Beyond Beverloo: Prediction of Solids Discharge Rates from Hoppers

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***Abstract*—The Johanson equation, which predicts the discharge rate of coarse solids from hoppers, was modified to account for the adverse pressure gradient that can develop when fine powders are handled. The modified equation agrees with published data.**

Keywords: discharge rate, powder, hopper.

Andrew Jenike's pioneering work on bulk solids handling fundamentals [1,2] allows designers of hoppers to specify minimum outlet dimensions that prevent obstructions to flow. However, the calculations do not reveal if a hopper outlet will provide the desired solids discharge rate.

The most common formula for calculating discharge rates from hoppers is the Beverloo equation [3]:

$$\dot{m}_s = C\rho_b g^{\frac{1}{2}}\left(B - kd_p\right)^{\frac{5}{2}} \tag{1}$$

where d_p is the particle diameter, ρ_b is the bulk density (at the hopper outlet), g is acceleration due to gravity, and C and k are empirical parameters. An empirical equation is philosophically unsatisfying, especially since a similar formula based on engineering fundamentals can be derived.

When deriving his formula for calculating solids discharge rates from hoppers, Jerry Johanson [4] began with a force balance:

$$-\frac{a}{g} = 1 - \frac{(m+1)f_C}{\rho_b g B} \tag{2}$$

where a is the acceleration of the solids, m = 0 or 1 for an elongated or round outlet, respectively, B is the width or diameter of the hopper outlet, and f_C is the unconfined yield strength. It can be shown that [4]

$$a = -\frac{2(m+1)v_o^2 \tan\theta'}{B} \tag{3}$$

where θ' is the hopper angle referenced from vertical and v_o is the solids velocity at the outlet. Equation 2 can then be rewritten as

$$\frac{2(m+1)\tan\theta'}{Bg}v_o^2 = 1 - \frac{(m+1)f_C}{\rho_{bo} g B} \tag{4}$$

Johanson elegantly recast Equation 4 as

$$\frac{2(m+1)\tan\theta'}{B}v_o^2 = g\left(1 - \frac{ff}{ff_a}\right) \tag{5}$$

where ff is the flow factor (the ratio of the major principal stress (σ_1) to the stress on the abutments of an arch) and ff_a is the actual flow function defined by

$$ff_a = \frac{\sigma_{1o}}{f_C} \tag{6}$$

where the solids stress of the outlet σ_{1o} was calculated from

$$\sigma_{1o} = ff\frac{\rho_{bo} g B}{m+1} \tag{7}$$

Figure 1 plots the flow factor as a function of the effective angle of friction δ, which is valid for wall friction angles (ϕ') greater than about 12° and hopper angles in the neighbourhood of the mass flow boundary [5].

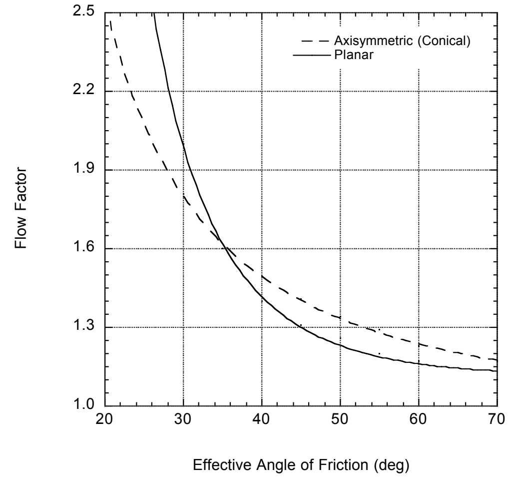

Figure 1. Johanson's flow factor [5].

Solving Equation 5 for v_o gives

$$v_o = \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\right)} \tag{9}$$

The mass discharge rate $\dot{m}_s$ is the product of the solids velocity and the cross-sectional area of the hopper outlet A_o:

$$\dot{m}_s = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\right)} \tag{10}$$

Equation 10 is called the Johanson equation. The Johanson equation can be used to determine the size of a hopper outlet required to provide the desired discharge rate of a coarse, cohesive bulk solid. It is similar to the Beverloo equation, but it was derived from first principles. Comparison of Equations 1 and 10 suggests that the term C is related to the hopper geometry and kd_p is related to the powder's cohesive strength.

Johanson assumed that the angle of the slope of the failing arch was equal to 45°. Jenike [1] noted that its angle is equal to $\beta + \theta'$, where

$$\beta = \frac{1}{2}\left[\phi' + \sin^{-1}\left(\frac{\sin\phi'}{\sin\delta}\right)\right] \tag{11}$$

Hence,

$$\frac{ff}{ff_a} = \frac{2(m+1)\cos\theta'\sin(\beta+\theta')f_C}{\rho_{bo} g B} \tag{12}$$

In addition, Jenike [1] modified Equation 7 to account for the non-uniformity of the arch:

$$\sigma_{1o} = ff\frac{\rho_{bo} g B}{H(\theta')} \tag{13}$$

where $H(\theta')$ is a function defined by Jenike [1,2] approximately equal to 1 for a planar hopper with a slotted outlet and 2 for a conical hopper.

For fine powders, gas-phase effects cannot be neglected as vacuum develops inside the hopper, and therefore a pressure gradient term should be included in the force balance:

$$\frac{2(m+1)\tan\theta'}{Bg} v_o^2 = 1 - \frac{ff}{ff_a} + \frac{1}{\rho_{bo} g}\left.\frac{dP}{dz}\right|_o \qquad (14)$$

where dP/dz is the interstitial gas pressure gradient.

Darcy's Law describes flow of gas through a bed of material:

$$u = -\frac{K}{\rho_b g}\frac{dP}{dz} \qquad (15)$$

where u is the gas slip velocity and K is the permeability. From continuity of the gas and solids, Gu [6] derived the following relationship between the solids velocity and the gas slip velocity:

$$u = v_o \rho_{bo}\left(\frac{1}{\rho_{bmp}} - \frac{1}{\rho_{bo}}\right) \qquad (16)$$

where the subscript mp denotes the location where the interstitial gas pressure is at a minimum and the pressure gradient is zero. The pressure gradient is therefore related to the solids velocity by:

$$\frac{dP}{dz} = \frac{v_o \rho_{bo}^2 g}{K_o}\left(\frac{1}{\rho_{bmp}} - \frac{1}{\rho_{bo}}\right) \qquad (17)$$

Substitution of Equation 17 into Equation 14 yields the following quadratic formula:

$$\left[\frac{2(m+1)\tan\theta'}{Bg}\right] v_o^2 + \left[\frac{1}{K_o}\left(1 - \frac{\rho_{bo}}{\rho_{bmp}}\right)\right] v_o + \frac{ff}{ff_a} - 1 = 0 \qquad (18)$$

from which the solids discharge rate can be calculated from

$$\dot{m}_s = \rho_{bo} A_o v_o \qquad (19)$$

The solids stress where the gas pressure is at a minimum is rather cumbersome to calculate. Kerry Johanson [7] noted that it is approximately equal to the maximum solids stress σ_1 in the cylinder section, which can be calculated from the Janssen equation:

$$\sigma_1 = \frac{\bar{\rho}_b g R_H}{k\tan\phi'}\left[1 - \exp\left(\frac{-k\tan\phi'}{R_H} z\right)\right] \qquad (20)$$

where $\bar{\rho}_b$ is the average bulk density, R_H is the hydraulic radius, k is the ratio of the horizontal and vertical solids stresses in the cylinder (approx. equal to 0.4), and z is the depth of solids in the cylinder. If the level of solids in the cylinder section is low, the maximum solids stress can be estimated from

$$\sigma_1 = \frac{\rho_b g D}{(m+1)\tan\theta'} \qquad (21)$$

where D is the diameter or diagonal of the cylinder. The solids stress at the outlet is determined from

$$\sigma_{1o} = ff\,\frac{\left(\rho_{bo} g + \left.\frac{dP}{dz}\right|_o\right) B}{H(\theta')} \qquad (22)$$

The solids velocity is calculated by first estimating the solids stress at the outlet and then using that value to calculate the bulk density, permeability, and unconfined yield strength at the outlet. The outlet solids velocity v_o can then be calculated by solving Equation 18. Knowing the velocity allows the pressure gradient to be calculated from Equation 17. An updated value of the solids stress at the outlet can then be calculated from Equation 22. The calculations are repeated until the correct value of σ_{1o} is found. The solids mass discharge rate is the product of the velocity, cross-sectional area, and bulk density at the solids stress at the outlet.

Figure 2 compares solids discharge rates measured by Gu [8] with those predicted from Equations 18, 21, and 22. The author provided the relationships between the fundamental flow properties (cohesive strength, internal friction, wall friction, compressibility, and permeability) and solids stress for ten powders and measured solids discharge rates from conical hoppers filled to various depths. The hopper consisted of a 15° (from vertical) cone with a 44.5- or 20-mm diameter outlet and a 145-mm diameter cylinder. The wall material of the hopper was polished galvanized steel.

The agreement is acceptable for design purposes, although a safety factor of, say, 25 per cent should be employed if specifying the size of a hopper outlet that will provide the desired maximum steady solids discharge rate is critical.

By measuring a bulk material's fundamental flow properties, *i.e.*, its unconfined yield strength, internal friction, compressibility, wall friction, and permeability, the size of the outlet of a hopper required to allow the desired discharge rate can be specified. While the solids discharge rate data were obtained from experiments on a small-scale hopper, the model, which is based on fundamental principles, can be expected to provide a reasonable estimate for larger hoppers, bin, and silos.

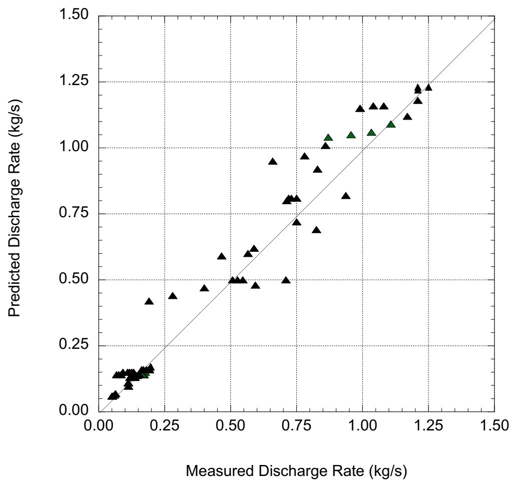

Figure 2. Comparison of observed and predicted solids discharge rates – Gu data [8].

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