Maximum Solids Discharge Rates from Hoppers

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Shear cell testers can be used to measure the fundamental flow properties of bulk materials, which are cohesive strength, internal friction, compressibility, and wall friction. From the test results, Andrew Jenike's analyses [1] can be used to design reliable hoppers, bins, and silos that will not have obstructions to flow, *i.e.*, arches and stable ratholes, or to predict the solids flow behavior in existing equipment.

Not only must the outlet of a hopper be large enough to prevent an obstruction to flow, it must also be sized to allow the desired discharge rate. It is well known that the velocity of a fluid through an orifice at the bottom of a tank is proportional to the square root of the depth h:

$$v = \sqrt{2gh} \tag{1}$$

where v is the velocity and g is acceleration due to gravity. For solids, stresses are not proportional to the depth but instead are proportional to the outlet dimension B. Therefore, one would expect:

$$v \propto \sqrt{gB} \tag{2}$$

Since the cross-sectional area A of a round outlet is proportional to the diameter B squared, the discharge rate from a conical hopper can be expected to be proportional to the outlet diameter to the 5/2 power, *i.e.*,

$$\dot{m}_s = A\rho_b v \propto B^2\sqrt{gB} \propto \sqrt{g}B^{\frac{5}{2}} \tag{3}$$

Experience has shown that this is not quite true, and several investigators therefore suggest using an effective outlet diameter. The most common formula for calculating discharge rates is the Beverloo equation [2]:

$$\dot{m}_s = C\rho_{bo}g^{\frac{1}{2}}(B - kd_p)^{\frac{5}{2}} \tag{4}$$

where C and k are empirical parameters. Such an approach is philosophically unsatisfying, especially since a similar formula based on engineering fundamentals can be derived.

If only inertial and gravitational forces are included, a force balance on a bulk solid in a converging hopper is

$$a = -g \tag{5}$$

where a is the acceleration of the solids. Defining time and spatial coordinates t and z, respectively, and employing some calculus gives

$$a = \frac{dv}{dt} = \frac{dz}{dt}\frac{dv}{dz} = v\frac{dv}{dz} \tag{6}$$

Equation 5 can then be rewritten as

$$v\frac{dv}{dz} = -g \tag{7}$$

From continuity (and assuming a constant bulk density),

$$\frac{d}{dz}(Av) = v\frac{dA}{dz} + A\frac{dv}{dz} = 0 \tag{8}$$

$$\frac{dv}{dz} = -\frac{1}{A}\frac{dA}{dz} \tag{9}$$

Substitution of Equation 9 into Equation 7 gives

$$\frac{v^2}{A}\frac{dA}{dz} = g \tag{10}$$

Now consider a conical hopper with walls sloped at an angle θ' from vertical (see Figure 1). For a conical hopper with a circular outlet,

$$A = \pi(z\tan\theta')^2 \tag{11}$$

$$\frac{dA}{dz} = 2\pi z\tan\theta' \tag{12}$$

$$A_o = \frac{\pi B^2}{4} \tag{13}$$

$$\left.\frac{1}{A_O}\frac{dA}{dz}\right|_o = \frac{4\tan\theta'}{B} \tag{14}$$

where the subscript o denotes the hopper outlet. Then

$$\frac{4v_o^2\tan\theta'}{B} = g \tag{15}$$

and solving for v_o gives

$$v_o = \sqrt{\frac{Bg}{4\tan\theta'}} \tag{16}$$

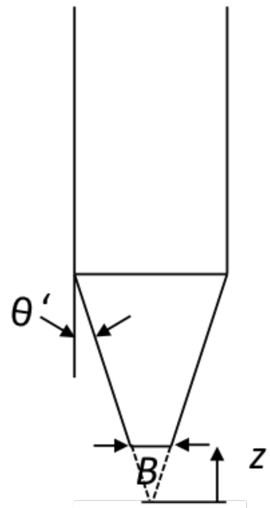

Figure 1. Hopper geometry.

The mass discharge rate $\dot{m}_s$ is equal to the product of the velocity, bulk density ρ_b, and cross-sectional area of the outlet:

$$\dot{m}_s = \rho_{bo} \frac{\pi B^2}{4} \sqrt{\frac{Bg}{4 \tan \theta'}} \tag{17}$$

which is consistent with the expected $B^{5/2}$ relationship between the outlet diameter and the solids mass discharge rate. Comparison of Equations 4 and Equation 17 suggests that

$$C \approx \frac{\pi}{4} \sqrt{\frac{1}{4 \tan \theta'}} \tag{18}$$

For a flow channel angle of 20°, Equation 18 gives a value of C equal to 0.65, which is comparable to the value of 0.58 in Beverloo's original paper.

For hoppers with slotted outlets having an outlet width equal to B, a similar analysis gives

$$v_o = \sqrt{\frac{Bg}{2 \tan \theta'}} \tag{19}$$

Therefore, in general

$$v_o = \sqrt{\frac{Bg}{2(m+1) \tan \theta'}} \tag{20}$$

and

$$\dot{m}_s = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1) \tan \theta'}} \tag{21}$$

where B is the diameter of a round outlet or the width of a slotted outlet, and m is equal to 1 for a circular opening and 0 for a slotted outlet.

Equations 20 and 21 do not account for the cohesive strength of the bulk solid. Jerry Johanson [3] included a cohesive strength term in his force balance:

$$-\frac{a}{g} = 1 - \frac{(m+1) f_C}{\rho_b g B} \tag{22}$$

from which the following can be derived:

$$\frac{2(m+1)\tan\theta'}{Bg} v_o^2 = 1 - \frac{(m+1) f_C}{\rho_b g B} \tag{23}$$

Johanson elegantly recast Equation 23 as

$$\frac{2(m+1)\tan\theta'}{B} v_o^2 = g\left(1 - \frac{ff}{ff_a}\right) \tag{24}$$

where ff is the flow factor (the ratio of the major principal stress (σ_1) to the stress on the abutments of an arch) and ff_a is the actual flow function defined by

$$ff_a = \frac{\sigma_{1o}}{f_C} \tag{25}$$

where the solids stress of the outlet σ_{1o} was calculated from

$$\sigma_{1o} = ff \frac{\rho_{bo} g B}{m+1} \tag{26}$$

Figure 2 plots the flow factor as a function of δ, which is valid for wall friction angles greater than about 12° and hopper angles in the neighborhood of the mass flow boundary [4].

Following the same steps as before yields

$$v_o = \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\right)} \tag{27}$$

and

$$\dot{m}_s = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\right)} \tag{28}$$

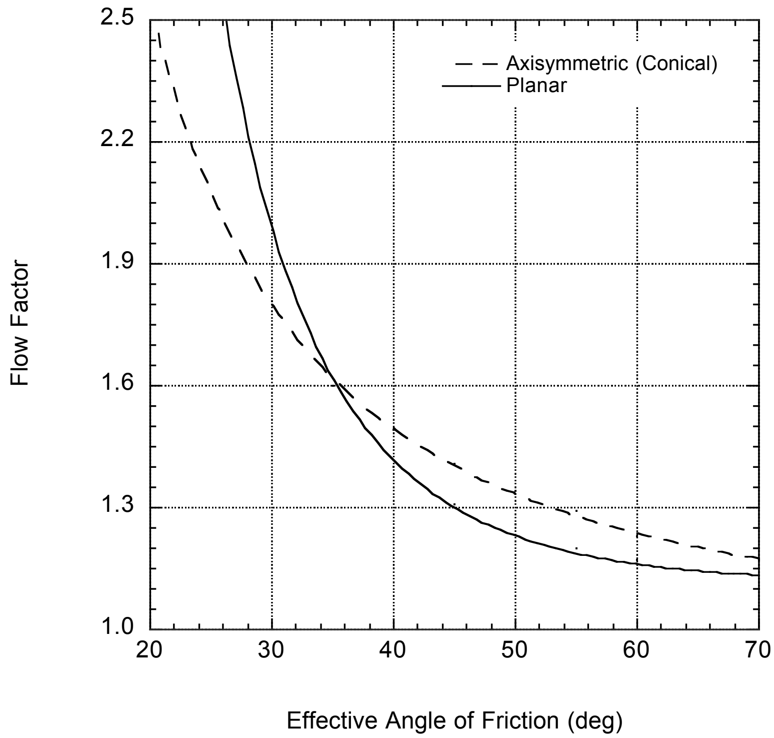

Figure 2. Johanson's flow factor.

Equation 28 is called the Johanson equation. The Johanson equation can be used to determine the size of a hopper outlet required to provide the desired discharge rate of a coarse, cohesive bulk solid. It is similar to the Beverloo equation, but it was derived from first principles. In the case of a conical hopper, comparison of Equations 4 and 28 suggests that the term kd_p is related to the powder's cohesive strength.

Johanson assumed that the angle of the slope of the failing arch was equal to 45°. Jenike [5] noted that its angle is equal to $\beta + \phi'$, where

$$\beta = \frac{1}{2}\left[\phi' + \sin^{-1}\left(\frac{\sin\phi'}{\sin\delta}\right)\right] \tag{29}$$

and ϕ' is the wall friction angle. In addition, Jenike [5] modified Equation 25 to account for the non-uniformity of the arch:

$$\sigma_{1o} = ff\,\frac{\rho_{bo} g B}{H(\theta')} \tag{30}$$

where $H(\theta')$ is a function defined by Jenike [1] and is shown in Figure 3. For a hopper with walls sloped at an angle equal to θ', Equations 22 and 23 can therefore be rewritten as

$$-\frac{a}{g} = 1 - \frac{2(m+1)\cos\theta'\sin(\beta+\theta')f_C}{\rho_b g B} \tag{31}$$

and

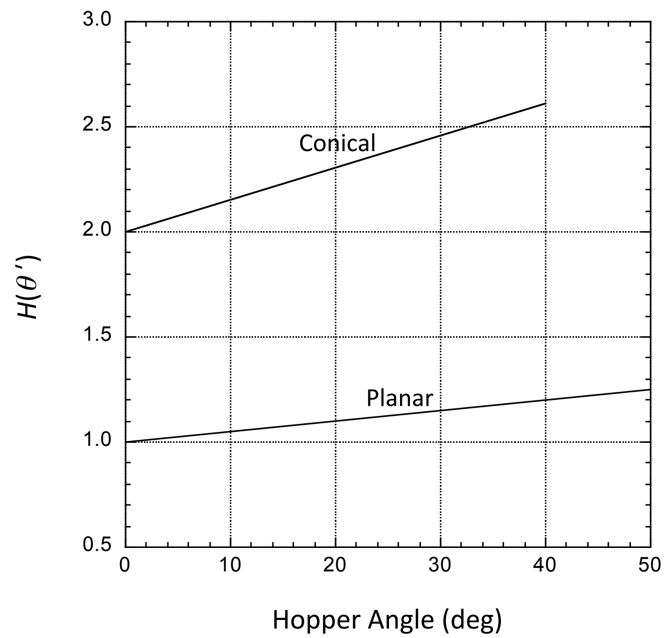

Figure 3. Jenike's geometry function H(θ').

$$\frac{2(m+1)\tan\theta'}{Bg}v_o^2 = 1 - \frac{2(m+1)\cos\theta'\sin(\beta+\theta')f_C}{\rho_b g B} \quad (32)$$

respectively.

For fine powders, gas-phase effects cannot be neglected, and a pressure gradient term should be included in the force balance:

$$\frac{2(m+1)\tan\theta'}{Bg}v_o^2 = 1 - \frac{2(m+1)\cos\theta'\sin(\beta+\theta')f_C}{\rho_b g B} + \frac{1}{\rho_{bo} g}\left.\frac{dP}{dz}\right|_o \quad (33)$$

Flow of gas through a bed of material is described by Darcy's Law:

$$u = -\frac{K}{\rho_b g}\frac{dP}{dz} \quad (34)$$

where u is the gas slip velocity, P is the interstitial gas pressure and K is the permeability. The permeability of a bulk material can easily be determined by measuring the pressure drop that results when a gas passes through a bed of solids. From continuity of the gas and solids, Gu [6] derived the following relationship between the solids velocity and the gas slip velocity:

$$u = v_o \rho_{bo}\left(\frac{1}{\rho_{bmp}} - \frac{1}{\rho_{bo}}\right) \quad (35)$$

where the subscript *mp* denotes the location where the interstitial gas pressure is at a minimum and the pressure gradient is zero. The pressure gradient is therefore related to the solids velocity by:

$$\frac{dP}{dz} = \frac{v_o \rho_{bo}^2 g}{K_o}\left(\frac{1}{\rho_{bmp}} - \frac{1}{\rho_{bo}}\right) \tag{36}$$

Substitution of Equation 36 into Equation 33 yields the following quadratic:

$$\left[\frac{2(m+1)\tan\theta'}{Bg}\right]v_o^2 + \left[\frac{1}{K_o}\left(1 - \frac{\rho_{bo}}{\rho_{bmp}}\right)\right]v_o + \frac{2(m+1)\cos\theta'\sin(\beta+\theta')f_C}{\rho_b g B} - 1 = 0 \tag{37}$$

from which the solids discharge rate can be calculated from

$$\dot{m}_s = \rho_{bo} A_o v_o \tag{38}$$

The solids stress where the gas pressure is at a minimum is difficult to calculate. Kerry Johanson [7] noted that it is approximately equal to the maximum solids stress σ_1 in the cylinder section, which can be calculated from the Janssen equation:

$$\sigma_1 = \frac{\bar{\rho}_b g R_H}{k\tan\phi'}\left[1 - \exp\left(\frac{-k\tan\phi'}{R_H}z\right)\right] \tag{39}$$

where R_H is the hydraulic radius, k is the ratio of the horizontal and vertical solids stresses in the cylinder (*i.e.,* the Janssen coefficient, which is equal to approximately 0.4), and z is the depth of solids in the cylinder. If the depth of solids in the cylinder section is low, the maximum solids stress can be estimated from

$$\sigma_1 = \frac{\rho_b g D}{(m+1)\tan\theta'} \tag{40}$$

where D is the diameter of the cylinder. The solids stress at the outlet is determined from

$$\sigma_{1o} = ff\,\frac{\left(\rho_{bo}g + \left.\frac{dP}{dz}\right|_o\right)B}{H(\theta')} \tag{41}$$

The solids velocity is calculated by first estimating the solids stress at the outlet and then using that value to calculate the bulk density, permeability, and unconfined yield strength at the outlet. The outlet solids velocity v_o is then calculated by solving Equation 37. Knowing the velocity allows the pressure gradient to be calculated from Equation 36. An updated value of the solids stress at the outlet can then be calculated from Equation 41. The calculations are repeated until the correct value of σ_{1o} is found. The solids mass discharge rate is the product of the velocity, cross-sectional area, and bulk density at the solids stress at the outlet.

Figure 4 compares solids discharge rates measured by Gu [8] with those predicted from Equations 37 and 38. The author provided the relationships between the fundamental flow properties (cohesive strength, internal friction, wall friction, compressibility, and permeability) and solids stress for ten powders and measured solids discharge rates from conical hoppers filled to various depths. The agreement is acceptable for design purposes, although a safety factor of, say, 20 percent should be employed if specifying the size of a hopper outlet that will provide the desired maximum steady solids discharge rate is critical.

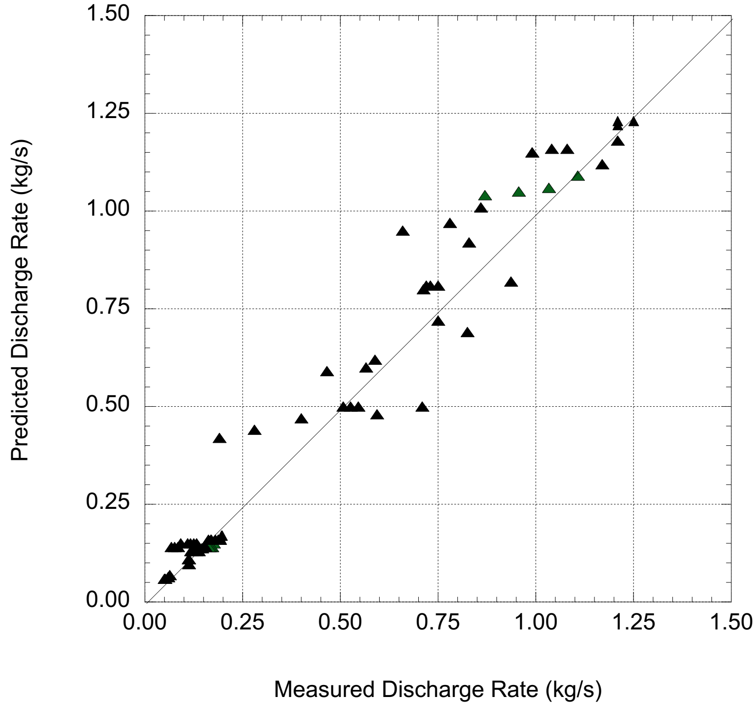

Figure 4. Comparison of observed and predicted solids discharge rates.

In the case of funnel flow, where the hopper walls are not steep enough given the wall friction to allow flow along the walls, the flow channel angle should be used in place of θ'. The flow channel angle θ_{fc} can be estimated from a relation adapted from Arnold [9]:

$$\theta_{fc} = \left[45° - 0.5\cos^{-1}\left(\frac{1-\sin\delta}{2\sin\delta}\right)\right]^{m}\left[65° - 0.5\cos^{-1}\left(\frac{1-\sin\delta}{\sin\delta}\right)\right]^{m-1} \quad (38)$$

By measuring a bulk material's fundamental flow properties, *i.e.*, its unconfined yield strength, internal friction, compressibility, wall friction, and permeability, the size of the outlet of a hopper, bin or silo required to allow the desired discharge rate can be specified. Likewise, whether or not an existing storage vessel can meet the solids discharge rate requirements can be determined.

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