 2F}}, \quad \underline{\underline{C = \frac{a}{c} F}}, \quad \underline{\underline{B = -\left(1 + \frac{a}{c}\right) F}} .$$

As a check, point A is chosen as the reference point instead of point 0. Then the moment equation

$$\sum M_{ix}^{(A)} = 0 : \quad -\frac{b}{2} B - \frac{b}{2} C - \frac{b}{2} F = 0$$

is used instead of the second equation above, which leads to the same solution.

3.3 Supplementary Problems

3.3

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

Example 3.11 A uniform pole (length l, weight W) leans against a corner as shown in Fig. 3.28. A rope S prevents the pole from sliding. All surfaces are smooth. E3.11

Determine the force S in the rope.

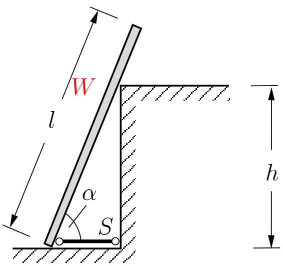


Fig. 3.28

Result: see (**B**) $S = \dfrac{Wl}{2h}\sin^2\alpha\cos\alpha\,.$

E3.12 **Example 3.12** A uniform beam (length l, weight W) is inserted into an opening (Fig. 3.29). The surfaces are smooth.

Calculate the magnitude of the force F required to hold the beam in equilibrium. Is the result valid for an arbitrary ratio a/l?

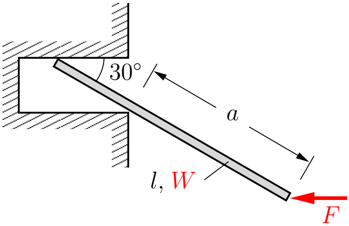


Fig. 3.29

Result: see (**B**) $F = \dfrac{\sqrt{3}}{6-8a/l}W\,.$

The result is valid only if the contact forces between the beam and the surfaces of the opening are positive, which leads to the requirement $3/8 < a/l < 3/4$.

E3.13 **Example 3.13** Two smooth rollers (each having weight W and radius r) are connected by a rope (length a) as shown in Fig. 3.30. A lever (length l) subjected to a vertical force F exerts contact forces on the rollers.

Determine the contact forces between the rollers and the horizontal plane.

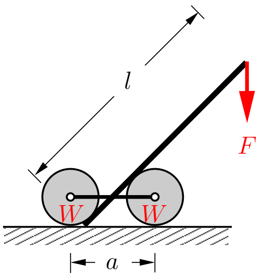


Fig. 3.30

Results: see (**A**)

$$N_1 = W - F\frac{l}{a}\sqrt{1-4(\frac{r}{a})^2}\,, \qquad N_2 = W + F\frac{l}{a}\sqrt{1-4(\frac{r}{a})^2}$$

where N_1 and N_2 are the contact forces acting on the left and on the right roller, respectively.

Example 3.14 Two smooth spheres (each having weight W and radius r) rest in a thin-walled circular cylinder (weight Q, radius $R = 4r/3$) as shown in Fig. 3.31.

Find the magnitude of Q required to prevent the cylinder from falling over.

E3.14

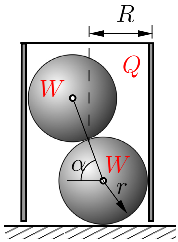


Fig. 3.31

Result: see (**A**) $\quad Q > W/2\,.$

Example 3.15 A rigid body is subjected to three forces: $\boldsymbol{F}_1 = F(-2,3,1)^T$, $\boldsymbol{F}_2 = F(7,1,-4)^T$, $\boldsymbol{F}_3 = F(3,-1,-3)^T$. Their points of application are given by the position vectors $\boldsymbol{r}_1 = a(4,3,2)^T$, $\boldsymbol{r}_2 = a(3,2,4)^T$, $\boldsymbol{r}_3 = a(3,5,0)^T$.

Determine the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ with respect to point A given by $\boldsymbol{r}_A = a(3,2,1)^T$.

E3.15

Results: $\quad \boldsymbol{R} = F(8,3,-6)^T\,, \quad \boldsymbol{M}_R^{(A)} = aF(-15,15,-4)^T\,.$

Example 3.16 A plate in the form of a rectangular triangle (weight negligible) is supported by six bars. It is subjected to the forces F and Q (Fig. 3.32).

Calculate the forces in the bars.

E3.16

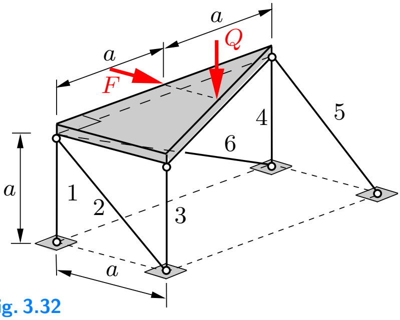


Fig. 3.32

Results: see (**A**) $\quad S_1 = F/2, \quad S_2 = S_5 = -\sqrt{2}\,F/2,$

$S_3 = -Q/2, \quad S_4 = (F-Q)/2, \quad S_6 = 0.$

E3.17 **Example 3.17** A homogeneous rectangular plate (weight W) is supported by six bars. The plate is subjected to a vertical load F (Fig. 3.33).

Calculate the forces in the bars.

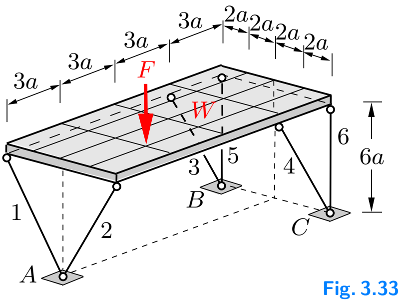


Fig. 3.33

Results: see (**B**) $S_1 = S_2 = -\sqrt{13}\,(2W + 3F)/24,$

$$S_3 = S_4 = 0, \quad S_5 = -(2W - F)/8, \quad S_6 = -(2W + 3F)/8.$$

E3.18 **Example 3.18** A rectangular plate of negligible weight is suspended by three vertical wires as shown in Fig. 3.34.

a) Assume that the plate is subjected to a concentrated vertical force Q. Determine the location of the point of application of Q so that the forces in the wires are equal.

b) Calculate the forces in the wires if the plate is subjected to a vertical constant area load p.

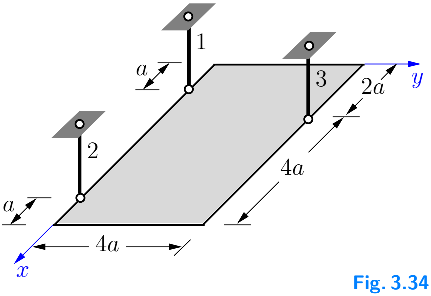


Fig. 3.34

Results: see (**A**) a) $x_Q = 8\,a/3\,, \quad y_Q = 4\,a/3.$

b) $S_1 = 3pa^2\,, \quad S_2 = 9pa^2\,, \quad S_3 = 12pa^2.$

E3.19

Example 3.19 The circular arch in Fig. 3.35 is subjected to a uniform tangential line load q_0.

Determine the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ with respect to the center A of the circle. If the load is reduced to a single force alone, find the corresponding line of action.

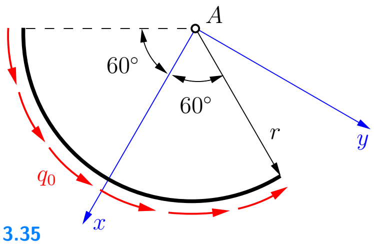


Fig. 3.35

Results: see (**B**) $R_x = 0\,, \quad R_y = \sqrt{3}\,q_0 r\,, \quad M_R^{(A)} = 2\pi q_0 r^2/3.$

$x_R = 1.21\,r\,, \quad y_R$ arbitrary.

E3.20

Example 3.20 A sphere (weight W_S) is held between a beam (weight W_B) and a wall as shown in Fig. 3.36. The surface of the sphere is smooth. The beam is supported by a hinge at A and a rope at B.

Calculate the force S in the rope.

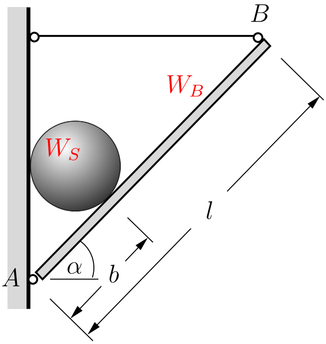


Fig. 3.36

Result: $$S = \frac{W_B}{2}\cot\alpha + \frac{b\,W_S}{l\,\sin\alpha\cos\alpha}\,.$$

3.4

3.4 Summary

- A couple consists of two forces having equal magnitudes, parallel action lines and opposite directions.
  The effect of a couple is uniquely given by its moment $\boldsymbol{M}$. The couple moment is determined by its magnitude $M = h\,F$ and its sense of rotation.
  A couple moment is not bound to an action line: it can be applied at arbitrary points of a rigid body without changing its effect on the body.
- The moment of a force $\boldsymbol{F}$ with respect to a point A is defined as $\boldsymbol{M}^{(A)} = \boldsymbol{r} \times \boldsymbol{F}$ where $\boldsymbol{r}$ is the vector pointing from point A to an arbitrary point on the action line of $\boldsymbol{F}$. The moment $\boldsymbol{M}^{(A)}$ has the magnitude $M^{(A)} = h\,F$ ($h =$ lever arm) and a sense of rotation.
  In the case of a coplanar system of forces, the moment vector has only one component $M^{(A)} = h\,F$ (perpendicular to the plane) and a sense of rotation about A.
- A general system of forces can be reduced to a resultant force $\boldsymbol{R}$ and a resultant moment $\boldsymbol{M}_R^{(A)}$ with respect to an arbitrary point A.
- A general system of forces is in equilibrium if the resultant force $\boldsymbol{R}$ and the resultant moment $\boldsymbol{M}_R^{(A)}$ vanish:

  $$\sum \boldsymbol{F}_i = \boldsymbol{0}\,, \qquad \sum \boldsymbol{M}_i^{(A)} = \boldsymbol{0}\,.$$

  These equations represent three force conditions and three moment conditions in spatial problems.
  In the case of a coplanar system of forces, the equilibrium conditions reduce to

  $$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,, \qquad \sum M_i^{(A)} = 0\,.$$

Chapter 4

Center of Gravity, Center of Mass, Centroids

4

4 Center of Gravity, Center of Mass, Centroids

Objectives: In this chapter, definitions of the center of gravity and the center of mass are given. It is shown how to determine the centroids of bodies, areas and lines. Various examples demonstrate how to apply the definitions to practical problems.

4.1 Center of Forces

In Section 3.1.3, it is shown that a system of coplanar forces that are not in equilibrium can be replaced by a *single* force, namely, the resultant $\boldsymbol{R}$, provided that the reduction does not lead to a couple. In the special case of a system of parallel forces, the direction of the resultant $\boldsymbol{R}$ coincides with the direction of the individual forces. The action line of the resultant $\boldsymbol{R}$ can be found from (3.11). If we introduce the force $\boldsymbol{H} = -\boldsymbol{R}$, which has the same action line as $\boldsymbol{R}$, then the given system of forces and the *single* force $\boldsymbol{H}$ are in equilibrium.

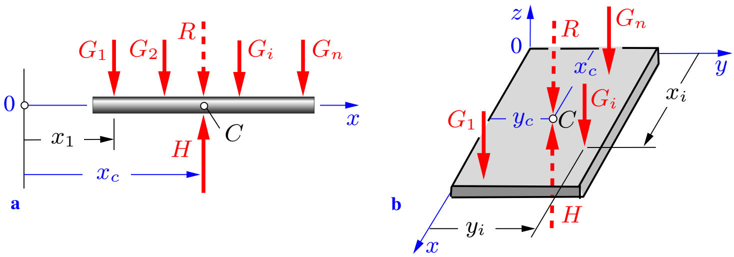


Fig. 4.1

As a simple example, consider a beam (weight neglected) that is loaded by a system of parallel forces G_i (Fig. 4.1a). In order to determine the location of the action line of the supporting force H (and therefore also of the resultant R), the coordinate x is introduced with an arbitrarily chosen origin 0. Applying the equilibrium conditions (3.12)

$$\uparrow: \quad H - \sum G_i = 0\,, \qquad \overset{\curvearrowleft}{0}: \quad x_c H - \sum x_i\, G_i = 0$$

the action line is found to be located at the distance

$$x_c = \frac{\sum x_i\, G_i}{\sum G_i} \tag{4.1}$$

from the origin 0 of the coordinate system. The corresponding point C (an arbitrary point on the action line of H) is called the *center of forces*.

The result (4.1) can be generalised in the case of a spatial force system where every force is parallel to the z-axis (Fig. 4.1b). The conditions of equilibrium (3.35)

$$\begin{aligned}
\sum F_{iz} = 0 \; : \qquad & H - \sum G_i = 0\,, \\
\sum M_{ix}^{(0)} = 0 \; : \qquad & y_c H - \sum y_i\, G_i = 0\,, \\
\sum M_{iy}^{(0)} = 0 \; : \qquad & -x_c H + \sum x_i\, G_i = 0
\end{aligned}$$

(signs of the moments according to the right-hand rule) yield the coordinates of the center of forces:

$$x_c = \frac{\sum x_i\, G_i}{\sum G_i}\,, \qquad y_c = \frac{\sum y_i\, G_i}{\sum G_i}\,. \tag{4.2}$$

The considerations that are used in the case of concentrated forces can also be applied to systems of continuously distributed loads. For this purpose, the line load $q(x)$ (dimension: force per unit length) is replaced according to Fig. 4.2a by a number of infinitely small single forces. The line load acting at a distance x on the infinitesimal element $\mathrm{d}x$ is replaced by the concentrated force $q(x)\,\mathrm{d}x$ (since $\mathrm{d}x$ is infinitesimal, the change in q can be neglected and q can be taken as a constant in this interval). Hence, the forces G_i in (4.1) are replaced by $q(x)\,\mathrm{d}x$ and the moment arms x_i by the coordinate x. In the limit $\mathrm{d}x \to 0$, the sums are replaced by integrals. Thus, the coordinate of the center of forces is obtained:

$$x_c = \frac{\int x\, q(x)\,\mathrm{d}x}{\int q(x)\,\mathrm{d}x}\,. \tag{4.3}$$

The integrations must be carried out over the entire length l on which the line load $q(x)$ acts.

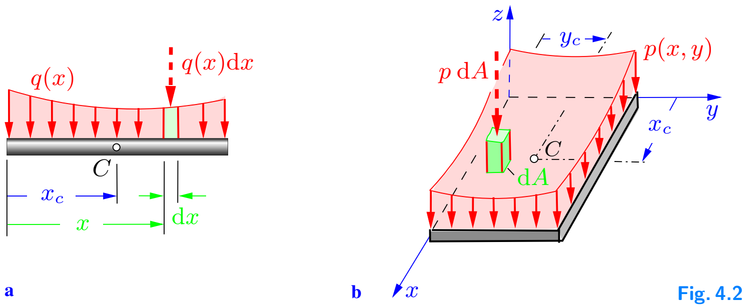


Fig. 4.2

In an analogous way, the area load $p(x,y)$ (dimension: force per unit area) acting at the infinitesimal area $\mathrm{d}A$, located at the point x, y, is replaced by the infinitesimal force $p(x)\,\mathrm{d}A$ (Fig. 4.2b). Integration leads to the coordinates of the center of forces:

$$x_c = \frac{\int x\,p(x,y)\,\mathrm{d}A}{\int p(x,y)\,\mathrm{d}A}\,, \quad y_c = \frac{\int y\,p(x,y)\,\mathrm{d}A}{\int p(x,y)\,\mathrm{d}A}\,. \tag{4.4}$$

It should be noted that the integration must be carried out over the entire loaded area A with both its directions x and y. For simplicity of notation only one integral sign is used instead of a double integral. The Examples 4.3 - 4.5 shall demonstrate how to practically carry out the integration.

Example 4.1 A beam is loaded by a triangular line load (Fig. 4.3a). Determine the resultant of the load and locate its action line. **E4.1**

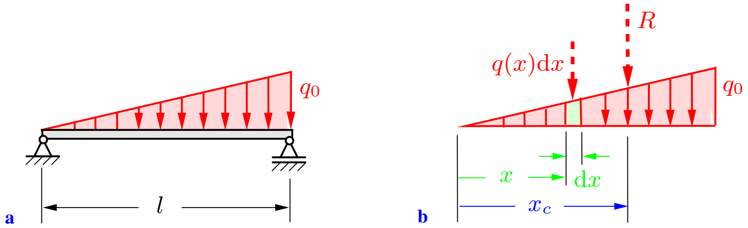


Fig. 4.3

Solution We introduce the coordinate x according to Fig. 4.3b. Then the triangular line load is described by

$$q(x) = q_0\,\frac{x}{l}\,.$$

The integration of $q(x)$ ($\widehat{=}$ sum of the forces $q(x)\,\mathrm{d}x$) yields the resultant

$$\underline{\underline{R}} = \int_0^l q(x)\,\mathrm{d}x = \int_0^l q_0\,\frac{x}{l}\,\mathrm{d}x = q_0\,\frac{x^2}{2\,l}\bigg|_0^l = \underline{\underline{\frac{1}{2}\,q_0\,l}}\,,$$

which is equivalent to the area of the triangle. With the numerator in (4.3),

$$\int x\,q(x)\,\mathrm{d}x = \int\limits_0^l x\,q_0\,\frac{x}{l}\,\mathrm{d}x = q_0\,\frac{x^3}{3\,l}\bigg|_0^l = \frac{1}{3}\,q_0\,l^2\,,$$

we obtain the coordinate of the center of forces, i.e., the location of the resultant's action line:

$$\underline{\underline{x_c}} = \frac{\int x\,q(x)\,\mathrm{d}x}{\int q(x)\,\mathrm{d}x} = \frac{\frac{1}{3}q_0\,l^2}{\frac{1}{2}q_0\,l} = \underline{\underline{\frac{2}{3}\,l}}\,.$$

The action of the concentrated force $R = q_0 l/2$, located at a distance $x_c = 2\,l/3$ from the left-hand side of the rigid beam, is statically equivalent to the action of the triangular line load; in rigid-body mechanics, the distributed load may be replaced by its resultant.

4.2

4.2 Center of Gravity and Center of Mass

Equations (4.4) can be generalised in the case of parallel body forces $f(x, y, z)$ (dimension: force per unit volume) that act on rigid bodies. Let the direction of the body forces be arbitrary and be given by the unit vector $\boldsymbol{e}$ (Fig. 4.4a). As before, the continuously distributed load is replaced by a system of infinitesimal concentrated forces. For this purpose, the infinitesimal volume element $\mathrm{d}V$, located at the point with the position vector $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$, is considered. This volume element is loaded by the infinitesimal force $\mathrm{d}\boldsymbol{G} = f(x, y, z)\,\mathrm{d}V\boldsymbol{e}$. Since the direction of the body forces is given by the vector $\boldsymbol{e}$ and is therefore the same for all volume elements and forces, respectively, the resultant of the body forces is obtained as

$$\boldsymbol{G} = \int \mathrm{d}\boldsymbol{G} = \left(\int f(x, y, z)\,\mathrm{d}V\right)\boldsymbol{e} = G\boldsymbol{e}$$

where

$$G = \int f(x, y, z)\,\mathrm{d}V\,.$$

The integration must be carried out over the entire volume V with its three directions x, y and z (triple integral).

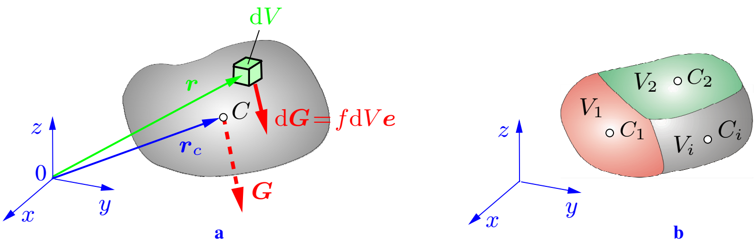


Fig. 4.4

The point of application C of the force $\boldsymbol{G}$, given by the position vector $\boldsymbol{r}_c$, follows from the condition that the moment of $\boldsymbol{G}$ with respect to the origin 0 of the coordinate system must be equal to the sum (i.e., the integral) of the moments of all the forces $\mathrm{d}\boldsymbol{G}$ with respect to the same point:

$$\boldsymbol{r}_c \times \boldsymbol{G} = \int \boldsymbol{r} \times \mathrm{d}\boldsymbol{G}\,.$$

Introducing $\boldsymbol{G}$ and $\mathrm{d}\boldsymbol{G}$ yields

$$\boldsymbol{r}_c \times G\boldsymbol{e} = \left(\int \boldsymbol{r} \times f(x,y,z)\,\mathrm{d}V\right)\boldsymbol{e}$$

$$\rightarrow \quad \left(\boldsymbol{r}_c\,G - \int \boldsymbol{r}\,f(x,y,z)\,\mathrm{d}V\right) \times \boldsymbol{e} = \boldsymbol{0}\,.$$

The direction of the vector $\boldsymbol{e}$ is arbitrary. Therefore, this equation is satisfied only if the expression in parentheses vanishes. This yields the center of the body forces

$$\boldsymbol{r}_c = \frac{\int \boldsymbol{r}\,f(x,y,z)\,\mathrm{d}V}{\int f(x,y,z)\,\mathrm{d}V} \tag{4.5a}$$

which reads in components

$$x_c = \frac{\int x\,f(x,y,z)\,\mathrm{d}V}{\int f(x,y,z)\,\mathrm{d}V}\,, \quad y_c = \frac{\int y\,f(x,y,z)\,\mathrm{d}V}{\int f(x,y,z)\,\mathrm{d}V}\,, \tag{4.5b}$$

$$z_c = \frac{\int z\, f(x,y,z)\, \mathrm{d}V}{\int f(x,y,z)\, \mathrm{d}V}\,. \tag{4.5b}$$

As a special case, a rigid body on the surface of the earth that is subjected to the action of the earth's gravitational field, is considered now. Here, the body force is given by $f(x,y,z) = \varrho(x,y,z)\, g$, where ϱ is the density of the material of the body and g is the gravitational acceleration. The density may be variable within the body, and the gravitational acceleration is assumed to be constant (uniform and parallel gravitational field). Thus, the gravitational acceleration g is a constant factor in both the numerators and the denominators of Equations (4.5b), and will cancel. This defines the *center of gravity* of the body as the point with the coordinates

$$x_c = \frac{\int x\, \varrho\, \mathrm{d}V}{\int \varrho\, \mathrm{d}V}\,, \quad y_c = \frac{\int y\, \varrho\, \mathrm{d}V}{\int \varrho\, \mathrm{d}V}\,, \quad z_c = \frac{\int z\, \varrho\, \mathrm{d}V}{\int \varrho\, \mathrm{d}V}\,. \tag{4.6}$$

The weight of the body is distributed over its entire volume V. In the case of a rigid body, the weight can be considered to be concentrated at the center of gravity without change in the static action.

It may be noted that the assumption of a uniform and parallel gravitational field is not satisfied exactly in reality. The directions and magnitudes of the gravitational forces for the various particles of a body differ slightly, since these forces are directed towards the center of attraction of the earth and their intensities depend on their distance from this center. However, in the case of sufficiently small bodies, the assumption is sufficiently accurate for engineering purposes.

The mass of the infinitesimal volume element $\mathrm{d}V$ is given by $\mathrm{d}m = \varrho\, \mathrm{d}V$, and the mass of the whole body is the sum (i.e., the integral) of the mass elements: $m = \int \varrho\, \mathrm{d}V = \int \mathrm{d}m$. This leads to the definition of the *center of mass*:

$$x_c = \frac{1}{m}\int x\, \mathrm{d}m\,, \quad y_c = \frac{1}{m}\int y\, \mathrm{d}m\,, \quad z_c = \frac{1}{m}\int z\, \mathrm{d}m\,. \tag{4.7}$$

The center of mass coincides with the center of gravity if the gravitational field is assumed to be uniform and parallel.

In the case of a body made of *homogeneous* material, the density ϱ is constant and will therefore cancel in (4.6). Thus,

$$x_c = \frac{1}{V}\int x\,\mathrm{d}V\,, \quad y_c = \frac{1}{V}\int y\,\mathrm{d}V\,, \quad z_c = \frac{1}{V}\int z\,\mathrm{d}V\,, \quad (4.8)$$

where $V = \int \mathrm{d}V$ is the volume of the whole body. Equations (4.8) define the *center of the volume*. If the density and the gravitational acceleration are constant, the center of gravity and the center of the volume coincide. Since Equations (4.8) define a purely geometrical property of the body, the center of gravity can be obtained by purely geometrical considerations in this case.

The term "center" (e.g., center of mass) refers to a real physical body. The term "centroid" is used when the density factors are omitted, i.e., when one is concerned with geometrical considerations only. If the density is uniform throughout the body, the positions of the center of mass and of the centroid of the body are identical. In the case of a variable density, the center of mass and the centroid are in general different points. If a body possesses a plane of geometrical symmetry, the centroid will lie in this plane.

In the following, a body is considered that is composed of several parts of simple shape (composite body). The volumes V_i, the constant densities ϱ_i and the coordinates x_i, y_i, z_i of the centers of gravity C_i of the individual parts are assumed to be known (Fig. 4.4b). The denominator in (4.6) can then be written in the form

$$\begin{aligned}\int_V \varrho\,\mathrm{d}V &= \int_{V_1} \varrho_1 \mathrm{d}V + \int_{V_2} \varrho_2 \mathrm{d}V + \ldots = \varrho_1 \int_{V_1} \mathrm{d}V + \varrho_2 \int_{V_2} \mathrm{d}V + \ldots \\ &= \varrho_1 V_1 + \varrho_2 V_2 + \ldots = \sum \varrho_i V_i\,,\end{aligned}$$

where the notation V_i at the integral signs indicates the range of the respective integration.

In order to simplify the numerators in (4.6), at first only the x-component is considered and the relation for the i-th body

$$x_i = \frac{1}{V_i}\int_{V_i} x\,\mathrm{d}V \quad \rightarrow \quad \int_{V_i} x\,\mathrm{d}V = x_i V_i$$

is used, which follows from the first equation of (4.8). If the integration over the entire volume is written again as the sum of the integrations over the individual volumes, the numerator of the first equation in (4.6) results in

$$\begin{aligned}\int_V x\varrho \,\mathrm{d}V &= \varrho_1 \int_{V_1} x \,\mathrm{d}V + \varrho_2 \int_{V_2} x \,\mathrm{d}V + \ldots \\ &= \varrho_1 x_1 V_1 + \varrho_2 x_2 V_2 + \ldots = \sum x_i \varrho_i V_i \,.\end{aligned}$$

Analogous equations are obtained for the coordinates y and z. Thus, the location of the center of gravity is determined by

$$x_c = \frac{\sum x_i \varrho_i V_i}{\sum \varrho_i V_i}\,, \quad y_c = \frac{\sum y_i \varrho_i V_i}{\sum \varrho_i V_i}\,, \quad z_c = \frac{\sum z_i \varrho_i V_i}{\sum \varrho_i V_i}\,. \tag{4.9}$$

The task to locate the center of gravity is therefore considerably simplified: no integration is necessary since finite sums replace the integrals.

If the density is constant within the whole body (homogeneous material, i.e., $\varrho_i = \varrho$), it will cancel and (4.9) reduces to

$$x_c = \frac{\sum x_i V_i}{V}\,, \quad y_c = \frac{\sum y_i V_i}{V}\,, \quad z_c = \frac{\sum z_i V_i}{V}\,. \tag{4.10}$$

Equations (4.9) and (4.10) can also be used in the case of bodies with holes or with regions having no material. The given body then is considered to be composed of the corresponding body without holes, and the holes are considered to be additional bodies with "negative" volumes. The volumes of the holes are thus inserted with negative signs in (4.9) and (4.10).

E4.2 **Example 4.2** A small cube (edge length $2a$) is removed from a large cube (edge length $4a$), as shown in Fig. 4.5a.

a) Locate the center of gravity of the remaining body.

b) Determine the center of gravity if the small cube is replaced by a cube with the same size but of different material (density $\varrho_2 = 2\varrho_1$), as shown in Fig. 4.5b.

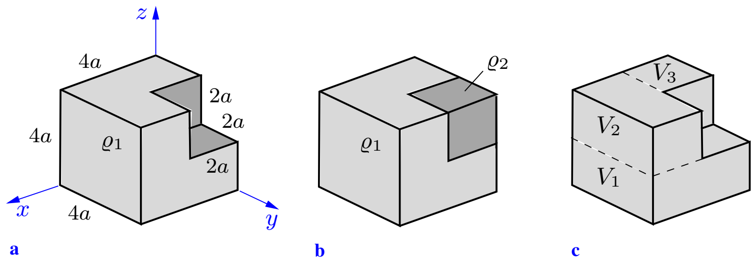


Fig. 4.5

Solution In both cases the body can be considered to be composed of several parts. In Problem a), the material is homogeneous (constant density ϱ_1). Therefore, the position of the center of gravity can be determined from (4.10). The three parts as indicated in Fig. 4.5c are used in the calculation. It is practical to carry it out in the form of the following table:

i	x_i	y_i	z_i	V_i	x_iV_i	y_iV_i	z_iV_i
1	$2a$	$2a$	a	$32a^3$	$64a^4$	$64a^4$	$32a^4$
2	$3a$	$2a$	$3a$	$16a^3$	$48a^4$	$32a^4$	$48a^4$
3	a	a	$3a$	$8a^3$	$8a^4$	$8a^4$	$24a^4$
		$\sum$		$56a^3$	$120a^4$	$104a^4$	$104a^4$

$$\rightarrow \quad \underline{\underline{x_c}} = \frac{120}{56}\,a = \underline{\underline{\frac{15}{7}\,a}}\,, \qquad \underline{\underline{y_c = z_c}} = \frac{104}{56}\,a = \underline{\underline{\frac{13}{7}\,a}}\,.$$

The same result can be obtained more easily if the complete cube (edge length $4a$) is taken as the volume V_1. The space from which the small cube is removed (the space that is void of material, i.e., negative volume V_2) is then the second part:

i	x_i	y_i	z_i	V_i	x_iV_i	y_iV_i	z_iV_i
1	$2a$	$2a$	$2a$	$64a^3$	$128a^4$	$128a^4$	$128a^4$
2	a	$3a$	$3a$	$-8a^3$	$-8a^4$	$-24a^4$	$-24a^4$
		$\sum$		$56a^3$	$120a^4$	$104a^4$	$104a^4$

In Problem b), the densities of the two parts of the body are different. Therefore, Equations (4.9) have to be used. The body as shown in Fig. 4.5a is chosen as the first part. Its center of gravity is known from Problem a). The second part is the small cube, having the density $\varrho_2 = 2\,\varrho_1$:

i	x_i	y_i	z_i	$\varrho_i V_i$	$x_i \varrho_i V_i$	$y_i \varrho_i V_i$	$z_i \varrho_i V_i$
1	$\frac{15}{7}a$	$\frac{13}{7}a$	$\frac{13}{7}a$	$56\varrho_1 a^3$	$120\varrho_1 a^4$	$104\varrho_1 a^4$	$104\varrho_1 a^4$
2	a	$3a$	$3a$	$16\varrho_1 a^3$	$16\varrho_1 a^4$	$48\varrho_1 a^4$	$48\varrho_1 a^4$
		$\sum$		$72\varrho_1 a^3$	$136\varrho_1 a^4$	$152\varrho_1 a^4$	$152\varrho_1 a^4$

$$\rightarrow \quad \underline{\underline{x_c}} = \frac{136}{72}\,a = \underline{\underline{\frac{17}{9}\,a}}\,, \qquad \underline{\underline{y_c = z_c}} = \frac{152}{72}\,a = \underline{\underline{\frac{19}{9}\,a}}\,.$$

4.3

4.3 Centroid of an Area

The location of the centroid of a plane area must be known for certain problems arising in engineering mechanics (see, for example, the chapter "Bending of Beams" in Volume 2). The coordinates of the centroid can be obtained from (4.8) if the body is considered to be a thin plate with a constant density and a constant thickness (Fig. 4.6) and $t \to 0$. Introducing the infinitesimal area $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ located at the point x, y, the entire area $A = \int \mathrm{d}A$, the volume element $\mathrm{d}V = t\,\mathrm{d}A$ and the entire volume $V = t\,A$ into (4.8) yields the coordinates of the *centroid of the area*:

$$x_c = \frac{1}{A}\int x\,\mathrm{d}A\,, \quad y_c = \frac{1}{A}\int y\,\mathrm{d}A\,. \tag{4.11}$$

The integration must be carried out over the entire area A (double integrals). Since $t \to 0$ and therefore $z \to 0$, the third equation

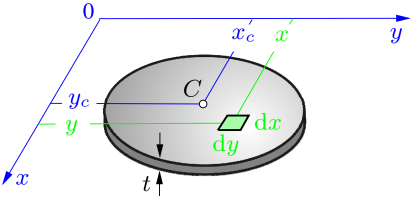


Fig. 4.6

in (4.8) results in $z_c \to 0$: the centroid of a *plane* area is located *in* its plane.

The center of mass of a thin plate coincides with the centroid of the surface area if the thickness ($t \to 0$) and density are constant over the entire area.

The integrals in (4.11) are called the *first moments of the area* with respect to the y- and the x-axis, respectively:

$$S_y = \int x \, \mathrm{d}A \,, \quad S_x = \int y \, \mathrm{d}A \,. \tag{4.12}$$

If the origin of the coordinate system is chosen to coincide with the centroid of the area, the coordinates x_c and y_c are zero. Thus, the integrals in (4.12) have to vanish:

The first moments of an area with respect to axes through its centroid are zero.

Finding the centroid of an area is simplified if an *axis of symmetry* exists. For example, Fig. 4.7 shows that for every infinitesimal area $\mathrm{d}A$ located at a positive distance x, there exists a corresponding element located at a negative distance. The integral $\int x \, \mathrm{d}A$ in (4.11) is therefore zero. Hence:

If the area has an axis of symmetry, the centroid of the area lies on this axis.

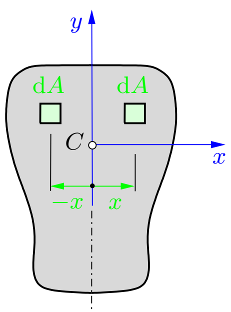


Fig. 4.7

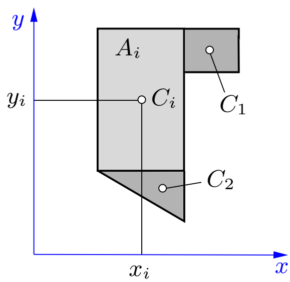


Fig. 4.8

In the case of two axes of symmetry, the centroid is determined by the point of intersection of these axes.

Let us now consider an area composed of several parts of simple shape (Fig. 4.8). The coordinates x_i, y_i of the centroids C_i and the areas A_i of the individual parts are assumed to be known. The first equation in (4.11) can then be written in the form

$$x_c = \frac{1}{A}\int x\,\mathrm{d}A = \frac{1}{A}\Big\{\int\limits_{A_1} x\,\mathrm{d}A + \int\limits_{A_2} x\,\mathrm{d}A + \ldots\Big\}$$
$$= \frac{1}{A}\{x_1\,A_1 + x_2\,A_2 + \ldots\}\,.$$

Thus, the integrals are replaced by sums (compare Section 4.2). With $A = \sum A_i$ the coordinates of the centroid are obtained as

$$x_c = \frac{\sum x_i\,A_i}{\sum A_i}\,, \qquad y_c = \frac{\sum y_i\,A_i}{\sum A_i}\,. \tag{4.13}$$

These equations can also be used for areas with holes or cut-out sections. The holes are then considered to be "negative" areas (see Example 4.7).

The method used to divide the whole area into several sections (method of composite areas) can also be applied to infinitely small sections. Therefore it is not necessary to choose in (4.11) an infinitesimal area $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$ as shown in Fig. 4.6. In many cases, it is more practical to choose a rectangular or a triangular element where one side, e.g. a, is finite: $\mathrm{d}A = a\,\mathrm{d}x$ (differential strip). This step reduces the double integrals to single integrals. Then, the quantities x and y in (4.11) refer to the coordinates of the centroid of the element. This method is demonstrated in the Examples 4.3 - 4.5.

E4.3 **Example 4.3** Locate the centroid of a rectangular triangle with baseline a and height h (Fig. 4.9a).

Solution The coordinates of the centroid can be determined from Equations (4.11):

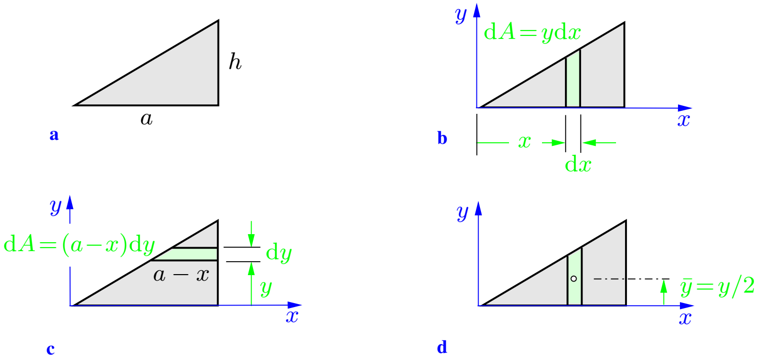


Fig. 4.9

$$x_c = \frac{1}{A}\int x\,\mathrm{d}A\,, \quad y_c = \frac{1}{A}\int y\,\mathrm{d}A\,.$$

First, we introduce the coordinate system as shown in Fig. 4.9b. According to Fig. 4.6, the quantity x in the first equation represents the distance of the infinitesimal element $\mathrm{d}A$ from the y-axis. To determine the coordinate x_c, however, it is more practical to choose the infinitesimal area $\mathrm{d}A = y\,\mathrm{d}x$ (see Fig. 4.9b) instead of the element shown in Fig. 4.6. With this choice, every point of the element (in particular its centroid) has the same distance x from the y-axis. Since this element incorporates the integration over y, the double integral reduces to a single integral. Introducing the equation $y(x) = h\,x/a$ for the inclined side of the triangle, we obtain the first moment of the area with respect to the y-axis:

$$\int x\,\mathrm{d}A = \int x\,y\,\mathrm{d}x = \int\limits_0^a x\,\frac{h}{a}\,x\,\mathrm{d}x = \frac{h}{a}\frac{x^3}{3}\bigg|_0^a = \frac{1}{3}\,h\,a^2\,.$$

With the area $A = a\,h/2$ of the triangle, we get

$$\underline{\underline{x_c}} = \frac{1}{A}\int x\,\mathrm{d}A = \frac{\frac{1}{3}h\,a^2}{\frac{1}{2}a\,h} = \underline{\underline{\frac{2}{3}\,a}}\,.$$

To determine the coordinate y_c of the triangle, we choose the infinitesimal strip $\mathrm{d}A = (a-x)\mathrm{d}y$ (see Fig. 4.9c). Here, every point of the element has the same distance y from the x-axis. With $x(y) = a\,y/h$, the first moment with respect to the x-axis becomes

$$\int y\,\mathrm{d}A = \int y\,(a - x)\,\mathrm{d}y = \int_0^h y\left(a - \frac{a}{h}\,y\right)\mathrm{d}y$$

$$= \left\{\frac{y^2}{2}\,a - \frac{a}{h}\frac{y^3}{3}\right\}\Bigg|_0^h = \frac{a\,h^2}{6}$$

and we obtain

$$\underline{\underline{y_c}} = \frac{1}{A}\int y\,\mathrm{d}A = \frac{\frac{1}{6}a\,h^2}{\frac{1}{2}a\,h} = \underline{\underline{\frac{1}{3}\,h}}\,.$$

One may also use the infinitesimal element as shown in Fig. 4.9b to determine the coordinate y_c. However, the points of this element do not have the same distance from the x-axis. As explained previously, the quantity y in (4.11) represents the coordinate of the centroid of the element. Therefore, one has to replace y in (4.11) by $\bar{y} = y/2$ (Fig. 4.9d):

$$y_c = \frac{1}{A}\int \bar{y}\,\mathrm{d}A = \frac{1}{A}\int \frac{y}{2}\,\mathrm{d}A\,.$$

With $\mathrm{d}A = y\,\mathrm{d}x$, $y = h\,x/a$, $A = ah/2$ and

$$\int y\,\mathrm{d}A = \int_0^a y^2\,\mathrm{d}x = \int_0^a \frac{h^2}{a^2}\,x^2\,\mathrm{d}x = \frac{1}{3}\,ah^2$$

we obtain again

$$y_c = \frac{\frac{1}{2}\,\frac{1}{3}\,ah^2}{\frac{1}{2}\,ah} = \frac{1}{3}\,a\,.$$

It should be noted that the infinitesimal area $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ according to Fig. 4.6 can also be used to determine the coordinates of the centroid. With this choice, the integration has to be carried out in both directions x and y (double integral). However, it is beyond the scope of this textbook to explain how double integrals are solved.

Example 4.4 Locate the centroid of the area that is bounded by a parabola (Fig. 4.10a). E4.4

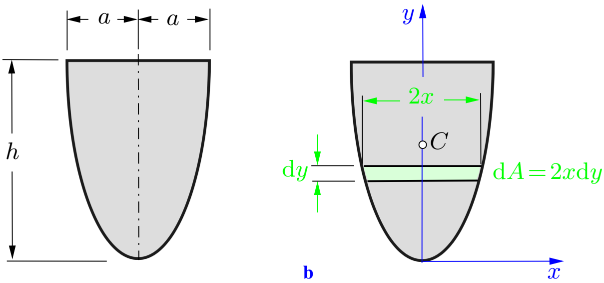


Fig. 4.10 a b

Solution We use the coordinate system as shown in Fig. 4.10b. Since the y-axis is an axis of symmetry, the centroid C lies on it: $x_c = 0$. To determine the coordinate y_c from

$$y_c = \frac{\int y \,\mathrm{d}A}{\int \mathrm{d}A},$$

we choose the differential strip $\mathrm{d}A = 2\,x\,\mathrm{d}y$ parallel to the x-axis. Every point of this element has the same distance y from the x-axis. The parabola is described by

$$y = \frac{h}{a^2}\,x^2 \quad \text{or} \quad x = \sqrt{\frac{a^2\,y}{h}}\,.$$

Therefore,

$$A = \int \mathrm{d}A = \int 2\,x\,\mathrm{d}y = 2\int_0^h \sqrt{\frac{a^2\,y}{h}}\,\mathrm{d}y = 2\sqrt{\frac{a^2}{h}}\,\frac{2}{3}\,y^{3/2}\Big|_0^h = \frac{4}{3}\,a\,h$$

and

$$\int y\,\mathrm{d}A = \int_0^h y\,2\sqrt{\frac{a^2\,y}{h}}\,\mathrm{d}y = 2\sqrt{\frac{a^2}{h}}\,\frac{2}{5}\,y^{5/2}\Big|_0^h = \frac{4}{5}\,a\,h^2\,.$$

This equation yields the result

$$\underline{\underline{y_c}} = \frac{\int y\,\mathrm{d}A}{\int \mathrm{d}A} = \frac{\frac{4}{5}a\,h^2}{\frac{4}{3}a\,h} = \underline{\underline{\frac{3}{5}\,h}}\,.$$

Note that the height y_c of the centroid does not depend on the width a of the parabola.

E4.5

Example 4.5 Locate the centroid of a circular sector (Fig. 4.11a).

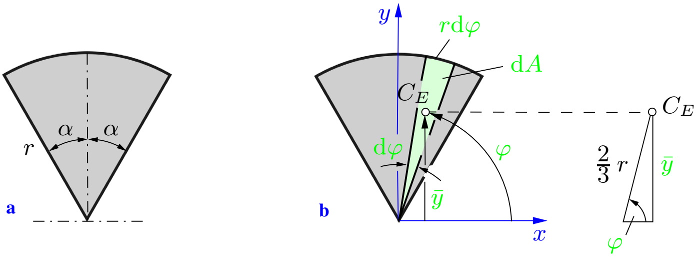


Fig. 4.11

Solution We choose the y-axis as the axis of symmetry, such that $x_c = 0$. In order to find y_c, we introduce the coordinate φ and the infinitesimal area $\mathrm{d}A$, as shown in Fig. 4.11b. Neglecting higher-order terms, the infinitesimal sector of the circle can be replaced by an infinitesimal triangle with a base $r\,\mathrm{d}\varphi$ and a height r. The centroid C_E of this triangle is located at 2/3 of its height. Therefore, it lies at a distance

$$\bar{y} = \tfrac{2}{3}\,r\sin\varphi$$

from the x-axis. With $\mathrm{d}A = \frac{1}{2}r\,\mathrm{d}\varphi\,r = \frac{1}{2}r^2\,\mathrm{d}\varphi$, we obtain

$$\underline{\underline{y_c}} = \frac{\int \bar{y}\,\mathrm{d}A}{\int \mathrm{d}A} = \frac{\displaystyle\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha} \tfrac{2}{3}\,r\sin\varphi\,\tfrac{1}{2}\,r^2\,\mathrm{d}\varphi}{\displaystyle\int_{(\pi/2)-\alpha}^{(\pi/2)+\alpha} \tfrac{1}{2}\,r^2\,\mathrm{d}\varphi}$$

$$= \frac{\dfrac{1}{3}\,r^3(-\cos\varphi)\Big|_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}}{\dfrac{1}{2}\,r^2\,\varphi\Big|_{(\pi/2)-\alpha}^{(\pi/2)+\alpha}} =$$

$$= \frac{1}{3}\,r\,\frac{\cos(\frac{\pi}{2}-\alpha)-\cos(\frac{\pi}{2}+\alpha)}{\alpha} = \underline{\underline{\frac{2}{3}\,r\,\frac{\sin\alpha}{\alpha}}}\,.$$

For a *semicircular area* ($\alpha = \pi/2$) this yields

$$\underline{\underline{y_c = \frac{4\,r}{3\,\pi}}}\,.$$

Example 4.6 Find the centroid of the L-shaped area in Fig. 4.12a. **E4.6**

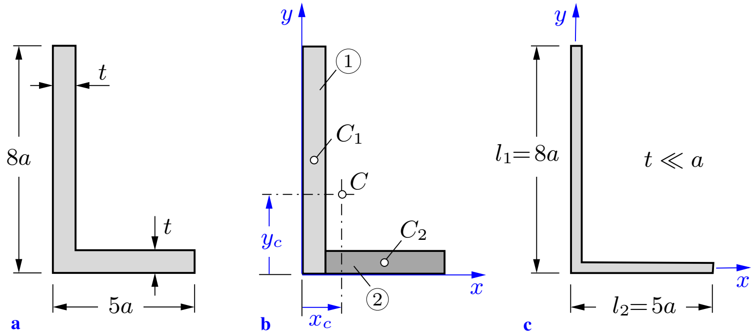


Fig. 4.12

Solution We choose a coordinate system and consider the area to be composed of two rectangles (Fig. 4.12b):

$$A_1 = 8\,a\,t\,, \quad A_2 = (5\,a - t)\,t\,.$$

The coordinates of their respective centroids are given by

$$x_1 = \frac{t}{2}\,, \quad y_1 = 4\,a\,, \quad x_2 = \frac{5\,a - t}{2} + t = \frac{5\,a + t}{2}\,, \quad y_2 = \frac{t}{2}\,.$$

Equations (4.13) yield

$$\underline{\underline{x_c}} = \frac{\sum x_i A_i}{\sum A_i} = \frac{\frac{t}{2}\,8\,a\,t + \frac{5\,a+t}{2}\,(5\,a-t)\,t}{8\,a\,t + (5\,a-t)t}$$

$$= \frac{4\,a\,t^2 + \frac{25}{2}\,a^2\,t - \frac{t^3}{2}}{8\,a\,t + 5\,a\,t - t^2} = \underline{\underline{\frac{25}{26}\,a\,\frac{1 + \frac{8}{25}\frac{t}{a} - \frac{1}{25}\left(\frac{t}{a}\right)^2}{1 - \frac{1}{13}\frac{t}{a}}}}$$

and

$$\underline{\underline{y_c}} = \frac{\sum y_i A_i}{\sum A_i}$$

$$= \frac{4\,a\,8\,a\,t + \frac{t}{2}\,(5\,a-t)\,t}{13\,a\,t - t^2} = \underline{\underline{\frac{32}{13}\,a\,\frac{1 + \frac{5}{64}\frac{t}{a} - \frac{1}{64}\left(\frac{t}{a}\right)^2}{1 - \frac{1}{13}\frac{t}{a}}}}\,.$$

In the special case of $t \ll a$ (Fig. 4.12c), the terms t/a and $(t/a)^2$ are negligible compared with 1. Then we obtain

$$x_c = \frac{25}{26}\,a\,, \quad y_c = \frac{32}{13}\,a\,.$$

E4.7

Example 4.7 A circle is removed from a triangle, as shown in Fig. 4.13a.

Locate the centroid C of the remaining area.

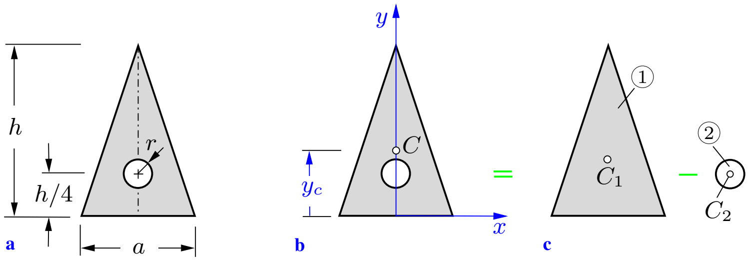


Fig. 4.13

Solution By symmetry, the centroid lies on the y-axis of the chosen coordinate system, i.e., $x_c = 0$ (Fig. 4.13b). We consider the remaining area as a composite area which consists of two parts: the triangle 1 and the circle 2. Since the region of the circle is void of material, this part has to be subtracted from the triangle (Fig. 4.13c). Introducing

$$A_1 = \frac{1}{2}\,a\,h\,, \quad y_1 = \frac{h}{3}\,, \quad A_2 = \pi\,r^2\,, \quad y_2 = \frac{h}{4}$$

into (4.13), we obtain

$$\underline{\underline{y_c}} = \frac{y_1\,A_1 - y_2\,A_2}{A_1 - A_2} = \frac{\frac{h}{3}\,\frac{1}{2}a\,h - \frac{h}{4}\pi\,r^2}{\frac{1}{2}a\,h - \pi\,r^2} = \underline{\underline{\frac{h}{3}\,\frac{1 - \frac{3}{2}\,\frac{\pi\,r^2}{a\,h}}{1 - \frac{2\pi\,r^2}{a\,h}}}}\,.$$

If the area of the circle is small compared with the area of the triangle ($\pi\,r^2 \ll a\,h/2$), the result is reduced to $h/3$.

Table 4.1 Location of Centroids

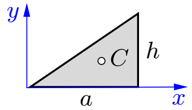
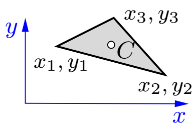
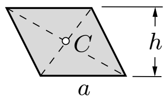
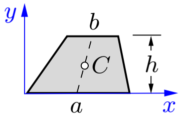
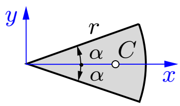
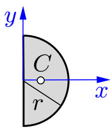
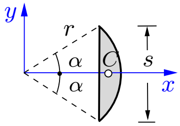
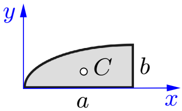
Area		Location of Centroid
Rectangular triangle 	$A = \frac{1}{2}\,ah$	$x_c = \frac{2}{3}\,a, \quad y_c = \frac{h}{3}$
Arbitrary triangle 	$A = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1)$ $-(x_3 - x_1)(y_2 - y_1)]$	$x_c = \frac{1}{3}(x_1 + x_2 + x_3)$ $y_c = \frac{1}{3}(y_1 + y_2 + y_3)$
Parallelogram 	$A = a\,h$	C is determined by the intersection of the diagonals
Trapezium 	$A = \frac{h}{2}(a + b)$	C is located at the median line $y_c = \frac{h}{3}\,\frac{a + 2\,b}{a + b}$
Circular sector 	$A = \alpha\,r^2$	$x_c = \frac{2}{3}\,r\,\frac{\sin\alpha}{\alpha}$
Semicircle 	$A = \frac{\pi}{2}\,r^2$	$x_c = \frac{4\,r}{3\,\pi}$

Table 4.1 Location of Centroids (cont'd)

	Area	Location of Centroid
Circular segment 	$A = \frac{1}{2}\, r^2 (2\,\alpha - \sin 2\,\alpha)$	$x_c = \frac{s^3}{12\,A}$ $= \frac{4}{3}\, r\, \frac{\sin^3 \alpha}{2\,\alpha - \sin 2\,\alpha}$
Quadratic parabola 	$A = \frac{2}{3}\, a\, b$	$x_c = \frac{3}{5}\, a$ $y_c = \frac{3}{8}\, b$

4.4

4.4 Centroid of a Line

Let us now consider a line that lies in a plane (Fig. 4.14). Its centroid C can be determined if in (4.11) the infinitesimal area $\mathrm{d}A$ is replaced by a line element $\mathrm{d}s$ and the area A is replaced by the length l of the line:

$$x_c = \frac{1}{l} \int x \,\mathrm{d}s\,, \qquad y_c = \frac{1}{l} \int y \,\mathrm{d}s\,. \tag{4.14}$$

The centroid of a straight line lies in its center, and the centroid of a curved line lies, in general, outside the line. Equations (4.14) can, for example, be applied to determine the centroid of a bent

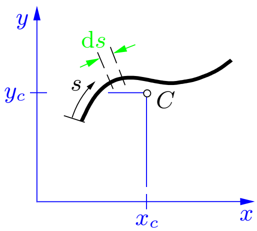


Fig. 4.14

uniform wire or to locate the action line of the resultant of forces that are evenly distributed along a line. The centroid of a line coincides with the center of mass of a uniform thin bar or wire. In the case of a variable density or a variable cross-section, the mass center and the centroid are different points.

Let a line be composed of several parts. The lengths l_i and the coordinates x_i, y_i of the individual centroids are assumed to be known. The integrals in (4.14) then reduce to sums (compare (4.13)), and Equations (4.14) are simplified to

$$x_c = \frac{\sum x_i\, l_i}{\sum l_i}\,, \quad y_c = \frac{\sum y_i\, l_i}{\sum l_i}\,. \tag{4.15}$$

As a simple example the centroid of the line shown in Fig. 4.12c is determined where $t \to 0$. Applying (4.15), we obtain

$$x_c = \frac{0 \cdot 8\,a + \frac{5}{2}a\,5\,a}{8\,a + 5\,a} = \frac{25}{26}\,a\,, \quad y_c = \frac{4\,a\,8\,a + 0 \cdot 5\,a}{8\,a + 5\,a} = \frac{32}{13}\,a\,.$$

This result coincides with the corresponding result derived in Example 4.6.

Example 4.8 A wire is bent into the shape of a circular arc with an opening angle 2α (Fig. 4.15a). **E4.8**

Locate its centroid.

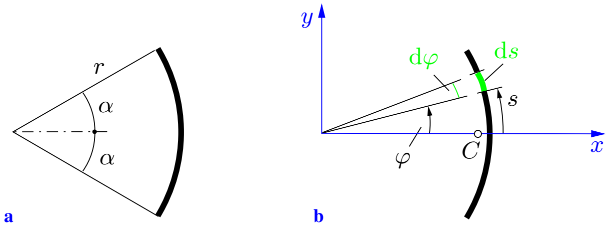


Fig. 4.15

Solution The coordinate system shown in Fig. 4.15b is chosen such that $y_c = 0$ (symmetry). Now the angle φ is introduced and an infinitesimal element of the arc is considered, whose length $\mathrm{d}s$ is given by $r\,\mathrm{d}\varphi$. Using the relation $x = r\cos\varphi$ for polar coordinates, (4.14) yields

$$\underline{\underline{x_c}} = \frac{\int x \,\mathrm{d}s}{\int \mathrm{d}s} = \frac{\int\limits_{-\alpha}^{+\alpha} r\cos\varphi \, r\,\mathrm{d}\varphi}{\int\limits_{-\alpha}^{+\alpha} r\,\mathrm{d}\varphi} = \frac{2\,r^2 \sin\alpha}{2\,r\,\alpha} = r\,\underline{\underline{\frac{\sin\alpha}{\alpha}}}\,.$$

For the special case of a *semicircular arc* (i.e., $\alpha = \dfrac{\pi}{2}$), we obtain

$$x_c = \frac{2\,r}{\pi}\,.$$

The centroid of a *semicircular area* ($x_c = 4\,r/(3\pi)$, see Table 4.1 and Example 4.5) is located much closer to the center of the circle than the centroid of a semicircular arc.

4.5

4.5 Supplementary Problems

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E4.9

Example 4.9 Locate the centroids of the profiles as shown in Fig. 4.16. The measurements are given in mm.

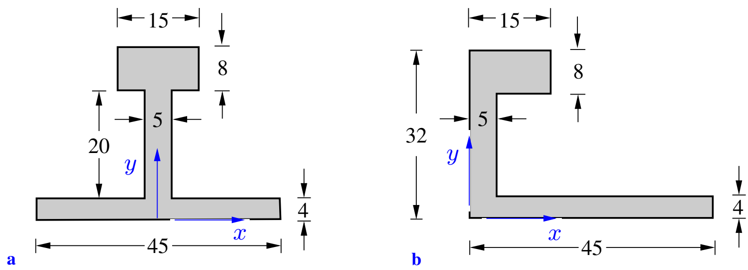


Fig. 4.16

Results: see (**A**) a) $x_c = 0\,,\quad y_c = 12.8$ mm,

b) $x_c = 13$ mm$\,,\quad y_c = 12.8$ mm.

Example 4.10 Locate the centroids of the thin-walled profiles ($t \ll a$) as shown in Fig. 4.17. **E4.10**

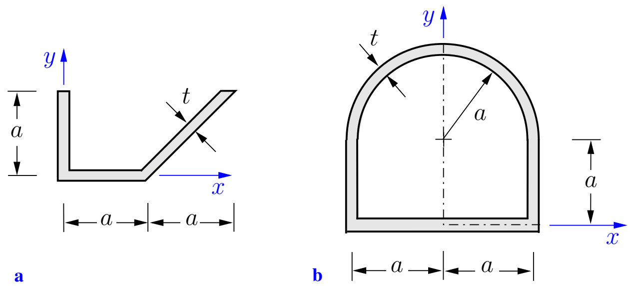


Fig. 4.17

Results: see (**B**) a) $x_c = (5\sqrt{2}/4 - 1)a\,, \quad y_c = \sqrt{2}\,a/4,$

$$\text{b)} \quad x_c = 0\,, \quad y_c = \frac{3+\pi}{4+\pi}\,a.$$

Example 4.11 Determine the coordinates of the centroid C of the number shown in Fig. 4.18. **E4.11**

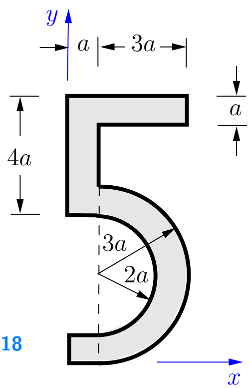


Fig. 4.18

Result: see (**B**) $x_c = 1.93\,a\,, \quad y_c = 4.89\,a.$

Example 4.12 A circular area is removed from a circle and an ellipse, respectively (Fig. 4.19). **E4.12**

Locate the centroids of the remaining areas.

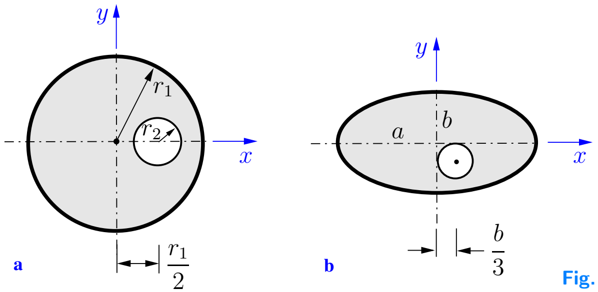


Fig. 4.19

Results: see (**B**) a) $x_c = -\dfrac{r_1 r_2^2}{2(r_1^2 - r_2^2)}\,, \quad y_c = 0\,,$

b) $x_c = -\dfrac{b^2}{3(9a-b)}\,, \quad y_c = -x_c\,.$

E4.13 **Example 4.13** A thin metal sheet of even thickness is bent into the shape shown in Fig. 4.20. It consists of a square and two triangles. The measurements are given in cm.

Determine the centroid.

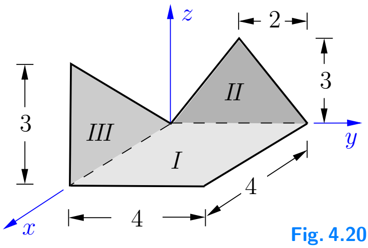


Fig. 4.20

Result: see (**A**) $x_c = 1.71\,\text{cm}\,, \quad y_c = 1.57\,\text{cm}\,, \quad z_c = 0.43\,\text{cm}\,.$

E4.14 **Example 4.14** The area shown in Fig. 4.21 is bounded by the coordinate axes and the quadratic parabola with its apex at $x = 0$.

Determine the coordinates of the centroid.

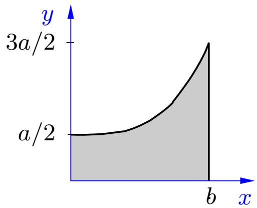


Fig. 4.21

Result: see (**A**) $x_c = 3\,b/5\,, \quad y_c = 47\,a/100\,.$

Example 4.15 Locate the the centroid of the area shown in Fig. 4.22. **E4.15**

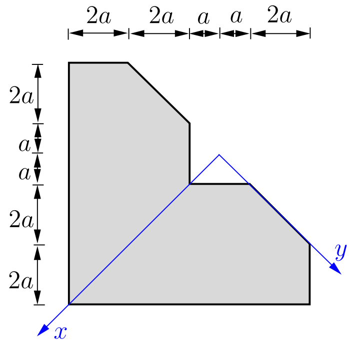


Fig. 4.22

Result: $x_c = 2.61\,a\,, \quad y_c = 0\,.$

Example 4.16 A thin wire has the shape of the function $y = a\cosh x/a$ (Fig. 4.23). **E4.16**

Find the centroid.

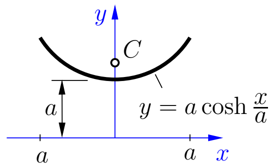


Fig. 4.23

Result: see (**A**) $\quad x_c = 0\,, \quad y_c = 1.197\,a\,.$

Example 4.17 Locate the centroid of a thin-walled spherical shell (radius R, Height H, thickness $t \ll R$) as shown in Fig. 4.24. **E4.17**

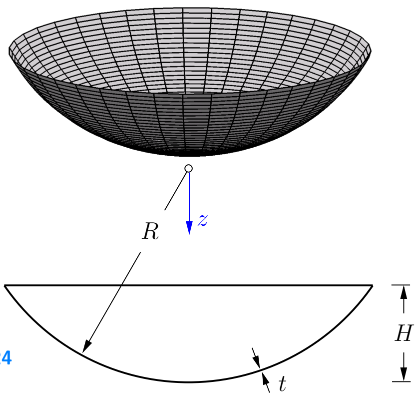


Fig. 4.24

Result: see (**B**) $\quad z_c = R - H/2\,.$

4.6

4.6 Summary

- The weight of a body is distributed over its entire volume. In the case of a rigid body it can be considered to be concentrated in the center of gravity without change in the static action. The coordinates of the center of gravity are defined by

$$x_c = \frac{\int x\,\varrho\,\mathrm{d}V}{\int \varrho\,\mathrm{d}V}\,, \quad y_c = \frac{\int y\,\varrho\,\mathrm{d}V}{\int \varrho\,\mathrm{d}V}\,, \quad z_c = \frac{\int z\,\varrho\,\mathrm{d}V}{\int \varrho\,\mathrm{d}V}\,.$$

  Analogous definitions are valid for the center of mass and the center of volume.
- The coordinates of the centroid of an area are defined by

$$x_c = \frac{1}{A}\int x\,\mathrm{d}A\,, \qquad y_c = \frac{1}{A}\int y\,\mathrm{d}A\,.$$

- The first moments of an area are defined by

$$S_y = \int x\,\mathrm{d}A\,, \qquad S_x = \int y\,\mathrm{d}A\,.$$

- The first moments of an area with respect to axes through its centroid are zero.
- If an area has an axis of symmetry, the centroid of the area lies on this axis.
- Let a body (area, line) be composed of several parts, the centroids of which are assumed to be known. The integrals are then replaced by sums. For example, the coordinates of the centroid of an area follow from

$$x_c = \frac{\sum x_i A_i}{\sum A_i}\,, \qquad y_c = \frac{\sum y_i A_i}{\sum A_i}\,.$$

  Analogous relations are valid in the case of a body or a line.
- In the case of bodies (areas) with holes, the spaces that are void of material are considered to be "negative" volumes (areas).

Chapter 5

Support Reactions

5

5 Support Reactions

Objectives: In this chapter, the most common kinds of supports of simple structures and the different connecting elements of multi-part structures are introduced. We will discuss their characteristic features and how they can be classified, so that the students will be able to decide whether or not a structure is statically and kinematically determinate. Students will also learn from this chapter how the forces and couple moments appearing at the supports and the connecting elements of a loaded structure can be determined. Here, the most important steps are the sketch of the free-body diagram and the correct application of the equilibrium conditions.

5.1 Plane Structures

5.1.1 Supports

Structures can be classified according to their geometrical shape and the loads acting on them. A straight slender structural element (cross-sectional dimensions much smaller than its length) that is loaded solely in the axial direction (tension or compression) is called a *bar* or a *rod* (see Section 2.4). If the same geometrical object is subjected to a load perpendicular to its axis, it is called a *beam*. A curved beam is usually designated as an *arch*. Structures consisting of inclined, rigidly joined beams are called *frames*. A plane structure with a thickness much smaller than its characteristic in-plane length is called a *disk* if it is solely loaded in its plane, e.g., by in-plane forces. If the same geometrical structure is loaded perpendicularly to its midplane it is called a *plate*. If such a structure is curved it is a *shell*.

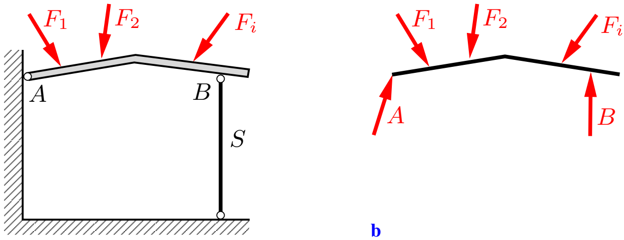


Fig. 5.1 **a** **b**

Structures are connected to their surroundings by *supports* whose main purpose is to fix the structure in space in a specific position. Secondly, supports transmit forces. As a simple example, consider the “roof” in Fig. 5.1a, loaded by external forces F_i, joined at A to a vertical wall by a pin, and supported at B by the strut S. Forces are transmitted to the wall and the ground via the supports A and B. According to the law of action and reaction (actio = reactio) the same forces are exerted in opposite directions from the wall and the ground onto the roof. These forces from the environment onto the structure are reaction forces (cf. Section 1.4), and are termed *support reactions*. They become visible in the free-body diagram (Fig. 5.1b), where they are generally denoted by the same symbols as the supports, i.e. by A and B in this example.

The following discussion is limited to single-part structures located and loaded in a plane. A free body with no restraints has three degrees of freedom, i.e., it can be independently displaced in three different ways: by two translations in different directions and by one rotation about an axis perpendicular to the plane (cf. Section 3.1.4). Supports (restraints) reduce the feasible displacements: each support reaction imposes a constraint. Let r be the number of support reactions. Then the number f of degrees of freedom of a body in a plane is given by

$$f = 3 - r \tag{5.1}$$

(for exceptions see Section 5.1.2).

We now will consider different types of supports and classify them by the number of support reactions involved.

Supports that can transmit only one single reaction ($r = 1$). Examples of this type of support are the roller support, the simple support and the support by a strut, cf. Fig. 5.2a–c. In this case, the direction of the reaction force is known (here vertical), whereas its magnitude is unknown.

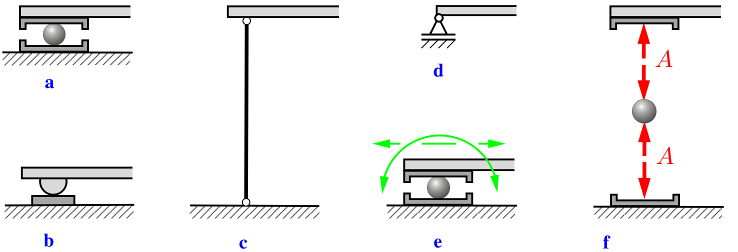


Fig. 5.2

Figure 5.2f shows the free-body diagram for the roller support. If the contact areas are assumed as frictionless, all contact forces can be considered to act perpendicular to the respective contact surfaces. With this assumption the action line of the reaction force A is determined. Figure 5.2e indicates the displacements that are unconstrained by the support: a horizontal translation and a rotation. A vertical translation is excluded through the support's restraint. If the support reaction A changes its sign, i.e., if it is reversed in the direction along the action line, a lift-off must be prevented by an appropriate support construction. From now on a simple support will be depicted by the symbol shown in Fig. 5.2d.

Supports that can transmit two reactions ($r = 2$). Examples of this type of support are the hinged support and the support by two struts (Fig. 5.3a, b), which are depicted symbolically in Fig. 5.3c.

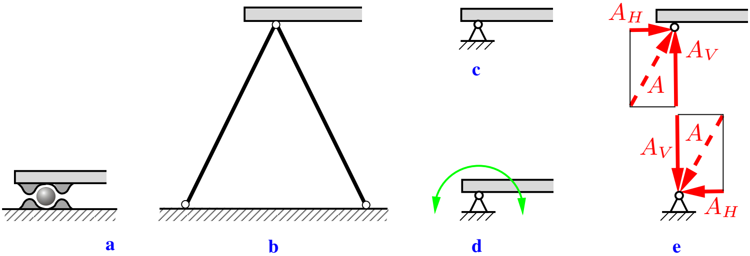


Fig. 5.3

As shown in Fig. 5.3d, the hinged support allows a rotation but not a displacement in any direction. Accordingly, it can transmit a reaction force A of arbitrary magnitude and arbitrary direction that can be resolved into its horizontal and vertical components A_H and A_V (Fig. 5.3e).

Additional variants of supports transmitting two reactions are the parallel motion and the sliding sleeve (Fig. 5.4a, b). Their free-body diagrams (Fig. 5.4c, d) show that in both cases one force and

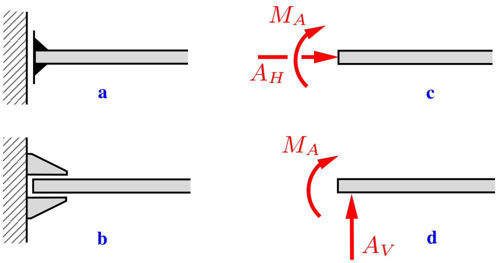


Fig. 5.4

one couple moment can be transmitted. A displacement in one single direction is possible in each case; a displacement in any other direction or a rotation are not possible.

The rotational degree of freedom disappears if a support by two struts is complemented by an additional, somewhat shifted, third strut (Fig. 5.5a): the structure becomes immobile. In addition to the two force components, the support can now also transmit a couple moment, i.e., in total *three reactions*: $r = 3$.

The same situation appears in the case of a clamped support (fixed support) according to Fig. 5.5b which symbolically is depicted in Fig. 5.5c. The free-body diagram in Fig. 5.5d shows that the clamped support can transmit a reaction force A of arbitrary magnitude and direction (or A_H and A_V) and a couple moment M_A.

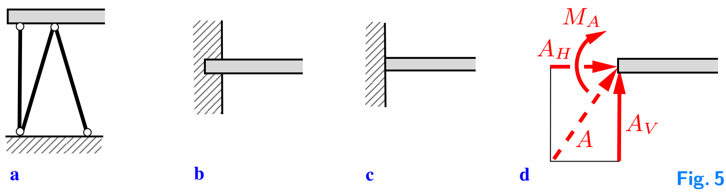


Fig. 5.5

5.1.2 Statical Determinacy

A structure is called *statically determinate* if the support reactions can be calculated from the *three* equilibrium conditions (3.12). Since the number of unknowns must coincide with the number of equations, *three* unknown reactions (forces or couple moments) must exist at the supports: $r = 3$. It will be explained later that this necessary condition may not be sufficient for the determination of the support reactions.

The beam in Fig. 5.6a is supported by the hinged support A and the simple support B. Accordingly, the three unknown support reactions A_H, A_V and B exist. Therefore, with $r = 3$ it follows from (5.1) that the beam is immobile: $f = 3 - r = 0$; it is statically determinate.

The support reactions of the clamped beam in Fig. 5.6b consist of the two force components A_H and A_V and the couple moment M_A. Figure 5.6c shows a disk supported by the three struts A, B and C, each transmitting one reaction. In both cases, with $r = 3$ and $f = 0$, the support is statically determinate.

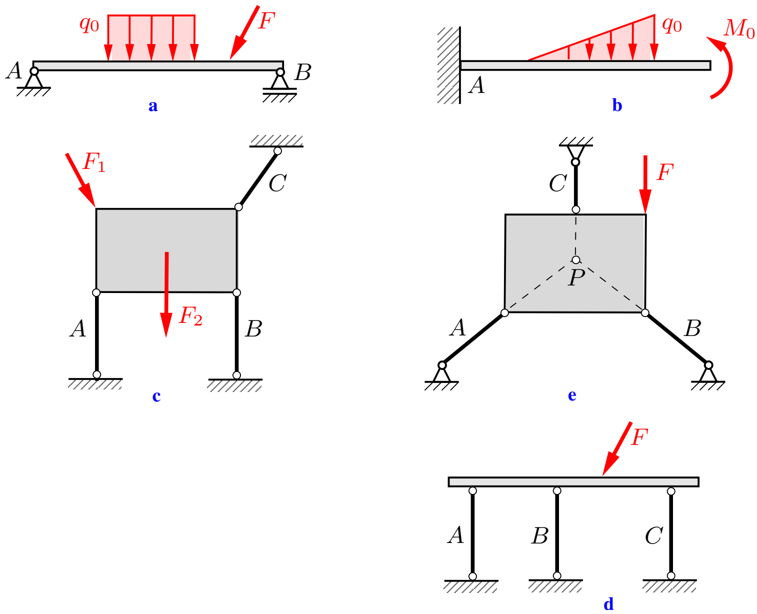


Fig. 5.6

In contrast, Fig. 5.6d shows a beam that is supported by three *parallel* struts A, B and C. Here too, the number of unknown support reactions coincides with the number of equilibrium conditions: the necessary condition for statical determinacy is satisfied. However, the reaction forces *cannot* be calculated from the equilibrium conditions. Here $r = 3$ does not imply $f = 0$ (exceptional case!): the beam can be displaced in a horizontal direction. Such exceptional cases must be excluded. A structure that may undergo finite or infinitesimal displacements is called *kinematically indeterminate* (cf. also Sections 5.3.4 and 6.1).

The disk in Fig. 5.6e is also kinematically indeterminate. Since the action lines of the reaction forces intersect at point P, the supports allow an infinitesimal rotation about this point. It can be seen immediately that the supports in Figs. 5.6d and e are not statically determinate. In the case of the beam, the equilibrium condition for the horizontal force components cannot be fulfilled ($\sum F_{iH} \neq 0$), whereas for the disk, the equilibrium of the moments with respect to P cannot be satisfied ($\sum M_i^{(P)} \neq 0$).

In the case of a plane problem, a structure is supported statically and kinematically determinate if it is immobile and exactly three support reactions appear. These may be

a) three forces which are not all parallel and not central,
b) two nonparallel forces and one moment.

It must be emphasized that the statical determinacy of a structure is solely dependent on the supports and not on the loading.

If additional supports are attached to a statically determinate structure, more than three support reactions exist, which can no longer be determined solely from the three equilibrium conditions. Such a structure is called *statically indeterminate*.

For example, if the clamped beam in Fig. 5.6b is additionally supported by the simple support B (see Fig. 5.7a), the number of unknown reactions increases from three to four. In this case, one redundant reaction (force or couple moment) is present. The beam is therefore statically indeterminate with *one degree of static indeterminacy*.

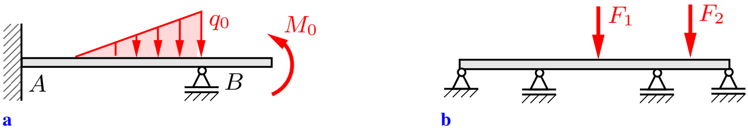


Fig. 5.7

Generally, a structure is statically indeterminate with a degree x of statical indeterminacy if the number of unknown support reactions exceeds the number of available equilibrium conditions by x. Consequently, for the beam in Fig. 5.7b, the degree of statical indeterminacy is equal to two, since $r = 2 + 3 \cdot 1 = 5$.

The support reactions of statically indeterminate structures can only be determined if they are not considered to be rigid but if their deformations are taken into account. The relevant methods will be discussed in Volume 2, *Mechanics of Materials*.

5.1.3 Determination of the Support Reactions

In order to determine the support reactions, the method of sections is applied (cf. Section 1.4): the body is freed from its supports and their action on the body is replaced by the unknown reactions.

As an example, consider the beam in Fig. 5.8a, which is supported by the strut A and the two simple supports B and C. The reaction forces become visible in the free-body diagram (Fig. 5.8b). Their sense of direction along the prescribed action lines can be chosen arbitrarily. However, for the strut the sign convention for rods is applied (see Sections 2.4 and 6.3.1) and it is assumed to be subject to tension. The assumptions are correct if the analysis yields positive values for the reaction forces, whereas a reaction force is oppositely directed in the case of a negative sign.

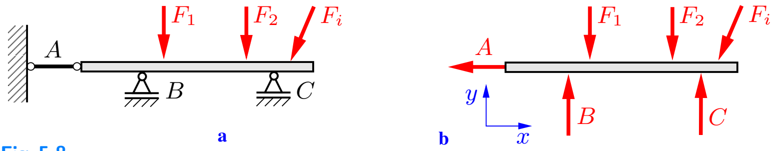


Fig. 5.8

All of the forces acting on the isolated body (i.e., active forces and reaction forces) must fulfill the equilibrium conditions (3.12):

$$\sum F_{ix} = 0\,, \quad \sum F_{iy} = 0\,, \quad \sum M_i^{(P)} = 0\,. \tag{5.2}$$

Here, P is a reference point that may be chosen arbitrarily. The support reactions can be calculated from (5.2).

Example 5.1 The beam shown in Fig. 5.9a is loaded by the force F which acts under an angle α. **E5.1**

Determine the reaction forces at the supports A and B.

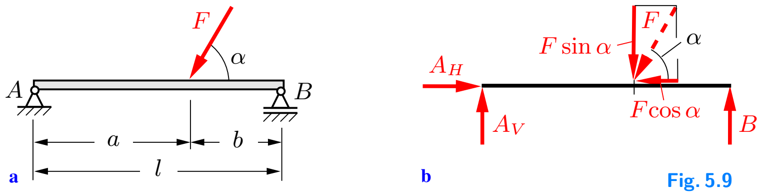


Fig. 5.9

Solution The beam is rigidly supported; the support A transmits two reactions and support B one reaction. In total, the three unknown reaction forces A_H, A_V and B exist: therefore, the beam is statically determinate. We free the beam from its supports and make the reaction forces visible in the free-body diagram (Fig. 5.9b) where we choose their senses of direction along the action lines freely. Hence, the equilibrium conditions are given by

$$\uparrow: \quad A_V - F\sin\alpha + B = 0\,, \tag{a}$$

$$\rightarrow: \quad A_H - F\cos\alpha = 0 \quad \rightarrow \quad \underline{\underline{A_H = F\cos\alpha}}\,,$$

$$\overset{\curvearrowleft}{A}: \quad -a\,F\sin\alpha + l\,B = 0 \quad \rightarrow \quad \underline{\underline{B = \frac{a}{l}\,F\sin\alpha}}\,. \tag{b}$$

Introducing B and the geometric relation $a + b = l$ into (a) yields

$$\underline{\underline{A_V}} = F\sin\alpha - B = \left(1 - \frac{a}{l}\right) F\sin\alpha = \underline{\underline{\frac{b}{l}\,F\sin\alpha}}\,.$$

As a check, the equilibrium condition for the couple moments about another reference point is applied:

$$\overset{\curvearrowleft}{B}: \quad -l\,A_V + b\,F\sin\alpha = 0 \quad \rightarrow \quad A_V = \frac{b}{l}\,F\sin\alpha\,.$$

This equation, in contrast to Equation (a), directly yields the reaction force A_V. Thus, application of the equilibrium conditions (3.14) instead of (3.12) would have been advantageous in this case.

E5.2 **Example 5.2** The clamped beam shown in Fig. 5.10a is loaded by the two forces F_1 and F_2.

Determine the reactions at the support.

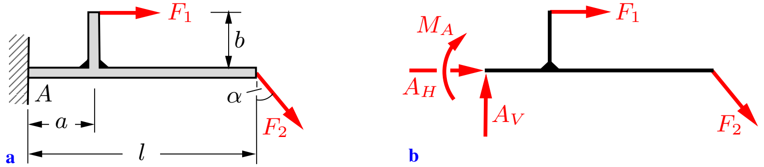


Fig. 5.10

Solution The fixed support A transmits three reactions: two force components A_H, A_V and the moment M_A. They are made visible in the free-body diagram (Fig. 5.10b) where their senses of direction have been chosen arbitrarily. Thus, the equilibrium conditions (5.2) yield

$$\uparrow: \quad A_V - F_2 \cos\alpha = 0 \quad \rightarrow \quad \underline{\underline{A_V = F_2 \cos\alpha}}\,,$$

$$\rightarrow: \quad A_H + F_1 + F_2 \sin\alpha = 0 \quad \rightarrow \quad \underline{\underline{A_H = -(F_1 + F_2 \sin\alpha)}}\,,$$

$$\overset{\curvearrowright}{A}: \quad M_A + b\,F_1 + l\,F_2 \cos\alpha = 0 \quad \rightarrow \quad \underline{\underline{M_A = -(b\,F_1 + l\,F_2 \cos\alpha)}}.$$

The negative signs of A_H and M_A indicate that these reactions in reality are directed oppositely to the directions chosen in the free-body diagram.

5.2 Spatial Structures

5.2

A body that can move freely in space has six degrees of freedom: three translations in x-, y- and z-direction and three rotations about the three axes. Supports constrain the possible displacements. As in the plane case, the different types of support are classified by the number of transferable support reactions.

The strut in Fig. 5.11a can transfer only *one* force in the direction of its axis. Therefore, $r = 1$ for spatial as well as for plane structures. In contrast, the hinged support in Fig. 5.11b transfers *three* force components in space (in x-, y- and z-direction), i.e., $r = 3$. The fixed support or clamping (Fig. 5.11c) transfers six reactions in space ($r = 6$): the force components in the three coordinate directions as well as the moments about the three axes. The sliding sleeve in Fig. 5.11d can transfer two moments and two force components, provided that the beam with a circular cross-section can rotate freely about its axis; for this type of support, $r = 4$ is valid. When the support and the beam have rectangular cross-sections, which makes a rotation impossible, moments about all three axes can be transferred. This leads to $r = 5$.

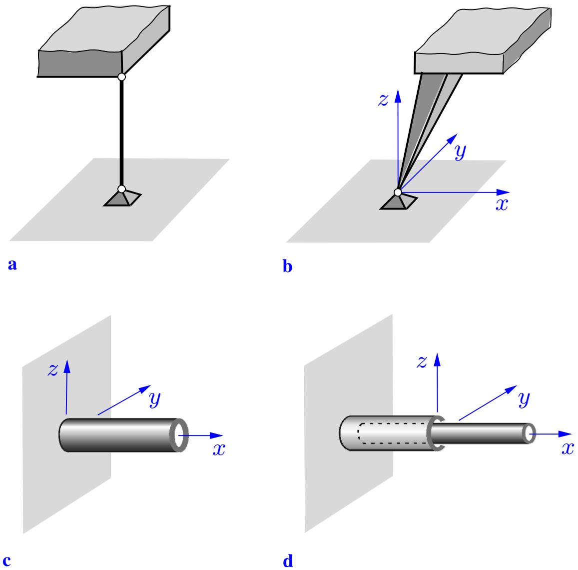


Fig. 5.11

A spatial structure is statically determinate when it is immobile and the support reactions can be calculated from the *six* equilibrium conditions (3.34), see also Section 5.3.4. Thus, in total *six* reactions must exist at the supports. As in the case of plane structures, these reactions are calculated by applying the method of sections.

E5.3 **Example 5.3** The rectangular lever which is clamped at A (Fig. 5.12a) is loaded by the line load q_0, two forces F_1, F_2 and the moment M_0.

Determine the support reactions.

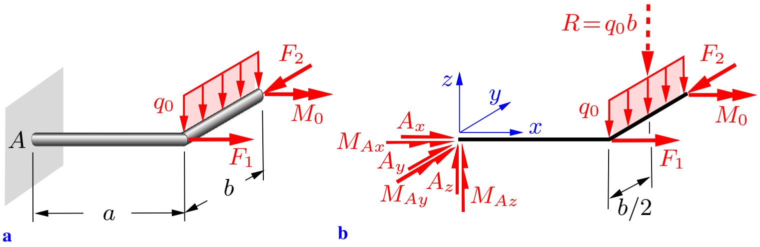


Fig. 5.12

Solution We free the lever from the fixed support and make the reactions visible in the free-body diagram. According to the clamped support, the three force components A_x, A_y, A_z and the three moment components M_{Ax}, M_{Ay}, M_{Az} exist (Fig. 5.12b). Their directions are chosen in such a way that they coincide with the positive coordinate directions. The line load can be replaced by its resultant $R = q_0\, b$. The equilibrium conditions (3.34) then yield

$$\sum F_{ix} = 0: \quad A_x + F_1 = 0 \quad \rightarrow \quad \underline{\underline{A_x = -F_1}}\,,$$

$$\sum F_{iy} = 0: \quad A_y - F_2 = 0 \quad \rightarrow \quad \underline{\underline{A_y = F_2}}\,,$$

$$\sum F_{iz} = 0: \quad A_z - q_0\, b = 0 \quad \rightarrow \quad \underline{\underline{A_z = q_0\, b}}\,,$$

$$\sum M_{ix}^{(A)} = 0: \quad M_{Ax} + M_0 - \frac{b}{2}(q_0\, b) = 0 \rightarrow \underline{\underline{M_{Ax} = \frac{q_0\, b^2}{2} - M_0}}\,,$$

$$\sum M_{iy}^{(A)} = 0: \quad M_{Ay} + a(q_0\, b) = 0 \quad \rightarrow \quad \underline{\underline{M_{Ay} = -\, q_0\, a\, b}}\,,$$

$$\sum M_{iz}^{(A)} = 0: \quad M_{Az} - a\, F_2 = 0 \quad \rightarrow \quad \underline{\underline{M_{Az} = a\, F_2}}\,.$$

Example 5.4 A spatial frame is supported at A, B and C (Fig. 5.13a). It is loaded by the line load q_0, the forces F_1, F_2 and the moment M_0. **E5.4**

Determine the support reactions.

Solution The hinged support A transfers the three force components A_x, A_y, A_z (Fig. 5.13b). At the support B, forces B_x and B_z act in the directions of the struts, and at the simple (moveable) support, force C acts perpendicularly to the horizontal plane, i.e.,

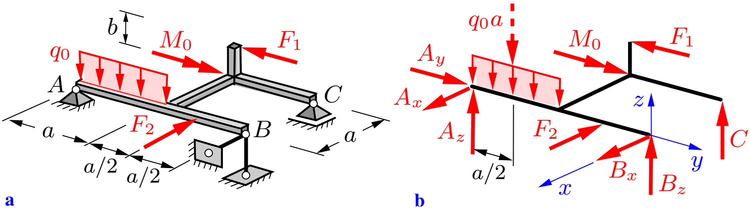


Fig. 5.13

in the direction of the z-axis. Hence, the equilibrium equations for the forces are

$$\sum F_{ix} = 0: \quad A_x + B_x - F_2 = 0\,, \qquad \text{(a)}$$

$$\sum F_{iy} = 0: \quad A_y - F_1 = 0 \qquad \rightarrow \quad \underline{\underline{A_y = F_1}}\,,$$

$$\sum F_{iz} = 0: \quad A_z + B_z + C - q_0\, a = 0\,. \qquad \text{(b)}$$

To formulate the equilibrium of moments we choose axes through the reference point B:

$$\sum M_{ix}^{(B)} = 0: -2\,a\,A_z + \frac{3}{2}a\,(q_0\,a) + b\,F_1 = 0 \rightarrow \underline{\underline{A_z = \frac{3}{4}q_0\,a + \frac{b}{2\,a}F_1}}\,,$$

$$\sum M_{iy}^{(B)} = 0: \quad a\,C + M_0 = 0 \qquad \rightarrow \underline{\underline{C = -\frac{1}{a}\,M_0}}\,,$$

$$\sum M_{iz}^{(B)} = 0: \quad 2\,a\,A_x + a\,F_1 - \frac{a}{2}F_2 = 0 \qquad \rightarrow \underline{\underline{A_x = -\frac{1}{2}F_1 + \frac{1}{4}F_2}}\,.$$

With the results for A_x, A_z and C, (a) and (b) yield

$$\underline{\underline{B_x}} = -\,A_x + F_2 = \underline{\underline{\frac{1}{2}\,F_1 + \frac{3}{4}\,F_2}}\,,$$

$$\underline{\underline{B_z}} = q_0\,a - A_z - C = \underline{\underline{\frac{1}{4}\,q_0\,a - \frac{b}{2\,a}\,F_1 + \frac{1}{a}\,M_0}}\,.$$

5.3

5.3 Multi-Part Structures

5.3.1 Statical Determinacy

Structures often consist not only of one single part but of a number of rigid bodies that are appropriately connected. The connecting members transfer forces and moments, respectively, which can be made visible by passing cuts through the connections. In the following the discussion is restricted to *plane* structures.

The connecting member between two rigid bodies ① and ② of a structure can be, for example, a strut S, a hinge G or a parallel motion P (Fig. 5.14a–c). The *strut* transfers only one single force S in its axial direction. In this case, the number v of joint reactions is $v = 1$. In contrast, a *hinge* can transfer a force in an arbitrary direction, i.e. the force components G_H and G_V. Since the hinge is assumed to be frictionless, it offers no resistance to a rotation: it cannot transfer a moment. Therefore, the number of joint reactions in this case is $v = 2$. The *parallel motion* prevents a relative rotation and a relative displacement in the horizontal direction of the connected bodies; however, it allows a vertical displacement. Therefore, only a horizontal force N and a moment M can be transferred: again $v = 2$. According to the principle actio = reactio, the joint reactions act in opposite directions on the two bodies.

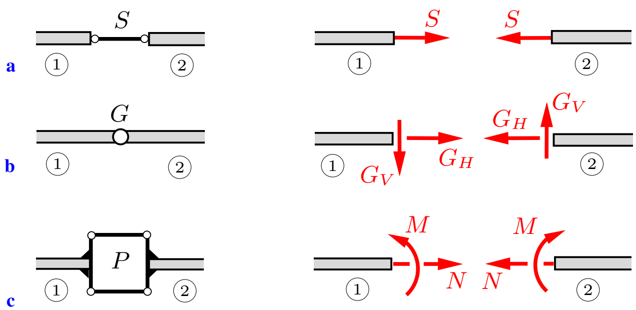


Fig. 5.14

In order to determine the support reactions and the forces and moments transferred by the connecting members the method of sections is applied: we free the different bodies of the structure by removing all of the joints and supports and replace them by the joint and support reactions.

Three equilibrium conditions can be formulated for each body of the structure. Therefore, there are in total $3\,n$ equations if the structure consists of n bodies. Let r be the number of support reactions and v be the number of transferred joint reactions. We call the multi-body structure statically determinate if the r support reactions and the v joint reactions can be calculated from the $3\,n$ equilibrium conditions. The necessary condition for statical determinacy is that the number of equations and the number of unknowns are equal:

$$r + v = 3\,n\,. \tag{5.3}$$

Moreover, if the structure is rigid, this condition is sufficient for statical determinacy. Condition (5.3) also includes the special case of a statically determinate single body where $n = 1$, $v = 0$ and $r = 3$ (cf. Section 5.1.2).

As examples, let us consider the multi-part structures depicted in Fig. 5.15. The structure shown in Fig. 5.15a consists of $n = 2$ beams ① and ②, connected by the hinge G, and it is supported by the fixed support A and strut B. Hinge G transfers $v = 2$ force components, and at the fixed support and the strut, $r = 3+1 = 4$ support reactions exist. Hence, since $4 + 2 = 3 \cdot 2$, the necessary condition (5.3) for statical determinacy is fulfilled. The structure in Fig. 5.15b consists of three beams ① - ③ and the disk ④; i.e., $n = 4$. The four hinges G_1 - G_4 transfer $v = 4 \cdot 2 = 8$ joint reactions. At the support A, two reactions exist and each of the

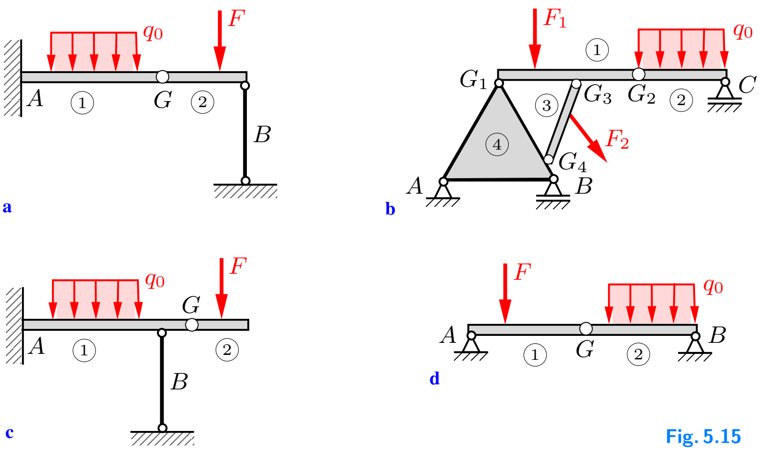


Fig. 5.15

supports B and C transfers one reaction; this results in a total of $r = 2+1+1 = 4$. Introducing these numbers into (5.3) shows that the necessary condition for statical determinacy is again satisfied: $4 + 8 = 3 \cdot 4$. Since both structures are rigid, they are statically determinate.

If the strut in Fig. 5.15a is attached to beam ① instead of beam ② as shown in Fig. 5.15c, the necessary condition for statical determinacy is still fulfilled. However, this structure is kinematically indeterminate (beam ② is moveable) and therefore useless. The structure in Fig. 5.15d is also kinematically indeterminate. Even though hinge G cannot undergo finite displacements, it can still be displaced infinitesimally upwards or downwards.

Example 5.5 The structure shown in Fig. 5.16a consists of the beam ① and the angled part ②, which are connected by the hinge G. The angled part is clamped at A and the beam is supported at B. The system is loaded by the force F. **E5.5**

Determine the support and joint reactions.

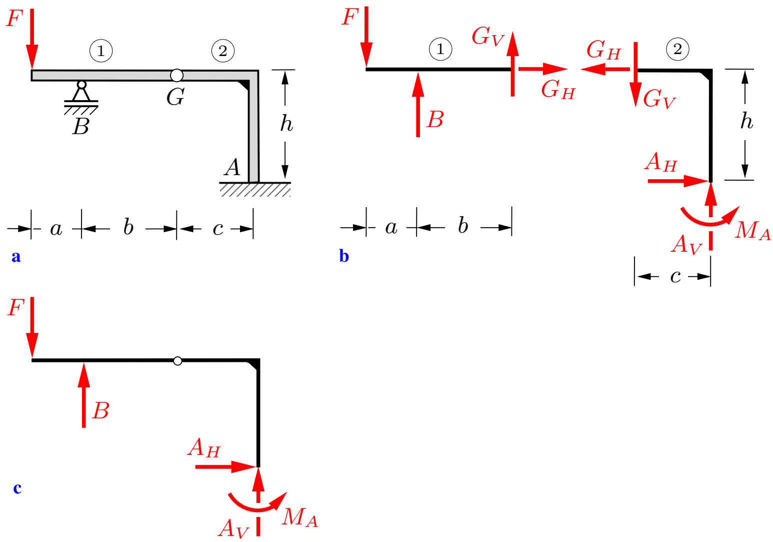


Fig. 5.16

Solution Since $r = 3 + 1 = 4$, $v = 2$ and $n = 2$, condition (5.3) is fulfilled: $4+2 = 3{\cdot}2$. Furthermore, since the structure is immobile, it is statically determinate.

We separate the bodies ① and ②, remove the supports and draw the free-body diagram (Fig. 5.16b). The directions of G_H and G_V can be chosen freely for one of the two bodies. Their directions for the second body are determined through the principle actio = reactio. The equilibrium conditions for body ① yield

$$\rightarrow: \quad \underline{\underline{G_H = 0}}\,,$$

$$\stackrel{\curvearrowleft}{G}: \quad (a+b)F - b\,B = 0 \quad \rightarrow \quad \underline{\underline{B = \frac{a+b}{b}\,F}}\,,$$

$$\stackrel{\curvearrowleft}{B}: \quad a\,F + b\,G_V = 0 \quad \rightarrow \quad \underline{\underline{G_V = -\,\frac{a}{b}\,F}}\,.$$

From the equilibrium conditions for body ② in conjunction with the results for G_H and G_V, we obtain

$$\uparrow: \quad -\,G_V + A_V = 0 \quad \rightarrow \quad \underline{\underline{A_V}} = G_V = \underline{\underline{-\,\frac{a}{b}\,F}}\,,$$

$$\rightarrow: \quad -\,G_H + A_H = 0 \quad \rightarrow \quad \underline{\underline{A_H}} = G_H = \underline{\underline{0}}\,,$$

$$\stackrel{\curvearrowleft}{A}: \quad M_A + h\,G_H + c\,G_V = 0 \quad \rightarrow \quad \underline{\underline{M_A}} = -h\,G_H - c\,G_V = \underline{\underline{\frac{a\,c}{b}\,F}}.$$

The negative signs of G_V and A_V indicate that their directions in reality are opposite to those assumed in the free-body diagram.

As a check, the equilibrium conditions are applied to the complete system (Fig. 5.16c), where the hinge G is assumed to be frozen:

$$\uparrow: \quad -\,F + B + A_V = 0 \quad \rightarrow \quad -\,F + \frac{a+b}{b}\,F - \frac{a}{b}\,F = 0\,,$$

$$\rightarrow: \quad A_H = 0\,,$$

$$\stackrel{\curvearrowleft}{B}: \quad a\,F + M_A + h\,A_H + (b+c)A_V = 0$$

$$\rightarrow \quad a\,F + \frac{a\,c}{b}\,F - (b+c)\frac{a}{b}\,F = 0\,.$$

Example 5.6 The symmetrical sawbuck in Fig. 5.17a consists of two beams connected at hinge C and fixed by the rope S. It is loaded with a frictionless cylinder of weight W. **E5.6**

Determine the support reactions at A and B, the force S in the rope and the joint reaction in C. The weight of the sawbuck can be neglected.

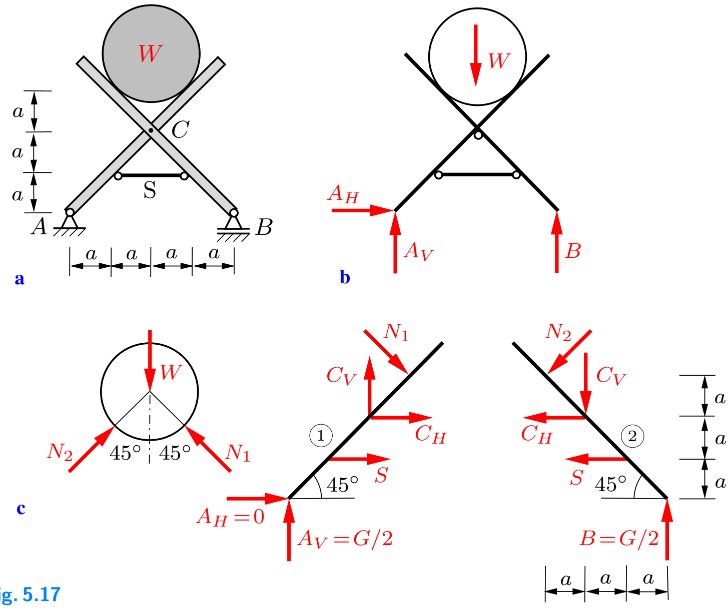


Fig. 5.17

Solution Since only three support reactions are present (see Fig. 5.17b), they can be determined by applying the equilibrium conditions to the complete system:

$$\rightarrow: \quad \underline{\underline{A_H = 0}}\,,$$

$$\overset{\curvearrowleft}{A}: \quad -2\,a\,W + 4\,a\,B = 0 \quad \rightarrow \quad \underline{\underline{B = W/2}}\,, \tag{a}$$

$$\overset{\curvearrowleft}{B}: \quad -4\,a\,A_V + 2\,a\,W = 0 \quad \rightarrow \quad \underline{\underline{A_V = W/2}}\,.$$

In order to determine the forces in the rope and the hinge, we separate the structure into its two parts ($n = 2$). In the hinge C and rope S, in total $v = 2 + 1 = 3$ forces are transferred (Fig. 5.17c). With $r = 3$, the necessary condition (5.3) for statical determinacy is fulfilled: $3 + 3 = 3 \cdot 2$.

Since the surface of the cylinder is assumed to be frictionless, the contact forces N_1 and N_2 between the beams and the cylinder act in directions normal to the respective contact planes. Therefore, with $\sin 45° = \sqrt{2}/2$, the equilibrium conditions for the cylinder read

$$\begin{aligned}
&\rightarrow: \quad \frac{\sqrt{2}}{2} N_2 - \frac{\sqrt{2}}{2} N_1 = 0 \qquad &&\rightarrow \; N_1 = N_2\,, \\
&\uparrow: \quad -W + \frac{\sqrt{2}}{2} N_2 + \frac{\sqrt{2}}{2} N_1 = 0 \quad &&\rightarrow \; N_1 = N_2 = \frac{\sqrt{2}}{2} W\,.
\end{aligned} \tag{b}$$

From the equilibrium conditions for the beam ② we obtain with (a) and (b):

$$\begin{aligned}
&\overset{\curvearrowleft}{C}: \quad \sqrt{2}\, a\, N_2 - a\, S + 2\, a\, B = 0 \quad &&\rightarrow \quad \underline{\underline{S}} = 2\, B + \sqrt{2}\, N_2 = \underline{\underline{2\, W}}\,, \\
&\uparrow: \quad -\frac{\sqrt{2}}{2} N_2 - C_V + B = 0 \quad &&\rightarrow \quad \underline{\underline{C_V}} = B - \frac{\sqrt{2}}{2} N_2 = \underline{\underline{0}}\,, \\
&\rightarrow: \quad -\frac{\sqrt{2}}{2} N_2 - C_H - S = 0 \quad &&\rightarrow \quad \underline{\underline{C_H}} = -\frac{\sqrt{2}}{2} N_2 - S = \underline{\underline{-\frac{5}{2} W}}.
\end{aligned}$$

The same result is obtained when the equilibrium conditions are applied to beam ①. By symmetry considerations, it can be concluded from Fig. 5.17c with no calculation that $N_1 = N_2$ and $C_V = 0$.

5.3.2 Three-Hinged Arch

The arch shown in Fig. 5.18a is statically determinate because it is immobile and in total three support reactions exist at A and B. In a real structure, the arch AB is not rigid but deforms under applied loads. If B is a roller support, this may lead to a large deformation that cannot be tolerated.

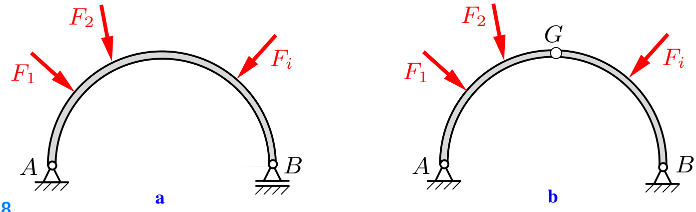


Fig. 5.18

Such a displacement is prevented if A and B are designed as hinged supports. As a consequence, the statical determinacy of the structure gets lost. However, statical determinacy can be re-established if an additional hinge G is introduced at an arbitrary location (Fig. 5.18b). Such a structure is called a *three-hinged arch*. It consists of $n = 2$ bodies connected by the hinge G, which transfers $v = 2$ joint reactions. Since the supports A and B transfer $r = 2 + 2 = 4$ support reactions, the condition for statical determinacy (5.3) is fulfilled: $4 + 2 = 3 \cdot 2$. Therefore, taking the immobility of the structure into account, the three-hinged arch is statically determinate.

The two bodies of a three-hinged arch need not necessarily be arch shaped. An arbitrary structure consisting of two bodies connected by a hinge and supported by two hinged supports (in total *three* hinges) is also called a three-hinged arch from now on. Two examples are shown in Fig. 5.19: a) a frame and b) a truss consisting of two single trusses (compare Chapter 6).

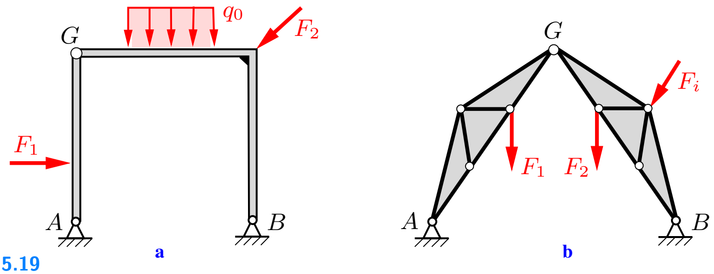


Fig. 5.19

In order to calculate the forces at the supports and the hinge, we isolate the two bodies ① and ② (cf. Fig. 5.20a,b) and apply the equilibrium conditions to each body. From the $2 \cdot 3 = 6$ equations, the unknowns A_H, A_V, B_H, B_V, G_H and G_V can be calculated. As a check, the equilibrium conditions can be applied to the complete system where the hinge is regarded as being frozen.

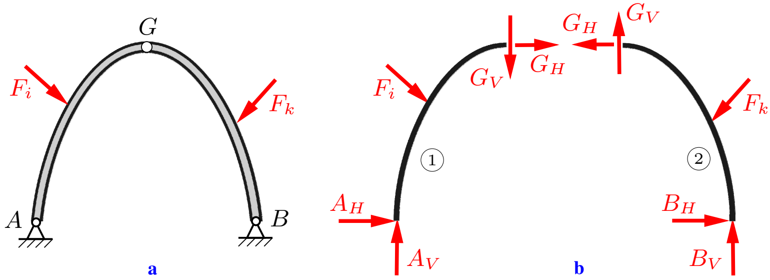


Fig. 5.20

E5.7

Example 5.7 The structure shown in Fig. 5.21a consists of two beams, joined by the hinge G and supported in A and B by hinged supports. The system is loaded by the forces $F_1 = F$ and $F_2 = 2\,F$.

Determine the forces at the supports and the hinge.

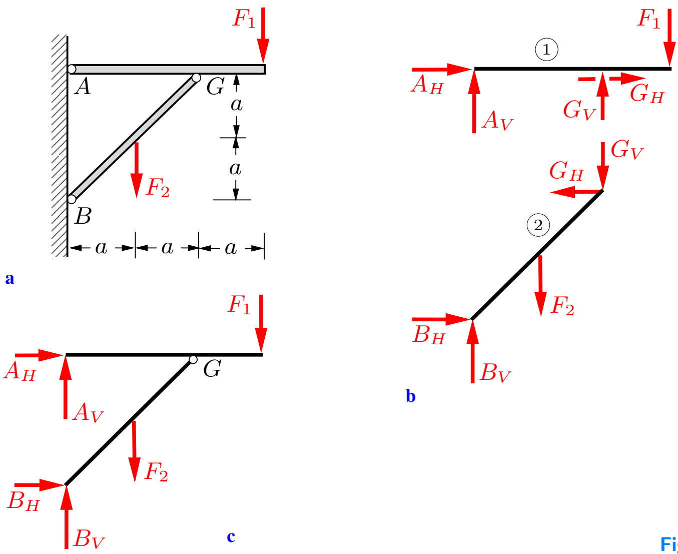


Fig. 5.21

Solution The structure is a three-hinged arch. In order to calculate the unknown forces we separate the bodies 1 and 2 and draw the free-body diagram (Fig. 5.21b). The equilibrium conditions for beam ① read

$$\overset{\curvearrowleft}{A}: \quad 2\,a\,G_V - 3\,a\,F_1 = 0 \quad \to \quad \underline{\underline{G_V}} = \frac{3}{2}\,F_1 = \underline{\underline{\frac{3}{2}\,F}}\,,$$

$$\overset{\curvearrowleft}{G}: \quad -2\,a\,A_V - a\,F_1 = 0 \quad \to \quad \underline{\underline{A_V}} = -\frac{1}{2}\,F_1 = \underline{\underline{-\frac{1}{2}\,F}}\,,$$

$$\to: \quad A_H + G_H = 0\,.$$

For beam ② we obtain

$$\overset{\curvearrowleft}{B}: \quad -a\,F_2 - 2\,a\,G_V + 2\,a\,G_H = 0\,,$$

$$\overset{\curvearrowleft}{G}: \quad 2\,a\,B_H - 2\,a\,B_V + a\,F_2 = 0\,,$$

$$\to: \quad B_H - G_H = 0\,.$$

Solving the system of equations yields

$$\underline{\underline{G_H}} = \frac{1}{2}\,F_2 + G_V = \underline{\underline{\frac{5}{2}\,F}}\,, \qquad \underline{\underline{B_H}} = G_H = \underline{\underline{\frac{5}{2}\,F}}\,,$$

$$\underline{\underline{B_V}} = \frac{1}{2}\,F_2 + B_H = \underline{\underline{\frac{7}{2}\,F}}\,, \qquad \underline{\underline{A_H}} = -\,G_H = \underline{\underline{-\frac{5}{2}\,F}}\,.$$

As a check we use the force equilibrium for the complete (frozen) system according to Fig. 5.21c:

$$\uparrow: \quad A_V + B_V - F_1 - F_2 = 0 \quad \to \quad -\frac{1}{2}\,F + \frac{7}{2}\,F - F - 2\,F = 0\,,$$

$$\to: \quad A_H + B_H = 0 \quad \to \quad -\frac{5}{2}\,F + \frac{5}{2}\,F = 0\,.$$

5.3.3 Hinged Beam

Structures with a wide span width are necessarily often supported by more than two supports. As an example, consider the beam shown in Fig. 5.22a. Since $r = 5$, the system is statically indeterminate with two degrees of statical indeterminacy (see Section 5.1.2). Therefore, the calculation of the support reactions solely from the equilibrium conditions is impossible.

A statically determinate multi-body structure can be obtained

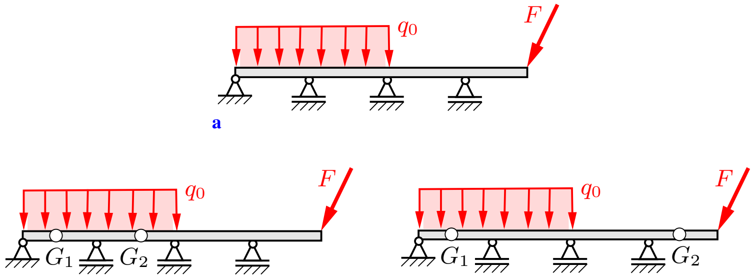


Fig. 5.22

if the continuous beam is divided into several parts by introducing an appropriate number of hinges. Such a structure is called a *hinged beam*.

If the number of these hinges is g, the continuous beam is divided into $n = g + 1$ parts. Since each hinge transfers two force components, the number of joint reactions is $v = 2\,g$. Therefore, according to (5.3) the necessary condition for statical determinacy takes the form

$$r + v = 3\,n \quad \rightarrow \quad r + 2\,g = 3(g + 1)\,. \tag{5.4}$$

Thus, the necessary number of hinges is given by

$$g = r - 3\,. \tag{5.5}$$

The beam in Fig. 5.22a has $r = 5$ support reactions. Therefore, according to (5.5) two hinges are necessary: $g = 5 - 3 = 2$. There are various possibilities for arranging the hinges; the support and joint reactions depend on their positions. One possibility is depicted in Fig. 5.22b. In contrast, Fig. 5.22c shows a hinge arrangement leading to a movable, i.e., kinematically indeterminate structure that is statically useless.

To determine the support and joint reactions, we first divide the hinged beam into its parts and subsequently apply the equilibrium conditions to each body.

E5.8

Example 5.8 The hinged beam shown in Fig. 5.23a is loaded by the single force F and the line load q_0.

Determine the support and hinge forces.

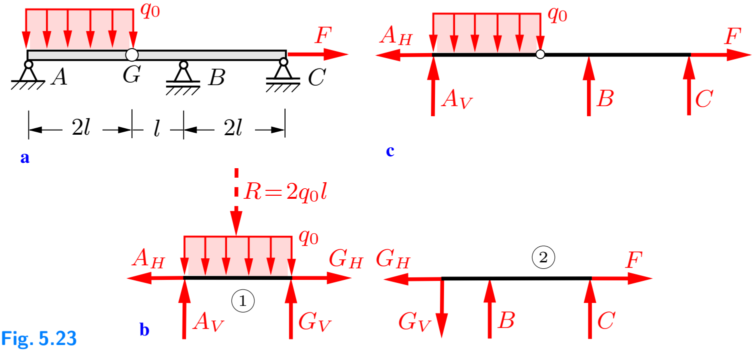


Fig. 5.23

Solution The system is statically determinate. We separate the two bodies and draw the free-body diagram (Fig. 5.23b). The line load can be replaced by its statically equivalent resultant force $R = 2\,q_0\,l$ acting in the middle of beam ①.

It is often advantageous to use moment equations with respect to the hinges and the supports. The unknowns can then be calculated successively from equations with only one unknown. Here, the equilibrium conditions for beam ① read

$$\overset{\curvearrowleft}{A}: \quad -l\,R + 2\,l\,G_V = 0 \quad \rightarrow \quad \underline{\underline{G_V}} = \frac{1}{2}R = \underline{\underline{q_0\,l}}\,,$$

$$\overset{\curvearrowleft}{G}: \quad -2\,l\,A_V + l\,R = 0 \quad \rightarrow \quad \underline{\underline{A_V}} = \frac{1}{2}R = \underline{\underline{q_0\,l}}\,,$$

$$\rightarrow: \quad -A_H + G_H = 0$$

and those for beam ② are

$$\overset{\curvearrowleft}{B}: \quad l\,G_V + 2\,l\,C = 0\,,$$

$$\overset{\curvearrowleft}{C}: \quad 3\,l\,G_V - 2\,l\,B = 0\,,$$

$$\rightarrow: \quad -G_H + F = 0 \quad \rightarrow \quad \underline{\underline{G_H = F}}\,.$$

The solution of the remaining system of equations is

$$\underline{\underline{A_H}} = G_H = \underline{\underline{F}}\,, \quad \underline{\underline{B}} = \frac{3}{2}G_V = \underline{\underline{\frac{3}{2}q_0\,l}}\,,$$

$$\underline{\underline{C}} = -\frac{1}{2}G_V = \underline{\underline{-\frac{1}{2}q_0 l}}\,.$$

As a check, the force equilibrium conditions for the complete system are used (Fig. 5.23c):

$$\begin{aligned}
\rightarrow: &\quad -A_H + F = 0 \quad \rightarrow \quad -F + F = 0\,,\\
\uparrow: &\quad A_V - 2\,q_0\,l + B + C = 0\\
&\qquad\qquad \rightarrow \quad q_0\,l - 2\,q_0\,l + \frac{3}{2}\,q_0\,l - \frac{1}{2}\,q_0\,l = 0\,.
\end{aligned}$$

5.3.4 Kinematical Determinacy

In this section, statical and kinematical determinacy or indeterminacy is discussed in more detail than in Section 5.3.1. Again, we will restrict the discussion to plane multi-part structures.

The number f of degrees of freedom of an n body system without any joints is given by $3n$ (3 degrees of freedom for each body). This number is reduced by the number of restraints r through supports and the number of restraints v through joints:

$$f = 3\,n - (r + v)\,. \tag{5.6}$$

Each restraint r and v, respectively, is associated with one support or joint reaction. Furthermore, the number of available equilibrium conditions is given by $3n$ (three equations for each body).

For $f > 0$ the system is movable. In contrast, for $f < 0$ the number $r + v$ of support and joint reactions exceeds the number $3n$ of equilibrium conditions by x. In this case, the system is statically indeterminate where the degree x of statical indeterminacy is given by

$$x = -f = r + v - 3\,n\,. \tag{5.7}$$

Even though it is impossible to determine *all* support and joint reactions of statically indeterminate systems solely from the equilibrium conditions, it may be possible to calculate some reactions in certain cases. For example, let us consider the system in Fig. 5.24a, where on account of $n = 2$, $r = 5$ and $v = 2$, the degree of statical indeterminacy is equal to one. Nevertheless, for beam $\overline{GC}$, the force components in the hinge G and the force in the simple support C can be calculated for a given loading from

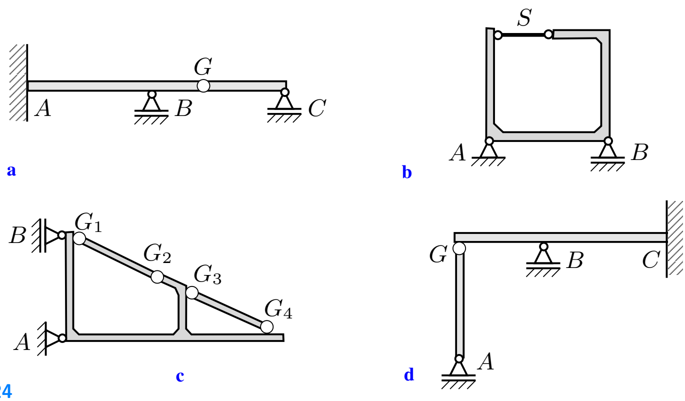


Fig. 5.24

the three equilibrium conditions. Two other examples of systems with one or two degrees of statical indeterminacy, respectively, are depicted in Figs. 5.24b and c. In both cases the support reactions can be calculated from the equilibrium conditions applied to the complete structures: the systems are *externally statically determinate*. However, the joint reactions (force in the strut, forces in the hinges) between the parts of the systems cannot be calculated. Such systems are called *internally statically indeterminate*.

Statically indeterminate systems in certain cases can undergo finite or infinitesimal displacements, i.e., they are then also kinematically indeterminate. As an example, the system shown in Fig. 5.24d is one-degree statically indeterminate because $n = 2$, $r = 5$ and $v = 2$. However, it can be seen that the system is not immovable since the vertical beam can rotate infinitesimally about G. It is evident that such a structure is not able to carry arbitrary loads.

Finally, for $f = 3n - (r + v) = 0$ the necessary condition for statical determinacy is fulfilled (cf. (5.3)). Then all support and joint reactions can be calculated except in the exceptional case of a movable system.

Now we will answer the question of how we can determine whether or not a multi-part structure is movable. Let us first consider only plane systems fulfilling the necessary condition for statical determinacy ($f = 0$). Whether the system is movable or not can always be formally decided by rewriting the equilibrium conditions in the form of a linear system of equations (cf. Appendix A.2):

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}\,. \tag{5.8}$$

Here, $\boldsymbol{b} = (b_1, \dots \dots, b_{3n})^T$ is given by the prescribed loading, $\boldsymbol{x} = (x_1, \dots, x_{3n})^T$ represents the unknown support and joint reactions, and the matrix $\boldsymbol{A}$ is given through the coefficients found through writing down the equilibrium conditions. The system of equations has a unique solution if the determinant of matrix $\boldsymbol{A}$ is nonzero:

$$\det \boldsymbol{A} \neq 0\,. \tag{5.9}$$

The multi-part structure with $f = 0$ is then not only statically but also kinematically determinate. This condition is very general; it also holds for an arbitrary spatial system.

E5.9

Example 5.9 Formulate the equilibrium conditions for the beam shown in Fig. 5.25a ($0 \le \alpha \le \pi$) in the form $\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}$. Calculate the determinant of the matrix $\boldsymbol{A}$. Is the system statically useful for every value of angle α?

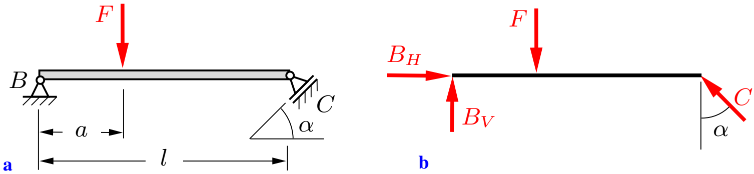


Fig. 5.25

Solution The equilibrium conditions (cf. Fig. 5.25b)

$$\begin{aligned} \rightarrow: \quad & B_H - C\sin\alpha = 0\,, \\ \uparrow: \quad & B_V + C\cos\alpha - F = 0\,, \\ \overset{\curvearrowright}{C}: \quad & lB_V - (l-a)F = 0 \end{aligned}$$

can be written in matrix representation as

$$\begin{pmatrix} 1 & 0 & -\sin\alpha \\ 0 & 1 & \cos\alpha \\ 0 & l & 0 \end{pmatrix} \begin{pmatrix} B_H \\ B_V \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ F \\ (l-a)F \end{pmatrix} \quad \text{i.e.,} \quad \boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}\,.$$

The determinant of $\boldsymbol{A}$ is evaluated through cofactor expansion along the first column:

$$\underline{\underline{\det \boldsymbol{A}}} = \begin{vmatrix} 1 & 0 & -\sin\alpha \\ 0 & 1 & \cos\alpha \\ 0 & l & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & \cos\alpha \\ l & 0 \end{vmatrix} = \underline{\underline{-\,l\,\cos\alpha}}\,.$$

Consequently, we find that

$$\det \boldsymbol{A} \begin{cases} \neq 0 & \text{for } \alpha \neq \pi/2\,, \\ = 0 & \text{for } \alpha = \pi/2\,. \end{cases}$$

Thus, the beam is supported kinematically determinate (immobile) for $\alpha \neq \pi/2$ and kinematically indeterminate solely for $\alpha = \pi/2$. In the latter case, the simple support C can move in the vertical direction. Then the beam may rotate infinitesimally about the hinged support B and is therefore statically useless.

It should be mentioned that even though the beam is formally useful for angles α near $\pi/2$, such a construction should be avoided from the technical point of view because the forces in the supports become very large in this case.

5.4 Supplementary Problems

5.4

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E5.10 **Example 5.10** The beam in Fig. 5.26 is supported by three struts and subjected to a triangular line load.

Determine the forces in the struts.

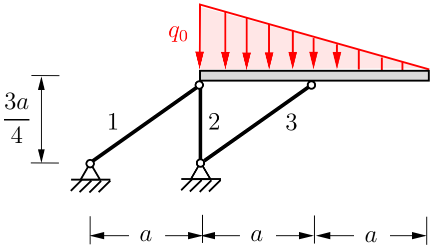


Fig. 5.26

Results: see (**B**) $S_1 = 10\,q_0 a/9, \quad S_2 = -\,q_0 a, \quad S_3 = -10\,q_0 a/9.$

E5.11 **Example 5.11** The structure shown in Fig. 5.27 consists of a beam and three bars. It carries a concentrated force F.

Determine the support reaction at A and the forces in the bars.

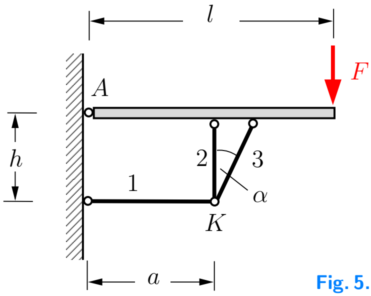


Fig. 5.27

Results: see (**B**) $A_V = F, \quad A_H = -\,lF/h,$

$S_1 = -\,lF/h, \quad S_2 = lF/(h\tan\alpha), \quad S_3 = -\,lF/(h\sin\alpha).$

E5.12 **Example 5.12** The simply supported beam (length $a = 1\,$m) shown in Fig. 5.28 is subjected to the three concentrated forces $F_1 = 4\,$kN, $F_2 = 2\,$kN, $F_3 = 3\,$kN, the line load $q_0 = 5\,$kN/m and the moment $M_0 = 4\,$kNm.

Calculate the support reactions.

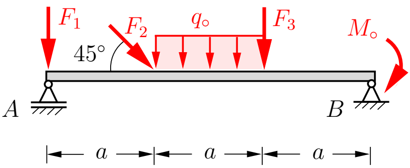


Fig. 5.28

Results: see (**A**) $A = 7.11\,\text{kN}, \quad B_H = 1.41\,\text{kN}, \quad B_V = 6.30\,\text{kN}.$

E5.13

Example 5.13 Find the support reactions for the hinged beam shown in Fig. 5.29.

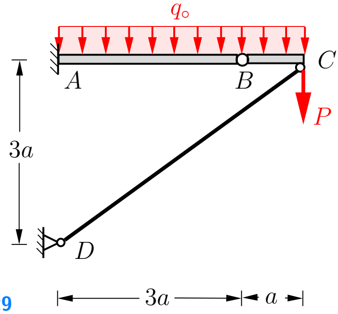


Fig. 5.29

Results: see (**A**) $A_V = 7\,q_0 a/2, \quad A_H = -4P/3 - 2\,q_0 a/3,$

$$D = 5P/3 + 5\,q_0 a/6, \qquad M_A = -3\,q_0 a^2.$$

E5.14

Example 5.14 The hinged beam in Fig. 5.30 carries a concentrated force and a triangular line load.

Determine the support reactions and the force in the hinge.

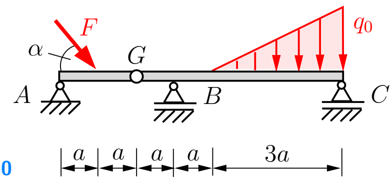


Fig. 5.30

Results: see (**B**) $B = (3\,q_0 a{+}5F\sin\alpha)/8, \quad C = (9\,q_0 a{-}F\sin\alpha)/8,$

$$A_V = F\sin\alpha/2, \quad A_H = F\cos\alpha, \quad G_V = F\sin\alpha/2, \quad G_H = 0.$$

E5.15

Example 5.15 Determine the support reactions for the structure shown in Fig. 5.31. The pulley is frictionless.

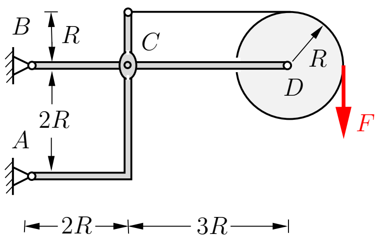


Fig. 5.31

Results: see (**A**) $A_H = 3F, \quad A_V = 5F/2,$

$$B_H = -3F, \quad B_V = -3F/2.$$

E5.16 **Example 5.16** A homogeneous beam (weight W) hangs on a crane (Fig. 5.32).

Determine the support reactions at A and B and the force at hinge C.

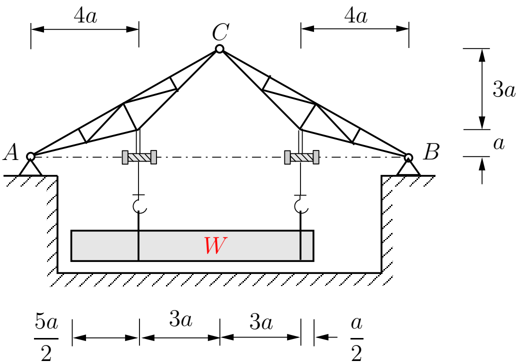


Fig. 5.32

Results: see (**B**) $A_V = 4W/7, \quad A_H = W/2, \quad B_V = 3W/7,$
$B_H = -W/2, \quad C_V = 2W/21, \quad C_H = -W/2.$

E5.17 **Example 5.17** A mast (weight W_1) has a hinged support (ball-and-socket connection) at A. In addition it is supported by two struts. Its upper end carries a weight W_2 (Fig. 5.33).

Determine the reaction force at A and the forces in the struts.

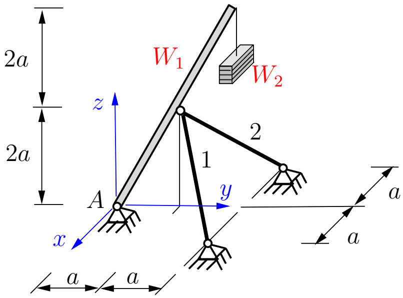


Fig. 5.33

Results: see (**B**) $A_x = 0, \quad A_y = (W_1 + 2W_2)/4, \quad A_z = W_1/2,$
$S_1 = S_2 = -\sqrt{6}\,(W_1 + 2W_2)/8.$

E5.18

Example 5.18 Determine the support reactions for the frame shown in Fig. 5.34.

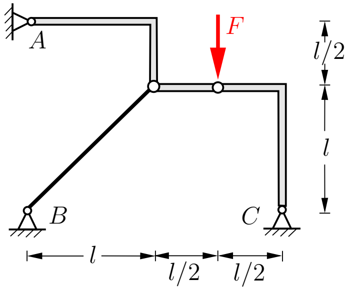


Fig. 5.34

Results: see (**A**) $A_H = F/3, \quad A_V = -F/6,$

$B = \sqrt{2}\,F/6, \quad C_H = F/2, \quad C_V = F.$

E5.19

Example 5.19 Calculate the support reactions for the spatial structure in Fig. 5.35.

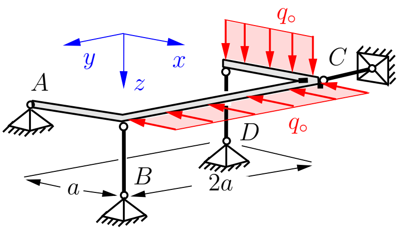


Fig. 5.35

Results: see (**A**) $A_x = -A_y = 2q_\circ a, \quad A_z = q_\circ a/2,$

$B_z = -q_\circ a/2, \quad C_y = 2q_\circ a, \quad D_z = -q_\circ a.$

5.5

5.5 Summary

- Supports and connecting elements, respectively, are classified according to the number of transferred reaction forces and couple moments. A simple support transfers one reaction, a hinged support two reactions etc. Analogously, a hinge as joining element transfers two reactions etc.
- A structure is statically determinate if all support and joint reactions can be calculated from the equilibrium conditions. This is the case if the number of unknown support and joint reactions is equal to the number of equilibrium conditions and the structure is immobile.
- A structure is kinematically determinate if it is immobile. A structure that can undergo finite or infinitesimal displacements is kinematically indeterminate.
- To calculate the support and joint reactions usually the following steps are necessary:
  - ◇ Removal of the supports from the structure and separation of the individual bodies.
  - ◇ Sketch of the free-body diagrams; all acting forces and couple moments as well as all reaction forces and couple moments must be drawn.
  - ◇ Formulation of the equilibrium conditions. In the plane case there are 3 equations for each body, e.g.

    $$\sum F_{ix} = 0\,, \qquad \sum F_{iy} = 0\,, \qquad \sum M_i^{(A)} = 0\,,$$

    where A is an arbitrary (appropriately chosen) reference point. In the spatial case there are 6 equilibrium conditions for each body.
  - ◇ Calculation of the unknowns by resolution of the system of equations. Note: the number of equilibrium conditions and the number of unknowns must be equal!
  - ◇ The system of equations has a unique solution if the determinant of the coefficient matrix is nonzero. Then the structure is statically and kinematically determinate.

Chapter 6

Trusses

6

6 Trusses

Objectives: A truss is a structure composed of slender members that are connected at their ends by joints. The truss is one of the most important structures in engineering applications. After studying this chapter, students should be able to recognise if a given truss is statically and kinematically determinate. In addition, they will become familiar with methods to determine the internal forces in a statically determinate truss.

6.1 Statically Determinate Trusses

A structure that is composed of straight slender members is called a *truss*. To be able to determine the internal forces in the individual members, the following assumptions are made:

1. The members are connected through smooth pins (frictionless joints).
2. External forces are applied at the pins only.

A truss that satisfies these assumptions is called an "ideal truss". Its members are subjected to tension or to compression only (two-force members, see Section 2.4).

In real trusses, these ideal conditions are not exactly satisfied. For example, the joints may not be frictionless, or the ends of the members may be welded to a gusset plate. Even then, the assumption of frictionless pin-jointed connections yields satisfactory results if the axes of the members are concurrent at the joints. Also, external forces may be applied along the axes of the members (e.g., the weights of the members). Such forces are either neglected (e.g., if the weights of the members are small in comparison with the loads) or their resultants are replaced by statically equivalent forces at the adjacent pins.

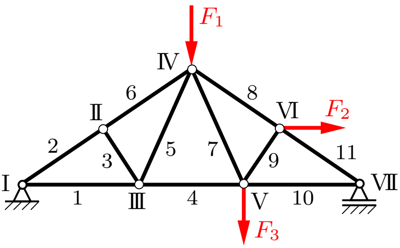


Fig. 6.1

In this chapter we focus on plane trusses; space trusses can be treated using the same methods. As an example, consider the truss shown in Fig. 6.1. It consists of 11 members which are connected with 7 pins (the pins at the supports are also counted). The members are marked with Arabic numerals and the pins with Roman numerals.

To determine the internal forces in the members we may draw a free-body diagram for every joint of the truss. Since the forces at the pins are concurrent forces, there are two equilibrium conditions at each joint (see Section 2.3). In the present example, we thus have $7 \cdot 2 = 14$ equations for the 14 unknown forces (11 forces in the members and 3 forces at the supports).

A truss is called *statically determinate* if all the unknown forces, i.e., the forces in the members and the forces at the supports, can be determined from the equilibrium conditions. Let a plane truss be composed of m members connected through j joints, and let the number of support reactions be r. In order to be able to determine the $m + r$ unknown forces from the $2\,j$ equilibrium conditions, the number of unknowns has to be equal to the number of equations:

$$2\,j = m + r\,. \tag{6.1}$$

This is a *necessary condition* for the determinacy of a plane truss. As we shall discuss later, however, it will not be sufficient in cases of improper support or arrangement of the members. If the truss is rigid, the number of support reactions must be $r = 3$.

In the case of a *space truss*, there exist three conditions of equilibrium at each joint, resulting in a total of $3\,j$ equations. Therefore,

$$3\,j = m + r \tag{6.2}$$

is the corresponding necessary condition for a space truss. If the truss is a rigid body, the support must be statically determinate: $r = 6$ (compare Section 3.2.2).

For the truss shown in Fig. 6.2a we have $j = 7$, $m = 10$ and $r = 2 \cdot 2$ (two pin connections). Hence, since $2 \cdot 7 = 10 + 4$, the necessary condition (6.1) is satisfied. The truss is, in addition, completely constrained against motion. Therefore, it is statically determinate.

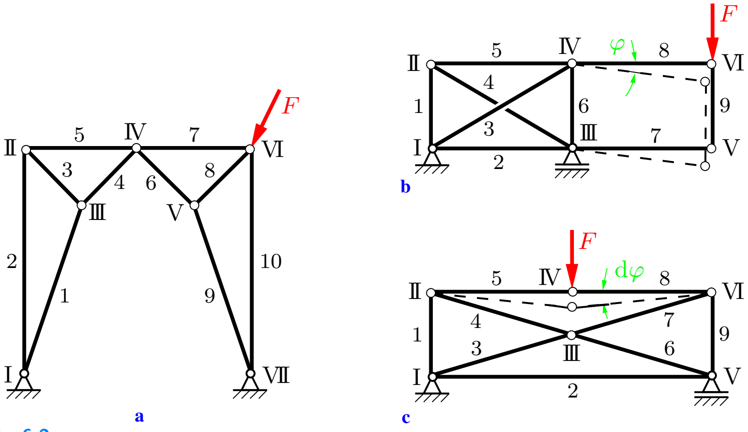


Fig. 6.2

A truss that is completely constrained against motion is called a *kinematically determinate* truss. In contrast, a truss that is not a rigid structure and therefore able to move is called kinematically indeterminate. This is the case if there are fewer unknowns than independent equilibrium conditions. If there are more unknowns than equilibrium conditions, the system is called statically indeterminate.

We shall only consider trusses that satisfy the necessary conditions (6.1) or (6.2), respectively. Even then, a truss will not be statically determinate if the members or the supports are improperly arranged. Consider, for example, the trusses shown in Figs. 6.2b and 6.2c. With $j = 6$, $m = 9$ and $r = 3$ the necessary condition (6.1) is satisfied. However, the members 7 and 8 of the truss in Fig. 6.2b may rotate about a finite angle φ, whereas the members 5 and 8 of the truss in Fig. 6.2c may rotate about an infinitesimally small angle $d\varphi$. Each of the improperly constrained trusses is statically indeterminate.

6.2 Design of a Truss 6.2

In the following, we shall discuss three methods for designing a statically and kinematically determinate plane truss.

Method 1: We start with a single bar and add two members to form a triangle (see Fig. 6.3). This basic element represents a rigid body. It may be extended by successively adding two members at a time and connecting them in such a way that the structure remains rigid (one has to avoid having two members along a straight line; such an improper arrangement is indicated by the dashed line in Fig. 6.3). A truss designed from a basic triangle as described is called a *simple plane truss*.

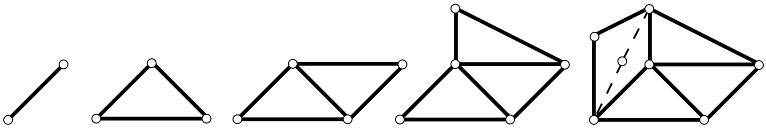


Fig. 6.3

As can be verified by inspection, the relation

$$2\,j = m + 3 \tag{6.3}$$

is satisfied by the trusses shown in Fig. 6.3. For every additional joint in a simple truss, there are two additional members. Therefore Equation (6.3) remains valid. If the truss is supported in such a way that there are $r = 3$ unknown reactions that completely constrain the truss, then it is statically determinate (see the necessary condition (6.1)).

Method 2: Two simple trusses are connected by three members (Fig. 6.4a). To ensure the rigidity of the system, the axes of the members must not be parallel or concurrent. The two simple trusses may also be connected by a joint and one member: the system in Fig. 6.4b has been obtained by replacing members 2 and 3 in Fig. 6.4a by the joint I.

If two simple trusses are connected through one joint only, the system is nonrigid. To obtain a statically and kinematically determinate system, an additional support must be introduced; an example is given in Fig. 6.4c. This system represents a three-hinged arch. It should be noted that instead of connecting the two simple

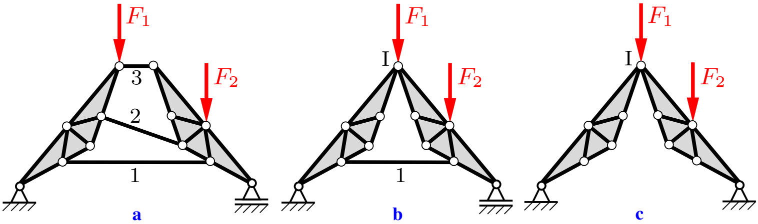


Fig. 6.4

trusses by one joint the trusses may be connected by two members that are not parallel and not concurrent.

It can easily be verified that for the three examples shown in Fig. 6.4, the necessary condition (6.1) is satisfied. Since these systems are completely constrained against motion, they are statically determinate.

Method 3: Consider a truss that has been designed according to the first or the second method. If we remove one member of the truss, the truss will become nonrigid. Therefore, we have to add one member at a different position in such a way that the truss will be rigid again. Since by doing so neither the number of the members nor the number of the joints is changed, condition (6.1) is still satisfied.

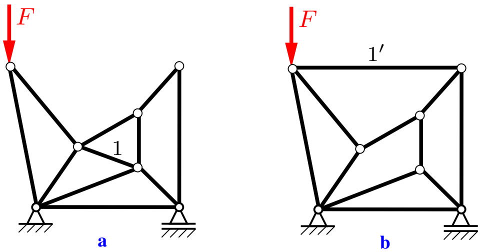


Fig. 6.5

An example of Method 3 is shown in Fig. 6.5. If we remove member 1 from the simple truss in Fig. 6.5a, the system is only partially constrained. Adding member $1'$ yields the statically determinate truss shown in Fig. 6.5b.

6.3

6.3 Determination of the Internal Forces

In the following, two methods to determine the internal forces in the individual members of a statically determinate truss will be discussed. In both methods, the conditions of equilibrium are applied to suitable free-body diagrams.

6.3.1 Method of Joints

The *method of joints* consists of applying the equilibrium conditions to the free-body diagram of each joint of the truss. It is a systematic method and can be used for every statically determinate truss.

In practice, it is often convenient first to identify those members of the truss that have a vanishing internal force. These members are called *zero-force members*. If the zero-force members are recognised in advance, the number of unknowns is reduced, which simplifies the analysis. It should be noted that the loading determines whether a member is a zero-force member or not.

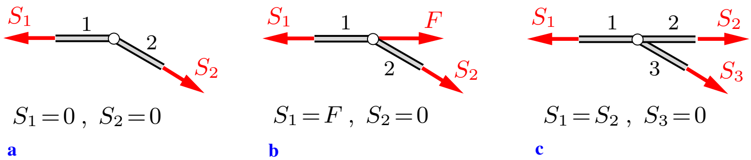


Fig. 6.6

The following rules are useful in identifying zero-force members.

1. If two members are not collinear at an *unloaded* joint (Fig. 6.6a), then both members are zero-force members.
2. Let two members be connected at a *loaded* joint (Fig. 6.6b). If the action line of the external force F coincides with the direction of one of the members, then the other member is a zero-force member.
3. Let three members be connected at an *unloaded* joint (Fig. 6.6c). If two members have the same direction, the third member is a zero-force member.

These rules can be verified by applying the equilibrium conditions to the respective joints.

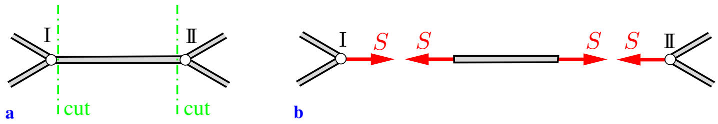


Fig. 6.7

To free a member of a truss from its constraints we cut it at both ends, i.e., at the adjacent joints (Fig. 6.7a). The corresponding free-body diagram is shown in Fig. 6.7b. The members in a truss are subject to tension or to compression; Fig. 6.7b shows a bar under tension. According to Newton's third law (actio = reactio), forces S of equal magnitude and opposite directions act at the pins I and II. As can be seen from Fig. 6.7b an arrow that points away from the pin (pull) indicates tension in the member, whereas an arrow that points toward the pin (push) indicates compression.

It is not always possible to determine by inspection whether a member is subject to tension or compression. Therefore, we shall always assume that all the members of a truss are under tension. If the analysis gives a negative value for the force in a member, this member is in reality subject to compression.

The $m+r$ unknown forces can be determined from the $2j$ equilibrium conditions for the j joints. One may also apply the three equilibrium conditions for the complete truss. These equations are not independent of the equilibrium conditions at the joints. Therefore they may provide a check on the correctness of the analysis. In practice, it may be more convenient to determine first the support reactions from the free-body diagram of the complete truss. Then three other equilibrium equations within the method of joints will serve as checks.

The method of joints can also be used to determine the forces in a space truss where there are three equilibrium conditions at each joint. If the truss is a rigid body, there must be six support forces to ensure a statically determinate support. The six equilibrium conditions for the whole truss can be used as a check. On the other hand, if the support forces are computed in advance, six other equilibrium equations within the method of joints may serve as checks.

E6.1

Example 6.1 The truss shown in Fig. 6.8a is loaded by an external force F.

Determine the forces at the supports and in the members of the truss.

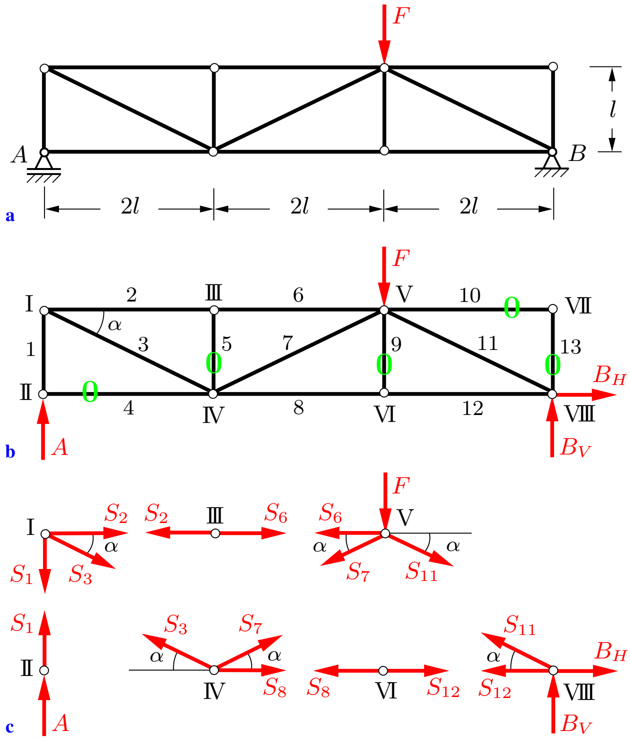


Fig. 6.8

Solution Fig. 6.8a represents a simple truss that is completely constrained against motion. Therefore, it is statically determinate.

The members of the truss are numbered in the free-body diagram of the complete truss (Fig. 6.8b). Zero-force members are identified by inspection and marked with zeroes: member 4 (according to Rule 2), the members 5 and 9 (Rule 3) and the members 10 and 13 (Rule 1).

To further reduce the number of unknown forces, we compute the support forces by applying the equilibrium conditions to the whole truss:

$$\rightarrow: \quad \underline{\underline{B_H = 0}}\,,$$

$$\overset{\curvearrowleft}{A}: \quad -4\,l\,F + 6\,l\,B_V = 0 \quad \rightarrow \quad \underline{\underline{B_V = \frac{2}{3}\,F}}\,,$$

$$\overset{\curvearrowleft}{B}: \quad -6\,l\,A + 2\,l\,F = 0 \quad \rightarrow \quad \underline{\underline{A = \frac{1}{3}\,F}}\,.$$

Fig. 6.8c shows the free-body diagrams of the joints. As previously stated, we assume that every member is subjected to tension. Accordingly, all of the corresponding arrows point away from the joints. Zero-force members are omitted in the free-body diagrams. Therefore, joint VII need not be considered. Applying the equilibrium conditions to each joint yields

$$\begin{aligned}
&\text{I)} && \rightarrow: && S_2 + S_3\cos\alpha = 0\,,\\
& && \downarrow: && S_1 + S_3\sin\alpha = 0\,,\\
&\text{II)} && \uparrow: && S_1 + A = 0\,,\\
&\text{III)} && \rightarrow: && S_6 - S_2 = 0\,,\\
&\text{IV)} && \rightarrow: && S_8 + S_7\cos\alpha - S_3\cos\alpha = 0\,,\\
& && \uparrow: && S_7\sin\alpha + S_3\sin\alpha = 0\,,\\
&\text{V)} && \rightarrow: && S_{11}\cos\alpha - S_6 - S_7\cos\alpha = 0\,,\\
& && \downarrow: && S_7\sin\alpha + S_{11}\sin\alpha + F = 0\,,\\
&\text{VI)} && \rightarrow: && S_{12} - S_8 = 0\,,\\
&\text{VIII)} && \rightarrow: && B_H - S_{11}\cos\alpha - S_{12} = 0\,,\\
& && \uparrow: && B_V + S_{11}\sin\alpha = 0\,.
\end{aligned}$$

These are 11 equations for the 8 unknown forces in the members and the 3 forces at the supports. Since the support forces have been computed in advance and are already known, the analysis is simplified, and three equations may be used as a check on the correctness of the results. Using the geometrical relations

$\sin\alpha = l/\sqrt{5\,l^2} = 1/\sqrt{5}\,,\quad \cos\alpha = 2\,l/\sqrt{5\,l^2} = 2/\sqrt{5}$, we obtain

$$\underline{\underline{S_1 = -\frac{1}{3}\,F}}\,,\quad \underline{\underline{S_2 = S_6 = -\frac{2}{3}\,F}}\,,\ \underline{\underline{S_3 = \frac{\sqrt{5}}{3}\,F}}\,,$$

$$\underline{\underline{S_7 = -\frac{\sqrt{5}}{3}\,F}}\,,\ \underline{\underline{S_8 = S_{12} = \frac{4}{3}\,F}}\,,\quad \underline{\underline{S_{11} = -\frac{2}{3}\sqrt{5}\,F}}\,.$$

It is useful to present the results in dimensionless form in a table, including negative signs:

Table of Forces

i	1	2	3	4	5	6	7	8	9	10	11	12	13
$\frac{S_i}{F}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{\sqrt{5}}{3}$	0	0	$-\frac{2}{3}$	$-\frac{\sqrt{5}}{3}$	$\frac{4}{3}$	0	0	$-\frac{2}{3}\sqrt{5}$	$\frac{4}{3}$	0

The negative values for the members 1, 2, 6, 7 and 11 indicate that these members are under compression.

E6.2

Example 6.2 Fig. 6.9 shows a spatial truss loaded by two external forces F at the joints IV and V.

Compute the forces in the members 1-6.

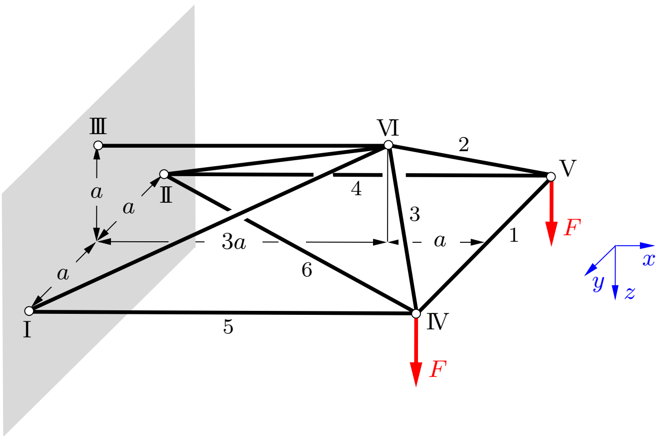


Fig. 6.9

Solution We free the joints V and IV by passing imaginary cuts through the bars, and we assume that the members 1-6 are in tension. The vector equations of equilibrium for these joints are given by

$$\begin{aligned}
\mathrm{V}: &\quad S_1\boldsymbol{e}_y + S_2\boldsymbol{e}_{\mathrm{V/VI}} - S_4\boldsymbol{e}_x + F\boldsymbol{e}_z = \boldsymbol{0}\,,\\
\mathrm{IV}: &\quad -S_1\boldsymbol{e}_y + S_3\boldsymbol{e}_{\mathrm{IV/VI}} - S_5\boldsymbol{e}_x + S_6\boldsymbol{e}_{\mathrm{IV/II}} + F\boldsymbol{e}_z = \boldsymbol{0}\,.
\end{aligned}$$

The initially unknown unit vectors can be determined from the vectors connecting adjacent joints, e.g., for $\boldsymbol{e}_{\mathrm{V/VI}}$ we obtain

$$\boldsymbol{e}_{\mathrm{V/VI}} = \frac{1}{\sqrt{a^2+a^2+a^2}}\begin{pmatrix}-a\\ a\\ -a\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ 1\\ -1\end{pmatrix}.$$

Similarly, the other unit vectors are

$$\boldsymbol{e}_{\mathrm{IV/VI}} = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\ -1\\ -1\end{pmatrix}, \quad \boldsymbol{e}_{\mathrm{IV/II}} = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\ -1\\ 0\end{pmatrix},$$

$$\boldsymbol{e}_x = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \qquad \boldsymbol{e}_y = \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}, \qquad \boldsymbol{e}_z = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}.$$

Introducing these into the two vector equations we get the six scalar equations

$$\begin{aligned}
\mathrm{V}: \quad -S_2\tfrac{1}{\sqrt{3}} - S_4 &= 0\,, & \mathrm{IV}: \quad -S_3\tfrac{1}{\sqrt{3}} - S_5 - S_6\tfrac{2}{\sqrt{5}} &= 0\,,\\
S_1 + S_2\tfrac{1}{\sqrt{3}} &= 0\,, & -S_1 - S_3\tfrac{1}{\sqrt{3}} - S_6\tfrac{1}{\sqrt{5}} &= 0\,,\\
-S_2\tfrac{1}{\sqrt{3}} + F &= 0\,, & -S_3\tfrac{1}{\sqrt{3}} + F &= 0\,.
\end{aligned}$$

Their solution yields the forces

$$\underline{\underline{S_2 = \sqrt{3}\,F}}\,, \qquad \underline{\underline{S_1 = -F}}\,, \qquad \underline{\underline{S_4 = -F}}\,,$$

$$\underline{\underline{S_3 = \sqrt{3}\,F}}\,, \qquad \underline{\underline{S_6 = 0}}\,, \qquad \underline{\underline{S_5 = -F}}\,.$$

6.3.2 Method of Sections

It is not always necessary to determine the forces in all of the members of a truss. If several forces only are of interest, it may be advantageous to use the *method of sections* instead of the method of joints. In this case, the truss is divided by a cut into two parts. The cut has to be made in such a way that it either goes through *three* members that do not all belong to the same joint, or passes through *one* joint and *one* member. If the support reactions are computed in advance, the free-body diagram for each part of the truss contains only three unknown forces that can be determined by the three conditions of equilibrium.

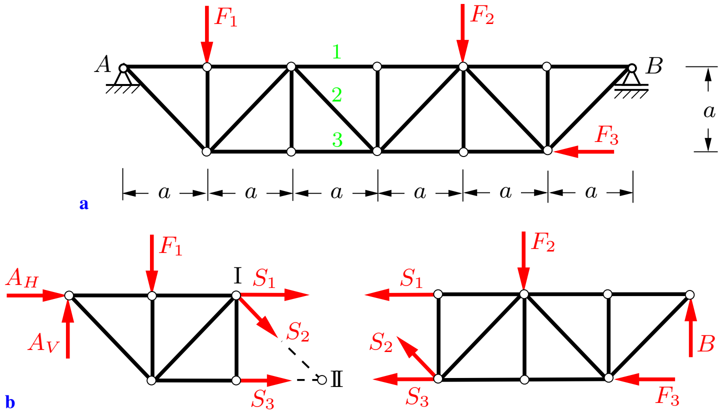


Fig. 6.10

To illustrate the method, we consider the truss shown in Fig. 6.10a with the objective of determining the forces in members 1-3. As a first step, the reactions at the supports A and B are computed by applying the conditions of equilibrium to the free-body diagram of the whole truss (not shown in the figure). In the second step, we pass an imaginary section through the members 1-3, cutting the truss into two parts. Fig. 6.10b shows the free-body diagrams of the two parts of the truss. The internal forces in members 1-3 act as external forces in the free-body diagrams; they are assumed to be tensile forces.

Both parts of the truss in Fig. 6.10b are rigid bodies in equilibrium. Therefore, either part may be used for the analysis. In practice, the part that involves a smaller number of forces will usually lead to a simpler calculation. We shall apply the equilibrium conditions to the free-body diagram on the left-hand side of Fig. 6.10b. It is advantageous to use moment equations about the points of intersection of two unknown forces. Each of the corresponding equations contains *one* unknown force only and can be solved immediately:

$$\overset{\curvearrowleft}{\mathrm{I}}: \quad -2\,a\,A_V + a\,F_1 + a\,S_3 = 0 \quad \rightarrow \quad S_3 = 2\,A_V - F_1\,,$$

$$\overset{\curvearrowleft}{\mathrm{II}}: \quad -3\,a\,A_V - a\,A_H + 2\,a\,F_1 - a\,S_1 = 0$$

$$\rightarrow \quad S_1 = 2\,F_1 - 3\,A_V - A_H,$$

$$\uparrow: \quad A_V - F_1 - \frac{1}{2}\sqrt{2}\,S_2 = 0 \quad \rightarrow \quad S_2 = \sqrt{2}\,(A_V - F_1)\,.$$

Since the support reactions have been computed in advance, the forces in members 1-3 are now known.

In many cases, the method of sections can be applied without having to determine the forces at the supports. Consider, for example, the truss in Fig. 6.11a. The forces in members 1-3 can be obtained immediately from the equilibrium conditions for the part of the truss on the right as shown in Fig. 6.11b.

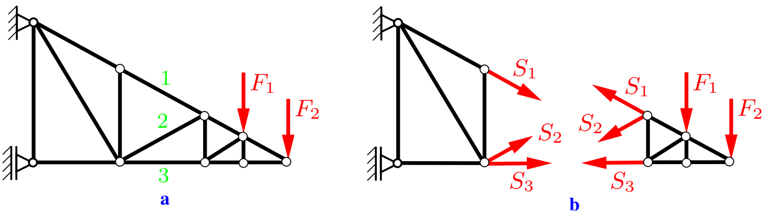


Fig. 6.11

The method of sections is also applicable to spatial trusses. Since there are six equilibrium conditions for a rigid body in the case of a spatial problem, the truss has to be divided by a cut through six members, or through three members and a pin.

E6.3

Example 6.3 A truss is loaded by two forces, $F_1 = 2\,F$ and $F_2 = F$, as shown in Fig. 6.12a.

Determine the force S_4.

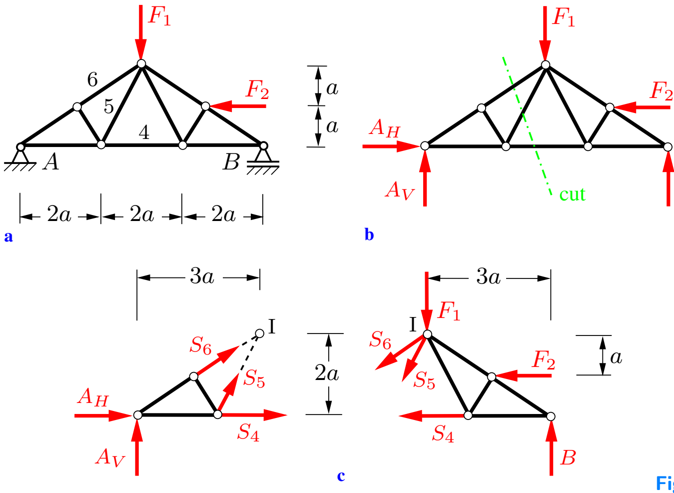


Fig. 6.12

Solution First, we determine the forces at the supports. Applying the equilibrium conditions to the free-body diagram of the whole truss (Fig. 6.12b) yields

$$\overset{\curvearrowleft}{A}: \quad -3\,a\,F_1 + a\,F_2 + 6\,a\,B = 0 \quad \rightarrow \quad B = \frac{3\,F_1 - F_2}{6} = \frac{5}{6}\,F\,,$$

$$\overset{\curvearrowleft}{B}: \quad -6\,a\,A_V + 3\,a\,F_1 + a\,F_2 = 0 \quad \rightarrow \quad A_V = \frac{3\,F_1 + F_2}{6} = \frac{7}{6}\,F\,,$$

$$\rightarrow: \quad A_H - F_2 = 0 \quad \rightarrow \quad A_H = F_2 = F\,.$$

Then we pass an imaginary section through the members 4-6 (Fig. 6.12c). The unknown force S_4 follows from the moment equation about point I (intersection of the action lines of the forces S_5 and S_6) of the free-body diagram on the left-hand side of Fig. 6.12c:

$$\overset{\curvearrowleft}{\mathrm{I}}: \quad 2\,a\,S_4 + 2\,a\,A_H - 3\,a\,A_V = 0$$

$$\rightarrow \quad \underline{\underline{S_4}} = \frac{1}{2}(3\,A_V - 2\,A_H) = \underline{\underline{\frac{3}{4}\,F}}\,.$$

The corresponding moment equation for the free-body diagram on the right-hand side may be used as a check:

$$\overset{\curvearrowleft}{\mathrm{I}}: \quad -2\,a\,S_4 + 3\,a\,B \;-\; a\,F_2 = 0$$

$$\rightarrow \quad S_4 = \frac{1}{2}(3\,B - F_2) = \frac{3}{4}\,F\,.$$

6.4 Supplementary Problems 6.4

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

Example 6.4 The truss shown in Fig. 6.13 carries the two forces $F_1 = F$ and $F_2 = 3F$. E6.4

Calculate the forces in the members 1, 2 and 3.

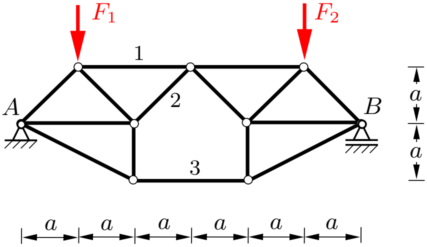


Fig. 6.13

Results: see (**B**) $S_1 = -2F/3\,, \quad S_2 = -\sqrt{2}\,F/3\,, \quad S_3 = F.$

Example 6.5 Determine the forces in the members 1, 2 and 3 of the truss shown in Fig. 6.14. E6.5

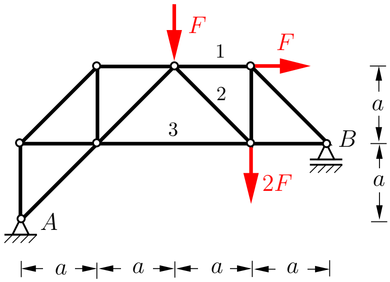


Fig. 6.14

Results: see (**A**) $S_1 = -3F/2\,, \quad S_2 = -\sqrt{2}\,F/2\,, \quad S_3 = 3F.$

E6.6

Example 6.6 The structure in Fig. 6.15 consists of a hinged beam AB and five bars. It is subjected to a line load q_0.

Determine the forces in the bars.

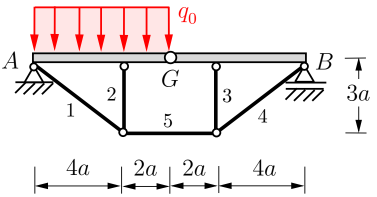


Fig. 6.15

Results: see (**B**) $S_1 = S_4 = 15\, q_\circ a/4\,, \quad S_2 = S_3 = -9q_\circ a/4\,,$
$S_5 = 3q_\circ a.$

E6.7

Example 6.7 Determine the forces in the members 1-7 of the truss shown in Fig. 6.16.

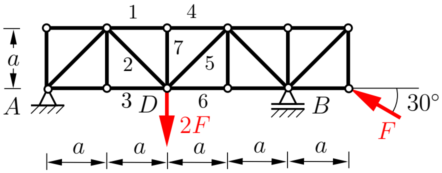


Fig. 6.16

Results: see (**A**) $S_1 = S_4 = -\dfrac{9}{4}\,F\,, \quad S_2 = \dfrac{9}{8}\sqrt{2}\,F\,, \quad S_7 = 0\,,$

$$S_3 = \left(\frac{9}{8} - \frac{\sqrt{3}}{2}\right)F\,, \quad S_5 = \frac{7}{8}\sqrt{2}\,F\,, \quad S_6 = \left(\frac{11}{8} - \frac{\sqrt{3}}{2}\right)F.$$

E6.8

Example 6.8 The truss shown in Fig. 6.17 carries the forces $F_1 = 10\,\text{kN}$ and $F_2 = 20\,\text{kN}$.

Calculate the forces in all the members.

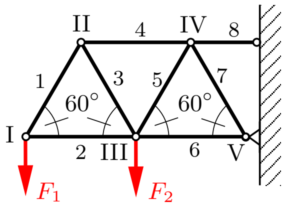


Fig. 6.17

Results: see (**A**) $S_1 = S_4 = -S_3 = 11.6\,\text{kN}\,, \quad S_2 = -5.8\,\text{kN}\,,$
$S_5 = -S_7 = 34.6\,\text{kN}\,, \quad S_6 = -28.9\,\text{kN}\,, \quad S_8 = 46.2\,\text{kN}.$

E6.9

Example 6.9 A rope is guided around the smooth pin C of the truss shown in Fig. 6.18. One end of the rope is connected to a rigid wall; the other end carries a box of weight W.

Calculate the force S_{11} in member 11.

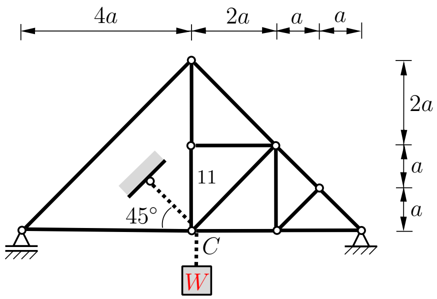


Fig. 6.18

Result: $S_{11} = (1 - \sqrt{2}/2)\, W\,.$

E6.10

Example 6.10 The truss shown in Fig. 6.19 carries the forces $F_1 = 10$ kN and $F_2 = 20$ kN (given: $a = 3$ m).

Calculate the forces in the members 6 and 8.

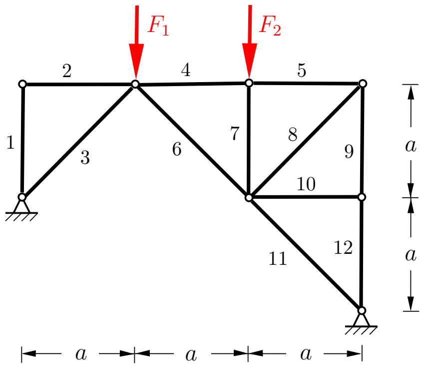


Fig. 6.19

Results: $S_6 = 0\,, \quad S_8 = 10\sqrt{2}$ kN.

E6.11

Example 6.11 Determine the support reactions and the forces in the members of the space truss shown in Fig. 6.20.

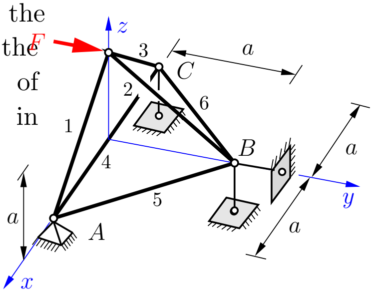


Fig. 6.20

Results: see (**A**)

$$A_x = A_y = 0\,, \quad A_z = C_z = -F/2\,,$$
$$B_z = -B_y = F\,, \quad S_1 = S_3 = F/\sqrt{2}\,,$$
$$S_2 = -\sqrt{2}\,F, \quad S_4 = -F/2\,, \quad S_5 = S_6 = 0.$$

E6.12

Example 6.12 Calculate the forces in all the members of the space truss in Fig. 6.21.

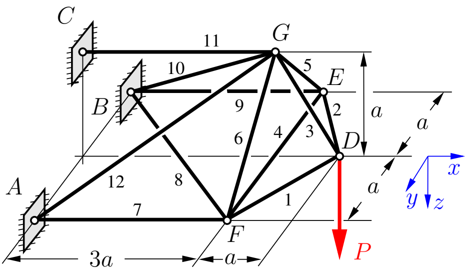


Fig. 6.21

Results: see (**A**)

$$S_1 = S_2 = -P/\sqrt{2}\,, \quad S_3 = \sqrt{2}\,P\,,$$
$$S_4 = -S_7 = -S_9 = P/2\,, \quad S_5 = S_6 = S_8 = 0\,,$$
$$S_{10} = S_{12} = -\sqrt{11}\,P/2\,, \quad S_{11} = 4\,P.$$

6.5 Summary

- A truss is a structure that consists of straight members connected at joints.
- A truss is statically determinate if the forces in the members and at the supports can be determined from the equilibrium conditions. This is the case if the number of the unknown forces equals the number of the independent equilibrium conditions and the system is rigid.
- A truss is kinematically determinate if it is completely constrained against motion. It is kinematically indeterminate if it is nonrigid and therefore can undergo a finite or an infinitesimal motion.
- The internal forces in the members and the support reactions can be determined with the method of joints:
  - ◇ Draw free-body diagrams of all the joints. Introduce the external loads, the support reactions and the forces in the members. Assume that the force in each member is a tensile force.
  - ◇ Write down the equilibrium conditions for the joints (2 equations for a plane truss, 3 equations for a spatial truss).
  - ◇ Solve the system of equations.
  - ◇ The system of equations has a unique solution if the determinant of the matrix of the coefficients is not equal to zero. Then the truss is statically and kinematically determinate.
- It is usually more practical to apply the method of sections instead of the method of joints if only several forces are to be determined.

Chapter 7

Beams, Frames, Arches

7

7 Beams, Frames, Arches

Objectives: Beams are among the most important elements in structural engineering. In this chapter it is explained how the internal forces in a beam can be made accessible to calculation.

The *normal force*, the *shear force* and the *bending moment* are introduced. Students will learn how to determine these quantities with the aid of the conditions of equilibrium. In addition, they will learn how to correctly apply the differential relationships between external loading and internal forces.

7.1 Stress Resultants

Beams are slender structural members that offer resistance to bending. They are among the most important elements in engineering. In this chapter the internal forces in structures composed of beams are analyzed. Knowledge of these internal forces is important in order to be able to determine the load-bearing capacity of a beam, to compute the area of the cross-section required to sustain a given load, or to compute the deformation (see Volume 2). For the sake of simplicity, the following discussion is limited to statically determinate plane problems, as indicated in Fig. 7.1a.

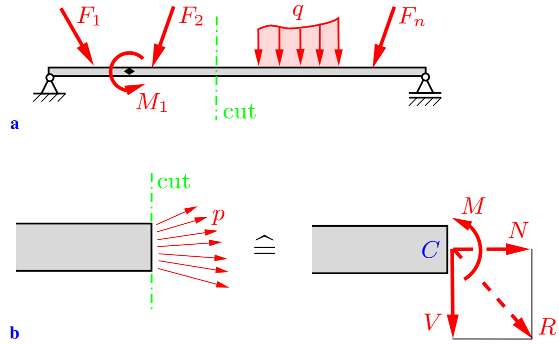


Fig. 7.1

According to Section 1.4, the internal forces in a beam can be made visible and thus accessible to calculation with the aid of a free-body diagram. Accordingly, we pass an imaginary section perpendicularly to the axis of the beam. The internal forces p (forces per unit area) acting at the cross-section are distributed across the cross-sectional area (Fig. 7.1b). Their intensity is called *stress* (see Volume 2). The actual distribution of the forces across the cross-section is unknown; it will be determined in Volume 2, Chapter 4. However, it was shown in Section 3.1.3 that any force system can be replaced by a resultant force R acting at an arbitrary point C and a corresponding couple $M^{(C)}$. When carrying this out, we choose the centroid C of the cross-sectional area as the reference point of the reduction. The reason for this particular choice will become apparent in Volume 2. In the following, we adopt the common practice of omitting the superscript C that refers to the reference point: instead of $M^{(C)}$, we simply write M. The resultant force R is resolved into its components N (normal to the cross-section, in the direction of the axis of the beam) and V (in the cross section, orthogonal to the axis of the beam). The quantities N, V and M are called the *stress resultants*. In particular,

> N is called the *normal force*, V is the *shear force* and M is the *bending moment*.

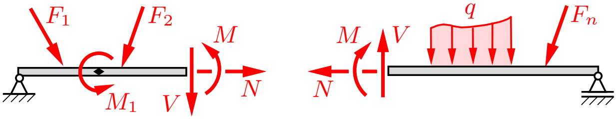


Fig. 7.2

In order to determine the stress resultants, the beam may be divided by a cut into two segments (method of sections). A free-body diagram of each part of the beam will include all of the forces acting on the respective part, i.e., the applied loads (forces and couples), the support reactions and the stress resultants acting at the cut sections. Because of Newton's third law (action equals reaction) they act in opposite directions at the two faces of the segments of the beam (compare Fig. 7.2). Since each part of the beam is in equilibrium, the three conditions of equilibrium for either part can be used to compute the three unknown stress resultants.

Before we can provide examples for the determination of the stress resultants, a sign convention must be introduced. Consider the two adjoining portions of the same beam shown in Fig. 7.3. The coordinate x coincides with the direction of the axis of the beam and points to the right; the coordinate z points downward. Accordingly, the y-axis is directed out of the x, z-plane (right-hand system, see Appendix A.1). By cutting the beam, a left-hand face and a right-hand face are obtained (see Fig. 7.3). They are characterized by a normal vector $\boldsymbol{n}$ that points outward from the interior of the beam. If the vector $\boldsymbol{n}$ points in the positive (negati-

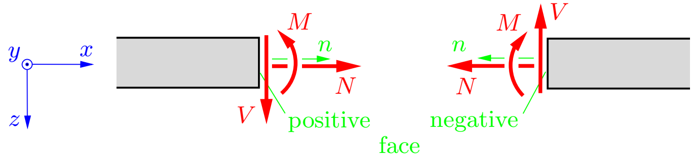


Fig. 7.3

ve) direction of the x-axis, the corresponding face is called positive (negative). The following sign convention is adopted:

> *Positive* stress resultants at a *positive (negative)* face point in the *positive (negative)* directions of the coordinates.

Here, the bending moment M has to be interpreted as a moment vector pointing in the direction of the y-axis (positive direction according to the right-hand rule). Fig. 7.3 shows the stress resultants with their positive directions. In the following examples, we shall strictly adhere to this sign convention. It should be noted, however, that different sign conventions exist.

In the case of a horizontal beam, very often only the x-coordinate is given. Then it is understood that the z-axis points downward. Sometimes it is convenient to use a coordinate system where the x-axis points to the left (instead of to the right) with the z-axis again pointing downward (compare Example 7.4). Then the y-axis is directed into the plane of the paper. In this case, only the positive direction of the shear force V according to Fig. 7.3 is reversed, the positive directions of M and N remain unchanged.

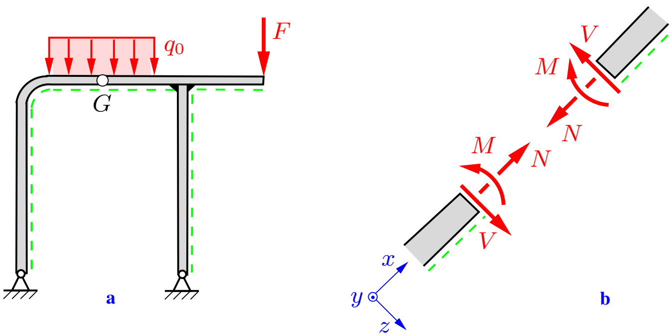


Fig. 7.4

The sign convention for frames and arches may be introduced by drawing a dashed line at one side of each part of the system (Fig. 7.4a). The side with the dashed line can then be interpreted as the "underneath side" of the respective part and the coordinate system can be chosen as the one for a beam: x-axis in the direction of the dashed line, z-axis toward the dashed line ("downward"). Fig. 7.4b shows the stress resultants with their positive directions.

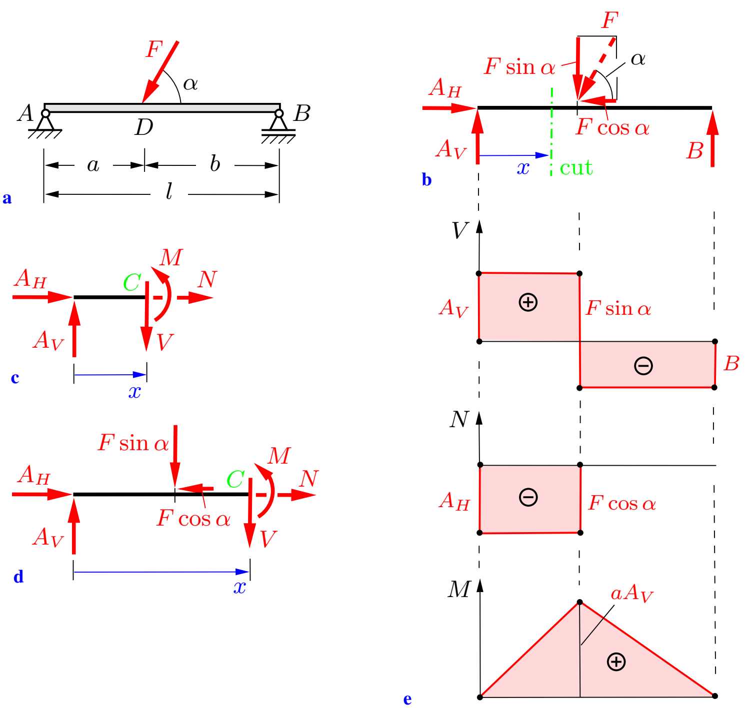


Fig. 7.5

We will now determine the stress resultants for the simply supported beam shown in Fig. 7.5a. First, the support reactions are computed from the equilibrium conditions for the free-body diagram of the beam as a whole (Fig. 7.5b). With $F_V = F \sin\alpha$ and $F_H = F\cos\alpha$ we obtain

$$\begin{aligned}
\rightarrow: &\qquad A_H - F_H = 0 \quad \rightarrow \quad A_H = F_H\,,\\
\overset{\curvearrowleft}{A}: &\qquad l\,B - a\,F_V = 0 \quad \rightarrow \quad B = \frac{a}{l}\,F_V\,,\\
\overset{\curvearrowleft}{B}: &\quad -\,l\,A_V + b\,F_V = 0 \quad \rightarrow \quad A_V = \frac{b}{l}\,F_V\,.
\end{aligned}$$

In a second step, a coordinate system is chosen and the beam is divided into two parts by a cut at an arbitrary position x between the points A and D ($0 < x < a$, Fig. 7.5b). The free-body diagram of the left-hand segment of the beam is depicted in Fig. 7.5c. We shall always represent the stress resultants at the cut section with their positive directions. If the analysis yields a negative value for a stress resultant, the resultant acts in opposite direction in reality. The equilibrium conditions for this part of the beam yield

$$\begin{aligned}
\rightarrow: &\qquad A_H + N = 0 \quad \rightarrow \quad N = -\,A_H = -\,F_H\,,\\
\uparrow: &\qquad A_V - V = 0 \quad \rightarrow \quad V = A_V = \frac{b}{l}\,F_V\,,\\
\overset{\curvearrowright}{C}: &\quad x\,A_V - M = 0 \quad \rightarrow \quad M = x\,A_V = x\,\frac{b}{l}\,F_V\,.
\end{aligned}$$

Now we cut the beam at an arbitrary position x between points D and B ($a < x < l$). The free-body diagram of the left-hand part is depicted in Fig. 7.5d. From the equilibrium conditions we obtain

$$\begin{aligned}
\rightarrow: &\qquad A_H - F_H + N = 0 \ \rightarrow \ N = F_H - A_H = 0\,,\\
\uparrow: &\qquad A_V - F_V - V = 0 \ \rightarrow \ V = A_V - F_V = \frac{b-l}{l}\,F_V\\
& \qquad\qquad\qquad\qquad\qquad\qquad = -\,\frac{a}{l}\,F_V = -\,B\,,\\
\overset{\curvearrowright}{C}: &\ x\,A_V - (x-a)F_V - M = 0 \ \rightarrow \ M = x\,A_V - (x-a)F_V\\
& \qquad\qquad\qquad\qquad\qquad\qquad = \left(1 - \frac{x}{l}\right) a\,F_V\,.
\end{aligned}$$

It should be noted that the equilibrium conditions for the corresponding right-hand parts yield the same results for the stress resultants. Usually, the part of the beam with a smaller number of forces will be chosen, since it allows for a simpler calculation of the results.

The stress resultants are functions of the coordinate x; they are shown graphically in Fig. 7.5e. These graphs are called the *shear-force*, *normal-force* and *bending-moment diagrams*, respectively. The shear-force and normal-force diagrams display a jump discontinuity at point D (point of application of the external force F). The jumps have the magnitude of the components F_V and F_H, respectively. The bending-moment diagram shows a slope discontinuity (kink) at D. The maximum bending moment is located at D. It is usually the most important value in the design of a beam (see Volume 2).

This example shows that, in order to determine the stress resultants, the beam may be sectioned at an arbitrary position x between two concentrated loads (external loads or support reactions). The process of sectioning has to be performed for each such span. Since there are discontinuities at the points of application of a concentrated load, these points should not be chosen for a section (compare however Section 7.2.5).

7.2

7.2 Stress Resultants in Straight Beams

Beams are usually subjected to forces perpendicular to their axes. If there are no components of forces (external forces or support reactions) in the direction of the axis of a beam, the normal force vanishes: $N = 0$. In the following subsections, we shall concentrate on such problems.

7.2.1 Beams under Concentrated Loads

In order to determine the stress resultants V and M we choose a coordinate system and imagine the beam cut at an arbitrary position x. The stress resultants are represented with their positive directions in the free-body diagrams; they can be computed from the equilibrium conditions for either portion of the beam. The results of the analysis are usually presented in a shear-force and a bending-moment diagram.

As an alternative to this elementary method, there exists another method to determine the stress resultants. It is based on the differential relationships between the load and the stress resultants and will be presented in the Sections 7.2.2 to 7.2.4.

For the sake of simplicity, we restrict the discussion in the following to beams that are subjected to concentrated loads and to couples. As an example we consider the simply supported beam shown in Fig. 7.6a.

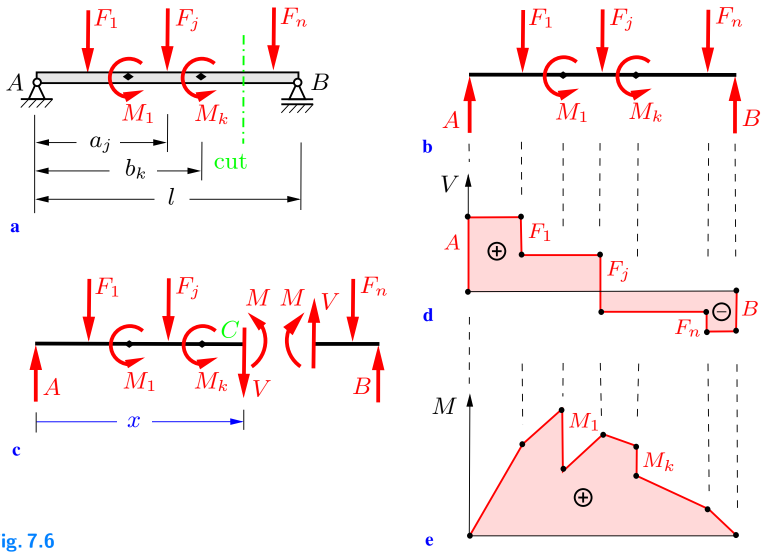


Fig. 7.6

The support reactions are obtained from the equilibrium conditions for the free-body diagram of the beam as a whole (Fig. 7.6b):

$$\overset{\frown}{A}: \; l\,B - \sum a_i\,F_i + \sum M_i = 0 \;\rightarrow\; B = \frac{1}{l}\left[\sum a_i\,F_i - \sum M_i\right],$$

$$\overset{\frown}{B}: \; -l\,A + \sum (l-a_i)F_i + \sum M_i = 0$$

$$\rightarrow\; A = \frac{1}{l}\left[\sum (l-a_i)F_i + \sum M_i\right].$$

Now, let us imagine the beam cut at an arbitrary position x (Fig. 7.6c). Since the normal force is equal to zero, it is not shown in the free-body diagram. The equilibrium conditions for the left-hand portion of the beam,

$$\uparrow: \quad A - \sum F_i - V = 0\,,$$

$$\stackrel{\curvearrowleft}{C}: \quad -x\,A + \sum (x - a_i)F_i + \sum M_i + M = 0\,,$$

yield the shear force and the bending moment:

$$V = A - \sum F_i\,, \tag{7.1}$$

$$M = x\,A - \sum (x - a_i)F_i - \sum M_i\,. \tag{7.2}$$

The summations in (7.1) and (7.2) include only the forces F_i and the couples M_i acting at the left-hand portion of the beam.

The stress resultants can also be computed with the equilibrium conditions for the right-hand part of the beam. Usually, the part of the beam that allows for a simpler calculation of the results is chosen.

The shear-force diagram is shown in Fig. 7.6d. According to (7.1), the shear force is piecewise constant. The shear-force diagram has jump discontinuities at the points of application of the concentrated forces F_i. The magnitude of a jump is equal to the magnitude of the respective force.

According to (7.2), the bending moment (Fig. 7.6e) is a piecewise linear function of the coordinate x. The diagram displays slope discontinuities (kinks) at the points of application of the forces F_i and jump discontinuities (magnitudes M_i) at the points of application of the external couples M_i. The supports A and B (hinge and roller support) cannot exert a moment. Therefore, the bending moment is zero at the end-points of a simply supported beam.

A relationship exists between the bending moment and the shear force. If the derivative of (7.2) with respect to x is calculated and (7.1) is applied, we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}x} = A - \sum F_i = V\,. \tag{7.3}$$

The slopes of the straight lines in the bending-moment diagram are thus given by the corresponding values of the shear force.

Example 7.1 The simply supported beam in Fig. 7.7a is subjected to the three forces $F_1 = F$, $F_2 = 2\,F$ and $F_3 = -F$. **E7.1**

Draw the shear-force and bending-moment diagrams.

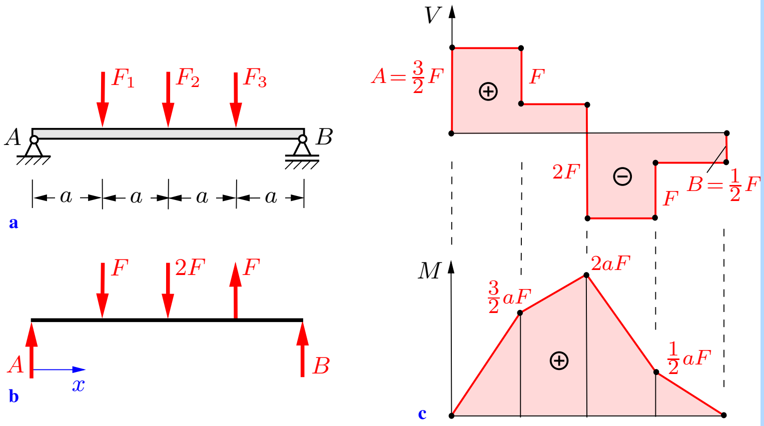


Fig. 7.7

Solution As a first step we draw the free-body diagram of the entire beam (Fig. 7.7b) and compute the support reactions A and B. The equilibrium conditions yield

$$\overset{\curvearrowleft}{A}: \quad -a\,F - 2\,a\,2\,F + 3\,a\,F + 4\,a\,B = 0 \quad \rightarrow \quad B = \frac{1}{2}\,F\,,$$

$$\overset{\curvearrowleft}{B}: \quad -4\,a\,A + 3\,a\,F + 2\,a\,2\,F - a\,F = 0 \quad \rightarrow \quad A = \frac{3}{2}\,F\,.$$

In the next step, we pass imaginary sections at arbitrary positions x in each span between two concentrated loads. The equilibrium of the forces for the left-hand parts of the beam yields the shear force (the corresponding free-body diagrams are not shown in Fig. 7.7):

$$V = A = 3F/2 \qquad \text{for} \quad 0 < x < a\,,$$

$$V = A - F = F/2 \qquad \text{for} \quad a < x < 2\,a\,,$$

$$V = A - F - 2\,F = -\,3F/2 \qquad \text{for} \quad 2\,a < x < 3\,a\,,$$

$$V = A - F - 2\,F + F = -\,F/2 \qquad \text{for} \quad 3\,a < x < 4\,a\,.$$

The bending moment is obtained from the equilibrium of the moments:

$$M = xA = \tfrac{3}{2}\,xF \qquad \text{for} \quad 0 \le x \le a\,,$$

$$M = xA - (x-a)F = (a + \tfrac{1}{2}\,x)F \qquad \text{for} \quad a \le x \le 2a\,,$$

$$M = xA - (x-a)F - (x-2a)2F = (5a - \tfrac{3}{2}\,x)F \qquad \text{for} \quad 2a \le x \le 3a\,,$$

$$\begin{aligned} M &= xA - (x-a)F - (x-2a)2F + (x-3a)F \\ &= (2a - \tfrac{1}{2}\,x)F \qquad \text{for} \quad 3a \le x \le 4a\,. \end{aligned}$$

It should be noted that it would have been simpler to use the right-hand parts instead of the left-hand parts for $x > 2a$.

The shear-force and bending-moment diagrams are shown in Fig. 7.7c. The bending-moment diagram has a positive (negative) slope in the regions of a positive (negative) shear force.

The values of the stress resultants at the right-hand end of the beam ($x = 4\,a$) may serve as checks:

– the shear force has a jump discontinuity of magnitude B (the diagram should close at $x = 4a$),
– the bending moment is zero (roller support at the end of the beam).

Since the beam in Fig. 7.7a is simply supported, the results for V and M are already given by (7.1) and (7.2).

E7.2 **Example 7.2** Determine the shear-force and bending-moment diagrams for the cantilever beam shown in Fig. 7.8a.

Solution First, the support reactions are calculated with the aid of the free-body diagram of the whole beam (Fig. 7.8b). The equilibrium conditions yield

$$\uparrow: \quad A - F = 0 \quad \rightarrow \quad A = F\,,$$

$$\overset{\curvearrowleft}{A}: \quad -M_A + M_0 - l\,F = 0 \quad \rightarrow \quad M_A = M_0 - l\,F = l\,F\,.$$

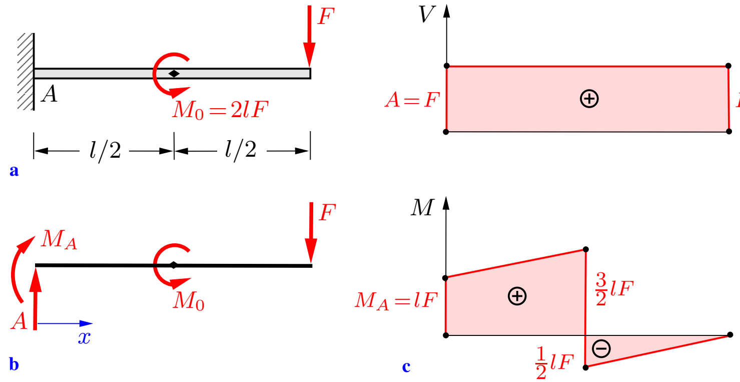


Fig. 7.8

In order to obtain the shear force, we section the beam at an arbitrary position x (since there is no concentrated load acting between both ends of the beam, only one region of x needs to be considered). The shear force follows from the equilibrium condition of the forces in the vertical direction:

$$\underline{\underline{V}} = A = \underline{\underline{F}} \quad \text{for} \quad 0 < x < l\,.$$

Because of the couple M_0 at the center of the beam, two regions of x must be considered when the bending moment is calculated. Accordingly, we pass a cut in the region given by $0 < x < l/2$ and another one in the span $l/2 < x < l$. The equilibrium of the moments yields

$$\underline{\underline{M}} = M_A + x\,A = \underline{\underline{(l+x)F}} \qquad \text{for} \quad 0 < x < \frac{l}{2}\,,$$

$$\underline{\underline{M}} = M_A + x\,A - M_0 = \underline{\underline{(x-l)F}} \quad \text{for} \quad \frac{l}{2} < x \le l\,.$$

The shear-force and bending-moment diagrams are shown in Fig. 7.8c. The shear force is constant over the entire length of the beam. The bending moment is a linear function of the coordinate x and has a jump of magnitude $M_0 = 2\,l\,F$ at the point of application ($x = l/2$) of the external couple M_0. The two straight lines in the regions $x < l/2$ and $x > l/2$ have the same slope since the shear force has the same value in both regions (see (7.3)).

It should be noted that in this example the support reactions need not be calculated in order to determine the shear force and the bending moment. If we apply the equilibrium conditions to the right-hand portions of the cut beam, the stress resultants are obtained immediately. The support reactions A and M_A can then be found from the diagrams: they are equal to the shear force and the bending moment, respectively, at $x = 0$.

E7.3

Example 7.3 Draw the diagrams of the stress resultants for the beam shown in Fig. 7.9a.

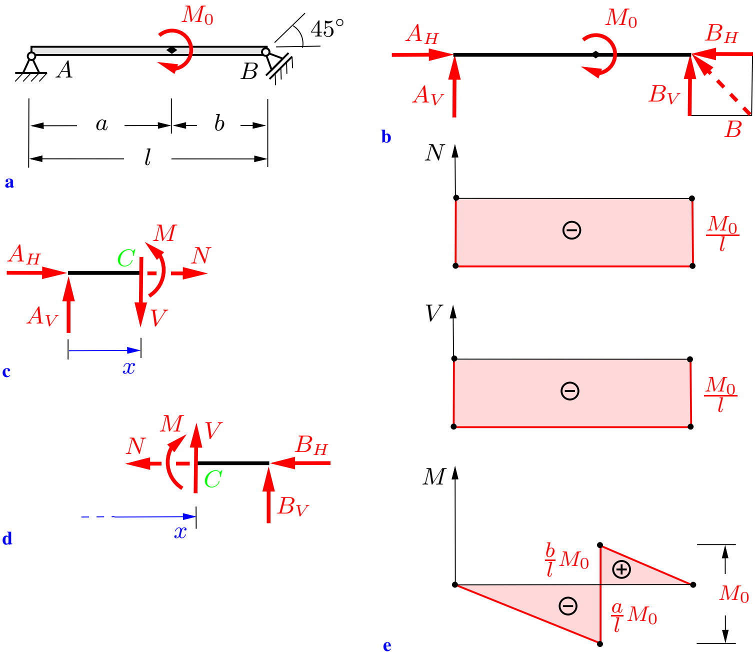


Fig. 7.9

Solution The support reactions follow from the conditions of equilibrium for the whole beam (Fig. 7.9b). With the components $B_H = B_V$ of the reaction force B (action line under 45° against the vertical axis) we obtain

$$\overset{\curvearrowright}{A}: \quad M_0 - l\,B_V = 0 \quad \rightarrow \quad B_V = B_H = \frac{M_0}{l}\,,$$
$$\overset{\curvearrowright}{B}: \quad l\,A_V + M_0 = 0 \quad \rightarrow \quad A_V = -\,\frac{M_0}{l}\,,$$
$$\rightarrow: \quad A_H - B_H = 0 \quad \rightarrow \quad A_H = \frac{M_0}{l}\,.$$

The forces A and B represent a couple with the moment M_0.

Since there is a discontinuity in the bending moment, two regions of x must be considered to describe it for the entire beam. First, we imagine the beam being cut in the region $x < a$. The equilibrium conditions for the left-hand portion of the beam (Fig. 7.9c) yield

$$\rightarrow: \quad A_H + N = 0 \quad \rightarrow \quad \underline{\underline{N}} = -\,A_H = \underline{\underline{-\,\frac{M_0}{l}}}\,,$$
$$\uparrow: \quad A_V - V = 0 \quad \rightarrow \quad \underline{\underline{V}} = A_V = \underline{\underline{-\,\frac{M_0}{l}}}\,,$$
$$\overset{\curvearrowleft}{C}: \quad -x\,A_V + M = 0 \quad \rightarrow \quad \underline{\underline{M}} = x\,A_V = \underline{\underline{-\,\frac{x}{l}\,M_0}}\,.$$

In order to obtain the stress resultants to the right of the applied couple, the beam is sectioned at a position $x > a$. Now it is simpler to use the free-body diagram of the right-hand portion of the beam (Fig. 7.9d). Notice the positive directions of the stress resultants (negative face!) in this diagram. The conditions of equilibrium yield

$$\leftarrow: \quad N + B_H = 0 \quad \rightarrow \quad \underline{\underline{N}} = -\,B_H = \underline{\underline{-\,\frac{M_0}{l}}}\,,$$
$$\uparrow: \quad V + B_V = 0 \quad \rightarrow \quad \underline{\underline{V}} = -\,B_V = \underline{\underline{-\,\frac{M_0}{l}}}\,,$$
$$\overset{\curvearrowright}{C}: \quad M - (l - x)B_V = 0 \quad \rightarrow \quad \underline{\underline{M}} = (l - x)B_V = \underline{\underline{\frac{l - x}{l}\,M_0}}\,.$$

The stress resultants are shown graphically in Fig. 7.9e. The moment diagram has a jump discontinuity at the point of application of the applied couple ($x = a$). The two straight lines in the regions $x < a$ and $x > a$ have the same slope, since the shear force has the same value in both regions (see also (7.3)). The normal force is induced by the support reactions and is constant in the entire beam.

It should be noted that the support reactions are independent of the point of application of the applied couple. The bending-moment diagram, however, depends on this point.

7.2.2 Relationship between Loading and Stress Resultants

A relationship between the shear force V and the bending moment M for beams under concentrated loads has already been given in (7.3). This result is now extended to a beam that is subjected to a load $q(x)$ (force per unit length) that varies continuously over the length of the beam (Fig. 7.10a). Fig. 7.10b shows the free-body diagram of a beam element of infinitesimal length $\mathrm{d}x$. The load q may be considered to be constant over the length $\mathrm{d}x$ since the effect of any change of q disappears in the limit $\mathrm{d}x \to 0$ (compare Section 4.1). The distributed load is replaced by its resultant $\mathrm{d}F = q\,\mathrm{d}x$. The shear force V and the bending moment M act at the location x. They are drawn in their positive directions (negative face!). Proceeding to the location $x + \mathrm{d}x$, the stress resultants have changed by an amount $\mathrm{d}V$ and $\mathrm{d}M$, respectively, to the values $V + \mathrm{d}V$ and $M + \mathrm{d}M$. They are also shown with their positive directions. The conditions of equilibrium yield

$$\uparrow: \quad V - q\,\mathrm{d}x - (V + \mathrm{d}V) = 0 \quad \to \quad q\,\mathrm{d}x + \mathrm{d}V = 0\,, \qquad (7.4)$$

$$\overset{\curvearrowleft}{C}: \quad -M - \mathrm{d}x\,V + \frac{\mathrm{d}x}{2}\,q\,\mathrm{d}x + M + \mathrm{d}M = 0$$

$$\to \quad -V\,\mathrm{d}x + \mathrm{d}M + \frac{1}{2}\,q\,\mathrm{d}x \cdot \mathrm{d}x = 0\,. \qquad (7.5)$$

From (7.4) we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q\,. \qquad (7.6)$$

Thus, the slope of the shear-force diagram is equal to the negative intensity of the applied loading.

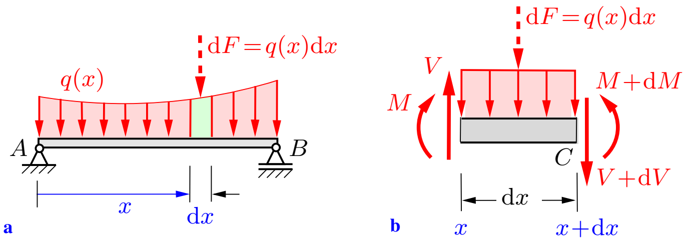


Fig. 7.10

The term in (7.5) containing $\mathrm{d}x \cdot \mathrm{d}x$ is "small of higher order" compared with $\mathrm{d}x$ or $\mathrm{d}M$. Therefore, it vanishes in the limit $\mathrm{d}x \to 0$ and (7.5) reduces to

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V\,. \tag{7.7}$$

The derivative of the bending moment with respect to x is equal to the shear force. This result is already known from the case of beams under concentrated forces (see (7.3)). It should be noted that the algebraic signs in (7.6) and (7.7) result from the sign convention for the stress resultants.

If (7.7) is differentiated and (7.6) introduced, we obtain

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -\,q\,. \tag{7.8}$$

The differential relations (7.6) and (7.7) may be used, for example, to determine qualitatively the stress resultants and can also serve as checks. For example, if $q = \text{const}$, then the shear force is a linear function of x according to (7.6) and the bending moment is represented by a quadratic parabola according to (7.7). The Table at the end of this section shows the relations between the loading and the stress resultants for several simple examples of q.

The most important value for the design of a beam is usually the magnitude of the maximum bending moment. The corresponding coordinate x for a relative maximum is characterized by the condition of a vanishing shear force (compare (7.7)). It should be noted, however, that the absolute maximum may be located at an end point of the beam or of a span (position of discontinuity).

q	V	M
0	constant	linear
constant	linear	quadratic parabola
linear	quadratic parabola	cubic parabola

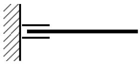
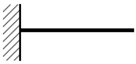
7.2.3 Integration and Boundary Conditions

The relations (7.6) and (7.7) may also be used to quantitatively determine the stress resultants for a given load $q(x)$. If we integrate (7.6) and (7.7), we obtain

$$V = -\int q \, \mathrm{d}x + C_1 \,, \tag{7.9}$$

$$M = \int V \, \mathrm{d}x + C_2 \,. \tag{7.10}$$

The constants of integration C_1 and C_2 can be calculated if the functions (7.9) and (7.10) for the stress resultants are evaluated at positions of x where the values of V or M are known. The corresponding equations are called *boundary conditions*. The following Table shows which stress resultant vanishes at a given support at the end of a beam. Statements $V \neq 0$ and/or $M \neq 0$ cannot be

support		V	M
pin		$\neq 0$	**0**
parallel motion		**0**	$\neq 0$
sliding sleeve		$\neq 0$	$\neq 0$
fixed end		$\neq 0$	$\neq 0$
free end		**0**	**0**

(7.11)

used as boundary conditions.

In contrast to the method of sections (see Section 7.2.1), the support reactions do *not* have to be computed in order to determine the stress resultants V and M: they are automatically obtained through the integration. If, on the other hand, some support reactions are known in advance, they can also be used to determine the constants of integration.

To illustrate the procedure, let us consider the three beams shown in Figs. 7.11a–c. They are subjected to the same load but have different supports. With $q = q_0 = \text{const}$, Equations (7.9) and (7.10) yield

$$
\begin{aligned}
V &= -\, q_0\, x + C_1\,, \\
M &= -\,\frac{1}{2}\, q_0\, x^2 + C_1\, x + C_2
\end{aligned}
$$

for each beam.

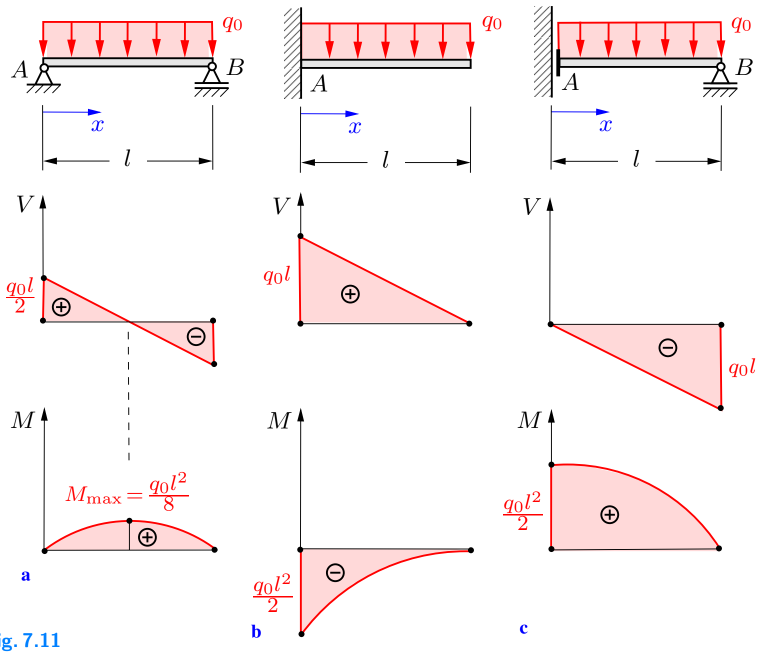


Fig. 7.11

The boundary conditions

$$\text{a) } M(0)=0\,,\quad M(l)=0\,,\qquad \text{b) } V(l)=0\,,\quad M(l)=0\,,\qquad \text{c) } V(0)=0\,,\quad M(l)=0$$

and thus the resulting constants of integration

$$\text{a) } 0=C_2\,,\qquad \text{b) } 0=-\,q_0\,l+C_1\,,\quad \text{c) } 0=C_1\,,$$

$$\text{a), b), c)}\quad 0=-\tfrac{1}{2}\,q_0\,l^2+C_1\,l+C_2$$

$$\to\quad \begin{cases} C_1=\frac{1}{2}\,q_0\,l\,,\\ C_2=0\,,\end{cases}\qquad \begin{cases} C_1=q_0\,l\,,\\ C_2=-\,\frac{1}{2}\,q_0\,l^2\,,\end{cases}\qquad \begin{cases} C_1=0\,,\\ C_2=\frac{1}{2}\,q_0\,l^2\end{cases}$$

are different for each of the cases a) to c). This leads to the following stress resultants (Figs. 7.11a–c):

$$\text{a) } V=\frac{1}{2}\,q_0\,l\left(1-2\,\frac{x}{l}\right),\qquad \text{b) } V=q_0\,l\left(1-\frac{x}{l}\right),$$

$$M=\frac{1}{2}\,q_0\,l^2\,\frac{x}{l}\left(1-\frac{x}{l}\right),\qquad M=-\,\frac{1}{2}\,q_0\,l^2\left(1-\frac{x}{l}\right)^2,$$

$$\text{c) } V=-\,q_0\,x\,,$$

$$M=\frac{1}{2}\,q_0\,l^2\left[1-\left(\frac{x}{l}\right)^2\right].$$

The results are written in such a way that the terms in parentheses are dimensionless. The maximum bending moment $M_{\max}=q_0\,l^2/8$ for the simply supported beam is located at the center of the beam ($x=l/2$: $V=0$).

The support reactions can be taken from the diagrams; they are equal to the values of the stress resultants at the endpoints of the beams:

$$\text{a) } A=V(0)=\tfrac{1}{2}\,q_0\,l\,,\qquad \text{b) } A=V(0)=q_0\,l\,,$$

$$B=-\,V(l)=\tfrac{1}{2}\,q_0\,l\,,\qquad M_A=M(0)=-\,\tfrac{1}{2}\,q_0\,l^2\,,$$

$$\text{c) } M_A=M(0)=\tfrac{1}{2}\,q_0\,l^2\,,$$

$$B=-\,V(l)=q_0\,l\,.$$

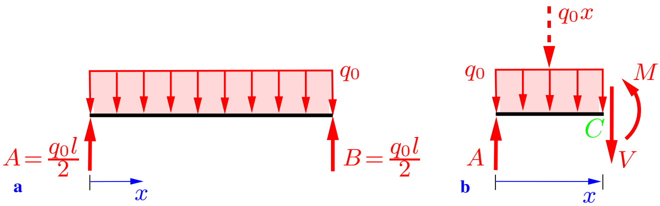


Fig. 7.12

The shear-force and bending-moment diagrams can also be obtained with the method of sections. In order to explain this method in the case of a distributed load, the beam in Fig. 7.11a is reconsidered. In a first step, the support reactions have to be calculated (Fig. 7.12a). Next, the beam is sectioned at an arbitrary position x (Fig. 7.12b). Then we replace the distributed load q_0 by its resultant $q_0 x$ (notice that the distributed load must not be replaced by its resultant for the entire beam, i.e., before the beam is divided by the cut). The equilibrium conditions for the left-hand portion of the beam (Fig. 7.12b) yield the stress resultants

$$\uparrow: \quad A - q_0\, x - V = 0$$

$$\to \quad V = A - q_0\, x = \frac{1}{2}\, q_0\, l \left(1 - 2\,\frac{x}{l}\right) ,$$

$$\overset{\curvearrowleft}{C}: \quad -x\, A + \frac{1}{2}\, x\, q_0\, x + M = 0$$

$$\to \quad M = x\, A - \frac{1}{2}\, q_0\, x^2 = \frac{1}{2}\, q_0\, l^2\, \frac{x}{l} \left(1 - \frac{x}{l}\right) .$$

This method is recommended only if the resultant of the distributed load acting at the cut beam and its line of action can easily be given.

Example 7.4 The cantilever beam in Fig. 7.13a is subjected to a triangular line load. **E7.4**

Determine the stress resultants through integration.

Solution In the coordinate system given in Fig. 7.13a, the load is described by the equation $q(x) = q_0(l - x)/l$. Integration leads to

$$V(x) = \frac{q_0}{2l}\,(l - x)^2 + C_1 \,, \quad M(x) = -\frac{q_0}{6l}\,(l - x)^3 + C_1\, x + C_2 \,,$$

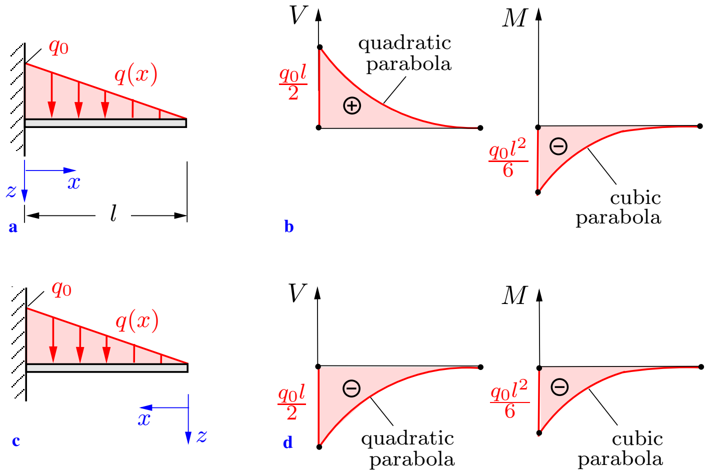


Fig. 7.13

(compare (7.9) and (7.10)). The boundary conditions $V(l) = 0$ and $M(l) = 0$ yield the constants of integration: $C_1 = 0$ and $C_2 = 0$. Hence,

$$\underline{\underline{V(x) = \frac{1}{2}\, q_0\, l \left(1 - \frac{x}{l}\right)^2}}\,, \quad \underline{\underline{M(x) = -\frac{1}{6}\, q_0\, l^2 \left(1 - \frac{x}{l}\right)^3}}\,,$$

see Fig. 7.13b. The shear-force diagram has a vanishing slope at $x = l$ since the distributed load is zero at the free end of the beam $(V'(l) = -q(l) = 0)$. Analogously, the bending-moment diagram has a vanishing slope at $x = l$ since the shear force is zero at the free end $(M'(l) = V(l) = 0)$.

In this example, it would have been more convenient to introduce the coordinate system according to Fig. 7.13c. Here, the x-axis points to the left. In this coordinate system, the triangular load is described by the simpler equation $q(x) = q_0\, x/l$, and the integration and boundary conditions $V(0) = 0$ and $M(0) = 0$ yield the stress resultants (Fig. 7.13d)

$$\underline{\underline{V(x) = -\frac{1}{2}\, q_0\, l \left(\frac{x}{l}\right)^2}}\,, \quad \underline{\underline{M(x) = -\frac{1}{6}\, q_0\, l^2 \left(\frac{x}{l}\right)^3}}\,.$$

Note that with this choice of the coordinate system the algebraic sign of the shear force is reversed (see Section 7.1).

It should also be noted that the stress resultants can easily be determined with the method of sections.

7.2.4 Matching Conditions

Frequently, the load $q(x)$ is given through different functions of x in different portions of the beam (instead of one function for the entire length of the beam). In this case, the beam must be divided into several regions and the integration of (7.6) and (7.7) must be performed separately in each of these regions.

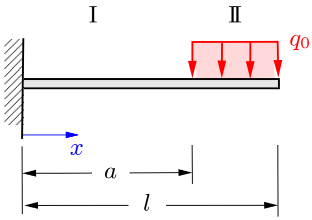


Fig. 7.14

To illustrate the method, the cantilever beam shown in Fig. 7.14 is considered. The load is given by

$$q(x) = \begin{cases} 0 & \text{for } 0 \le x < a\,, \\ q_0 & \text{for } a < x < l\,. \end{cases}$$

Integration in region I ($0 \le x < a$) and region II ($a < x < l$) yields

$$\begin{aligned} \text{I:}\quad q_{\text{I}} &= 0\,, & \text{II:}\quad q_{\text{II}} &= q_0\,, \\ V_{\text{I}} &= C_1\,, & V_{\text{II}} &= -\,q_0\,x + C_3\,, \\ M_{\text{I}} &= C_1\,x + C_2\,, & M_{\text{II}} &= -\,\tfrac{1}{2}\,q_0\,x^2 + C_3\,x + C_4\,. \end{aligned} \tag{7.12}$$

The *two* boundary conditions

$$V_{\text{II}}(l) = 0\,, \quad M_{\text{II}}(l) = 0 \tag{7.13}$$

are not sufficient to determine the *four* constants $C_1 - C_4$ of integration. Therefore, two additional equations must be used. They describe the behaviour of the stress resultants at the point $x = a$ (point of transition from region I to region II). These equations are called *matching conditions*.

The beam is not subjected to a concentrated force or to an external couple at $x = a$. Therefore, there are no jumps in the shear-force or bending-moment diagrams (since $\mathrm{d}V/\mathrm{d}x = -q$ and q has a jump at $x = a$, the shear-force diagram has a jump in the slope). Hence, the matching conditions are

$$V_{\mathrm{I}}(a) = V_{\mathrm{II}}(a)\,, \quad M_{\mathrm{I}}(a) = M_{\mathrm{II}}(a)\,. \tag{7.14}$$

Introducing (7.12) into the boundary conditions (7.13) and the matching conditions (7.14) yields the constants of integration:

$$\begin{aligned} C_1 &= q_0(l-a)\,, \quad & C_2 &= -\tfrac{1}{2}\,q_0(l^2-a^2)\,, \\ C_3 &= q_0\,l\,, & C_4 &= -\tfrac{1}{2}\,q_0\,l^2\,. \end{aligned}$$

As a second example, we consider the beam in Fig. 7.15. It is subjected to a concentrated force F at $x = a$ and an external couple M_0 at $x = b$. Then the shear-force diagram exhibits a jump of magnitude F at $x = a$, whereas the bending-moment diagram is continuous at this point (it has a jump in the slope). The matching conditions for the values of the stress resultants at the transition from region I to region II are therefore

$$V_{\mathrm{II}}(a) = V_{\mathrm{I}}(a) - F\,, \quad M_{\mathrm{II}}(a) = M_{\mathrm{I}}(a)\,. \tag{7.15}$$

The external couple M_0 at $x = b$ causes a jump in the bending-moment diagram; the shear-force diagram is continuous. Hence, the matching conditions at the transition from region II to region

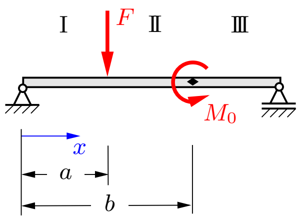
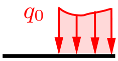
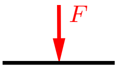
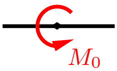


Fig. 7.15

III are given by

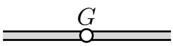
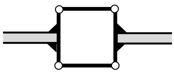
$$V_{\mathrm{III}}(b) = V_{\mathrm{II}}(b)\,, \quad M_{\mathrm{III}}(b) = M_{\mathrm{II}}(b) - M_0\,. \tag{7.16}$$

The following Table shows which loads cause jumps in the stress resultants or in the slopes of the diagrams.

Load	V	M
	kink	–
	jump	kink
	–	jump

If a beam has to be divided into n regions, the integration in each region yields a total of $2n$ constants of integration. They can be determined from $2n-2$ matching conditions and 2 boundary conditions.

Let us now consider structures composed of several beams that are connected by joints. Since an internal pin cannot exert a moment, the bending moment is zero at the pin: $M = 0$. The shear force is in general not equal to zero at this point: $V \neq 0$. In contrast, at a parallel motion $V = 0$ and $M \neq 0$ are valid. These statements concerning the stress resultants at a connecting member are displayed in the following Table.

connecting member	V	M
	$\neq 0$	$\mathbf{0}$
	$\mathbf{0}$	$\neq 0$

(7.17)

If an internal pin or a parallel motion exists in a structure, a matching condition is replaced by *one* of the following conditions: bending moment or shear force equal to zero.

A division into regions at a connecting member is not necessary if no concentrated force or external couple acts on the element. Also, no division into regions is required in the case of a distributed load that is described by the same function to the left and to the right of the element.

If a beam must be divided into many regions, a system of equations with many unknowns has to be solved in order to obtain the constants of integration. Therefore, this method is recommended only for beams with very few regions.

E7.5

Example 7.5 A simply supported beam is subjected to a concentrated force and a triangular line load (Fig. 7.16a).

Determine the stress resultants.

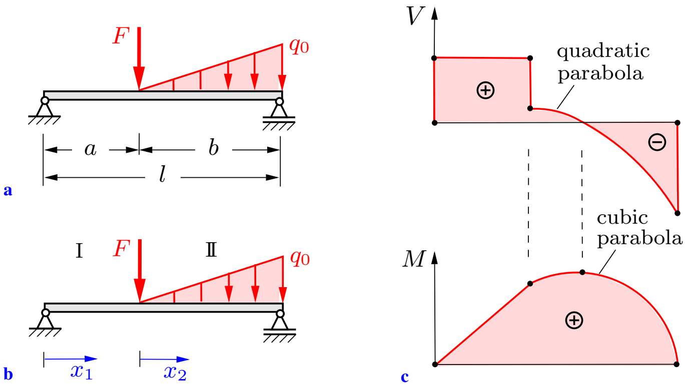


Fig. 7.16

Solution The beam is divided into the two regions I and II according to Fig. 7.16b. We use the coordinate x_1 in the region I and coordinate x_2 in region II (instead of the coordinate x for the entire length of the beam). Integration according to (7.9) and (7.10) in both regions yields

$$\begin{aligned} &\text{I:} \quad q_\text{I} = 0\,, && \text{II:} \quad q_\text{II} = q_0\,\frac{x_2}{b}\,,\\ &V_\text{I} = C_1\,, && V_\text{II} = -\,q_0\,\frac{x_2^2}{2\,b} + C_3\,,\\ &M_\text{I} = C_1\,x_1 + C_2\,, && M_\text{II} = -\,q_0\,\frac{x_2^3}{6\,b} + C_3\,x_2 + C_4\,. \end{aligned}$$

The boundary conditions and the matching conditions are

$$M_{\mathrm{I}}(x_1=0)=0\,,\quad M_{\mathrm{II}}(x_2=b)=0\,,$$

$$V_{\mathrm{II}}(x_2=0)=V_{\mathrm{I}}(x_1=a)-F\,,\ M_{\mathrm{II}}(x_2=0)=M_{\mathrm{I}}(x_1=a)\,.$$

They lead, after some calculation, to the constants of integration:

$$C_1=\Big(\frac{1}{6}\,q_0\,b+F\Big)\frac{b}{l}\,,\qquad C_2=0\,,$$

$$C_3=\Big(\frac{1}{6}\,q_0\,b-\frac{a}{b}\,F\Big)\frac{b}{l}\,,\quad C_4=\Big(\frac{1}{6}\,q_0\,b+F\Big)\frac{a\,b}{l}\,.$$

Hence, we obtain the stress resultants (Fig. 7.16c)

$$\underline{\underline{V_{\mathrm{I}}}}=\Big(\frac{1}{6}\,q_0\,b+F\Big)\frac{b}{l}\,,$$

$$\underline{\underline{V_{\mathrm{II}}}}=-\,q_0\,\frac{x_2^2}{2\,b}+\Big(\frac{1}{6}\,q_0\,b-\frac{a}{b}\,F\Big)\frac{b}{l}\,,$$

$$\underline{\underline{M_{\mathrm{I}}}}=\Big(\frac{1}{6}\,q_0\,b+F\Big)\frac{b}{l}\,x_1\,,$$

$$\underline{\underline{M_{\mathrm{II}}}}=-\,q_0\,\frac{x_2^3}{6\,b}+\Big(\frac{1}{6}\,q_0 b-\frac{a}{b}\,F\Big)\frac{b}{l}\,x_2+\Big(\frac{1}{6}\,q_0\,b+F\Big)\frac{a\,b}{l}\,.$$

Example 7.6 Determine the stress resultants for the compound beam in Fig. 7.17a. **E7.6**

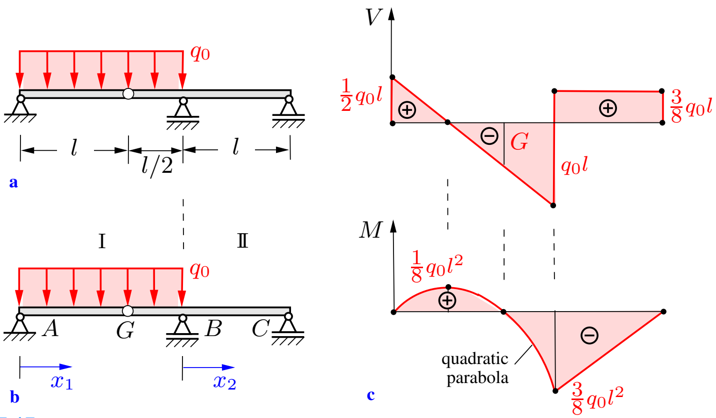


Fig. 7.17

Solution The beam must be divided into two regions at the location of support B (support reaction B and discontinuous load q!). A division into regions at the internal pin G is not required since $q = q_0$ to the left and to the right of the pin. We use the coordinates x_1 and x_2 in regions I and II, respectively (Fig. 7.17b). Integration leads to

$$\begin{aligned} \text{I:}\quad q_{\mathrm{I}} &= q_0\,, & \text{II:}\quad q_{\mathrm{II}} &= 0,\\ V_{\mathrm{I}} &= -\,q_0\,x_1 + C_1\,, & V_{\mathrm{II}} &= C_3\,,\\ M_{\mathrm{I}} &= -\,\tfrac{1}{2}\,q_0\,x_1^2 + C_1\,x_1 + C_2\,, & M_{\mathrm{II}} &= C_3\,x_2 + C_4\,. \end{aligned}$$

The four conditions

$$M_{\mathrm{I}}(x_1=0)=0\,,\quad M_{\mathrm{II}}(x_2=l)=0\quad\text{(boundary conditions)}\,,$$

$$M_{\mathrm{I}}(x_1=\tfrac{3}{2}l)=M_{\mathrm{II}}(x_2=0)\quad\text{(matching condition)}\,,$$

$$M_{\mathrm{I}}(x_1=l)=0\quad\text{(zero bending moment at internal pin } G)$$

yield the four constants of integration:

$$C_1=\tfrac{1}{2}\,q_0\,l\,,\quad C_2=0\,,\quad C_3=\tfrac{3}{8}\,q_0\,l\,,\quad C_4=-\,\tfrac{3}{8}\,q_0\,l^2\,.$$

Thus, we obtain the stress resultants (Fig. 7.17c)

$$\underline{\underline{V_{\mathrm{I}}}} = -\,q_0\,x_1 + \frac{1}{2}\,q_0\,l\,,\qquad \underline{\underline{V_{\mathrm{II}}}} = \frac{3}{8}\,q_0\,l\,,$$

$$\underline{\underline{M_{\mathrm{I}}}} = -\,\frac{1}{2}\,q_0\,x_1^2 + \frac{1}{2}\,q_0\,l\,x_1\,,\quad \underline{\underline{M_{\mathrm{II}}}} = \frac{3}{8}\,q_0\,l(x_2-l)\,.$$

As a check, the support reactions are taken from the shear-force diagram:

$$A = V_{\mathrm{I}}(x_1=0)=\tfrac{1}{2}\,q_0\,l\,,$$

$$B = V_{\mathrm{II}}(x_2=0)-V_{\mathrm{I}}(x_1=\tfrac{3}{2}\,l)=\tfrac{11}{8}\,q_0\,l\,,$$

$$C = -\,V_{\mathrm{II}}(x_2=l) = -\,\tfrac{3}{8}\,q_0\,l\,.$$

They are in equilibrium with the resulting external load $3\,q_0\,l/2$.

7.2.5 Pointwise Construction of the Diagrams

To apply the theory to practical problems, it is not always necessary to give the stress resultants as functions of x over the length of the beam. Frequently, it suffices to calculate the stress resultants at several specific points only. The values of the stress resultants at these points are then connected with the curves that are associated with the respective load.

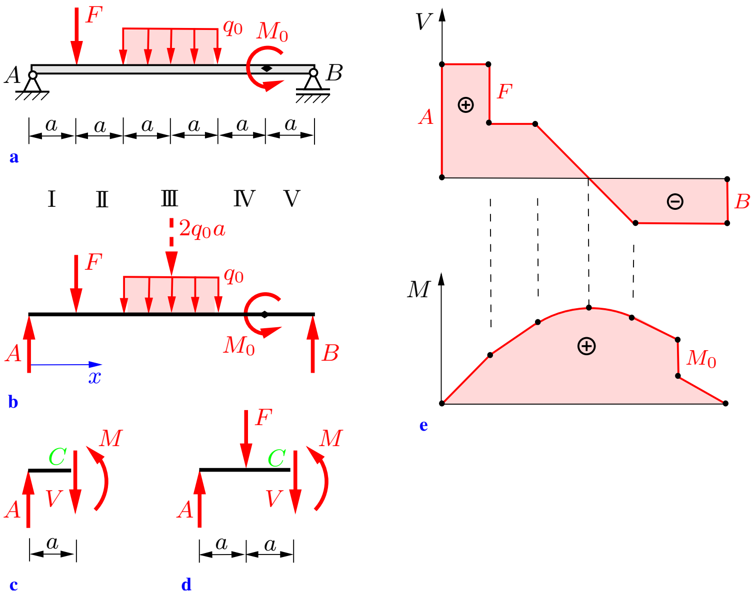


Fig. 7.18

To illustrate the method, the simply supported beam in Fig. 7.18a is considered. As a first step, we compute the support reactions with the conditions of equilibrium for the free-body diagram of the entire beam (Fig. 7.18b):

$$\overset{\curvearrowleft}{B}: \quad -6\,a\,A + 5\,a\,F + 3\,a\,2\,q_0\,a + M_0 = 0$$

$$\rightarrow \quad A = \frac{1}{6}\left(5\,F + 6\,q_0\,a + \frac{M_0}{a}\right),$$

$$\overset{\curvearrowleft}{A}: \quad -a\,F - 3\,a\,2\,q_0\,a + M_0 + 6\,a\,B = 0$$
$$\rightarrow \quad B = \frac{1}{6}\left(F + 6\,q_0\,a - \frac{M_0}{a}\right).$$

The stress resultants exhibit jumps, or jumps in the slopes, at $x = a$, $2\,a$, $4\,a$ and $5\,a$, respectively. At these specific points, the stress resultants are computed using the method of sections. If we cut the beam at $x = a$ immediately to the left of the force F (Fig. 7.18c), we obtain

$$\uparrow: \quad A - V = 0 \qquad \rightarrow \quad V(a) = A = \frac{1}{6}\left(5\,F + 6\,q_0\,a + \frac{M_0}{a}\right)$$
$$\text{to the left of } F\,,$$
$$\overset{\curvearrowleft}{C}: \quad -a\,A + M = 0 \quad \rightarrow \quad M(a) = a\,A$$
$$= \frac{1}{6}(5\,a\,F + 6\,q_0\,a^2 + M_0)\,.$$

A cut at $x = 2a$ (Fig. 7.18d) yields

$$\uparrow: \quad A - F - V = 0 \quad \rightarrow \quad V(2\,a) = \frac{1}{6}\left(-F + 6\,q_0\,a + \frac{M_0}{a}\right),$$
$$\overset{\curvearrowleft}{C}: \quad -2\,a\,A + a\,F + M = 0$$
$$\rightarrow \quad M(2\,a) = \frac{1}{3}(2\,a\,F + 6\,q_0\,a^2 + M_0)\,.$$

Similarly, we find

$$V(4\,a) = \frac{1}{6}\left(-F - 6\,q_0\,a + \frac{M_0}{a}\right),$$
$$M(4\,a) = \frac{1}{3}\,(a\,F + 6\,q_0\,a^2 + 2\,M_0)\,,$$
$$V(5\,a) = V(4\,a)\,,$$
$$M(5\,a) = \frac{1}{6}\,(a\,F + 6\,q_0\,a^2 - M_0) \quad \text{to the right of } M_0\,.$$

The distributed load is zero in regions I, II, IV and V. Therefore, the shear force is constant in each of these regions. In region III, the shear force varies linearly since $q = q_0 = \text{const}$. The shear-force diagram has a jump of magnitude F at $x = a$.

Accordingly, the bending moment varies linearly in the regions I, II, IV and V, and it is described by a quadratic parabola in region III (compare the Table in Section 7.2.2). The diagram shows a jump in the slope at $x = a$. Since the shear-force diagram is continuous at $x = 2\,a$ and $x = 4\,a$, the moment diagram has no jumps in the slopes at these points ($V = \mathrm{d}M/\mathrm{d}x$). The external couple M_0 causes a jump in the diagram at $x = 5\,a$.

Fig. 7.18e shows the diagrams of the stress resultants. The maximum bending moment is located at the position of the vanishing shear force.

Example 7.7 Draw the diagrams of the stress resultants for the structure in Fig. 7.19a ($a = 0.5\,\mathrm{m}$, $q_0 = 60\,\mathrm{kN/m}$, $F = 80\,\mathrm{kN}$, $M_0 = 10\,\mathrm{kNm}$). E7.7

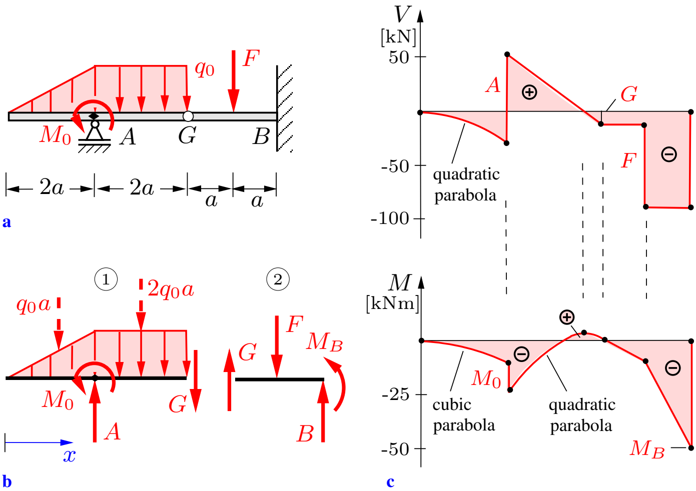


Fig. 7.19

Solution First, we compute the support reactions and the force in the internal pin G (the horizontal components are zero). The equilibrium conditions for the free-body diagrams of the two portions of the structure (Fig. 7.19b) yield

$$① \quad \overset{\curvearrowleft}{A}: \quad \frac{2}{3}\,a\,q_0\,a + M_0 - a\,2\,q_0\,a - 2\,a\,G = 0$$

$$\rightarrow \quad G = -\frac{2}{3} q_0 a + \frac{M_0}{2a} = -10\,\text{kN}\,,$$

$$\overset{\curvearrowleft}{G}: \quad \frac{8}{3} a\, q_0\, a + M_0 - 2\,a\,A + a\,2\,q_0\,a = 0$$

$$\rightarrow \quad A = \frac{7}{3} q_0 a + \frac{M_0}{2a} = 80\,\text{kN}\,,$$

$$② \ \overset{\curvearrowleft}{B}: \quad -2\,a\,G + a\,F + M_B = 0$$

$$\rightarrow \quad M_B = -\frac{4}{3} q_0 a^2 + M_0 - a\,F = -50\,\text{kNm}\,,$$

$$\uparrow: \quad G - F + B = 0$$

$$\rightarrow \quad B = \frac{2}{3} q_0 a - \frac{M_0}{2a} + F = 90\,\text{kN}\,.$$

The stress resultants at specific points of the beam are obtained using the method of sections:

$$V(2a) = \begin{cases} -q_0\,a = -30\,\text{kN} & \text{to the left of } A\,, \\ -q_0\,a + A = 50\,\text{kN} & \text{to the right of } A\,, \end{cases}$$

$$V(4a) = -q_0\,a + A - 2\,q_0\,a = -10\,\text{kN}\,,$$

$$V(5a) = \begin{cases} G = -10\,\text{kN} & \text{to the left of } F\,, \\ G - F = -90\,\text{kN} & \text{to the right of } F\,, \end{cases}$$

$$V(6a) = -B = -90\,\text{kN}\,,$$

$$M(2a) = \begin{cases} -\dfrac{2}{3} a\,q_0\,a = -10\,\text{kNm} & \text{to the left of } A\,, \\ -\dfrac{2}{3} a\,q_0\,a - M_0 = -20\,\text{kNm} & \text{to the right of } A\,, \end{cases}$$

$$M(5a) = a\,G = -5\,\text{kNm}\,,$$

$$M(6a) = M_B = -50\,\text{kNm}\,.$$

In addition, $V(0) = 0$, $M(0) = 0$ and $M(4\,a) = 0$.

The values of the stress resultants are now connected with the appropriate curves (straight lines or parabolas, Fig. 7.19c). Note that

– at $x = 0$, the quadratic parabola for V (because of $q(0) = 0$) and the cubic parabola for M (because of $V(0) = 0$) have horizontal slopes,
– at $x = 4\,a$, the slope of the moment diagram has no jump (because of the continuous shear force).

7.3 Stress Resultants in Frames and Arches

The methods for determining the stress resultants will now be generalized to frames and arches. Note that the differential relations derived in Section 7.2.2 can be applied to straight portions of a frame only; they are *not valid for arches.*

In this section, the discussion is limited to plane (i.e., coplanar) problems and focused on the pointwise construction of the stress-resultants diagrams. According to this method (see Section 7.2.5), the stress resultants are computed at specific points of the structure with the aid of the method of sections. The algebraic signs of the stress resultants are defined using dashed lines (Section 7.1). A frame, in general, experiences also a normal force, even if the external loads act perpendicularly to its members. Therefore, we will always calculate all three stress resultants: bending moment, shear force and normal force.

At the corners of frames where two straight beams are rigidly joined, the equilibrium conditions reveal how the stress resultants

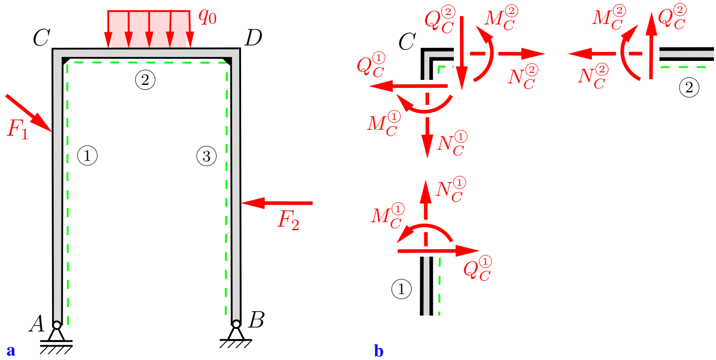


Fig. 7.20

are transferred. As an example, consider the externally unloaded rectangular corner C of the frame shown in Fig. 7.20a. If we free the corner by appropriate cuts, the equilibrium conditions yield (Fig. 7.20b)

$$N_C^{①} = -V_C^{②}\,, \quad V_C^{①} = N_C^{②}\,, \quad M_C^{①} = M_C^{②}\,. \tag{7.18}$$

Whereas the bending moment is transferred unaltered from part ① to part ②, the normal force becomes the shear force and the shear force becomes the normal force. If the beams are joined at a corner under an arbitrary angle, the transferred stress resultants depend on this angle.

E7.8 **Example 7.8** Determine the stress resultants for the frame in Fig 7.21a.

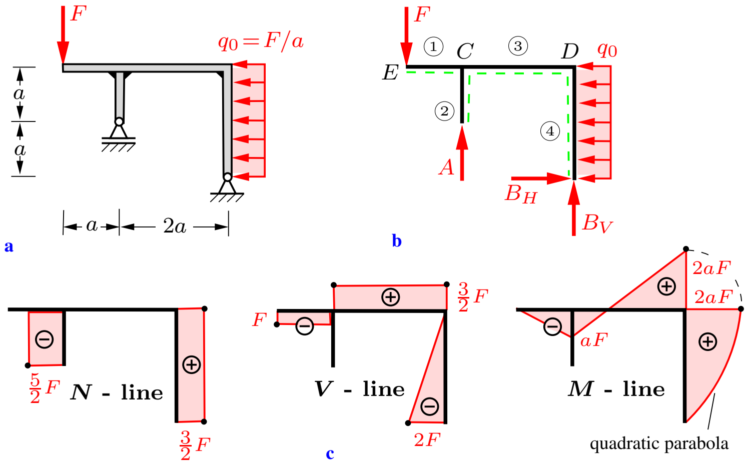


Fig. 7.21

Solution The support reactions are obtained from the equilibrium conditions for the frame as a whole (Fig. 7.21b):

$$A = \frac{5}{2}\,F\,, \quad B_V = -\frac{3}{2}\,F\,, \quad B_H = 2\,F\,.$$

In order to define the algebraic signs of the stress resultants, we introduce the dashed lines according to Fig. 7.21b. The following stress resultants are calculated using the method of sections:

$$N_C^{①} = 0\,, \qquad V_C^{①} = -\,F\,, \qquad M_C^{①} = -\,aF\,,$$

$$N_C^{②} = -\,A = -\,\frac{5}{2}\,F\,, \qquad V_C^{②} = 0\,, \qquad M_C^{②} = 0\,,$$

$$N_D^{③} = 0\,,\;\; V_D^{③} = -F + A = \frac{3}{2}\,F\,,\;\; M_D^{③} = -3\,a\,F + 2\,a\,A = 2\,a\,F.$$

The equilibrium conditions at the freed corner D, where parts ③ and ④ are rigidly connected, yield in analogy to (7.18)

$$N_D^{④} = V_D^{③} = \frac{3}{2}\,F\,, \quad V_D^{④} = -\,N_D^{③} = 0\,, \quad M_D^{④} = M_D^{③} = 2\,a\,F\,.$$

With the following values at the ends of the members

$$N_E^{①} = 0\,, \quad V_E^{①} = -\,F\,, \quad M_E^{①} = 0\,,$$
$$N_A^{②} = -\,A = -\,\frac{5}{2}\,F\,, \quad V_A^{②} = 0\,, \quad M_A^{②} = 0\,,$$
$$N_B^{④} = -\,B_V = \frac{3}{2}\,F\,, \quad V_B^{④} = -\,B_H = -\,2\,F\,, \quad M_B^{④} = 0$$

we obtain the stress resultants displayed in Fig. 7.21c.

The equilibrium conditions for the freed bifurcation point C (cuts adjacent to C) may serve as a check. As an example, we consider the stress resultants in the vertical direction showing that the equilibrium condition is fulfilled:

$$V_C^{①} - V_C^{③} - N_C^{②} = 0 \quad \rightarrow \quad -F - \frac{3}{2}\,F + \frac{5}{2}\,F = 0\,.$$

Example 7.9 Determine the stress resultants for the members of the structure in Fig. 7.22a. **E7.9**

Solution First we compute the support reactions. With the components $A_V = A_H$ of the reaction at A, the equilibrium conditions for the entire structure (Fig. 7.22b) yield

$$\overset{\curvearrowright}{B}: \quad 2a\,A_V + 2a\,A_H + 2a\,F = 0 \quad \rightarrow \quad A_V = A_H = -\,\tfrac{1}{2}\,F\,,$$

$$\rightarrow: \quad A_H + B_H = 0 \quad \rightarrow \quad B_H = \tfrac{1}{2}\,F\,,$$

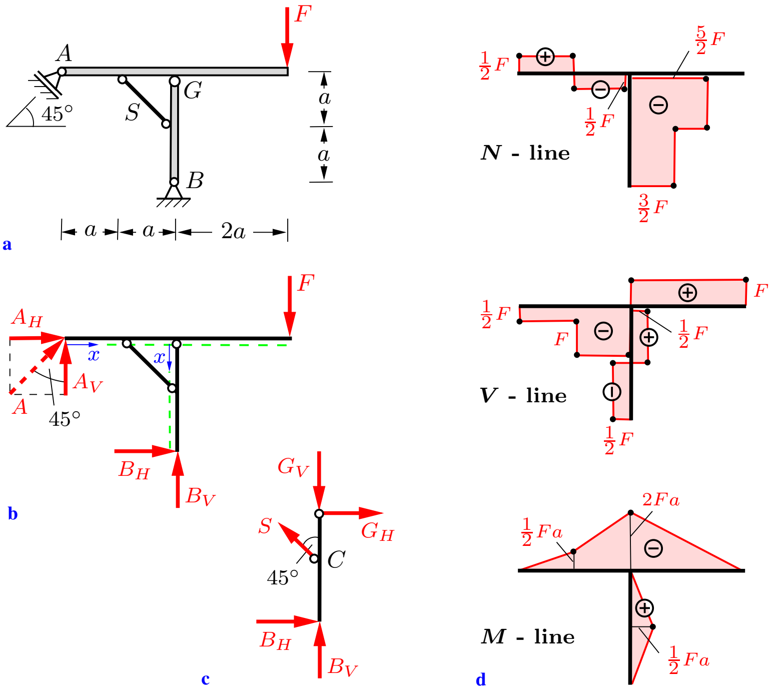


Fig. 7.22

$$\overset{\curvearrowleft}{A}: \quad 2a\, B_V + 2a\, B_H - 4a\, F = 0 \quad \rightarrow \quad B_V = \tfrac{3}{2}\, F\,.$$

Then the force in bar S (= normal force S) and the force in pin G (components G_H and G_V) are calculated using the equilibrium conditions for the vertical beam (Fig. 7.22c):

$$\overset{\curvearrowright}{C}: \qquad a\, G_H - a\, B_H = 0 \quad \rightarrow \quad G_H = \frac{F}{2}\,,$$

$$\overset{\curvearrowleft}{G}: \quad 2a\, B_H - a\,\frac{\sqrt{2}}{2}\, S = 0 \quad \rightarrow \quad S = \sqrt{2}\, F\,,$$

$$\uparrow: \quad B_V + \frac{\sqrt{2}}{2}\, S - G_V = 0 \quad \rightarrow \quad G_V = \frac{5}{2}\, F\,.$$

In order to define the algebraic signs of the stress resultants, we introduce the dashed lines according to Fig. 7.22b. The normal-force diagram and the shear-force diagram can be drawn without further computation (note the jumps in the diagrams due to the concentrated forces S, G_H and G_V). The bending-moment diagram can be constructed with the aid of the values at specific points (Fig. 7.22d).

Example 7.10 The circular arch in Fig. 7.23a is subjected to a concentrated force F. **E7.10**

Draw the diagrams of the stress resultants.

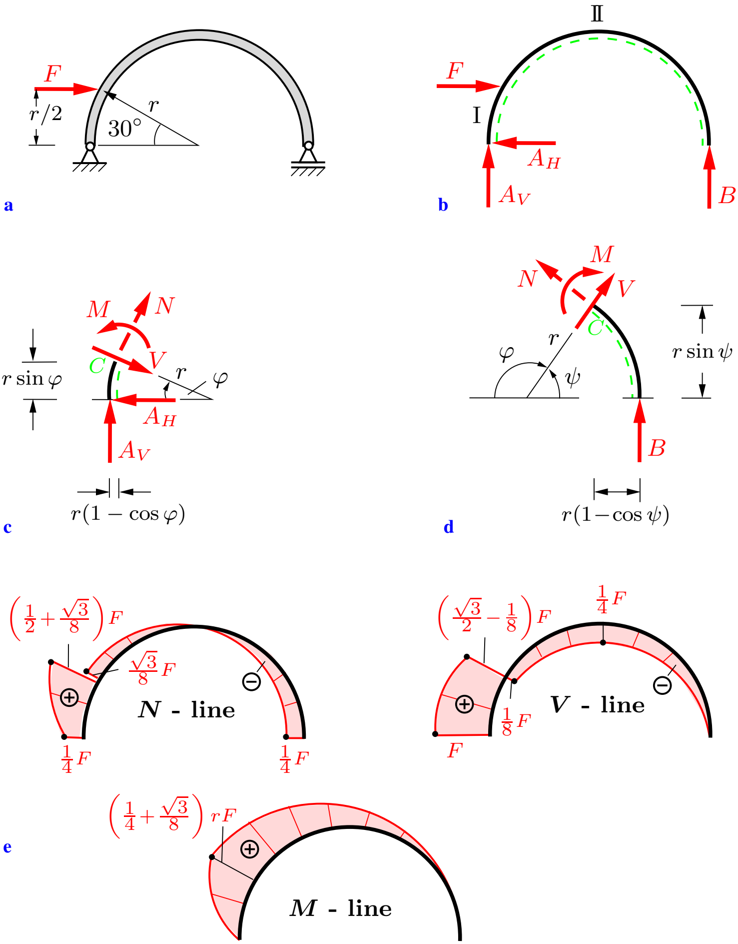


Fig. 7.23

Solution The reactions at the supports follow from the equilibrium conditions for the the entire arch (Fig. 7.23b):

$$B=\frac{1}{4}\,F\,,\quad A_V=-\,B=-\,\frac{1}{4}\,F\,,\quad A_H=F\,.$$

We introduce the dashed line and cut the arch at an arbitrary position φ in region I ($0<\varphi<30°$). The free-body diagram of the corresponding part of the arch is shown in Fig. 7.23c. The equilibrium conditions lead to the stress resultants:

$$\nearrow:\quad N+A_V\cos\varphi-A_H\sin\varphi=0$$

$$\rightarrow\quad \underline{\underline{N}}=\left(\sin\varphi+\tfrac{1}{4}\cos\varphi\right)F\,,$$

$$\searrow:\quad V-A_V\sin\varphi-A_H\cos\varphi=0$$

$$\rightarrow\quad \underline{\underline{V}}=\left(\cos\varphi-\tfrac{1}{4}\sin\varphi\right)F\,,$$

$$\overset{\curvearrowleft}{C}:\quad M-r\sin\varphi\,A_H-r(1-\cos\varphi)A_V=0$$

$$\rightarrow\quad \underline{\underline{M}}=\left(\sin\varphi+\tfrac{1}{4}\cos\varphi-\tfrac{1}{4}\right)r\,F\,.$$

Since the arch is sectioned at an arbitrary position φ, these equations describe the variations of the stress resultants in region I. Similarly, introducing the angle $\psi=\pi-\varphi$, the stress resultants in region II ($30°<\varphi<180°$) are obtained from the equilibrium conditions for the free-body diagram in Fig. 7.23d:

$$\nwarrow:\quad N+B\cos\psi=0$$

$$\rightarrow\quad \underline{\underline{N}}=-\,\tfrac{1}{4}\,F\cos\psi=\tfrac{1}{4}\,F\cos\varphi\,,$$

$$\nearrow:\quad V+B\sin\psi=0$$

$$\rightarrow\quad \underline{\underline{V}}=-\,\tfrac{1}{4}\,F\sin\psi=-\,\tfrac{1}{4}\,F\sin\varphi\,,$$

$$\overset{\curvearrowleft}{C}:\quad -\,M+r(1-\cos\psi)B=0$$

$$\rightarrow\quad \underline{\underline{M}}=\tfrac{1}{4}\,(1-\cos\psi)r\,F=\tfrac{1}{4}\,(1+\cos\varphi)r\,F\,.$$

The stress resultants are drawn perpendicularly to the axis of the arch in Fig. 7.23e. The jumps $\Delta N=F/2$ in the normal force and $\Delta V=\sqrt{3}\,F/2$ in the shear force at $\varphi=30°$ are equal to the components of F tangential and orthogonal to the arch, respectively.

7.4 Stress Resultants in Spatial Structures

The discussion to this point has been limited to plane problems, i.e., plane structures that are subjected to loads acting in the same plane. Now we extend the investigation of the stress resultants to *spatial structures* and to three-dimensional load vectors.

As a simple example, consider the cantilever beam in Fig. 7.24a, which is subjected to the concentrated forces $\boldsymbol{F}_j$ and the external couples $\boldsymbol{M}_j$ acting in arbitrary directions. As in the case of plane problems, we cut the beam at an arbitrary position x (compare Section 7.1). The internal forces acting in the cross-section are replaced by the resultant $\boldsymbol{R}$ (acting at the centroid C of the cross-section) and the corresponding couple $\boldsymbol{M}^{(C)}$. Again, the superscript C is omitted: instead of $\boldsymbol{M}^{(C)}$ we simply write $\boldsymbol{M}$ (Fig. 7.24b). Vectors $\boldsymbol{R}$ and $\boldsymbol{M}$ have, in general, components in all *three* directions of the coordinate system:

$$\boldsymbol{R} = \begin{pmatrix} N \\ V_y \\ V_z \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} M_T \\ M_y \\ M_z \end{pmatrix}. \tag{7.19}$$

The component of the resultant $\boldsymbol{R}$ in the x-direction (*normal* to

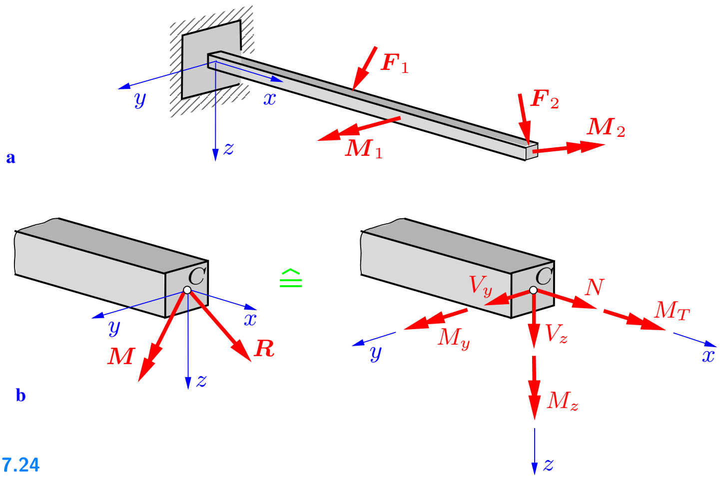


Fig. 7.24

the cross-section) is the normal force N (compare Section 7.1). The components in the y- and z-directions (*perpendicular* to the axis of the beam) are the shear forces V_y and V_z, respectively.

The component M_T of the moment $\boldsymbol{M}$ in the x-direction is called *torque*. In an elastic beam, the torque causes a twist about the longitudinal axis (see Volume 2). The components in the y- and z-directions are the bending moments M_y and M_z, respectively.

The sign convention for the stress resultants coincides with the sign convention for plane problems (see Section 7.1): positive stress resultants at a positive (negative) face point in the positive (negative) directions of the coordinates. Fig. 7.24b shows the stress resultants with their positive directions. In structures composed of several members having different directions, it is helpful to use a different coordinate system for each member.

The stress resultants are determined using the method of sections, i.e., from the equilibrium conditions for a portion of the structure.

E7.11 **Example 7.11** The spatial structure in Fig. 7.25a is subjected to a concentrated force F.

Determine the stress resultants.

Solution We cut the structure at arbitrary positions in the regions ① – ③. If we apply the equilibrium conditions to the cut-off portions of the structure, the stress resultants can be determined without having calculated the support reactions.

To define the algebraic signs of the stress resultants in the three regions, we use three coordinate systems (Fig. 7.25a). First, a cut is passed at an arbitrary position x in region ①. The stress resultants are drawn in the free-body diagram (Fig. 7.25b) with their positive directions (negative face!). The equilibrium conditions (3.34) yield

$$\sum F_{iy} = 0: \qquad F - V_y = 0 \quad \rightarrow \quad \underline{\underline{V_y = F}}\,,$$

$$\sum M_{iz} = 0: (c-x)F - M_z = 0 \quad \rightarrow \quad \underline{\underline{M_z = (c-x)F}}\,.$$

The other stress resultants are zero in region ①.

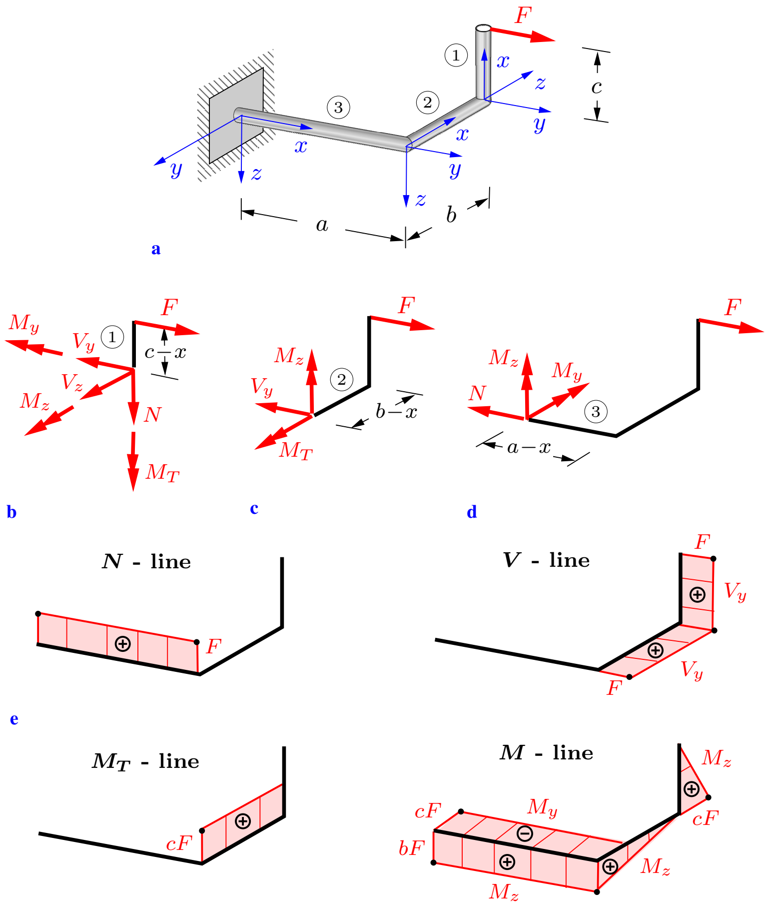


Fig. 7.25

The stress resultants in regions ② and ③ are also obtained using the method of sections. The free-body diagram for region ② (Fig. 7.25c) leads to

$$\sum F_{iy} = 0: \quad F - V_y = 0 \quad \rightarrow \quad \underline{\underline{V_y = F}},$$

$$\sum M_{ix} = 0: \quad c\,F - M_T = 0 \quad \rightarrow \quad \underline{\underline{M_T = c\,F}},$$

$$\sum M_{iz} = 0: \quad (b-x)F - M_z = 0 \quad \rightarrow \quad \underline{\underline{M_z = (b-x)F}},$$

and for region ③ (Fig. 7.25d) we obtain

$$\sum F_{ix} = 0: \quad F - N = 0 \quad \rightarrow \quad \underline{\underline{N = F}},$$

$$\sum M_{iy} = 0: \quad -c\,F - M_y = 0 \quad \rightarrow \quad \underline{\underline{M_y = -c\,F}},$$

$$\sum M_{iz} = 0: \quad b\,F - M_z = 0 \quad \rightarrow \quad \underline{\underline{M_z = b\,F}}.$$

The stress resultants that are equal to zero are omitted in these free-body diagrams for the sake of clarity of the figures.

The stress resultants are presented in Fig. 7.25e. The support reactions are equal to the values of the stress resultants at the fixed end.

E7.12 **Example 7.12** The clamped circular arch in Fig. 7.26a is subjected to a concentrated force F that acts perpendicularly to the plane of the arch.

Determine the stress resultants.

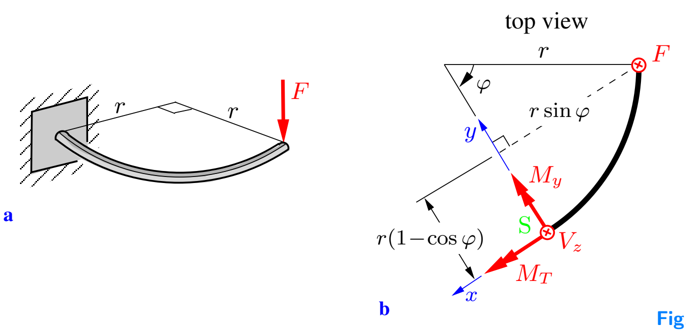


Fig. 7.26

Solution We section the arch at an arbitrary position φ and consider the cut-off portion of the arch (Fig. 7.26b). To define the algebraic signs of the stress resultants, a local x, y, z-coordinate system is used. At the position S given by φ, the shear force V_z, the torque M_T and the bending moment M_y are introduced with their positive directions (positive face). The other stress resultants are zero; they are omitted in the free-body diagram.

The equilibrium conditions yield

$$\sum F_{iz} = 0: \qquad V_z + F = 0 \quad \rightarrow \quad \underline{\underline{V_z = -F}},$$

$$\sum M_{ix}^{(S)} = 0: \qquad M_T + F\,r(1 - \cos\varphi) = 0$$

$$\rightarrow \quad \underline{\underline{M_T = -F\,r(1 - \cos\varphi)}},$$

$$\sum M_{iy}^{(S)} = 0: \qquad M_y + F\,r\sin\varphi = 0$$

$$\rightarrow \quad \underline{\underline{M_y = -F\,r\sin\varphi}}.$$

7.5 Supplementary Problems

7.5

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E7.13

Example 7.13 A crab on two wheels can move on a beam (weight negligible). Its weight W is linearly distributed as indicated in Fig. 7.27.

Determine the value $\xi = \xi^*$ for which the bending moment attains its maximum value $M_{\max}$. Calculate $M_{\max}$.

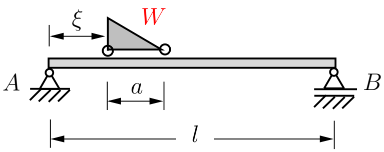


Fig. 7.27

Results: see (**B**) $\quad \xi^* = (3l - a)/6\,, \qquad M_{\max} = \dfrac{1}{36}\Big(3 - \dfrac{a}{l}\Big)^2 W l.$

E7.14

Example 7.14 Determine the bending moment for a cantilever subjected to a sinusoidal line load (Fig. 7.28).

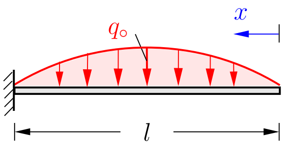


Fig. 7.28

Result: see (**A**) $\quad M(x) = -\dfrac{q_\circ l^2}{\pi}\Big(\dfrac{x}{l} - \sin\dfrac{\pi x}{l}\Big).$

E7.15

Example 7.15 The structure in Fig. 7.29 consists of a hinged beam and five bars. It is subjected to two concentrated forces.

Determine the forces in the bars and the bending moment in the beam.

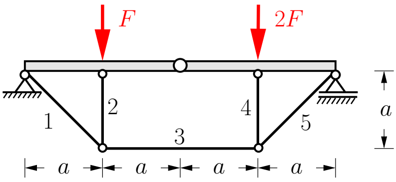


Fig. 7.29

Results: see (**A**) $S_1 = S_5 = \frac{3}{2}\sqrt{2}F\,, \quad S_2 = S_4 = -S_3 = -\frac{3}{2}F\,,$
$M_{\max} = aF/4\,.$

E7.16

Example 7.16 A simply supported beam carries a linearly varying line load as shown in Fig. 7.30.

Calculate the location and the magnitude of the maximum bending moment for $q_1 = 2q_0$.

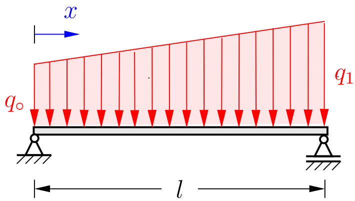


Fig. 7.30

Results: see (**A**) $x^* = 0.53\,l\,, \qquad M_{\max} = 0.19\,q_\circ l^2.$

E7.17

Example 7.17 Draw the shear-force and bending-moment diagrams for the hinged beam shown in Fig. 7.31.

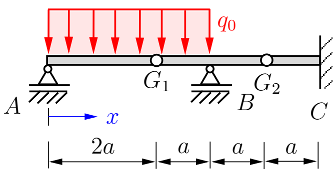


Fig. 7.31

Result: see (**B**) Selected values:

$$V(0) = q_\circ a\,, \quad V(a) = 0\,, \quad V(5a) = 3q_\circ a/2\,,$$
$$M(a) = q_\circ a^2/2\,, \quad M(3a) = -3q_\circ a^2/2\,, \quad M(5a) = 3q_\circ a^2/2\,.$$

Example 7.18 The beam shown in Fig. 7.32 carries a uniformly distributed line load $q_\circ$ and a couple $M_0 = 4q_\circ a^2$.

E7.18

Draw the shear-force and bending-moment diagrams.

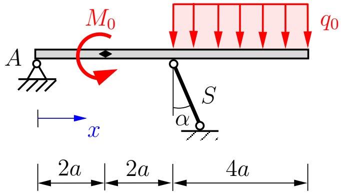


Fig. 7.32

Result: see (**B**) Selected values: $V(a) = -q_\circ a$, $M(4a) = -8q_\circ a^2$.

Example 7.19 Determine the distance a of hinge G from the support B (Fig. 7.33) so that the magnitude of the maximum bending moment becomes minimal.

E7.19

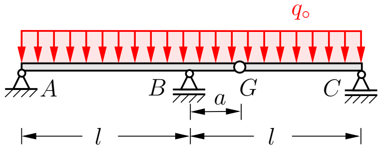


Fig. 7.33

Result: see (**A**) $a = (3 - \sqrt{8})l = 0.172\,l$.

Example 7.20 Determine the stress resultants for the clamped angled member shown in Fig. 7.34.

E7.20

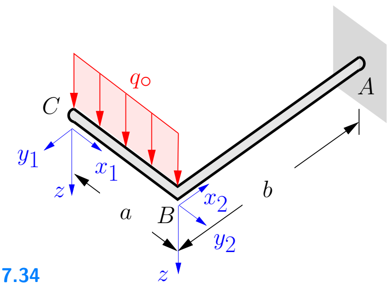


Fig. 7.34

Results: see (**A**)

$$\overline{CB}: \quad V_z = -q_\circ x_1, \quad M_y = -q_\circ x_1^2/2,$$

$$\overline{BA}: \quad V_z = -q_\circ a, \quad M_T = q_\circ a^2/2, \quad M_y = -q_\circ a x_2.$$

E7.21 **Example 7.21** Determine the distance a of the hinge G (Fig. 7.35) so that the magnitude of the maximum bending moment becomes minimal.

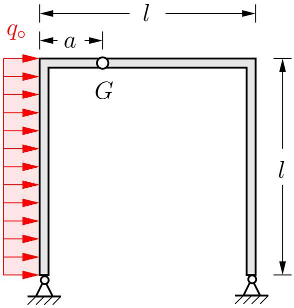


Fig. 7.35

Result: see (**A**) $a = (2\sqrt{3} - 3)l = 0.464\,l$.

E7.22 **Example 7.22** Draw the shear-force and bending-moment diagrams for the frame shown in Fig. 7.36.

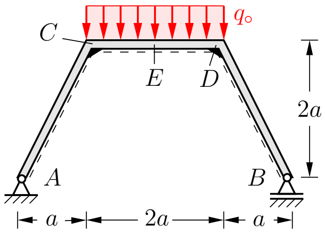


Fig. 7.36

Result: see (**A**) Selected values:

$$V(A) = -V(B) = q_\circ a/\sqrt{5}\,, \quad V(C) = -V(D) = q_\circ a\,,$$
$$M(C) = M(D) = q_\circ a^2\,, \quad M(E) = 3q_\circ a^2/2\,.$$

E7.23 **Example 7.23** The arch shown in Fig. 7.37 carries a constant line load $q_\circ$.

Calculate the maximum values of the normal force and the bending moment.

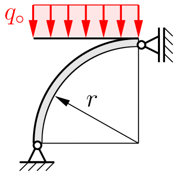


Fig. 7.37

Results: see (**A**) $N_{\max} = 17\,q_\circ r/16\,, \quad M_{\max} = -q_\circ r^2/8.$

E7.24

Example 7.24 A clamped arch in the form of a quarter-circle (weight negligible) supports a line load $q_\circ$ (Fig. 7.38).

Determine the stress resultants as functions of the coordinate φ.

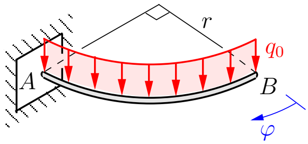


Fig. 7.38

Results: see (**B**) $V_z = -q_\circ r\varphi\,, \quad M_y = -q_\circ r^2(1-\cos\varphi)\,,$

$$M_T = -q_\circ r^2(\varphi - \sin\varphi)\,.$$

E7.25

Example 7.25 Draw the shear-force and bending-moment diagrams for the hinged beam shown in Fig. 7.39.

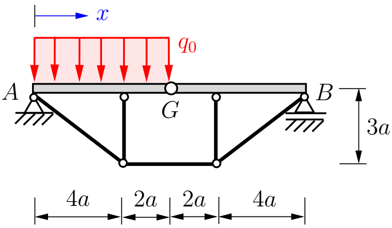


Fig. 7.39

Results: see (**B**) Selected values:

$$V(0) = \tfrac{9}{4}q_0 a\,, \quad V(6a) = -\tfrac{3}{2}q_0 a\,, \quad V(12a) = \tfrac{3}{4}q_0 a\,,$$

$$M(9a/4) = \tfrac{81}{32}q_0 a^2\,, \quad M(4a) = q_0 a^2\,, \quad M(8a) = -3q_0 a^2\,.$$

7.6

7.6 Summary

- In plane problems, the stress resultants in a beam, frame or arch are the normal force N, the shear force V and the bending moment M.
- Sign convention for stress resultants: positive stress resultants at a positive (negative) face point in the positive (negative) directions of the coordinates.
- The stress resultants can be determined using the method of sections:
  - ◇ Pass an imaginary cut through the beam (frame, arch).
  - ◇ Choose a coordinate system.
  - ◇ Draw a free-body diagram of a portion of the structure (stress resultants acting in their positive directions).
  - ◇ Formulate the equilibrium conditions (3 equations in a plane problem, 6 equations in a spatial problem).
  - ◇ Solve the equations.
  - ◇ The system of equations has a unique solution if the structure is (externally and internally) statically determinate.
- The differential relationships

  $$V' = -q\,, \qquad M' = V$$

  are valid for beams and for the straight parts of a frame (not for arches). If the applied load q is known, the stress resultants can be obtained through integration. The constants of integration are determined from boundary conditions or from boundary conditions and matching conditions.
- Frequently, it suffices to compute the stress resultants at several specific points only. The curves between these points are determined by the corresponding loads. Note the relationships between q, V and M.

Chapter 8

Work and Potential Energy

8

8 Work and Potential Energy

Objectives: Students will become familiar with the concepts of *work*, *conservative forces* and *potential energy*. In addition, they will become acquainted with the *principle of virtual work*. After studying this chapter, students should be able to correctly apply this principle in order to determine equilibrium states in nonrigid systems as well as support reactions and internal forces and moments. Finally, it will be shown how to investigate the *stability* of equilibrium states of conservative systems with one degree of freedom.

8.1 Work and Potential Energy

The concept of work involves displacements and therefore belongs to the field of kinetics (see Volume 3) since in the field of statics no movement occurs. However, as will be shown in Section 8.2, problems in the field of statics can also be solved with the aid of work concepts. For this purpose, first the mechanical term “work” is introduced.

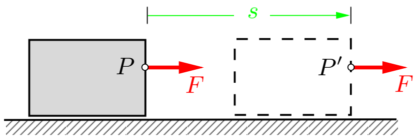


Fig. 8.1

Fig. 8.1 shows a body which is displaced a distance s by a constant force F acting in the direction of the displacement s. Here the *work* of the force F is defined as the product of F and the displacement s of the point of application P:

$$U = F\,s\,.$$

This definition can be generalized with the aid of vector calculus. In Fig. 8.2a the point of application P of a force $\boldsymbol{F}$ moves along an arbitrary path. Let us now consider an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$ from the current position given by vector $\boldsymbol{r}$ to a neighboring position. The infinitesimal work $\mathrm{d}U$ done by the force $\boldsymbol{F}$ is then defined as the scalar product

$$\mathrm{d}U = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}\,. \tag{8.1}$$

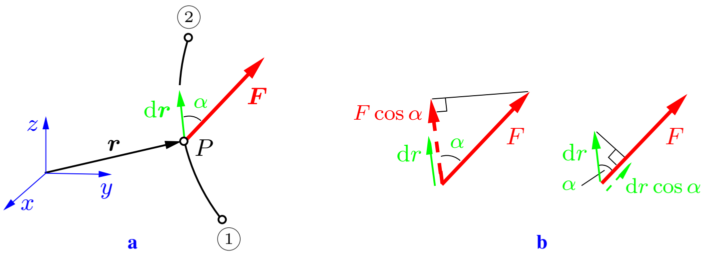


Fig. 8.2

According to (A.19), this product of the vectors $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ is given by the scalar value

$$\mathrm{d}U = |\boldsymbol{F}||\mathrm{d}\boldsymbol{r}|\cos\alpha = (F\cos\alpha)\mathrm{d}r = F(\mathrm{d}r\cos\alpha)\,. \tag{8.2}$$

Thus, the work $\mathrm{d}U$ is the displacement $\mathrm{d}r$ multiplied by the force component $F\cos\alpha$ in the direction of the displacement. Alternatively, it may be interpreted as the product of the force F and the component of the displacement $\mathrm{d}r\cos\alpha$ in the direction of the force (Fig. 8.2b). If force and displacement are orthogonal ($\alpha = \pi/2$), then no work will be done: $\mathrm{d}U = 0$.

The work done along an arbitrary path (Fig. 8.2a) from point ① to point ② can be determined from the line-integral

$$U = \int \mathrm{d}U = \int_{①}^{②} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}\,. \tag{8.3}$$

Work has the dimension $[F\,l]$ and is given in the unit named after the physicist James Prescott Joule (1818–1889):

$$1\,\mathrm{J} = 1\,\mathrm{Nm}.$$

The magnitude and direction of the force $\boldsymbol{F}$ in (8.3) may depend on the position vector $\boldsymbol{r}$, i.e., $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r})$. Therefore, the expression "work = force × distance" is only valid when both vectors $\boldsymbol{F}$ and $\mathrm{d}\boldsymbol{r}$ permanently have the same direction ($\alpha = 0$), and the magnitude of $\boldsymbol{F}$ is constant.

As an example, let us consider a system of two bodies with weights W and Q, respectively, connected by an unstretchable cable (Fig. 8.3). If the body on the left side undergoes a downward displacement of length $\mathrm{d}s$, then the body on the right side will be pulled the same length upward along the inclined plane. The work done by weight W is then given by $\mathrm{d}U_W = W\,\mathrm{d}s$, since the directions of the force and the displacement coincide. Only the component $Q\sin\alpha$ in the direction of the displacement is taken into account for the work done by Q, since the component of a force perpendicular to the direction of the displacement does no

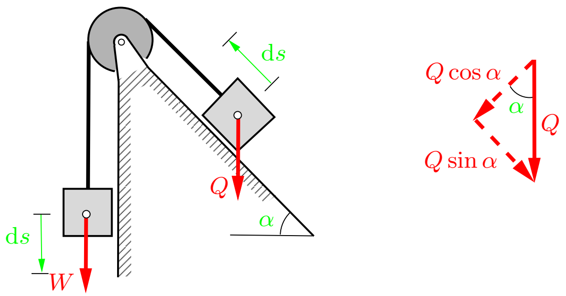


Fig. 8.3

work. As the working component acts in the opposite direction of the displacement, its work is negative: $\mathrm{d}U_Q = -Q\sin\alpha\,\mathrm{d}s$.

As a further example, consider a two-sided lever to which the forces F_1 and F_2 are applied (Fig. 8.4a). During an infinitesimal rotation $\mathrm{d}\varphi$ about the supporting point A, force F_1 does the work (Fig. 8.4b)

$$\mathrm{d}U = F_1\,\mathrm{d}s_1 = F_1\,a\,\mathrm{d}\varphi\,.$$

The product of force F_1 and length a of the lever arm is, according to (3.5), the moment M_1 of the force F_1 about A. Therefore, the work can also be expressed by

$$\mathrm{d}U = M_1\,\mathrm{d}\varphi\,.$$

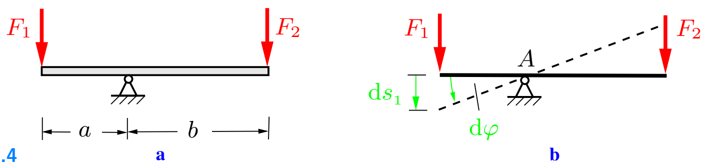


Fig. 8.4

In order to generalize the results obtained so far, we introduce an infinitesimal rotation vector $\mathrm{d}\boldsymbol{\varphi}$, whose direction coincides with the axis of rotation and whose magnitude is the angle $\mathrm{d}\varphi$. Therefore, the work done by a moment vector $\boldsymbol{M}$ during an infinitesimal rotation is given by

$$\mathrm{d}U = \boldsymbol{M}\cdot\mathrm{d}\boldsymbol{\varphi}\,. \tag{8.4}$$

The work done by a moment vector $\boldsymbol{M}$ during a finite rotation of the body on which it acts, follows from integration:

$$U = \int \mathrm{d}U = \int \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{\varphi} \,. \tag{8.5}$$

This equation is analogous to the expression (8.3) for the work of a force. If $\boldsymbol{M}$ and $\mathrm{d}\boldsymbol{\varphi}$ are parallel, then it follows from (8.4) that $\mathrm{d}U = M \mathrm{d}\varphi$. If, in addition, M is constant during a finite rotation φ, then (8.5) yields $U = M\,\varphi$.

Since an angle is dimensionless, moment and work (although they are different physical quantities) have the same dimension $[F\,l]$.

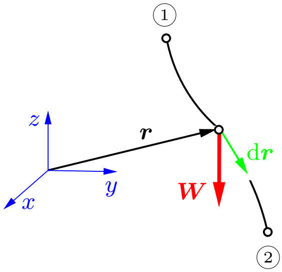


Fig. 8.5

Now a special case of a constant force is considered, namely, the weight W of a body (dead load) in the vicinity of the earth's surface (Fig. 8.5). Let the coordinate z point perpendicularly outward from the earth's surface, then the force vector is given by

$$\boldsymbol{W} = -W\,\boldsymbol{e}_z \,.$$

With the change of the position vector

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}x\,\boldsymbol{e}_x + \mathrm{d}y\,\boldsymbol{e}_y + \mathrm{d}z\,\boldsymbol{e}_z$$

and

$$\boldsymbol{e}_z \cdot \boldsymbol{e}_x = \boldsymbol{e}_z \cdot \boldsymbol{e}_y = 0 \,, \quad \boldsymbol{e}_z \cdot \boldsymbol{e}_z = 1 \quad \text{(see (A.22))}$$

we obtain from (8.1)

$$\mathrm{d}U = \boldsymbol{W} \cdot \mathrm{d}\boldsymbol{r} = -W\,\boldsymbol{e}_z \cdot (\mathrm{d}x\,\boldsymbol{e}_x + \mathrm{d}y\,\boldsymbol{e}_y + \mathrm{d}z\,\boldsymbol{e}_z) = -W\,\mathrm{d}z\,.$$

Therefore, according to (8.3), the work done by the weight along the path from ① to ② is given by

$$U = \int \mathrm{d}U = -\int_{z_1}^{z_2} W\,\mathrm{d}z = W\,(z_1 - z_2)\,. \tag{8.6}$$

The work U depends only on the location of the endpoints. Hence, the work done by the weight along arbitrary paths is unchanged as long as the endpoints ① and ② are the same, i.e., the work is *path-independent*.

Forces whose work is path-independent are called *conservative forces* or *potential forces*. These forces, and these forces only, can be derived from a *potential* V, which is defined as

$$V = -U = -\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,. \tag{8.7}$$

The quantity V is also referred to as *potential energy*.

As a first example, we consider again the weight W in Fig. 8.6a. With $z_2 = z$, Equation (8.6) yields

$$V(z) = -U = W\,(z - z_1) = W\,z - W\,z_1\,. \tag{8.8a}$$

In Volume 3 it will be shown how a conservative force may be derived from its potential. In our case, the weight W is obtained by calculating the negative derivative of V with respect to the coordinate z:

$$-\frac{\mathrm{d}V}{\mathrm{d}z} = -W\,. \tag{8.8b}$$

The minus sign in front of W indicates that the weight acts in the negative z-direction.

According to (8.8a), the potential depends on the choice of the coordinate system (location of the origin). It is determined up to an arbitrary additive constant (here $W z_1$). However, this constant does not enter into the calculation of the force W, cf. (8.8b). It also has no influence on the work because the work depends only on the height difference $z - z_1$. Therefore, the location of the coordinate system can be chosen arbitrarily. It is often useful to choose the coordinate system in such a way that the potential is zero at $z = 0$: $V(0) = 0$ (zero level).

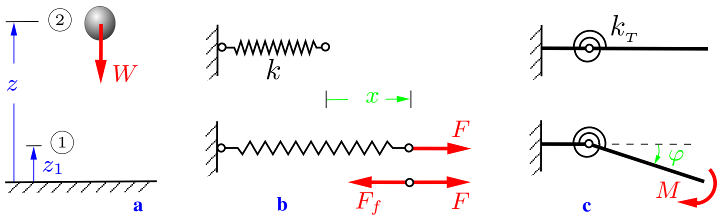


Fig. 8.6

As a further example, the potential of the force in a spring is considered. The spring depicted in Fig. 8.6b is elongated by a length x from its unstretched length through an external force F. It is known from experiments that the linear relation $F = k\,x$ exists between the force F and the spring elongation x, provided the elongation of the spring remains sufficiently small. The spring constant k is a measure of the stiffness of the spring. It has the dimension force divided by length, and therefore the unit $\mathrm{N\,cm^{-1}}$. The spring force F_f (restoring force) is the reaction force associated with F and points in the opposite direction of the elongation. Thus, calculation of the work done by the spring force during elongation yields

$$U = -\int_0^x F_f\,\mathrm{d}\bar{x} = -\int_0^x k\,\bar{x}\,\mathrm{d}\bar{x} = -\frac{1}{2}\,k\,x^2\,.$$

According to (8.7), the potential of the spring force is given by

$$V(x) = \frac{1}{2}\,k\,x^2\,. \tag{8.9}$$

The potential represents the energy stored in the spring due to its elongation. The spring force F_f also follows from the negative derivative of V with respect to the coordinate x:

$$F_f = -\frac{\mathrm{d}V}{\mathrm{d}x} = -\,k\,x\,.$$

Consider now a linear torsion spring as depicted in Fig. 8.6c. There is a linear relation $M = k_T\varphi$ between the moment M and the angle of rotation φ. This relation is analogous to $F = k\,x$ in the foregoing case. The torsion spring constant k_T has the dimension $[Fl]$ and therefore the unit N cm or a multiple of it. If in Equation (8.9), k is replaced by k_T and x by φ, the potential of the moment of a torsion spring is obtained:

$$V = \frac{1}{2}\,k_T\varphi^2\,. \qquad (8.10)$$

8.2 Principle of Virtual Work

8.2

Up to this point in the chapter we have calculated the work done by a force when the point of application actually moves along a path. However, the concept of work can also be applied to statics, where *no displacements occur*. In this framework, actual displacements must be replaced by *virtual displacements*. Virtual displacements are displacements (or rotations) that are

a) fictitious, i.e., do not exist in reality,
b) infinitesimally small,
c) geometrically (kinematically) admissible, i.e. consistent with the constraints of the system.

In order to distinguish virtual displacements from real displacements $\mathrm{d}\boldsymbol{r}$, we denote virtual displacements as $\delta\boldsymbol{r}$, i.e., with the δ-symbol, taken from the calculus of variations. Accordingly, the *virtual work* done by forces or moments during a virtual displacement is written as

$$\delta U = \boldsymbol{F}\cdot\delta\boldsymbol{r} \qquad \text{(compare (8.1))},$$

$$\delta U = \boldsymbol{M}\cdot\delta\boldsymbol{\varphi} \qquad \text{(compare (8.4))}.$$

We now reconsider the two-sided lever (Fig. 8.7a) and calculate the work done during a virtual displacement. A virtual displacement, i.e., a deflection that is consistent with the constraints of the system, is a rotation with an angle $\delta\varphi$ about the support A (Fig. 8.7b). The total virtual work δU done by the two forces F_1 and F_2 is

$$\delta U = F_1\, a\, \delta\varphi - F_2\, b\, \delta\varphi = (F_1\, a - F_2\, b)\, \delta\varphi\,.$$

The minus sign in the second term takes into account that the force F_2 acts in opposite direction of the virtual deflection $b\,\delta\varphi$. In the equilibrium state, the expression in parentheses disappears due to the equilibrium condition of moments $F_2\, b = F_1\, a$ (Archimedes' law of the lever). Therefore, in this example, the virtual work vanishes in the equilibrium position: $\delta U = 0$. It should be noted that only the *external* forces F_1 and F_2 (cf. Section 1.4) enter into the virtual work, whereas the reaction force in A does not contribute.

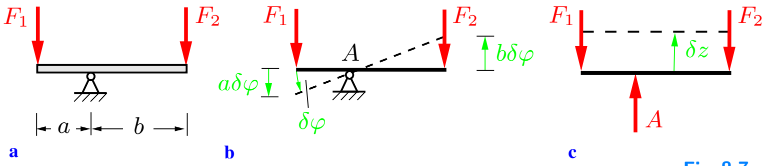


Fig. 8.7

The aforementioned results can be generalized. We postulate as an axiom that for an arbitrary system with an arbitrary number of external forces $\boldsymbol{F}_i^{(e)}$ and external moments $\boldsymbol{M}_i^{(e)}$ the entire virtual work must disappear in an equilibrium state:

$$\delta U = \sum \boldsymbol{F}_i^{(e)} \cdot \delta \boldsymbol{r}_i + \sum \boldsymbol{M}_i^{(e)} \cdot \delta \boldsymbol{\varphi}_i = 0\,. \qquad (8.11)$$

Since this equilibrium axiom provides a statement on the work done during virtual displacements, it is called the *principle of virtual work*. It may be expressed as follows:

A mechanical system is in equilibrium if the virtual work of the external loads (forces and moments) vanishes during an arbitrary virtual displacement.

The axiom of the principle of virtual work $\delta U = 0$ is also often referred to as the *principle of virtual displacements* . In the mechanics of deformable solids, an extended version of this principle has a significant meaning (cf. Volume 2).

The equilibrium conditions can be derived from the principle of virtual work; conversely, the principle of virtual work can be derived from the equilibrium conditions, which also have an axiomatic character. Therefore, the entire field of statics can be based either on the equilibrium conditions or on the principle of virtual work. From a practical point of view, the principle of virtual work offers the great advantage that the number of unknowns in the equations can often be reduced through an appropriate choice of virtual displacements. The drawback is that complicated kinematic conditions may have to be formulated.

The principle of virtual work cannot only be applied to movable systems, but also to systems that are rigidly supported, i.e., immobile. In the case of a rigid system, one or several supports are removed and replaced by the support reactions. These reactions then will be considered to be external loads and consequently taken into account in the principle of virtual work. For example, if we remove in Fig. 8.7b the pin joint at A, it is replaced by the force A (Fig. 8.7c). This force does the virtual work $A\delta z$ during a virtual displacement δz. Hence, the principle of virtual work reads

$$\delta U = A\,\delta z - F_1\,\delta z - F_2\,\delta z = (A - F_1 - F_2)\delta z = 0\,.$$

Since the virtual displacement δz is nonzero, the term in parentheses must vanish, yielding the support force $A = F_1 + F_2$.

8.3 Equilibrium States and Forces in Nonrigid Systems 8.3

In the following we consider systems of rigid bodies that are incompletely constrained and therefore able to move. If the applied forces are prescribed, the corresponding equilibrium configuration can be determined with the aid of the principle of virtual work. On the other hand, if an equilibrium position is prescribed, the principle of virtual work yields the necessary forces. There are two ways to formulate the principle of virtual work:

a) We can draw the system in an arbitrary configuration and in an adjacent configuration. The magnitudes and directions of the displacements associated with the applied forces can be taken from the drawn figure. The corresponding terms for the virtual work (including the algebraic signs) are inserted into the principle of virtual work.
b) We can choose a coordinate system and describe the coordinates of the point of application of each force in this coordinate system. The virtual displacements are equal to infinitesimal changes of the coordinates. They are obtained formally through differentiation (the δ-symbol can be treated as a differential). For example, if $\boldsymbol{r}$ is a function of a coordinate α, i.e., $\boldsymbol{r} = \boldsymbol{r}(\alpha)$, then $\delta\boldsymbol{r} = (\mathrm{d}\boldsymbol{r}/\mathrm{d}\alpha)\delta\alpha$. The differentiation automatically yields the correct algebraic sign for each term.

Both methods will now be applied to the simple example of a rod (weight negligible) of length l, supported by a pin joint at A (Fig. 8.8a). The rod is loaded by a horizontal force P and a vertical force Q at the free end B. Our aim is to determine the angle α in the equilibrium position. Applying the first method, the rod is considered to be infinitesimally rotated by $\delta\alpha$ from an arbitrary, yet unknown position α (Fig. 8.8b). The point of application of the forces then is displaced by an amount $l\,\delta\alpha\sin\alpha$ upward, and by an amount $l\,\delta\alpha\cos\alpha$ to the left. The principle of virtual work

$$\delta U = P\,l\,\delta\alpha\cos\alpha - Q\,l\,\delta\alpha\sin\alpha = (P\cos\alpha - Q\sin\alpha)\,l\,\delta\alpha = 0$$

yields the equilibrium position ($\delta\alpha \neq 0!$):

$$P\cos\alpha - Q\sin\alpha = 0 \quad \to \quad \tan\alpha = \frac{P}{Q}\ .$$

In the second, more formal method, we describe the position of point B (point of application of the forces) with the aid of the position vector $\boldsymbol{r}$ that points from the fixed support A to B. In the coordinate system shown in Fig. 8.8c, the coordinates of $\boldsymbol{r}$ are $x = l\sin\alpha$ and $y = l\cos\alpha$. The virtual displacements are obtained

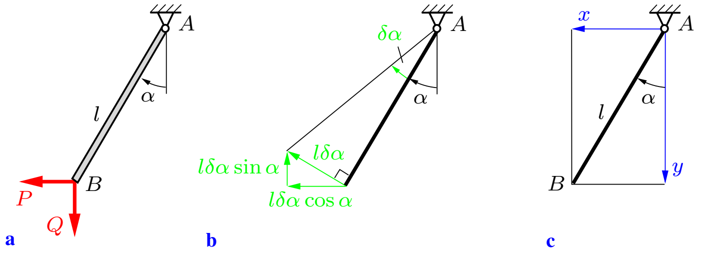


Fig. 8.8

through differentiation:

$$\delta x = \frac{\mathrm{d}x}{\mathrm{d}\alpha}\,\delta\alpha = l\cos\alpha\,\delta\alpha\,, \quad \delta y = \frac{\mathrm{d}y}{\mathrm{d}\alpha}\,\delta\alpha = -\,l\sin\alpha\,\delta\alpha\,.$$

Inserting these relations into the principle of virtual work (both forces point in the directions of the *positive* coordinate axes)

$$\begin{aligned}\delta U &= P\,\delta x + Q\,\delta y = P\,l\cos\alpha\,\delta\alpha - Q\,l\sin\alpha\,\delta\alpha \\ &= (P\cos\alpha - Q\sin\alpha)\,l\,\delta\alpha = 0\end{aligned}$$

leads to the same result as before.

The advantage of the formal method now becomes clear: in the first method the algebraic signs of the virtual displacements have to be determined from observation, whereas the second method automatically gives the correct signs ($\delta\alpha > 0$ yields $\delta y < 0$). The second method is preferable in the case of complicated kinematics (geometry), since it is not always possible to rely on observation. It should be noted that α can also be determined by formulating the moment equilibrium condition $\overset{\curvearrowright}{A}$: $P\,l\cos\alpha - Q\,l\sin\alpha = 0$.

If a system has several independent possibilities of movement (degrees of freedom), the position $\boldsymbol{r}(\alpha,\beta,\ldots)$ of the point of application of a force is given by several independent coordinates $\alpha,\beta,\ldots$. The virtual displacements can then be found analogous to the total differential of a function of several variables:

$$\delta\boldsymbol{r} = \frac{\partial\boldsymbol{r}}{\partial\alpha}\,\delta\alpha + \frac{\partial\boldsymbol{r}}{\partial\beta}\,\delta\beta + \ldots\,. \tag{8.12}$$

E8.1

Example 8.1 A drawbridge of weight W can be raised with the aid of a cable (weight neglected) and a counter-weight Q (Fig. 8.9a).

Determine the positions of equilibrium.

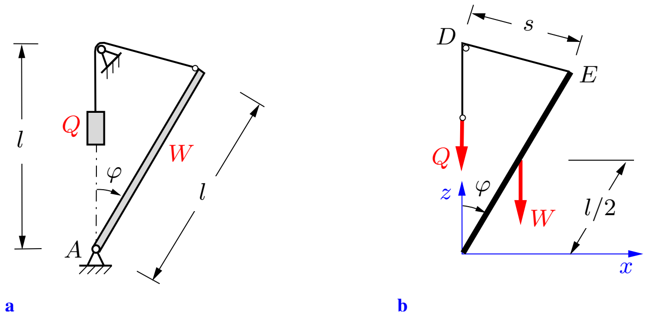


Fig. 8.9

Solution Since the angle φ uniquely describes the position of the system, the system has one degree of freedom.

If the arbitrary position given by φ is changed by a virtual displacement $\delta\varphi$, the points of application of the forces W and Q are displaced. In order to determine the virtual work of W, we have to consider only the change of its elevation, since the weight does no work in a horizontal displacement. With the coordinate system chosen in Fig. 8.9b (origin at the fixed support A), we obtain

$$z_W = \frac{l}{2}\cos\varphi \quad \rightarrow \quad \delta z_W = \frac{\mathrm{d}z_W}{\mathrm{d}\varphi}\,\delta\varphi = -\,\frac{l}{2}\sin\varphi\,\delta\varphi\,.$$

Determining the virtual displacement of the point of application of Q is more difficult. For this purpose, we introduce the auxiliary coordinate s, i.e., the distance between the fixed point D and point E (Fig. 8.9b):

$$s = 2\,l\sin\frac{\varphi}{2}\,.$$

A virtual displacement $\delta\varphi$ causes a virtual change δs of s, which is equal to the virtual displacement δz_Q of the weight Q (inextensible cable!):

$$\delta z_Q = \delta s = \frac{\mathrm{d}s}{\mathrm{d}\varphi}\,\delta\varphi = l\cos\frac{\varphi}{2}\,\delta\varphi\,.$$

Since both forces point in the direction of the *negative* z-axis, the principle of virtual work yields

$$\delta U = -W\,\delta z_W - Q\,\delta z_Q = \left(W\,\frac{l}{2}\sin\varphi - Q\,l\cos\frac{\varphi}{2}\right)\delta\varphi = 0\,.$$

With $\sin\varphi = 2\sin(\varphi/2)\cos(\varphi/2)$ and $\delta\varphi \neq 0$, we obtain

$$\cos\frac{\varphi}{2}\left(W\sin\frac{\varphi}{2} - Q\right) = 0\,.$$

The first solution $\cos(\varphi/2) = 0$ leads to $\varphi = \pi$. Therefore, the only remaining solution is

$$\underline{\underline{\sin\frac{\varphi}{2} = \frac{Q}{W}}}\,.$$

For φ lying within the technically reasonable region $0 < \varphi < \pi/2$, the inequality $\sin(\varphi/2) < \sin(\pi/4) = \sqrt{2}/2$ must be satisfied. This requirement leads to the following condition for the load Q:

$$\frac{Q}{W} < \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\,.$$

Example 8.2 A car-jack loaded with a weight W is schematically depicted in Fig. 8.10. The height of the screw-thread is h. **E8.2**

What torque M must be applied to the car-jack in order to keep the jack in equilibrium? The jack-screw, when turned, moves without friction.

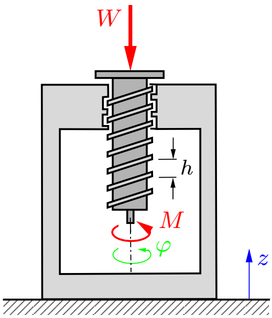


Fig. 8.10

Solution According to the principle of virtual work, the system is in equilibrium when the total work done by force W and torque M vanishes:

$$\delta U = M\delta\varphi - W\,\delta z = 0\,.$$

Here, the directions of M and φ coincide, whereas W and z have opposite directions (Fig. 8.10). The virtual displacements $\delta\varphi$ and δz are not independent. A rotation of $\Delta\varphi = 2\pi$ will raise the jack-screw by the height $\Delta z = h$ of the screw-thread. Therefore, a rotation of $\delta\varphi$ results in $\delta z = (h/2\,\pi)\delta\varphi$. Thus, the principle of virtual work yields

$$\delta U = \left(M - \frac{h}{2\pi}\,W\right)\delta\varphi = 0\,.$$

With $\delta\varphi \neq 0$ we obtain the torque necessary for equilibrium:

$$\underline{\underline{M = \frac{h}{2\,\pi}\,W\,.}}$$

It can be recognized that a large weight W can be raised with a small force K by an externally applied torque $M = l\,K$ if $l \gg h$.

E8.3 **Example 8.3** Two pin-jointed rods of weights W_1 and W_2 are displaced from the vertical position by a horizontal force F (Fig. 8.11a).

Determine the equilibrium configuration.

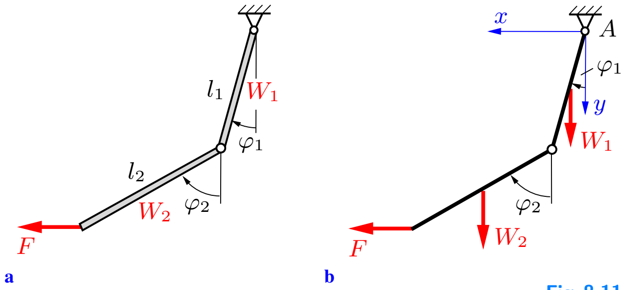


Fig. 8.11

Solution The configuration of the system is uniquely determined by the two angles φ_1 and φ_2. We use a coordinate system that has its origin at the fixed support A (Fig. 8.11b). Then we obtain the following coordinates of the points of application of the forces:

$$y_1 = \frac{l_1}{2}\cos\varphi_1\,, \quad y_2 = l_1\cos\varphi_1 + \frac{l_2}{2}\cos\varphi_2\,,$$
$$x_F = l_1\sin\varphi_1 + l_2\sin\varphi_2\,.$$

According to (8.12), the virtual displacements are given by

$$\delta y_1 = -\frac{l_1}{2}\sin\varphi_1\,\delta\varphi_1\,, \quad \delta y_2 = -\,l_1\sin\varphi_1\,\delta\varphi_1 - \frac{l_2}{2}\sin\varphi_2\,\delta\varphi_2\,,$$
$$\delta x_F = l_1\cos\varphi_1\,\delta\varphi_1 + l_2\cos\varphi_2\,\delta\varphi_2\,.$$

Therefore, the principle of virtual work leads to

$$\begin{aligned}\delta U &= W_1\,\delta y_1 + W_2\,\delta y_2 + F\,\delta x_F\\ &= W_1\Big(-\frac{l_1}{2}\sin\varphi_1\,\delta\varphi_1\Big) + W_2\Big(-l_1\sin\varphi_1\,\delta\varphi_1 - \frac{l_2}{2}\sin\varphi_2\,\delta\varphi_2\Big)\\ &\quad + F(l_1\cos\varphi_1\,\delta\varphi_1 + l_2\cos\varphi_2\,\delta\varphi_2)\\ &= \Big(F\,l_1\cos\varphi_1 - W_1\,\frac{l_1}{2}\sin\varphi_1 - W_2\,l_1\sin\varphi_1\Big)\delta\varphi_1\\ &\quad + \Big(F\,l_2\cos\varphi_2 - W_2\,\frac{l_2}{2}\sin\varphi_2\Big)\delta\varphi_2 = 0\,.\end{aligned}$$

Since the system has two degrees of freedom, there are two virtual displacements, $\delta\varphi_1$ and $\delta\varphi_2$. They are independent of each other and not equal to zero at the same time. Therefore, the virtual work vanishes only if the expressions in both parentheses are zero:

$$F\,l_1\cos\varphi_1 - W_1\,\frac{l_1}{2}\sin\varphi_1 - W_2\,l_1\sin\varphi_1 = 0$$
$$\rightarrow \quad \underline{\underline{\tan\varphi_1 = \frac{2\,F}{W_1 + 2\,W_2}}}\,,$$
$$F\,l_2\cos\varphi_2 - W_2\,\frac{l_2}{2}\sin\varphi_2 = 0 \quad \rightarrow \quad \underline{\underline{\tan\varphi_2 = \frac{2\,F}{W_2}}}\,.$$

8.4 Reaction Forces and Stress Resultants

8.4

Structures such as beams, frames or trusses are rigidly supported. In order to calculate a reaction force (moment) with the aid of the principle of virtual work, the corresponding support must be removed and replaced by the support reaction. The point of application of the support reaction is then able to move. Therefore, the reaction force (moment) is considered to be an external force (moment) in the principle of virtual work. Similarly, the force in an internal pin or the stress resultants in beams can be determined by cutting the system appropriately. Due to this imaginary cut, the system becomes movable and the corresponding internal force does virtual work during a virtual displacement. The following examples will explain the procedure.

E8.4 **Example 8.4** Determine the force in the pin G of the structure shown in Fig. 8.12a.

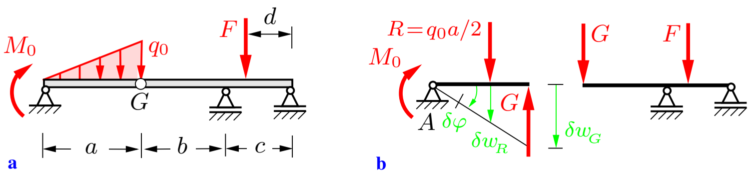


Fig. 8.12

Solution The horizontal component of the force in the pin is zero. In order to determine the vertical component, we divide the structure into two parts by a cut through the pin. The left-hand part is then able to rotate about the point A, whereas the right-hand portion cannot move in the vertical direction (Fig. 8.12b). Furthermore, the triangular load is replaced by its resultant

$$R = q_0\, a/2$$

which is located at a distance $2\,a/3$ from A. With the virtual displacements

$$\delta w_R = \frac{2}{3}\, a\, \delta\varphi\,, \quad \delta w_G = a\, \delta\varphi$$

the principle of virtual work yields

$$\begin{aligned}\delta U &= R\, \delta w_R + M_0\, \delta\varphi - G\, \delta w_G = \frac{q_0\, a}{2}\frac{2}{3}\, a\, \delta\varphi + M_0 \delta\varphi - G\, a\, \delta\varphi \\ &= \left(q_0\, \frac{a^2}{3} + M_0 - G\, a\right) \delta\varphi = 0\,.\end{aligned}$$

Since $\delta\varphi \neq 0$, the force in pin G becomes

$$\underline{\underline{G = \frac{q_0\, a}{3} + \frac{M_0}{a}}}\,.$$

The force F acting on the right-hand section has no influence on the force in pin G.

Example 8.5 The structure in Fig. 8.13a consists of three beams that are pin-connected at G_1 and G_2. It is subjected to a concentrated force F and a constant line load $q_0 = F/(3\,a)$. E8.5

Determine the support reactions at A with the aid of the principle of virtual work.

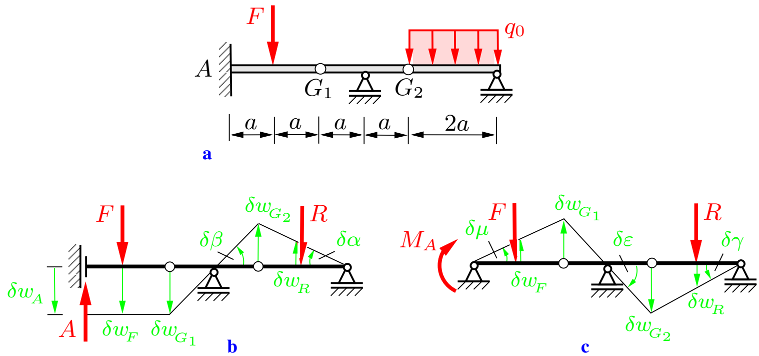


Fig. 8.13

Solution In order to determine the vertical support reaction A, the clamped end of the structure has to be replaced by a parallel motion according to Fig. 8.13b. Then point A can move in the vertical direction and the support reaction A has to be treated as an external load in the principle of virtual work. Since no rotation is possible at A, the moment M_A is a reaction moment and does no work. From Fig. 8.13b we obtain the following relations between the virtual displacements:

$$\delta w_A = \delta w_F = \delta w_{G_1} = a\,\delta\beta\,, \qquad \delta w_R = a\,\delta\alpha\,.$$

The angles $\delta\alpha$ and $\delta\beta$ are mutually dependent. Considering the displacement of pin G_2, we find

$$\delta w_{G_2} = 2\,a\,\delta\alpha = a\,\delta\beta \quad \to \quad \delta\alpha = \tfrac{1}{2}\,\delta\beta\,.$$

From the principle of virtual work

$$\begin{aligned}\delta U &= -A\,\delta w_A + F\,\delta w_F - R\,\delta w_R = -A\,a\,\delta\beta + F\,a\,\delta\beta - \tfrac{2}{3}\,F\,a\,\delta\alpha \\ &= (-A + F - \tfrac{2}{3}\,F\,\tfrac{1}{2})\,a\,\delta\beta = 0\end{aligned}$$

and $\delta\beta \neq 0$, we obtain the support reaction

$$\underline{\underline{A = \frac{2}{3}\,F\,.}}$$

In order to calculate the moment M_A we replace the clamped support by a smooth pin (Fig. 8.13c). Then the left part of the structure can rotate about A and the moment M_A enters into the principle of virtual work as an external moment. Now the point A cannot be vertically displaced. Therefore, force A is a reaction force and does no work. The three angles in Fig. 8.13c are mutually dependent:

$$\begin{aligned}&\delta w_{G_1} = 2\,a\,\delta\mu = a\,\delta\varepsilon\,, \quad \delta w_{G_2} = a\,\delta\varepsilon = 2\,a\,\delta\gamma \\ &\qquad \to \quad \delta\varepsilon = 2\,\delta\mu \quad \text{and} \quad \delta\gamma = \delta\mu\,.\end{aligned}$$

Considering the algebraic signs (F acts in the opposite direction to δw_F and M_A acts in the opposite direction to $\delta\mu$) the principle of virtual works yields

$$\begin{aligned}\delta U &= -\,M_A\,\delta\mu - F\,\delta w_F + R\,\delta w_R \\ &= -\,M_A\,\delta\mu - F\,a\,\delta\mu + \frac{F}{3\,a}\,2\,a\,a\,\delta\gamma \\ &= \Big(-\,M_A - Fa + \frac{2}{3}\,Fa\Big)\delta\mu = 0 \quad \to \quad \underline{\underline{M_A = -\,\frac{1}{3}\,Fa\,.}}\end{aligned}$$

Both calculations show the advantage of the principle of virtual work: the forces G_1 and G_2 in the pins and the remaining support reactions, which appear in the classical calculation (cf. Section 5.3.3), do no work and therefore need not be taken into account when applying the principle of virtual work.

Example 8.6 Determine the force in member 5 of the truss shown in Fig. 8.14a. **E8.6**

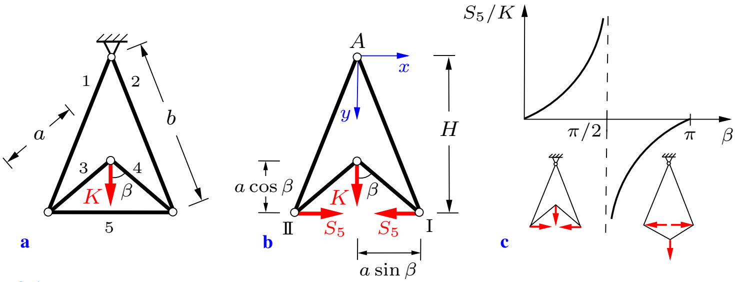


Fig. 8.14

Solution We remove member 5 from the truss. Then the internal forces S_5 at pins I and II act as external forces (Fig. 8.14b). In order to determine the positions of the points of application of the various forces, an x, y-coordinate system is introduced with its origin at the fixed point A. The point of application of force K has the y-coordinate

$$y_K = H - a\cos\beta = \sqrt{b^2 - a^2\sin^2\beta} - a\cos\beta\,.$$

Pin I (point of application of S_5) has the x-coordinate

$$x_\mathrm{I} = a\sin\beta\,.$$

A virtual displacement (i.e., a small change of angle β) yields

$$\delta y_K = \frac{\mathrm{d}y_K}{\mathrm{d}\beta}\,\delta\beta = \Big(\frac{-a^2\,2\sin\beta\cos\beta}{2\sqrt{b^2-a^2\sin^2\beta}} + a\sin\beta\Big)\delta\beta\,,$$

$$\delta x_\mathrm{I} = \frac{\mathrm{d}x_\mathrm{I}}{\mathrm{d}\beta}\,\delta\beta = a\cos\beta\,\delta\beta\,.$$

Due to the symmetry of the system and the loading, the displacement of pin II (to the right) is equal to the displacement of pin I (to the left). Therefore, the total virtual work performed by the forces is

$$\begin{aligned}\delta U &= K\delta y_K - 2\,S_5\,\delta x_\mathrm{I}\\ &= \Big[Ka\sin\beta\Big(1 - \frac{a\cos\beta}{\sqrt{b^2-a^2\sin^2\beta}}\Big) - 2\,S_5\,a\cos\beta\Big]\delta\beta\,.\end{aligned}$$

Since $\delta U = 0$ and $\delta\beta \neq 0$, we obtain

$$\underline{\underline{S_5 = \frac{1}{2} K \tan\beta \left(1 - \frac{a\cos\beta}{\sqrt{b^2 - a^2\sin^2\beta}}\right)}}\,.$$

Fig. 8.14c qualitatively depicts the ratio S_5/K versus β. Because of $b > a$, force S_5 is positive for $0 \leqq \beta < \pi/2$ and negative for $\pi/2 < \beta \leqq \pi$. This result may be verified by inspection.

8.5

8.5 Stability of Equilibrium States

In Section 8.3 the equilibrium states were determined by applying the principle of virtual work $\delta U = 0$. However, experience teaches that there are "different types" of equilibrium states. To characterize these states the concept of *stability* must be introduced. In the following, the discussion is limited to conservative forces and systems with only *one* degree of freedom. In this case the potential V depends only on *one* coordinate. Fig. 8.15 shows two examples where the only applied force is the weight. The first example in Fig. 8.15a shows a sphere (weight W) that lies at the lowest point of a concave surface (equilibrium position). If a small disturbance is imposed, e.g., a small lateral deflection x, the sphere is lifted by an amount Δz and the potential is increased by

$$\Delta V = W\,\Delta z > 0\,.$$

In the second example, a rod supported by a frictionless pin at its upper end is shown in the equilibrium position (vertical position). A small disturbance (small angle φ) raises the center of gravity of the rod (point of application of W) and therefore increases the potential. In both cases, the respective body returns to its equilibrium position when left to itself. Such equilibrium positions are referred to as *stable*.

Now we consider a sphere lying on a horizontal plane and a rod supported at its center of gravity (Fig. 8.15b). The elevation of the points of application of the weights remains unchanged in

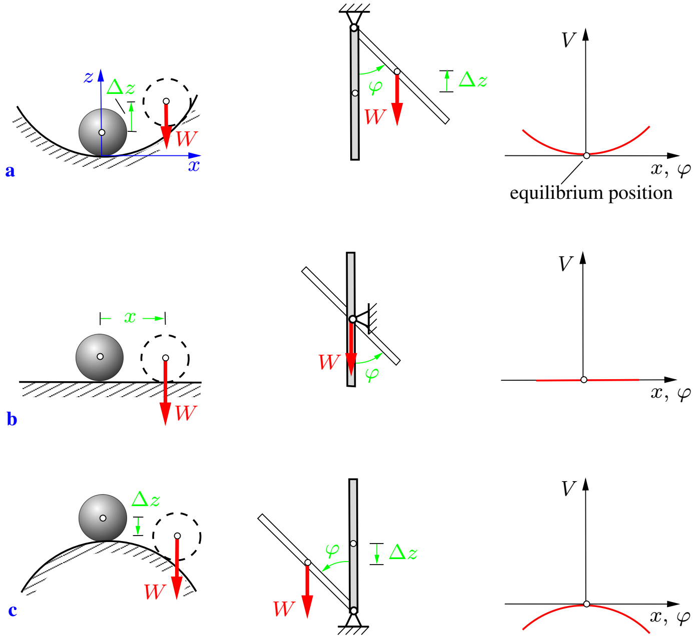


Fig. 8.15

the case of a displacement x or a rotation φ. Therefore, there is no change in the potential:

$$\Delta V = W\,\Delta z = 0\,.$$

The displaced positions are also equilibrium positions. The bodies remain at rest when left to themselves. Such equilibrium states are called *neutral*.

If the sphere is in equilibrium at the highest point of a convex surface or if the rod is supported at its lower end (Fig. 8.15c), then the potential decreases due to a displacement:

$$\Delta V = W\,\Delta z < 0\,.$$

When the sphere or the rod are left to themselves in the displaced position, they will move further away from the equilibrium state. Such equilibrium states are referred to as *unstable*.

On the right-hand side of Fig. 8.15 the graph of the potential is qualitatively depicted as a function of x (sphere) or as a function of φ (rod) for the three cases. It can be recognized from the diagrams that the potential in each equilibrium state has an extreme value. This extreme value is equivalent to the principle of virtual work that, according to (8.11), requires for equilibrium

$$\delta V = -\,\delta U = 0\,.$$

If the potential depends on one coordinate only, e.g., $V = V(x)$, then

$$\delta V = \frac{\mathrm{d}V}{\mathrm{d}x}\,\delta x = 0\,.$$

Since $\delta x \neq 0$, this equation leads to the following condition for an equilibrium state:

$$\frac{\mathrm{d}V}{\mathrm{d}x} = V' = 0\,. \tag{8.13}$$

The graph of the potential function has a horizontal tangent at the point corresponding to an equilibrium position.

The *criterion of stability* follows from the behavior of the potential curve in the vicinity of the considered equilibrium state. In case a), the potential increases due to a displacement x or a rotation φ; in case c) the potential decreases. Therefore, taking into account the above-mentioned concepts, it can be stated:

$$\Delta V \left\{ \begin{array}{ll} > 0 & \text{stable} \\ \equiv 0 & \text{neutral} \\ < 0 & \text{unstable} \end{array} \right\} \text{ equilibrium}\,. \tag{8.14}$$

Accordingly, the potential has a minimum in a stable equilibrium state and a maximum in an unstable equilibrium state. Consequently, the stability criterion (8.14) can also be formulated in terms of the second derivative of the potential function $V(x)$, characterizing a minimum and a maximum:

$$\begin{aligned} V''(x_0) > 0 \quad &\rightarrow \quad \text{minimum} \quad \rightarrow \quad \text{stable}\,, \\ V''(x_0) < 0 \quad &\rightarrow \quad \text{maximum} \quad \rightarrow \quad \text{unstable}\,. \end{aligned} \tag{8.15}$$

Here, the position x_0 denotes the equilibrium state.

In the case $V''(x_0) = 0$, further investigations are necessary. Let $V(x_0) = V_0$ be the value of the potential in the equilibrium state x_0. The potential in a neighboring position $x_0 + \delta x$ can be expressed by the Taylor-Series

$$\begin{aligned} V(x_0 + \delta x) = V_0 + V'(x_0)\,\delta x + \tfrac{1}{2}V''(x_0)\,(\delta x)^2 \\ + \tfrac{1}{6}V'''(x_0)\,(\delta x)^3 + \ldots\,. \end{aligned}$$

Thus, the change in the potential is

$$\begin{aligned} \Delta V &= V(x_0 + \delta x) - V_0 \\ &= V'(x_0)\,\delta x + \tfrac{1}{2}V''(x_0)\,(\delta x)^2 + \tfrac{1}{6}V'''(x_0)\,(\delta x)^3 + \ldots\,. \end{aligned} \tag{8.16}$$

According to (8.13), the first derivative of the potential vanishes in the equilibrium state, i.e. $V'(x_0) = 0$. Therefore, the algebraic sign of the second derivative $V''(x_0)$ of the potential determines whether ΔV is greater or less than zero and hence, whether the equilibrium state is stable or unstable. If $V''(x_0)$ and *all* higher derivatives disappear, then $\Delta V \equiv 0$ according to (8.16). This corresponds to a *neutral* equilibrium state. On the other hand, if $V''(x_0) = 0$ holds and higher derivatives are nonzero, the algebraic sign of the lowest non-vanishing derivative of the potential determines the type of equilibrium.

Example 8.7 Three cogged wheels are supported without friction at their centers. Three mass points (weights W_1, W_2 and W_3) are mounted eccentrically on the wheels as shown in Fig. 8.16a. **E8.7**

Determine the equilibrium states and investigate their type of stability (i.e., stable or unstable) for $W_1 = W_3 = 2\,W$, $W_2 = W$ and $x = \sqrt{3}\,r$.

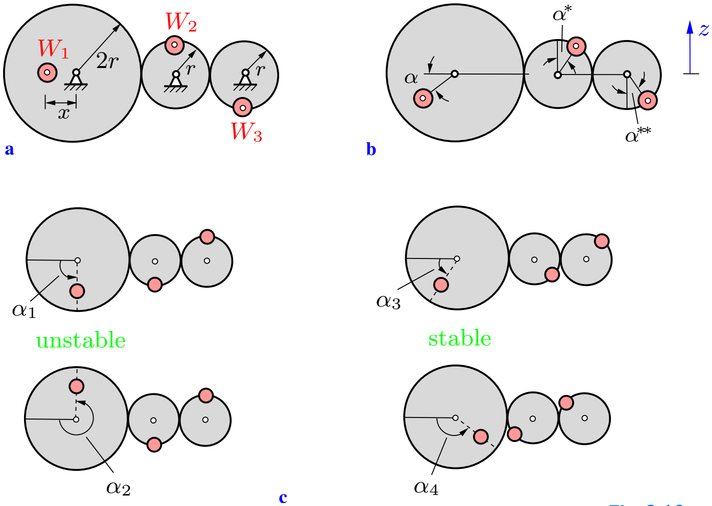


Fig. 8.16

Solution To formulate the potential we introduce the coordinate z as shown in Fig. 8.16b. Let the large wheel be rotated by an arbitrary angle α from the initial position (Fig. 8.16b). This rotation causes the other wheels also to rotate. The arc lengths at the contact points of the wheels must be the same for every wheel. With the given radii, we obtain

$$2\,r\,\alpha = r\,\alpha^* = r\,\alpha^{**} \quad \to \quad \alpha^* = \alpha^{**} = 2\,\alpha\,.$$

The coordinates of W_1, W_2 and W_3 are given by

$$z_1 = -\,x\sin\alpha\,, \quad z_2 = r\cos\alpha^* = r\cos 2\,\alpha\,,$$
$$z_3 = -\,r\cos\alpha^{**} = -\,r\cos 2\,\alpha\,.$$

Thus, the total potential can be expressed as

$$\begin{aligned} V &= W_1\,z_1 + W_2\,z_2 + W_3\,z_3 \\ &= 2\,W(-\,x\sin\alpha) + W\,r\cos 2\,\alpha + 2\,W(-\,r\cos 2\,\alpha) \\ &= -\,W\,r(2\sqrt{3}\sin\alpha + \cos 2\,\alpha)\,. \end{aligned}$$

According to (8.13), the equilibrium states are determined from

$$
\begin{aligned}
V' = \frac{\mathrm{d}V}{\mathrm{d}\alpha} &= -W\,r(2\sqrt{3}\cos\alpha - 2\sin 2\,\alpha) \\
&= -2\,W\,r\cos\alpha(\sqrt{3} - 2\sin\alpha) = 0\,.
\end{aligned}
$$

A first solution is obtained by setting the first factor to zero:

$$\cos\alpha = 0\,.$$

The corresponding equilibrium positions are

$$\underline{\underline{\alpha_1 = \pi/2}}\,, \qquad \underline{\underline{\alpha_2 = 3\,\pi/2}}\,.$$

A second solution follows from the disappearance of the bracketed term:

$$\sqrt{3} - 2\sin\alpha = 0 \quad \rightarrow \quad \sin\alpha = \sqrt{3}/2\,.$$

The corresponding equilibrium positions are

$$\underline{\underline{\alpha_3 = \pi/3}}\,, \qquad \underline{\underline{\alpha_4 = 2\,\pi/3}}\,.$$

According to (8.15), the type of equilibrium of the four different positions can be determined with the aid of the second derivative of the potential:

$$
\begin{aligned}
V'' = \frac{\mathrm{d}^2V}{\mathrm{d}\alpha^2} &= -W\,r(-2\sqrt{3}\sin\alpha - 4\cos 2\,\alpha) \\
&= 2\,W\,r(\sqrt{3}\sin\alpha + 2\cos 2\,\alpha)\,.
\end{aligned}
$$

Inserting the solutions α_i yields:

$$
\begin{aligned}
V''(\alpha_1) &= V''(\tfrac{1}{2}\pi) = 2\,W\,r(\sqrt{3} - 2) < 0 && \rightarrow \quad \text{unstable,} \\
V''(\alpha_2) &= V''(\tfrac{3}{2}\pi) = 2\,W\,r(-\sqrt{3} - 2) < 0 && \rightarrow \quad \text{unstable,} \\
V''(\alpha_3) &= V''(\tfrac{1}{3}\pi) = 2\,W\,r\left[\sqrt{3}\tfrac{1}{2}\sqrt{3} + 2\left(-\tfrac{1}{2}\right)\right] = W\,r > 0 && \\
& && \rightarrow \quad \text{stable,} \\
V''(\alpha_4) &= V''(\tfrac{2}{3}\pi) = 2\,W\,r\left[\sqrt{3}\tfrac{1}{2}\sqrt{3} + 2\left(-\tfrac{1}{2}\right)\right] = W\,r > 0 && \\
& && \rightarrow \quad \text{stable.}
\end{aligned}
$$

Fig. 8.16c shows the four equilibrium configurations.

E8.8

Example 8.8 A weightless rod is subjected to a vertical force F and held on each side by a spring (Fig. 8.17a). The spring constant of each spring is k. An appropriate support keeps the springs in a horizontal position during a rotation of the rod.

Determine the type of stability of the equilibrium positions.

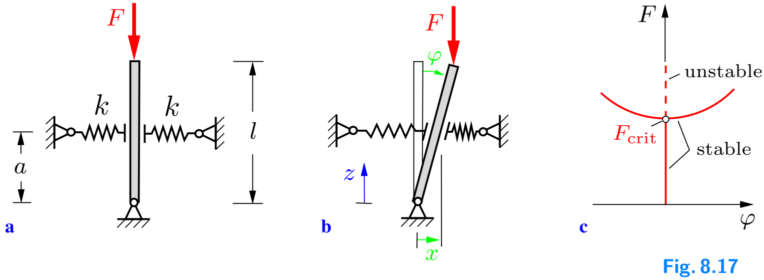


Fig. 8.17

Solution We introduce a coordinate system with its origin at the fixed support (Fig. 8.17b). To determine the potentials of the force F and the spring forces we consider an arbitrary position of the rod. The zero-level of the potential of force F (dead load, cf. Section 8.1) is chosen to be at $z = 0$. Thus, its potential is

$$V_F = F\, z_F\,.$$

According to (8.9) the potential of a spring (spring constant k) due to an elongation x is given by

$$V_f = \tfrac{1}{2}\, k\, x^2\,.$$

The system has one degree of freedom. Therefore, the total potential depends only on a single coordinate; it is useful to choose the angle φ (Fig. 8.17b). With the geometrical relations $z_F = l\, \cos\varphi$ and $x = a\, \tan\varphi$, the total potential becomes

$$V(\varphi) = F\, l \cos\varphi + 2\, \tfrac{1}{2}\, k\, (a \tan\varphi)^2\,.$$

We determine the equilibrium positions using condition (8.13):

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\varphi} = -\,F\, l \sin\varphi + 2\, k\, a^2 \frac{\tan\varphi}{\cos^2\varphi} = 0$$

$$\rightarrow \quad \sin\varphi\left(-F\,l + 2\,k\,a^2\frac{1}{\cos^3\varphi}\right) = 0\,.$$

A first equilibrium position is obtained from

$$\sin\varphi = 0 \quad \rightarrow \quad \underline{\underline{\varphi_1 = 0}}\,.$$

Further solutions follow from the term in parentheses:

$$-F\,l + 2\,k\,a^2\frac{1}{\cos^3\varphi} = 0 \quad \rightarrow \quad \cos^3\varphi_2 = \frac{2\,k\,a^2}{F\,l}$$

$$\rightarrow \quad \underline{\underline{\varphi_2 = \arccos\sqrt[3]{\frac{2\,k\,a^2}{F\,l}}}}\,. \qquad \text{(a)}$$

These positions exist only for $Fl > 2k\,a^2$.

In order to investigate the type of stability, the second derivative of the total potential is calculated:

$$V'' = \frac{\mathrm{d}^2V}{\mathrm{d}\varphi^2} = -F\,l\cos\varphi + 2\,k\,a^2\,\frac{\cos^2\varphi\,\dfrac{1}{\cos^2\varphi} + \tan\varphi\;2\cos\varphi\;\sin\varphi}{\cos^4\varphi}$$

$$= -F\,l\cos\varphi + 2\,k\,a^2\,\frac{1 + 2\sin^2\varphi}{\cos^4\varphi}$$

$$= -F\,l\cos\varphi + 2\,k\,a^2\,\frac{3 - 2\cos^2\varphi}{\cos^4\varphi}\,. \qquad \text{(b)}$$

Inserting the first solution $\varphi_1 = 0$ yields

$$V''(\varphi_1) = (-F\,l + 2\,k\,a^2) = 2\,k\,a^2\left(1 - \frac{F\,l}{2\,k\,a^2}\right)\,. \qquad \text{(c)}$$

The algebraic sign of V'' (and thus the stability of this equilibrium position) depends on the parameters that appear in parentheses. From (c), we obtain

$$V''(\varphi_1) > 0 \quad \text{for} \quad \frac{F\,l}{2\,k\,a^2} < 1 \quad \rightarrow \quad \text{stable position,}$$

$$V''(\varphi_1) < 0 \quad \text{for} \quad \frac{F\,l}{2\,k\,a^2} > 1 \quad \rightarrow \quad \text{unstable position.}$$

The special case

$$\frac{F\,l}{2\,k\,a^2} = 1 \quad \rightarrow \quad F = 2\,\frac{k\,a^2}{l} = F_{\text{crit}} \tag{d}$$

characterizes the "critical load", since at this point of the load-displacement curve in Fig. 8.17c, the stable position $\varphi = 0$ (for $F < F_{\text{crit}}$) changes to an unstable position (for $F > F_{\text{crit}}$).

The second derivative $V''(\varphi_1)$ is equal to zero for $F = F_{\text{crit}}$. To investigate the type of stability at this particular point, it would be necessary to consider higher derivatives of V.

If the angle φ_2 of the second equilibrium position is inserted into (b) and the relation $2\,k\,a^2 = F\,l\cos^3\varphi_2$ is used, we obtain for the second derivative of V

$$\begin{aligned}
V''(\varphi_2) &= -\,F\,l\cos\varphi_2\left(1 - \cos^2\varphi_2\,\frac{3 - 2\cos^2\varphi_2}{\cos^4\varphi_2}\right)\\
&= -\,F\,l\cos\varphi_2\left(1 - \frac{3}{\cos^2\varphi_2} + 2\right)\\
&= 3\,F\,l\cos\varphi_2\left(\frac{1}{\cos^2\varphi_2} - 1\right).
\end{aligned}$$

For $0 < \varphi_2 < \pi/2$, we have $\cos\varphi_2 < 1$. Hence, $V''(\varphi_2) > 0$, i.e., the equilibrium position $\varphi = \varphi_2$ is stable. Note, since $\cos\varphi$ is an even function, a second solution $\varphi_2^* = -\varphi_2$ exists. As a consequence, according to (a), $2\,k\,a^2/F\,l < 1$ holds and thus, according to (d), $F > F_{\text{crit}}$. In the special case of $\varphi_2 = 0$ we obtain $V''(\varphi_2) = 0$ and $F = F_{\text{crit}}$.

The result is depicted in Fig. 8.17c: for $F < F_{\text{crit}}$ there exists only one equilibrium position, i.e., $\varphi = 0$; this position is stable. For $F > F_{\text{crit}}$ this position becomes unstable, and two new stable equilibrium positions $\pm\varphi_2$ appear. Hence, three different equilibrium positions exist in this region.

Since a bifurcation of the solution appears for the critical load $F = F_{\text{crit}}$, this particular value in load-displacement diagrams is called the "bifurcation point". The critical load and the bifurcation of a solution play an important role in the analysis of the stability behavior of *elastic* structures (cf. Volume 2).

Example 8.9 The homogeneous body shown in Fig. 8.18a (density ϱ) consists of a half circular cylinder (radius r) and a rectangular block (height h).

E8.9

Determine the value of h for which the body is in neutral equilibrium at an arbitrary position.

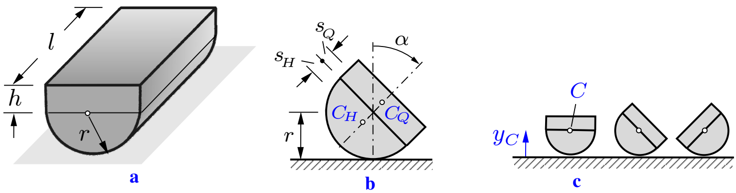


Fig. 8.18

Solution The rectangular block has the weight

$$W_Q = 2\,r\,h\,l\,\varrho\,g\,,$$

and the distance of its center of gravity measured from the separating surfaces of the two parts of the body is

$$s_Q = h/2$$

(Fig. 8.18b). The corresponding values for the half cylinder are

$$W_H = \frac{1}{2}\,\pi\,r^2\,l\,\varrho\,g$$

and

$$s_H = \frac{4}{3}\,\pi\,r$$

(see Table 4.1).

If we choose the surface of the base as the reference level for the potential, we obtain the following expression for an arbitrary position α:

$$V = \varrho\,g\,l\left[\frac{\pi\,r^2}{2}\left(r - \frac{4\,r}{3\,\pi}\cos\alpha\right) + 2\,r\,h\left(r + \frac{h}{2}\cos\alpha\right)\right].$$

The equilibrium positions are determined from

$$V' = \frac{\mathrm{d}V}{\mathrm{d}\alpha} = \varrho\, g\, l \sin\alpha \left[\frac{2}{3}\, r^3 - r h^2\right] = 0\,.$$

A first equilibrium position follows from

$$\sin\alpha = 0 \quad \rightarrow \quad \alpha_1 = 0\,.$$

Therefore, the reference configuration (characterized by $\alpha = 0$) in which the centers of gravity C_Q and C_H lie over each other on a vertical line is an equilibrium position. The derivative V' also vanishes if the term in the brackets becomes zero:

$$\frac{2}{3}\, r^3 - r h^2 = 0 \quad \rightarrow \quad \underline{\underline{h = \sqrt{\frac{2}{3}}\, r}}\,.$$

Equilibrium positions with arbitrary α exist for this specific value of h only.

To apply the stability criterion, we consider the second derivative

$$V'' = \varrho\, g\, l \cos\alpha \left[\frac{2}{3}\, r^3 - r h^2\right]\,.$$

For an arbitrary value of α and $h = \sqrt{2/3}\, r$, the second derivative V'' and all higher derivatives are equal to zero. The equilibrium positions are therefore neutral. The body is then in equilibrium in every arbitrary position, as shown in Fig. 8.18c.

This problem may also be solved without using the potential and its derivatives. An equilibrium position is neutral, if the center of gravity remains at the same elevation during a rotation ($\Delta V \equiv 0$). Therefore, the center of gravity C of the whole body must have the constant elevation r above the base for arbitrary α. Using (4.13), the center of gravity is given by

$$y_C = \frac{\dfrac{\pi\, r^2}{2}\left(r - \dfrac{4\, r}{3\,\pi}\right) + 2\, r h \left(r + \dfrac{h}{2}\right)}{\dfrac{\pi\, r^2}{2} + 2\, r h}\,.$$

After some algebraic manipulations, h follows from the condition $y_C = r$. This value is identical to that calculated above.

Example 8.10 Determine the type of stability of the equilibrium position in Problem 8.1. **E8.10**

Solution We choose the zero level of the potential of force W at the level of the support A (Fig. 8.9a). The zero level of the potential of force Q is chosen at the height at which the counterweight is located when the drawbridge is in the vertical position ($\varphi = 0$). With $s = 2l\sin(\varphi/2)$, the total potential as a function of the coordinate φ becomes

$$V(\varphi) = \frac{W\,l}{2}\cos\varphi + Q\,s = \frac{W\,l}{2}\cos\varphi + 2Q\,l\sin\frac{\varphi}{2}\,.$$

Using the relation $\sin\varphi = 2\sin(\varphi/2)\cos(\varphi/2)$, the derivatives are obtained as

$$V' = -\frac{W\,l}{2}\sin\varphi + Q\,l\cos\frac{\varphi}{2} = \Big(-W\sin\frac{\varphi}{2} + Q\Big)\,l\cos\frac{\varphi}{2}\,,$$

$$V'' = -\frac{W\,l}{2}\cos\varphi - \frac{Q\,l}{2}\sin\frac{\varphi}{2}\,.$$

From $V' = 0$, we again obtain the equilibrium position

$$\sin(\varphi/2) = Q/W.$$

Without further calculations, it can be seen that V'' is negative for $\varphi < \pi/2$. Therefore, the equilibrium position determined in Problem 8.1 is unstable.

8.6 Supplementary Problems

8.6

The detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E8.11

Example 8.11 The mechanism shown in Fig. 8.19 is subjected to a force F and a moment M_0. The weights of the three links may be neglected.

Apply the principle of virtual work to find the equilibrium position $\varphi = \varphi^*$.

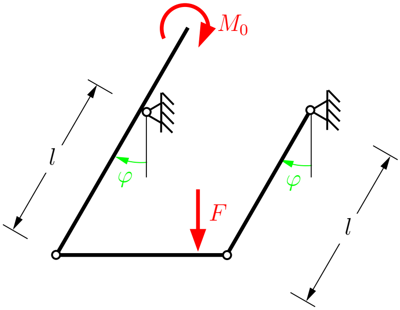


Fig. 8.19

Result: see (**B**) $\quad \varphi^* = \arcsin \dfrac{M_0}{F\,l}$.

E8.12

Example 8.12 A hinged beam is subjected to a line load q_0 and a concentrated force F (Fig. 8.20).

Determine the support reaction B with the aid of the principle of virtual work.

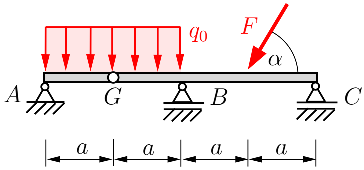


Fig. 8.20

Result: see (**B**) $\quad B = 2\,q_0 a + \frac{1}{2} F \sin\alpha$.

E8.13

Example 8.13 The system in Fig. 8.21 is held by a spring (stiffness k) and a torsion spring (stiffness k_T). The force in the spring and the moment in the torsion spring are zero in the equilibrium position shown in the figure. This equilibrium position is unstable if the applied force F exceeds a critical value F_{crit}.

Find F_{crit}.

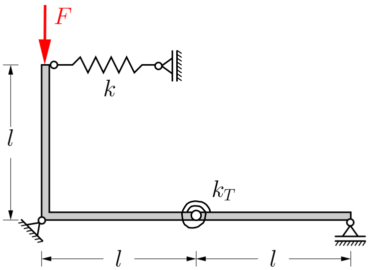


Fig. 8.21

Results: see (**A**) $\quad F_{\text{crit}} = kl + 4k_T/l$.

E8.14

Example 8.14 A hydraulic ramp is schematically depicted in Fig. 8.22. The two beams (each length l) are pin-connected at their centers M. A car (weight W) stands on the ramp.

Determine the force F which has to be generated in the hydraulic piston and applied to the lever of length a in order to keep the system in equilibrium.

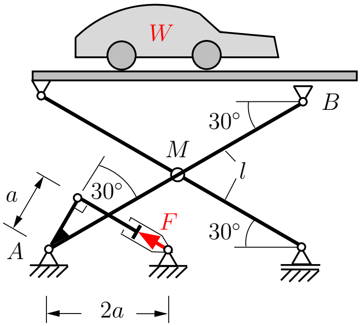


Fig. 8.22

Result: see (**B**) $F = \dfrac{\sqrt{3}\,l}{2\,a}\,W$.

E8.15

Example 8.15 A wheel (weight W, radius r) rolls on a circular cylinder (radius R) without sliding. It is connected to a wall by a spring (stiffness k). The spring is kept in a horizontal position by the support; the force in the spring is zero in the position shown in Fig. 8.23.

Determine the equilibrium positions and investigate their stability.

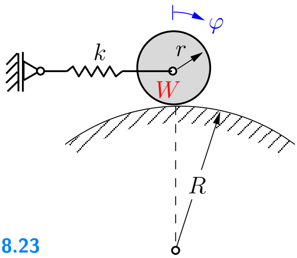


Fig. 8.23

Results: see (**B**)

$$\varphi_1 = 0 \quad (W < k(R+r)\text{: stable}),$$
$$\qquad\quad (W \geq k(R+r)\text{: unstable}),$$
$$\varphi_{2,3} = \pm \arccos \frac{W}{k(R+r)} \quad \text{(unstable)}.$$

E8.16

Example 8.16 The slider crank mechanism shown in Fig. 8.24 consists of the crank $\overline{AC}$ and the connecting rod $\overline{BC}$. Their weights can be neglected in comparison with the force F acting at B.

Determine the moment $M(\alpha)$ which is necessary to keep the system in equilibrium at an arbitrary angle α.

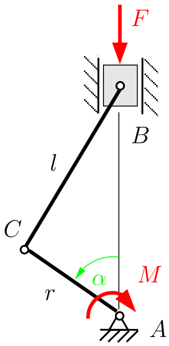


Fig. 8.24

Result: see (**A**) $M = Fr\,\sin\alpha\Big(1 + \dfrac{r\,\cos\alpha}{\sqrt{l^2 - r^2\sin^2\alpha}}\Big).$

E8.17

Example 8.17 Calculate the force S_1 in member 1 of the structure in Fig. 8.25.

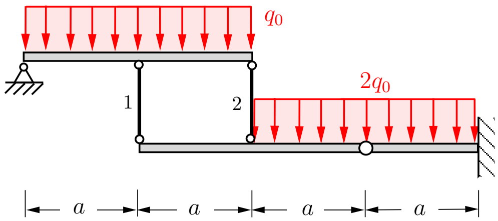


Fig. 8.25

Result: see (**A**) $S_1 = \frac{4}{3}\,q_0 a.$

E8.18

Example 8.18 A concentrated mass m is attached to a circular disk (radius R, mass M) as shown in Fig. 8.26. The disk can roll on an inclined plane (no sliding!).

Determine the positions of equilibrium and investigate their stability.

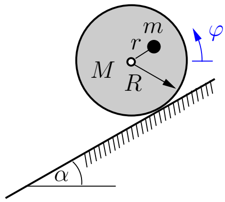


Fig. 8.26

Results: see (**B**) $\varphi_1 = 0 \quad (\kappa = 1,\ \text{unstable}),$

$\varphi_2 = \pm\arccos\kappa \quad (\kappa < 1,\ \text{stable}),$

$\kappa = \cos\varphi = (M+m)R\sin\alpha/(mr).$

Example 8.19 Figure 8.27 shows schematically the door CD (weight W, height $2r$) of a garage. It is supported by a lever BC and a spring AB (stiffness k). The spring is unstretched for $\alpha = \pi$. The distance between the points B and M is denoted by a.

Investigate the stability of the equilibrium configurations for the case $a \ll r$ and $Wr/ka^2 = 3$.

E8.19

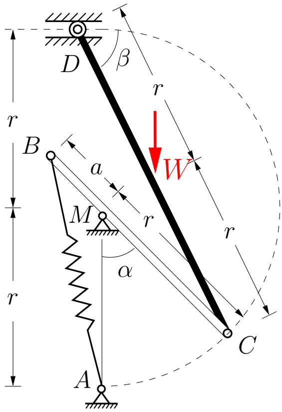


Fig. 8.27

Results: see (**A**) $\alpha_1 = 0$ (unstable), $\alpha_2 = \pi/3$ (stable), $\alpha_3 = \pi$ (unstable).

8.7

8.7 Summary

- The work done by a force $\boldsymbol{F}$ during an infinitesimal displacement $\mathrm{d}\boldsymbol{r}$ of the force application point is $\mathrm{d}U = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$. The work done during a finite displacement is given by

$$U = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}\,.$$

  Remark: The expression "work = force x displacement" is valid if and only if the directions of the force and of the displacement coincide and the force is constant.
- The work U of a conservative force is path-independent. A conservative force can be derived from a potential (potential energy) $V = -U$.
- The potentials of a weight W, a spring force and a torsion-spring moment, respectively, are given by

$$V = W\,z\,, \qquad V = \frac{1}{2}\,k\,x^2\,, \qquad V = \frac{1}{2}\,k_T\,\varphi^2\,.$$

- Virtual displacements $\delta\boldsymbol{r}$ or rotations $\delta\boldsymbol{\varphi}$ are infinitesimal, imaginary (they do not really exist) and kinematically possible. The virtual work of a force $\boldsymbol{F}$ is given by $\delta U = \boldsymbol{F} \cdot \delta\boldsymbol{r}$, and the virtual work of a moment $\boldsymbol{M}$ by $\delta U = \boldsymbol{M} \cdot \delta\boldsymbol{\varphi}$.
- A mechanical system is in equilibrium if the work done by the external forces disappears during a virtual displacement from this position:

$$\delta U = 0\,.$$

- In a conservative system, the total potential has an extreme value in an equilibrium position. The equilibrium position x_0 of a system with one degree of freedom is characterized by

$$V'(x_0) = 0\,.$$

- An equilibrium state is stable (unstable), if its total potential has a minimum (maximum):

$$V''(x_0) > 0 \quad \rightarrow \ \text{stable}\,, \qquad V''(x_0) < 0 \quad \rightarrow \ \text{unstable}\,.$$

Chapter 9

9

Static and Kinetic Friction

9 Static and Kinetic Friction

Objectives: Bodies in contact exert a force on each other. In the case of ideally smooth surfaces, this force acts perpendicularly to the contact plane. If the surfaces are rough, however, there may also be a tangential force component. Students will learn that this tangential component is a reaction force if the bodies adhere, and an active force if the bodies slip. After studying this chapter, students should be able to apply the Coulomb theory of friction to determine the forces in systems with contact.

9.1 Basic Principles

In this textbook so far it has been assumed that all bodies considered have *smooth* surfaces. Then, according to Chapter 2.4, only forces perpendicular to the contact plane can be transferred between two bodies in contact. This is a proper description of the mechanical behavior if the tangential forces occurring in reality due to the *roughness* of the surfaces can be neglected. In this chapter, we will address problems for which this simplification is not valid. First, let us consider the following simple example.

Fig. 9.1a shows a box with weight W resting on a rough horizontal surface. If a sufficiently small horizontal force F is applied to the box, it can be expected to stay at rest. The reason for this behavior is the transfer of a tangential force between the base and the box due to the surfaces' roughness. This tangential force is frequently called *static friction force* H.

Using the notation given in the free-body diagram (Fig. 9.1b), the equilibrium conditions for this system lead to the following relations:

$$\uparrow: \quad N = W\,, \qquad \rightarrow: \quad H = F\,. \tag{9.1}$$

The equilibrium of moments would additionally yield the location of N, which is not needed here.

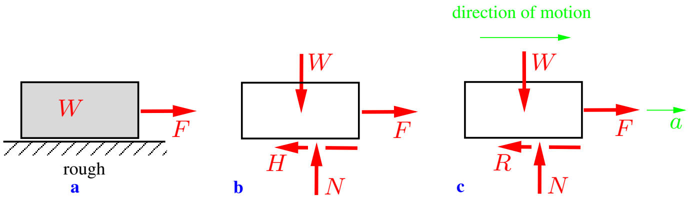


Fig. 9.1

If the force F exceeds a certain limit, the box slips on the base (Fig. 9.1c). Again, a force is transferred between the box and the base due to the roughness of the surfaces. This tangential force is commonly called *kinetic friction force* R. Since it tends to prevent the movement, its orientation is opposed to the direction of the motion. Assuming the acceleration a to be positive to the right, the second equilibrium condition (9.1) is replaced by Newton's second law (cf. Volume 3)

$$\text{mass times acceleration} = \sum \text{forces}\,,$$

i.e., in the current example,

$$m\,a = F - R\,. \tag{9.2}$$

Here, the kinetic friction force R is as yet unknown.

Even though static and kinetic friction are caused by the roughness of the surfaces, their specific nature is fundamentally different. The static friction force H is a *reaction force* that can be determined from the equilibrium conditions for statically determinate systems without requiring additional assumptions. On the other hand, the kinetic friction force R represents an *active force* depending on the surface characteristics of the bodies in contact. In order to keep this fundamental difference in mind, it must be carefully distinguished between *static friction* corresponding to friction in a position of rest and *kinetic friction* related to the movement of bodies in contact. Accordingly, we distinguish precisely between the corresponding *static friction forces* and the *kinetic friction forces*.

Friction forces are altered strongly if other materials are placed between the bodies considered. Every car driver or bicyclist knows the differences between a dry, wet, or even icy road. Lubricants can significantly decrease friction in the case of moveable machine parts. Due to the introductory nature of this chapter, "fluid friction" and related phenomena will not be addressed.

The following investigations are restricted to the case of so-called *dry* friction occurring due to the roughness of any solid body's surface.

Static and kinetic friction are of great practical relevance. It is static friction that enables motion on solid ground. For instance, wheels of vehicles adhere to the surface of the road. Forces needed for acceleration or deceleration are transferred at the contact areas. If these forces cannot be applied, for example in the case of icy roads, the wheels slide and the desired state of motion cannot be attained.

Screws and nails are able to perform their tasks due to their roughness. This effect is reinforced in the case of screw anchors with the increased asperity of their surfaces.

On the other hand, kinetic friction is often undesirable due to the resulting loss of energy. In the contact areas, mechanical energy is converted to thermal energy, resulting in a temperature increase. While one tries to increase static friction on slippery roads by spreading sand, the kinetic friction of rotating machine parts is reduced by lubricants, as mentioned before. Again, it becomes obvious that static and kinetic friction need to be distinguished carefully.

9.2 Coulomb Theory of Friction

9.2

Let us first consider static friction, using the example in Fig. 9.1b. As long as F is smaller than a certain limit F_0, the box stays at rest and equilibrium yields $H = F$. The tangential force H attains its maximum value $H = H_0$ for $F = F_0$. Charles Augustine de Coulomb (1736–1806) showed in his experiments that this limit force H_0 is in a first approximation proportional to the normal force N:

$$H_0 = \mu_0 \, N \, . \tag{9.3}$$

The proportionality factor μ_0 is commonly referred to as the *coefficient of static friction*. It depends solely on the roughness of surfaces in contact, irrespective of their size. Table 9.1 shows several numerical values for different configurations. Note that coefficients derived from experiments can only be given within certain tolerance limits; the coefficient for "wood on wood", for example, strongly depends on the type of wood and the treatment of the surfaces. It should also be noted that (9.3) relates the tangential force and the normal force only in the limit case when slip is impending; it is *not* an equation for the static friction force H.

A body adheres to its base as long as the *condition of static friction*

$$H \leq H_0 = \mu_0 \, N \tag{9.4a}$$

Table 9.1

	coefficient of static friction μ_0	coefficient of kinetic friction μ
steel on ice	0.03	0.015
steel on steel	$0.15\ldots0.5$	$0.1\ldots0.4$
steel on Teflon	0.04	0.04
leather on metal	0.4	0.3
wood on wood	0.5	0.3
car tire on snow	$0.7\ldots0.9$	$0.5\ldots0.8$
ski on snow	$0.1\ldots0.3$	$0.04\ldots0.2$

is fulfilled. The orientation of the friction force H always opposes the direction of the motion that would occur in the absence of friction. For complex systems, this orientation is often not easily identifiable, and must therefore be assumed arbitrarily. The algebraic sign of the result shows if this assumption was correct (compare e.g., Section 5.1.3). In view of a possible negative algebraic sign of H, we generalise condition (9.4a) as follows:

$$|H| \leq H_0 = \mu_0\, N\,. \tag{9.4b}$$

The normal force N and the static friction force H can be assembled into a resultant force K (Fig. 9.2a). Its direction is defined by the angle φ, which can be derived from

$$\tan\varphi = \frac{H}{N}\,.$$

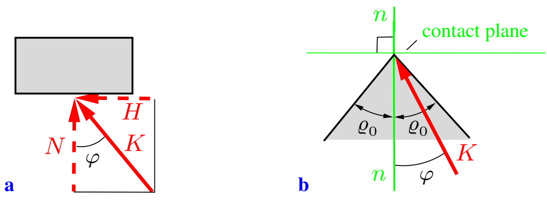


Fig. 9.2

Referring to the limit angle φ_l as ϱ_0 (in the case $H = H_0$) yields

$$\tan \varphi_l = \tan \varrho_0 = \frac{H_0}{N} = \frac{\mu_0 N}{N} = \mu_0 \,.$$

This so-called "angle of static friction" ϱ_0 is related to the coefficient of static friction:

$$\tan \varrho_0 = \mu_0 \,. \qquad (9.5)$$

A "static friction wedge" for a plane problem (Fig. 9.2b) is obtained by drawing the static friction angle ϱ_0 on both sides of the normal n. If K is located within this wedge, $H < H_0$ is valid and the body stays at rest.

In three-dimensional space, the static friction angle ϱ_0 can also be interpreted graphically. A body stays at rest if the reaction force K corresponding to an arbitrarily oriented external load is located within the so-called "static friction cone". This cone of revolution around the normal n of the contact plane has an angle of aperture of $2\varrho_0$. If K is located within the static friction cone, $\varphi < \varrho_0$ and consequently $|H| < H_0$ holds (Fig. 9.3).

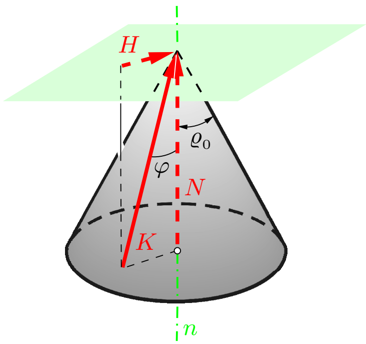


Fig. 9.3

If K lies outside of this cone, equilibrium is no longer possible: the body starts to move. We now will discuss the friction phenomena occurring in this case. Again, Coulomb demonstrated experimentally that the friction force R related to the movement is (in a good approximation)

a) proportional to the normal force N (proportionality factor μ) and
b) oriented in the opposite direction of the velocity vector while being independent of the velocity.

Consequently, the *law of friction* can be stated as follows:

$$R = \mu N . \tag{9.6}$$

The proportionality factor μ is referred to as the *coefficient of kinetic friction*. In general, its value is smaller than the formerly introduced coefficient of static friction μ_0 (compare Table 9.1).

When considering the direction of $\boldsymbol{R}$ by means of a mathematical formula, a unit vector $\boldsymbol{v}/|\boldsymbol{v}|$ oriented in the direction of the velocity vector $\boldsymbol{v}$ must be introduced. Coulomb's friction law then reads

$$\boldsymbol{R} = -\mu N \frac{\boldsymbol{v}}{|\boldsymbol{v}|} ,$$

with the minus sign indicating that the friction force acts in the opposite direction of the velocity vector. In contrast to static friction forces, the sense of direction of kinetic friction forces therefore cannot be assumed arbitrarily.

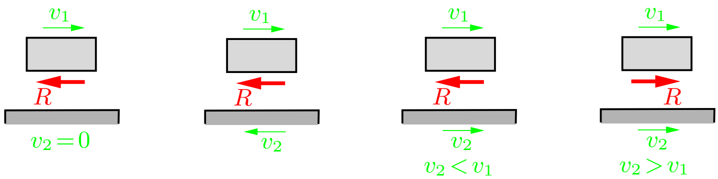


Fig. 9.4

If both the body and the base move (e.g., bulk goods slipping on a band-conveyor) then the direction of the kinetic friction force depends on the relative velocity, i.e., on the difference of the velocities v_1 and v_2 (cf. Volume 3). In Fig. 9.4, the resulting direction of the kinetic friction force exerted on the body is illustrated for different configurations.

In summary, the following three cases have to be distinguished:

a)	"Static friction" $H < \mu_0 N$	The body stays at rest; the corresponding static friction force H can be calculated from the equilibrium conditions.
b)	"Limiting friction" $H = \mu_0 N$	The body is still at rest but on the verge of moving. After a disturbance, the body will be set in motion due to the fact that $\mu < \mu_0$.
c)	"Kinetic friction" $R = \mu N$	If the body slips, the kinetic friction force R acts as an active force.

When investigating friction phenomena, one has to distinguish between statically determinate and statically indeterminate systems, respectively. In the first case, the reaction forces H and N can be calculated from the equilibrium conditions in a first step. Subsequently, fulfillment of the static friction condition (9.4b) can be checked. In statically indeterminate problems, determining the reaction forces H and N is not possible. In this case, one can only formulate equilibrium conditions as well as static friction conditions at those locations where the bodies adhere. Then a system of algebraic equations and inequalities needs to be solved. However, in this case it is often easier to investigate only the limiting friction case.

Example 9.1 A block with weight W resting on a rough inclined plane (angle of slope α, coefficient of static friction μ_0) is subjected to an external force F (Fig. 9.5a). **E9.1**

Specify the range of F such that the block stays at rest.

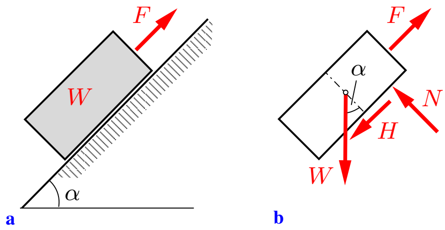


Fig. 9.5

Solution If F is a sufficiently large positive force, the block would move upwards without static friction. The static friction force H then is oriented downwards (Fig. 9.5b). From the equilibrium conditions

$$\nearrow: \quad F - W\sin\alpha - H = 0\,, \qquad \nwarrow: \quad N - W\cos\alpha = 0$$

the static friction force and the normal force can be determined:

$$H = F - W\,\sin\alpha\,, \qquad N = W\,\cos\alpha\,.$$

Inserting these forces into (9.4a) establishes the range of F fulfilling the static friction condition:

$$F - W\sin\alpha \le \mu_0\, W\cos\alpha \quad \to \quad F \le W(\sin\alpha + \mu_0\cos\alpha)\,.$$

Using (9.5), we can reformulate the preceding relation in terms of the static friction angle ϱ_0:

$$F \le W\,(\sin\alpha + \tan\varrho_0\cos\alpha) = W\,\frac{\sin(\alpha+\varrho_0)}{\cos\varrho_0}\,. \tag{a}$$

On the other hand, if F is too small, the block may slip downwards due to its weight. The static friction force preventing this motion is then oriented upwards according to Fig. 9.5c. In this case, from the equilibrium equations

$$\nearrow: \quad F - W\sin\alpha + H = 0\,, \qquad \nwarrow: \quad N - W\cos\alpha = 0$$

and the static friction condition

$$H \le \mu_0\, N\,,$$

we obtain the following inequality:

$$W\sin\alpha - F \le \mu_0\, W\cos\alpha\,.$$

This result can be formulated in terms of the static friction angle:

$$F \ge W\,(\sin\alpha - \mu_0\cos\alpha) = W\,\frac{\sin(\alpha-\varrho_0)}{\cos\varrho_0}\,. \tag{b}$$

Summarizing the results (a) and (b) yields the following admissible range for the force F:

$$\underline{\underline{W\,\frac{\sin(\alpha-\varrho_0)}{\cos\varrho_0}\le F\le W\,\frac{\sin(\alpha+\varrho_0)}{\cos\varrho_0}}}\,. \tag{c}$$

Assuming, for example, the case "steel on steel", friction coefficient $\mu_0 = 0.15$ from Table 9.1 yields $\varrho_0 = \arctan 0.15 = 0.149$. Choosing furthermore $\alpha = 10° \mathrel{\widehat{=}} 0.175\,\text{rad}$, we obtain from (c)

$$W\,\frac{\sin(0.175-0.149)}{\cos 0.149}\le F\le W\,\frac{\sin(0.175+0.149)}{\cos 0.149}$$

or

$$0.026\,W \le F \le 0.32\,W\,.$$

In this numerical example, the block stays at rest provided that F is in the range between approximately 3% and 30% of its weight W. If $\alpha < \varrho_0$, F can also take on negative values according to (c).

In the case $\alpha = \varrho_0$, the lower limit of the range of F equals zero. Therefore, the slope of the inclined plane is a direct measure of the coefficient of static friction. A body under the action of only its own weight (i.e., $F = 0$) stays at rest on an inclined plane as long as $\alpha \le \varrho_0$ holds.

Example 9.2 A man with weight Q stands on a ladder as depicted in Fig. 9.6a. E9.2

Determine the maximum position x he can reach on the ladder if a) *only* the floor and b) the floor *and* the wall have rough surfaces. The coefficient of static friction in both cases is μ_0.

Solution a) If the wall surface is smooth, the ladder is subjected only to a normal force N_B at B (Fig. 9.6b). At A, we have a normal force N_A and a static friction force H_A (opposing the movement that would occur without static friction). From the equilibrium conditions

$$\rightarrow:\; N_B - H_A = 0\,,\qquad \uparrow:\; N_A - Q = 0\,,\qquad \overset{\curvearrowleft}{A}:\; x\,Q - h\,N_B = 0$$

the normal force and static friction force at point A can be calculated:

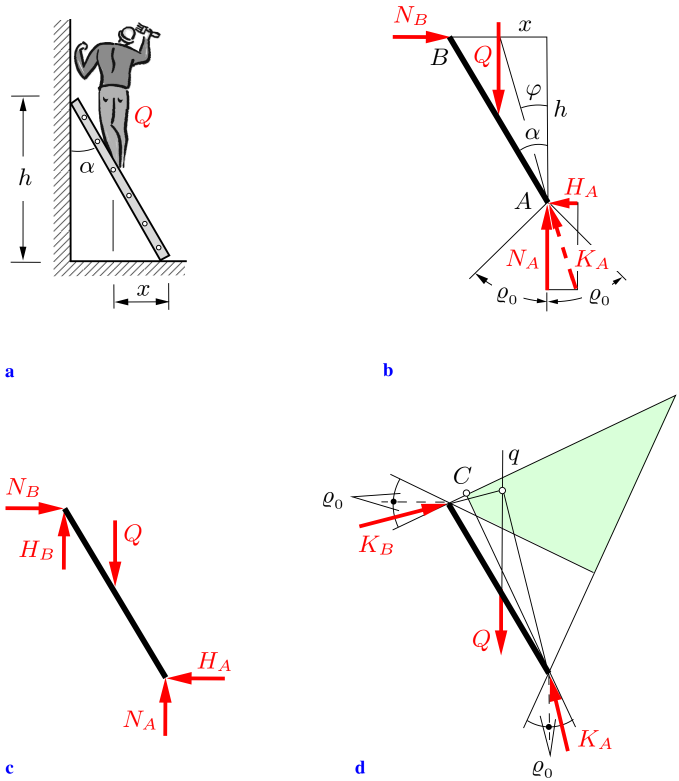


Fig. 9.6

$$H_A = \frac{x}{h}\, Q\,, \qquad N_A = Q\,.$$

Insertion into the condition of static friction

$$H_A \leq \mu_0\, N_A$$

yields the solution

$$\frac{x}{h}\, Q \leq \mu_0\, Q \quad \rightarrow \quad \underline{\underline{x \leq \mu_0\, h}}\,.$$

This result can also be obtained in the following way: in equilibrium, the lines of action of the three forces Q, N_B, and K_A (resultant of N_A and H_A) have to intersect at *one* point (Fig. 9.6b). Thus,

$$\tan\varphi = \frac{H_A}{N_A} = \frac{x}{h}\,.$$

The line of action of the reaction force K_A must be located within the static friction cone ($\varphi \leq \varrho_0$). Consequently, the ladder stays at rest provided that

$$\frac{x}{h} = \tan\varphi \leq \tan\varrho_0 = \mu_0 \quad \to \quad x \leq \mu_0\, h\,.$$

For $\alpha \leq \varrho_0$, the stability of the ladder is ensured for all possible values of x, due to $x \leq h\tan\alpha$.

b) If the surface of the wall is also *rough*, four unknown reaction forces are exerted on the ladder, according to Fig. 9.6c. However, only three equilibrium equations are available. Hence, these forces cannot be determined unambiguously and the problem is *statically indeterminate*. Nevertheless, one can calculate the admissible range of x from the equilibrium equations

$$\to: \quad N_B = H_A\,, \qquad \uparrow: \quad N_A + H_B = Q\,,$$

$$\stackrel{\curvearrowleft}{A}: \quad x\,Q = h\,N_B + (h\tan\alpha)\,H_B$$

and the conditions of static friction

$$H_B \leq \mu_0\, N_B\,, \quad H_A \leq \mu_0\, N_A\,.$$

Since the solution of the system of equations and inequalities is not straightforward, we prefer the graphical approach illustrated in Fig. 9.6d. For this purpose, the static friction cones are drawn at both points of contact. If the line of action q of the load Q is located within the domain of the overlapping cones marked in green, a multitude of possible reaction forces is conceivable; an example of *one* combination is illustrated in Fig. 9.6d. The ladder starts to slide if q is located to the left of C, since in this case the required static friction force cannot be applied anymore. Obviously, the danger of slipping is decreased or even eliminated in the case of a steeper position of the ladder.

E9.3

Example 9.3 A screw with a flat thread (coefficient of static friction μ_0, pitch h, radius r) is subjected to a vertical load F and a moment M_d as shown in Fig. 9.7a.

State the condition for equilibrium if the normal forces and the static friction forces are uniformly distributed over the screw thread.

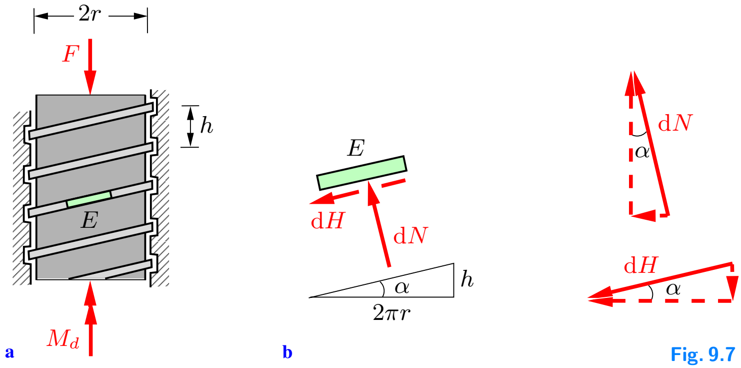


Fig. 9.7

Solution We decompose the normal force dN and the static friction force dH exerted on an element E of the thread into their horizontal and vertical components, as illustrated in Fig. 9.7b. The corresponding angle α can be determined from the pitch h and the unrolled perimeter $2\,\pi\,r$ as $\tan\alpha = h/2\,\pi\,r$. The resultant, i.e., the integral of the vertical components, has to equilibrate the external load F:

$$F = \int \mathrm{d}N \cos\alpha - \int \mathrm{d}H \sin\alpha = \cos\alpha \int \mathrm{d}N - \sin\alpha \int \mathrm{d}H\,. \quad \text{(a)}$$

Furthermore, M_d has to be in equilibrium with the moment resulting from the horizontal components:

$$M_d = \int r\,\mathrm{d}N \sin\alpha + \int r\,\mathrm{d}H \cos\alpha = r \sin\alpha \int \mathrm{d}N + r\cos\alpha \int \mathrm{d}H\,.$$

In combination with (a) we obtain

$$\int \mathrm{d}N = F\cos\alpha + \frac{M_d}{r}\sin\alpha\,, \qquad \int \mathrm{d}H = \frac{M_d}{r}\cos\alpha - F\sin\alpha\,.$$

Introducing the condition of static friction

$$|\mathrm{d}H| \leq \mu_0 \, \mathrm{d}N \quad \text{or} \quad \int |\mathrm{d}H| \leq \mu_0 \int \mathrm{d}N$$

yields

$$\left| \frac{M_d}{r} \cos\alpha - F \sin\alpha \right| \leq \mu_0 \left(F \cos\alpha + \frac{M_d}{r} \sin\alpha \right) .$$

If $M_d/r > F \tan\alpha$, this inequality results in

$$\left| \frac{M_d}{r} - F \tan\alpha \right| = \frac{M_d}{r} - F \tan\alpha \leq \mu_0 \left(F + \frac{M_d}{r} \tan\alpha \right)$$

or, using (9.5) and the addition theorem of the tangent function

$$\frac{M_d}{r} \leq F \frac{\tan\alpha + \mu_0}{1 - \tan\alpha \, \mu_0} = F \frac{\tan\alpha + \tan\varrho_0}{1 - \tan\alpha \tan\varrho_0} = F \tan(\alpha + \varrho_0) .$$

If, on the other hand, $M_d/r < F \tan\alpha$, we analogously obtain from

$$\left| \frac{M_d}{r} - F \tan\alpha \right| = F \tan\alpha - \frac{M_d}{r} \leq \mu_0 \left(F + \frac{M_d}{r} \tan\alpha \right)$$

the condition

$$\frac{M_d}{r} \geq F \tan(\alpha - \varrho_0) .$$

Hence, the screw is in equilibrium provided that

$$\underline{\underline{F \tan(\alpha - \varrho_0) \leq \frac{M_d}{r} \leq F \tan(\alpha + \varrho_0)}}$$

holds. In particular, if $\alpha \leq \varrho_0$ (i.e., $\tan\alpha \leq \mu_0$), the load F is supported only by the static friction forces and no additional external moment is required for equilibrium ($M_d = 0$). In this case, the screw is “self-locking”.

9.3 Belt Friction

9.3

If a rope wrapped around a rough post is subjected to a large force on one of its ends, a small force on the other end may be able to

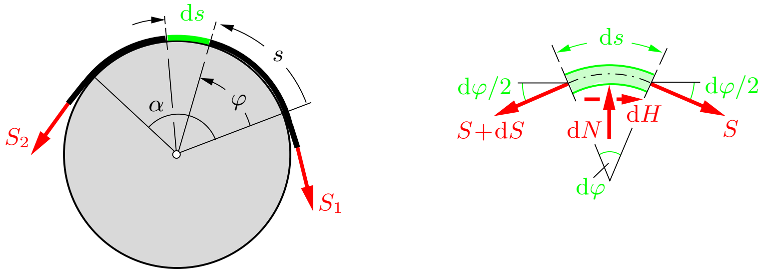


Fig. 9.8

prevent the rope from slipping. In Fig. 9.8a, the rope is wrapped around the post at an angle α. It is assumed that the force S_2 applied to the left end of the rope is larger than the force S_1 exerted on the right end. In order to establish a relation between these forces, we draw the free-body diagram shown in Fig. 9.8b and apply the equilibrium conditions to an element of the rope with length $\mathrm{d}s$. In this context, we take into account that the tension is changing by the infinitesimal force $\mathrm{d}S$ along $\mathrm{d}s$. Since $S_2 > S_1$ holds, the rope would slip to the left in a case without friction; i.e. slipping can be prevented only if the static friction force $\mathrm{d}H$ is oriented to the right. The equilibrium conditions are

$$\rightarrow: \quad S\cos\frac{\mathrm{d}\varphi}{2} - (S+\mathrm{d}S)\cos\frac{\mathrm{d}\varphi}{2} + \mathrm{d}H = 0\,,$$

$$\uparrow: \quad \mathrm{d}N - S\sin\frac{\mathrm{d}\varphi}{2} - (S+\mathrm{d}S)\sin\frac{\mathrm{d}\varphi}{2} = 0\,.$$

Since $\mathrm{d}\varphi$ is infinitesimally small, we obtain $\cos(\mathrm{d}\varphi/2) \approx 1$ and $\sin(\mathrm{d}\varphi/2) \approx \mathrm{d}\varphi/2$; furthermore, the higher order term $\mathrm{d}S(\mathrm{d}\varphi/2)$ is small and can be neglected in the following. Therefore, the above relations simplify to

$$\mathrm{d}H = \mathrm{d}S\,, \quad \mathrm{d}N = S\,\mathrm{d}\varphi\,. \tag{9.7}$$

Obviously, the three unknowns H, N, and S cannot be determined from these two equations: the system is statically indeterminate. Therefore only the limiting friction case is considered, i.e., when slippage of the rope is impending. In this case (9.3) gives

$$\mathrm{d}H = \mathrm{d}H_0 = \mu_0\,\mathrm{d}N\,.$$

Applying (9.7) yields

$$\mathrm{d}H = \mu_0\, S\, \mathrm{d}\varphi = \mathrm{d}S \quad \rightarrow \quad \mu_0\, \mathrm{d}\varphi = \frac{\mathrm{d}S}{S}\,.$$

Integration over the domain of rope contact produces

$$\mu_0 \int\limits_0^{\alpha} \mathrm{d}\varphi = \int\limits_{S_1}^{S_2} \frac{\mathrm{d}S}{S} \quad \rightarrow \quad \mu_0\, \alpha = \ln \frac{S_2}{S_1}$$

or

$$S_2 = S_1\, \mathrm{e}^{\mu_0 \alpha}\,. \tag{9.8}$$

This formula for belt friction is commonly named after Leonhard Euler (1707–1783) or Johann Albert Eytelwein (1764–1848).

If, in contrast to the initial assumption, $S_1 > S_2$ holds, one simply has to exchange the subscripts to obtain

$$S_1 = S_2\, \mathrm{e}^{\mu_0 \alpha} \qquad \text{or} \qquad S_2 = S_1\, \mathrm{e}^{-\mu_0 \alpha}\,. \tag{9.9}$$

For a given S_1, the system is in equilibrium provided that the value of S_2 remains within the limits given in (9.8) and (9.9):

$$S_1\, \mathrm{e}^{-\mu_0 \alpha} \le S_2 \le S_1\, \mathrm{e}^{\mu_0 \alpha}\,. \tag{9.10}$$

The rope slips to the right if $S_2 < S_1\, \mathrm{e}^{-\mu_0 \alpha}$, whereas it slips to the left for $S_2 > S_1\, \mathrm{e}^{\mu_0 \alpha}$.

The following numerical example gives a sense of the ratio between the two forces. We assume the rope to be wrapped n-times around the post; the coefficient of static friction is given by $\mu_0 = 0.3 \approx 1/\pi$. In this case we obtain

$$\mathrm{e}^{\mu_0 2n\pi} \approx \mathrm{e}^{2n} \approx (7.5)^n \qquad \text{and} \qquad S_1 = \frac{S_2}{\mathrm{e}^{\mu_0 \alpha}} = \frac{S_2}{(7.5)^n}\,.$$

Consequently, the more the rope is wrapped around the support, the smaller is the force S_1 that is required to equilibrate the larger force S_2. This effect is taken advantage of when, for example, mooring a boat.

The Euler-Eytelwein formula can be transferred from static to kinetic belt friction by simply replacing the static coefficient of friction μ_0 with the corresponding kinetic coefficient μ. Kinetic friction may occur if either the rope slips over a fixed drum or the drum rotates while the rope is at rest. The direction of the friction force R is opposite to the direction of the relative velocity (compare Fig. 9.4). When we know in which direction R acts, we also know which of the forces S_1 or S_2 is the larger one. Then

$$\begin{aligned} &\text{for } S_2 > S_1: \quad && S_2 = S_1\,\mathrm{e}^{\mu\alpha}\,, \\ &\text{for } S_2 < S_1: \quad && S_2 = S_1\,\mathrm{e}^{-\mu\alpha}\,. \end{aligned} \tag{9.11}$$

E9.4

Example 9.4 The cylindrical roller shown in Fig. 9.9a is subjected to a moment M_d. A rough belt (coefficient of static friction μ_0) is wrapped around the roller and connected to a lever.

Determine the minimum value of F such that the roller stays at rest (strap brake).

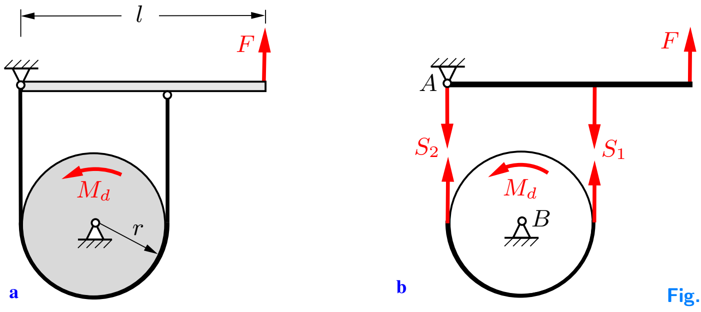


Fig. 9.9

Solution The free-body diagram of the system is given in Fig. 9.9b. The equilibrium of moments for both lever and roller yields:

$$\overset{\curvearrowleft}{A}: \qquad l\,F - 2\,r\,S_1 = 0 \quad \rightarrow \quad S_1 = \frac{l}{2\,r}\,F\,,$$

$$\overset{\curvearrowleft}{B}: \quad M_d + (S_1 - S_2)r = 0 \quad \rightarrow \quad S_1 = S_2 - \frac{M_d}{r}\,.$$

Obviously, $S_2 > S_1$ is valid for equilibrium due to the orientation of M_d. Introducing the angle of wrap $\alpha = \pi$, the condition for limiting friction follows from (9.8):

$$S_2 = S_1 \, e^{\mu_0 \pi} .$$

Hence,

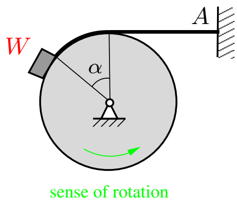
$$S_1 = S_1 \, e^{\mu_0 \pi} - \frac{M_d}{r} \quad \rightarrow \quad S_1 = \frac{M_d}{r(e^{\mu_0 \pi} - 1)} ,$$

and we obtain the required force

$$\underline{\underline{F}} = \frac{2\,r}{l} \, S_1 = \underline{\underline{2 \, \frac{M_d}{l} \frac{1}{e^{\mu_0 \pi} - 1}}} .$$

Example 9.5 A block with weight W lies on a rotating drum. It is held by a rope which is fixed at point A (Fig. 9.10a). **E9.5**

Determine the tension at A if friction acts between the drum and both the block and the rope (each with a coefficient of kinetic friction μ).



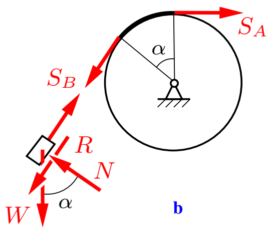


Fig. 9.10

Solution First, the bodies are separated. Due to the movement of the drum, a friction force R is exerted on the block. Its direction is given in the free-body diagram shown in Fig. 9.10b. Furthermore, $S_A > S_B$ holds. From the equilibrium conditions for the block the forces S_B and N are determined as

$$\nearrow : \quad S_B = W \sin\alpha + R , \qquad \nwarrow : \quad N = W \cos\alpha .$$

Introducing them into the friction laws for the rope (9.11) and for the block (9.6)

$$S_A = S_B\,\mathrm{e}^{\mu\alpha}\,, \qquad R = \mu\,N$$

we obtain

$$\underline{\underline{S_A}} = (W\sin\alpha + R)\,\mathrm{e}^{\mu\alpha} = \underline{\underline{W(\sin\alpha + \mu\cos\alpha)\,\mathrm{e}^{\mu\alpha}}}\,.$$

9.4

9.4 Supplementary Problems

Detailed solutions to most of the following examples are given in (**A**) D. Gross et al. *Formeln und Aufgaben zur Technischen Mechanik 1*, Springer, Berlin 2011 or (**B**) W. Hauger et al. *Aufgaben zur Technischen Mechanik 1-3*, Springer, Berlin 2011.

E9.6

Example 9.6 A sphere (weight W_1) and a wedge (weight W_2) are jammed between two vertical walls with rough surfaces (Fig. 9.11). The coefficient of static friction between the sphere and the left wall and between the wedge and the right wall, respectively, is μ_0. The inclined surface O of the wedge is smooth.

Determine the required value of μ_0 in order to keep the system in equilibrium.

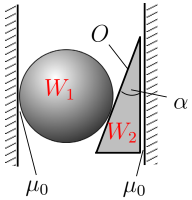


Fig. 9.11

Result: see (**B**) $\mu_0 \geq (1 + W_2/W_1)\tan\alpha$.

E9.7

Example 9.7 The excentric device in Fig. 9.12 is used to exert a large normal force onto the base. The applied force F, desired normal force N, coefficient of static friction μ_0, length l, radius r and angle α are given.

Calculate the required eccentricity e.

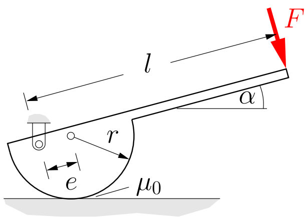


Fig. 9.12

Result: see (**A**) $e > \dfrac{l\,\dfrac{F}{N} - \mu_0\,r}{\cos\alpha - \mu_0\sin\alpha}.$

E9.8

Example 9.8 A horizontal force F is exerted on a vertical lever to prevent a load (weight W) from falling downwards (Fig. 9.13). The drum can rotate without friction about point B; the coefficient of static friction between the drum and the block is μ_0.

Determine the magnitude of the force F needed to prevent the drum from rotating.

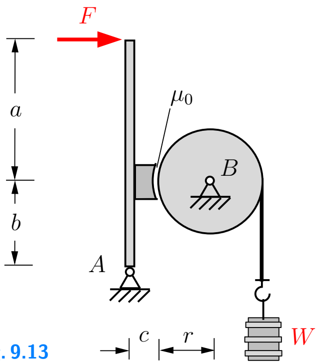


Fig. 9.13

Result: see (**B**) $F \geq \dfrac{b + \mu_0\, c}{\mu_0 (a + b)} W$.

E9.9

Example 9.9 A wall and a beam (weight $W_2 = W$) keep a roller (weight $W_1 = 3W$) in the position as shown in Fig. 9.14. The beam adheres to the rough base; all the other areas of contact are smooth.

Determine the minimum value of the coefficient of static friction μ_0 between the base and the beam in order to prevent slipping.

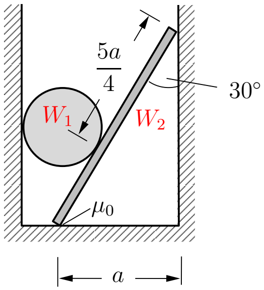


Fig. 9.14

Result: see (**B**) $\mu_0 \geq \sqrt{3}/3$.

E9.10

Example 9.10 A block (weight W_2) is clamped between two cylinders (each weight W_1) as shown in Fig. 9.15. All the surfaces are rough (coefficient of static friction μ_0).

Find the maximum value of W_2 in order to prevent slipping.

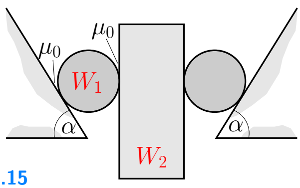


Fig. 9.15

Result: see (**A**) $W_2 < \dfrac{2\mu_0 \sin\alpha}{\cos\alpha - \mu_0(1 + \sin\alpha)} W_1$.

E9.11 **Example 9.11** A peg A that can rotate without friction and a fixed peg B are attached to a curved member (weight W) as depicted in Fig. 9.16. The rope supports a load (weight $W_K = W/5$).

Determine the number of coils of the rope around peg B that are needed to prevent slipping. Calculate the angle β in the equilibrium position.

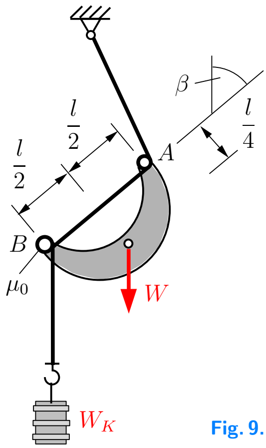


Fig. 9.16

Results: see (**B**) 3 coils are sufficient, $\beta = 19.7°$.

E9.12 **Example 9.12** A block with weight W can move vertically between two smooth walls. It is held by a rope which passes around three fixed rough pegs (coefficient of static friction μ_0) as shown in Fig. 9.17.

Calculate the force F which will ensure that the block remains suspended. Find the forces N_1, N_2 which are exerted from the block onto the walls.

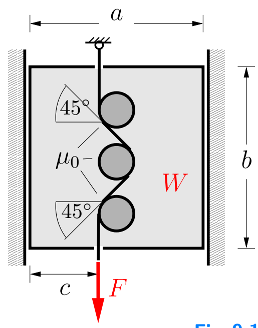


Fig. 9.17

Results: see (**A**) $F > \dfrac{W}{e^{\mu_0 \pi} - 1}$, $N_1 = N_2 = W\dfrac{a - 2c}{2b}$.

E9.13

Example 9.13 Three cylinders (each radius r, weight W) are arranged as shown in Fig. 9.18. The surfaces of all contact planes are rough (coefficient of static friction μ_0).

Determine the minimum value of μ_0 in order to prevent slipping.

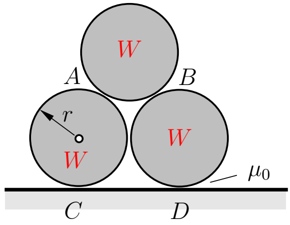


Fig. 9.18

Result: $\mu_0 \geq 0.268$.

E9.14

Example 9.14 A rotating drum (weight W_1) exerts a normal force and a kinetic friction force on a wedge (Fig. 9.19). The wedge lies on a rough base (coefficient of static friction μ_0).

Find the value of the coefficient of kinetic friction μ between the drum and the wedge that is required to move the wedge to the right.

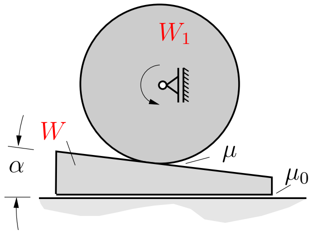


Fig. 9.19

Result: see (**A**) $\quad \mu = \dfrac{\mu_0(1 + W/W_1) + \tan\alpha}{1 - \mu_0(1 + W/W_1)\tan\alpha}$.

E9.15

Example 9.15 A beam (length $2a$, weight W) rests on support A. The triangle attached to its right end touches a rotating drum (Fig. 9.20). The coefficient of static friction μ_0 at A and the coefficient of kinetic friction μ at B are given.

a) Calculate the maximum allowable value of x in order to prevent slipping at A.

b) Determine the necessary value of μ_0 so that the beam does not slip for arbitrary values of x ($0 \leq x \leq a$).

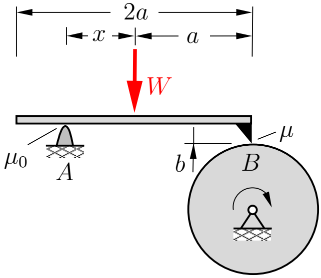


Fig. 9.20

Results: a) $x \leq \dfrac{\mu_0(a + \mu b)}{\mu}$, b) $\mu_0 \geq \dfrac{\mu a}{a + \mu b}$.

E9.16

Example 9.16 The rotating drum in Fig. 9.21 is encircled by a break band that is tightened by the applied force F. The coefficient of kinetic friction between the drum and the band is μ.

Calculate the magnitude of F that is necessary to induce a given breaking moment M_B if the rotation of the drum is clockwise (c) and if it is counterclockwise (cc).

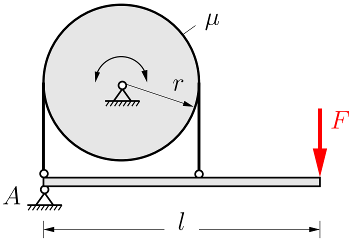


Fig. 9.21

Results: see (**A**) $F_c = \dfrac{2M_B}{l\,(\mathrm{e}^{\mu\pi} - 1)}$, $F_{cc} = \dfrac{2M_B\,\mathrm{e}^{\mu\pi}}{l\,(\mathrm{e}^{\mu\pi} - 1)}$.

9.5 Summary

- The static friction force H is a reaction force that can be determined directly from the equilibrium conditions in the case of statically determinate systems. Note that the orientation of H can be assumed arbitrarily in the free-body diagram.
- The absolute value of the static friction force H cannot exceed a certain limit friction force H_0. A body adheres to another body if the condition of static friction

  $$|H| \leq H_0 = \mu_0\, N$$

  is fulfilled.
- The kinetic friction force R is an active force given by Coulomb's friction law

  $$R = \mu\, N.$$

  The force R is oriented in the opposite direction of the (relative) velocity vector.
- In the case of static belt friction, the limiting case can be stated with the aid of the Euler-Eytelwein formula:

  $$S_2 = S_1 \mathrm{e}^{\mu_0 \alpha}\,.$$

- In the case of kinetic belt friction, the forces are also related by means of the Euler-Eytelwein formula if the coefficient of static friction μ_0 is replaced with the kinetic coefficient μ:

  $$S_2 = S_1 \mathrm{e}^{\mu \alpha}\,.$$

Appendix

A

Vectors, Systems of Equations

A Vectors, Systems of Equations

A.1

A.1 Vectors

Scalar physical quantities (e.g., time, mass, density) possess only magnitude. *Vectors* are physical quantities (e.g., force, velocity, acceleration) that possess magnitude and direction. A vector is represented geometrically by an arrow. The direction of the arrow coincides with the direction of the vector; the length of the arrow is a measure of the magnitude of the vector (Fig. A.1). It is common practice to denote a vector by a boldfaced letter, for example $\boldsymbol{A}$. The *magnitude* of a vector is denoted by $|\boldsymbol{A}|$ or by A. It is a positive number or zero. A vector with magnitude "1" is called a unit vector, and denoted by $\boldsymbol{e}$. A vector with magnitude zero is called a zero vector. The negative of a vector $\boldsymbol{A}$ is the vector $-\boldsymbol{A}$; it has the same magnitude as $\boldsymbol{A}$ but the opposite direction. Two vectors are equal if they have the same magnitude and the same direction.

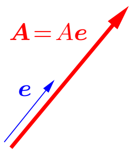


Fig. A.1

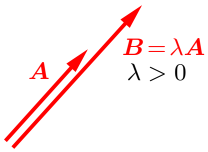


Fig. A.2

If a vector $\boldsymbol{A}$ is multiplied by a scalar quantity λ, the vector $\boldsymbol{B} = \lambda\,\boldsymbol{A}$ is obtained (Fig. A.2); it has the magnitude $|\boldsymbol{B}| = |\lambda||\boldsymbol{A}|$. The direction of $\boldsymbol{B}$ coincides with the direction of $\boldsymbol{A}$ if $\lambda > 0$, whereas it is oppositely directed if $\lambda < 0$. Accordingly, any vector may be written as the product of its magnitude and a unit vector with the same direction (Fig. A.1):

$$\boldsymbol{A} = A\,\boldsymbol{e}\,. \tag{A.1}$$

The addition of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ yields the vector

$$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}\,. \tag{A.2}$$

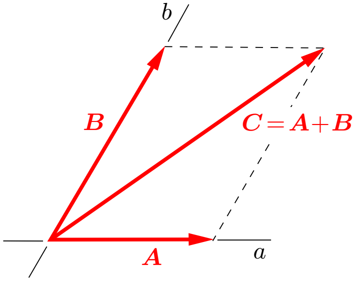


Fig. A.3

It may be obtained graphically: the sum of the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is the diagonal of the parallelogram of which $\boldsymbol{A}$ and $\boldsymbol{B}$ are adjacent sides (Fig. A.3).

This parallelogram may also be interpreted in the following way: a given vector $\boldsymbol{C}$ is resolved into the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ having the lines of action a and b. Vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are then called *components* of the vector $\boldsymbol{C}$ in the directions a and b. In two dimensions, the resolution of a vector into *two* different directions is unique. Similarly, the resolution into *three* directions (which are not coplanar) is uniquely possible in three-dimensional problems.

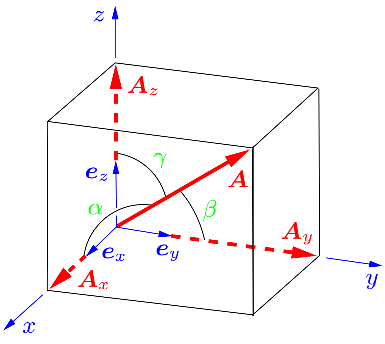


Fig. A.4

For the convenience of calculation, vectors are usually resolved in a Cartesian coordinate system (Fig. A.4). The components are then called rectangular components. The mutually perpendicular unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ (called base vectors) point in the positive x, y and z directions, respectively. The vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$, in this order, form a right-handed system: the thumb, the forefinger and the middle finger, in this order, can represent the directions of $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$.

The vector $\boldsymbol{A}$ may be resolved into its rectangular components $\boldsymbol{A}_x$, $\boldsymbol{A}_y$ and $\boldsymbol{A}_z$ in the directions x, y and z, respectively (Fig. A.4):

$$\boldsymbol{A} = \boldsymbol{A}_x + \boldsymbol{A}_y + \boldsymbol{A}_z \,. \tag{A.3}$$

According to (A.1), the components can be written as

$$\boldsymbol{A}_x = A_x\,\boldsymbol{e}_x\,, \quad \boldsymbol{A}_y = A_y\,\boldsymbol{e}_y\,, \quad \boldsymbol{A}_z = A_z\,\boldsymbol{e}_z\,. \tag{A.4}$$

Hence, we obtain from (A.3)

$$\boldsymbol{A} = A_x\,\boldsymbol{e}_x + A_y\,\boldsymbol{e}_y + A_z\,\boldsymbol{e}_z\,. \tag{A.5}$$

The quantities A_x, A_y and A_z are called *coordinates* of the vector $\boldsymbol{A}$. Note that they are often also called the components of $\boldsymbol{A}$ (even though, strictly speaking, the components are the vectors $\boldsymbol{A}_j$ $(j = x, y, z)$).

One may arrange the coordinates in the form of a column:

$$\boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} . \tag{A.6}$$

This representation of the vector $\boldsymbol{A}$ is called a column vector. Frequently, it is more appropriate to arrange the coordinates in a row instead of a column. If the columns and the rows are interchanged, the vector is "transposed"; this is indicated by the superscript "T". Hence, vector $\boldsymbol{A}$ may also be written in the form of a so-called row vector:

$$\boldsymbol{A} = (A_x, A_y, A_z)^T \,. \tag{A.7}$$

A vector is uniquely determined by its three coordinates. Its magnitude follows from the theorem of Pythagoras:

$$|\boldsymbol{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}\,. \tag{A.8}$$

The direction of $\boldsymbol{A}$ is determined by the three angles α, β and γ.

The direction cosines

$$\cos\alpha = \frac{A_x}{A}\,, \quad \cos\beta = \frac{A_y}{A}\,, \quad \cos\gamma = \frac{A_z}{A} \tag{A.9}$$

are taken from Fig. A.4. Hence, from (A.8)

$$\frac{A_x^2}{A^2} + \frac{A_y^2}{A^2} + \frac{A_z^2}{A^2} = 1 \tag{A.10}$$

and therefore

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1\,. \tag{A.11}$$

Thus, the three angles α, β and γ are not independent.

The vector equation

$$\boldsymbol{A} = \boldsymbol{B} \tag{A.12}$$

is equivalent to the three scalar equations

$$A_x = B_x\,, \quad A_y = B_y\,, \quad A_z = B_z\,. \tag{A.13}$$

Hence, two vectors are equal if the three coordinates of both vectors coincide.

In the following, some elements of vector algebra are given using the representation of the vectors in a Cartesian coordinate system.

A.1.1 Multiplication of a Vector by a Scalar

Using (A.3) and (A.4), the multiplication of a vector $\boldsymbol{A}$ by a scalar λ (Fig. A.2) yields the vector

$$\begin{aligned}\boldsymbol{B} &= \lambda\,\boldsymbol{A} = \boldsymbol{A}\,\lambda = \lambda(\boldsymbol{A}_x + \boldsymbol{A}_y + \boldsymbol{A}_z)\\ &= \lambda\,A_x\,\boldsymbol{e}_x + \lambda\,A_y\,\boldsymbol{e}_y + \lambda\,A_z\,\boldsymbol{e}_z\,.\end{aligned} \tag{A.14}$$

Hence, every coordinate of the vector is multiplied by the scalar λ. If λ is positive, $\boldsymbol{B}$ has the same direction as $\boldsymbol{A}$; if λ is negative, $\boldsymbol{B}$ points in the opposite direction. In the special case of $\lambda = -1$, the negative of vector $\boldsymbol{A}$ is obtained: $\boldsymbol{B} = -\boldsymbol{A}$. For $\lambda = 0$, the zero vector $\mathbf{0}$ is obtained.

A.1.2 Addition and Subtraction of Vectors

The sum of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is given by

$$\begin{aligned}\boldsymbol{C} &= \boldsymbol{A}+\boldsymbol{B} = (A_x\,\boldsymbol{e}_x+A_y\,\boldsymbol{e}_y+A_z\,\boldsymbol{e}_z)+(B_x\,\boldsymbol{e}_x+B_y\,\boldsymbol{e}_y+B_z\,\boldsymbol{e}_z)\\ &= (A_x+B_x)\,\boldsymbol{e}_x+(A_y+B_y)\,\boldsymbol{e}_y+(A_z+B_z)\,\boldsymbol{e}_z \\ &= C_x\,\boldsymbol{e}_x+C_y\,\boldsymbol{e}_y+C_z\,\boldsymbol{e}_z\,.\end{aligned} \tag{A.15}$$

Hence,

$$C_x = A_x + B_x\,,\quad C_y = A_y + B_y\,,\quad C_z = A_z + B_z\,. \tag{A.16}$$

Two vectors are added by adding the corresponding coordinates.

To subtract a vector $\boldsymbol{B}$ from a vector $\boldsymbol{A}$ is equivalent to adding the negative of $\boldsymbol{B}$:

$$\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})\,. \tag{A.17}$$

Therefore,

$$C_x = A_x - B_x\,,\quad C_y = A_y - B_y\,,\quad C_z = A_z - B_z\,. \tag{A.18}$$

A.1.3 Dot Product

Consider two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ that include the angle φ (Fig. A.5a). The *dot product* (so called because of the notation "·") or *scalar product* is defined as

$$\boldsymbol{A}\cdot\boldsymbol{B} = A\,B\cos\varphi\,. \tag{A.19}$$

The scalar product yields a scalar (hence the name), not a vector! It may be interpreted in one of the following ways (Fig. A.5b):

a) product of the magnitudes of $\boldsymbol{A}$ and $\boldsymbol{B}$ multiplied by the cosine of the angle φ,

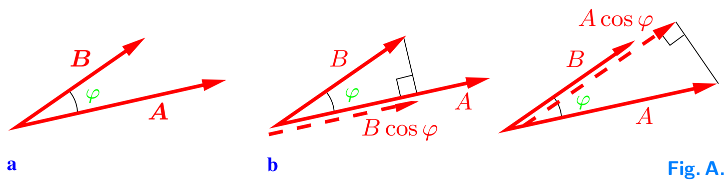


Fig. A.5

b) magnitude of $\boldsymbol{A}$ multiplied by the component $B\cos\varphi$ of $\boldsymbol{B}$ in the direction of $\boldsymbol{A}$,
c) magnitude of $\boldsymbol{B}$ multiplied by the component $A\cos\varphi$ of $\boldsymbol{A}$ in the direction of $\boldsymbol{B}$.

The scalar product is positive if the two vectors include an acute angle ($\varphi < \pi/2$) and negative if the angle is obtuse ($\varphi > \pi/2$). In the special case of two orthogonal vectors ($\varphi = \pi/2$), the scalar product is zero.

From the definition (A.19) of the scalar product, it follows that

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}\,. \tag{A.20}$$

Hence, the order of the factors may be interchanged: the commutative law is valid.

Introducing the vector components, the scalar product can be written as

$$\boldsymbol{A} \cdot \boldsymbol{B} = (A_x\,\boldsymbol{e}_x + A_y\,\boldsymbol{e}_y + A_z\,\boldsymbol{e}_z)\cdot(B_x\,\boldsymbol{e}_x + B_y\,\boldsymbol{e}_y + B_z\,\boldsymbol{e}_z)\,. \tag{A.21}$$

Since

$$\begin{aligned} \boldsymbol{e}_x \cdot \boldsymbol{e}_x &= \boldsymbol{e}_y \cdot \boldsymbol{e}_y = \boldsymbol{e}_z \cdot \boldsymbol{e}_z = 1\,, \\ \boldsymbol{e}_x \cdot \boldsymbol{e}_y &= \boldsymbol{e}_y \cdot \boldsymbol{e}_z = \boldsymbol{e}_z \cdot \boldsymbol{e}_x = 0\,, \end{aligned} \tag{A.22}$$

we obtain

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z\,. \tag{A.23}$$

In the special case of $\boldsymbol{B} = \boldsymbol{A}$, the angle φ is zero. Then, (A.19) yields the magnitude of vector $\boldsymbol{A}$:

$$\boldsymbol{A} \cdot \boldsymbol{A} = A^2 \quad \text{or} \quad A = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}}\,. \tag{A.24}$$

A.1.4 Vector Product (Cross-Product)

The *vector product* (because of the notation "$\times$" also called the *cross-product*) of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ yields a vector. It is written as

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} \tag{A.25}$$

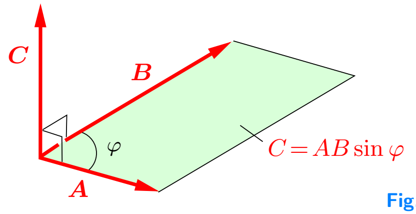


Fig. A.6

and is defined as follows:

a) The vector $\boldsymbol{C}$ is perpendicular to both vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ (Fig. A.6).
b) The magnitude of $\boldsymbol{C}$ is numerically equal to the area of the parallelogram formed by $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$|\boldsymbol{C}| = C = A\,B \sin\varphi\,. \tag{A.26}$$

c) The vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ in this order form a right-handed system.

Therefore,

$$\boldsymbol{A} \times \boldsymbol{B} = -\,\boldsymbol{B} \times \boldsymbol{A}\,. \tag{A.27}$$

Hence, the commutative law does not hold for a vector product.

According to b), the vector product is zero for two parallel vectors ($\varphi = 0$).

From the definition of the cross product we have

$$\begin{aligned}
&\boldsymbol{e}_x \times \boldsymbol{e}_x = \boldsymbol{0}\,, && \boldsymbol{e}_x \times \boldsymbol{e}_y = \boldsymbol{e}_z\,, && \boldsymbol{e}_x \times \boldsymbol{e}_z = -\boldsymbol{e}_y\,,\\
&\boldsymbol{e}_y \times \boldsymbol{e}_x = -\boldsymbol{e}_z\,, && \boldsymbol{e}_y \times \boldsymbol{e}_y = \boldsymbol{0}\,, && \boldsymbol{e}_y \times \boldsymbol{e}_z = \boldsymbol{e}_x\,,\\
&\boldsymbol{e}_z \times \boldsymbol{e}_x = \boldsymbol{e}_y\,, && \boldsymbol{e}_z \times \boldsymbol{e}_y = -\boldsymbol{e}_x\,, && \boldsymbol{e}_z \times \boldsymbol{e}_z = \boldsymbol{0}\,.
\end{aligned} \tag{A.28}$$

Hence,

$$\begin{aligned}
\boldsymbol{C} = \boldsymbol{A}\times\boldsymbol{B} &= (A_x\,\boldsymbol{e}_x + A_y\,\boldsymbol{e}_y + A_z\,\boldsymbol{e}_z)\times(B_x\,\boldsymbol{e}_x + B_y\,\boldsymbol{e}_y + B_z\,\boldsymbol{e}_z)\\
&= (A_yB_z - A_zB_y)\,\boldsymbol{e}_x + (A_zB_x - A_xB_z)\,\boldsymbol{e}_y\\
&\quad + (A_xB_y - A_yB_x)\,\boldsymbol{e}_z\,,
\end{aligned} \tag{A.29}$$

which yields the coordinates of vector $\boldsymbol{C}$:

$$\begin{aligned} C_x &= A_y B_z - A_z B_y\,, \\ C_y &= A_z B_x - A_x B_z\,, \\ C_z &= A_x B_y - A_y B_x\,. \end{aligned} \tag{A.30}$$

Note that C_y and C_z can be obtained from C_x also by *cyclic permutation*: x is replaced by y, y is replaced by z and z by x.

The vector product may also be written in the form of the determinant

$$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\,. \tag{A.31}$$

The first row is given by the unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$. The coordinates of vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are written in the second and the third row, respectively. Expanding the determinant yields (compare (A.29))

$$\begin{aligned} \boldsymbol{C} &= \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \boldsymbol{e}_x - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \boldsymbol{e}_y + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \boldsymbol{e}_z \\ &= (A_y B_z - A_z B_y)\,\boldsymbol{e}_x + (A_z B_x - A_x B_z)\,\boldsymbol{e}_y \\ &\quad + (A_x B_y - A_y B_x)\,\boldsymbol{e}_z\,. \end{aligned} \tag{A.32}$$

The triple vector product $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C})$ (the parentheses are needed!) is a vector that lies in the plane defined by $\boldsymbol{B}$ and $\boldsymbol{C}$. Applying (A.31) yields the relation

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}\,. \tag{A.33}$$

A.2 Systems of Linear Equations

A.2

Frequently, the treatment of a problem in mechanics leads to a system of linear equations. Examples are the determination of the support reactions of a structure or of the forces in the members of a truss. In the case of a beam in a plane problem, the equilibrium conditions yield three equations for the three unknown support reactions. For a space truss with j joints, there are $3j = m + r$ equations to calculate the m unknown forces in the members and the r support reactions.

Let us consider the system

$$\begin{aligned}
&a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1n}\,x_n = b_1\,,\\
&a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2n}\,x_n = b_2\,,\\
&\ldots\ldots\ldots\\
&a_{n1}\,x_1 + a_{n2}\,x_2 + \ldots + a_{nn}\,x_n = b_n
\end{aligned} \tag{A.34}$$

of n linear inhomogeneous equations for the n unknowns $x_1, x_2, \ldots$ $\ldots, x_n$ (for example, the support reactions and/or forces in the members of a truss). The coefficients a_{jk} and the "right-hand sides" b_k are known. In matrix notation

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{A.35}$$

equation (A.34) may be written in the form

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b}\,. \tag{A.36}$$

If the determinant of matrix $\boldsymbol{A}$ is nonzero, i.e.,

$$\det \boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \neq 0\,, \tag{A.37}$$

then the n equations (A.34) are linearly independent. In this case, the system of equations (A.36) has the unique solution

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\,\boldsymbol{b}\,. \tag{A.38}$$

Matrix $\boldsymbol{A}^{-1}$ is called the *inverse* of $\boldsymbol{A}$. It is defined by $\boldsymbol{A}^{-1}\,\boldsymbol{A} = \mathbf{1}$, where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \tag{A.39}$$

is the unit matrix. It is often rather time consuming to calculate the inverse with pencil and paper. However, it can be very easily determined with the aid of computer programs, e.g., MATLAB or MATHEMATICA.

Practical methods to solve a system of equations include the *Gaussian elimination* (Carl Friedrich Gauss, 1777-1855) (also called *Gauss's algorithm*) and *Cramer's rule* (Gabriel Cramer, 1704-1752). If we apply Gauss's algorithm, the unknowns are systematically eliminated to obtain from (A.34) the equivalent system

$$\begin{aligned} a'_{11}\,x_1 + a'_{12}\,x_2 + \ldots + a'_{1n}\,x_n &= b'_1\,, \\ a'_{22}\,x_2 + \ldots + a'_{2n}\,x_n &= b'_2\,, \\ \ldots\ldots\ldots \\ a'_{nn}\,x_n &= b'_n\,. \end{aligned} \tag{A.40}$$

Beginning with the last equation, the unknowns can be computed successively.

As an example, consider the system

$$\begin{array}{rcrcrcrcl} 2\,x_1 &+& 5\,x_2 &+& 8\,x_3 &+& 4\,x_4 &=& 3\,, \\ 6\,x_1 &+& 16\,x_2 &+& 22\,x_3 &+& 13\,x_4 &=& 9\,, \\ 4\,x_1 &+& 14\,x_2 &+& 28\,x_3 &+& 10\,x_4 &=& 4\,, \\ 10\,x_1 &+& 23\,x_2 &+& 84\,x_3 &+& 25\,x_4 &=& 22 \end{array}$$

of four equations for the four unknowns $x_1, \ldots, x_4$. The first equation (row) is multiplied by -3 and added to the second row. Also, the first row is multiplied by -2 and added to the third one, etc. Thus, the unknown x_1 is eliminated from the second, third and fourth equation:

$$\begin{aligned} 2\,x_1 \;+\; 5\,x_2 \;+\; 8\,x_3 \;+\; 4\,x_4 &= 3\,, \\ x_2 \;-\; 2\,x_3 \;+\; x_4 &= 0\,, \\ 4\,x_2 \;+\; 12\,x_3 \;+\; 2\,x_4 &= -2\,, \\ -\; 2\,x_2 \;+\; 44\,x_3 \;+\; 5\,x_4 &= 7\,. \end{aligned}$$

Similarly, the unknowns x_2 and x_3 are subsequently eliminated. It is convenient to arrange the algorithm according to the following scheme, where only the coefficients a_{jk} and the right-hand sides b_j are written down:

x_1	x_2	x_3	x_4	$\boldsymbol{b}$	
2	5	8	4	3	(a)
6	16	22	13	9	
4	14	28	10	4	
10	23	84	25	22	
0	1	−2	1	0	(b)
0	4	12	2	−2	
0	−2	44	5	7	
0	0	20	−2	−2	(c)
0	0	40	7	7	
0	0	0	11	11	(d)

With the coefficients (a)–(d), we obtain the "staggered" system (compare (A.40))

$$\begin{aligned} 2\,x_1 \;+ 5\,x_2 \;+\; 8\,x_3 \;+\; 4\,x_4 &= 3\,, \\ x_2 \;-\; 2\,x_3 \;+\; x_4 &= 0\,, \\ 20\,x_3 \;-\; 2\,x_4 &= -2\,, \\ 11\,x_4 &= 11\,. \end{aligned}$$

Beginning with the last row, we get successively

$$x_4 = 1\,, \qquad x_3 = 0\,, \qquad x_2 = -1\,, \qquad x_1 = 2\,.$$

According to Cramer's rule, the unknowns are given by

$$x_k = \frac{\det(\boldsymbol{A}_k)}{\det \boldsymbol{A}}, \qquad k = 1, \ldots, n\,. \tag{A.41}$$

The determinant $\det(\boldsymbol{A}_k)$ is obtained by replacing the k-th column of the determinant of $\boldsymbol{A}$ by $\boldsymbol{b}$.

As an example, consider the system

$$\begin{aligned} a_{11}\,x_1 + a_{12}\,x_2 &= b_1\,, \\ a_{21}\,x_1 + a_{22}\,x_2 &= b_2\,. \end{aligned}$$

The unknowns are found to be

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}\,,$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11} b_2 - b_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}\,.$$

Note that Cramer's rule is suitable only for systems with two (at most three) unknowns. For larger systems, Gaussian elimination is recommended. It should also be noted that for large systems, a loss of accuracy may occur due to round-off errors.

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