Maximum Steady Solids Discharge Rates from Mass Flow Hoppers

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The steady-state discharge rate of a coarse powder from a hopper can be determined by a force balance. Consider a hopper with the geometry shown in Figure 1. If a coarse powder is handled, an equilibrium force balance on the powder inside the hopper is:

$$a = -g \tag{1}$$

where a is the powder's acceleration and g is the acceleration due to gravity. Since

$$a = \frac{dv}{dt} = \frac{dz}{dt}\frac{dv}{dz} = v\frac{dv}{dz} \tag{2}$$

where r is the radial coordinate, v is the velocity of the powder, and t denotes time. Equation 1 can be rewritten as

$$v\frac{dv}{dz} = -g \tag{3}$$

Continuity of the solids stream can be expressed as

$$d\left(\rho_b v A\right) = 0 \tag{4}$$

where ρ_b is the bulk density of the powder, and A is the cross-sectional area. Neglecting changes in bulk density,

$$\frac{dv}{dz} = -\frac{v}{A}\frac{dA}{dz} \tag{5}$$

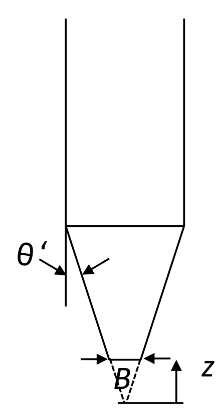

Figure 1. Hopper geometry.

and therefore,

$$\frac{v^2}{A}\frac{dA}{dz} = g \tag{6}$$

For round outlets,

$$A = \pi r^2 = \pi(z\tan\theta')^2 \tag{7}$$

$$\frac{dA}{dz} = 2\pi z\tan\theta' \tag{8}$$

where z is the distance from the hopper vertex. At the hopper outlet,

$$\frac{1}{A}\frac{dA}{dz} = \frac{4\tan\theta'}{B} \tag{9}$$

where B is the outlet diameter, θ' is the hopper angle (from vertical), and the subscript o denotes the outlet. Hence, from Equation 6,

$$\frac{4v_o^2\tan\theta'}{B} = g \tag{10}$$

and

$$v_o = \sqrt{\frac{Bg}{4\tan\theta'}} \tag{11}$$

The discharge rate $\dot{m}_S$ is the product of the velocity, the bulk density at the outlet ρ_{bo}, and the cross-sectional area of the outlet A_o, *i.e.*,

$$\dot{m}_S = \rho_{bo}A_o\sqrt{\frac{Bg}{4\tan\theta'}} \tag{12}$$

For planar geometries and slotted outlets, a similar analysis yields

$$v_o = \sqrt{\frac{Bg}{2\tan\theta'}} \tag{13}$$

and

$$\dot{m}_S = \rho_{bo}A_o\sqrt{\frac{Bg}{2\tan\theta'}} \tag{14}$$

For wedge-shaped and transition hoppers with slotted outlets, B denotes the outlet width. The general form of the solids discharge mass flow rate is therefore given by

$$\dot{m}_S = \rho_{bo}A_o\sqrt{\frac{Bg}{2(m+1)\tan\theta'}} \tag{15}$$

where m is equal to 1 for conical hoppers and equal to 0 for hoppers with straight walls and slotted outlets and B is the diameter of the outlet of a conical hopper or the width of a slotted outlet beneath a planar-flow hopper.

The maximum flow rate of a fine powder can be several orders of magnitude lower than that of coarser materials. Two-phase flow effects are significant due to the movement of interstitial gas as the powder compresses and expands during flow. Figure 2 illustrates solids and gas pressure profiles in bins for coarse (high permeability) and fine (low permeability) powders.

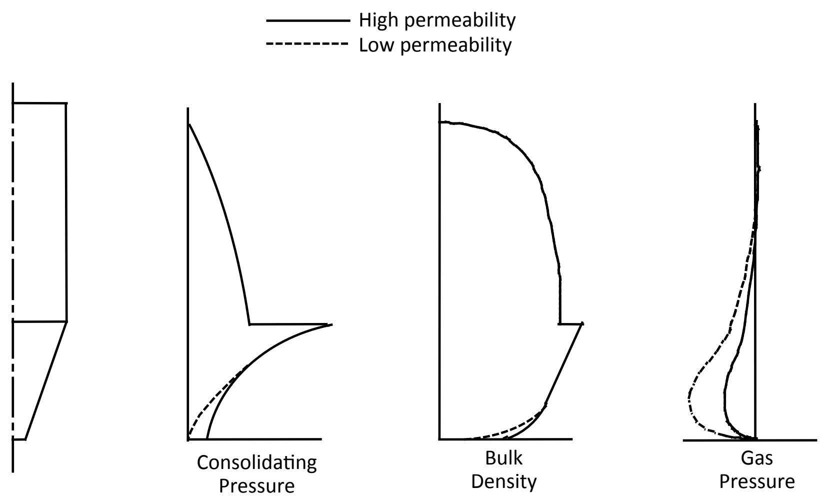

Figure 2. Consolidating pressure, bulk density, and gas pressure profiles for coarse (high permeability) and fine (low permeability) powders.

In the straight-walled section of a bin, the stress level increases with depth, causing the bulk density of the material to increase and its void fraction to decrease, squeezing out a portion of the interstitial gas. This gas leaves the bulk material through its top free surface. In the hopper section of the bin, the consolidated material expands as it flows toward the outlet, reducing its bulk density and increasing its void fraction. This expansion results in a reduction of the interstitial gas pressure to below atmospheric (*i.e.*, vacuum), causing gas counter flow through the outlet if the pressure below the outlet is atmospheric. At a critical solids discharge rate, the solids contact pressure reduces to zero, and efforts to exceed this limiting discharge rate will result in erratic flow.

For fine powders, Equation 1 should be rewritten as

$$\rho_b a = -\rho_b g - \frac{dP}{dz} \tag{16}$$

where P is the interstitial gas pressure (and dP/dz is the gas pressure gradient). Following the same analysis as before yields

$$\dot{m}_S = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 + \frac{1}{\rho_{bo} g}\left.\frac{dP}{dz}\right|_o\right)} \tag{17}$$

Because the pressure gradient at the outlet $dP/dz|_o$ is often less than zero for fine powders, Equation 17 shows they can have discharge rates dramatically lower than those of coarse powders.

The pressure gradient is related to the material's permeability and the rate of air counter flow by Darcy's law. Using Jenike's convention[1], Darcy's law is given by

$$u_g = -\frac{K}{\rho_b g}\frac{dP}{dz} \tag{18}$$

where u_g is the superficial gas velocity and K is the permeability of the bulk material. (Defined this way, when $dP/dz = \rho_b g$, $K = u_g$, *i.e.,* the drag forces are equal to the body forces. Hence, K is related to the powder's minimum fluidization velocity.)

Applying continuity to the gas phase, Gu *et al*. [*Powder Techn.,* 72, 39 (1992)] derived a relationship between the air and solids flow rates that when combined with Darcy's law gives:

$$\left.\frac{dp}{dz}\right|_o = \frac{v_o \rho_{bo}^2 g}{K_o}\left(\frac{1}{\rho_{bmp}} - \frac{1}{\rho_{bo}}\right) \tag{19}$$

where K_o is the permeability of the powder at the hopper outlet, ρ_{bo} is its bulk density at the outlet, and ρ_{bmp} is the bulk density at a location inside the hopper where the pressure gradient is equal to zero (*i.e.*, the gas pressure is at a minimum). A value of ρ_{bmp} equal to its bulk density at the solids stress at the cylinder-hopper junction is often used for design purposes [Johanson, K., "Successfully Dealing with Erratic Flow Rates", *Powder Pointers*, 3, A (2009)]. The solids stress at the cylinder-hopper junction can be calculated using the Janssen equation:

$$\sigma_1 = \frac{\rho_b g R_H}{k\tan\phi'}\left[1 - \exp\left(\frac{-k\tan\phi'}{R_H}z\right)\right] \tag{20}$$

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1 Frequently, Darcy's law is expressed as $u_g = -\frac{k}{\eta}\frac{dP}{dz}$ where η is the gas viscosity and k is a permeability constant with units of area squared. K and k are related by: $K = \frac{\rho_b g}{\eta}k$. Jenike's definition (Equation 18) is useful because K is theoretically equal to a powder's minimum fluidization velocity.

where R_H = the hydraulic radius of the cylinder, z is the Janssen equation (typically assumed equal to 0.4), and z is the height of bulk material inside the cylinder. Because there is a peak stress at the hopper-cylinder junction and since the solids stress decreases as the powder dilates as it flows downward, the bulk density where the gas pressure gradient is zero is not dramatically different from that at the peak stress.

Noting that $v_o = \dot{m}/(\rho_{bo}A_o)$ and combining Equations 17 and 19 yields a quadratic:

$$\left[\frac{2(m+1)\tan\theta'}{Bg}\right]v_o^2 + \left[\frac{1}{K_o}\left(1-\frac{\rho_{bo}}{\rho_{bmp}}\right)\right]v_0 - 1 = 0 \qquad (21)$$

Solution of Equation 21 also requires values of K and ρ_b at the hopper outlet. At the minimum discharge rate, the solids stress is equal to zero. Because the powder is somewhat aerated at the free boundary of the powder, it is prudent to assume that the bulk density is approximately 80 percent of the powder's minimum bulk density. The permeability of the powder depends on its bulk density and its relationship can be described by the following equation (Gu *et al.*, *Powder Techn.,* 72, 39 (1992)):

$$K_o = K_{\min}\left(\frac{\rho_{b\min}}{\rho_b}\right)^a \qquad (22)$$

where K_{min} is the permeability of the powder at its minimum bulk density ρ_{bmin} and a is an empirical constant determined by regression.

The results of the analysis were compared to published data given by Gu *et al.* (see Figure 4). The model tends to under-predict actual discharge rates, which is acceptable for design purposes. Note that the powders were non-cohesive. The analysis is valid if the size of the outlet is significantly greater than the critical arching dimension, *i.e.*, $B >> f_C/\rho_b g$ where f_C is the material's cohesive strength at zero solids stress.

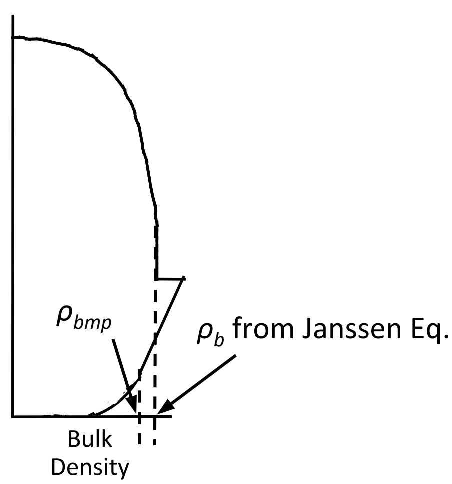

Figure 3. Comparison of ρ_{bmp} and the bulk density at the cylinder-hopper junction.

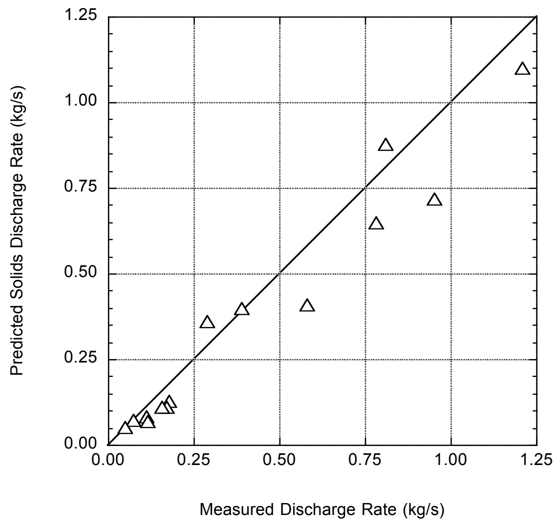

Figure 4. Comparison of measured and predicted solids discharge rates.

The relaxation of the boundary condition at the outlet suggests that the discharge rate calculated from Equation 21 is unsteady. A steady discharge rate can be expected when the pressure and gravity forces are equal. Hence, using the minimum bulk density for ρ_b and using the value of K obtained at the powder's minimum bulk density is a conservative means for determining the size of a hopper outlet required to achieve a desired steady solids discharge rate.