## Solids Discharge Rates

Not only must the outlet of a hopper be large enough to prevent an arch, it should also be sized to allow the desired discharge rate. It is well known that the velocity of a fluid through an orifice at the bottom of a tank is proportional to the square root of the depth:

$$v = \sqrt{2gh} \tag{1}$$

Nomenclature is given in Table 1. For solids, stresses are not proportional to the depth; instead they are proportional to the outlet diameter, and therefore one would expect:

$$v \propto \sqrt{gB} \tag{2}$$

Since the cross-sectional area of a round outlet is proportional to the diameter squared, a discharge rate proportional to the outlet diameter to the 5/2 power can be expected, *i.e.*,

$$\dot{m}_s = A\rho_b v \propto B^2\sqrt{gB} \propto \sqrt{g}B^{\frac{5}{2}} \tag{3}$$

Experience has shown that this is not quite true, and several investigators therefore suggest using an effective outlet diameter. The most common formula for calculating discharge rates is the Beverloo Equation:

$$\dot{m}_s = C\rho_{bo}g^{\frac{1}{2}}(B - kd_p)^{\frac{5}{2}} \tag{4}$$

The Beverloo equation [1] has two empirical parameters. Such an approach is philosophically unsatisfying, especially since a formula based on engineering fundamentals can readily be derived.

If only inertial and gravitational forces are included, a force balance on a bulk solid in a converging hopper yields

$$a = -g \tag{5}$$

Employing some calculus:

$$a = \frac{dv}{dt} = \frac{dz}{dt}\frac{dv}{dz} = v\frac{dv}{dz} \tag{6}$$

Equation 5 can be rewritten as

$$v\frac{dv}{dz} = -g \tag{7}$$

From continuity,

$$\frac{d(Av)}{dz} = 0 \tag{8}$$

and therefore

$$\frac{dv}{dz} = -\frac{v}{A}\frac{dA}{dz} \tag{9}$$

Substitution of Equation 9 into Equation 7 gives

$$\frac{v^2}{A}\frac{dA}{dz} = g \tag{10}$$

For circular outlets (see Figure 1),

$$A = \pi(z\tan\theta')^2 \tag{11}$$

$$\frac{dA}{dz} = 2\pi z\tan^2\theta' \tag{12}$$

$$A_o = \frac{\pi B^2}{4} \tag{13}$$

$$\left.\frac{dA}{dz}\right|_o = \frac{\pi B}{\tan\theta'} \tag{14}$$

Therefore, at the outlet,

$$\frac{4v_o^2\tan\theta'}{B} = g \tag{15}$$

Solving for $v_o$ gives

$$v_o = \sqrt{\frac{Bg}{4\tan\theta'}} \tag{16}$$

The mass discharge rate is equal to the product of the velocity, bulk density, and cross-sectional area of the outlet:

$$\dot{m}_s = \rho_{bo}\frac{\pi B^2}{4}\sqrt{\frac{Bg}{4\tan\theta'}} \tag{17}$$

which shows a $B^{5/2}$ relationship between the outlet diameter and the solids mass discharge rate. Beverloo's original work was based on measurements of solids discharge rates from flat-bottomed silos with round outlets. Comparison of Equations 2 and Equation 17 suggests that

$$C = \frac{\pi}{4}\sqrt{\frac{1}{4\tan\theta'}} \tag{18}$$

For a flow channel angle of 20°, Equation 18 gives a value of $C$ equal to 0.65, which is comparable to the value of 0.58 in Beverloo's original paper.

For hoppers with slotted outlets, a similar analysis gives

$$v_o = \sqrt{\frac{Bg}{2\tan\theta'}} \tag{19}$$

and therefore in general,

$$v_o = \sqrt{\frac{Bg}{2(m+1)\tan\theta'}} \tag{20}$$

and

$$\dot{m}_s = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1)\tan\theta'}} \tag{21}$$

where $m$ is equal to 1 for a circular opening and equal to 0 for a slotted outlet.

Equations 20 and 21 do not account for the cohesive strength of the bulk solid. Johanson [2] included cohesive strength in his force balance:

$$\frac{2(m+1)\tan\theta'}{B} v_o^2 = g\left(1 - \frac{f_C(1+m)}{\rho_b g B}\right) \tag{22}$$

Johanson noted that

$$\frac{f_C(1+m)}{\rho_b g B} = \frac{ff}{ff_a} \tag{23}$$

where

$$ff_a = \frac{\sigma_{1o}}{f_C} \tag{24}$$

and Equation 22 could be elegantly rewritten as

$$\frac{2(m+1)\tan\theta'}{B} v_o^2 = g\left(1 - \frac{ff}{ff_a}\right) \tag{25}$$

The unconfined yield strength is determined by the bulk material's flow function at a stress at the outlet, which according to Johanson was equal to

$$\sigma_{1o} = ff \frac{\rho_{bo} g B}{m+1} \tag{26}$$

Following the same steps as before yields

$$v_o = \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\right)} \tag{27}$$

and

$$\dot{m}_s = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\right)} \tag{28}$$

Equation 29 is called the Johanson equation. Note that when $B$ is equal to the critical arching outlet dimension, the actual flow factor is equal to the critical flow factor, and the solids discharge velocity is zero. The Johanson equation can be used to determine the size of a hopper outlet required to provide the desired discharge rate of a course, cohesive bulk solid. It is similar to the Beverloo equation, but it was derived from first principles.

Johanson assumed that the hopper angle and the angle of the slope of the failing arch were equal to 45°. A more rigorous analysis yields the following force balance:

$$\frac{2(m+1)\tan\theta'}{B} v_o^2 = g\left(1 - \frac{ff \sin\beta \cos\theta'}{ff_a}\right) \tag{29}$$

where

$$\beta = \frac{1}{2}\left[\phi' + \sin^{-1}\left(\frac{\sin\phi'}{\sin\delta}\right)\right] \tag{30}$$

and Equations 25 and 26 can be rewritten as

$$v_o = \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\sin\beta\cos\theta'\right)} \tag{31}$$

and

$$\dot{m}_s = \rho_{bo} A_o \sqrt{\frac{Bg}{2(m+1)\tan\theta'}\left(1 - \frac{ff}{ff_a}\sin\beta\cos\theta'\right)} \tag{32}$$

Following Jenike [3], the solids stress at the hopper outlet should be calculated using Equation 33:

$$\sigma_{1o} = ff\,\frac{\rho_{bo} g B}{H(\theta')} \tag{33}$$

For fine powders, gas-phase effects are not negligible, and a pressure gradient term must be added to the force balance:

$$\frac{2(m+1)\tan\theta'}{Bg} v_o^2 = 1 - \frac{ff \sin\beta\cos\theta'}{ff_a} + \frac{1}{\rho_{bo} g}\left.\frac{dP}{dz}\right|_o \tag{34}$$

Flow of gas through a bed of material is described by Darcy's Law:

$$u = -\frac{K}{\rho_b g}\frac{dP}{dz} \tag{35}$$

From continuity, it can be shown that the pressure gradient is related to the solids velocity by [4]

$$\frac{dP}{dz} = \frac{v_o \rho_{bo}^2 g}{K_o}\left(\frac{1}{\rho_{bmp}} - \frac{1}{\rho_{bo}}\right) \tag{36}$$

Substitution of Equation 33 into Equation 31 yields the following quadratic:

$$\left[\frac{2(m+1)\tan\theta'}{Bg}\right]v_o^2 + \left[\frac{1}{K_o}\left(1 - \frac{\rho_{bo}}{\rho_{bmp}}\right)\right]v_o + \frac{ff}{ff_a}\sin\beta\cos\theta' - 1 = 0 \tag{37}$$

from which the solids discharge rate can be calculated from

$$\dot{m}_s = \rho_{bo} A_o v_o \tag{38}$$

The solids stress where the gas pressure is at a minimum is difficult to calculate. Gu *et al*. [4] noted that it is approximately equal to the maximum solids stress in the cylinder section, which can be calculated from the Janssen equation.

$$\sigma_1 = \frac{\bar{\rho}_b g R_H}{k\tan\phi'}\left[1 - \exp\left(\frac{-k\tan\phi'}{R_H}z\right)\right] \tag{39}$$

The solids stress at the outlet is determined from

$$\sigma_{1o} = ff\,\frac{\left(\rho_{bo} g + \left.\frac{dP}{dz}\right|_o\right)B}{H(\theta')} \tag{40}$$

By measuring a bulk material's fundamental flow properties, *i.e.*, its unconfined yield strength, internal friction, compressibility, wall friction, and permeability, the size of the outlet of a hopper, bin or silo can be determined. Knowing that the design equations are based on engineering fundamentals, an engineer can be confident in his or her design.

## References


1. Beverloo, W.A., H.A. Leniger, and T. Van de Velde, *Chem. Engr. Sci.,* 15 (1961), 260.
2. Johanson, J., "Method of Calculating Rate of Discharge from Hoppers and Bins", *Trans. SME,* 232 (March 1965).
3. Jenike, A., "Storage and Flow of Solids – Bulletin 123", University of Utah, Salt Lake City, 1964.
4. Gu., Z.H., P.C. Arnold, and A.G. McLean, "Prediction of the Flowrate of Bulk Solids from Mass Flow Bins with Conical Hoppers", *Powder Techn.* 72 (1992), 157.
5. Gu., Z.H., P.C. Arnold, and A.G. McLean, "Modeling of Air Pressure Distributions in Mass Flow Bins", *Powder Techn.* 72 (1992), 121.

**Table 1**
**Nomenclature**

| | |
|---|---|
| $A$ | area |
| $A_o$ | outlet cross-sectional area |
| $B$ | outlet diameter or width |
| $C$ | Beverloo equation empirical parameter |
| $d_P$ | particle diameter |
| $f_C$ | unconfined yield strength |
| $ff$ | critical flow factor |
| $ff_a$ | actual flow factor |
| $g$ | acceleration due to gravity |
| $h$ | depth |
| $H(\theta')$ | Jenike's geometry function |
| $k$ | Beverloo empirical constant or Janssen coefficient |
| $K$ | permeability |
| $m$ | constant = 1 for a round outlet, = 0 for a slotted outlet |
| $\dot{m}_s$ | solids mass discharge rate |
| $P$ | gas pressure |
| $R_H$ | hydraulic radius |
| $u$ | gas velocity |
| $v_o$ | solids velocity at outlet |
| $z$ | vertical coordinate |
| $\beta$ | arch angle |
| $\phi'$ | wall friction angle |
| $\theta'$ | hopper angle referenced from vertical |
| $\rho_b$ | bulk density |
| $\rho_{bmp}$ | bulk density in hopper where interstitial gas pressure is at a minimum |
| $\rho_{bo}$ | bulk density at outlet |
| $\sigma_1$ | major principal stress |
| $\sigma_{1o}$ | major principal stress at outlet |

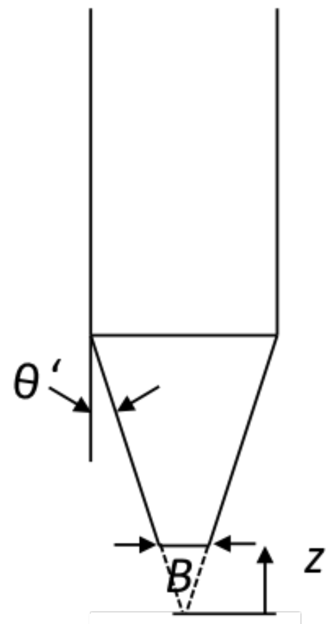


**Figure 1. Hopper geometry.**